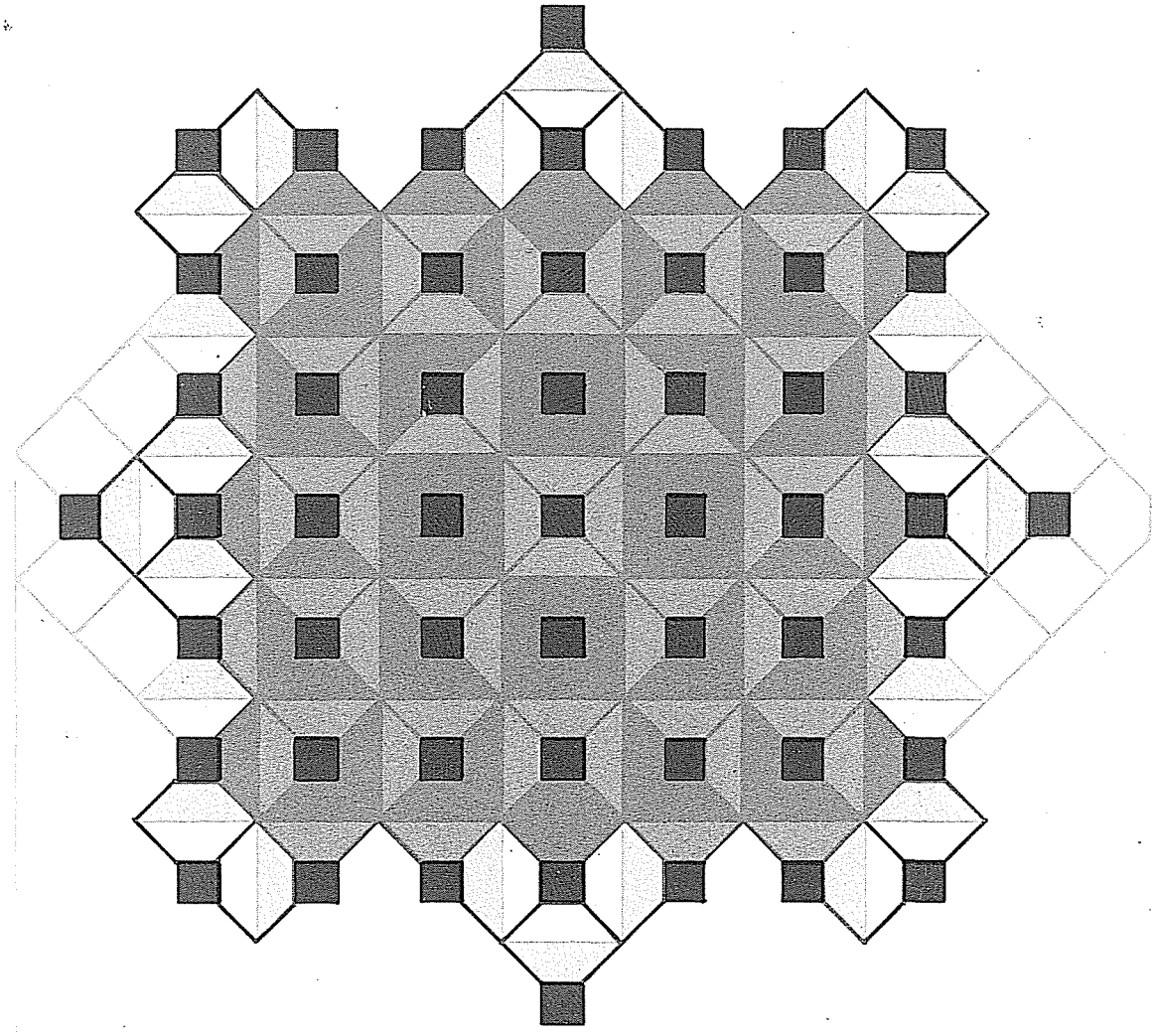
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L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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EDITORIAL

After three years as editor of your magazine, Arn Harris has laid down his editorial pen. During this time he focused our attention on the issues of the day ... from the **Report on Basic Educational Skills** to **Recommendations for Intermediate Division Mathematics in Ontario.** With the special issue on Jean Piaget he reminded us of the necessity to examine primary research as well as to be concerned about the problems of curriculum revision. Although he retired from active teaching in 1976 he still enjoys returning to the classroom, even if it is but for a few hours. Such is the man! He is now looking forward to editing the proceedings of the May Conference. All success to you Arn in this new endeavour. If, in the future, you feel the desire to take up your pen and editorialize once again, we would welcome your contribution.

It is my hope that the good work done in the past will continue. The change of editorship has been relatively painless as the dedicated groups that bring you your magazine have agreed to carry on. For this, I thank them. It is our desire to serve all sectors of the association. Perhaps it is worthwhile to remind ourselves that it is good to read about mathematics and about the work of O.A.M.E./A.O.E.M. but, it is also worthwhile to become a participant. Mathematics is not chiefly a spectator sport. Take pen in hand and ...

By the time this issue reaches you the new guidelines for Intermediate Division Mathematics may have been issued. As these guidelines were the subject of the March issue one year ago, the editorial board suggests we reexamine them in the March issue of 1979. In this issue we have printed a statement adopted by the Mathematical Association of America and the National Council of Teachers of Mathematics because it is felt that the recommendations are highly relevant here in Ontario. Although the report is titled **Preparation of High School Students for College Mathematics Courses**, such recommendations as the following: the need for proficiency in mathematics, the need for cumulative evaluations, the maintenance of proper grading standards, and the dangers of advancement of students without appropriate achievement, might well be considered by all. As you make your summative evaluations in June consider the issues as the September magazine will deal with evaluation.

Hope to see you at the Fifth Annual Conference in May at McMaster University whose colours have been used on the cover all this year. Does anyone wish to join Hugh Allen and me on a jog around the campus? It might help to transform your day.

Bob

LETTERS TO THE EDITOR

Mathematics Department,
Hamilton Board of Education,

Hi Bob 1978 01 26

Here is an editorial which appeared in The Spectator. It is one of the best editorials I have read re: Calculators.

Thought you might like to share this with the "readership"

Bob Robinson

P.S. I put the circulation on for interest. The Spectator is a Hamilton newspaper which gives the "Golden Horseshoe".

The Spectator
1978 01 14

Circulation: 140,000

It all adds up

Pocket calculators, properly used, can take much of the time-consuming drudgery out of mathematics. The Hamilton board of education math department's proposal to allow students to use calculators even during exams, makes sense.

It also makes sense to be certain students have a solid grounding in the mathematical basics — not only to get the most out of their portable computers but to prepare them to cope with mathematical problems they might face in life when equipped with nothing more sophisticated than a pencil and paper.

Using a calculator without knowing mathematics is like using a typewriter without knowing how to write or spell; the machine may be able to improve on human speed and efficiency but it can't compensate for a shortage of knowledge.

Since the calculator has become a useful and popular device in business, industry, the home and in education itself, it's education's function to see that people learn to use the machines properly. The first step is to make certain that young people learn fundamental mathematics.

Otherwise the calculators will be no more useful to students than a cello would be to a bullfrog.

London, Ontario
November 15, 1977

November 15, 1977

Dear Editor,

While musing in the shower the other day I hit upon this gem. You will be pleased to know that I managed to control my excitement to the extent that I did not there and then run through the streets shouting Greek obscenities.

T.J. Griffiths

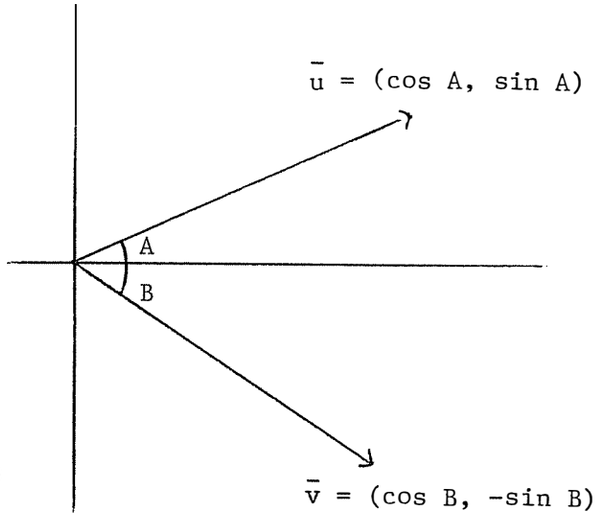
$$\cos\Theta = \frac{\bar{u} \bullet \bar{v}}{|\bar{u}|\,|\bar{v}|}$$

$$|\bar{u}| = \sqrt{\cos^2 A + \sin^2 A} = 1$$

$$|\bar{v}| = \sqrt{\cos^2 B + \sin^2 B} = 1$$

$$\bar{u} \bullet \bar{v} = (\cos A, \sin A) \bullet (\cos B, -\sin B)$$
$$= \cos A \bullet \cos B - \sin A \bullet \sin B$$

$$\cos(A+B) = \cos A \bullet \cos B - \sin A \bullet \sin B$$



Dear Mr. Smith,

VERIFYING YOUR SIN(S)

While visiting my neighbourhood Brewer's Retail Store, the manager presented me with a page from a memo that was sent to him from his Head Office. As the title, "Social Insurance Number Verification" indicates, the memo illustrates the method used to verify the accuracy of a S.I.N.

I found this an interesting arithmetic problem, and I was amused at some of the terminology used, such as "standard modulus 10 check digit formula".

The sentence before the example is an understatement to say the least!

Others may wish to use this exercise as a computational problem in their classroom, or, at least, to verify their own SIN(S)

Yours sincerely,
Patrick A. Gibbons, B. Math, B.Ed.

Editors Note: As we all have a S.I.N. we now have a method to check it.

SOCIAL INSURANCE NUMBER VERIFICATION

The Social Insurance Number contains nine digits. The number consists of eight assigned digits, with the ninth digit being constructed from the first eight and acting as a check digit. The accuracy of transcription of the number may be verified using the standard modulus 10 check digit formula.

Briefly, the check digit is established by considering the unit's digit of the **basic** number and each alternate digit to the left to be a whole number. This number is multiplied by two and the digits in the product and the intervening digits which were not inflated are cross added. The sum is subtracted from the next highest number ending in zero. The difference is the check digit.

For the purpose of verifying the Social Insurance Number, you must consider the **basic** social number which consists of the first eight digits. The example below may better illustrate the verification method.

	Basic #	
Social Insurance Number provided in your employee's record	123 456 78	2 Check Digit
Make a number from units position and each alternate position to the left	2 4 6 8	
Add number to itself	2 4 5 8	
Sum	4 9 3 6	
Cross add the digits in the sum (4+9+3+6) =	22	
Cross add the intervening digits (1+3+5+7) =	16	
Total	38	
Subtract from next highest number ending in zero	40 −38	
Check digit	2	

Dear Professor Harris:

I found your two questions (**Farmer Harris Asks,** the **Gazette**, Sept. 77) highly provocative. My guess on the first one was that the function is linear because linear regression is what everybody thinks of first when he's curve-fitting, and its easy for the farmer to compute! But that's unsatisfactory because it doesn't deal with the question why does a linear function work? I'm sure that an experienced beef farmer has information about where on the animal the weight (between 700 and 1200 lbs) is gained - perhaps the steers being compared are not geometrically similar, even up to a reasonable approximation. (?)

As to the second, my first guess is that the hay bales tend to stay as they land, their distribution then reflecting a bias in the baler, while perhaps the oat straw bales are ''bouncier'' so assume a ''shaken-down'', low energy distribution. But there may be hidden parameters: Is the geometry of the two bales (hay, straw) the same? Are the straw bales ''loaded'' - all stem on one side, head on the other?

I hope you'll enlighten us, eventually - but the fun will have been in the meanwhile!

Sincerely,
L. Terrell Gardner.

Editors Note: We are sorry to report that Farmer Harris is buried in a snowbank somewhere west of London and has not been mathematizing lately. The conjectures above look reasonable. It is true that straw bales are ''bouncier''; however, the distribution of contents is somewhat random.

With regard to the linear function, perhaps we are looking for something too complicated. You may know the story of the little boy who asked his mother ''where he came from''. If you do not, do not expect this magazine to enlighten you.

1978 01 25

Bob Smith,
London, Ont.

Bob,

Correction

Terry Cherry was the coach of the Thornlea S.S. winners of the Junior Math Contest. We apologize for this error.

Thanks.

Neil Williamson

RECOMMENDATIONS FOR THE PREPARATION OF HIGH SCHOOL STUDENTS FOR COLLEGE MATHEMATICS COURSES

The following statement, adopted by the Board of Governors of the Mathematical Association of America and the Board of Directors of the National Council of Teachers of Mathematics, is a brief outline of the basic ingredients of adequate preparation for collegiate level mathematics.* The statement does not break new ground; if reflects standards that have been generally accepted for over a decade. It is intended to support the continuing efforts of conscientious teachers everywhere to provide their students with sound and stimulating mathematical training. It is specifically designed to provide a benchmark for our efforts and those of others to assess and react to recent reports of a general decline in the performance of students in mathematics.

A joint committee of the Mathematical Association of America and the National Council of Teachers of Mathematics consulted with secondary school and college teachers in various parts of the country to study recent trends in the preparation of students. The comments from these consultations on which there was strongest consensus are the basis for this statement and its ten recommendations.

The Mathematical Association of America and the National Council of Teachers of Mathematics wish to emphasize that the statement and recommendations, as they refer to secondary school programs, are addressed only to those programs for students planning to go to college and that they are not intended to be more comprehensive. During the past twenty years many important changes have taken place in both the content and teaching of mathematics at the secondary school level. Many excellent new programs were adopted and taught effectively by teachers in elementary and secondary schools. Nevertheless, any consideration of the relative merits of new vs. traditional school curricula has been deliberately avoided. A study of this issue would have exceeded both the charge to the committee and its limited resources. This statement and these recommendations incorporate many of the best features of both of these curricula and are addressed to all mathematics programs regardless of pedagogical heritage.

* Collegiate mathematics refers to courses in calculus (or calculus and analytic geometry), probability and statistics, finite mathematics, and higher level mathematics courses.

NECESSARY COURSES WORK

Mathematics is a highly structured subject in which various concepts and techniques are highly dependent upon each other. The concepts of arithmetic and algebra, however, are basic to all of mathematics. Further work in mathematics and in all areas in which mathematics is used as a tool requires correct performance, with understanding, of basic arithmetic operations, manipulation of algebraic symbols and understanding of what the manipulation mean.

Any student who is unable to perform arithmetic calculations and algebraic operations with accuracy and reasonable speed, to understand which operations to use in a given problem, and to determine whether the results have meaning is severely handicapped in the study and applications of mathematics. The prevalence of inexpensive pocket calculators makes the performance of complicated calculations less tedious, but the use of calculators does not lessen the need for students to understand which concepts and operations are needed to solve a problem, to make sensible estimates, and to analyze their results.

For further work in mathematics, and in many other areas, from business to psychology, from biology to engineering, the ability to use algebra with skill and understanding is also essential. Having a passing grade in algebra is not enough. Both understanding and competence in the skills of algebra are necessary. Neither conceptual understanding alone nor technical skill alone will suffice in today's world, let alone in tomorrow's. Algebra is a useful subject which will help to solve problems in the real world. Opportunities to apply algebraic skills should be provided whenever possible, especially to problems that show the utility of mathematics.

Algebra courses in secondary school should include, in addition to the basic topics:

A) polynomial functions,

B) properties of logarithms,
C) exponential and logarithmic functions and equations,
D) arithmetic and geometric sequences and series,
E) the binomial theorem,
F) infinite geometric series,
G) linear and quadratic inequalities.

For most students adequate coverage of the topics in algebra requires at least two years of study.

Students who will take calculus - and this now includes very many students who will take college work in business, premedicine, economics, biology, statistics, engineering, and physical science - may or may not need trigonometry, depending on the type of calculus course appropriate for their particular programs, but they will need a good deal of what is often called "pre-calculus", including especially a sound understanding of the concept of a function, which is also fundamental for work beyond the most elementary level in probability and computing.

Those students needing trigonometry should study:
A) trigonometric functions and their graphs,
B) degree and radian measure,
C) trigonometric identities and equations,
D) inverse trigonometric functions and their graphs.

For such students, the equivalent of one semester should be devoted to the study of the topics in trigonometry.

All students who go on to take collegiate mathematics will find their college work easier if they have been introduced to some axiomatic system and to deductive reasoning. Traditionally, this has been accomplished in a geometry course. Geometry courses in secondary school should include, in addition to basic topics:
A) fundamental properties of geometric figures in three dimensions,
B) applications of formulas for areas and volumes,
C) experience in visualizing three-dimensional figures.

Other courses** beyond algebra, trigonometry, and geometry should be available to students who have the adequate background and the time to take them. A course in coordinate (or analytic) geometry is ideal since it combines algebra with geometry and provides a useful preparation for calculus. In addition to coordinate geometry, courses in the following topics are valuable: probability, statistics, elementary finite mathematics (or linear algebra), an introduction to computers and computing, and applications of mathematics.

If coordinate geometry is offered, it should include in addition to the basic topics:
A) conic sections,
B) rational functions and their graphs,
C) polar coordinates,
D) parametric equations and their graphs.

Inductive as well as deductive reasoning, techniques of estimation and approximation, and an awareness of problem solving techniques, with special emphasis on the transition from the verbal form to the language of mathematics, should be emphasized in all courses.

Calculus, where offered in secondary schools, should be at least a *full year* course and be taken only by those students who are strongly prepared in algebra, geometry, trigonometry, and coordinate geometry.

We recognize that many secondary schools have a curriculum similar to that outlined above. We emphasize again that, in order to be properly prepared for collegiate level courses in mathematics, students need to develop skills.
i) in applying standard techniques and
ii) in understanding of important concepts.

**The word "course" refers here and elsewhere in this statement to a semester course unless otherwise noted.

RECOMMENDATIONS

The Board of Governors of the Mathematical Association of America and the Board of Directors of the National Council of Teachers of Mathematics make the following recommendations:

1. Proficiency in mathematics cannot be acquired without individual practice. We, therefore, endorse the common practice of making regular assignments to be completed outside of class. We recommend that parents encourage their children to set aside sufficient time each day to complete these assignments and that parents actively support the request of the teachers that homework be turned in. Students should be encouraged to develop good study habits in mathematics courses at all levels and should develop the ability to read mathematics.
2. Homework and drill are very important pedagogical tools used to help the student gain understanding as well as proficiency in the skills of arithmetic and algebra; but students should not be burdened with excessive or meaningless drill. We, therefore, recommend that teachers and authors of textbooks step up their search for interesting problems that provide the opportunities to apply these skills. We realize that this is a difficult task, but we believe that providing problems that reinforce manipulative skills as a by-product should have high priority, especially those that show that mathematics helps solve problems in the real world.
3. We are aware that teachers must struggle to maintain standards of performance in courses at all levels from kindergarten through college and that serious grade inflation has been observed. An apparent growing trend to reward effort or attendance rather than achievement has been making it increasingly difficult for mathematics teachers to maintain standards. We recommend that mathematics departments review evaluation procedures to insure that grades reflect student achievement. Further, we urge administrators to support teachers in this endeavor.
4. In light of 3 above, we also recognize that advancement of students without appropriate achivement has a detrimental effect on the individual student and on the entire class. We, therefore, recommend that school districts make special provisions to assist students when deficiencies are *first* noted.
5. We recommend that cumulative evaluations be given throughout each course, as well as at its completion, to ***all*** students. We believe that the absence of cumulative evaluation promotes short-term learning. We strongly oppose the practice of exempting students from evaluations.
6. We recommend that computers and hand calculators be used in imaginative ways to reinforce learning and to motivate the student as proficiency in mathematics is gained. Calculators should be used to supplement rather than to supplant the study of necessary computational skills.
7. We recomend that colleges and universities administer placement examinations in mathemátics prior to final registration to aid students in selecting appropriate college courses.
8. We encourage the continuation or initiation of joint meetings of college and secondary school mathematics instructors and counselors in order to improve communication concerning mathematics prerequisites for careers, preparation of students for collegiate mathematics courses, joint curriculum coordination, remedial programs in schools and colleges, an exchange of successful instructional strategies, planning of in-service programs, and other related topics.
9. Schools should frequently review their mathematics curricula to see that they meet the needs of their students in preparing them for college mathematics. School districts that have not conducted a curriculum analysis recently should do so now, primarily to identify topics in the curriculum which could be either omitted or de-emphasized, if necessary, in order to provide sufficient time for the topics included in the above statement. We suggest that, for example, the following could be de-emphasized or omitted if now in the curriculum:

   A) logarithmic calculations that can better be handled by calculators or computers,
   B) extensive solving of triangles in trigonometry,
   C) proofs of superfluous or trivial theorems in geometry.

10. We recommend that algebraic concepts and skills be incorporated wherever possible into geometry and other courses beyond algebra to help students retain these concepts and skills.

DIAGNOSING AND CORRECTING PUPIL DIFFICULTIES IN ELEMENTARY MATHEMATICS

Harold H. Lerch, University of Illinois, Urbana, Illinois

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The topic of diagnosing and correcting pupil difficulties in elementary mathematics is of such magnitude, has so many inter-related factors, and has been so maltreated in publications and efforts of application that one cannot possibly deal with all of its ramifications in the time and space permitted here. However, some attention can be given to the breadth and depth of this major area of concern by briefly reacting in a very general manner to four questions:

1. What is the nature of pupil difficulties in elementary mathematics?
2. What are the causes of these difficulties?
3. What should be the nature of diagnostic efforts?
4. What should be the instructional procedures of correction?

What is the nature of pupil difficulties in elementary mathematics?

Although children may encounter or exhibit extreme difficulties with any topic at any level in elementary mathematics, concern or attention often is not paid to children's problems until or unless those difficulties pertain to or are exhibited in the computational operations of addition, subtraction, multiplication or division with whole numbers, common fractions or decimal fractions. Usually, the first correctional concerns occur when children are having difficulties with exercises in computation at some level beyond establishing the basic facts.

Consider the following examples of common error types in computational exercises with whole numbers. Other examples of types of errors in computation with whole numbers or errors in computation with common or decimal fractions could be cited, but these few will serve for the purpose of this discussion. Assume that these are not one-time errors of carelessness or distraction, but that certain children are making one or more of these errors repeatedly. In each instance, it is not difficult to determine the nature of the procedural error in computation merely by observing the response.

47	36	73	35	26	62
+ 27	+23	−18	x 3	x 6	x34
613	95	65	915	306	248
					186
					434

Although any one of these examples might be the first overt sign of a pupil's difficulties that any teacher might observe, each error is indicative of earlier difficulties, misconceptions, misunderstandings, or omissions in the child's arithmetical learning activities. These earlier difficulties are more grave, more influential for future learning, and more highly related to important arithmetical ideas than the errors in the illustrations. The difficulties referred to pertain to the child's inability to effectively communicate about number (in regard to the given examples, whole numbers) and to the child's lack of understanding of the meaning and the use of the particular operations. Would a child who knew the meanings of the numerals and the operational symbols utilized in the preceding examples to the extent that she/he could show them through selection and manipulation of real physical object models of number respond in the manner shown? Would a child with those capabilities accept those responses? The point to be made is that the real nature of a pupil's difficulties in elementary mathematics is not often discernable in responses to computational exercises.

What are the causes of pupil difficulties in elementary mathematics?

Lists of causes of pupil difficulties, in elementary mathematics as well as in other areas, often include such factors as emotional instability; lack of maturity, poor home environment, and social maladjustment. While such factors cannot be disregarded in professional and social interactions

with children, those phrases should not be used as excuses or "cop-outs" for lack of success in helping children grow and develop in mathematics learning. Neither should teacher-learner situations in elementary mathematics be conducted in such ways that these maladjustments are worsened. Professional responsibilities in elementary mathematics learning situations can be conducted in such ways as to help children grow and develop emotionally, socially and intellectually with major concerns directed toward those causes of difficulty which possibly can be remediated within the domain of instructional responsibilities.

Most children who are having pronounced difficultues in elementary mathematics are the products of improper and/or poor teacher-learner interactions in previous learning experiences. As a result of these poor past experiences, the children come to various points in their elementary mathematics programs where blatant deficiencies emerge. The initial injuries are often compounded by not ascertaining their causes and trying to correct or alleviate those causes. Instead, attempts are made to charge through or plow over the difficulty by treating the symptom of the difficulty in a rather superficial manner to effect a rather temporary patching job. A child may simply be told that she/he is not getting the correct answers because she/he is not using the right rules or procedures. Then a proper rule or procedure for a particular type of exercise is given for the child to remember and apply. For example, a child making the type of error illustrated in the first of the preceding examples may simply be told that to get the right answer, "Just put down the 3 and carry the 1 to the next column, then add the 1 to the 4 and the 2." This is a superficial, temporary correctional procedure because it does not attack the root of the difficulty.

Referring again to the illustrations of pupils' errors in the computational operations with whole numbers, several possible reasons or causes may be given for why these difficulties have now emerged. In the various exercises, the difficulties may have become more apparent because the child does not know the facts; does not know the rules of the operation; cannot visualize or structure the situation; does not understand the nature of the operation or its use; cannot "read" the problem or example; or, does not understand the nature of numbers or numerals. All of these possible causes of difficulty are support for the premise that the majority of pupil difficulties in elementary mathematics are due to a lack of ***readiness*** to undertake the content, major idea, skills, or operation now being encountered.

The lack of readiness to undertake a new concept, a higher difficulty level of an idea or topic, or the extension of a skill may be due to any of the following:

1. There may be gaps in a child's knowledge, skill or understanding due to:
   a. extended absences at some period. (Some elementary mathematics programs are so structured by textbook coverage and the classroom that any long period of absence may cause a child to miss the entire introduction and development of a bit of content.)
   b. the omission of absolute prerequisites for the idea or skill. (Select any one of the illustrated exercises and consider the exercise type and associated problems as a new bit of content. Then list the requisite performance skills that are absolutely necessary for the undertaking of this type of exercise).
   c. a previous emphasis upon something other than the major ideas which are now needed. (For example, emphasis may have been upon saying, reading, writing, numerals in order and sequence rather than upon the meaning of number names, base and positional value).
2. At the introductory and developmental stages of the concept or skill, the procedures may have been developed through abstractions rather than through concrete representations. (With references only to written exercises, words and rules were stated and demonstrations given which involved such abstractions as "carry", "borrow", "bring down", "move over", etc. with little or no pupil involvement with manipulating counters, bundles of sticks, or base ten blocks).
3. No relationships were developed or established between real or contrived problem situations and the abstract exercises. (Computational forms, terminology, and rules were used to find correct answers without reference to or use of physical object models for number in problem situations).

What should be the nature of diagnostic procedures?

Diagnostic procedures should entail more than an analysis of what children can or cannot do in regard to solving problems or performing computational operations. Results from commercially prepared standardized "achievement" tests and "diagnostic" tests give neither a picture of a child's true achievement nor a picture of a child's real difficulties in elementary mathematics. An analysis of the breadth and depth of the performance tasks required on these instruments will reveal the inadequacies for obtaining such total information. Perusal of an individual child's performance on either type of test will give a general indication of what the child can and cannot do in response to abstract situations in relation to her/his current level of schooling. While knowing what a child can do might be of assistance in planning, selecting and presenting future learning opportunities, knowing what a child cannot do (particularly in relation the current level of schooling) is merely indicative of a present difficulty - probably a symptom of a more serious deficiency.

Diagnostic procedures should entail more than a search for patterns in errors made by children on certain types of computational exercises. Error patterns such as those previously illustrated can easily be ascertained through teacher made and administered survey type instruments. These survey instruments usually deal with only one computational operation and would contain at least three exercises or examples at each difficulty or "new-step" level. An analysis of the errors made by a child is made to ascertain whether there is a pattern of incorrect facts or rules being applied by the child. If a pattern exists, the analyzer is able to determine what incorrect procedures the child is using. However, knowledge of a child's error pattern does not indicate what the real or major deficiency might be. Quite often, errors of a pattern are corrected, one at a time, through a direction or a rule that is given to the child in an expository manner. It would be quite easy to give a child who is exhibiting any one of the error patterns previously illustrated directions or rules of procedure which would lead to correct responses. Those rules are used by many adults, even teachers, who have learned to perform the operations in rote, mechanical manners. Such correctional procedures often do not alleviate the major deficiency or difficulty of the erring child.

The act of diagnosing pupil difficulties is a *search* for reasons WHY the child is having difficulties. What is the cause? What is at the root of the problem? The search begins when a general area of difficulty is exhibited by a child and that difficulty is confirmed by repeated similar actions on the part of the child to stimuli presented by the investigating teacher. Those stimuli may include a brief survey, paper and pencil, instrument made and administered by the teacher. After the difficulty is confirmed, an analysis of the absolute prerequisites for the difficult item or idea under consideration is made. Information in regard to the child's knowledge of and skill with these prerequisites must be obtained. This information may be obtained by developing and administering a survey instrument which deals comprehensively with the prerequisite knowledge and skills. The survey may also contain various aspects of the operation or idea in question. Once such a survey instrument is developed, it may be used with other children who exhibit similar difficulties. Information about prerequisite knowledge and skills can also be obtained from a child through a more informal personal individual conference in which questions similar to those on a survey would be asked and in which real physical object models can be used.

The individual conference with the child who is having difficulty can make use of a diagnostic schema. The schema is nothing more than a planned sequence of questions and stimuli based on the major ideas inherent in the prerequisites and in the operation or idea. The conference should be conducted with the presence of physical object materials and centered around the child responding with those materials. Through the child's responses to the questions and stimuli, the investigator-teacher should be able to determine whether or not certain deficiencies or unusual misconceptions or misunderstandings exist. It should not be considered unusual to find one or more basic deficiencies or misunderstandings in a child's background of mathematical knowledge that would most surely contribute to a display of difficulties in more than one operation or at several levels in the same operation.

What should be the instructional procedures of correction?

Correctional treatments should be directed toward the causes or roots of the difficulties exhibited rather than concentrated on the symptoms. Treating symptoms such as correcting error patterns with new rules, is a superficial approach which often requires treatment of one symptom after

another. While the treatment of a specific error or symptom may give the appearance of correction, the reappearance of the difficulty in another form or at a later level will indicate that the current teacher and the child were only temporarily fooled into believing that the idea or operation had been mastered by the child.

Appropriate and desirable correctional procedures are actually nothing more than what the previous learning experiences should have been. It is unreasonable to believe that a rehash or recoverage of the content ideas using the same approaches and the same materials that were used when the roots of the difficulty were being established will be a successful correctional attempt. In most instances the correctional procedures should be much more informal and more directly individualized than were the child's initial contacts with the mathematical ideas under consideration.

If deficiencies or misunderstandings in the absolute prerequisites for the content in which difficulties became apparent are observed, those prerequisites must be reintroduced and developed in a most real and concrete manner. Concrete physical object models of the mathematical ideas should be used by the child until she/he becomes facile with the ideas and the models. Communication should be limited to the child's showing and doing, listening and speaking, until it is evident that printed or written symbols will be meaningful. While tasks or learning opportunities should be kept simple, emphasis should be placed upon major ideas rather than upon isolated singular skills. Introductory, developmental and practice learning activities should be interesting, realistic and meaningful to the child. Game-like situations are quite appropriate.

In no instance should a child be expected to perform operations on numbers using numerals until she/he has mastered the communication skills in regard to the numbers-numerals involved. For example, before beginning any of the operations previously illustrated, a child should be able to show a number using a physical object model when given an oral number name; to give an oral number name when shown a number with a physical object model; to show a number using a physical object model when given a written numeral; and, to write a numeral when shown a number with a physical object model. In the preceding expectations, physical object models to be used could include chips or counters, sticks and bundles of ten sticks, and base ten blocks.

The meaning and use of computational operations should be introduced and developed in a similar manner. Problems, activities and games involving combining, separating and comparing real quantities (physical object models for number) can be presented as introductory and developmental learning opportunities. Initially, the tasks would involve only the correct or proper manipulation of the physical object models to arrive at correct solutions. Later, the participants may be asked to keep a written record requiring numerals and operational symbols of the manipulations as they are being made. Still later, the tasks may involve giving the child a written exercise or computational example and requiring that the example be solved through manipulation of a model. When these tasks are mastered, the child will probably be ready to solve similar computational exercises or problems using only paper and pencil, without reliance upon a physical model.

In summary, diagnosing and correcting pupil difficulties in elementary mathematics is a major professional teaching responsibility. Such efforts must be well planned and carefully conducted. The act of diagnosing is an effort to ascertain the real causes and roots of the errors and difficulties displayed by children. Correctional procedures should deal with those real causes and roots in a comprehensive manner with the hope that alleviating the root of one specific difficulty of a child may actually erase, prevent or soften other possible difficulties.

PATTERNS

Multiply 11 x 11
Multiply 111 x 111
Multiply 1111 x 1111

Multiply 6 x 7
Multiply 66 x 67
Multiply 666 x 667

What do you observe about the products?

Continue the pattern.

Write the next three questions and their products.

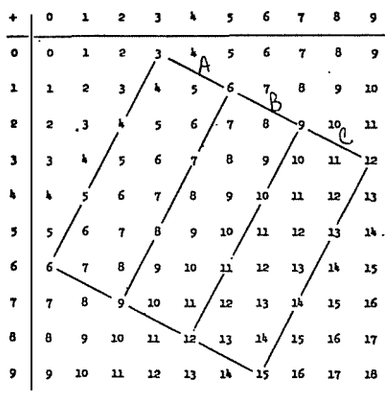
OBLIQUE RECTANGLES AND THE ADDITION TABLE

David R. Duncan and Bonnie H. Litwiller, Professors of Mathematics, University of Northern Iowa, Cedar Falls, Iowa

Students usually need practice in computation, discovering patterns, making conjectures and verifying them. This activity provides such practice on the addition table.

Consider oblique rectanges A, B, and C on the addition table as shown in Figure 1.

Figure 1



+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

1. Add the 4 numbers which represent the 4 vertices of each oblique rectangle; call this sum V.
2. Add the numbers which lie on the perimeter; call this sum P. Note that in these examples 8 numbers lie on each perimeter.

3. Add the numbers which lie in the interior of the oblique rectangle; call this sum I. In these examples there are 12 interior numbers.

4. Find the ratios V/P and P/I.

Oblique Rectangle	V	P	I	V/P	P/I
A	24	48	72	$\frac{24}{48} = \frac{4}{8}$	$\frac{48}{72} = \frac{8}{12}$
B	36	72	108	$\frac{36}{72} = \frac{4}{8}$	$\frac{72}{108} = \frac{8}{12}$
C	48	96	144	$\frac{48}{96} = \frac{4}{8}$	$\frac{96}{144} = \frac{8}{12}$

V/P = 4/8; there are 4 numbers which represent the vertices and 8 numbers which lie on the perimeter.

P/I = 8/12; there are 8 numbers which are on the perimeter and 12 numbers which lie in the interior. Check this pattern for other oblique rectangles of this size drawn elsewhere on this table.

Does the pattern hold for different sized oblique rectangles? Use Figure II to check. Computation yields these results: V = 30, P = 75, I = 195. Thus V/P = 30/75 = 2/5 = 4/10 and P/I = 75/195 = 5/13 = 10/26.

In this example there were 10 numbers on the perimeter and 26 numbers in the interior.

FIGURE II

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Conjecture that in any oblique rectangle V/P represents the ratio of the number of vertices to the number of numbers which lie on the perimeter and P/I represents the ratio of the number of numbers which lie on the perimeter to the number of interior numbers.

Proof (for a specific size oblique rectangle): Figure III is a general addition table.

FIGURE III

+	0	1	. . .	b			
0							
1							
.							
.							
.							
a	a + b	a+(b+1)	a+(b+2)	a+(b+3)	a+(b+4)	a+(b+5)	a+(b+6)
	(a+1)+b	(a+1)+(b+1)	(a+1)+(b+2)	(a+1)+(b+3)	(a+1)+(b+4)	(a+1)+(b+5)	(a+1)+(b+6)
	(a+2)+b	(a+2)+(b+1)	(a+2)+(b+2)	(a+2)+(b+3)	(a+2)+(b+4)	(a+2)+(b+5)	(a+2)+(b+6)
	(a+3)+b	(a+3)+(b+1)	(a+3)+(b+2)	(a+3)+(b+3)	(a+3)+(b+4)	(a+3)+(b+5)	(a+3)+(b+6)
	(a+4)+b	(a+4)+(b+1)	(a+4)+(b+2)	(a+4)+(b+3)	(a+4)+(b+4)	(a+4)+(b+5)	(a+4)+(b+6)
	(a+5)+b	(a+5)+(b+1)	(a+5)+(b+2)	(a+5)+(b+3)	(a+5)+(b+4)	(a+5)+(b+5)	(a+5)+(b+6)
	(a+6)+b	(a+6)+(b+1)	(a+6)+(b+2)	(a+6)+(b+3)	(a+6)+(b+4)	(a+6)+(b+5)	(a+6)+(b+6)

The Sum V;

$$\begin{array}{r} a+b+2 \\ a+b+8 \\ a+b+9 \\ \underline{a+b+3} \\ 4a+4b+22 \end{array}$$

The Sum P:

$$\begin{array}{r} a+b+2 \\ a+b+5 \\ a+b+8 \\ a+b+9 \\ a+b+6 \\ \underline{a+b+3} \\ 6a+6b+33 \end{array}$$

The Sum I:

$$\begin{array}{r} a+b+3 \\ a+b+4 \\ a+b+4 \\ a+b+5 \\ a+b+6 \\ a+b+7 \\ a+b+7 \\ \underline{a+b+8} \\ 8a+8b+44 \end{array}$$

Then $\frac{V}{P} = \frac{4a + 4b + 22}{6a + 6b + 33} = \frac{2(2a + 2b + 11) = 2 = 4}{3(2a + 2b + 11) \quad 3 \quad 6.}$

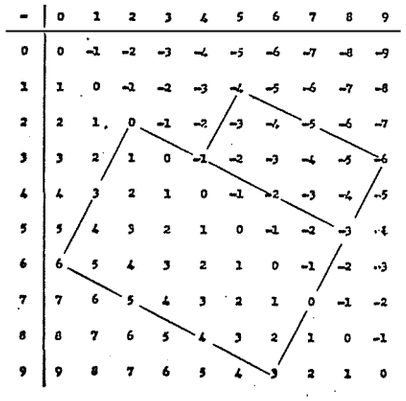
In this case there were 4 vertices and 6 numbers on the perimeter.

$\frac{P}{I} = \frac{6a + 6b + 33}{8a + 8b + 44} = \frac{3(2a + 2b + 11) = 3 = 6}{4(2a + 2b + 11) \quad 4 \quad 8..}$

In this case there were 6 numbers on the perimeter and 8 in the interior.

Question for the reader: Given figure IV, does the pattern observed here for the ratios V/P and P/I still hold?

Figure IV



THE D.L.
A SIMPLE STRATEGY TO UNDERSTANDING

Sam Pravica, Faculty of Education, University of Western Ontario

Do your students have difficulty understanding or applying the distribution law (The D.L.)? Well, here is a sure fire method that will not only give them a measure of understanding but also a simple reminder of the strategy which produces quick recall when they forget.

Oh, I know you have already tried umpteen methods; why yet another, you may ask? Well, for two reasons. Firstly, acquiring an applied understanding of the distributive law is vital to conceptual development in the four fundamental operations. Dr. C. Carlow of O.I.S.E. suggests that this concept is what he terms an "anchor point", these concepts are the building stage of later cognitive development. He is convinced that if "anchored" in the students framework of understanding number properties, the student will have little difficulty in succeeding mathematical pursuits.

Secondly, successful teachers are always looking for new ideas and strategies. Another approach may be just what the doctor ordered for some struggling student.

The distributive law is first encountered incidentally in the primary grades when students are first exposed to multiplication of a double digit number by a single digit number. It appears again when students encounter fractions, specifically the multiplication of a mixed number by a whole number. Most teachers avoid the issue by having children memorize the algorithm; change the mixed number to an improper fraction, then multiply. It also appears incidentally in geometry when students are asked to find the perimeter of a rectangular figure. To avoid the difficulty, most students memorize that $P = 2l + 2w$ rather than $P = 2(l + w)$. Is it any wonder that we continue

to search for that magic formula! Well, I won't say that I have found it, but I will say that this method gave me the greatest success. It is probably because I begin with first principals, with concepts that students have already mastered and work from there. My approach eliminates the need for memorization, just a fundamental understanding of multiplication. I will attempt to develop the concept through a series of four stages. The first two stages must be carefully developed and understood by the student. If this is done, the last two stages take care of themselves.

Stage 1

The concept that is critical to the understanding of this method is the meaning given to multiplication in the primary grades; that is, 3 x 4 means that the answer is obtained by adding three groups of four objects together. It is important that the students recognize that the multiplier "3" tells us what ever follows is summed 3 times. I will pursue this idea until the student clearly recognizes the role of the multiplier, i.e. that the 4 in "4 x b" tells us that b is to be summed 4 times and that $4 \times b = b + b + b + b$.

Stage 2

In this stage I provide the students with an exercise to reinforce the concepts:

(i) $3 \times 5 = 5 + 5 + 5$
and $4 \times b = b + b + b + b$, and conversely that,
(ii) $6 + 6 + 6 + 6 + 6 + 6 + 6 = 7 \times 6$
and $d + d + d + d + d = 5 \times d$

Stage 3

This stage actually constitutes the main lesson. However, I find that if the students grasp stages 1 and 2 well, this stage and the one following are anti-climactic. I first have the students recall the role that 4 plays in 4 x b (I choose the variable "b" purposely), I then extend the concept by using the expression "4 x brackets" and again ask the students "what does the multiplier "4" tell us to do?" I will reinforce this by writing the expansion on the

4 x brackets = brackets + brackets + brackets + brackets

I then replace the word "brackets" by ( ) and repeat the question to get

$4 \times (\ ) = (\ ) + (\ ) + (\ ) + (\ )$

and then finally ask the students "how can 4 x (3 + 5) be written in expanded form?" With the expansion $4 \times (3 + 5) = (3 + 5) + (3 + 5) + (3 + 5) + (3 + 5)$ written on the board, I then focus the class's attention to the expansion on the R.H.S. How many 3's are being added? How many 5's are being added? How can the repeated sums be simplified' The class quickly sees that the R.H.S. condenses to $4 \times 3 + 4 \times 5$

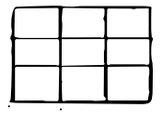
Stage 4

I now begin to summarize the development with a series of questions followed by appropriate responses. In the expression 6 x (4 + 7), how many brackets are being summed? Therefore how many 4's are being summed, and how many 7's are being summed? The students quickly recognize that 6 x (4 + 7) = 6 x 4 + 6 x 7. I then follow this by including variables into the expression such as 48 x (a + b) until eventually I spring the expression a x (b + c) on them. By this stage the students do not even hesitate. It is obvious to them that it cannot be anything else but $a \times (b + c) = a \times b + a \times c$.

With this technique I can present the students with any number of terms in the bracketed expression without encountering any difficulty. The student has virtually nothing to memorize other than recognizing the role of the multiplier in the multiplication question.

MAGIC SQUARE

Make a magic square using the numerals 3, 4, 5, 6, 7, 8, 9, 10, 11



The sums must be the same in all directions.

MATHEMATICS OR ENGLISH - WHICH IS BEING TAUGHT?

John Kay, Lambton County Board of Education

Education in the Primary and Junior Divisions states, "Every opportunity should be used to enrich the child's vocabulary with regard to shape, size, number, distance and 'scale' words such as bigger and smaller".

The following article outlines the use of attribute blocks with deaf children to develop mathematical language.

A WORD OF INTRODUCTION

When a student shows a marked degree of linguistic retardation, a means of presenting various syntactical structures in a practical way is often most advantageous. Attribute blocks provide a medium through which this can be done with considerable success.

The various strategies outlined below have been used successfully with hearing impaired students and have helped both to organize their written and spoken language and to develop their vocabularies in certain areas.

What follows is not an exhaustive exposition of the idea, but it is hoped that some readers will find it sufficiently interesting to encourage them to experiment and develop it according to the specific needs of their own classrooms.

PREPOSITIONS

It is possible, through the manipulation of the blocks, to demonstrate in a visual way, the physical meaning of many prepositions. Those prepositions which I have taught by this method are: in, on, out (using a box), under, over, above, below, between, beside, behind, next to, up, down, across, opposite (position), in the middle of, among.

WORD ORDER (Adjectives)

The physical attributes of the blocks make it possible to emphasize correct word order, e.g. a large, thin, triangular, red block.

COMPARISON OF ADJECTIVES

The concepts of larger, smaller, thicker, thinner are difficult for the student to grasp. The use of attribute blocks has proved invaluable in helping to ensure that the concept is properly understood. The word 'than' becomes meaningful when introduced by inclusion at this stage. By making blocks of intermediate size and thickness, it is possible, at the appropriate stage, to introduce both comparative and superlative forms, e.g. this is bigger than that. By using piles of blocks of differing numbers, it is possible to demonstrate the meanings of less, least, more and most.

ASSERTIVE SENTENCES

These sentences can be built up by the manipulation of the blocks and the introduction, at the appropriate time, in both oral and written form of the required verb, e.g. This block is a triangle.
A triangle has three sides.

INTERROGATIVE SENTENCES

It is possible here to contrive these sentences so that inversions, adverbs and interrogative pronouns may be used, e.g. Is this block blue? (inversion) How many sides has a square? (adverb) Which block has three sides? (interrogative pronoun)

IMPERATIVE SENTENCES

Put the block down!

EXCLAMATORY SENTENCES

Pretend to lose a block and then find it saying, "Here it is!" There are many other situations which can be contrived to demonstrate the use of exclamatory sentences.

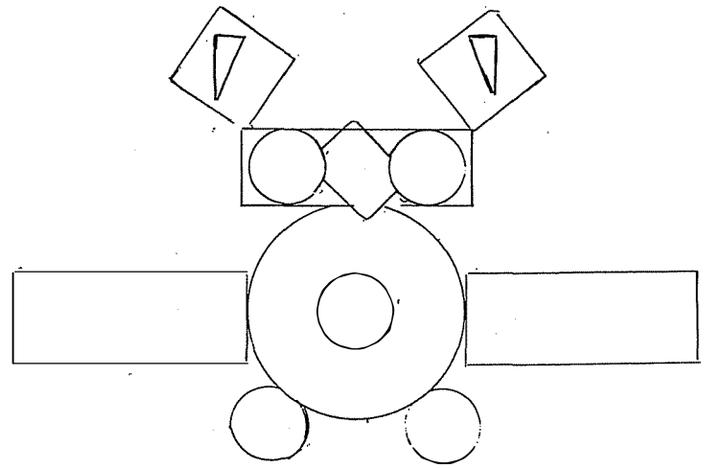
TENSE

It is possible to introduce the idea of tenses in a practical form. Put a block on a desk. Ask a student to fetch it to you. When you have the block ask, "Where was the block?" and "Where is the block now?"

It is possible to develop activities and games which will provide opportunities for the use of spontaneous speech and written language. One idea is to provide one of each shape and size of block and tell the student to draw an animal by using the blocks as templates. The more imaginative the animal, the better will be the results. Having produced the drawing the following instructions are given:

> "Think of names for the animal and its baby like cow and calf.
> Tell what colours it is. Say where it lives and what it eats."

An example produced by this activity is shown below. This was the uncorrected work of a severely deaf Grade 4 boy.



The Square-headed Loe

A square-headed loe is a bird. The babies are called
loeys.
They eat plants. They lay three eggs. They live on a
mountaintop in a hole.
They don't have any daddy loes.
They sleep through the winter,

A surprising amount of vocabulary can be developed through the use of attribute blocks.

> shape, circle, triangle, square, rectangle, hexagon, circular, round, triangular, rectangular, oblong, corner, point, angle, right angle, equal, the same as, side, parallel, surface, flat, circumference, perimeter, colour, blue, red, yellow, different, size, big, large, little, small, than, bigger, larger, smaller, biggest, largest, smallest, thickness, thick, thin, thicker, thinner, thickest, thinnest, roll, balance, fall over, slope, slide.

AN APPLICATION OF VECTORS

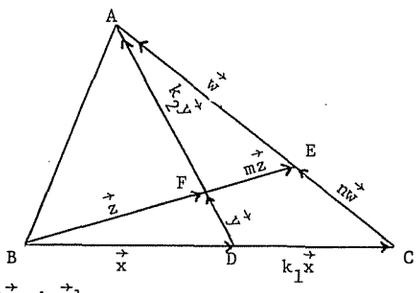
Henry Dernyck, Westgate Collegiate & Vocational Institute, Thunder Bay, Ontario.

"The line joining the mid-point of a median of a triangle to a vertex trisects the side opposite that vertex."

(Problem 15e P 221 Algebra 13 by Coleman et al)

The solution to the above problem led to the more difficult problem which follows:

Given: triangle ABC with D in BC so that
BD : DC = 1 : k_1 and with F in AD so that
AF : FD = k_2 : 1 BF produced meets AC at E.



Required: Express the ratios AE : EC and
BF : FE in terms of k_1 and k_2.

Solution: In triangle BFD, $\vec{z} = \vec{x} + \vec{y}$ (1)

In figure FDCE, $m\vec{z} = -\vec{y} + k_1\vec{x} + n\vec{w}$ (2)

In triangle ADC, $(n+1)\vec{w} = (k_2+1)\vec{y} - k_1\vec{x}$ (3)

Substitute (3) in 2 $m\vec{z} = -\vec{y} + k_1\vec{x} + \frac{n}{n+1}[(k_2+1)\vec{y} - k_1\vec{x}]$

or $$m\vec{z} = \left[\frac{n(k_2+1)}{n+1} - 1\right]\vec{y} + \left[k_1 - \frac{k_1 n}{n+1}\right]\vec{x} \quad (4)$$

$$m\vec{z} = m\vec{y} + m\vec{x} \quad (5)$$

Subtract to get

$$0 = \left[\frac{n(k_2+1)}{n+1} - 1 - m\right]\vec{y} + \left[k_1 - \frac{k_1 n}{n+1} - m\right]\vec{x}$$

$\therefore$ $\vec{x}$ and $\vec{y}$ are linearly independent

$$\frac{n(k_2+1)}{n+1} = m + 1, \qquad \frac{k_1 n}{n+1} = k_1 - m$$

Solving for m and n we have $n = \frac{k_1+1}{k_2}$ and $m = \frac{k_1 k_2}{k_1+k_2+1}$

AE : EC = $k_2 : (k_1+1)$ and BF : FE = $(k_1+k_2+1) : k_1 k_2$

FUN MATHEMATICS

1. If you went to bed at 8 o'clock at night and set the alarm to get you up at 9 o'clock in the morning, how many hours sleep would you get?
2. Do they have a 4th of July in England?
3. How many birthdays does the average man have?
4. Why can't a man living in Winston, Salem, N.C., be buried west of the Mississippi River?
5. If you had only one match and entered a room in which there was a kerosene lamp, an oil heater, and a woodburning stove, which would you light first?
6. Some months have 30 days, some have 31, how many have 28?
7. If a doctor gave you nine pills and told you take one every half hour, how long would they last you?
8. A man builds a house with four sides to it and it is a rectangular shape; each side has a southern exposure. A big bear wanders by - what color is the bear?

Answers on page 64.

SIMPLIFICATION OF EXPRESSIONS IN SET ALGEBRA USING McCULLOCK DIAGRAMS

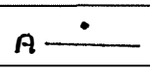
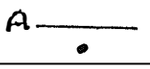
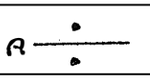
David J. Laurenson, University of Toronto Schools

When teaching introductory work in set theory Venn diagrams are often used as illustrations. For more than three sets however, Venn diagrams often become unmanageable.

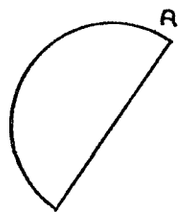
An alternative diagramatic system is that which I have called McCullock diagrams, after W.S. McCullock who developed the system for use in logic.

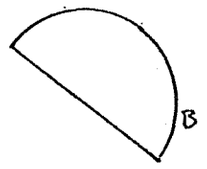
To represent one formula, McCullock used a closed semi-circle in which a dot represented truthhood, and a blank falsehood. The semi-circle has been abbreviated to a bar which provides two sectors, each of which can be either dotted or blank to denote truthhood or falsehood.

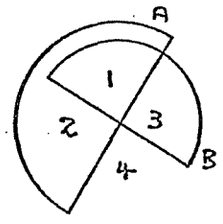
The interpretation of these diagrams for a formula A in terms of the symbols U, Ø and ′, where U, Ø, ′take on the meanings universe, emptiness and complement is:

McCullock diagram		Interpretation
initial	Abbreviated	
		A is true, A' is false
		A is false, A' is true
		Ø
		U

For two forumulae, A and B, two single diagrams are superimposed forming 4 quadrants:



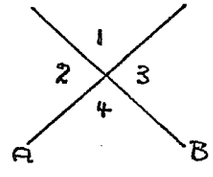




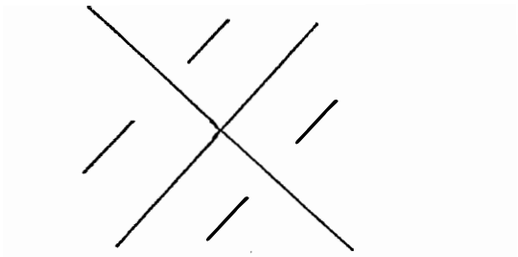
The four quadrants then represent:

1. A and B both true.
2. A true, B false.
3. A false, B true.
4. A and B both false.

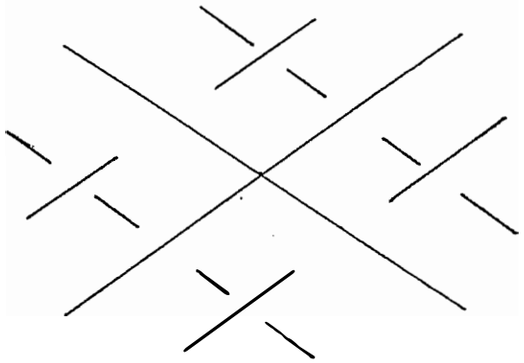
This diagram is abbreviated to



The McCullock diagram for three formulae is constructed by inserting a small slant, /, in each quadrant of the two formulae diagram giving 8 compartments.

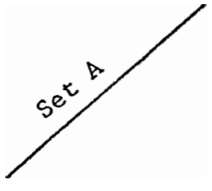


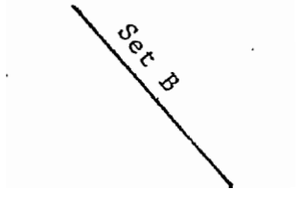
This process may be continued for any number of formulae. In general, an n-formula diagram will have 2^n compartments. The 4 formula diagram is drawn below.

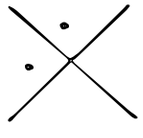


A convention which is followed for diagrams representing sets is that the upper side of a bar /, indicates that portion of the universe which is designated as a set (say A) and the lower half indicates the complement of this set (A').

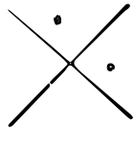
First consider the diagram for two sets A and B that sets A and B are designated as:



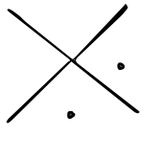




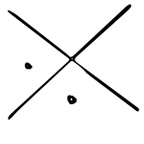
represents set A



represents set B

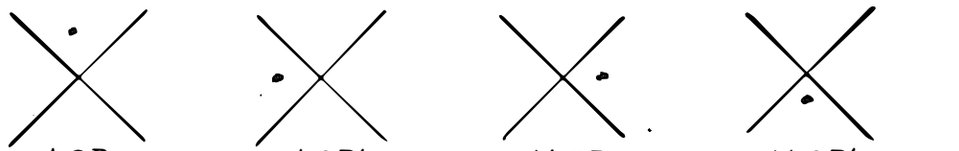


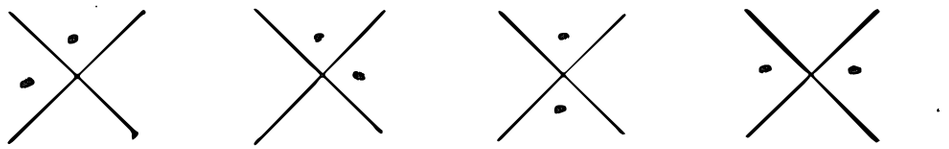
represents set A'

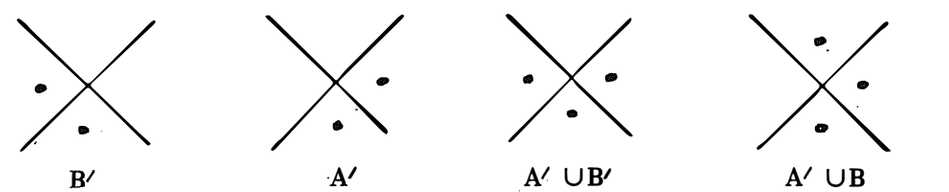


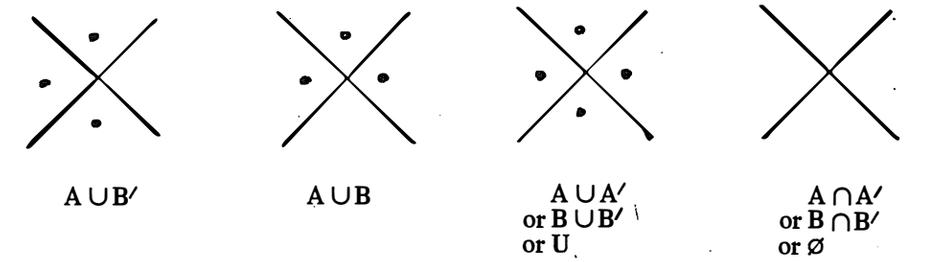
represents set B'

The dots represent subsets of elements of the sets A and B. Since there are 4 compartments, each of which may have a dot or be blank, there are 2 combinations of dots and blanks possible. These 16 combinations with their interpretation are given below, taking the sets A and B as in Section 4.21 above.





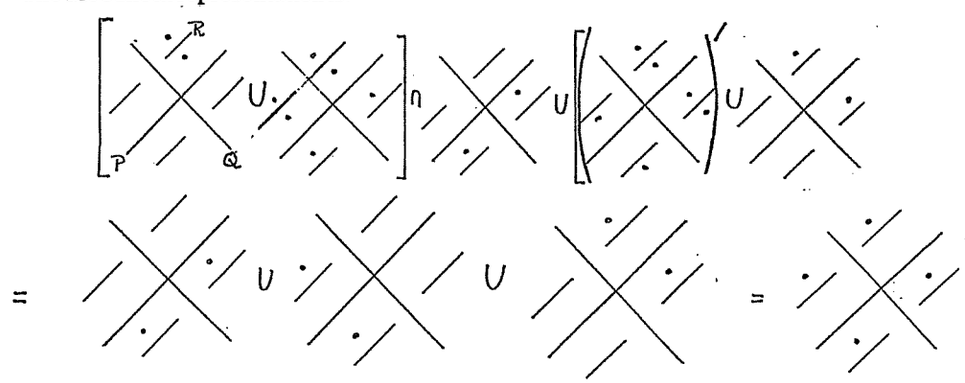




Examples of the use of McCullock diagrams in simplifying expressions in set algebra.

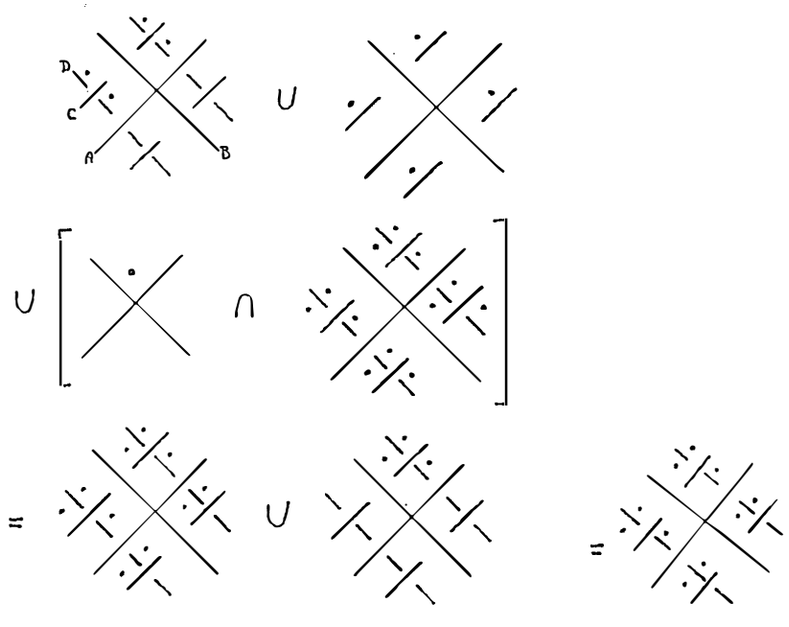
1. Simplify $[(P \cap Q) \cup (P \cup R)] \cap (P' \cap R) \cup [(Q \cup R')' \cup (Q \cap R)]$

The McCullock representation is:



∴ The given expression simplifies to R

2. Simplify $(A \cap D) \cup C \cup [(A \cap B) \cap (C \cup D)]$



The given expression simplifies to $(A \cap D) \cup C$.

It is interesting to note that the simple set operations on the set of dots correspond to logical operations on the propositional interpretations.

TRANSFORMATIONS ON GRIDS

Peter Weygang and Tom Griffiths

There are three main approaches to the transformation geometry proposed for and being used in schools at the present time. They are the experimental or experiential approach, the synthetic or Euclidean approach and the graphing or analytic approach. These are not distinct branches in the sense that the basic concepts pervade all three, but the last one can be considered separately in as far as it is concerned primarily with graphs and their equations.

This article will concentrate on some of the developments of notations and concepts which can make the study of transformations on a grid simple and the most effective method of studying graphs. These developments are:

1. The use of vectors or 2 × 1 matrices to define translations.
2. The use of 2 × 2 matrices to define some reflections, rotations and magnifications.
3. The construction of 2 × 2 matrices for specified transformations.
4. The use of the parametric form of equations in transforming graphs.

The article will also discuss some of the results which are simplified by this approach.

1. In order to translate a point right 2 and down 3 it is necessary to add 2 to the x-co-ordinate and subtract 3 from the y-co-ordinate. This can be shown by the addition of the vector [2,-3]. (Square brackets can be used to avoid confusing a vector with a point.)

Example: Translate (1,4) by [2,-3]

$(1,4) + [2,-3] = (3,1)$

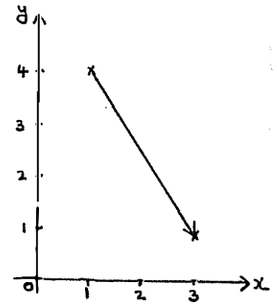


Fig. 1

Using this notation and the same vector on several points, concepts such as the invariance of slope and magnitude of translations can be studied, as well as the invariant properties of figures under translations. This also gives a good basis for the study of vectors.

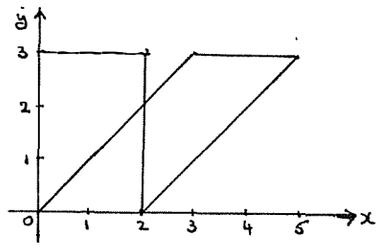
2. The other transformations to be considered at the high school level, those of rotation, reflection, dilation, magnification and shear can all be represented by 2 × 2 matrices, which when multiplied by a point give image points.

Example: Using the matrix $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ transform the points (2,3), (2,0), (0,0) and (0,3). These points can be transformed individually or all together by matrix multiplication.

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix} = \begin{pmatrix}5\\ 3\end{pmatrix}$$

or
$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 2 & 2 & 0\\ 0 & 0 & 3 & 3\end{pmatrix} = \begin{pmatrix}0 & 2 & 5 & 3\\ 0 & 0 & 3 & 3\end{pmatrix}$$

Fig. 2



3. Using this notation leads to a very useful bonus. Transformation matrices can be constructed relatively easily. This is accomplished by considering transformations on the points (0,1) and (1,0). Consider a transformation matrix acting on these two points so $T\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = T.I = T$, where I is the identity matrix under multiplication. Therefore if we know the images of (1,0) and (0,1) under a transformation, we can construct the transformation matrix.

Example: Construct the transformation matrix which rotates points by 90° about (0,0).

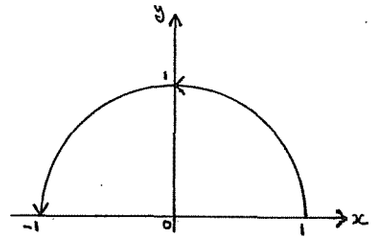


Fig. 3

$(1,0) \rightarrow (0,1)$ and $(0,1) \rightarrow (-1,0)$

$$\therefore \quad R_{90°} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\therefore \quad R_{90°} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Example: Construct the matrix for the shear shown in the graph below.

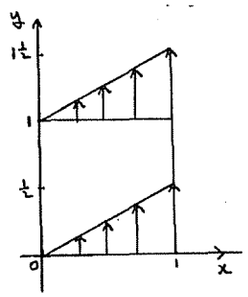


Fig. 4

$(1,0) \rightarrow (1,1\frac{1}{2})$ and $(0,1) \rightarrow (0,1)$. The transformation matrix is $\begin{pmatrix} 1 & 0 \\ 1\frac{1}{2} & 1 \end{pmatrix}$

Example: Construct the transformation matrix which magnifies by a factor of 2 in the x-direction and 3 in the y-direction.

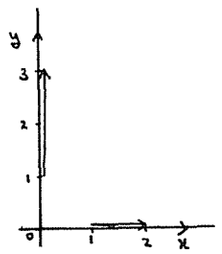


Fig. 5

Therefore $(1,0) \rightarrow (2,0)$ and $(0,1) \rightarrow (0,3)$. The matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

4. A matrix transforms points. Equations may be expressed as points in a parametric form. Thus matrices acting on parametric points will transform equations.

The following examples illustrate the above statement, and show how the parametric approach to equations may be combined with matrices to form a powerful tool for transforming equations. Parametric equations and matrices are both in the new Intermediate Division Guidelines.

Example: Translate the parabola $y = x^2$ by $[2,-3]$

The parametric form of $y = x^2$ is $x = t$; $y = t^2$ or (t,t^2).

$(t,t^2) + [2,-3] = (t + 2,\ t^2 - 3)$

$x = t + 2$; $y = t^2 - 3$

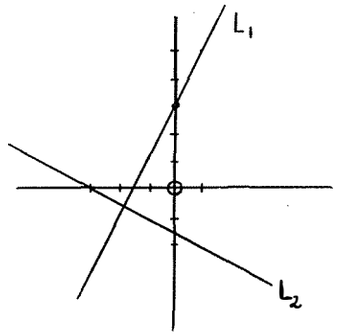
$x - 2 = t$; $y + 3 = t^2$

$(y + 3) = (x - 2)^2$ is the required equation.

Example: The line $y = 2x + 3$ is rotated 90° counterclockwise about the origin. Find the equation of the line in its new position.

The equation $y = 2x + 3$ may be written in terms of a parameter 't' as

$$\left.\begin{aligned} y &= t \\ y &= 2t + 3 \end{aligned}\right\}$$



The rotation about 0 by 90° may be written as:

$$R_{0,90°} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The desired transformation is expressed as

$R_{0,90°} * L_1 \rightarrow L_2$ * means 'acting on'

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} t \\ 2t + 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -2t - 3 \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

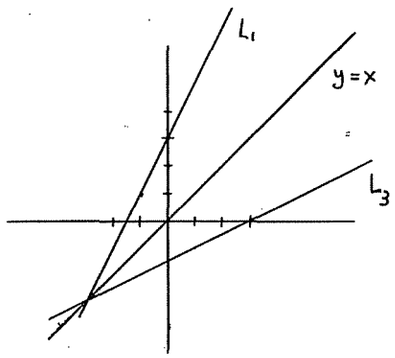
Thus the parametric form for L_2 is $\left.\begin{aligned} x &= -2t - 3 \\ y &= t \end{aligned}\right\}$

The equation may be left in this parametric form; or may be expressed in a more usual form by doing a little algebra. Since we want the final equation to start with "y = ..." the trick is make sure that 't' is isolated from the equation which starts "x = ..."

In this example $x = -2t - 3$

$$\Rightarrow \quad t = \frac{-(x+3)}{2}$$

whence $$y = -\frac{x}{2} - \frac{3}{2}$$



Example: Find the equation of the inverse of $y = 2x + 3$. The line L_3 is the image of L_1 reflected in the line $y = x$.

Reflection in $y = x$ may be written as:

$$\mathcal{R}_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

The desired transformation is expressed as:

$\mathcal{R}_{y=x} * L_1 \rightarrow L_3$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} t \\ 2t + 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2t + 3 \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus the parametric form of L_3 is $\left.\begin{aligned} x &= 2t + 3 \\ y &= t \end{aligned}\right\}$

Changing this into the usual form we proceed as follows:

$$x = 2t + 3$$

$$\Rightarrow \quad t = \frac{x - 3}{2}$$

whence $$y = \frac{x}{2} - \frac{3}{2}$$

Example

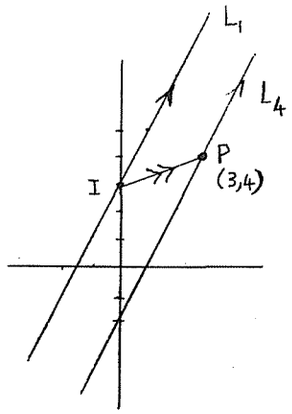
Find the equation of the line parallel to $y = 2x + 3$ which passes through (3,4).

The line L_4 is the image of L_1 formed by the translation from I to P.

This translation may be written as:

$$T = \begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}0\\3\end{pmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

Note the style; round brackets for the points and square brackets for the translation vector.



The desired transformation is expressed as:

$$T * L_1 \rightarrow L_4$$

$$\begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}t\\2t+3\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{bmatrix}t+3\\2t+4\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

Thus the parametric form for L_4 is $\left.\begin{aligned}x &= t + 3\\ y &= 2t + 4\end{aligned}\right\}$

Again we may re-arrange this into the usual form

$$x = t + 3$$

$$\Rightarrow \quad t = x - 3$$

Thus $y = 2(x - 3) + 4$

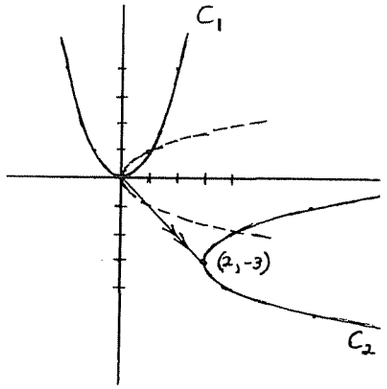
whence $y = 2x - 2$

Example: Find the equation of a parabola which is congruent to $y = x^2$, opens to the right and has its vertex as (2,-3).

The parametric form for this parabola is

$$\left.\begin{aligned}x &= t\\ y &= t^2\end{aligned}\right\}$$

It is apparent that the transformation consists of two steps. First the inverse is formed by reflection in $y = x$, which is the dotted parabola, then this image is translated to its final position.



The whole process may be written as:

$$T * \mathcal{R}_{y=x} * C_1 \rightarrow C_2$$

$$\begin{bmatrix}2\\-3\end{bmatrix} + \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \cdot \begin{bmatrix}t\\t^2\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{bmatrix}2\\-3\end{bmatrix} + \begin{bmatrix}t^2\\t\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

Thus the parametric form for C_2 $\left.\begin{aligned}x &= t^2 + 2\\ y &= t - 3\end{aligned}\right\}$

To obtain the results in the more usual form we proceed as follows, always taking care to start with the 'x' parameter

$$x = t^2 + 2$$

$$\Rightarrow \quad t = \pm\sqrt{x - 2}$$

$$\text{Whence} \qquad y = \pm\sqrt{x - 2} - 3$$

Example: Magnify the unit circle by 3 in the x-direction, and by 2 in the y-direction. A parametric form of the circle is $x = \cos\theta$; $y = \sin\theta$ or $(\cos\theta, \sin\theta)$. The transformation matrix is $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. The new equation is given by $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 3\cos\theta \\ 2\sin\theta \end{pmatrix}$

$\therefore$ $x = 3\cos\theta$; $y = 2\sin\theta$

$\therefore$ $\left(\frac{x}{3}\right) = \cos\theta$; $\left(\frac{y}{2}\right) = \sin\theta$

but $(\cos\theta)^2 + (\sin\theta)^2 = 1$.

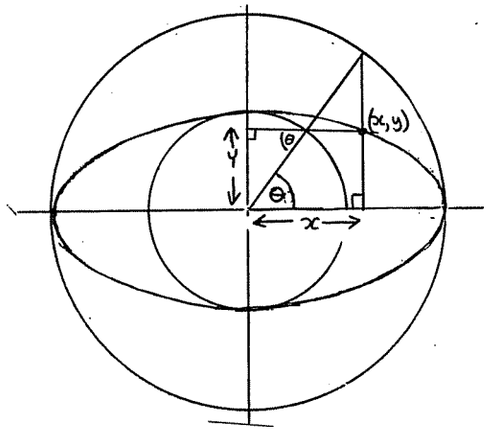
Since $(\cos\theta, \sin\theta)$ lies on $x^2 + y^2 = 1$

$\therefore$ $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

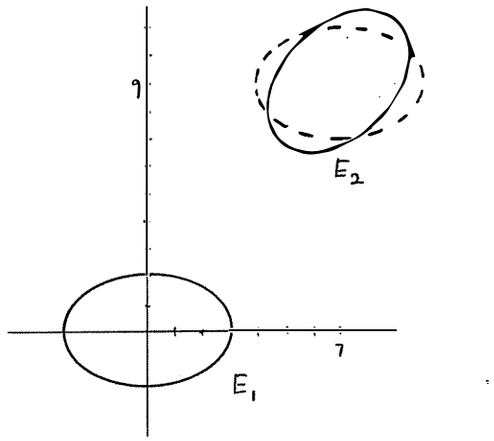
$\therefore$ $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Example: It follows that this method may be applied to the transformation of any relationship; provided that the relationship can be expressed in parametric form.

The ellipse has a very convenient parametric form, which may be understood by examining the following diagram.



If the radius of the large circle is 'a' then $x = a\cos\theta$. If the radius of the small circle is 'b' then $y = b\sin\theta$. These then represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in a parametric form. The ellipse $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$ is moved so that its centre is at the point (7,9), and the major axis is inclined at 45° to the X axis. What is the equation of the ellipse in its new position?



The matrix which represents a counterclockwise rotation by an angle 'A' is given by:

$$R_{O,A} = \begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}$$

Since $\cos 45° = \sin 45° = \frac{1}{\sqrt{2}}$

$$R_{O,45°} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The translation is given by $T = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ and the parametric equation for the ellipse is

$\left.\begin{aligned} x &= 3 \cos \theta \\ y &= 2 \sin \theta \end{aligned}\right\}$. The required transformation may be written:

$$T * R_{O,45°} * E_1 \rightarrow E_2$$

$$\begin{bmatrix} 7 \\ 9 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \cos \theta \\ 2 \sin \theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ 9 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 3 \cos \theta - 2 \sin \theta \\ 3 \cos \theta + 2 \sin \theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

and thus the parametric equations for E_2 are given by

$$x = \frac{3 \cos \theta - 2 \sin \theta}{\sqrt{2}} + 7$$

$$y = \frac{3 \cos \theta + 2 \sin \theta}{\sqrt{2}} + 9$$

These equations may be solved quite readily in order to put the equation in its usual form:

$$3 \cos \theta - 2 \sin \theta = \sqrt{2}(x - 7)$$

$$3 \cos \theta + 2 \sin \theta = \sqrt{2}(y - 9)$$

whence:

$$\cos \theta = \frac{\sqrt{2}[(x + y) - 16]}{6}$$

$$\sin \theta = \frac{\sqrt{2}[(y - x) - 2]}{4}$$

Note that $\sin^2\theta + \cos^2\theta = 1$ and the next step is obvious. By and large it is simplest to use the parametric form! With this approach many results can be greatly simplified. The following examples occur in the grade 13 course.

Example: Rotate the point (x,y) by an angle θ about (0,0). The matrix for this rotation is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. This result is obtained by considering the effect of rotating the points (1,0) and (0,1) around the unit circle, together with the definition of $\cos\theta$ and $\sin\theta$. The new points are given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

or $(x,y) \to (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$.

Example: A rotation of angle B on the point (cos A, sin A) will give a point whose coordinates are (cos (A+B), sin (A+B)). Performing this transformation by matrix multiplication gives

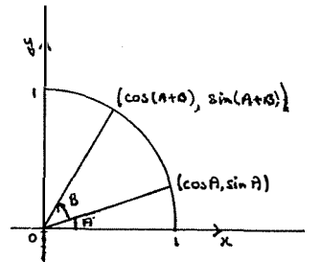


Fig. 6

$$\therefore \begin{pmatrix} \cos B - \sin B \\ \sin B + \cos B \end{pmatrix}\begin{pmatrix} \cos A \\ \sin A \end{pmatrix} = \begin{pmatrix} \cos(A+B) \\ \sin(A+B) \end{pmatrix}$$

$$\therefore \begin{pmatrix} \cos A \cdot \cos B - \sin A \cdot \sin B \\ \sin B \cdot \cos A + \sin A \cdot \cos A \end{pmatrix} = \begin{pmatrix} \cos(A+B) \\ \sin(A+B) \end{pmatrix}$$

which is a much neater method than the usual one for obtaining the angle sum formulae.

Example: (grade 11) The negative reciprocal property of the slopes of perpendicular lines. The general straight line $y = mx + b$ can be translated to the parallel line $y = mx$ without changing results concerning slopes, since slope is invariant under translation. The parametric form of the line $y = mx$ is (t,mt). The rotation matrix for 90° is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The equation of the rotated line if obtained as follows $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} -mt \\ t \end{pmatrix}$

$\therefore \quad x = -mt;\ y = t$

$\therefore \quad \frac{-1}{m}x = t;\ y = t$

$\therefore \quad y = \frac{-1}{m}x$

$\therefore$ the slope of the line perpendicular to $y = mx$ is $-1/m$.

Many results can be obtained by this approach. They include "the side splitting" theorem through dilation, the mid-point of a line segment by considering a "half" translation, and dot-product of vectors by considering a rotation of θ from one vector to another.

We hope you and your students have as much fun finding new results as we are having with ours.

HYPERBOLIC ROTATIONS

Che-Keo Fong

1. Introduction. For many years, the mysterious addition formulae for trigonometric functions

$$\sin (s + t) = \sin s \cos t + \cos s \sin t$$

$$\cos (s + t) = \cos s \cos t - \sin s \sin t$$

were not properly explained, or even digested. It was a significant advance when the realization came that these formulae followed directly from the intuitively clear identity $R_t R_s = R_{t+s}$ for rotations. From then on many educators considered the idea of replacing some traditional approaches by so-called "transformation mathematics". In support of this educational trend, we present here another fascinating but lesser known class of linear transformations. We thereby illustrate the intrinsic beauty and usefulness of "transformation mathematics".

We will examine those transformations known as hyperbolic rotations; they are intimately related to the hyperbolic functions sinh and cosh just as ordinary rotations are related to the trigonometric functions. Usually, hyperbolic functions are treated in calculus courses, but their important role in geometry is often ignored. We think it more appropriate to introduce them as follows. However, since this material is rather advanced for the average high school student, we do not recommend it for inclusion in the curriculum; but it should be more widely known to teachers and educators.

I would like to thank Barry Monson for his help and suggestions in preparing the manuscript.

2. Preliminaries: Rotations and One-Parameter Groups. Since hyperbolic rotations have many properties similar to those of ordinary rotations, we start with a brief review of some well-known facts about ordinary rotations.

The plane rotation R_t through an angle t is described (for Cartesian coordinates) by the matrix

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} . \tag{1}$$

We shall denote the above matrix by the same symbol R_t. It is intuitively clear that the motion of rotating first through the angle s and then through the angle t is the same as the motion of the rotation through s+ t. As a mathematical equation, this reads:

$$R_t R_s = R_{t+s} \tag{2}$$

By means of (1), (2) and the familiar method of multiplying two matrices, we immediately get the addition formulae for sine and cosine.

Also note that the determinant of (1) is +1:

$$\det \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \cos^2 t + \sin^2 t = 1.$$

Geometrically this means that rotations preserves both area and orientation. In general, if A is a linear transformation, F is a region in the plane and G is the image of F under A, then

$$\text{area of } G = |\det A| \times \text{area of } F. \tag{3}$$

(This is a very useful principle, whose proof is elementary. For some interesting applications, see item IV in [1].)

For a vector $x = (x_1, x_2)$ in the plane, the length of x is $(x_1^2 + x_2^2)^{1/2}$. We shall write $q_0(x)$ for $x_1^2 + x_2^2$, that is, the square of the length of x. Since rotations do not

change the length of a vector, we have

$$q_0(R_t x) = q_0(x) \tag{4}$$

for each vector x and each rotation R_t. Thus we say that R_t preserves the quadratic form q_0. It happens that if a linear transformation preserves q_0 and has determinant +1, then it must be a rotation. (It is possible to have a length-preserving transformation which is not a rotation. For example, the transformation determined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note, however, that det $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1$.)

Let us now introduce the following convenient concepts:

Definition: By a one-parameter group (of transformations) we mean a family $\{T_t: -\infty < t < +\infty\}$ of linear transformations such that $T_0 = I$. and $T_t T_s = T_{s+t}$ for all reals s and t. For a one-parameter group $\{T_t\}$ and a vector x, the set $\{T_t x: -\infty < t < +\infty\}$ is called the orbit of $\{T_t\}$ through x.

Thus the family $\{R_t: -\infty < t < +\infty\}$ of rotations forms a one-parameter group and the orbit of $\{R_t\}$ through a non-zero vector is a circle. (Figure 1) Another example of a one-parameter group is the family of shears $\{S_t: -\infty < t \infty +\infty\}$ where S_t is given by the matrix $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The reader can check that $S_t S_s = S_{t+s}$. An orbit of $\{S_t\}$ is either a horizontal straight line or a single point on the horizontal axis. (Figure 2)

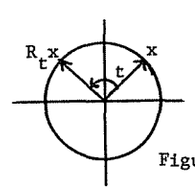


Figure 1

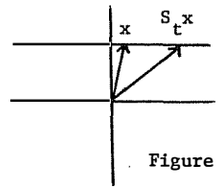


Figure 2

Suppose $\{T_t\}$ is a one-parameter group. Then, from $T_t T_s = T_{t+s}$, we obtain $(\det T_t)(\det T_s) = \det T_{s+t}$. In particular, we have $(\det T_t)(\det T_{-t}) = \det T_0 = \det I \neq 0$. Hence $\det T_t$ is never zero. Also, for any t, $\det T_t = \det (T_{t/2} T_{t/2}) = (\det T_{t/2})^2 > 0$. Hence $\det T_t$ is always positive.

3. Hyperbolic Rotations. In the previous section we mentioned that ordinary rotations preserve the quadratic form $q_0(x) = x_1^2 + x_2^2$. Because of this and because the equation $q_0(x) = c$ describes a circle where c is a positive constant, an orbit of $\{R_t\}$ is, in general, a circle (except that the orbit through the origin is a single point.)

Now let us replace the quadratic form q_0 by q defined by $q(x) = x_1^2 - x_2^2$ and let us look for a one-parameter group $\{H_t: -\infty < t < +\infty\}$ of transformations which preserve q, that is $q(H_t x) = q(x)$ for all t and x. Since $q(x) = c$ (where c is a positive constant), describes a hyperbola, we expect that the orbit of H_t through a vector x will be in general a branch of a hyperbola. (Figure 3)

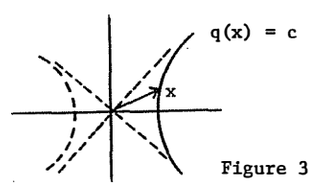


Figure 3

We remark as an aside that our consideration of q is not merely out of curiosity. In fact, in the theory of special relativity, one studies those linear transformations

preserving the Lorentz metric $x_1^2 - c^2t^2$ (where t is the time parameter and c is the speed of light in vacuo) which is the same as q if we substitute x_2 for ct. In this section we will essentially derive an explicit form for Lorentz transformations.

So let H be a linear transformation which preserves the quadratic form q: q(Hx) = q(x). Suppose the matrix of H is

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

If $x = (x_1,x_2)$, $y = (y_1,y_2)$ and y = Hx, then

$$y_1 = ax_1 + bx_2$$

$$y_2 = cx_1 + dx_2 .$$

Let u = (1,1) and v = (1,-1). Then Hu = (a+b, c+d) and Hv = (a-b, c-d). Since q(u) = q(v) = 0, we have q(Hu) = q(Hv) = 0. Therefore

$$(a + b)^2 - (c + d)^2 = 0 \qquad \text{(i)}$$

$$(a - b)^2 - (c - d)^2 = 0 \qquad \text{(ii)}$$

From (i), we have either

$$a + b = c + d \qquad \text{(i)'}$$

or $$a + b = -(c+d). \qquad \text{(ii)'}$$

From (ii), we have either

$$a - b = c - d \qquad \text{(i)"}$$

or $$a - b = -(c-d). \qquad \text{(ii)"}$$

We must consider four cases:

Case 1: (i)' and (i)" hold. In this case, a = c and b = d.

Case 2: (i)' and (ii)" hold. Here a = d and b = c.

Case 3: (i)" and (ii)' hold. Here a = -d and b = -c.

Case 4: (ii)' and (ii)" hold. Here a = -c and b = -d.

Now let e = (1,0). Then He = (a,c). Since q(e) = 1, we have $q(He) = a^2 - c^2 = 1$. Hence a = c and a = -c are impossible. Therefore cases 1 and 4 cannot occur. In cases 2 and 3, we have either c = b or c = -b. Hence, $a^2 - c^2 = 1$ becomes $a^2 - b^2 = 1$.

Thus the matrix H equals

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ -a & -b \end{pmatrix}$$

corresponding to cases 2 and 3 respectively.

Which matrix is the "right" one? Consider their determinants:

$$\det \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a^2 - b^2 = 1$$

$$\det \begin{pmatrix} a & b \\ -a & -b \end{pmatrix} = -a^2 + b^2 = -(a^2 - b^2) = -1 .$$

Recall that if a matrix belongs to a one-parameter group, then its determinant must be positive. Since we want to have H in a one-parameter group, we wish H to have positive determinant. Hence, finally, we let

$$H = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \qquad (5)$$

with $\det H = a^2 - b^2 = 1$. The reader should check that such matrices preserve q. They are called <u>hyperbolic rotations.</u>

4. <u>The One-Parameter Group of Hyperbolic Rotations.</u> A rotation R_t is determined by a real number t. This number is chosen to be the angle between e = (1,0) and its image

under the rotation, that is, $R_t e$. (Figure 4) In the case of a hyperbolic rotation H, how can we choose an appropriate t so that we can write $H = H_t$? As with the ordinary rotation, one might guess that t should be the angle between e and He. However, when $t = \pi/4$, we have an embarrassing situation: the ray determined by the angle $\pi/4$ does not touch the hyperbola $q(x) = x_1^2 - x_2^2 = 1$ and hence $H_{\pi/4}$ is not defined if, we choose the parameter t in that way. (See Figure 5)

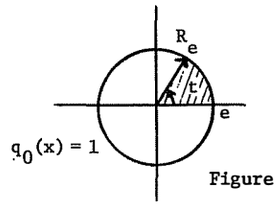


Figure 4

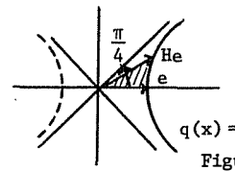


Figure 5

One reason for choosing an angle as the parameter for ordinary rotations is that angles are preserved under them. But the same thing does not hold for hyperbolic rotations. Nevertheless, areas are preserved under a hyperbolic rotation H because det H = 1. Therefore we try to choose the area of a suitable region as the parameter for H. Returning to the case of ordinary rotations, we find, from Figure 4, that the angle t (in radians) is twice the oriented area of the shaded fan-shape region. This suggests that

the parameter t of a hyperbolic rotation should be twice the area of the region bounded be e, He and the hyperbola $q(x) \equiv x_1^2 - x_2^2 = 1$ (Figure 5), with positive sign if He is above e and negative sign if He is below e. This t can be any real number because, as the angle between e and He nears $\pi/4$, t will be as large as we please. (This fact will be clear later on.)

Suppose $H = H_t$ and H is given by (5). The reader can check that a and b are determined by t. We define the functions cosh and sinh by $\cosh t = a$ and $\sinh t = b$. Thus we have

$$H_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \tag{6}$$

with $\det H_t = \cosh^2 t - \sinh^2 t = 1$.

Next we prove that $\{H_t: -\infty < t < +\infty\}$ is a one-parameter group.

For two vectors x and y satisfying $q(x) = q(y) = 1$, we write a(x,y) for the area of the region bounded by x, y and the hyperbola $q(x) = x_1^2 - x_2^2 = 1$, with positive sign if y is above x and negative sigh if y is below x. Note that $a(e, H_t e) = t/2$. Next lemma says that the same thing holds if e is replaced by any "unit" vector.

Lemma. If $q(x) = 1$, then $a(x, H_t x) = t/2$.

Proof: From Figure 6 we see that

$$a(x, H_t x) = a(H_t e, H_t x) - a(H_t e, x).$$

Since the transformation H_t is area preserving, we have $a(H_t e, H_t x) = a(e,x)$. Hence

$$\begin{aligned} a(x, H_t x) &= a(e, x) - a(H_t e, x) \\ &= a(e, H_t e) = t/2. \end{aligned}$$

The proof is complete.

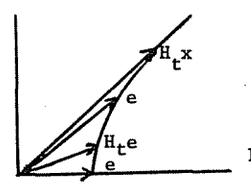


Figure 6

Theorem. $\{H_t: -\infty < t < +\infty\}$ is a one-parameter group.

Proof: It is obvious that $H_0 = I$. We have to show that, for all real t and s, $H_tH_s = H_{t+s}$. Since both H_t and H_s preserve the quadratic form q and have determinant +1, so is H_tH_s. Hence H_tH_s is a hyperbolic rotation. Therefore it remains to show that the parameter for this hyperbolic rotation is t+s. From Figure 7 and the above lemma we see that it should be

$$2a(e, H_tH_se) = 2(a(e, H_te) + a(H_te, H_tH_se))$$
$$= 2(t/2 + s/2) = t + s.$$

This completes the proof.

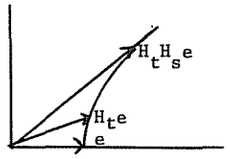


Figure 7

5. Hyperbolic Functions and the Function exp. By using the explicit expression for H_t, namely, the identity (6), the equality $H_tH_s = H_{t+s}$ becomes

$$\begin{pmatrix}\cosh(t+s) & \sinh(t+s)\\ \sinh(t+s) & \cosh(t+s)\end{pmatrix} = \begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}\begin{pmatrix}\cosh s & \sinh s\\ \sinh s & \cosh s\end{pmatrix}$$

By the matrix multiplication rule, we obtain

$$\cosh(t+s) = \cosh t \cosh s + \sinh t \sinh s$$
$$\sinh(t+s) = \cosh t \sinh s + \sinh t \cosh s$$

These are called the addition formula for hyperbolic functions. Adding both sides of the above two identities, we get

$$\cosh(t+s) + \sinh(t+s)$$
$$= \cosh t(\cosh s + \sinh s) + \sinh t(\sinh s + \cosh s)$$
$$= (\cosh t + \sinh t)(\cosh s + \sinh s)$$

We define the function exp by $\exp t = \cosh t + \sinh t$. Then we get

$$\exp(t+s) = (\exp t)(\exp s)$$

We deduce next the derivatives of hyperbolic functions. The procedure is similar to that of trigonometric functions, but we start with some general notions.

Let $\{T_t: -\infty < t < +\infty\}$ be a one-parameter group. Then the derivative $\frac{d}{dt}T_t$ is defined to be the limit $\lim_{h\to 0} h^{-1}(T_{t+h} - T_t)$. Indeed this is the matrix obtained by differentiating each entry of T_t. Note that this limit is equal to

$$\lim_{h\to 0} h^{-1}(T_hT_t - T_t) = (\lim_{h\to 0} h^{-1}(T_h - I))T_t$$
$$= (\frac{d}{dt}T_t\Big|_{t=0})\ T_t.$$

The matrix $A = \frac{d}{dt}T_t\Big|_{t=0}$ is called the infinitesmal generator of the one-parameter group $\{T_t\}$. Thus we have

$$\frac{d}{dt}T_t = AT_t \qquad (7)$$

The reader should check that the infinitesmal generator of the one-parameter group of ordinary rotations is $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and the infinitesmal generator of $\{D_t\}$ is $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$.

(From the theory of ordinary differential equations, one can show that, given any matrix A, there exists a unique one-parameter group $\{T_t\}$ which is a solution to the

differential equation (7).)

To find the infinitesmal generator for $\{H_t\}$, we have to know $\frac{d}{dt}\cosh t$ and $\frac{d}{dt}\sinh t$ at $t = 0$.

The derivative of sinh t at t = 0 is, by definition, $\lim_{t\to 0} (\sinh t)/t$. We make an estimate of $(\sinh t)/t$ as follows. From Figure 8, we have: area of $\Delta OA'B' \le \frac{1}{2}t \le$ area of ΔOAB.

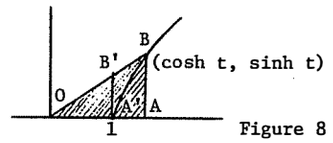


Figure 8

Now, OA' = 1, OA = cosh t and AB = sinh t. Since OA' : OA = A'B' : AB, we get A'B' = sinh t/cosh t. Hence, comparing areas, $\frac{1}{2}\cdot\frac{\sinh t}{\cosh t} \le \frac{1}{2}t \le \frac{1}{2}\sinh t \cdot \cosh t$. Therefore $\cosh t \le \sinh t/t \le 1/\cosh t$. Since $\lim_{t\to 0}\cosh t = 1 = \lim_{t\to 0} 1/\cosh t$, we must have $\lim_{t\to 0}\sinh t/t = 1$. Hence $\frac{d}{dt}\sinh t\Big|_{t=0} = 1$. Now

$$\frac{d}{dt}\cosh t\Big|_{t=0} = \lim_{t\to 0}\frac{\cosh t - 1}{t} = \lim\frac{\cosh^2 t - 1}{t(\cosh t + 1)}$$

$$= \lim\frac{\sinh t}{t}\cdot\frac{\sinh t}{\cosh t + 1} = 0 .$$

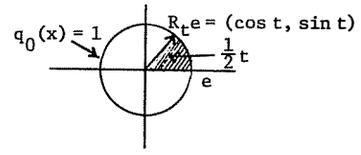
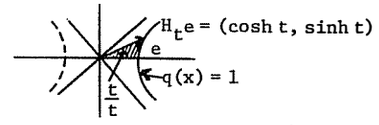
since $\sinh t/t \to 1$, $\sinh t \to 0$, $\cosh t + 1 \to 2$ as $t \to 0$. Now we see that the infinitesmal generator of $\{H_t\}$ is $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

The identity (7) becomes

$$\begin{pmatrix}\frac{d}{dt}\cosh t & \frac{d}{dt}\sinh t\\ \frac{d}{dt}\sinh t & \frac{d}{dt}\cosh t\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix} .$$

Thus $\frac{d}{dt}\cosh t = \sinh t$ and $\frac{d}{dt}\sinh t = \cosh t$. Recall that $\exp t = \cosh t + \sinh t$. Hence $\frac{d}{dt}\exp t = \exp t$. A function with derivative equal to itself!

6. Conclusion. We summarize the results we obtained and compare them with those ordinary rotations and trigonometric functions in the following list:

rotations R	hyperbolic rotations H_t
trigonometric functions sin, cos	hyperbolic functions sinh, cosh
$R_t = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$	$H_t = \begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}$
	
$R_tR_s = R_{t+s}$	$H_tH_s = H_{t+s}$
$\text{cis } t = \cos t + i \sin t$	$\exp t = \cos t + \sin t$
$\text{cis}(t+s) = (\text{cis } t)(\text{cis } s)$	$\exp(t+s) = (\exp t)(\exp s)$
$\cos^2 t + \sin^2 t = 1$	$\cosh^2 t - \sinh^2 t = 1$
$\det R_t = 1$	$\det H_t = 1$

R_t preserves q_0	H_t preserves q
$q_0(x) = x_1^2 + x_1^2$	$q(x) = x_1^2 - x_2^2$
infinitesmal generator of $\{R_t\}$: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	infinitesmal generator of $\{H_t\}$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\frac{d}{dt} \sin t = \cos t$	$\frac{d}{dt} \sinh t = \cosh t$
$\frac{d}{dt} \cos t = -\sin t$	$\frac{d}{dt} \cosh t = \sinh t$

References: 1. A. S. Cooper and C. Fong, A proposal for a course of linear algebra for high school students, Ontario Math. Gazette, (1975), 22-28.
2. S. Lang, Algebra, Addison-Wesley Pub. Co. Inc., Reading, Mass. (1965).

VECTOR PROOFS IN GEOMETRY

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This is the second part of an article which began in the December, 1977 issue.

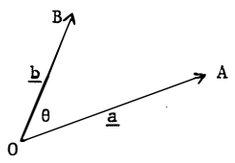
PART II

5. Dealing With Angles. The only way to deal with angles in vector proofs is with dot products.

The dot product of two vectors

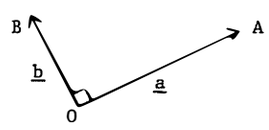
$$\underline{a} \cdot \underline{b} = |\underline{a}|\ |\underline{b}| \cos\theta$$

$$\therefore \cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\ |\underline{b}|}$$

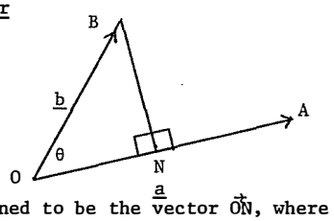


The condition for perpendicularity

$OA \perp OB$ if and only if $\underline{a} \cdot \underline{b} = 0$



The projection of a vector on a vector



The projection of $\underline{b}$ on $\underline{a}$ is defined to be the vector $\vec{ON}$, where $ON \perp NB$. Therefore the projection of $\underline{b}$ on $\underline{a}$ is a multiple of $\underline{a}$. To find which multiple it is we proceed as follows:

$$ON = OB \cos\theta$$
$$= |\underline{b}| \cos\theta$$
$$= |\underline{b}| \left(\frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|} \right)$$
$$= \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|}$$

Since $OA = |\underline{a}|$,

$$\frac{ON}{OA} = \frac{\dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}|}}{|\underline{a}|}$$
$$= \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|^2}$$
$$= \frac{\underline{a} \cdot \underline{b}}{\underline{a} \cdot \underline{a}}$$

$$\therefore \quad \vec{ON} = \left(\frac{\underline{a} \cdot \underline{b}}{\underline{a} \cdot \underline{a}} \right) \vec{OA}$$

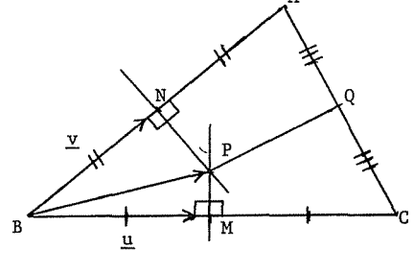
6. Further Examples. We are now ready to solve problems involving angles. Our strategy is the same as before--choose convenient "basis vectors" and express other vectors as linear combinations of them. This time we will need to use dot products and their properties.

Since the dot product expression includes a cosine, vector methods work best for problems involving perpendicularity. Vector solutions to problems involving angles other than 90° are usually inelegant.

Example 7: Prove that the right bisectors of the sides of a triangle are concurrent

Given: PM ⊥ BC, BM = MC
PN ⊥ BA, BN = NA
AQ = QC

Prove: PQ ⊥ AQ



Strategy

1) The only way to prove PQ ⊥ AQ is to prove that $\vec{PQ} \cdot \vec{AQ} = 0$. Therefore we must find expressions for $\vec{PQ}$ and $\vec{AQ}$ in terms of other vectors. Convenient choices for these vectors are $\vec{BM}$ and $\vec{BN}$, since these are the sides of right angled triangles.
2) Let $\vec{BM} = \underline{u}$ and $\vec{BN} = \underline{v}$. Then $\vec{AQ}$ can be found from ΔABC.
3) To find $\vec{PQ}$, we must first determine $\vec{BP}$. We do this by letting $\vec{BP} = s\,\underline{u} + t\,\underline{v}$. To find s and t, we let $\vec{MP} \cdot \underline{u} = 0$ and $\vec{NP} \cdot \underline{v} = 0$.
4) We can now find $\vec{PQ}$ by using quadrilateral PBAQ, and the solution will conclude by evaluating $\vec{PQ} \cdot \vec{AQ}$. This dot product should have a value of 0. It has to, or the result would not be true!

Solution:

Let $\vec{BM} = \underline{u}$ and $\vec{BN} = \underline{v}$.

$$\vec{AQ} = \frac{1}{2} \vec{AC}$$
$$= \frac{1}{2} (\vec{BC} - \vec{BA})$$
$$= \frac{1}{2} (2\,\underline{u} - 2\,\underline{v})$$
$$\therefore \quad \vec{AQ} = \underline{u} - \underline{v}$$

Let $\vec{BP} = s\,\underline{u} + t\,\underline{v}$.

$\therefore\ \vec{MP} = \vec{BP} - \vec{BM}$ and $\vec{NP} = \vec{BP} - \vec{BN}$

$= s\,\underline{u} + t\,\underline{v} - \underline{u}$ $= s\,\underline{u} + t\,\underline{v} - \underline{v}$

$\therefore\ \vec{MP} = (s-1)\,\underline{u} + t\,\underline{v}$ $\therefore\ \vec{NP} = s\,\underline{u} + (t-1)\,\underline{v}$.

Since $\vec{MP} \cdot \underline{u} = 0$ and $\vec{NP} \cdot \underline{v} = 0$

$(s-1)\,\underline{u} \cdot \underline{u} + t\,\underline{u} \cdot \underline{v} = 0$ (1)

$s\,\underline{u} \cdot \underline{v} + (t-1)\,\underline{v} \cdot \underline{v} = 0$. (2)

Rather than solve (1) and (2) for s and t, we will use these equations later.

$\vec{PQ} = \vec{PB} + \vec{BA} + \vec{AQ}$

$= (-s\,\underline{u} - t\,\underline{v}) + 2\,\underline{v} + (\underline{u} - \underline{v})$

$\therefore\ \vec{PQ} = (1-s)\,\underline{u} + (1-t)\,\underline{v}$.

$\therefore\ \vec{PQ} \cdot \vec{AQ} = [(1-s)\,\underline{u} + (1-t)\,\underline{v}] \cdot [\underline{u} - \underline{v}]$

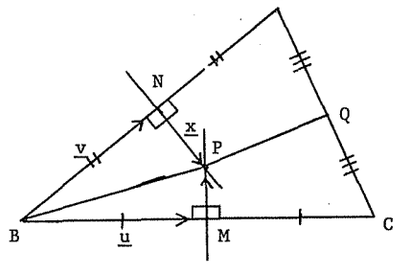
$= (1-s)\,\underline{u} \cdot \underline{u} - (1-s)\,\underline{u} \cdot \underline{v} + (1-t)\,\underline{u} \cdot \underline{v} - (1-t)\,\underline{v} \cdot \underline{v}$

$= [(1-s)\,\underline{u} \cdot \underline{u} - t\,\underline{u} \cdot \underline{v}] + [s\,\underline{u} \cdot \underline{v} + (t-1)\,\underline{v} \cdot \underline{v}]$

$= 0$ (using equations (1) and (2))

$\therefore\ PQ \perp AQ$ q.e.d.

The strategy used in solving this problem was direct and simple. However, the algebraic steps were far more complicated than they need to be. A much easier solution is obtained by introducing single symbols for the vectors $\vec{MP}$ and $\vec{NP}$.



Alternate Solution

Let $\vec{BM} = \underline{u}$, $\vec{BN} = \underline{v}$, $\vec{MP} = \underline{w}$ and $\vec{NP} = \underline{x}$.

$\therefore$ from triangles BMP and BNP,

$\vec{BP} = \underline{u} + \underline{w}$ and $\vec{BP} = \underline{v} + \underline{x}$

$\therefore\ \underline{u} + \underline{w} = \underline{v} + \underline{x}$ (1)

Since $MP \perp BM$ and $NP \perp BN$,

$\therefore\ \underline{w} \cdot \underline{u} = 0$ (2)

and $\underline{x} \cdot \underline{v} = 0$ (3)

$\vec{PQ} = \vec{PB} + \vec{BA} + \vec{AQ}$

$= (-\underline{x} - \underline{v}) + 2\,v + (\underline{u} - \underline{v})$

$\therefore\ \vec{PQ} = \underline{u} - \underline{x}$

$\therefore\ \vec{PQ} \cdot \vec{AQ} = (\underline{u} - \underline{x}) \cdot (\underline{u} - \underline{v})$

$= \underline{u} \cdot \underline{u} - \underline{u} \cdot \underline{v} - \underline{x} \cdot \underline{u} + \underline{x} \cdot \underline{v}$

$= \underline{u} \cdot (\underline{u} - \underline{v} - \underline{x}) + \underline{x} \cdot \underline{v}$

$= \underline{u} \cdot (-\underline{w}) + \underline{x} \cdot \underline{v}$ (using (1))

$= 0$ (using (2) and (3))

$\therefore \quad \overrightarrow{PQ} \perp \overrightarrow{AQ}$ q.e.d.

Example 8: Prove that the angle in a semi-circle is a right angle.

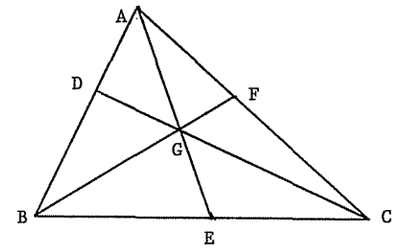
Example 9: Prove that the midpoint of the hypotenuse of a right angled triangle is equidistant from the three vertices. (Converse of Example 8.)

Example 10: Prove that the altitudes of a triangle are concurrent.

Example 11:

Given: BD : DA = a : b
BE : EC = c : d

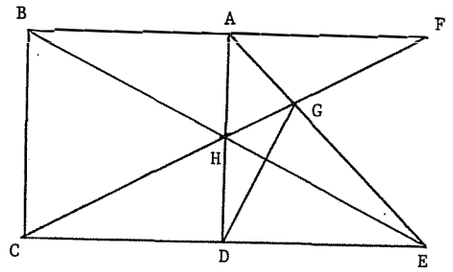
Find: CF : FA
CG : GD
AG : GE
BG : GF



Example 12:

Given: Square ABCD
H is any point on AD.

Prove: DG ⊥ BE



Students should realize that vector methods can be used to solve problems involving such things as overlapping triangles, midpoints, ratios of line segments, parallelism, and perpendicularity. It is hoped that the strategies given in this article will help them learn how to solve these kinds of problems.

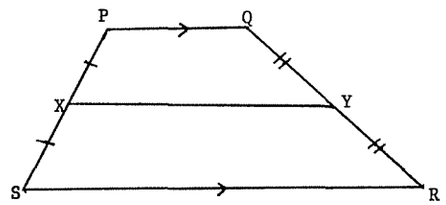
Appendix: The following additional examples may be used to illustrate the strategies in Part I of this article (December 1977).

Laws of Addition Used Twice

Example 13:

Given: PX = XS
QY = YR
$\overrightarrow{SR} = 3\overrightarrow{PQ}$

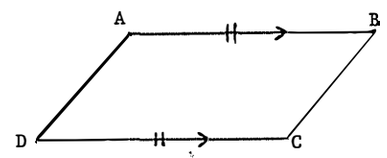
Prove: $\overrightarrow{XY} = 2\overrightarrow{PQ}$



Example 14:

Given: AB = DC
AB ∥ DC

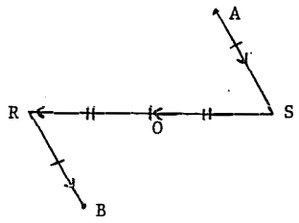
Prove: AD = BC
AD ∥ BC



Example 15:

Given: $\vec{AS} = \vec{RB}$
$\vec{SO} = \vec{OR}$

Prove: A, O, B are collinear

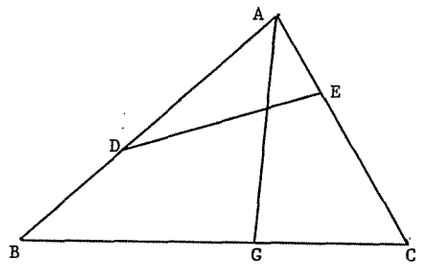


Choose Two Non-Parallel Vectors and Express Others in These Terms

Example 16:

Given: AD : DB = 3 : 2
AE : EC = 1 : 2
DF : FE = 3 : 1

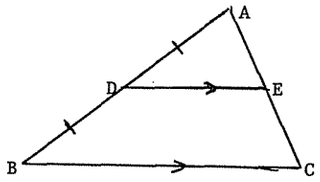
Find: AF : FG
BG : GC



Example 17:

Given: AD = DB
DE ∥ BC

Prove: AE = EC

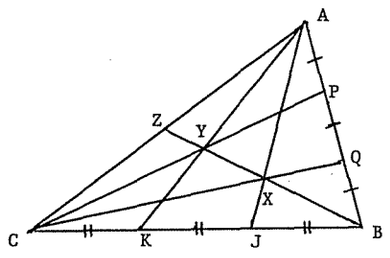


Example 18:

Given AP = PQ = QB
BJ = JK = KC

Prove: B, X, Y collinear

Find: AZ : ZC
BX : XY : YZ



FUN MATHEMATICS

9. How far can a dog run into the woods?
10. What is the minimum number of active baseball players "ON THE FIELD" during any part of an inning? How many outs in each inning?
11. I have in my hand only two U.S. coins which total 55c in value. One is not a nickel. Please bear this in mind. What are the two coins?
12. A farmer had 17 sheep. All but 9 died. How many did he have left?
13. Divide 30 x ½ and add ten. What is the answer?
14. Two men were playing checkers. Each played five games and each man won the same number of games. No draws. How can you figure this?
15. Take two apples from three apples and what do you have?
16. An archaeologist claimed he found some coins of gold dated 46 B.C. Do you think he really did?

Answers on page 64.

THE TEACHING OF MATHEMATICS A FRENCH EXPERIMENT IN CONTINUOUS TEACHER TRAINING

Jean G. Dhombres, Scientific Counsellor, Embassy of France to Canada

We had hoped to print the following article in French. However, it was received in English so we have printed it as is.

All countries, whether developed or developing, are experiencing upheavals, sometimes violent - in fundamental teaching theories and methods. Nowhere is this more evident than in the domain of mathematics.

When faced with the complexity and rapidity of evolution in the discipline, legitimate concerns have been raised about the more or less traditional pedagogic methods which has approached almost crisis proportions. The introduction of the so-called "New Math" is a striking aspect of these frantic developments.

There seem to be two separate positions, which are intimately entwined, each requiring individual analysis, specifically the pedagogical issue (the function of teaching itself) as well as the mathematical issue.

Now, the arguments over mathematics, may be tainted with either intentional emotional humanism or with uncompromising technocratic efficiency but they cannot disguise the fact that the true debate is essentially a sociological one.

The Elimination Process

In the French school system, from nursery school through secondary school, the teaching of mathematics is openly based on the elimination process - selection by failure. In our grandparents' day, high schools reserved the teaching of Latin for the few. Some learned people were actually serious when they argued that this dead language was indispensable in the teaching of medicine. Today, the role, once reserved for Latin, has been transferred to mathematics. However, an insidious psychological modification has surreptitiously occurred. In grandfather's day an ignorance of Latin indicated one's non-membership in certain social strata. Whereas today, the same kind of ignorance, but in mathematics, can wrongfully undermine a young students' confidence to the point of persuading him so deeply of his intellectual incapacity that he doubts the role he can ever occupy in society.

To be clear from the beginning of this discussion, in the French student's day to day school work, mathematics occupies an inescapable and dominant position.

The special status of mathematics taught in French schools, even if it appears far removed from the objectives of mathematics as a science, weighs heavily on every teacher. Indeed, this is without a doubt, of concern to every social class in every country.

Who decides and about what?

In making decisions about content or simply in wrestling with one's prejudices, a first if not unique area of consideration must be the scope of mathematics itself: What should be taught? an axiomatic method, for example? ... or, experimental geometry? .. or, discursive geometry?

A second level of issue raises pedagogical queries - whether they concern either the mathematics program itself or its integration into the overall course of study: How to get across the ideas of a given program? Which mathematical elements best suit the student's age level, tastes and interest? How should mathematics be related to social sciences, to biology or to politics (gallups and modes of election as well)?

Yet a third level of concern to be studied embeds the role of mathematics teaching within the

educational system itself or within the society at large: Should we teach in order to dispense knowledge or should we teach a way of life? Should we teach to select the best students? Should we teach in order to turn out citizens who can cope with the technological and evolving society of the 1980's or should we teach for the sheer cultural pleasure and self-fulfilment that learning brings?
The complexity of the issue is such that opposite or parallel ideologies will, therefore, compete in day to day practice, whether they be ideologies arising spontaneously from mathematics itself, from the teaching experience, or from the theoretical science of education. (L'Emile has his devotees, but then so has Illich!)

The intent of this article

Faced by such an avalanche of complex questions, I would like to emphasize the unpretentious nature of the ideas I am going to put forward.

Firstly, I cannot pretend to fully discuss the reasons for the new role being played by mathematics which doubtless stems from the intensive development of this science in the positivist philosophy of our times. The extension of education to all levels of society is, of course, another contributing factor.

Secondly, I do not wish to comment critically on the various questions raised in the debate in order to favor one position over another, rather, I wish to outline a particular French experiment in the methodology of teaching mathematics, and the continuing training program for teachers in the subject area. This experiment has now been tested for over ten years, which I hope, may brush aside the possible reproaches of both egoism or foolhardiness. Moreover, who said ideas do not come from heaven? So, let us describe something existing.

Since the French educational system is very different from that of English speaking countries, I will be obliged to sometimes point out in the briefest way certain French peculiarities.

The French Irem Experiment

The gigantic shift in the significance of mathematics training in the 60's so profoundly affected the French educational structure that the Ministry of National Education, a centralized pyramid-shaped structure, was obliged to concomitantly react while setting in place a new mathematical program for secondary schools. The Ministry decided in early 1968 to create in each educational zone, called an academy [1] an Institute for Research in Mathematics Education. These institutes are referred to by their initials under the acronym IREM.

At the beginning of the October '74 school year, each academy had such an Institute (IREM), each with a university affiliation, a team of educators and its own rather comfortable [2] budget provided by the Ministry of Education.

IREM: To do what?

The role of an IREM is threefold:

Firstly, it consists of assuring the continuing education of mathematics teachers by organizing on-the-job training.

Secondly, it undertakes research on possible methods for teaching mathematics, implements them in the classroom, then evaluates their suitability and effectiveness.

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[1] France is divided into 24 "academies". At the head of each academy a rector appointed by the Government, has the responsibility for all educational questions from nursery school through the primary, secondary and university levels. He (but seldom she) is chancellor of the universities and is also responsible for some professional schools.

[2] An always rather subjective statement, not to be pronounced too loudly if things are to go on.

Thirdly, it encourages multidisciplinary teams to coordinate the logical sequence of instruction in various subject matters in the same grade.

Thus, by its very nature, an IREM is involved in educational projects at the level of teachers while accenting in-service training and method's assessment.

IREMS certainly maintain the spirit of Professeur A. Lichnerowicz, President of the Mathematics Great Commission in the Ministry of Education. The following quotation from this famous geometrician roughly describes the IREM philosophy:

"Mathematics, in the eyes of all thinking men who must act, enjoys a particular status among the sciences. It cannot be defined as the study of a certain group of phenomena. It matters even to people who are the least concerned with technological or scientific knowledge of our physical universe. It matters because it bears constant witness to the way in which our minds function and the way in which we communicate accurately with each other.

For anyone belonging to the last third of the 20th century, it is not only a tool of his trade, but also, amongst other things an essential element of his culture".

The IREM Structure

1) **Reduction of teaching time.**

Each IREM is assigned by the Ministry of Education a certain "number of teaching hours per week". At the beginning of the school year, the IREM managing team assigns these hours to secondary and sometimes primary level teachers.

This means the teachers in question[3] are officially relieved each week of 1, 2, even 3 hours of courses in order to pursue IREM activities.

These IREM trainees then meet weekly or fortnightly for an afernoon (3 or 4 hours) during a one year period. Under the guidance of a group leader, they work together on a predetermined theme. It is important to note that being part of the IREM team carries no special merit and has no influence on either the trainee's salary or grade.

Each group always has as its leader a high school teacher who is allowed substantial relief (3 to 9 hours weekly) from his teaching reponsibilities which would ordinarily average 17 hours a week. In addition to this high school teacher group leader, another research consultant from a university closely follows the projects activities.

In 1975-1976 for example, the IREM of Nantes Academy[4] employed 13 research consultants from universities, 34 group leaders from universities, and had 366 trainees.

The group leaders meet frequently for brief periods and at least twice yearly hold two-day seminars, if possible in some secluded country village, so that everybody is free from the daily burden of class activities.

2. **The IREM trainees**

It may be useful to appreciate the composition of trainee groups since the French teaching system is sometimes accused of maintaining a mandarin cast structure.

Of the 366 trainees in Nantes in 1975-1976 there were:

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[3] for simplicity sake, I will call these teachers IREM trainees.

[4] I have chosen this particular example not only because I was part of it, as its Director but also because it is a typical medium-sized institute.

- 6 agreges [5]
- 170 certified teachers [6]
- 190 P.E.G.C. [7]

Let me specify that in the Nantes Academy area there are about 1800 secondary level mathematics teachers. Under the current regulations, financial aid is unfortunately not provided for supplementary teachers. These teachers, however, can participate in IREM activities and about 40 such teachers were enrolled last year.

The trainees are divided into many groups, 29 in 1975-1976, each group averaging about a dozen participants. The activity centers are scattered throughout the geographical area of which there are eleven regional centers in the Nantes Academy. Travel expenses are reimbursed by IREM.

The Pedagogical Equipment

Thanks to its generous budget, IREM has had access to sufficient funds to purchase books, (1,000 volumes at Nantes on Mathematics and teaching methods; even more volumes at IREM Paris), pocket - calculators programmable with magnetic cards, and a library of relevant application programs. (15 HPG 5's and 20 HPz 5's in Nantes), two advanced programmable calculators 2200S, [8] a magnetoscope and an audio-visual tape system. Due to inflation, the ability of IREM to up-date its equipments is unfortunately declining.

The Topics under study in an IREM.

1) **Continuous training to maintain teacher knowledge.**

This aspect of the program demands more resources that any other for three main reasons:

a) the necessity to train teachers in parts of elementary mathematics, which only recently have been introduced into the university curriculum [9]: linear algebra, vectorial geometry, topology, logic, combinatorics, probability theories, statistics and data processing, are the most common examples. It does not imply that such subjects must **stricto sensu** all be taught in classes. They ought not only to be known but also to be mastered by teachers; the reason being that such fields, constitute the unifying pattern by which one grasps mathematics as a whole.
b) the moral obligation, of keeping teachers abreast of current mathematical research, and
c) the need to supplement and increase the theoretical training of teachers, even amongst the most recent graduates. It is painful to insist on this point, but one finds in the CES [10], many

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[5] **Agregation** (or affiliation) is the most sought after recruitment competitive examination and therefore the most difficult, for secondary school teachers. Coming after a master's degree, its preparation requires no activity in mathematical research and up to now, only minimum pedagogical knowledge.

[6] The secondary school teaching certificate is similar to the "agregation" but is one notch lower in difficulty and demands some pedagogical training.

[7] The Teachers of the General Education Colleges are former primary school instructors who have obtained secondary school teaching certificates after some internal competition. They do not, however, have any specialized mathematical training. Included in this group are their colleagues, the technical school teachers.

[8] The Nantes IREM, as well as others, pays for a certain number of hours of computer time at the University data processing centre.

[9] In addition, the new programs are embroiled in jargon. As an example, consider this "official" definition of a straight line for 8th grade students (12-13 years of age): "A straight line is defined as a set of elements called points equipped with a bijection g from D onto R and with all those which arise from f in the following manner, using a real arbitrary number a.

$$f(M) = g(M) + a, \text{ and } f(M) = g(M) - a$$

The family of all bijections f is called an Euclidean structure".

[10] Junior High Schools (College Enseignment Secondaire).

mathematics teachers who have never taken maths at a university level and so never went into a complete mathematical theory. This situation pertains because promotions within the Ministry of Education, albeit deserved, have been based on considerations other than mathematical training.

2. **Research and evaluation**

Research and evaluation are the most creative and the most necessary tasks of the IREM. Here, obviously the need for the non-mathematician is felt: psychologists, specialists in education, semeiologists, epistemologists, etc. .. The asset of having an IREM associated with a university should be apparent.

What is happening today is that faculties of teacher education are establishing programs on the didactics of mathematics: at Strasbourg, Paris and Bordeaux, for example. Rather than going into the details of such research and training programs, it is preferable to underline the fact of an evolving new attitude toward teacher development in mathematics by the universities and mention that it is also penetrating other levels of the educational system. This new perception in the long run will have an important impact on education.

The confrontation of ideas between academic theorists, and the classroom experts who are the teachers in primary, secondary or technology schools must, and will, lead to progress in the field of teaching.

At the primary level, for example, certain IREMS are associated with experimental schools (such as those in Grenoble, Bordeaux, Paris), and have acquired an international reputation (if one can judge by the number of invitations their members have received to attend seminars on pedagogy all over the world).

Many IREM centers study the impact of data processing on teaching not only because of the speed of calculating, even with small office calculators, but because in one's dealing with modern hardware, one is exposed to the internal structure of program languages and consequently the logical thinking which this imposes on the mind. As a result realistic teaching experiments can be assisted by computers.

In mentioning audio visual methods and their development by IREM, it should be noted that the availability and good use of video cassettes, cameras and playback projectors has been obviously useful. But is it not still more useful to make teachers realize the psychological and structural changes brought about by this audio-visual arsenal in the teaching profession? This awareness, which no text will explain, can be developed in work groups by presenting actual real-life confrontations and experiences.

Other teams study the way some children react when faced with a situation more or less mathematical in nature: what languages they use; which words or what attitudes divert them from an interesting solution; which group leaders grasp or reject good ideas. Heuristic or problem-solving research, which recalls such innovators as G. Polya, requires groups well-equipped with gifted observers capable of sensitive analysis and the ability to clearly interpret their perceptions. Many mathematical games have been catalogued this way.

Finally, other groups (in some IREMS) deal with failures in mathematics and with the social role of both teacher and the teaching of mathematics. Being a group leader under such scrutiny, is not an easy task, particularly if one wants to avoid pitfalls such as prejudging situations whether mathematical, political or social. But is not the very existence of such groups a testimony to a certain new maturity in our attitude?

3. **The multidisciplinary teams**

There are many multidisciplinary groups that unite teachers of mathematics with teachers of applied sciences, physics, history and geography, philosophy, French, natural sciences, etc. It is not enough to juxtapose these disciplines.

Careful reflection on the objectives, an awareness of the weaknesses and pitfalls of a multidisciplinary group, concentration on what should constitute the essential ingredient of each subject (what might be called its internal structure) and close contacts of the day to day life of classes, are the composite elements which bring life to new ideas and moreover sustain the necessary stamina for such groups. The National Experimental Scientific Teaching or E.S.E. (Enseignement Scientifique Experimental) is worth mentioning. Located at Grenoble, Rennes and Toulouse, its aim is to study the possibility of integrating scientific teaching into the 7th grade program while evaluating with minute precision the progress in a given number of pilot classes.

The following is a random list of the subjects studied by the groups at the Nantes IREM;

- Plane curves, initiation to algebraical topology and analysis,
- Introduction to programming,
- Approximation and Data Processing,
- Up-dating linear algebra and affine geometry,
- Initiation to Data Processing,
- Programmable calculators,
- Analysis and topology,
- Probability,
- Teaching mathematics at primary level,
- Links between primary and secondary education,
- Mathematics at professional schools,
- Physics-Mathematics Relationships,
- History of mathematics: the continuum through history,
- Failure in mathematics,
- Mathematical games.

4. **Publications**

Many IREMS issue a trimestrial bulletin and there is also an Inter IREM Bulletin. In the latter bulletin, international items are being discussed and a special issue (no. 14, March 1977) was devoted to the Quebec in-training of mathematics teachers (PERMAMA): another one to the Social Functions of Mathematics Teaching (no. 13, October 1976)

As an example for the reader, here follows a list of some Nantes IREM publications, printed under the generic name of **Nanta Iremica**; each work contains between 100 and 300 pages.

NANTA IREMICA

Volume no. 1	**Introduction à la logique**	Mr. DURAND Mr. VAN DEN BOSSCHE
Volume no. 2	**Introduction à la théorie des ensembles**	Mr. DURAND Mr. VAN DEN BOSSCHE
Volume no. 3	**Etude épistémologique et historique de la notion de nombre, de mesure et de continu**	J. DHOMBRES
Volume no. 4	**Documents historiques relatifs au volume 3**	J. DHOMBRES
Volume no. 5	**Le langage BASIC**	Mr. BELHACHE
Volume no. 6	**Echec en mathématiques**	A. BIGARD
Volume no. 7	**Eléments d'analyse fonctionnelle**	J. DHOMBRES
Volume no. 8	**Exercices d'analyse fonctionnelle**	J. DHOMBRES
Volume no. 9	**Algèbre linéaire et géométrie vectorielle**	R. SEROUX
Volume no. 10	**Analyse et topologie: convergence, les réels et les rationnels**	Miss VENARD J. DHOMBRES
Volume no. 11	**Méthodes mathématiques modernes utilisées en théorie de l'approximation**	J. DHOMBRES
Volume no. 12	**Géométrie et axiomatique**	J.P. LETOURNEUX
Volume no. 13	**Informatique et enseignement**	J. BETREMA
Volume no. 14	**Le langage PL/1**	Jr. BELHACHE

IREM as an Innovator

As strange as it may seem, the most important innovation of IREM is certainly the fact that it has become a university department. Until 1969, in spite of the initiatives of the Mathematics Professors' Association and personal efforts of certain well-known French academics, the bonds between the teaching staff of the university and those of other levels of education, were purely juridical. In some cases, they did not even share the same initial teacher training. An IREM, by contrast, unites university teachers with those of all other educational levels.

A second innovation, and a very real one, is the absence of bureaucracy where secondments to the IREM are concerned. Teachers are taking care themselves of the IREM and there does not exist a permanent and cumbersome staff to tell us what ought to be done.

A third novel aspect of IREMs leave to each center the task of determining its own work policies so that it is not locked into the narrow and rigid straight jacket of an overall program. In this way, each IREM can grasp the essence of the educational problem in its own area. Whether one approves or not, from Montpellier to Rennes, through Caen, Paris, Grenoble, or Lyon, IREMS have their own individual characteristics. In any event, the numerous inter-IREM meetings permit an exchange of various points of view in an atmosphere of research, free of hierarchical or bureaucratic formalities.

IREMS certainly are not a unique example in the world of in-house training, but they do stress the principle that teacher training in mathematics cannot be limited to a single initial period; it is a continuing process.

INTERVIEW

Taped Talks with Bob Smith

• • • John Del Grande

John Del Grande needs no introduction to an Ontario audience. For those outside the province, let me introduce him as an outstanding teacher of mathematics, a well-known author, and the Coordinator of Mathematics for the Borough of North York. It was my pleasure to interview him in his office overlooking the snowy site of North York's municipal complex at 5050 Yonge Street

R.S. John, has mathematics always been a first love?

J.D.G. Yes, I seemed to have a natural aptitude for mathematics. I remember as a little boy that I was able to add up a column of figures faster than Sister Mary could carry the pointer up to the board. I was always singled out to show my mathematics prowess when the Bishop visited the class.

R.S. I understand that Professor Coxeter has been a real influence on you.

J.D.G. To know Doctor Coxeter is to know a real human being, a man with a great heart and a great soul. When I was a student at the University of Toronto, almost thirty-five years ago, he was my professor. The geometry he was doing at that time, was very much the geometry of the Greeks. It has only been in the last ten years of my association with Doctor Coxeter that I have come to realize he is no longer using the old approach to Euclidean Geometry. It is now very much the modern approach, the approach through Transformation Geometry.

R.S. Unfortunately, many teachers place geometry at the end of their priority list. Parents want their children to be skillful in the basics. How do you propose to convince them that transformation geometry is necessary?

J.D.G. Let me answer that with a question. "How Do Children Learn Geometry?" This is a question that was not answered until about ten years ago when Freudenthal, an outstanding geometer and mathematician, brought the research of Van Hiele to light. The Russians were interested and have spent five years checking to see how accurate the research was. Some glaring errors in their pedagogy have been found. For example, you do not start teaching geometry in Grade 10 as the course requires a student be a level three, according to Van Hiele. Van Hiele states that experience at levels one and two are necessary prerequisites. This is different from the Piagetian states of learning, where given time, a pupil can often mature through the stages. At one stage, a child can not conserve number, but if you wait long enough he will be able to do so. In geometry, you do not mature through level one. You must experience level one, then level two, etc. Geometry of level 1 and 2 requires active hands-on experience, - experience which in our present system can only be obtained in the elementary school. Geometry **must be** started at an early age.

R.S. Is this why you are advocating that by 1980, here in North York, at least one quarter of the time given to mathematics in the primary grades will be on geometry.

J.D.G. Yes. In fact, we have proposed one third of the program for the kindergarten be spent on geometry. I know this upsets a lot of people, and a lot of people shake their heads and say it can't be done. My reaction to them is ... YOU DON'T KNOW.

R.S. I have found the primary teachers to be very flexible and prepared to try new things. Where is the problem?

J.D.G. It seems to revolve around irresponsible statements such as "You can not do transformations in the elementary school." There was a similar reaction a few years ago when it was felt that calculus was not a fit topic for the Secondary School. I felt very strongly that it should be. I recall the resentment I felt when I discovered that first year University calculus had no relationship to the Grade 13 course in mathematics. There was a gap and I felt the bridge should properly be made at the secondary school level. The same thing applies now with transformations.

R.S. Perhaps the word geometry, with its memories of deductions, constructions, and proof is the problem. Could we examine with you the question: What is Geometry?

J.D.G. If we could talk about the study of space, I think we would have more acceptance for the subject. Particularly when teachers realize concepts such as proof have meaning to a Grade 1 child. Proof is not the horrendous thing that teachers experienced in school. In fact, teachers soon learn we are not talking about the same kind of geometry that they studied. Specific guidelines for teachers are essential.

R.S. What are your feelings about the proposed guidelines for Intermediate Division Mathematics?

J.D.G. I can support the guidelines from a philosophical point of view. The changes they suggest have been coming about slowly in Ontario through the work of interested teachers. However, the way in which they were presented to the province for validation was terrible. Further, no effort was made to contact schools to find out what teachers have been doing, to discover what they have found to be successful parts in their programs. As an example, I know of a school which has an excellent Grade 10 program in statistics, but the present guidelines would prohibit teaching that course. Over the last few years Boards of Education were encouraged to build their own curricula and the Ministry should take our experience into account. It is my hope that we might have a mathematics curriculum that is interesting for students, for it is through interest that you obtain commitment to learn. I believe that the coordinators as a group have much to offer in this regard. To my knowledge the mathematics coordinators were not given the opportunity to make a significant contribution.

R.S. The calculator has been spoken of as a device that can either harm or help a program. How do

you see it being put to good use?

J.D.G. We do want pupils to calculate well. If the calculator can help in meeting this goal, we suggest the teacher use it. Some work on number patterns and number puzzles demonstrates the usefulness of the calculator. I see the opportunity it provides to investigate problems as its greatest advantage. For example, Bill Bissett of A.Y. Jackson Secondary School wondered what would happen if he punched 0 to the 0 power. The answer was 1, and that led to an interesting little investigation. We looked at x to the x power and let x approach 0. Without the calculator, the work involved would have been prohibitive. With the calculator we did the investigation easily. Why don't you try it?

R.S. We will.

I DIDN'T KNOW THAT

TRICKS — GAMES — PUZZLES

Dr. H. Laurence Ridge, Faculty of Education, University of Toronto

GETTING TO THE "ROOT" OF THE PROBLEM

Here are a few old-timer number curiosities set up for classroom consumption. They all depend on the same principle. If your calculator's handy, be our guest. We've given you and your students enough clues in the "Explanations" for you to figure out why these 'tricks' work. Following that, more advanced generalizations are given for those of you who really want to dig in.

A. **Divisible by 3?**

1. Which of these numbers are divisible by 3 (zero remainder)?
   13, 57, 75, 108, 243, 324, 572, 573

2. Now add the digits of each number. If the sum is ten or more add the digits again.
   (Example: For 85, $8 + 5 = 13$, $1 + 3 = 4$)
   4 is called the DIGITAL SUM or DIGITAL ROOT of 85.

3. Which of the numbers in question 1 have a DIGITAL ROOT that is divisible by three?

4. What's the significance of the result in question 3?

B. **Divisible by 9?**

1. Which of these numbers are divisible by 9?
   13, 57, 75, 108, 243, 324, 512, 573

2. What appears to be true about the digital root of a number divisible by 9?

3. What is the significance of the result in question 2?

Explanation for A and B

Testing for divisibility by 3 or 9 by finding the DIGITAL ROOT of a number is more popularly called CASTING OUT NINES. Now here's a big clue as to why it works. The details are for you or your students to complete.

$378 = 3x____ + 7x____ + 8$
$= 3x(__+1) + 7x(__+1) + 8$
$= 3x__ + 7x__ + 3 + 7 + 8$

Since $3x____ + 7x____$ is divisible by 9 (or 3) then 378 is divisible by 9 (or 3) if (____) is divisible by 9 (or 3). (Note that $3 + 7 + 8 = 18$ and then $1 + 8 = 9$).

C. **What's the Difference?**

1. Take any four digit number (or three digit or any number of digits) (Eg. 6783)

2. Make a different number by mixing up the order of the digits of your number in question 1. (Eg. 3867)

3. Find the difference of the numbers. (6783 — 38687 = 2916)

4. Test the difrerence for divisibility by 9. (2 + 9 + 1 + 6 = 18, 1 + 8 = 9)

5. Try it with different numbers of digits.

6. What's your conclusion?

Explanation for C:

$6783 = 6x____ + 7x____ + 8x____ + 3x____ + (6+7+8+3)$
(Look back at procedure (*) in the Explanation for A and B if you're stuck).

$3867 = 3x____ + 8x____ + 6x____ + 7x____ + (3+8+6+7)$

$6783 - 3867 = (6-3)x____ + (7-8)x____ + (8-6)x____ + (3-7)x____$
$= 3x__ - 1x__ + 2x__ - 4x$
which must be divisible by 9. Why?

D. **Cheques and Balances**

If bank tellers find their 'balance' doesn't agree with the true 'balance', they find the difference then add the digits of the difference. If the sum of the digits is divisible by 9 what kind of error have they probably made in their calculations?

Explanation for D.

The difference between any number and a number formed by changing the order of any of its digits is divisible by 9 (according to part C). Tellers would look for a 'transposition' of digits in their copying of figures. (Eg. $214.73 instead of $241.73.)

E. **Mathmatch Game '78**

Now you're ready to mystify and amaze all and sundry with the MATH MATCH GAME and know why it works!

Tell your audience to make up any four digit number where the first and last digits differ by at least 2. (Eg. **5137,** but not **5136**)

Reverse the order of the digits. (7315)

Subtract the smaller, number from the larger (Eg. 7315 — 5137 = 2178)

Now have each person read you this difference replacing any **one** of the digits by the word "BLANK" (as in the television Match Game). (Eg. for 2178, they might read "2 1 BLANK 8", or "BLANK 1 7 8").

You can now miraculously match their blank with the correct digit (at least nine out of ten times correctly).

Here's how:

For 2 1 BLANK 8, add 2 + 1 + 8 to get 11. To get to the closest (greater) multiple of 9 (18 in this case), you need to add 7. This means that "BLANK" must be '7' (because the difference is divisible by 9 according to part C and so the digits must add to a multiple of 9 according to part B).

For you and your audience to investigate:

1. Why should the first and last digit differ by at least 2?

2. For a four digit number, is the sum of the digits of the difference always 18? (More generally, how is the sum of the digits of the difference related to the number of digits?)

3. Why can you make a match reliably only nine out of ten times? (Note that 9887 — 7889 = 1998 but 9787 — 7879 = 1908).

4. For added amazement and amusement have your audience multiply the difference by their favourite number (such as their age); then read out the resulting digits including a "BLANK". The same principle still applies (Why?).
   (Even more amazing to the audience will be your 'power' to tell them when they've made an error in calculation. This occurs when the sum of their given digits is not a multiple of 9) as it was in question 3) but they claim that you have not matched their "Blank digit.

F. **The Multiplier's Delight**

Here's a way of checking multiplication that's right out of the ark. Maybe you can figure out now why it works.

3175 → 3 + 1 + 7 + 5 = 16, 1 + 6 = 7
429 → 4 + 2 + 9 = 15, 1 + 5 = 6
} 7 x 6 = 42, 4 + 2 = (6)

28575
6350
12700
1362075 → 1 + 3 + 6 + 2 + 0 + 7 + 5 = 24, 4 + 2 = [6]

If the DIGITAL ROOTs in O and □ agree, the product is most likely correct. If they do not agree, the product is certainly incorrect (assuming the digital roots have been calculated properly).

Explanation for F.

35 → 3 x 9 + (3 + 5) = 3 x 9 + 8
x 23 → 2 x 9 + (2 + 3) = 2 x 9 + 5
5 x (3 x 9) + 5 x 8
(2x9) x (3x9) + 8 x (2 x 9)
2 x 3 x 9 x 9 + 5 x 3 x 9 + 8 x 2 x 9 + 5 x 8

The DIGITAL ROOT is the remainder when a number is divided by 9.

The only part of the product that can give a non-zero remainder when divided by 9 ix 5 x 8 or 40. This shows that the digital root of 5 x 8 is also the digital root of the entire product (35 x 23 = 805, 8 + 0 + 5 = 13, 1 + 3 = 4).

In general, the digital root of the product of two numbers is the product of their digital roots.

Generalizations (for A-E)

Any positive integer, N, can be expressed as a base ten numeral in the form:

$$a_0 10^n + a_1 10^{n-1} + a_2 10^{n-2} + \cdots + a_{n-2} 10^2 + a_{n-1} 10 + a_n$$

or $\sum_{k=0}^{n} a_k \, 10^{n-k}$ where $0 \le a_k \le 9$

Hence $$N = \sum_{k=0}^{n} a_k \, 10^{n-k}$$

$$= \sum_{k=0}^{n} [a_k (10^{n-k} - 1) + a_k]$$

$$= \sum_{k=0}^{n} a_k (10^{n-k} - 1) + \sum_{k=0}^{n} a_k$$

Since 9 (evenly) divides $10^r - 1$ for all $r \ge 0$, $r \in I$, 9 divides N if and only if 9 divides $\sum_{k=0}^{n} a_k$, the base ten digital root of N.

(In terms of modular arithmetic $N \equiv \sum_{k=0}^{n} a_k \pmod 9$).

Also, any positive integer, N, can be expressed as a base b numeral, $b \ge 2$, $b \in I$, in the form

$$a_0 b^n + a_1 b^{n-1} + a_2 b^{n-2} + \cdots + a_{n-2} b^2 + a_{n-1} b + a_n$$

or $\sum_{k=0}^{n} a_k \, b^{n-k}$ where $0 \le a_k \le b - 1$

Hence $$N = \sum_{k=0}^{n} a_k \, b^{n-k}$$

$$= \sum_{k=0}^{n} [a_k (b^{n-k} - 1) + a_k]$$

$$= \sum_{k=0}^{n} a_k (b^{n-k} - 1) + \sum_{k=0}^{n} a_k$$

Just as 9 divides $10^r - 1$ for all $r \ge 0$, $r \in I$, $b-1$ divides $b^r - 1$ for all $r \ge 0$, $r \in I$ since $\frac{b^r - 1}{b - 1}$ is the sum of the geometric series $1 + b + b^2 + \cdots + b^{r-1}$ which is an integer.

Hence $b - 1$ divides N if and only if $b - 1$ divides $\sum_{k=0}^{n} a_k$, the base b digital root of N.

(Example: 7 divides 105 since

$105_{ten} = 151_{eight}$

and $(1 + 5 + 1)_{eight} = 7_{eight}$).

Generalization for F

If m and n are positive integers with (base ten) digital roots a and b, then $m \equiv a \pmod 9$ and $N \equiv b \pmod 9$,

that is

$m = 9p + a$ and $n = 9q + b$, $p, q \in I$.

Hence, $mn = (9p + a)(9q + b)$
$= 81pq + 9pb + 9aq + ab$
$= 9(9pq + pb + aq) + ab$
and $mn \equiv ab \pmod 9$

thereby establishing that the digital root of a product of two numbers is the product of their digital roots.

There are many interesting patterns involving digital roots of certain classifications of numbers. Let us know if you're interested in having such material in this column.

Let's have news and views of your reactions and classroom trials of any of this 'stuff' as well as suggestions for other topics. We'll work as many of your ideas as we can into future issues.

Recreationally yours,
H.L.R.

BOOK REVIEWS AND RESEARCH

CALCULATORS IN THE MATHEMATICS CLASSROOM: A SHORT SURVEY OF WARES, HARD AND SOFT

Dr. William Higginson, Queen's University

> "With the decrease in cost of the minicalculator, its accessibility to students at all levels is increasing rapidly. Mathematics teachers should recognize the potential contribution of this calculator as a valuable instructional aid. In the classroom, the mini-calculator should be used in imaginative ways to reinforce learning and to motivate the learner as he becomes proficient in mathematics."

In September of 1974 the National Council of Teachers of Mathematics endorsed this position as an official policy statement. Looking back some three years later, one cannot quarrel with either of the first two premises. Calculators consistently have been one of the very few products to counter the inflationary price spirals of recent years and they are now easily within financial reach of almost all students. Furthermore, there is no doubt that teachers **should** recognize the potential contribution of this technological development as a valuable instructional aid. However, from this point, things get rather sticky. How can we move from "should" to "can" and "do?" "Imaginative ways" do not miraculously appear to teachers. Where do these ideas come from? Are calculators to be to this decade what television was to the last? There, too, we had a series of striking technological advances which were rapidly and almost universally integrated into the general culture. And yet, despite great initial optimism and statements of the "should recognize the potential contribution" type, the reality ten years later is that television, by and large, has not greatly influenced standard classroom experience or curricula. If this has been the educational fate of television in a time of financial expansion, can we expect a different result for calculators in a period of financial stringency? (These comments should not be taken as a criticism of the endeavours of the Ontario Educational Communications Authority which has an excellent, justly deserved, reputation in international circles; it is intended rather to illustrate the high resistance of classroom practice to technological change. Computers are still another example of this phenomenon.)

While it is still too early to make a definite conclusion about this issue, there are reasons for thinking that calculators may be different in enough important ways to have this potential shift in classroom practice become a reality. Two important characteristics of calculators in this regard are related to economics and curriculum focus. The economic argument is straightforward. The unit cost of calculators is relatively small and the option of having students provide their own is a feasible one. (Try extending the 'running shoes' argument to television sets!) From a curriculum view-point it seems clear that while calculators have applications in many different areas, their major academic 'home' is mathematics. For better or worse, mathematics teachers will be seen to have the responsibility of effectively utilizing these devices in school settings.

Several features of the calculator phenomenon are most unusual from an educational viewpoint. Paramount among these is the rate of technological change. It seems that one has only to purchase a calculator to find that it has been rendered 'obsolete' by the introduction of a newer model which does twice as much for half the price. (Some early purchasers of Texas Instruments' 'Fifty' models are still bitter about the $165 they paid to 'rent' their machines for two years - a $215 model in 1975 had become a $50 model in 1977.) The number of companies entering and leaving the marketplace is very high and there is considerable confusion about issues like notation, format, and logic. Many publications of dubious worth have appeared to exploit the new market. On a per word basis some of these calculator handbooks, with their wide margins, large type, and thick paper, must be among the most expensive publications on the market.

The remainder of this article is devoted to a brief overview of some of the better calculator hardware and software currently available for use in mathematics classes. Of necessity this survey will be arbitrary and incomplete. Because of the nature of the product, the prices listed in the references at the end of the article should be taken as rough approximations. Hardware, or machines, will be considered first, with comments on support materials, or software, following.

One common classification of calculators sorts them according to the range of available functions. Four categories of interest to teachers are Four-Function, Slide-Rule, Programmable, and Instructional-Recreational. Four-Function calculators have the capacity to do addition, subtraction, multiplication, and division. 'Deluxe' models at this level may also provide keys for one or more of memory, per-cent, square root, or constant. It is at this level that rock-bottom prices are common. The highly serviceable Novus 750, for example, frequently sells for less than five dollars. The day of the 'throw-away' calculator (the battery costs more than the rest of the components), undesirable as it may be in the eyes of some people, is probably not far away with this level of machine.

Slide-Rule calculators add a minimum of trigonometric, logarithmic, and exponential functions to the basic repertoire. Models in this class, which seem destined to be the standard high school level calculators for at least the near future, are available in the twelve to fifteen dollar range. The outstanding machine in this range, in the opinion of many mathematics educators, is the Texas Instrument TI-30 which sells for approximately twenty dollars. A remarkably powerful machine (48 functions), it is attractively designed and capable of doing almost any calculation which might arise in a secondary school mathematics curriculum.

At the programmable calculator level, the boundary between calculators and computers begins to blur, for these calculators have the capacity to 'learn' and 'remember' keyed-in sequences of operations that can be played back on demand. With this class of machine only the industry giants produce a range of models and the prices increase considerably. Texas Instruments and Hewlett-Packard dominate the market with National Semiconductor (Novus) and Sinclair providing some competition. Attractive small machines of this type are the Hewlett-Packard 25, and the Texas Instruments SR-56. While these models can be purchased for just over one hundred dollars, their 'big brothers' with magnetic-card programme libraries and printing capacity, can cost in the neighbourhood of a thousand dollars.

Perhaps of particular interest to teachers are the Instructional-Recreational calculators. The first two models of this type to appear were the Quizkid (Winkey the Owl by Novus) and the Little Professor (by Texas Instruments). On the Quizkid, Winkey's green eye lights up for a correct statement such as $7 \times 13 = 91$, while a response of 90 to the same question elicits a wink from the red eye. Contrived in some ways but quite clever and effective in others. With the alternate owl, academic status has been purchased at the cost of freedom for all questions on the Little Professor

are preprogrammed. Students merely select one of the operations and one of four levels of difficulty and they are presented with a sequence of ten questions. Division at the third level of difficulty, for example, brings one questions such as 30 ÷ 5, 1 ÷ 1, 16 ÷ 2, 8 ÷ 2, and 0 ÷ 2. Three tries are allowed for any one question and after three errors the correct answer is shown. One's score out of ten is flashed at the end of the sequence. Prices for these machines are in the twenty-dollar range.

Obviously influenced by the success of **Star Wars** in 1977, Texas Instruments has brought out a much more powerful Instructional-Recreational calculator called Dataman. ["Somewhere out among the stars, trouble was brewing" . . . Commander Number Fun versus the evil Antimath . . .] It is undoubtedly not accidental that Dataman bears a great resemblance to a pocket-sized version of the "loveable" (?) robot R2D2, or Artoo Deetoo as one reviewer would, perhaps unconsciously, have him (it?) in a more humanized form. One could do without the space motif, but there is no doubt that the innards of Dataman are a bargain for twenty-five dollars since the machine is preprogrammed to play a number of interesting mathematical games with names like Number Guesser and Force Out. Dataman incorporates a timer so that the standard sequences of ten questions on basic operations can be a challenge for even the best students. With Dataman we probably have a taste of what is to come in the future and it seems intriguing.

In the area of software the most basic materials are those published by the manufacturers. Instruction manuals vary greatly in length and complexity. Included with four-function calculators are at most a page or two of comments while the Owner's Handbooks and Applications Programs of the programmable calculators are substantial documents which are indispensable to the full utilization of the machines. Several companies also provide a range of supplementary handbooks which are of general interest. Texas Instruments, for instance, publishes **Fintertip Math, The Great International Math on Keys Book,** and **Calculating Better Decisions.** These books are reasonably priced and can serve as a useful source for teachers desiring curriculum ideas.

Several useful documents in this area are available through the NCTM. For a consideration of calculators in the elementary school, the Special Issue of the **Arithmetic Teacher** of November 1976 is a basic reference. The thematic essay of the 1977 NCTM Yearbook, entitled **Hand-held Calculators: Past, Present, and Future,** is an excellent survey. Joseph Caravella's **Minicalculators in the Classroom** is a somewhat bloated but generally helpful publication with sections on 'Selecting and Using Minicalculators' and 'Activities'. A comprehensive and frequently revised bibliography, **Minicalculator Information Resources,** can be had, free of charge from NCTM headquarters.

The different production timelines for the electronics and publishing industries has led to some awkward software situations. While we are perhaps heading for fourth-generation calculators, we are just starting to get first-generation calculator curriculum materials. To this point, one of the most attractive of these publications would seem to be J. Norman C. Sharp's **The Calculator Work Box.** This collection of eighty assignment or work cards is divided into five sections: Calculator, Numbers, Measurement, Real-Life Math, and Extra Points. The activities are attractive, well chosen, and mathematically sound. Each box comes with a Teacher's Guide and a set of Student Record charts. Unfortunately the price for this unit seems rather hefty at $40.00. For those who might be interested in this publication, a 'sampler' is available from the publisher for $1.00.

There is a large number of calculator tricks and games books available. Many of these are of questionable value (and taste - to find the secret of Stella Starlet's movie fame find 3 x.19336.306 and read the answer from your upside-down calculator). The best of the range - they at least have the virtue of being inexpensive - are Wallace Judd's **Games Calculators Play** and Edwin Schlossberg and John Brockman's **The Calculator Game Book.**

For a more serious study of calculators and their workings, Jon M. Smith's **Scientific Analysis on the Pocket Calculator** and the excellent article by Eugene W. McWhorter "The Small Electronic Calculator" in **Scientific American** of March 1976 are very good starting points.

The report of a major study on **Electronic Hand Calculators: The Implications for Pre-College Education,** sponsored by the National Science Foundation in the United States and edited by Marilyn Suydam, is available through ERIC. For groups of teachers starting to think about ways in which calculators might be integrated into Ontario curricula, the work already done by some boards of education in the province might prove to be among the most valuable of available resources.

Boards which have done extensive work of high quality in this area include Hamilton, Thunder Bay, Etobicoke, and North York.

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Hewlett-Packard (Canada) Ltd.
6877 Goreway Drive
Mississauga, Ontario L4V 1L9
(416-678-9430)

National Semiconductor - Novus
N.C.P.S.
286 Wildcat Road
Downsview, Ontario M3J 2N5

Texas Instruments Inc.
Centre Street East
Richmond Hill, Ontario L4C 1B1
(416-889-7373)

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BOOK REVIEW

Iverson, K.E. **Algebra: An Algorithmic Treatment.** Pleasantville, N.Y. 10570 (Box 378): APL Press, 1976. Originally published by Addison-Wesley, 1972.

It is unusual to prepare a review of a book that has been available for 5 years. At least two explanations are possible: that the book has just been "discovered" and that the book is judged to be of such value that a form of public awareness is considered desirable. The first explanation is a weak excuse at best. Fortunately, we shall resort to the second explanation.

This is not a typical textbook. In fact, a quick skimming of its contents is likely to confuse the one-time browser. There is little that even looks familiar. Absent are the familiar algebraic expressions and graphs. Instead one sees strange combinations of symbols such as:

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1.  X ← 1 2 3 4 5
2.  X∘.×X
3.  +/X
4.  2 3 ⍴ X
5.  R ← 9 - ⍳ 10
6.  R∘. = (X- 3) × (X- 5)
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In spite of this apparent anomaly, the book is suitable for either high school or undergraduate students. However, it must be emphasized that this is not a standard textbook for a standard mathematics course.

The basic difference between Iverson's approach and virtually all other mathematics texts is the notational sytem. Iverson has developed a different method of writing expressions. This notational system is referred to as APL, an acronym whose meaning is left to the interested reader. Why a new notational system? Our present system has worked with only minor adjustments for almost 2,000 years. Then again, where would we be today if the Romans had adopted a similar posture? Sometimes the advantages are more subtle. How much do we learn about say, English, by learning French? Also, is a fish aware of the water it swims in?

The book represents a powerful departure from the way many of us were taught mathematics. It is a book that requires a deliberate effort on the part of the reader - not because it is difficult (I have taught aspects of APL to Grade 6 students) but because it is new. It requires the reader to view mathematics through new glasses, an experience that may lead to slight periods of dizziness and disequilibrium but may also lead to crystal clear pictures of the beauty and structure of mathematics. It is not often that a teacher is given an opportunity to engage in the process of mathematics. This book provides one such opportunity. Perhaps you will share the exhiliration of learning to "think" in APL, an experience shared by the many mathematicians and computer scientists who annually attend APL conferences.

Finally, a brief word about computer support is appropriate. Although APL is a mathematical notation, it is now becoming available on an increasing number of computer systems. This permits the evaluation of complex expressions or expressions involving large amounts of data using exactly the same statements that one has in one's notebook. Access to such a system greatly expands the use of an APL approach while also providing a genuine mathematics laboratory where the students can explore and discover the characteristics of various functions.

A detailed review of the contents of each chapter is not likely to be appreciated by one unfamiliar with APL. The purpose of this review is to introduce one to a book that outlines a new mathematical notation.

It is not clear whether this book represents a new way to teach mathematics or whether it is a new mathematics. It certainly has the potential to catalyse a re-examination of what is really important in mathematics education.

J. Dale Burnett, Ph.D., Queen's University



To: The Mathematics Department at Kapuskasing District High School and E. Dukovac.

It held a poetry contest - yes, that is correct - a poetry contest during Education Week 1977.

The response was overwhelming for the simple reason the students had no choice but write a poem.

The poems took the following form:

> The first line is composed of one word that names a mathematical concept. The second line consists of two words that describe the first line. The third line is made up of three words that tell what the first line is doing. The fourth line is personal --- it consists of four words that tell how the author feels about the first line. The fifth line is composed of one word that renames the concept on the first line.

The poems were judged in four categories:

Category A: Grade **13, grade 12 enriched**
Category B: Grade **11, 12 general**
Category C: Grade **9, 10, 11 enriched**
Category D: Grade **9, 10 general**

Category

A

Tangents
Special secants
Touching of Adjacents
Contact point creating excitement
Kiss
Francois Lachance
P.S. Kiss is a navigational term
Meaning "To touch in tangency
as The Limb of the sun to the
Horizon line in the making of a
sextant observation".

Absolute values
Always positive
Forcing a plus
Trapped between two lines
Affirmative
Colleen Zagrodney

B

Mid point
Divides evenly
Solves many problems
Equals out the distance
Halfway
Carole Carriere

Circles
Terrestrial globe
Round and round
The most beautiful creation
Infinite
Doreen Greene

C

Reciprocals
Bottoms up
Spinning me around
What's up, is down
Switch

Nicole Piche

Mass
Continually changing
Gram, kilogram, tonne
I have too much
Volume

Wendy Swain

D

Zero
Small number
Oval in shape
Easy to draw
Nothing

Deirdra Gallagher

Binary
Zero one
Seems slightly crazy
But its great fun
Computers

Donna Sherval

You might consider such a contest for this year's Education Week - April 17, 1978.

ANNOUNCEMENT

TENTH CANADIAN MATHEMATICAL OLYMPIAD - 1978
Patron: The Governor-General of Canada

DIXIEME OLYMPIADE CANADIENNE DE MATHEMATIQUES, 1978
Sous le haut patronage du gouverneur-general du Canada.

Date of Examination

The Tenth Olympiad will be written on Wednesday, May 3, 1978, from 9:00 a.m. to 12:00 noon at schools throughout Canada. The examination papers will be marked in May, and the results will be circulated to schools as soon as possible thereafter.

Date du concours

La dixième Olympiade aura lieu le mercredi, 3 mai 1978, de neuf heures à midi dans les école de tout le Canada. La correction sera effectuée dès le mois de mai et les résultats seront communiqués aux écoles le plus tôt possible.

Conditions of Examination

All examination materials, whether submitted for marking or otherwise, are the property of the Canadian Mathematical Congress. Decisions of the examiners are final. Cash prizes and scholarships may be awarded at the discretion of the examiners, who may choose to subdivide or otherwise alter the terms of awards.

Eligibility of Candidates

Candidates must be Canadian citizens or landed immigrants, and must be registered in a high school or its equivalent.

Nomination of Candidates

All candidates of the Olympiad must be nominated by the chairman of a provincial competition or by a school principal. Copies of nomination forms can be obtained from:

Professor W.J. Blundon
Chairman, Tenth Canadian Mathematics Olympiad
Department of Mathematics, Statistics and Computer Science
Memorial University of Newfoundland
St. John's, Newfoundland. A1B 3X7

Professeur W.J. Blundon, Président national
Dixieme Olympiade canadienne de mathématiques
Département de mathématiques, statistiques et informatique
Memorial University of Newfoundland
Saint-Jean, Terre-Neuve, A1B 3X7

(The Ontario chairman is Prof. K.D. Fryer, Department of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1 and the Ontario quoto of candidates is 50.)

Lès directeurs d'écoles secondaires ou de CEGEP peuvent nommer jusqu'à toirs candidats; choisis parmi ceux qui n'ont pu partiper à un concours provincial. Les frais d'inscription sont de $5.00 per nomination.

The First Seven Olympiads

A booklet containing the questions of the first seven Olympiads, solutions to the questions, and the names of the prize winners, has been prepared and may be obtained from:
Secretariat,
Canadian Mathematical Congress

Prix principaux

Premier prix : $1,200
Deuxième prix : 900.00
Troisième prix : 600.00
Quatrièmes prix: cinq prix de $200.00 chacun
Cinquièmes prix: huit prix de $100.00 chacun.

INTERNATIONAL MATHEMATICS EDUCATION

O.A.M.E./A.O.E.M. 5th Annual Meeting, May 5, 6, 7, 1978, Hamilton, Ontario.
N.C.T.M. 56th Annual Meeting, April 12 - 15, 1978, San Diego, California
O.A.M.E./A.O.E.M. 6th Annual Meeting, May , 1978, Peterborough, Ontario
N.C.T.M. 57th Annual Meeting, April 18 - 21, 1979, Boston, Massachusetts
O.A.M.E./A.E.O.M. 7th Annual Meeting, May , 1980, Toronto, Ontario
N.C.T.M. 58th Annual Meeting, April 16 - 19, 1980, Seattle, Washington.

Groupe International de Recherce in Pedagogie de la Mathematique, 7th Annual Meeting, Glasgow, Scotland, July 15-23, 1978.

Editor's Note: David Kent of Lower Canada College, Montreal . . . Where are you? Our letter to you regarding a seat on the charter flight to the N.C.T.M. Annual Meeting has gone unanswered. Has anyone else managed a seat?

O.A.M.E./A.O.E.M. FUTURE MEETINGS

Meeting Dates effective May 1978 following Hamilton Conference.

Executive Meeting
First Friday after May 24th weekend holiday.
Council Meeting
Second Saturday after May 24th weekend holiday
Executive Meeting
Second Friday after Labour Day
Council Meeting
First Saturday after Thanksgiving
Executive Meeting
First Friday after November 11
Council Meeting
Second Saturday of February
Executive Meeting
First Friday of April unless it is Good Friday - then one week later

PUBLICATIONS AVAILABLE

TO O.A.M.E. MEMBERS

	Regular Price	O.A.M.E. Member Price
Handbook of Trigonometry (1975)	$2.00	N/C
A Preparedness Test for Grade Nine Students (1973)	2.00	N/C
Sample Grade 13 Examinations	1.00	N/C
Grade 11 (Level 5) Sample Examinations	1.00	N/C
Reprints of 1974 & 1975 Gauss Contest Papers (Elementary)	.10	N/C

To receive these please contact:

Mr. R. G. Scoins,
Faculty of Mathematics,
University of Waterloo,
WATERLOO, Ontario.

Boxes. Squares and Other Things (Elementary)		
A leather guide for a unit in informal geometry	1.80	N/C

Available from: Mr. M. MacGregor,
Secretary-Treasurer O.A.M.E.

IN THE UNITED STATES

NATIONAL WEATHER SERVICE REVISES CONVERSION REPORTING PLANS

The National Weather Service (NWS) has proposed that mid-1980 (previousy 1 January 1979) be set as the date when the news media will begin reporting weather entirely in metric and that after that date all NWS announcements to the public will be completely in metric units.

A comprehensive program of public awareness will be undertaken by the NWS beginning in June 1979, concentrating first on temperature conversion, with conversion of other weather elements following at three-month intervals.

O A M E
1 9 7 8
7 8 1 9
8 7 9 1
9 1 8 7
HAMILTON

Fifth Annual Conference

Ontario Association for Mathematics Education

OAME

Association Ontarienne pour l'Enseignement des Mathématiques

PROGRAM (from Friday noon to Sunday noon)

Seventy speakers including:
Mary Baratta - Borton, California
Carole Greenes, Boston, Mass.
James Gray, San Antonio, Texas

REGISTRATION

Accommodation including breakfast in McMaster's modern air-conditioned residences.

Registration & information:

Dr. J. Stewart,
Dept. of Mathematics,
McMaster University,
Hamilton, Ontario

O A M E
1 9 7 8
7 8 1 9
8 7 9 1
9 1 8 7
A O E M

SOCIAL

Friday night reception
Guest program
Saturday Luncheon
President's dinner
and dance

May 5, 6, 7, 1978
McMaster University
Hamilton, Ontario

THE ANNUAL CONFERENCE PROCEEDINGS ANNOUNCEMENT

Reserve your copy of the

"PROCEEDINGS OF THE ANNUAL CONFERENCE"

Vous serez heureux de recevoir les

"TRAVAUX DE LA CONFERENCE ANNUELLE"

If you attend the Annual Conference regularly you know that there are many excellent presentations. If you have not been able to participate, wouldn't you like to know what you are missing?

This is your opportunity to obtain a worthwhile record of the Proceedings of the Fifth Annual Conference. Arn Harris (Gazette Editor 1975-78) will be the Editor of the Hamilton proceedings, and we can look forward to a valuable publication. The Council, at its November meeting, approved this publication as a one year experiment with the proviso that it pay for itself. The price of $3.25 (includes postage) has been set on the basis of approximately 350 subscribers. Therefore, if you, like the Council, believe that it is a worthwhile venture for OAME, send in your order and $3.25 to Morley as soon as possible. The last date to order a copy is Sunday, May 7, 1978 - the closing date of the Fifth Annual Conference.

- -

Please Print

NAME ______________________________

ADDRESS: ______________________________
Street **Apt.**

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City **Province**

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____
Postal Code

Send with $3.25 to M.B. MacGregor

FUN MATHEMATICS

Answers

1. 13 hrs.
2. Yes.
3. One
4. He is still living
5. The match
6. They all do
7. 4 hrs. (take one pill now)
8. White (polar bear - northpole)
9. ½ way, then he's running out.
10. 10; 6
11. 50c, 5c, (the **other** coin is a nickel)
12. 9
13. 70
14. They were not playing each other.
15. 2 apples
16. No - you can't date a coin before Christ was born - you wouldn't know when he was to be born.

MEMBERSHIP RENEWAL NOTICE

The Members of OAME have (in the past years) done an excellent job of renewing their membership at the Annual Meeting. With postage rates and stationary at present high levels, a renewal bought at the membership desk brings a considerable savings to the Association.

We realize the absolute due date is August 31st, but we appreciate the early renewal. It assists us in arranging volume postage for periodicals.

We look forward to meeting you at Hamilton and presenting to you your 1978-79 seal and pocket card.

If you cannot attend the Annual at Hamilton, perhaps you would care to clip off the bottom of this page and return it to the Sarnia office.

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

MEMBERSHIP APPLICATION

DATE .....

Dr. ☐ Mrs. ☐ .....
Mr. ☐ Miss ☐ (Please print) NEW MEMBER ☐
Apartment No. ..... RENEWAL ☐

Street or R.R. .....

Town or City ..... Postal Code .....

School .....
Chapter Affiliation
(if applicable) .....

PLEASE ENCLOSE CHEQUE OR MONEY ORDER
FOR $13.00, PAYABLE TO OAME

NINE CHAPTERS OF OAME/AOEM

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Representative - L. Lefebvre

COMA (Carleton - Ottawa)

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S.E.Y.M.A. (Scarborough - East York)

President - Paul Sherk
Representative - Paul Zolis,

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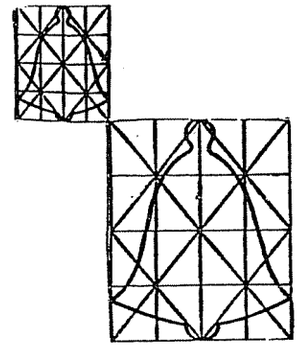
President - Barbara Palmer, Laurier S.S.
Representative - Gary Peters,

CHAPTER ITEMS

Comment s'y prendre...?
pour agrandir ou reduire un diagramme
en se servant de diagonales.

Les dessins à l'échelle font plus figure qu'auparavant, avec l'arrivée des nouvelles directives du Ministère. La méthode traditionnelle d'agrandir ou de réduire un diagramme fait appel à un quadrillage qu'il faut reproduire, à une échelle différente. La méthode décrite ci-dessous dépend des propriétés des diagonale d'un rectangle, et peut s'avérer plus exacte.

- Encadrer le diagramme d'un rectangle. Tracer un deuxième rectangle, de différentes dimensions mais de mêmes proportions, de sorte que les deux se touchent à un sommet (voir l'illustration). Tracer les deux diagonales pour chaque rectangle.



- Par le centre de chaque rectangle, fair passer un segment vertical et un segment horizontal, de facon à séparer chaque rectangle en quatre rectangles égaux. Dans chacun de ceux-ci, trouver le centre, comme auparavant, en tracant lès diagonales.Séparer en quatre autres rectangles.

- Copier les parties correspondantes du dessin dans le nouveau rectangle.

At the COMA/AMOC Council meeting of 1977 11 14 the following motion was passed -

> Whereas it appears that the Ministry of Colleges and Universities and the Ministry of Education feel that some form of ongoing program and student evaluation is required at the interface level between secondary and post-secondary institutions, it is moved
>
> "that COMA Council appoint a committee of persons, representative of the secondary and post-secondary schools in the Carleton-Ottawa region, to examine ways and means of increasing the exchange of information across this interface regarding programs and student achievement in mathematics."

A committee was appointed and held two meetings under the chairmanship of Don Fox. Much information was exchanged directly among the members; and some initial plans were made for classroom visitations across the interface, involvement in the preparation of the Ottawa Board-wide Calculus and Relations and Functions exams for this spring, and possible future workshops. However, of particular significance to COMA (and OAME) was the recommendation that a Mathematics Examining Board be established for at least the Ottawa-Carleton region. This recommendation is to be considered at the February 20th meeting of the Council of our local chapter. If approval is given to the concept, then teachers and administrators of the secondary schools will proceed to more detailed planning with representatives of the three local post-secondary institutions.

QUINTE-ST. LAWRENCE

Report of the Feb. 15th dinner meeting held at Queens University. After J. Whitley spoke on Man and Measurement and H. Allen spoke on Splitting the Triangle, the following topics were offered:

Loyalist C.V.I. - Games for Seniors
R. Davies - Transformation Games
R. Ripley - Aids for Learning
E. Harvey - Equation Solving with a Difference

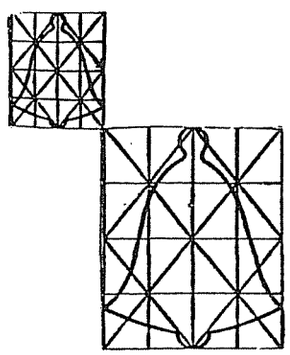
Trick of the Trade
How to Enlarge and Reduce Diagrams
The Diagonal Method

This method of enlarging and reducing diagrams is perhaps not as well-known as the "grid" or "squaring-off" method, but it will certainly be useful in the elementary or intermediate divisions, what with the new emphasis on scale diagrams in the recent Ministry guidelines. This "diagonal" method is somewhat better than the "grid" method in that there is a better chance that all corresponding parts remain in proportion.

- Draw a rectangle to fit around your diagram. Immediately next to and below this rectangle, draw a second rectangle of same proportions to fit the desired diagram. Draw diagonals from corner to corner of each rectangle, as illustrated.

- Draw one horizontal and one vertical line to divide each rectangle into four equal rectangles. In each of these, draw diagonals to find the centre, as before, and divide into four more rectangles.

- Copy the segments of the original diagram in corresponding divisions of the second rectangle.



SUBSCRIPTION RATES

$6.00 FOR ONE YEAR'S SUBSCRIPTION
$2.50 FOR A SINGLE COPY

O.A.M.E. members receive the Gazette as an adjunct to their membership.

Invoices will be sent 30 days prior to expiry of a subscription, and 60 days is allowed for payment.

Orders for subscription and single copies should be sent to the Secretary-Treasurer.

M.B.McGregor

O.A.M.E. COMMITTEES AT WORK

The effectiveness of our Association depends on the participation of the members in policy formation. The following committees will be at work during 1977-78: The Ad-hoc Committee on Teacher Qualifications, the Interface Committee, the French Language Committee and the Evaluation Committee. Submissions to these committees are welcomed and may be sent to the Secretary at the above address.