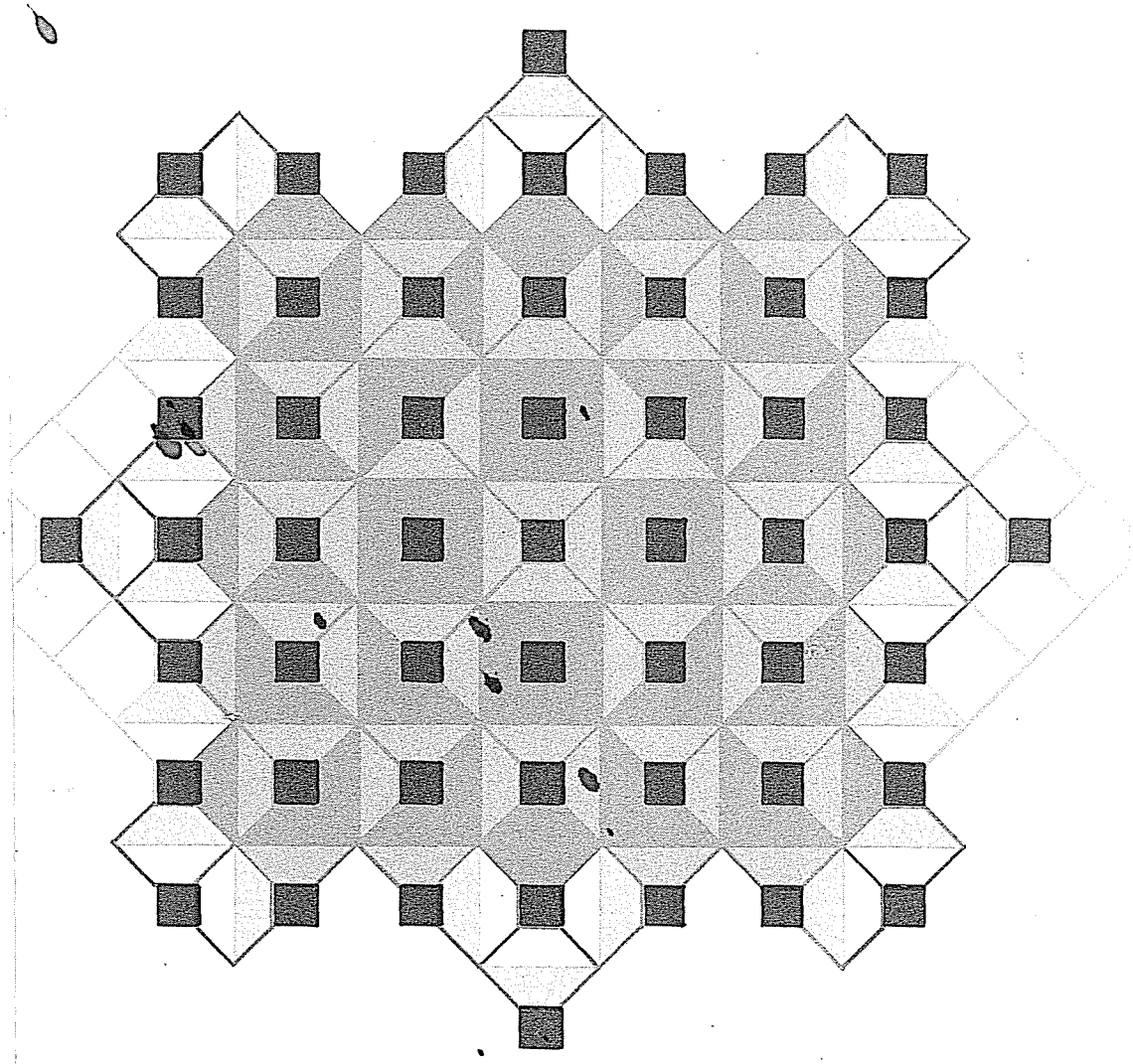
# ontario mathematics gazette

**VOLUME 17** **NUMBER 1** **SEPTEMBER, 1978**



THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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# EDITORIAL

Another school year has begun. May it be a successful and rewarding one for each of you.. As suggested in the last editorial, this issue contains articles dealing with evaluation. "Why evaluation as I start a new year?" you might well ask, "evaluation comes at the end of a term not at its beginning". Yet, I am sure we would all agree that to evaluate progress either of an individual student or of a group of students, reference must be made to the goals and specific objectives of a program. These goals, I believe, must be detailed with sufficient precision to permit the development of tests and/or other strategies of evaluation that relate closely and directly to them. Evaluation must be considered at the same time as you consider your goals, for evaluation should be applied continually from the time the goals and objectives are formulated (are they appropriate?) to the time of final evaluation (have the desired goals and objectives been achieved?). Evaluation should consider pupil achievement, pupil attitudes, teacher effectiveness and program effectiveness.

Evaluation is far more than paper-and-pencil tests. Testing is only one component of evaluation. I would offer as a definition of evaluation the following: the making of judgements about educational practices, such judgements being dependent upon the analysis and interpretation of relevant information gathered by appropriate means.

Evaluation is required for many different purposes and thus requires different means. Consider how different would be the gathering of relevant information for the following purposes:
- screening teachers or administrators for tenure or promotion;
- screening students for university admission;
- monitoring provincial performance to meet accountability needs;
- testing to establish county or provincial norms of performance;
- monitoring locally constructed mathematics course guidelines;
- screening students for promotion;
- administering tests for individual diagnosis of pupil difficulty.

The lists of purposes seems to divide into two kinds. One is to obtain data on the achievement levels of a given pupil population as a basis for rendering accountability so that decisions may be made. As mathematics teachers this is not our pressing concern. The second kind of purpose is to obtain data that will facilitate the teaching/learning process. Here we are concerned with individual student needs; information which will provide the basis for continuation, remediation or enrichment of the topic being presented.

Let us look at the gathering of relevant information for a moment. Rather than my providing a list, I prefer to refer you to a report, **Evaluation In The Mathematics Classroom.**[1] It provides a good list of evaluation techniques with comments. It contains most useful practical suggestions to help in the assessment of your students, from non-paper-and-pencil procedures to paper-and-pencil instruments. A section follows which suggests ways to make your tests as good as possible. The remaining third of this sixty page report is an annotated listing of selected references.

Perhaps it is worthwhile to remind ourselves that the very process of preparing for and taking a test helps the student to organize and use what he has been studying. The test is and should be a learning situation. It is, if there is not a great stress on the results as a final report. The responses should help the teacher to diagnose a weakness or confirm an observation. A test worth giving should be a test worth reviewing and discussing in class. From the teacher's point-of-view, a simple item analysis does two things: a) it immediately identifies the strong or weak pupils in the class, b) it gives a measure of difficulty of each item.

In Benjamin S. Bloom's **Taxonomy of Educational Objectives,** knowledge, which emphasizes the remembering, either by recognition or recall, is the lowest level of his six categories. The other categories should also be considered as we set our objectives and formulate our evaluative instruments. The measurement of content goals is comparatively easy. Baur[2] has suggested four levels of mathematics learning which may be characterized as follows:

1 knowledge or rote learning — involved such areas as facts, terminology, information from memory.

| | |
|---|---|
| 2 concept learning | mental structures that represent a class of experiences. |
| 3 principle learning | a mental structure involving two or more concepts. |
| 4 problem solving | concepts and principles accommodated and used. |

Do read Mike McKenna's suggested four levels of behaviour in his article on **Evaluation.** Measuring these other goals is not so easy, but should be attempted. Nor should the affective component of learning be forgotten, for if we are interested and enjoy mathematics we learn more easily and so it is with our pupils. Evaluation in this area will require you to collect evidence in a variety of ways: observation, anecdotal reports, samples of work, questionnaires or attitude scales, seminars or projects.

In the report to the Minister of Education of the **Work Group on Evaluation and Reporting,** several recommendations are worth reviewing.

1. Every curriculum committee, whether board or Ministry, have as part of its responsibility to develop criteria whereby the effectiveness of the guideline may be evaluated."[3] A representative to the Ministry field testing committee on the Intermediate Division Mathematics Guidelines was appointed at the last council meeting.

2. "There be developed a bank of specific objectives and corresponding test items and instruments based on curriculum objectives that would be available to teachers for their use in the diagnostic testing of students.?[4] An annotated catelogue of locally developed evaluation instruments is being written now at MECA in London, a contractural project of the Research and Evaluation Branch of the Ministry of Education.

3. "When tests are used as part of the process of program evaluation, every effort be made to have such tests go beyond the most easily tested areas (i.e. the lower level cognitive skills) and examine as many other types of objectives (i.e. affective, or higher cognitive) as possible."[5]

4. "As research suggests that pencil and paper standardized tests have limited value in the primary division, their use at that level not be encouraged."[6] Be sure to read **Measuring Achievement At The Primary and Junior Levels.**[7]

These recommendations as well as others which suggest funds be made available for professional development programs in the fields of evaluation, testing and reporting are interesting, but our immediate goal surely is to improve instruction. Let us consider our goals carefully as we start the teaching year. Are they realistic, achievable, appropriate? And as we come to measure the results of our teaching remember that a test only indicates we will have to do something as a result of what the pupil has or has not learned.

Bob Smith

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1. Suydam, Marilyn N., **Evaluation in the Mathematics Classroom**
2. Baur, G.R. & George, L.O. **Helping Children Learn Mathematics**, p. 387
3. **Report to the Minister of Education the Hon. Thomas L. Wells of the Work Group on Evaluation and Reporting** (issued by the Ministry of Education) p. 12 #3
4. Ibid. p. 16 #2
5. Ibid. p. 16 #5
6. Ibid. p. 16 #7
7. **Measuring Achievement At The Primary and Junior Levels** (issued by the Ministry of Education) O.I.S.E. c 1977.




OAME - 1979, May 13 - 15
Sixth Annual Conference
The Peterborough Conference
by
P.R.M.A. (Pine Ridge)

# WHAT'S NEW AT THE GAZETTE

The editorship of your journal continues in London, but relies on help from contributors about the province. As it is the Associate Editors who are largely responsible for ensuring the supply of quality articles, I would draw your attention to the two new Associate Editors, Dr. Larry Ridge, professor in the Mathematics Education Department, U. of T. and Walker Schofield, Mathematics Head at Oakridge Secondary School, London. Walker was the co-editor of the conference proceedings in 1975 and has done the same job again this year. Larry has become a regular contributor, writing about recreational mathematics.

The Advisory Board is representative of the mathematics groups we serve and functions not only to advise but also to read articles as they are submitted. Dr. Clive Reis has taken his sabbatical so will be replaced this year by Dr. Lai Chan who has been associated with Dr. MacNeill in the Statistics Survey done in 1975. Mrs. Barbara Palmer, H. B. Beal Secondary School, London and her husband Dave of Saunders Secondary School, have also joined us. This group is most helpful and although not the reason for their selection, all are close enough for me to drop articles in at their homes as I run by. There is one exception - your vice-president, Alex Norrie, Co-ordinatior of Mathematics for the Peel Board of Education.

The technical side of the journal continues to be handled easily and well at J. Pole Printing in Forest. I am grateful that Dorothy Elliot and Norah Logan of North Lambton Secondary School will continue to proof-read. Business and mailing arrangements are efficiently handled by the McGregors.

I welcome two new features to the Gazette, "The Coordinators' Corner," and a "Problems Section." The latter is not really new. What is new is the way that its return and continuation (we hope) was brought about. In January, I wrote to the co-ordinators group suggesting that they undertake a regular column. They accepted and John Del Grande has written the first article. I also suggested that they would be the obvious group to recommend each year or so persons capable of composing the problems section. Michael Silbert suggested Dave McKay and John McKnight from Westdale Secondary School, Hamilton. We hope this arrangement continues and wish Dave and John all success in their endeavour.

I trust you will notice the suggestion on the inside back cover about preparation of articles. Of course we will read what is sent, and we are pleased to receive all contributions. We recognize many do not have secretaries to do professional typing. However, these guidelines may prove helpful, and will certainly lighten our load. Do not hesitate to submit articles to any one of the Associate Editors or to me.



# LETTERS TO THE EDITOR

Marymount College
Sudbury, Ontario P3C 5E7

April 10, 1978

Dear Prof. Smith,

I read with great interest the section you had in the March issue of the Gazette that dealt with the poetry contest held at Kapuskasing District High School.

In December 1977 a few enterprizing young women here at Marymount College published the first issue of the school newspaper, "Etc." and in it appeared two mathematical sections,

"Mathematical Melodies" and "Polygonal Poetry". The submissions to the paper were made primarily by some grade 13 students in my Functions and Relations class. I hope you enjoy them.

Gen Waszczylo
Marymount College

**MATHEMATICAL LIMMERICK**

There was a young man who enjoyed conics,
Who worked at them in jets supersonic,
Parabolas and hyperbolas he enjoyed
And with ellipses and circles he toyed
But his favourite were degenerates and tonic.

Darlene Osborne
Grade 13

I'm really not co-ordinated
In equating involving absolute value
All my unions concern credit
My intersections involve streets
I'm going nowhere with my slopes
Unless I'm on a ski-hill,
ATT means nothing to me but
having A Terrible Time
The only fun I ever have concerns my substitution
When its time for homework, I substitute with television.

Nancy San Cartier
Grade 11

**Editors Note:** Would be glad to hear from any other budding poets.

Laura Secord Secondary School
St. Catharines

February 7, 1978

Dear Professor Smith,

I have been using a game similar to the New Year Game (March 1977 issue of Ont. M. Gazette), for a number of years. Students are given the 4 digits to be used early in each month, and at the end of month, hand in solutions for as many integers as they can form. I give a monthly prize to the winner, marking them only until I find the first error.

This game teaches more order of operations, operations with decimals, powers and roots, and factorials, than any textbook exercises. I am enclosing two winning entries, by Grade 9 students Jeff Jaeger and Larissa Kozub. I feel that Jeff's is especially worthy of praise.

Yours truly,

James A. Sernasie

P.S. I use this almost exclusively in Grade 9 classes.

**Editor's Note:** It would appear Jeff Jaeger might well have been inspired by the note in the September 1977 issue, which suggested producing the natural numbers using four 4's.
e.g. $21 = 4! - 4 - (4/4)$

$1 = 4 - \sqrt{4} - 4 \div 4$
$2 = 4 - \sqrt{4} \times (4 \div 4)$
$3 = 4 - \sqrt{4} - 4 + \sqrt{4}$
$4 = 4 + \sqrt{4} - 4 + \sqrt{4}$
$5 = \sqrt{4} + \sqrt{4} + 4 - 4$
•
•

•
•
$66 = 4! \div \bullet 4 + 4 + \sqrt{4}$
$67 = \dfrac{4! + \sqrt{4} + \sqrt{4}}{.4}$
$68 = 4! \div \bullet 4 + 4 + 4$
$69 = 4! \div \bullet 4 + 4/.4$
$70 = 4! \div \bullet 4 + 4/.4$

Jeff Jaeger

The Romans thought that they had troubles with their numerical system!

Look what would happen if instead of 0, 1, 2, 3, 4, 5, 6. 7, 8, 9, numbers there were only four:

6, 7, 7, 9

Then zero could look like this:

$0 = [(7! \div 6!) \div 7] - .\overline{9}$
$1 = [(7! \div 6!) \div 7] \times .\overline{9}$
$2 = [(7! \div 6!) \div 7] + .\overline{9}$
$3 = \sqrt{9}\,[(7! \div 6!) \div 7]$
$4 = \sqrt{9} + [(7! \div 6!) \div 7]$
$5 = [(9! \div 7!) \div 6] - 7$
•
•

•
•
$96 = 97 + 6 - 7$
$97 = 96 \div (7 \div 7)$
$98 = (7 \bullet 7) \bullet (6 \div \sqrt{9})$
$99 = 96 + \sqrt{7} \div .7$
$100 = 7(7 + 6) + 9$

Larissa Kozub

Waterloo, Ontario
N2L 1W6

Editor, Ontario Mathematics Gazette
Faculty of Education, U. W. O.,
London, Ontario.

Dear Sir:

May I say a word regarding the letter in the March issue from a gentleman who made a mathematical discovery while in the shower, and refrained from rushing out shouting Greek obscenities.

Presumably the reference is to Archimedes. Archimedes' shout of ηὕρηκα was not an obscenity but an exclamation of jubilation.

Yours truly,

G. S. Prunner




Don't forget to read Eureka, now "Crux Mathematicorum" - it's getting better every month. For special rates for COMA members, phone Fred Maskell at 225-8333. (P.S. That's Archimedes in the tub, not Fred ... although the resemblance is there...)

# USING CRITERION REFERENCED OBJECTIVES OR HOW I STOPPED WORRYING AND LEARNED TO LOVE MASTERY TEACHING

**Wes Keener, Mathematics Consultant, London Board of Education**

The grade five teacher's problem was one to which we can all relate. "I've just finished teaching rounding to the nearest thousand, hundred, ten and one but the kids are at different levels in their understanding and performance".

In order to help, I volunteered to take over the class for a few days and decided to try to incorporate a mastery teaching approach.

**Mastery Teaching - What's It About?**

If a person were to teach using a pure mastery approach, grades become unnecessary since performance criteria are specified in advance (in our system our Mathematics Curriculum Guide does specify the objectives) and what is assessed is the achievement or non-achievement of mastery. In a true mastery approach an "A" represents the mastery of a prespecified set of objectives. There is no concern for normal distributions since almost all students will achieve an "A" with time being the main variable. There is, therefore, no real place for norm referenced grading (i.e. comparing students one to another).

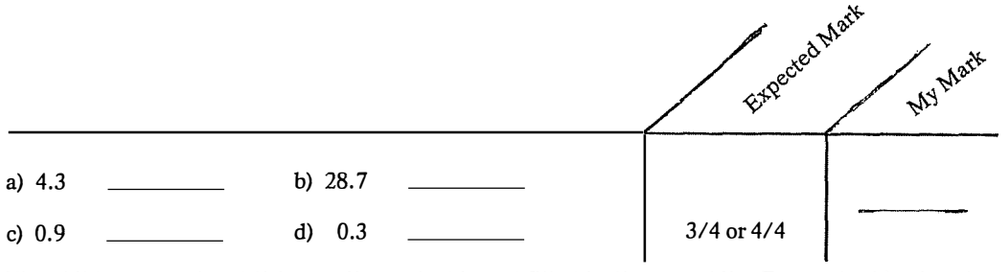
**The Reality:**

The reality of the matter is that our system does have grades and report cards reflect this. I was not as concerned about this aspect as I was interested in seeing if a mastery approach could mean better learning on a student's part. I wanted to get a feeling for a form of mastery teaching.

**Procedure:**

A one page dittoed copy (Pretest) was prepared for each pupil with four questions on each of the objectives (a total of 16 questions). The definition of mastery we used was 3 out of 4 or 4 out of 4 correct for each objective.

eg. Round off each of the following to the nearest unit (one)

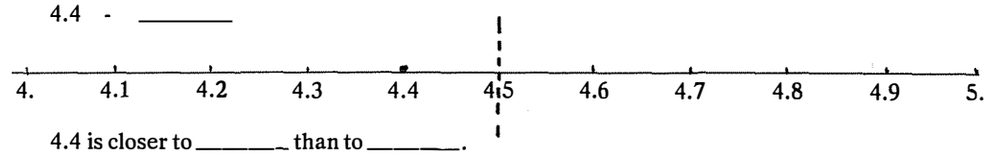



The children completed "the test", marked it and filled in "my mark". For each objective the number of children who failed to achieve a minimum of 3 out of 4 was quickly tabulated. From these results small groups of children who had not obtained the mastery expectation for each objective were called up and retaught.

eg. Eleven children received less that 3 out of 4 on "rounding to the nearest unit (one)" so they were called up to the front and retaught this objective using a number line.

Round off to the nearest unit (one)

4.4 - _____




4.4 is closer to _____ than to _____.

So 4.4 rounded to the nearest "one" is _____.

A special point was made to teach for understanding without teaching a formal rule. (The teacher had not yet got to the point of rounding to the nearest tenth so I simply avoided the issue of rounding used in statistics.

i.e. 7.35 = 7.4 (rounded to nearest tenth) - "if digit preceding 5 is odd round up"

7.45 = 7.4 (rounded to nearest tenth) - "if digit preceding 5 is even leave it alone"

but 7.451 = 7.5 (rounded to nearest tenth)

- a technique that I don't feel is appropriate or necessary for a grade 5 student to function anyway). As each child's work was finished it was marked immediately. Where it was occasionally necessary, the strategy was again presented. (This was a sort of post test but I thought the immediate feedback to the student was important at this point). A similar procedure was followed for each of the other three objectives (i.e. where a child did not obtain a minimum of 3 out of 4 he was retaught in smaller groups).

**What Were the Other Kids Doing?**

Before any of the reteaching began I briefly explained my procedure and what we hopefully could accomplish. A number of enrichment materials were shown that they could work on when they weren't being retaught (games, self checking puzzles, etc.).

**Post Test:**

Having retaught all the objectives to the appropriate students the children were given a "post test" which was parallel in design to the "pretest". Kids were expected to write only those objectives on which they had not previously achieved mastery. The results were very satisfying. All the students but one had achieved the minimum level of mastery on all the objectives. (even with the one student minor reclarification solved his problems).

**Some Advantages to Mastery Teaching:**

- achievement becomes almost a constant and time becomes the major variable;
- students experience success before they are faced with new tasks;
- increased confidence, more positive attitudes seemed apparent in students (however, I was only in this class for a limited time frame);
- increased opportunity for reteaching (if we all agree that this is indeed an advantage).

**Some Disadvantages:**

- have to develop a tremendous number of alternative tests;
- must provide numerous enrichment materials;
- a possibility that a very good student might "let down" a little;
- used in totality, there could be problems reporting student progress if a system's reporting procedures are based on norm-referenced testing;
- a need to reteach units (for those whom I failed to convince under advantages).

**Gut Level Feelings: A Mastery Approach**

- No claim to an original idea or teaching strategy is suggested since for years many good teachers

have retaught and retested those kids who needed help;
- I felt good about the achievement of the pupils and that I was reteaching only those pupils who needed it (as opposed to putting everyone through the hoop again);
- I believe the kids were enthusiastic about their own achievement (based on casual remarks by students after class);
- I was working with a group of kids who weren't discipline problems and who really involved themselves with the enrichment materials **but** would these same kids be this motivated throughout the year to work independently on enrichment?
- What would I do if there were several discipline problems?
- Could I continue to come up with suitable enrichment materials for each new topic, make up all the pre tests and post tests and still have time for the other subject areas (this was in an elementary school where most teachers are generalists).

My feelings were positive towards this experiment but reserved - mostly from the aspect of teacher mental health. If i were going to incorporate a mastery approach I would "hurry slowly". I would probably experiment with one unit per year gradually building up all the test and enrichment material. In some situations a team teaching approach might be suitable - one person in charge of reteaching, pre and post tests, while the other teacher creates the enrichment materials and helps kids who are working independently. These roles could be interchangeable whenever convenient.

One cannot assume that a mastery teaching approach is a panacea for all our pedogogical problems - nor should he/she assume that it just won't work in his/her particular teaching situation without first giving it a try. After reading the research I wanted to give it a go and at the same time offer no hesitation or apology to adapting and modifying the pure model so that it could blend with my own teaching style. After working in this class I stopped worrying and learned to love (almost) mastery teaching.

# EVALUATION: CLOSE ENCOUNTERS OF THE FOURTH KIND

**John R. Bramwell, Chief of Service, Measurement, Evaluation & Computing Applications (MECA) with the London Board of Education**

Encounters in evaluative research are inevitable. Educators are acquainted with the difficulties that result when attempts are made to report the findings from evaluative research and testing programmes. It is assumed in this short article that we, as educators, are aware of and interested in the contributions of evaluative research; therefore, it is incumbent upon us as professionals to engage in 'Encounters of all four kinds', in order to improve our basis for decision-making.

Gathering information to determine the status of a child, class, school, programme, or system is being done and facilitates the first encounter, "Where are we now?". After collecting the data, secondary encounters follow and suggest the question, "Where do we go now?", in the planning of a programme for a child, class, school, programme or system. This obviously leads to goal or outcome specifications. The third encounter, "How do we get there?" (change or intervene), follows naturally in order to achieve our stated goals or outcomes. The author is confident that the first three encounters are practised by many educators.

The fourth encounter eludes many for a variety of reasons; however, it is the key step in the process that can contribute to rebirth, renewal, and growth. We ask ourselves, "Are we having an effect?" Did the programme we set in motion result in desirable ends? In other words, "Have we arrived?" We, as educators, come close to dealing with this fourth encounter; but, not close enough in my opinion.

More active encounters of the fourth kind must become one of the integral activities of those involved in all levels of public education. Educators must apply more effort, in order to focus increasingly on the outcomes of the programmes we set in motion. If we do make the fourth encounter an integral part of the planning process, the author suggests that the public interest in, and perceived value of, education will increase. Surely, many practices and much information exist that permits glimpses into whether or not we have achieved desirable ends. If the recent trends

toward increased use of tests, matrix testing, creation of locally-developed tests, and new scoring techniques help us to achieve encounters of the fourth kind, these activities are justified and should be continued. These practices are particularly useful if they fill a void in the acquisition and sharing of information.

In order to facilitate the fourth encounter, several steps are suggested. The first, **stock-taking,** involves assembling known data and opinions on whatever entity is under scrutiny, **summarizing** the trends found and recording trends and findings in simplified form. The next phase may sound like heresy to some; however, it is essential. **Assemble interested persons,** such as teachers, administrators, trustees, citizens, university and college staff (depending on the entity being examined), then provide a time to talk about the trends and findings. Following discussion and immersion in the trends or findings, facilitate a **valuing process** where the acceptability of the situation that is indicated by the findings, and the criteria for acceptability, are examined. Finally, the group(s) would be invited to suggest action areas or next steps for consideration.

The process is summarized below:

1) Stock-taking
2) Summarizing
3) Assembling consultative group(s)
4) Setting valuing criteria
5) Generating action statements

The author is convinced that an approach such as this will result in many benefits. In order to progress, it is necessary to engage skillfully and willingly in encounters of the first, second, and third kinds; thus developing the basis for active and fruitful encounters of the fourth kind. We will have arrived when these encounters become the standard of the day.

In retrospect, the management of action and change based on evaluative data has been elusive, sensitive, sometimes embarrassing and sometimes rewarding. If open approaches such as those suggested are practised, evaluation will serve one of its major purposes as suggested by Stufflebeam, 1971, **"The purpose of evaluation is not to prove, but to improve".**

Reference
Stufflebeam, Danial I. (1971) **Education evaluation**

# EVALUATION - A GENERAL SCHEMA

**M. J. McKenna, Ministry of Education, Province of Ontario**

Education is a process which changes the learner. Within this context maturation, growth, the individual experience of the learner, the school, the curriculum and instruction interact to produce these changes. Evaluation must be an attempt to describe, appraise and influence these agents of change. Specifically, evaluation should be viewed as a systematic collection of data to improve the teaching and learning process. Indeed evaluation must assume the following roles:

- a method of acquiring and processing evidence to improve teaching and learning
- an aid to clarify significant goals and objectives of mathematics education
- a process to determine the extent and direction of student development in mathematics
- a system of quality control on the teaching-learning process
- a program-review process to determine the degree of achievement of stated goals and objectives for a particular mathematics course

Each of the stated roles focus upon obtaining information for instructional decision-making. There are three major phases of instruction which require decisions: the state of the learner, the structure and organization of the learning process, and the extent to which the intended learning has taken place. In turn each phase requires that certain questions are asked, specific data is gathered and decisions made. Table I suggests an outline for obtaining information needed in an evaluation program to aid in decision-making for the three phases.

**TABLE I**

| | THE STATE OF THE LEARNER | THE STRUCTURE AND ORGANIZATION OF THE LEARNING PROCESS | THE EXTENT OF INTENDED LEARNING |
|---|---|---|---|
| MAJOR DECISIONS | • readiness for learning tasks<br>• point of entry into learning sequence<br>• the affective orientation of the learner | • structure of the learning tasks and units<br>• best sequence of tasks and units<br>• progress of learner with each task or unit<br>• prescription of alternate tasks, as necessary, to facilitate desired outcome | • what is to be learned?<br>• to what extent has learning taken place?<br>• has the instructional process been effective? |
| SPECIFIC TYPES OF INFORMATION NEEDED | • do students have any mastery of course content?<br>• what has been studied previously?<br>• are course prerequisites satisfied?<br>• do certain topics have special interest and meaning? | • do students have partial mastery of some topics or units?<br>• what type of instruction is most suitable?<br>• are students mastering topics and units as they progress? | • have students retained the essential skills? |
| SOURCES OF INFORMATION | • record of previous achievements<br>• course pretest<br>• placement tests<br>• aptitude tests relevant to the choice of alternative learning approaches<br>• standardized tests | • interview with student<br>• unit pretest<br>• quiz covering a given objective or topic<br>• unit post test | • end of course examination<br>• projects<br>• observations<br>• standardized test |
| | (INITIAL EVALUATION) | (FORMATIVE EVALUATION) | (SUMMATIVE EVALUATION) |

**The State of the Learner.** The teacher must be able to identify the relevent characteristics of the learner at the point of entering the program. This mode of evaluation has student placement as its primary function and may take several forms. It may seek to determine whether or not a student possesses prerequisite skills, it may establish whether a student already has mastery of certain units and it may assess the attitude and self image of the learner. It should take place prior to instruction and focus upon the student's present status thus classifying the student relative to alternate modes of instruction. In addition, this initial evaluation should determine, where necessary, underlying causes of repeated learning difficulties. Instrumentation should include teacher made pretests related to general topic or unit objectives, observation and checklists regarding performance under different teaching-learning environments and techniques, and attitudinal surveys.

**The Structure and Organization of the Learning Process.** The most important aspect of evaluation used in this context is the aid it can give students **during** the learning of a particular topic or unit. Such evaluation is referred to as formative, taking place during the learning process. Specifically, student learning can be paced according to the mastery of a sequence of mathematical skills and/or concepts. As well the student should receive rather immediate feedback on what has been mastered and what still needs to be learned.

In these regards, formative evaluation tests should be relatively short and emphasize the most important objectives under immediate study. As a key result of such testing, the student and teacher can more easily analyze the causes of difficulties in the sense of both why and where.

Finally, if these short tests or similar evaluation techniques are well designed, it is possible to relate the diagnostic aspect of the tests to alternate ways in which a student can overcome difficulties. This mode of operation will lead to a library of alternate remedial materials related to specific objectives and which can be developed over a period of time so as to improve the scope of the teaching-learning process, particularly from the teacher.

Instrumentation should ultimately include a specific sample of all related tasks in the hierarchy of objectives for a particular unit.

**The Extent of the Intended Learning.** Evaluation in this content is directed to a much more general assessment of the degree to which larger outcomes have been attained over the entire course or some significant part of it. Often, this process is referred to as summative evaluation. The general process for the structuring of test items for this purpose is to establish a "specifications matrix" and then develop specific items which fit the applicable cells of the matrix. Generally, a specifications matrix has content as rows and behaviour as columns. A general model is given as follows in Table II.

**TABLE II**

| | COMPUTATION | COMPREHENSION | APPLICATION | ANALYSIS |
|---|---|---|---|---|
| directed line segments | | | | |
| equivalent vectors | | | | |
| multiplication by a scalar | | | | |

The four levels of behaviour are chosen as particularly appropriate for the Intermediate Divisions. Fundamental definitions are given as follows:

- computation includes recall of basic facts and terminology according to the rules learned
- comprehension extends recall to demonstrating understanding of concepts and their relationships
- application requires recall of relevant knowledge, selection of appropriate operation and the performance of operations in a specific context which has usually been practiced
- analysis requires non routine applications of concepts

Instrumentation in this regard is generally in the format of formal tests either at the end of a major topic or series of topics and given at the end of a term or a year. It should be understood that summative instruments, will produce data that should only affect part of the decision making concerning student achievement and growth.

## EVALUATION-TECHNIQUES FOR COGNITIVE GROWTH AREAS

Although the "specifications matrix" concept was introduced as part of the summative decision making process, the matrix is useful as an organizer for the initial evaluation and formative evaluation stages. A delineation of techniques for the four behaviour categories follows:

**Computation (Recall).** In developing tests and observation techniques for objectives at this level, the teacher must always keep in mind observable behaviour which demonstrate "recognizing", "recalling", and "knowing". Further, there are two important characteristics category: the situation must be very similar to the level used in the original teaching-learning situation and the situation must not be described in terms or settings which are new to the student. A variety of data gathering techniques include check lists of observable behaviour items which require filling in blanks (supplying the required term, fact, etc.), multiple choice items, matching items and true-false questions.

**Comprehension.** Comprehension is best defined in three stages: translation, which requires a known concept to be put in different words; interpretation which requires the student to see an interrelation between parts; extra-polation which requires making inferences about consequences. An example of translation occurs when a student puts into words the parts of a histogram which shows trends in the cost of living index. Interpretation is required when the student is asked to relate the graph to actual day-to-day events. The ability to extrapolation can be measured by requiring the student to suggest possible meanings of the graph for different situations such as wages or taxes. Evaluation of comprehension must not be in the same terms or in the same situation as used in the original teaching-learning mode but it should have similar characteristics in terms of language. Data gathering techniques similar to those listed for recall are suitable at this level and can be extended to include evaluation of oral or written reports based upon relatively structured activities and laboratory approaches to the learning of mathematics.

**Application.** Simply, application requires the ability to apply principles and generalizations to new problems and situations. Several aspects in this category can be considered as appropriate for data gathering:

- determining which principles are applicable in dealing with a new problem
- stating alternative possible approaches to a problem
- specifying the limits within which generalizations are relevant
- explaining new phenomena in terms of known principles
- predicting what will happen in new situations
- justifying the choice of specific alternative routes to a solution

The data gathering techniques used to gather information for decision making in this area should satisfy at least two criteria: the problem situation must be new or in some way different from those used in the original instructional process and the problem must be solvable in terms of principles and generalizations which were directly a part of the teaching-learning process.

This behaviour area can be evaluated in terms of information gathered by techniques already described in the previous two situations, as well as in terms of more open activities and lab. approaches and print problems.

**Analysis.** This behaviour is indicative of the ability to break down a new situation into constituent parts, analyze the relation between these parts, and put the parts into some formal hierarchy which will lead to effective decision making. Although many students in the final years of the Intermediate Division will demonstrate little development in this area, the following indicators of the ability to analyze are included for consideration:

- classification of information according to rather complex criteria
- can infer assumptions, laws or principles which are implicit in a problem
- can use criteria to discern patterns, orders and arrays; given particular data
- can recognize generalizations arising from particular situations

Complex activity-oriented situations as well as print problems serve as data gathering mechanisms in this area.

In summary, the teacher who plans data gathering situations for decision making using the described behaviour in a Matrix such as in the example above will be able to cope with initial, formative and summative evaluation of cognitive growth in an organized fashion.

On this basis, every teacher-student interaction can be based on some implicit correction about what changes in the learner are possible and what are desirable.

## EVALUATION-AFFECTIVE COMPONENTS

Earlier it has been stated that education is a process which changes the learner and that evaluation is a process of making a decision based upon data to improve the teaching-learning process. In evaluating at the entry level, or for formative or summative purposes, emphasis has been traditionally on cognitive performance.

Although mathematics educators may intellectually recognize that anxiety, inhibitions, loss of self esteem and a self defeating attitude may be significant contributors to failure in mathematics, seldom are such characteristics analyzed at entry points and in formative evaluations. When these characteristics are ignored, feedback to the student concerning progress to the attainment of objectives is incomplete.

Research indicates that an intimate relationship exists between cognitive and affective behavioural changes. Attitudinal surveys, observation and continual dialogue with the student should be used to establish growth along the following continuum:

- the student develops a tolerance for mathematics
- the student participates actively in mathematical situations
- the student develops some positive feelings toward mathematics
- the student initiates mathematical pursuits and activities
- the student applies mathematical skills and concept as a part of the daily decision making processes

The constant monitoring required in this area will have significant influence in improving interpersonal relationships and in creating a positive environment for learning for students of all abilities.

The teacher must diagnose with care, plan for an extensive range of individual differences and mediate between the intellect on one hand and the affections of the learner on the other. There is strong evidence that underachievement and low achievement is as much a personality problem as an intellectual problem. The student thus must feel socially and emotionally secure in the classroom environment and the teacher must speak to the feelings of the learner. Continual monitoring in this area must become a regular part of the analysis of learning that takes place in mathematics classrooms and mathematical situations throughout the entire learning experience.

# THE PRINCIPAL AND THE MATHEMATICS PROGRAM

**by John J. Stapleton. Dr. Stapleton, who has taught Mathematics in British Columbia, Ontario, and Newfoundland, now is a member of the Department of Educational Administration in the Faculty of Education at U. W. O.**

In the early 1960's, I was a high school mathematics teacher in a school of about 800 students. My principal who had been an outstanding English and music teacher played an active part in the operations of the mathematics department. He did this, not by visiting classrooms for in the five years in which I taught in this school, I cannot recall a single visit to my class, but by scrutinizing examination results thoroughly. These were still the days of the departmentals, and the principal insisted that Christmas and Easter examinations be conducted as trial runs. To that end, all students were marched into the gym to write exams, and heaven help the teacher whose test had to be publicly corrected over the loudspeaker. When marking was completed, he called a faculty meeting to examine the number of honours, passes, and failures for every section in every grade. Teachers whose classes deviated significantly from the norm were called upon to explain the discrepancies. He wasn't adverse to making changes based upon the results either. On one occasion, he introduced team teaching primarily to lessen the impact of a weak teacher in the department. But it was the final grades to which he ascribed most weight. He pored over these with the result that those teachers whose classes earned marks below the levels the principal deemed to be appropriate found themselves assigned different teaching duties in September. This principal was a formidable man, much loved by staff and students, and on the basis of pass percentages and scholarships earned by students during his tenure as principal, the school was judged to be a good one.

Schools of the late 1970s differ from the schools of the early 1960s. As Hall and Carlton have observed so ably, countervailing pressures to those exerted by strong willed principals exist.[1] The large influx of students of widely differing abilities, the consumer power of students, the attractiveness of outside interests, the removal of external standards of performance, the increased power of teachers, and the legacy of elementary school innovations make it possible for schools to differ widely in their quality. Yet despite the reduced discretionary power of the principal, research findings tend to support the frequently voiced opinion of trustees and senior administrators that a major factor in whether a school is either a "beacon of brilliance" or a "pothole of pestilence" is the leadership ability of the principal.[2]

Many factors affect the role of the principal. Certainly the impact of the acts, regulations, collective agreements, federation policies, and legal precedents cannot be overestimated. Equally important is the set of expectations held by the local district about its principals' behaviors. In some systems, the principal is a plant manager whose tasks are to ensure that the paper flow continues at a steady rate and to see to it that no serious complaints are made about the school by the community. In other systems, he is expected to be an instructional leader, one who deserves a high salary because he helps teachers become better teachers, parents become better teammates, students become better learners, central office personnel become better colleagues, and citizens become better supporters.[3] Establishing, implementing, and evaluating programs to accomplish such lofty objectives are the tasks which these systems expect of their principals.

The purpose of this paper is to support the contention that the principal should be the instructional leader of the school, and to suggest strategies to allow the incumbent to be more effective in this dimension of his or her job.

It is clear that many Ontario principals and vice-principals see instructional leadership as an important part of their role. In a recent study conducted by the researchers of the Faculty of Education at the University of Toronto, administrators were asked to describe the skills they thought necessary for effective job performance.[4] The analysis of the open ended responses yielded four sets of skills. First, the administrators stressed the importance of developing one's own philosophy of education and of making more explicit one's own value system. Two illustrations underscore the need for these personal traits. First, when Ontario elementary schools became "curriculum building units" in 1972, local programs had to be developed in the absence of prescriptive guidelines. The philosophical views of the principal were crucial in these developments. Secondly, in the current debates about the role of the secondary school[5], the principal must be sure of his own stand before he can participate in an effective manner.

The second set of skills concerned the administrative and organizational abilities that are needed to deploy time, people, and facilities effectively and efficiently.

The third skill, and the one identified most frequently by the respondents, was skill in human relations, the ability to be effective in face to face encounters with individuals and small groups.

And fourthly, curricular skills were identified. Being able to supervise teachers constructively, to develop programs, and to evaluate these when implemented, was prized by the administrators. The frequency with which Ontario administrators identified these skills is a measure of their desire to be more active instructional leaders. Further, the data suggests that teacher supervision and program review are two of the most promising areas in which a principal can make his influence felt.

The following extended example of a program review mechanism describes the procedures followed by an Ontario school board which has about 30 elementary schools, 400 teachers, and approximately 10,000 students. Initiated in 1972, this review has been in operation for the past six years, although like a car model, it has had improvements during that time.

A school is reviewed every third year. First, the principal writes a profile of the school's community. Then he organizes an internal study of the school's programs. Normally, the staff is deeply involved in this aspect. Data are collected by means of a questionnaire based on the goal statements and learning opportunities of **The Formative Years.** The following example taking from the manual illustrates the approach.

**Primary Division - Communication Evaluation**

**Arithmetic**

1. The child will understand and use numbers and operation. 1 (2) 3 4 5 6 7

   Commentary:

   Applying arithmetic to everyday problems requires improvement in our opinion. We do not feel our present text resources are adequate and we should like to develop this learning opportunity further.

   Our "low-average" pupils require special attention and we require advice on organizing our program to achieve this better.

The manual includes a code for use by the staff. Thus a rating of 1 indicates that the goal is absent from present school offerings, 4 indicates that the goal is present in school program but one learning opportunity is deficient, and 7 means that the goal and learning opportunities are present and that pupil achievement is from good to excellent. In the example, circling 2 meant that the staff perceived that the goal was present but that the teachers were unsure of its appropriate application in the program.

When the internal review has been completed, the different coordinators from the board office examine the programs. Using the internal review and their own observations, they write separate reports. They are specifically enjoined not to evaluate teachers, which is one of the weaknesses of the process when problems are those of personnel rather than of program. Coordinators prepare a list of recommendations and a statement of the help which they are prepared to offer in the next three years. This report is then given to the principal with a copy filed at central office.

The program superintendent then visits the school to record his impressions about the total program and to review the program leadership of the principal. His report is also given to the principal as well as filed.

The principal then synthesizes the internal and external reports and generates a set of ordered recommendations to guide program development during the next three years. Staff involvement in this phase depends on the discretion of the principal.

All reports then go to the director who chairs the formal cyclic review. Attending this meeting are the principal, the director, the superintendents, and occasionally, a coordinator who might be asked to enlarge upon a specific recommendation. Outcomes from this meeting include a ratification of the directions to be taken in the next three years, a listing of the resources that will be made available to the school, and the dates of the follow-up visits when the quality of the implementation will be examined. A summary of the review is written by the director and communicated to the board.

In the research which I conducted on the process, several flaws were uncovered, particularly in the amount of paper work and in the coordinator's role. But despite these, a majority of the system's teachers, principals, coordinators, senior administrators, and trustees wanted to keep a modified format. As for the mathematics program in particular, the principals and teachers were asked if they could identify any benefits in program objectives, program content, instructional procedures, evaluation procedures, awareness of student achievement, and in pointing out future curriculum directions. The results were as follows.

**TABLE 1**

Percentages of System Respondents Perceiving Benefits in Mathematics Program as a Result of Cyclic Review 6

| Item | Principals | | | Teachers | | |
|---|---|---|---|---|---|---|
| | No. Ben. | N.O. | Some Ben. | No Ben. | N.O. | Some Ben |
| | | (N=28) | | | (N=77) | |
| 1. Program Objectives | 35.7 | 3.6 | 60.7 | 42.3 | 23.7 | 33.7 |
| 2. Program Content | 21.4 | 3.6 | 75.0 | 50.0 | 23.3 | 33.7 |
| 3. Instructional Procedures | 46.4 | 3.6 | 50.0 | 51.6 | 28.3 | 19.9 |
| 4. Evaluation Procedures | 39.3 | 3.6 | 57.1 | 42.3 | 48.0 | 16.8 |
| 5. Awareness of Student Achievement | 57.1 | 3.6 | 39.3 | 51.6 | 33.3 | 14.9 |
| 6. Pointing out Future Math. Directions | 28.6 | 0.0 | 71.4 | 46.6 | 28.3 | 24.8 |

Note. N.O. means No Opinion

Although this table is open to several interpretations, the system concluded that while the principals were able to use the data from cyclic review to make program alterations, a large number of teachers did not realize the uses to which the process was put.

The point of this rather elaborate example is that the principal does not have to be an expert in each subject area. However, he does have to have a set of procedures which can identify strengths and weaknesses in the current program as well as point to steps that can be taken by the school to make the necessary modifications. He can get assessments of this kind from his system, from the ministry of education, or from a university. When future directions have been identified either through the application of evaluative procedures or through an externally mandated change, the principal must concern himself with implementation and assessment procedures. If others are subject specialists, then he should be an expert on the change process, that is, he must know how to get consensus if not unanimity on preferred directions, how to secure necessary human and material resources, and how to collect the data that will be used to assess the impact of change.

So much for program. The other point of leverage for the principal is teacher supervision. I am prepared to argue that this task has never been more demanding. Rare is the system which does not require the principal to engage in formal evaluation of tenured teachers. Given reduced teacher mobility, it is not unusual for the principal to have to observe a teacher who has been teaching Math 251 for ten or more years. Faced with such a challenge, the principal can and should give the moral support that every teacher needs. But he has an obligation to help the teacher retain his creativity and his enthusiasm. Although the principal has a number of tactics that he can apply, there is little doubt that he has to become more knowledgeable about classroom visitations.

A teacher should expect that a principal will establish an initial conference prior to the visitation to mutually determine when the visit will take place, what will be observed, and what instrumentation will be used. Following the visit, the teacher should expect an intensive post-observation conference in which the two jointly work through the data and generate future directions. To be an effective supervisor, the principal will need to become familiar with observational technology and he will have to become more skillful in providing helpful feedback to teachers.

All principals cannot be expected to know the difference between a transformation and a transversal. They must know, however, what the goals of their schools are, and they must understand the contribution that the mathematics program makes in accomplishing those goals. We expect principals to be involved in the setting of objectives in the mathematics department, in procuring required resources, in deploying these resources efficiently, in supporting the teachers while the program is underway, and in establishing the procedures which assess the extent to which the objectives are met.

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**Notes**

1 Oswald Hall and Richard Carlton, **Basic Skills at School and Work** (Toronto: Ontario Economic Council, 1977).
2 Keith Goldhammer **et al, Elementary School Principals and Their Schools** (Eugene, Oregon: Center for the Advanced Study of Educational Administration, 1971).
3 Madeleine Hunter, "What's going on around here?", **The National Elementary Principal,** March, 1978, p. 11.
4 V. K. Gilbert **et al, Perceptions of Educational Leadership** (Toronto: Ministry of Eduation, 1977).
5 Hall and Carlton, Chapter VII.
6 Those readers interested in a fuller description of the project are invited to contact the author at The Faculty of Education, University of Western Ontario.

# RENEWAL NOTICES

**TO MEMBERS OF ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION**

Each year in late August members of OAME receive a renewal notice and remittance envelope. If the membership is not renewed a similar second mailing is done in October. The President and the office of OAME, due to increasing paper and postage costs, are asking that members consider this notice which is appearing in the September Gazette and the September Abacus as their second notice reminder. If you have misplaced your application form, one is printed on pg. 32. Please mail to Ontario Association for Mathematics Education, 247 Bright St., Sarnia, Ont. N7T 4E9.

# DIVIDE TO CONQUER

**H. Grant Jones, Mathematics Resource Teacher, The Peel Board of Education**

I was quite surprised, and disappointed, to see a new edition of a very popular American text book series revert back to a 'gazinta' form of division. Let me temper my criticism by pointing out that they do encourage the child to estimate the final answer, e.g. between 60 and 70, before actually doing the question. However, the format throughout the series is as follows:

A.

```
   33 R 1
8 ) 265
    24
    25
    24
     1
```

Before anyone shouts "Eureka", let us consider what has happened over the last 10 or 12 years, and most of all, let's not throw out the baby with the bath water.

For many years a system comparable to the following has been developed in most text book series. It has often been referred to as the subtractive method.

B.

```
8 ) 265
    80 | 10
   185
    80 | 10
   105
    80 | 10
    25
    24 |  3
     1 | 33
```

leads to

```
8 ) 265
   240 | 30
    25
    24 |  3
     1 | 33
```

leads to

```
        3
       30
8 ) 265
   240
    25
    24
     1
```

33 R1

leads to

```
   33
8 ) 265
    24
    25
    24
     1
```

Let us admit that this method has not met with the success hoped for initially. We could conclude perhaps, that is has failed miserably, if the new series mentioned above is our gauge.

BUT WAIT! If it did, WHY did it fail? Did it have any value? Can we learn anything from the experience?

I believe that the subtractive method has some degree of failure for at least two reasons. First, many teachers did not completely understand its development as shown above in B. We often were quoted to Grade 5 or 6 children as saying, "Now I'll show you ANOTHER way, a quicker way to divide." If you have said that, it is possible that you did not use the progression as illustrated. This progression was also spread over two, three and even four grades, therefore, it was not easy to continue a specific sequence.

Parents who have always wanted to assist their children, wanted to help teach division, because, "It was always hard for me!" You know the rest! Your opportunity to develop a method was lost because, "My Dad showed me a faster way."

What, if any, were the advantages of the subtractive method? Perhaps its greatest advantage over the way "we were taught", is that it helps to explain what you are doing, and why you are doing it. Compare that to the method, "if it won't go into 2, try 26". Division is really grouping or sharing, often described as rapid subtraction. The 'gazinta' method, even with good materials and good teaching methods did not make that same impression.

A second advantage is that it not only allows, but encourages, individual differences. 'Gazinta' cannot make such a claim because it must be done one way without variations. The subtractive method encourages estimation of partial quotients, if necessary. These estimations are really the development of an understanding of division, which at the same time, allows individual

differences.

Before this becomes a dissertation on subtractive versus gazinta, let me consider the best of both worlds and suggest an alternative. This alternative may appear simply as a stacking method, which it obviously is, but its development brings in much more than that. It gives considerable thought to the 'size of the quotient'. This is as critical as the format you choose to use.

Let me illustrate with an example. Most text book series, have titles such as, 'three-digit numbers divided by one-digit numbers'. Then within the seatwork provided on that page, have questions ranging from $6\overline{)105}$ to $6\overline{)584}$. One quotient is in the teens, the other in the nineties. Estimation skills needed to answer those questions are vastly different. Because division IS difficult to learn there needs to be more control, i.e. control of the quotients, as outlined in the steps below.

This development seems to work at Grade 4, where previous grades have concentrated on the concept of division rather than the operation.

Step 1 **Quotients greater than 10, less than 20, one-digit divisors**

```
    3
   10   13 R 1
6 ) 79
    60
   ---
    19
    18
   ---
     1
```

The zero in the 10 should be used at first, for the same reason that the zero is usually used in beginning two-digit multipliers, e. g.

```
   16
x  12
  ---
   32
  160
  ---
  192
```

Both zero's are very meaningful to someone trying to understand the process. Step 1, using quotients less than 20 should be mastered before Step 2 begins. This is where we keep the advantage of allowing for individual differences. Some children will speed through these steps, while others will not.

Step 2 **Quotients less than 30, one-digit divisors**

Oral multiplication of one-digit numbers by 20 must be taught just prior to Step 2, e.g. 10 x 6; 20 x 6; 10 x 7; 20 x 7; etc.

a)

```
    2
   20   22 R 3
6 ) 135
    120
   ---
     15
     12
   ---
      3
```

b)

```
    2
   10   22 R 3
   10
6 ) 135
    60
   ---
    75
    60
   ---
    15
    12
   ---
     3
```

Comparing a) and b) is vital in my recommendation. a) is strongly recommended over b). If a child continually uses b), then Step 1 with quotients less than 20 should have been maintained.

By mixing quotients less than 20 with quotients in the twenties, it is easy to tell whether a a child is ready to proceed to Step 3. A child must have mastered Step 2 a) before going on to Step 3.

Step 3 **Quotients less than 40, one-digit divisors**

Oral multiplication of one-digit numbers by 30 must precede Step 3,
e.g. 10 x 6; 20 x 6; 30 x 6; etc.

```
    4
   30  34 R 4
8 ) 276
   240
   ---
    36
    32
    --
     4
```

Testing a child's ability to estimate is vital before going on to Step 4. By presenting questions with quotients from 10 to 39, the child will demonstrate his or her competency. If partial quotients such as 20 + 10 are used instead of 30, do not proceed to Step 4, and perhaps even revert to Step 2. Mastery to this point must be achieved.

Step 4 **Quotients less than 50, one-digit divisors**

Once again oral multiplication of one-digit numbers by 40 must precede Step 4,
e.g. 10 x 6; 20 x 6; 30 x 6; 40 x 6; etc.

```
    5
   40  45 R 5
7 ) 320
   280
   ---
    40
    35
    --
     5
```

Step 5 **Quotients less than 60, one-digit divisors**

or: **Quotients less than 100, one-digit divisors**

At this point a lock-step development of quotients less than 100 is not necessary for most children. If they are able to cope with multiplication of one-digit numbers by 50, 60, 70, 80 and 90, they should be able to estimate quotients less than 100 quite accurately. Again let me point out that this accuracy is the key to this stacking process.

Step 'n'

If you have developed the method this far, the question now becomes, "What's next?" There are at least two different answers.

1) three-digit quotients, one-digit divisors

2) two-digit quotients, two-digit divisors

Either answer would be acceptable, as long as very careful development is used.

However, another question arises! Prior to accepting either answer above, should we omit the zero's in the quotient, then in the products throughout the question? e.g.

$$\begin{array}{r} 9 \\ 80 \\ 8\overline{)713} \\ 640 \\ \hline 73 \\ 72 \\ \hline 1 \end{array}$$ 89 R 1 leads to $$\begin{array}{r} 89 \\ 8\overline{)713} \\ 640 \\ \hline 73 \\ 72 \\ \hline 1 \end{array}$$ leads to $$\begin{array}{r} 89 \\ 8\overline{)713} \\ 64 \\ \hline 73 \\ 72 \\ \hline 1 \end{array}$$

There are two reasons why the zero matter should be given strong consideration prior to two-digit divisors. First, if a child 'understands' division to this point, why not proceed to a shortened method? Furthermore, is it possible to comprehend 876 groups of 79? Secondly, metrication makes greater use of the decimal, therefore division of decimals is introduced somewhat earlier than it was.

Note: As we proceed forward to more difficult questions, continue to illustrate problems concretely or pictorially. Base ten blocks do this quite well.

**Summary**

In summary, let me state the advantages I see in the 'controlled quotient stacking' method, which if used at Grade 4 should move smoothly into the 'gazinta' method at Grade 5.

1) It continues the place value columns upwards, that are established when you write the original dividend, rather than creating a new column to the right of the question. (which many students have trouble keeping in order, so they can add properly)

2) Many teachers do not like the subtractive method which has been promoted over the last ten years. The result is that they often rush back to the way they were taught (gazinta). They lose the logical development, and therefore, understanding.

3) It moves more quickly towards 'gazinta' than the subtractive method, yet still emphasizes understanding.

4) Parents may understand it more than the subtractive method, since it looks more like the way they were taught. Mathematically, this appears to be an absurd reason, but in the case of division, past history has proved that it is critical.

# STOP - START METHOD

**Robert Mealey, Head of Mathematics Department, Edith Cavell School, Moncton**

***Reprinted with permission from Mathematics, New Brunswick Teachers Association, 1977***

Teacher: "Remember that you do multiplication and division in the order that they appear and then you do the addition and subtraction in the order in which they appear."

Student: "Does that mean that you do all the multiplying first and then do the division? Then you do all the adding and then the subtraction?"

Does this seem familiar to you? The rules are simple but many students do not seem to be able to use them.

For many years I fought with the pupils over the order of operations until I developed a new method for them which we call the "Stop-Start Method".

The basic rules are as follows:

1. Keep brackets together

2. We will stop before any adding or subtracting signs (unless in brackets)
3. We will do what is in EACH block (term) working from left to right
4. IN the brackets we will stop-start as well.

Example:

Calculate: $2 + 3 \times 5 - 4 \div 2 + 7 - (2+4)$

left ⟶ right

| = 2 | + 3 X 5 | - 4 ÷ 2 | + 7 | - (2 + 4) |
|---|---|---|---|---|
| | ↑ stop | ↑ stop | ↑ stop | ↑ stop |

| = 2 | + 15 | - 2 | + 7 | - (6) |
|---|---|---|---|---|

| = 17 | -2 | +7 | -6 |
|---|---|---|---|

| = 15 | +7 | -6 |
|---|---|---|

| = 22 | -6 |
|---|---|

= 16

If you are dealing with Integers then you can add and/or substract in any order

2 -2 (may cancel the 2's)

+15 -6

+7

+24 -8 = 16

When working with Integers or Rationals there are certain questions to ask for each block (term).

A. Is there any work to be done in the block?

Response:

1. No - then go to next block

2. Yes - (a) What is the sign of the term? (b) Work the question out from left to right.

Once the blocks are worked the following questions are asked:

B. SSA? (If the SIGNS of the terms are the SAME you ADD them. The sign of largest term added is used for the sign of the sum.)

C. SDS? (If the SIGNS are DIFFERENT you SUBTRACT. The sign of the largest term used is the sign of the difference.)

When the pupils reach the point where they picture the terms (blocks) automatically then the lines or blocks may be dropped.

The word "block" is used in grade seven and the early part of grade eight. Toward the end of

grade eight "term" is used as well as in grade nine.

I have found this to be most successful as it helps the pupils to separate factors from terms and this helps them to realize what rules to use for each section of the question.

Teacher: Now students, calculate the following using the "stop-start method".

$$\frac{2 \text{ X } 2 \div 1}{4 \text{ X } -2} - \frac{-3 \text{ X } 2}{12 \div 1} + \frac{3 + 2 \text{ X } -4}{-11 + 6}$$

Solution:

$$= \frac{2 \text{ X } 2 \div 1}{4 \text{ X } -2} \Bigg| - \frac{-3 \text{ X } 2}{12 \div 1} \Bigg| + \frac{3 + 2 \text{ X } -4}{-11 + 6}$$

$$= \frac{4}{-8} \Bigg| + \frac{6}{12} \Bigg| + \frac{5}{5}$$

$$= \frac{-1}{2} \Bigg| + \frac{1}{2} \Bigg| + 1$$

Cancel

$$= 1$$

$$\frac{2 \text{ X } 2 \div 1}{4 \text{ X } -2} = \frac{4 \div 1}{-8} = \frac{4}{-8}$$

$$- \frac{-3 \text{ X } 2}{12 \div 1} = - \frac{-6}{12} = + \frac{6}{12}$$

$$\frac{3 + 2 \text{ X } -4}{-11 + 6} = \frac{3 - 8}{-5} = \frac{-5}{-5} = + \frac{5}{5}$$

# WHAT ARE WE DOING FOR THE BRIGHT PUPILS?

**M. S. Cohen, D. Guthrie, E. R. Muller and P. Thomas**

This is the second year that Brock University has run an enrichment program for selected students from Grades 6 to 8 in the Lincoln County School Board. It is the first year that Mathematics and Computer Science were involved with the project. Based upon the experience of other departments in last year's program, the following conditions were imposed:

(i) not more than forty students were to be admitted into each of two groups;

(ii) the first group, composed entirely of Grade 6 students, were to meet first for five consecutive Saturday mornings from 9:30 to 11:45 a.m.;

(iii) the second group composed of Grade 7 and 8 students would follow in the next five consecutive Saturdays.

The selection of students was done by the Board using the following:

(i) Entrance Criteria (Grade 6)

(a) tested IQ of 135+

(b) working comfortably 3 grade levels above placement on a standardized test

(c) showing social maturity

(ii) Continuation Criteria (Grades 7 and 8)

(a) admission to the previous year's program (as above)

(b) attended at least 80% of the previous year's sessions.

Following discussions with some experienced mathematics teachers* and the two departments involved, it was decided to run a concurrent program where students would be solving realistic mathematics problems, while at the same time they would be learning to program. During the problem-solving sessions special emphasis was paid to the steps which the students took to attain a solution, thereby underlining the instructions which would be given to the computer in order to reach the same conclusion.

On the first Saturday the Grade 6 group was faced with the following set of problems:

**Problems on Carpeting Costs**

**Problem 1A**

A contractor is hired to carpet a house (see plan). Calculate the total cost based on the following:

(a) All carpeting is available in 3 metre widths.

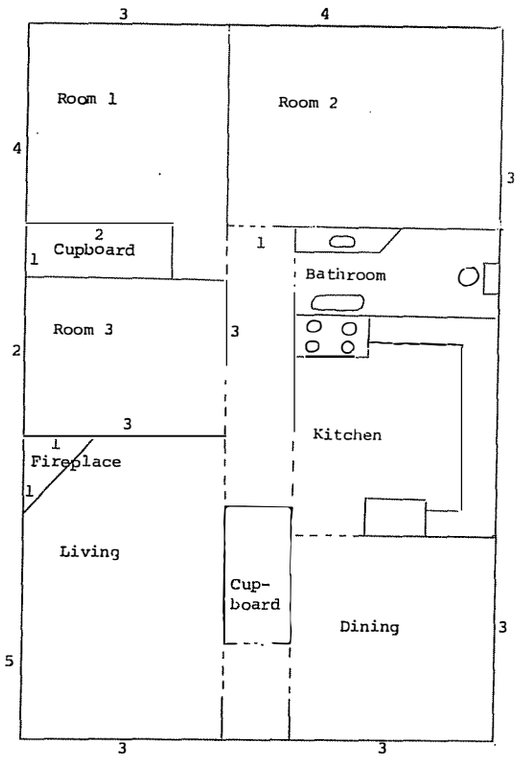
(b) Rooms 1, 2 and 3 are carpeted with carpet which costs $10.75 per square metre.

(c) Dining room and living room are carpeted with carpet which costs $15.25 per square metre.

(d) Hallway between rooms is carpeted with carpet which costs $7.50 per square metre.

(e) The contractor charges an installation cost of $8.50 per square metre.

**NOTE:** Both the carpet costs and the installation charges quoted by the contractor are based on the full area of each room, for example, the living room is considered a 3 x 5 room as though it did not have a fireplace.




**Problem 2A**

Calculate the cost of the wasted carpet which was cut out for the fireplace in the living room and for the cupboard in Room 1.

**Problem 3A**

Calculate the total cost of carpeting the house if all but condition (e) of Problem 1 hold and this condition is replaced by:

---

*The authors wish to thank Al Penteliuk and Oleg Bezotosny for their time and ideas.

(e) the contractor charges an installation fee of $8.50 per square metre for the areas greater or equal to 7 square metres and $10.50 per square metre for areas less than 7 square metres.

**Problem 4A**

**Financing the costs calculated in Problem 1A**

The owner of the house wishes to pay half the total costs at the end of the job, and the other half exactly one year after the job has been completed. The contractor agrees to these terms provided the owner pays a simple interest of 18% per year on the outstanding balance.

Calculate:

(a) the interest on the outstanding balance;

(b) the final payment due one year after the job has been completed.

**Problem 5A**

**Different financing arrangements to cover costs of Problem 1A**

Because the contractor is charging a higher interest rate than the bank the owner decides to go and see his bank manager. He is able to secure a loan of $500 at 12% simple interest per year which he adds to his first payment of half the total cost.

Calculate:

(a) how much he will have to pay the bank at the end of the year;

(b) how much he will have to pay the contractor at the end of the year;

(c) how much he has saved by borrowing from the bank rather than from the contractor.

After completing the first three problems the students were given the necessary instructions to run a prewritten program (stored on disk). This exercise got them used to accessing the machine and the program gave them possible solutions to the problems they had solved earlier. In order to run this program the students had to keypunch their name and the dimensions of each room together with the carpeting cost per $m^2$ . For homework students had to check the output against their numerical calculations and to pinpoint the errors. (There were some errors in the program!) The following four Saturdays were devoted to developing and applying the following:

1. simple basic ideas about the computer;

2. the simple Fortran statements: READ, PRINT, =, +, -, *, /, REAL, INTEGER, END, STOP, DATA

3. the Fortran statements: IF and DO.

By the end of the five week period the Grade 6 students had written and run programs for the Problems 1A to 3A. Some of the students who worked faster than the rest (about 10 percent of the class) programmed problems 4A and 5A.

It is interesting to note that the above work load which took five weeks with the Grade 6 was completed in three weeks with the Grade 7 and 8 group. This was possible because these students were mathematically more mature and we found it possible to introduce the DO and IF statements early. This allowed them to program the whole house without having to repeat cards for each room, thus minimizing keypunch time. On the fourth Saturday the Grade 7 and 8 group were given the following problems.

**Problems on Replacing the Furnace**

**Problem 1B**

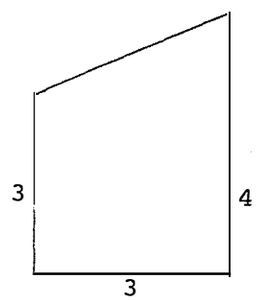
The individual who owns the house which has been carpeted is advised that he should think about replacing his furnace as it is very inefficient. The gas company recommends sizes of furnaces based on the volume of air to be heated. They are:

| Furnace | Volume ($m^3$ ) | Cost (includes Installation) |
|---|---|---|
| A | less than 220 | $1700 |
| B | between 220 and 250 | $2000 |
| C | over 250 | $2500 |

The height of the ceiling in the house is 3 meters. Calculate the total volume of the house and select the right furnace.

**Problem 2B**

The living room of the house has a cathedral ceiling height 4 metres at the cupboard side (see diagram). Calculate the new total volume of the house. Must he select a new furnace?




Vertical cross section of living room

The following two problems are particularly well suited to computer application.

**Problem 3B**

The owner of the house is prepared to pay $200 per month. The gas company is willing to finance the installation of the furnace at 12 percent per year compounded monthly. If the first payment is made a month after the furnace has been installed, complete the following list for each month until the furnace has been paid off.

| payment | interest paid | loan reduced by | value of loan |
|---|---|---|---|

**Problem 4B**

Instead of buying the furnace straight away the owner decides to put $200 per month in his savings account where the bank pays interest at the rate of 6 percent per year compounded monthly. Complete the following table until he has accumulated sufficient money to buy the furnace outright.

| payment | interest paid | total as of previous month | total to date |
|---|---|---|---|

By the end of the fifth week all Grade 7 and 8 students had programmed (with a little help) Problem 3B. The better students (25% of the group) programmed Problem 4B without supervision.

We believe that this programme was rewarding both for the students and the teachers. We have seldom seen such an enthusiastic lot (most days it was impossible to get them away!).

We conclude by noting the following impressions.

1. The Grade 6 group did not go far enough in the time available to experience the computational advantages of the computer.

2. Very few, if any, of the Grade 6 students really understood the abstract formulation of a problem in terms of variables.

3. For most of the Grade 6 group the program was an experience rather than an intellectual enrichment. Nevertheless, we hope that new horizons may have been opened for some of them.

4. Our impression of the Grade 7 and 8 group was different, as we believe that the length and the level of the programme was such that it did extend the logical and mathematical competence of the majority of the students.




The editors of the Gazette have asked the Ontario Mathematics Coordinators to write a short article for each issue of the Gazette. In this article, I will outline the nature of our organization and give a short report on my activities. In succeeding issues the other coordinators have agreed to report in turn.

The Ontario Mathematics Coordinators Association started fourteen years ago as a casual meeting of friends with the sole purpose of discussing issues and airing concerns. This group has grown over the years and the meetings are still informal - but very informative. There are now 14 active members of this coordinator group. Any coordinator with a K-13 responsibility can become a member of the group.

Curriculum and in-service takes up much of our time. The proposed Intermediate Guidelines have encountered some problems and the mathematics coordinators have agreed to assist the Ministry by making specific proposals and developing try-out materials.

There are four main areas to which we are giving priority.

1. **Statistics** - There has been some controversy about statistics in our school curriculum. There is a joint American Statistical Association - N.C.T.M. committee now in existence in the U.S.A. dealing with this question and we have set up a comparable one called the Pre College Statistics Committee. Two of the A.S.A. - N.C.T.M. members are also members of this group. As well the group includes interested university professors, a number of teachers who teach statistics and two mathematics coordinators.

   The committee is funded by the Samuel Beatty Fund and will meet regularly until its task is completed. The task is to develop a program of statistics for grades 7 - 12 inclusive. Units of material are being written and will be offered to schools across Canada on an experimental basis.

2. **Geometry** is a contentious issue and has become the source of much heated discussion in relation to the Intermediate Guidelines. In education there are three questions we must always keep in front of us.

   (a) **Why** are we teaching a subject or topic?
   (b) **What** should be the content of the subject or topic?
   (c) **How** do we teach the content?

   Most Ontario educators have not addressed themselves to the question of why we teach Geometry. Those that have, know that it is absolutely essential to start the study of Geometry

as early as possible - even before Kindergarten. By grade 7, students should have many geometric concepts that are currently not even taught in our schools today. The **why** leads to the **what.** In light of the preceding it is not surprising that many statements about geometry made by well-meaning educators are both uninformed and irresponsible.

3. **Computer Assisted Instruction (C.A.I.)**

   To ignore the impact of modern technology in education is to bury one's head in the sand. Industry has discovered that computers can be very effectively used to train and retrain their people. This new technology is proving to be very popular and has great implications for educators.

   For the past three years we have been developing a High School Mathematics Program to be implemented by computers. This project is being done jointly with O.I.S.E. and North York. As of May 1st of this year the Ministry has given O.I.S.E. a contract to develop a test item pool through C.A.I. The principal investigators are O.I.S.E. and North York people. Three other Boards have been invited and have agreed to participate in writing of C.A.I. materials. They are Hastings, Carlton and Thunder Bay. More Boards may soon be added to this list in the near future.

4. **Testing** is becoming very popular in many Boards. As indicated in the preceding description of C.A.I., the Ministry is taking a very serious look at testing. Many coordinators are actively involved in this area and our plans are to make sure that testing, should it come, will be controlled by the teaching community.

   In North York we have declining enrollment, multicultural classes, inner city schools and all the problems they bring. How to provide for Basic Level students has become a vital concern in many of our schools.

   The North York Coordinator of Mathematics has full time responsibility for all facets of the subject. However, three program leaders (consultants) deal almost exclusively with mathematics but are assigned to three regions into which the Borough is divided. Most of the in-service is done in workshops either at lunchtime or after school although Professional Activity days are frequently used for this purpose. Working cooperatively with the coordinator, the program leaders assist in the writing of support materials such as units for elementary geometry and measurement throughout the grades. A new focus on how to teach computation using Korean finger counting is currently being considered.

One of the important functions of a Coordinator of Mathematics is to anticipate the needs for the future in mathematics education and to take steps to provide for those needs. The different coordinators foresee different needs and therefore have different priorities. In succeeding issues of the Gazette you will hear from these people.



*Congratulations to Hamilton. A Great Conference!*

# INTERMEDIATE DIVISION MATHEMATICS GUIDELINE

**Excerpts from talk given by George Scroggie,
Ministry of Education, at Math '78, York University, June 2, 1978.**

## TIME-EXTENSION FOR FIELD TESTING AND VALIDATION

As you are aware, the timeline for study, discussion, and classroom testing of topics from the draft guideline has been extended by almost one year, until April 30, 1979. By that time we expect to have received from the boards of education reactions that reflect the experiences of their teachers based on classroom trials of the ideas presented in the draft guideline.

A new committee is being formed in the summer of 1978, with representatives from OFT, OSSHC, OAME, OACM, a Faculty of Education the former committee, and the Ministry of Education. It will be responsible for planning strategies for field-testing the guideline and for providing channels through which the results of the field-testing will be available to the committee. It will also be responsible for studying the implications of the guideline for changes in the Senior Division, for planning the validation and making recommendations for revision of the guideline, and for monitoring the changes that are to be made in the summer of 1979 by a writing team that is to be appointed in the near future.

It is anticipated that individual teachers of the province will welcome the chance of investigating some of the proposed changes at the classroom level, in order that reactions can be based on personal experiences rather than on emotional reaction to change. The committee hopes that teachers of Ontario will view the new curriculum from the standpoint of their students. Do the ideas hold the potential for a more interesting and relevant mathematics program - a program which fosters problem solving related to practical instances of real-world situations. If so, then the effort to press on to an implementation of the program, as is or in modified form, will be justified.

## THREE STRANDS: NUMERICAL METHODS, ALGEBRA, GEOMETRY

In the space that remains, I would like to make some comments on the rationale for each of the strands in which the Outlines of Topics are organized. These comments will reflect the opinions of committee members during the developmental stages of the program.

### i) Numerical Methods

From the outset, the committee believed that the program must place a strong emphasis on numerical methods in all grades.

The research project **Applications of Mathematics - A Nationwide Survey** reveals that arithmetic techniques using computers, calculators, and programmable calculators are used extensively in solving the major problems of business, industry, and government. Major policy decisions are based on mathematical models, usually algebraic in nature but solved by numeric methods with computers. The traditional techniques for solving systems of two or three algebraic equations in two or three unknowns are ineffective for solving systems that represent the realities of business, industry, and government. These systems involve n equations in n variables where n is much larger than 3 and can only be solved in realistic time by numeric techniques on computers. Numerous situations exist that justify a resurgence of skills for thinking numerically and for solving problems by numeric methods. In recent years, the increased efficiency of calculating and computing machines coupled with their reduced costs have made it necessary to explore the potential of these devices in designing a more vital curriculum for the immediate future. Projections for the near future indicate that many small businesses will make extensive use of programmable calculators and micro-computers for organizing their business data and for making key decisions. Many households may even use these machines to establish a more efficient operation.

For the above reasons, the committee felt that students should continue to use numbers as a means of solving problems throughout the secondary school program. They need to increase their number

sense and develop confidence in using numbers as a prime technique of solving problems. Thus the revised program has been aimed at identifying numerical techniques that can be coupled with the algebraic techniques of the traditional programs -- systematic trial; nested multiplication; statistical data gathering, representation, and processing; probability; matrices and their applications; coordinate geometry; mappings in the coordinate plane; the use of calculators, etc.

**ii) Algebra**

The committee felt that greater emphasis should be given to the development of algebraic expressions, equations, and formulae as generalizations of numerical patterns and as models of real-world situations. Projecting into the future of applied mathematics it is likely that more and more emphasis will be given to generalizing numerical situations by algebraic models and then solving the models numerically with computers and/or calculators.

Numerical techniques for solving equations provide the only sensible ways of solving equations of order greater than 2, and systems of equations with more than three variables. The technique of systematic trial is presently used in the elementary school to solve equations such as $3\square - 2 = 16$ by essentially a 'too big -- too small' technique. The special techniques for solving linear and quadratic equations do not apply to cubic or higher order equations. The over-riding pattern for solving equations should be numerical, with algebraic techniques for the special cases.

Expressions should be evaluated for specific values of the variable; however, this work should include cases where the value of the variable is not simple, say 5.8029, and where the use of a calculator and nested multiplication form for the polynomial greatly simplifies the calculation (calculator with 'left to right' logic).

The program places less emphasis than before on factoring and on operations with rational expressions as core topics in the Intermediate Division. These techniques are not significant to real world applications at this level; their applications are mathematical in nature and occur in the Grade 13 Calculus and Algebra programs. These topics should be developed in sufficient depth in the Senior Division program for the students who elect to study advanced and general level programs as preparation for further studies.

**iii) Geometry**

In the past, the purpose of the geometry program seems to have been to prepare students for a formal study of Euclidean Geometry from an axiomatic viewpoint. However, there is a strong body of opinion today that it is fruitless at the high school level to teach geometry in this manner. The program should rather evolve from a variety of informal experiences in which the fundamental properties of geometry are established inductively. These properties should form the basis for gradually developing patterns of sequential (or deductive) reasoning and for introducing notions of coordinates, vectors, and matrices in later years.

Ideas of congruence, size, shape, similarity, scale drawings, motion, and symmetry are a part of the P1J1 program. These ideas are extended in the geometry strand of the draft program; opportunities are given to consolidate and systematize these earlier experiences and to use these as the basis for accurate constructions, classification, learning of new properties, transformations as mappings, and use of these experiences as the basis for developing new ideas and properties - and eventually deductive reasoning.

Symmetry and transformations occur early in the intellectual development of children, and are fundamental in their visual perceptions. Children see the relationship of two figures very early in their lives - one fits on the other (congruence by slide, flip, or turn), one is the same shape as the other (similarity), the figures are different (distortions, non-congruence, non-similar). The notions of symmetry are central in the thinking of most mathematicians. Yet the programs of the past have treated these ideas as self-evident and have failed to capitalize on them as one of the basic building blocks for the development of geometry. The proposed program indicates that we should look at real objects in our environment and represent what we see by geometric drawings, then look at properties of the figures in the drawings and relate these back to the objects of the real world. Skills for making accurate drawings are important in this process.

Accurate drawings enable the student to classify figures by their properties (symmetry properties,

and/or measurement properties). Activities with tilings, tangrams, polyominoes, geoboards, dot and grid paper, paper folding, and tracing paper help students to acquire a feeling for geometry and its properties before they meet these ideas formally. These properties become the basis for introducing simple lines of deductive reasoning. The experiences with geoboards, dot paper and grids (non coordinated), scale drawings, and mappings lead naturally into the coordinate plane - first in an informal way, then as the basis for fundamental ideas of coordinate geometry.

**UNIFIED VIEW OF MATHEMATICS**

The program is not intended to be taught as a collection of isolated topics -- it is intended to be taught with a unified view of mathematics, in which ideas of number, algebra and geometry are used to reinforce one another, with applications from real life and techniques of problem solving as the unifying forces. Only when this is done, will students experience mathematics as a cohesive force that provides a sense of reason and understanding to our complex world. If an idea is important enough to teach, then it should be important enough to use. Once simple ideas of graphing, or coordinates, or statistics, or symmetry, or ratio, or integers, or equations, and so on are introduced, we should find many ways of continually recalling and using them in other sections of the course. This is ideally what the committee would like to see. We hope that teachers will recognize that this view of mathematics is vital and will work in the direction of building their programs along these lines.

# PUBLICATIONS AVAILABLE

| | Price | O.A.M.E. G.V.M.A. |
|---|---|---|
| **Handbook of Trigonometry (1975)** | **$2.00** | **$1.50** |
| Handbook of Trigonometry (1975) | $2.00 | $1.50 |
| A Preparedness Test for Grade Nine Students (1973) | $2.00 | $1.50 |
| Sample Grade 13 Examinations | $1.00 | $ .75 |
| Grade 11 (Level 5) Sample Examinations | $1.00 | $ .75 |
| Reprints of 1974 to 1977 Gauss Contest Papers (Elementary) | $1.25 per bundle of 10 | |
| Recreational Math Problems for High Students | $2.00 | $1.50 |
| **Available May 1** - Solutions to Handbook of Trigonometry | $2.50 | $2.25 |

**Coming Soon** - A Numerical Approach to Euclidean Geometry

To Receive these please contact:
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# INTERVIEW

## Taped Talks with Bob Smith

### Kathy Hart

Kathy Hart is the leader of the mathematics team for 'Concepts in Secondary Mathematics and Science Project' (C.S.M.S.) located at Chelsea College, University of London, U.K. It was my pleasure to meet her first in January 1974 in London, and to renew acquaintances at the N.C.T.M. meeting in San Diego, where she was giving a report of her work to the research section. After watching Peter Hilton's and George Polya's motion picture debut at noon hour, Kahty and I met for lunch. While I had brunch, Kathy chatted about her work.

**R.S.** Do give us some background about C.S.M.S.

**K.H.** C.S.M.S. is a five year project (1974-1979), funded by the Social Science Research Council which finances psychology projects. Thus we are limited to pure research, to reporting findings and giving advice to curriculum developers, teachers, authors, etc. There are two sections at work. The science team has been conducting a wide scale survey of Piagetian stages, using class-test versions of Piagetian tasks. They tested ten thousand children, assigned them levels on the basis of the results, and found that most were at the concrete level even at age 16. They are now looking at the science curricula trying to point out at which level, which bit of the curriculum is; e.g. the teaching of light. With this information, the classroom teacher may teach more appropriate science particularly if they are aware of the child's level of ability.

**R.S.** I am more interested in the mathematics section. What is your direction?

**K.H.** The Mathematics team consists of three people, but we have graduate students and a fair number of foreign visitors some of whom, like yourself, stay to help. We likewise are doing a class-test version (of Piagetian tasks). But our main aim is to draw up a hierarchy for the mathematics curriculum. You will recall the work done by the British Nuffield Mathematics Project under Professor Matthews. It is based on a concept hierarchy derived from the work of Piaget and the Geneva School and has already had a significant effect on the primary school mathematics curriculum. However, mathematics in secondary school has experienced relatively little impact from cognitive psychology. One of our aims is to produce assessment instruments which will tell us as accurately as possible which stage a particular pupil has attained, so that curriculum material can be best matched to the needs of individuals. Perhaps we will be able to complete the analysis of cognitive demand levels begun by Piaget and continued by such persons as Collis, Taback, Orton, Fischbein and Bell. The most important aim of all is to help teachers gain insight into the nature of children's learning difficulties in mathematics.

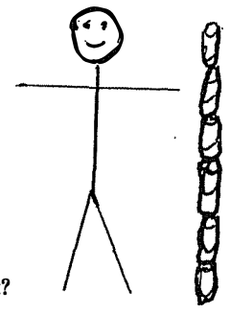
**R.S.** Would you tell us something about your operational methods?

**K.H.** For topics such as algebra, graphs, fractions, measurement, transformations, rate and ratio we have written paper and pencil tests. These are on key concepts and we hope, test understanding. They are nearly all problems as opposed to pure calculation or items to test skills. Some thirty-five interviews are recorded, each lasting up to one hour, with children across the full ability range at all ages in question. This is not only to modify the question but to find out how the children do the questions. We also may find an initial picture of various typical levels of development in that topic.

**R.S.** When you and I interviewed children on rate and ratio, this was stage two. You then have to try out your test instrument in the format of a class test and modify it until a final version is settled on. At this point about 1000 children in each of the age groups under consideration complete each test. What were the surprises with the rate and ratio test?

**K.H.** Not a single child used an algorithm, despite the fact that these are taught in the schools. They had their own methods. Unfortunately these methods fell down when the questions got harder. The incorrect strategies we think are important, because when we are testing a large sample, we find that the children very seldom give us the working they've gone through. Thus we hope to connect an incorrect answer with a particular sort of reasoning.

**R.S.** It is time for a particular example. Describe your findings for the question about Mr. Short and Mr. Tall.



Mr. Short's height is shown measured in paperclips. Mr. Short has a friend Mr. Tall. When we measure their height with matchsticks:

Mr. Short's height is 4 matchsticks.
Mr. Tall's height is 6 matchsticks.

How many paperclips are needed for Mr. Tall's height?

**K.H.** This item is adapted from Karplus (Proportional Reasoning and Control of Variables in 7 Countries, 1975) who identified four levels of understanding. One was intuitive, in which the child just guesses ... "since Mr. Tall is taller, just give him some more paperclips". The second level, not necessarily a progression, but a second common error, was that of "addition". Mr. Tall needs 2 more matchsticks, so he will need 2 more paperclips, 2+6=8. Karplus then had a third level, which he called transitional, but I found this very difficult to define. I looked instead at children who doubled. These children really feel that if you want something larger, you double it and if you want something smaller, you halve it, and they don't move beyond this. And then, of course, there is the child who gets the true ratio.

**R.S.** What did you find about adders from your ratio and proportion paper given to children ages 12 - 15?

**K. H.** We had four questions on the ratio and proportion paper which appeared to attract the addition strategy, thus looking at the difference as opposed to looking at the fraction. The four questions were really very difficult but what was interesting was that they were attracting the addition strategy at the 50% level. 50% of my population of 2000 children were using this addition strategy on these difficult questions, and when I looked at them more closely, there are 30% of the 2000 children who use it consistently during these four questions. It's not "Oh, they do it on one, and not on the other". It is used consistently. Of course you obtain the wrong number. It is an attractive method, because you feel that you have a method. You're not guessing. We have found in a longitudinal study that the child sticks to this method. Surprisingly, they're not the least intellingent children. They are children who can do a lot of the test paper. In the longitudinal survey, of 31 who were adding in '76, 18 of them were still adding in '77, and only 2 were now getting the questions right; the others were getting some right, some wrong, or getting them all wrong in a different way; and this is despite the fact that the teachers had been notified which children were using this strategy. So it's not only **consistent**, it's **persistent** as well.

**R. S.** Did you find any students using the ratio algorithm?

**K. H.** Yes, we went through all 2,257 papers individually to find whether children were using $\frac{x}{a} = \frac{y}{b}$ (which I think is called the rule of three ... You know three of these values and you can find the fourth by cross-multiplication). We found twenty children who used this method.

**R.S.** That is a sophisticated method.

**K. H.** Yes, but it's the one that is in the text books though and of these twenty students, fifteen were in the same school. One hundred children wrote the test in that school and these fifteen happened to be the top group. I suspect that they understood what they were doing (as well as remembering the algorithm), whereas the other eight-five, who were taught the same algorithm

and had the same test, neither remembered it, nor used it.

**R. S.** What information do you return to the teachers?

**K. H.** Every school that we test is in fact a volunteer. We send back results in the form of a statistical analysis so hardest questions can be identified; the sort of error most commonly made can be identified; and even to identify which children are exhibiting this error. We are speaking at teacher centres and at conferences as assessment seems to be much in demand right now. And of course we are writing program reports and articles for the journals.

**R. S.** Once you have found the facility of items, how do you hope to use this information in your project?

**K. H.** Recall that one of our aims is to produce an ordered hierarchy of operations within secondary mathematics. Since we are looking for a hierarchy, every child we test does two test papers, one on algebra and one on ratio, we'll say. For each paper, in our analysis we will be able to group items by the facility levels using various homogeneity coefficients. We also give a mathematical description to this group of items. With these levels we have formed vertical strands, one for ratio, one for algebra. If the levels are reasonable, a child who can do ratio level 5, the hardest group of items, should also be able to do ratio level 2. At the present moment we are finding less than 5% of the children do not fit this progression. What we would then hope to do is to look at the children who did algebra and ratio (we have 1500). Let us label the child as a level one ratio (80% facility) and see what he does in algebra. Perhaps he has not even been able to start the algebra paper.

**R. R.** But eventually you want the hierarchy to stand against Piagetian stages.

**K. H.** That is the final aim, yes. We will hopefully manage to get a large enough sample of children to whom we can assign Piagetian stages from our Piagetian tasks test instrument. We then will see if the majority of pupils at the early concrete stage are the ones who do our level one items, or the majority of late concrete stage pupils are the ones who do our level one items, and early concrete stage pupils can't do any of the items. This will give us a wall if you like on which to line up the tree, but if we can't line up the tree against the wall I think the tree might be quite good on its own.

**R. S.** Kathy, it has been great talking to you. We look forward to seeing you in London England next year. Do send us an article when you get back as we would like to hear more.

**K. H.** Love to Bob, but it must wait until my return from Malaysia where I am running a three week seminar on research in mathematics education.

# SOME THOUGHTS ON MATH ANXIETY

**Peter Hilton**

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***Part 1 of a two-part article***

## Thoughts on Diagnosis

Five years ago the phrases 'math anxiety', 'mathophobia', 'math avoidance' were virtually unknown. Today they are commonplaces of the mathematics education scene. Many concerned people--among them some of the brightest and most innovative of our educators--are devoting themselves almost exclusively to attempts to remedy a situation in which so many of our citizens seek to avoid any contact with mathematical activity. There is the Math Anxiety Clinic at Wesleyan University, founded by Sheila Tobias (who, I believe, invented the term 'math anxiety') and run by her in conjunction with her colleagues Jean Smith and Bonnie Donady. Wellesley College has introduced a special course, 'Discovery in Elementary Math and its Applications', under the direction of Alice Schafer. Knowles Daugherty and Joel Schneider, of CEMREL (St. Louis) have been holding math anxiety workshops in various parts of the United States; the enrollments have

been impressive.

One could extend this list of activities almost indefinitely. Certainly a major thrust of the remedial work has been the mathematical rehabilitation of women, but nobody suggests that the problem is confined to the female sex--it is simply that there are especially adverse features of our social and cultural environment which militate against a woman's success in mathematics. These, though of enormous importance, will not be the subject of this article. We will concern ourselves only with those sources of math anxiety, avoidance, and incompetence which are to be found in the standard curriculum and pedagogy of mathematics itself.

Math avoidance has always existed; it is an urgent problem now because mathematics is so important today to the citizen of an industrialized nation--in his (or her) daily life, in his job opportunities, in his understanding of the world around him. Math avoidance stultifies and impoverishes; mathophobia may be compared with the loss of one of the primary senses.

In some cases this avoidance is the deliberate, conscious and well-worked-out procedure of a mature individual who has coolly assessed his or her capacities; sometimes at the other extreme of the spectrum, mathophobia is a panic reaction. Treatments must therefore vary substantially and the problem is plainly a very complicated one. (Math anxiety, as I have indicated, is not to be identified with math avoidance nor with math incompetence.) The literature thus abounds in reports of theoretical and practical studies of the problem, and in descriptions of its sources and its possible cures. I would have to be far more of an authority on this problem than I am to be able to give a comprehensive citation of that literature.[1]

My limited objective in this article is, first, to attempt to locate math anxiety within the various categories of our school leavers and to explain, on the basis of that analysis, why the problem is so widespread and why, in my view, it is liable to get worse before it gets better. I will also have something to say about the dynamic of the spread of this anxiety. However, my most serious purposes are to examine certain cardinal features of the present-day mathematical curriculum and pedagogy and to identify those features which are most responsible for math avoidance; and then to describe those parts of mathematics which should be emphasized in a sound programme. For I wish my thesis to be absolutely clear. Math avoidance is no pathology; the experiences which most children have of mathematics through their mathematics lessons and through the mathematics texts which they use are, in my judgment, very likely to lead them to seek to avoid mathematics. Moreover, the spread of awareness of the importance of mathematics will naturally lead to anxiety among those who realize that they have virtually no understanding of it. We are dealing here with a mass infection caused by the virus of bad mathematics teaching.

Having identified certain infected parts of the traditional curriculum, I would propose their replacement by healthy material. I intend my remarks to be taken as a contribution to a discussion, and not as definitive. For I wish to emphasize that I do not regard myself as expert in the problem. Nevertheless, I believe that it is encumbent on those of us who enjoy mathematics and appreciate its importance to speak up about the contamination of the educational process from currupt sources.

I must insert here another important disclaimer. I identify bad teaching, bad texts and bad educational instruments (e. g., the standardized test) as among the principal causes of math incompetence and math avoidance. The reader is not to infer that I regard *all* teaching as bad, *all* texts as bad, *all* instruments as deleterious to education. Most especially, I know very well that there are many wonderful teachers, at the elementary and secondary level. However, even where the student is fortunate to have a fine teacher, he (or she) may well be handicapped, along with the teacher, by a dull, misguided text. And even where teacher and text are both conducive to good learning, the exigencies of the educational system may still induce so drastic a distortion of the natural rhythm of mathematical instruction that the student suffers severely -- and mathophobia results.

## 1. The Location and Spread of Anxiety

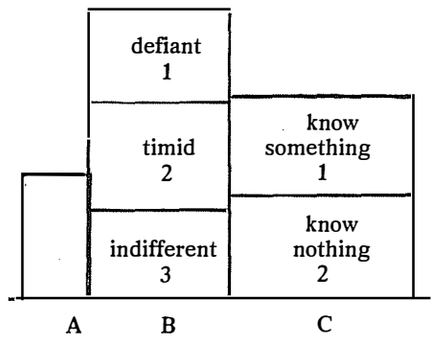



**Figure 1**

In Figure 1, I attempt to describe the way in which a mathematical education separates those who have experienced it into certain clearly-defined classes. In this nomogram I distinguish three primary classes: Class A consists of those who as a result of their mathematics education know some mathematics; Class B consists of those who know that they don't know any mathematics; and Class C consists of those who don't know they don't know any mathematics. I am frankly guessing as to the sizes of these three classes, but I have no doubt that Class A is, by a very considerable margin, the smallest.

I further subdivide Class B into three subsclasses. Class B1 consists of those who react to their awareness of their ignorance of mathematics with defiance. They challenge the central role of mathematics in education. They maintain that they are well educated despite their ignorance. It is very likely that their defiance is reinforced by the fact that in some area far removed from mathematics their education has been extremely successful. We are likely to find among them many of the cultural leaders of our community. They avoid mathematics but have no consequent anxiety; generally they will not seek to overcome their ignorance. Class B2 consists of those rendered timid by their awareness of their mathematical ignorance. It is in Class B2, then, that we typically find those who are both anxious about mathematics and seek to avoid it. These people are unable to function in any situation in which some understanding of mathematics and ability to use it are required, and feel disquiet as a consequence. On the other hand, they are likely to be sufficiently aware of the importance of knowing some mathematics that they will seek help. Their willingness to do so depends largely on whether their timidity is a general characteristic of their personality or is specifically expressed in their mathophobia. Class B3 consists of those who are indifferent to the fact that they know no mathematics. They are aware but they do not care. Traditionally, women expecting to pass the most significant years of their lives in a matriarchal role in the family would enter this class. With the very welcome societal changes giving women the formal, legal opportunity, if not always the means to choose lifestyles from a far broader spectrum of possibilities, we find women far more prone to membership of Class B2. We also find in Class B3 those who have no conception whatsoever of the value of education. People who view their entire educational experience, perhaps even going as far as higher education, as nothing but a means to the end of job security may very well be indifferent to their ignorance of mathematics if they find themselves in a job congenial[2] to them despite having learnt no mathematics.

I subdivide Class C into two subclasses. Class C1 consists of people who, although they don't know that they don't have mathematics, do know something. Class C2 consists of those who know nothing at all. Here I use the word 'know' in a very stringent sense. To know something is to have gone through the discipline of learning and understanding, acquiring knowledge and being able to use that knowledge. If a person has been through that discipline in connection with some subject, then he or she is liable to become aware of his ignorance of mathematics. Such a person is therefore prone, through this awareness, to leave Class C1 and enter Class B2, the class of the timid and frightened. Class C1 is thus unstable. Class C2, on the other hand, is highly stable. The members of this class tend to be entirely insulated from awareness of their ignorance. They are likely to be confident people and, in adult life, they are apt to be found in many positions of authority. Protected from an awareness of the abyss of ignorance in which they pass their lives, they will not hesitate to take decisions affecting the lives of other people as well as themselves.

Their ranks may also be swollen from time to time by the capture of erstwhile members of Class B3, the indifferent ones. These people, insensitive to the consequences of their ignorance of mathematics, may lose their capacity to apprehend and recognize the existence of this state of ignorance and share the confident attitude of the genuine 'know-nothings'.

In this admittedly somewhat superficial analysis of the categories into which people fall as a result of their mathematical education, I have attempted to adopt a dynamic rather than a static approach. To reinforce this dynamic approach, I would like to devote a little attention to the question of how anxiety is transmitted in the course of the educational process. Here again, I ask the reader not to suppose that my analysis is profound, nor to suppose that I really can attach quantities to the rates of flow between the categories I am discussing. I wish to consider teachers (T), students (S), and citizens (C). In connection with each of these groups, I wish to consider the attributes, math anxiety (A) and lack of mastery of mathematics (L). Thus the notation TL refers to the set of teachers of mathematics lacking mastery in mathematics. I find then that there is a flow of attributes A and L in the following directions. There are flows from SL to CL and SA to CA representing the passage of time. The flow from SA to TA represents the teacher's unhappy awareness of his or her failure to teach effectively. The flow from CA to SA represents the influence of parents on children. What this flow pattern omits, probably among other features, is the very important fact that anxiety reinforces itself. In this respect the spread of anxiety is to be contrasted sharply with, for example, the spread of rumour. For where two people, both of them anxious about mathematics, meet and communicate their anxiety, they will almost certainly stabilize and reinforce each other's anxiety. Where two people, each anxious to communicate the same rumour, meet, the awareness that the rumour is already shared will lead to a catastrophic loss of interest in the rumour itself. Thus rumours die out, while anxiety spreads rapidly. The attempts that so many people (let me instance as a fine example the Math Anxiety Clinic at Wesleyan University) are making to deal with this problem are, in my judgment, very much handicapped by countervailing natural forces. This of course makes their effort the more, not the less, important and urgent.

## 2. The Sources of Anxiety

So much for airy generalizations. Let me now adopt a much more serious and much more responsible attitude towards the problem and indicate by examples what are, in my judgment, those defects of mathematics education which are most conducive to the spread of anxiety. The rationale for my thinking is the following. The learning and the doing of mathematics should fit into the natural intellectual rhythms of the child. That is to say, the mathematics should be clearly seen to be relevant to the life of the child. It should be motivated as responding to his natural curiosity and desire to understand and master the world around him. Of course there is no conflict here with an appeal to the child's natural delight in fantasy, so long as the fantasy is honestly presented as such, and attractively presented to the child. Moreover a desire to know should not be severed from a desire to understand. This means that the child should not have to depend upon memory as a substitute for understanding. Mathematics should not be boring since, if it is, it is an unnatural activity for the child. By a similar argument, the teaching of mathematics should not be dominated by the frequent incidence of standardized tests--we will certainly have more to say about this later. For, if the education is so dominated, then both student and teacher will inevitably conclude that success in the tests not merely provides the criterion, but is actually the objective of the educational process. Everything in mathematics education must be done honestly, and there must be no spurious sales talk. Ideas must be presented at a rate natural to the student; here we should include, with ideas, notational innovations which may very well tax the child quite as much as a new concept.

I am acutely aware, as I enunciate the above principles, that they are banal. It is difficult to imagine any serious person adopting a point of view contrary to any of those stated above. Yet it is not difficult to specify familiar features of the curriculum which demonstrate a voilation of those self-evident principles. Here are some important examples of those features.

(a) *Rote calculations.* These ugly excrescences on the mathematical body are familiar to everybody. Dreary long multiplications and long divisions disfigure every curriculum. The child is drilled in these calculations in the hope that he will be able to execute them accurately. No justification is given beyond the gratuitous and utterly misleading assignment to them of the epithet 'basic'. Parents and legislators, distressed at the inability of children to execute the basic algorithms, blame the new math and recommend a return to the old. They bewail a lost skill. It is

perfectly true that, with the advent of the hand calculator, many children and adults will lose the skill to execute these useless computations. So they should. In the 19th century, women and children were employed as coal miners (at least, in Great Britain). Today, they are not. Women and children have lost a skill. Who will deplore it? Some argue that it would be a mistake to become excessively dependent on the hand calculator. What, they ask, if the hand calculator is not available? What if the battery is dead? My response is, get a new battery. I have yet, in thirty years in the profession, to meet a situation in which a long multiplication was so urgent that one could not postpone it for the time it would take to purchase a new battery or borrow a functioning calculator.

Let me make it plain that I certainly advocate the mastery of what are called multiplication facts; and I certainly do not advocate slavish dependence on the hand calculator. Optimal use of the hand calculator requires in the user the ability to estimate the answers to fairly complicated arithmetical problems. This is a basic human skill. The calculator renders it even more important. On the other hand there are many situations in which an estimate of the answer suffices. In these situations the hand calculator may well be superfluous. The ability to recognize such situations is one which our education must foster. Rote calculations have nothing to do with this important human skill.

(b) *Memory dependence.* The child learns early in his traditional exposure to so-called mathematics that he must remember a great deal. He quickly concludes that memory is the only important mental device to employ in doing mathematics. There thus begins an inexorable process in which the reasoning faculty lapses into disuse and the memory becomes overloaded. The effects of this abuse of the system are sometimes slow to reveal themselves, but are very commonly there already by the junior high school level. It is a supreme irony that the measure of progress in any science is its independence of the memorization of brute fact, yet such memorization is endemic in current education.

It is deploarable that intelligent people do not understand that 'learning mathematics' means 'learning to do mathematics' and not 'learning (skills, facts, proofs) by heart'. Harry Schwarz, reviewing *Asimov on Numbers* in the New York Times Book Review of April 24, 1977, writes, "As conventionally taught, especially since the introduction of the intellectual disaster and pedagogical catastrophe known as the 'new math', math has consistently scared off and soured millions of intelligent youngsters, many of whom are now adults suffering both math phobia and math ignorance ... But in all fairness one must remember that the teacher is responsible for inculcating a certain amount of technical knowledge--to have his students memorize multiplication tables or to have them learn the proof of the Pythagorean theorem ...". Here we see a genuine and worthy concern that mathematics education should be made exciting and significant for the child, embedded in a farrago of prejudice and misunderstanding of the nature of mathematical education. There is no evidence that I know of that the new math has increased the rate at which students are scared off mathematics. That the new math can be criticized is beyond dispute. But that it makes mathematics less attractive to the student is not one of the known charges against it.[3] However my principal point here is to take issue with the end of the quotation above. By singling out as characteristic responsibilities of the teacher the requirement to have his students memorize multiplication tables and the proof of Pythagoras' theorem is to show a total lack of understanding of what mathematics is. There is absolutely no point in learning the proof of Pythagoras' theorem if one does not understand the proof; and if one understands the proof, there is practically no point in learning it. The reason why students are asked to learn the proof is clear to everybody--it is because they can then be tested. It is as if we were to measure the moral worth of a man by his ability to recite the Ten Commandments.

(c) *Unmotivated problems.* These abound in the mathematical education of the child. Let me give one typical example. Frequently the child is called upon to execute complicated computations with fractions. There is no indication of any situation in which such a skill would be necessary or even desirable. Moreover there is an intrinsic defect in such skills which damns them. For the child should always be able to check the reasonableness of his answer. For example, an addition involving decimals, 25.37 + 106.86, lends itself to the immediate check that the answer should be more than 120 and less than 140. On the other hand, the apparently equally simple problem 1/7 + 2/5 lends itself to no such check. The child could have no feel for the answer on the basis of an order-of-magnitude estimate. Indeed, even if the child got the answer right, he would not invoke at any stage of the calculation any awareness of approximately how big either the summands or the answer are.

(d) *Spurious applications.* This typical defect of a traditional education is closely related to that above. For, aware that it is today necessary to justify the presence in the curriculum of some mathematical skill on the basis of its utility, and unwilling for a variety of reasons to discard, for example, the addition of fractions, many curricula motivate this fairly useless technique through phony applications. I have met the following problem, "John swims 22 3/5 metres on Monday and 23 4/7 metres on Tuesday. How far has he swum on the two days?" This is intended to motivate the addition of fractions; in any intelligent child it should only motivate the question--how must a swimming bath be calibrated in order that John could acquire such remarkable information?

I would like to give another example of this very ubiquitous source of discomfort with mathematics. In another much used elementary text we find the following question, "Johnny goes to the fair. His mother gives him $1.50 and he spends $0.89 at the fair. How much does he have left?" We are intended to see here an application of the subtraction algorithm. In fact it is entirely phony. Let us imagine the situation. Johnny returns in the late afternoon from the fair and his mother asks him, "Did you enjoy yourself at the fair, Johnny?" "Yes, mom, it was great." "How much money have you got left, Johnny?" "Well, mom, how much did you give me?" "I gave you $1.50." "Well, mom, I spent $0.89 at the fair, so just let me have a piece of paper and a pencil, and I will work out for you how much I have left." How absurd! Plainly, this question "How much have you left?" is not naturally conducive to a subtraction model. Johnny will answer the question by putting his hands in his pockets, taking out his money and counting it. He almost certainly would not know how much he spent at the fair, since he will have spent money on several items. Thus the interesting question that his mother might pose would be, "How much did you spend at the fair, Johnny?" And Johnny would reply to this question by asking how much his mother had given him, and then subtracting the amount of money he finds he has left by putting his hands in his pockets. The reason for preferring the question in the text to the question here proposed is clear. The word 'left' is the signal to the child to execute a subtraction. Thus the designers of the text can spuriously compliment themselves, and the teacher, and the child, on being able to recognize subtraction situations. A genuine ability to recognize subtraction situations would be represented by the successful attack on the question "How much did you spend at the fair?" For this question contains no key word indicating that a subtraction is involved. Indeed it is barely possible that Johnny might do the problem by an addition if he could remember all the items he sampled at the fair and what each individual item cost.

(e) *Authoritarianism.* Typically in a child's traditional mathematics education, he (or she) is told what to do, told that it will turn out to be useful, told that it works, told that he will understand it later. He learns early on the advantage to himself of accepting argument by authority. However, this experience runs counter to so much else that he is simultaneously learning, and creates conflict. In other parts of his education and his experience, he is learning to take responsibility, to understand the significance of what he sees going on around him, to think, to reason.

It is a superb irony that it is in mathematics education, so-called, that he learns to discount the value of his reasoning process. It is natural that this authoritarianism breeds anxiety and distaste. When, later on, that anxiety is reinforced by the increasing frequency of memory failure, one can say that mathophobia has been firmly established. Let me give one example of authoritarianism, again taken from a very recent and commercially successful elementary text. We are given a 3 x 3 table (see Figure 2).

| | | |
|---|---|---|
| 37 | 14 | (23) |
| 21 | 6 | (15) |
| (16) | (8) | (8) |

**Figure 2**

The child is asked to carry out two horizontal subtractions obtaining 23 and 15, and then to carry out the vertical subtraction obtaining 8. He is then invited to carry out the two vertical subtractions, obtaining 16 and 8, and then carry out the horizontal subtraction, again obtaining 8 in the bottom right-hand corner. The Teachers' Guide instructs the teacher to "*explain* to the children that they get the same answer either way (my italics)." I would have thought that the

word 'explain' should be followed by 'why' rather than 'that'. Here, again, we have a deliberate deception. The word 'explain' is very popular today in educational circles. Everything must be explained; the child must be treated with respect. So the teacher is instructed to 'explain that ...'. The more accurate term is 'tell the children that ...'. But this smacks too much of authority. So the child does problems of this kind and does not understand at all why they work. I take leave to doubt whether many children ever will.

(f) *Tests.* If I were to be asked to single out the one most baleful influence in the whole of education today, I would unhesitatingly cite the appalling prevalence of tests--mechanical, standardized evaluations of student progress. They pervade and currupt the entire educational process; 4 but nowhere are they more pernicious than in mathematics. They are based on a set of principles for none of which I can find sympathy or justification. They assume that mathematics can be divided into tiny water-tight compartments so that one can test individual items of mathematics education and mastery of mathematics is just a set of masteries of such items. They presume that competency in mathematics can be measured--in this way they have reduced the word 'competency' to jargon. They presume that where measurement is possible, this measurement can be achieved significantly by a single score; they thus overlook the many-faceted nature of genuine mathematical learning, and perpetuate the atrocious educational fallacy that students are linearly ordered with respect to ability. They presume that the evidence for the success of an educational experience can be obtained immediately on the completion of that experience, thus putting a premium on rote efficiency and memory. Tests are administered under conditions which violate almost every principle for effectively doing mathematics. Tests generate anxiety and establish a stable association between education and anxiety.

Tests corrupt the curriculum. Inevitably teachers teach with a view to success in the tests--their own reputations, let alone the 'success of the child' depend on the outcome of these tests. The child is aware of a test looming up and will consequently concentrate effort, with the teacher's encouragement, on those parts of what he is learning on which he is likely to be tested. I speak of the child; but all of us teaching at the college and university level know that the problem persists far beyond childhood. Often, when I have been describing a piece of mathematics and have called for questions in order that I might be able to find out whether students are having difficulties, the only question I have received has been "Will we get it on the test?" The more successful students learn to adapt themselves to the system, and the supreme irony is that many of these students are then rendered anxious when the teacher, rebelling against the slavish acceptance of the tyranny of the tests, tries genuinely to teach them. The student then finds himself adrift in a totally unfamiliar situation--he is learning in order to understand and to do, instead of learning in order to reproduce a facsimile of what he has learned in the forthcoming standardized test. It may be argued that the tests are not so serious in their consequences in the early grades. I will leave that fine point of discussion to those with more experience than myself. I believe it to be true that, as the student advances up the educational ladder and mathematics becomes more highly differentiated, more articulated, so it becomes increasingly ridiculous to suppose that one can test success in the way in which it is now done.

A specially ridiculous type of test is the multiple choice question. I do not suggest that the multiple choice question plays any particular role in generating math anxiety or math avoidance. But I cannot resist this opportunity to declare that, of all types of test question, the multiple choice question is the most absurd. For I take it that a justification for a test item must be that it asks the sort of question which the student is likely to meet when he is doing mathematics. In 35 years of doing mathematics as a professional, I have never yet met a situation in which I knew that the answer was one of four possibilities and I simply had to make the correct choice. Moreover, were I ever placed in such a situation artificially, it is highly unlikely that I would use as a method of solving the problem the method appropriate to a situation in which I had not been told that the correct answer figured in a set of four potential solutions.

I have given about six especially pernicious features of a traditional curriculum. I do not wish to weary the reader by prolonging the list, but I do want to say that the list is by no means complete. Nor, indeed, are the examples of these pedagogical errors confined to the elementary or even the pre-college level. I would draw your attention to the gratuitous introduction of unfamiliar symbolism and to the simultaneous introduction of many different new concepts as examples of bad pedagogy that one finds frequently at the undergraduate or even the graduate level. Let this list, however, suffice to establish my thesis that math avoidance is a natural reaction to so much of the experience of the child in situations euphemistically described as the learning of mathematics.

1 The best treatment of the problem, and source of available literature, will undoubtedly be the forthcoming book by Sheila Tobias, **Math Anxiety: What it is, why you have it and what you can do about it**, to be published by W. W. Norton, New York.

2 The type of job an individual finds congenial is an indicator of whether he (or she) belongs to class B1 or class B3.

3 Such evidence as there is seems to point the other way. Efficiency in the 'old math' was very often accompanied by distaste; but this is rarely so with the new.

4 A vigorous, and invigorating, attack on the use of certain kinds of test in educational assessment is to be found in **Postscript to 'Standards and Criteria'** by Gene V. Glass, published by the Center for the Study of Evaluation, UCLA (January, 1978).

# BIORYTHMES ET TRIGONOMETRIE PHYSIQUE EMOTIF INTELLECTUEL

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La théorie des biorythmes est une théorie très en vogue, de nos jours, et nous, en tant que professeurs de mathématiques, devrions en profiter davantage. Cet exposé a comme but d'expliquer ce que la théorie propose, et de démontrer comment on peut s'en servir dans la salle de classe, en trigonométrie.

Le mot biorythmes vient de deux mots grecs: BIOS - vie, et RYTHMOS - un mouvement régulier (mesuré). La théorie des biorythmes suggère que notre comportement est déterminé par trois cycles: les cycles physique, émotif et intellectuel. Ces trois cycles débutent à la naissance, et se répètent constamment durant toute notre vie.

Le cycle physique se répète à chaque 23 jours. Durant cette période, on voit des variations dans le montant d'énergie, d'aggressivité, de confiance, de force, d'endurance, et de résistance aux maladies. Le cycle émotif dure 28 jours. Cette période voit des changements dans les aspects émotifs de notre comportement; entre autres, on discerne: la sensibilité, l'humeur, le tempérament artistique, l'intuition et la créativité. Notre cycle intellectuel a une période de 33 jours. Durant cette période, nos capacités de concentration, de logique, de mémoire, de raisonnement et de résolution de problèmes sont affectés.

Du point de vue graphique, chacun de ces cycles ressemble à un sinus, puisque les stages successifs dans chaque période sont toujours: neutre, maximum, neutre, minimum, neutre. Les périodes entre les points extrêmes et les points d'inflexion ne sont pas quantifiables; on ne peut que les qualifier comme étant positives ou négatives. Par définition, les jours "critiques" sont ces jours-là où le graphe traverse l'axe horizontal: c'est alors qu'il faut exercer un peu plus de précaution, puisque notre comportement est en période de transition.

On dit souvent que les biorythmes "prédisposent" au lieu de "prédéterminer". Mais, malgré tous les faits à l'appui, cette théorie est encore au stage de la recherche, et le temps nous dira si elle a sa raison d'être. Cependant, rien ne nous empêche d'utiliser cette "théorie" dans la salle de classe, puisqu'elle fournit une application intéressante des courbes trigonométriques, en donnant une approche personnalisée à une matière parfois trop aride. De plus, l'élève a intérêt à voir des fonctions trigonométriques que dépendent de nombres entiers (des jours), au lieu de multiples de $\pi$.

## COMMENT CALCULER VOS VALEURS BIORYTHMIQUES ET TRACER VOS COURBES BIORYTHMIQUES POUR CE MOIS

A. Calculez le nombre de jours que vous avez vécus jusqu'au premier jour (inclusivement) de ce mois:

1. Multipliez votre âge (en années entières) par 365.25

   a = 365.25 _____ = _____ (arrondir à une journée près)

2. Additionnez le nombre de jours écoulés depuis votre dernier anniversaire de naissance jusqu'au premier jour de ce mois, inclusivement (N.B. votre anniversaire compte pour 1)

   b = _____

3. Le nombre de jours que vous avez vécus est

   j = a + b = _____

B. Divisez j, le nombre de journées vécues, par 23, 28 et 33, et gardez les restes (servez-vous de la division longue, au lieu d'une calculatrice).

1. $\frac{j}{23}$ = , reste (PHYSIQUE)

2. $\frac{j}{28}$ = , reste (EMOTIF)

3. $\frac{j}{33}$ = , reste (INTELLECTUEL)

Chacun de ces restes représente le jour dans le cycle, lors du premier jour du mois.

C. Servez-vous de papier soie et des graphes fournis à la page suivante pour tracer vos trois courbes biorythmiques pour le mois.

1. Tracez un système d'axes comme suit (ajoutez les jours, s'il y a plus de 28 jours dans le mois):



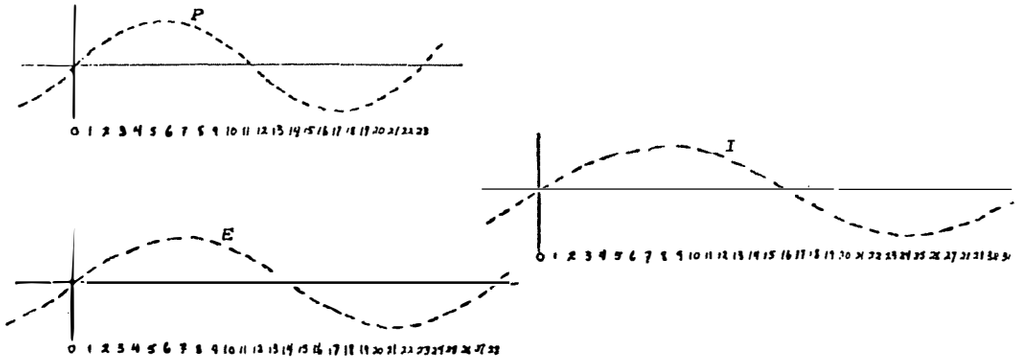
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28

2. Pour tracer chaque courbe, alignez l'axe vertical du mois avec "le premier jour dans le cycle" (les restes de la partie B).
3. Tracez les trois courbes sur le même système d'axes.

D. Quelles sont vos journées "critiques" ce mois-ci? Le tableau ci-dessous donne une interprétation pour les journées "non-critiques". Consultez votre graphe et le tableau: y a-t-il concordance?

| Physique | Emotif | Intellectuel | Interprétation |
|---|---|---|---|
| + | + | + | Meilleure période possible pour planifier et accomplir tous ses buts. |
| + | + | - | Bon temps pour accomplir des taches difficiles et cultiver des amitiés. |
| + | - | - | Bonne habileté physique; plutôt irritable; raisonnement peu vague. |
| - | - | - | Temps de repos, d'activités routinières, qui ne demandent pas trop d'effort. |
| - | - | + | Bon temps pour planifier; évitez la négotiation. |
| - | + | + | C'est le temps d'être créateur et de résoudre des problèmes. |
| + | - | + | Attention aux conflits de personnalité; vaut mieux travailler avec ses mains et garder ses pensées pour soi-même. |
| - | + | - | Il faut éviter de trop s'exercer physiquement ou mentalement; sur le plan émotif, c'est un temps favorable. |

LES COURBES DE BASE




Problèmes

1. Soit x le nombre de jours vécus, p(x) le cycle physique, e(x) le cycle émotif et i(x) le cycle intellectuel.
   (a) Trouvez une expression en fonction de sinus pour p(x), e(x) et i(x). (Posez l'amptitude = 1)
   (b) Exprimez chacune des fonctions p(x), e(x) et i(x) en fonction de cosinus.
2. (a) En vous servant de l'axe des y comme premier jour du mois, trouvez une équation pour chacune de vos courbes biorythmiques pour ce mois-ci.
   (b) Utilisez les équations de la partie (a) pour déterminer vos journées

"critiques" pour ce mois. Vérifiez vos réponses avec votre graphe.

3. Les fonctions biorythmiques de M. X pour le mois sont:

$$p(x) = \sin\left\{\frac{2\pi}{23}(x - 15.5)\right\}$$

$$e(x) = \sin\left\{\frac{\pi}{14}(x - 1)\right\}$$

$$i(x) = \sin\left\{\frac{2\pi}{33}(x + 12.5)\right\}$$

   (a) Combien de "jours dans le cycle" sont écoulés pour chaque fonction, au début du mois?

   (b) Faites le graphe des trois fonctions, et déterminez les journées critiques pour M. X.

   (c) Servez-vous de trigonométrie pour vérifier vos réponses pour (b).

4. Pas besoin de trigo, ici! A quel âge les trois cycles recommencent-ils tous ensemble, comme à votre naissance?

5. Complétez le tableau suivant, en vous servant de vos connaissances de biorythmes, trigonométrie et calcul différentiel (p représente la fonction physique):

| x<br>date du mois | p | p' | p" | Observations |
|---|---|---|---|---|
| 3.75 | -1 | 0 | | vous êtes fatigué |
| 5 | | + | | les choses s'améliorent |
| 9.5 | | | | journée "critique" |
| 12 | + | | | bon temps pour amorcer un projet |
| | | 0 | | c'est le temps d'accomplir des taches difficiles |
| 17 | | - | | relâchez, un peu |
| | | | | journée "critique" |
| 23 | | | + | vous sentez venir la fatigue |
| | | | | l'épuisement, encore |

Ressources


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# A SHORT, SIMPLE PROOF OF EULER'S FORMULA

**Elizabeth Ruppel**

$e^{ix} \doteq \cos x + i \sin x$

Two textbooks are used in Ontario for the teaching of Grade 13 Calculus: Calculus by Del Grande, et al. (Gage), and Calculus, Complex Numbers and Polar Coordinates by Elliot, et al. (Holt, Rinehart and Winston). The first textbook introduces the expression $e^{ix}$ in a chapter on Taylor and MacLaurin series and, after developing the separate power series for sin x, cos x and $e^{ix}$, among other things, eventually shows that $e^{ix} = \cos x + i \sin x$. Mr. Elliot, et al., on the other hand, either thought the Grade 13 Calculus course long enough without an extensive treatment of power series or perhaps that the topic was too difficult for that level. In any case, since the ministry guidelines do not mention power series, neither does this text. Unfortunately, the authors decided to exclude Euler's formula completely. Instead, they introduced their own expression cis for cos x + i sin x. Unable to decide whether to pronounce it kis or sis (after all, sis may imply sin x + i sin x to the confused student), I luckily came upon a simple proof for Euler's formula which took about 10 minutes in class so that even the weakest student had no trouble understanding. Since the formula is frequently used in higher levels of mathematics, it is certainly to the advantage of the student to gain an early familiarity with it. Also, Euler's formula facilitates the proof of De Moivre's Theorem--something that is included in the ministry guidelines. The only background information needed is a very elementary knowledge of differentiation including thetranscendental functions which is usually covered long before the topic of complex numbers is approached in Grade 13 calculus. This short, simple proof is a useful addition to the topic of complex numbers in the course.

Proof: Consider $y = e^{-ix} (\cos x + i \sin x)$

$y' = -ie^{-ix} (\cos x + i \sin x) + e^{-ix} (-\sin x + i \cos x)$

$= -e^{-ix} (-\sin x + i \cos x) + e^{-ix} (-\sin x + i \cos x) = 0.$

Since the derivative of $e^{-ix} (\cos x + i \sin x) = 0$, then $e^{-ix} (\cos x + i \sin x) = k$, $k \in R$

What is the value of k? Since this is an identity in x,

let x = 0. $\therefore e^{0} (\cos 0 + i \sin 0) = 1 = k$

$\therefore e^{-ix} (\cos x + i \sin x) = 1$

$\therefore e^{ix} = \cos x + i \sin x.$ Q.E.D.

# BRAIN TEASERS

1. Each time the pump handle is pushed down a quart and a pint of water flows from the spout. How many strokes will be necessary to fill a 4½ gallon tub if every fifth stroke fails to produce any water?
2. Two trains each going 60 miles an hour leave Toronto and Hamilton, 40 miles apart, at the same time and travel toward each other. A bird flying ninety miles an hour flies back and forth between the engines. If the bird wastes no time in turning how far will it have flown when the trains meet?

Answers on Pg. 59

# SERIES EXPANSIONS FOR INTEGRALS

**RESEARCH PAPERS BY GRADE 13 STUDENTS**

***with an Introduction by Gordon Nicholls, Northview Heights, Willowdale***

The inclusion of a Calculus course at the Grade 13 level and flexibility in Curriculum Design has given every teacher of senior level Mathematics in the province an opportunity to extend the mathematical experience available to his students in a variety of ways. Over the past few years I have been trying the idea of a major research assignment in Integration. The purposes of the assignment were as follows: (1) To test the knowledge of basic techniques taught in the course; (2) to give students a chance to work without the time pressure of a test; (3) to give the opportunity for students to write a major paper in mathematics (anywhere from 15 to 50 pages) that was properly annotated and carefully written; and (4) to encourage students, especially brighter students, to go beyond the depth offered in the regular course and learn some mathematics on their own.

This year I received some exceptionally fine work on a Research Topic that was titled: "An Investigation of the Use of Maclaurin's Series to solve more difficult Integrals". Some hints were given as to the type of problem that could not be solved by the standard techniques of Substitution, Partial Fractions or Integration by Parts. After that the students had to begin researching for further ideas and approaches.

The two articles were written by students who exhibited both above average (but not exceptional) Mathematical maturity and a capacity for very hard work.

The calculation of an elliptical integral is not discussed very thoroughly in most texts. Courant and Count give it the most thorough attention of any books available in our library, but even then it is far less intensive than the treatment given below. The important functions used in university level statistics often seem to appear with a short note that says use the tables for evaluation. A good question for a Grade 13 student is "How do you get the values in the tables?" The work below sets forth the approach for two of the standard Statistical Functions.

At a time when a lot of negative comments are being made about the capacities of Grade 13 graduates, I hope that the work of Owen Pellew, Alexander Shafarenko and Daljit Samagh will provide some concrete evidence for optimism if individual teacher initiative at the local level is encouraged, and if the freedom to modify curriculum is left at the school level.

**Editors Note:** All of us might well consider this kind of assessment. It is not often used. It is indeed rewarding to see what students can do on their own, when challenged.

# ELLIPTIC INTEGRALS

**Owen Pellew, Alexander Shafarenko, Northview Heights Secondary School**

In our grade 13 Calculus course this year we were assigned a project in integral calculus.

One of the problems which we had to deal with in calculus was finding the length of arc of a curve.

We found this problem to be rather deceptive - the theory was easy, but many of the specific cases could not be found.

We wish to show here one of those deceptively "easy" cases - how to find the perimeter of the ellipse.

$$b^2x^2 + a^2y^2 = a^2b^2 \qquad 1$$

Originally this problem was approached by Kepler when he was calculating the length of the orbit of planets around the sun. Our approach started by using the general formula for arc length:

$$A.L. = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \qquad 2$$

To get the total length of the perimeter of an ellipse with the general formula we will multiply the arc length of the ellipse in the first quadrant by four.

Solving this problem by substituting for f'(x) led to a function that we could not integrate by standard methods. Taking the derivatives with respect to x of 1 gives

$$dy/dx = -b^2x/a^2y \qquad 3$$

$\therefore$ the perimeter of the ellipse is

$$4\int_0^a \sqrt{1+\frac{b^4x^2}{a^4y^2}}\ dx$$

But

$$a^2y^2 = a^2b^2 - b^2x^2 \qquad 4$$

$\therefore$ the perimeter is

$$4\int_0^a \sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}\ dx$$

Maclaurin's series would give an expansion, but this would involve a lot of work. Expanding by the extended binomial theorem will give an expansion quicker. However, nothing but the first two terms of this expansion can be integrated, the rest get too complicated.

Thus we used a different approach using the substitutions

$x = a\cos\theta$ which relates the abscissa of any point on the ellipse to a point on the circle having the same radius as a semi-major axis of ellipse, and 5

$y = b\sin\theta$ which relates the ordinate of any point on the ellipse to a point on the circle having the same radius as a semi-minor axis of the ellipse. (6)

Now taking the differentials of (5) and (6) we obtain

$$\left.\begin{aligned} dx &= -a\sin\theta\, d\theta \\ dy &= b\cos\theta\, d\theta \end{aligned}\right\} \text{ and } \quad dy/dx = f'(x) = dy/d\theta \times d\theta/dx = -b\cos\theta/a\sin\theta \qquad (7)$$

When $x = 0$, $\theta = \pi/2$ and when $x = a$, $\theta = 0$. Making the substitutions we get the following:

$$\text{Perimeter} = 4\int_{\frac{\pi}{2}}^{0} \sqrt{1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}}\ (-a\sin\theta\, d\theta)$$

$$= -4\int_{\frac{\pi}{2}}^{0} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\ d\theta \qquad (8)$$

(Using the substitution $\sin^2\theta = 1 - \cos^2\theta$)

$$= -4\int_{\frac{\pi}{2}}^{0} \sqrt{a^2 - (a^2-b^2)\cos^2\theta}\ d\theta \qquad (9)$$

Now for an ellipse $\quad \therefore b^2 = a^2(1-e^2)$

$$\text{Perimeter} = -4a\int_{\frac{\pi}{2}}^{0} (1-e^2\cos^2\theta)^{\frac{1}{2}}\, d\theta \qquad (10)$$

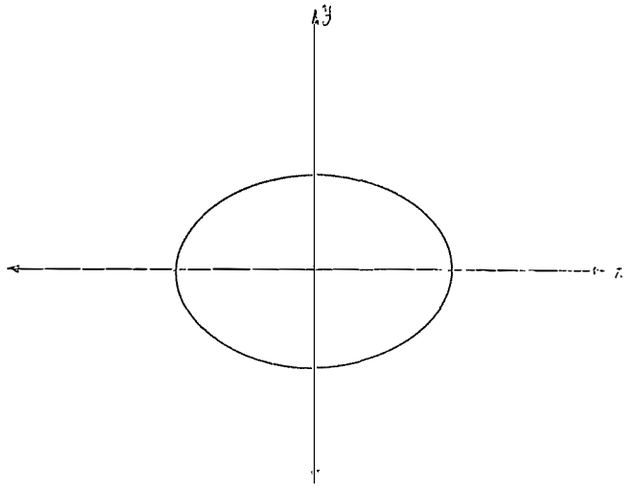
Now expanding this by the binomial theorem we get

$$\text{Perimeter} = 4a\int_0^{\frac{\pi}{2}} [1 - \tfrac{1}{2}e^2 \cos^2\theta - \tfrac{1}{8}e^4 \cos^4\theta - \tfrac{1}{16}e^6 \cos^6\theta - \tfrac{5}{128}e^8 \cos^8\theta - \cdots]\, d\theta.$$

Integrating this using substitutions for $\cos^2\theta$, $\cos^4\theta$, etc. and simplifying the general equation becomes

$$\text{Perimeter} = 2a\pi(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - \tfrac{5}{256}e^6 - \tfrac{175}{10384}e^8 - \cdots)\ . \qquad (11)$$

For the ellipse $16x^2 + 25y^2 = 400$, the Perimeter by the formula (11) equals $4 \times (7.0908) = 28.3632$, which is close to the value obtained from tables.



# MATHEMATICS AND MUSIC
# SYMMETRY TRANSFORMATIONS

**Ed Barbeau, University College, University of Toronto**

1. Because the composition of music involves the use of symmetry, it is natural that we have the same sort of transformations that occur in mathematics, and which are the subject matter of group theory. A theme or succession of notes may be repeated in a number of transformed ways:

(a) it may be inverted or repeated upside down;
(b) it may be retrograded or run backwards;
(c) it may be repeated at a different pitch, or translated tonally;
(d) it may be augmented or diminished, i.e. slowed down or speeded up;
(e) in part work, it may be translated in time, as in a fugue, round, or canon.

If we look at tonal translation, we may consider it to be chromatic or diatonic. In the chromatic case, if the translation is repeatedly made up or down an interval, we will arrive back at the same pitch after 12 translations (and fewer, if the interval of translation is a whole tone, a minor third, a major third or an augmented fourth). In the case of diatonic translation, the same pitch is retrieved after seven translations.

The alert mathematician-musician will find ample example of these in music of different periods. I will give only one or two here. In Chapter 14 of DALLIN's book, the author discusses the transformation of a twelve-tone row used in the Concerto for Violin, Op. 36 by Schoenberg.

Bartok's Mikrokosmos, Vol. 1, Op. 29, is an example of a theme which is inverted and run against itself a bar later. Churchgoers will be familiar with Tallis' Canon (Anglican-United Hymn Book, #363; compare soprano and tenor lines), and, if Anglican, might know Crotch's chant (Canadian Psalter, Ps. 45, Ch. 116) in which the third and fourth versets are retrograde repetitions of the first and second respectively. Other Anglican chants illustrate lesser degrees of such symmetry. The most striking example of tonal translation I know of occurs in J.S. Bach's Dorina Toccata in D Minor for Organ (BWV538). Look particularly at bars 65-71 and at bars 73-79; in the first case, a motive almost makes its way through a whole octave by rising through diatonic fourths, and in the second more than an octave is covered by a succession of diatonic fifth drops (over a steadily descending pedal passage), modulo octaves. Those who retained the catalogue from the display on Deceptions at the Ontario Science Centre in the fall of 1977 should look up the piece, "The Way of the World" by Moscheles; the music reads the same when placed upsidedown. Unfortunately, it does not sound very well. Bach's "Well-Tempered Clavichord" presents lots of material; look particularly at the Fugue in E flat minor of Book 1. Chopin's Valse in C sharp minor (Op. 64, No. 2) is a particularly attractive piece illustrating effective use of tonal translations. One could go on, and doubtless the reader and his students will produce more favorite examples for themselves. Consult BUDDEN's book for a wider discussion, and some more examples, including rounds.

2. The English have been particularly noted for the use of bells in public worship, and have developed a distinctive form of celebration in the form of change-ringing. A "peal" of 5-12 bells of different pitches is lodged in a tower, each to be rung by a separate individual pulling on a rope. The idea is to ring through the bells in all possible orders (120 permutations for 5 bells, 720 for 6, etc.) This apparently arose about the sixteenth century, went into a decline and was revived in the nineteenth. The flavour of this undertaking is captured nicely in a good mystery by Dorothy Sayers called "Nine Tailors Make a Man". To illustrate, consider the following ringing of the changes on a peal of 4 bells, numbered 1, 2, 3, 4 (in descending order of pitch). The order of ringing is indicated by letters A, B, C, D.

Each order is derived from its predecessor by interchanging the positions of one or more pairs of adjacent bells. The pairs interchanged are indicated by pairs of letters in brackets. The basic rule in this example is to alternate interchanging the two outer pairs with interchanging the inner pair. However, if this is persevered with, we return to the original order of the bells after 8 changes. Consequently, we introduce an extraordinary change (*) to avoid this; this extraordinary change will be brought in again to avoid a premature repetition.

| A | B | C | D | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | (AB) (CD) |
| 2 | 1 | 4 | 3 | (BC) |
| 2 | 4 | 1 | 3 | (AB) (CD) |
| 4 | 2 | 3 | 1 | (BC) |
| 4 | 3 | 2 | 1 | (AB) (CD) |
| 3 | 4 | 1 | 2 | (BC) |
| 3 | 1 | 4 | 2 | (AB) (CD) |
| 1 | 3 | 2 | 4 | (CD) * |
| 1 | 3 | 4 | 2 | (AB) (CD) |
| 3 | 1 | 2 | 4 | (BC) |
| 3 | 2 | 1 | 4 | (AB) (CD) |
| 2 | 3 | 4 | 1 | (BC) |
| 2 | 4 | 3 | 1 | (AB) (CD) |
| 4 | 2 | 1 | 3 | (BC) |
| 4 | 1 | 2 | 3 | (AB) (CD) |
| 1 | 4 | 3 | 2 | (CD) * |
| 1 | 4 | 2 | 3 | (AB) (CD) |
| 4 | 1 | 3 | 2 | (BC) |
| 4 | 3 | 1 | 2 | (AB) (CD) |
| 3 | 4 | 2 | 1 | (BC) |
| 3 | 2 | 4 | 1 | |

In designing a series of changes, certain rules should be followed:

(a) move from one change to the next by transposing the order of pairs of adjacent bells;

(b) repeat no order more than once in progressing through the changes;

(c) let each bell do the same work (i.e. each number progresses through the positions in essentially the same way);

(d) let no bell remain in the same position for more than three consecutive blows.

The mathematical problem is to design sequences of permutations which will achieve this aim in such a way that the generating permutations (e.g. (AB) (CD), (BC)) are kept as simple and as few as possible. Again, the reader is referred to BUDDEN for discussion in much greater detail.

3. **Problems for students:**

(1) Is there a twelve-tone row which is equal to its inversion? to its retrogradation? to its retrograde inversion? Explain your answers.
(2) In the Steadman Triples method of ringing the changes, we have 7 bells. Using numbers and letters as in Section 2, let **c** denote the change (AB) (DE) (FG), **a** the change (BC) (DE) (FG) and **g** the change (AB) (CD) (EF). We start by continually repeating the following sequence of twelve changed: c a g c a c a c g a c a. For example, we start off:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| A | B | C | D | E | E | G | | |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | (AB) (DE) (FG) | c |
| 2 | 1 | 3 | 5 | 4 | 7 | 6 | (BC) (DE) (FG) | a |
| 2 | 3 | 1 | 4 | 5 | 6 | 7 | (AB) (CD) (EF) | g |
| 3 | 2 | 4 | 1 | 6 | 5 | 7 | (AB) (DE) (FG) | c |
| 2 | 3 | 4 | 6 | 1 | 7 | 5 | (BC) (DE) (FG) | a |
| 2 | 4 | 3 | 1 | 6 | 5 | 7 | (AB) (DE) (FG) | c |
| 4 | 2 | 3 | 6 | 1 | 7 | 5 | (BC) (DE) (FG) | a |
| 4 | 3 | 2 | 1 | 6 | 5 | 7 | | |

and so on. If this process is followed, how many changes are possible before returning to the original order? Design a system of extraordinary changes to insert just before an undesired repetition, which will permit a ringing through of all 50 40 changes. (Conjecture: Steadman Triples requires at least two singles (a change in which exactly one pair of positions is interchanged to complete 50 40 changes without repetition).

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# BOOK REVIEWS AND RESEARCH



**Dr. W. Higginson, Queen's University**

## WHY JOHNNY CAN'T INTEGRATE: MORRIS KLINE ON UNIVERSITY MATHEMATICS and ITS TEACHING

Dr. Morris Kline is Professor Emeritus at the Courant Institute of Mathematical Sciences, New York University. In a long and distinguished career he has made extensive contributions to three areas of his chosen discipline: applications, history, and pedagogy.

Professor Kline's status as an applied mathematician would seem to be quite secure. He has held a position of leadership in a major research institute over a long period of time and has made significant contributions to electromagnetic theory. In the last quarter century he has perhaps been the most prolific of historians of mathematics. Between 19 53 and 19 72 he published five

major books in this area, beginning with **Mathematics in Western Culture,** and ending with **Mathematical Thought from Ancient to Modern Times.** The latter massive book (1238 pages) is a major work against which any future comprehensive histories of mathematics written in this century will be compared. Nevertheless, Dr. Kline is not without critics in this area. Some of the 'purer' historians of mathematics contend that the scale of his work is such that it must necessarily be superficial and to some extent misleading. (This is not a criticism unique to Kline's work; the publications of E.T. Ball, another research mathematician who writes extensively in the history of mathematics, are thought by many historians of mathematics to be highly romanticized versions of reality. Those among us whose interest in the history of mathematics was first kindled by Bell's **Men of Mathematics** might give as a rejoinder Whitehead's statement that "it is more important that something be interesting than that it be true"). Kline's heavy dependence upon secondary sources is particularly noted by his critics. In trying to take an objective view of this dispute it is hard to avoid the conclusion that what is at issue is not so much Kline's sholarship but the question of the necessary limitations of popularization and large-scale historical surveys (the Toynbee phenomenon).

What is clear, however, is that the waves, of the historical dispute are but mere ripples compared to the whitecaps generated by Kline in the area of mathematics education. As an applied mathematician Kline was disturbed by what he saw as the excessive purism of school curricula developed in the nineteen-sixties. After many years of verbal debates, journal exchanges, hot-line interviews, and conference papers, Professor Kline summarized his position in 1973 with the book **Why Johnny Can't Add: The Failure of the New Math.** A classic polemic, this book was acerbic, intemperate, witty, and filled with points of substance which even the most ardent advocates of curriculum reform found hard to adequately rebut. While admitting the numerous weaknesses of traditional programmes, Kline attacked the new math curricula for being overly formal, essentially elitist, deficient in applications, and devoid of historical links. Despite his tendency to consistenly oversimplify and overstate his case, and to push analogies past all reasonable limits, the essential soundness of many of Kline's arguments was evident, and a substantial percentage of the mathematical community has come to hold a Klinean position on pre-college mathematics curricula.

Some university professors of mathematics who supported Dr. Kline in his attacks on school mathematics may now find themselves in a somewhat awkward position. This situation arises because of contentions in his recently published work entitled **Why the Professor Can't Teach: Mathematics and the Dilemma of University Education.** In this book, Dr. Kline--anyone with a taste for psychohistory would be sorely tempted to speculate that as a young child, Morris Kline was fascinated by the fable of the Emperor's New Clothes--applies an argument isomorphic to the one he used in **Why Johnny Can't Add.** This time, however, the application is to the university level of instruction.

On the evidence of this book Kline would not seem to have mellowed with age. One basic tenet of his position is that "mathematics education has been a debacle" (p.2). A second fundamental assumption is that "the greatest threat to the life of mathematics is posed by the mathematicians themselves and their most potent weapon is their poor pedagogy. Mathematics as part of liberal arts education may disappear just as Greek and Latin have disappeared and as modern foreign language studies are now gradually disappearing" (p.5).

Kline finds three aspects of the contemporary college and university mathematics scene especially distressing: The extreme emphasis on research and publication at the cost of teaching, the separation from science and real world applications in most curricula, and the low quality of most textbooks. With reference to the second of these, for instance, he claims:

> What is wrong, then, is not that professors do not know what they are teaching but that they do not know how to relate their subject to the rest of knowledge and to life. Education is evaded within academic walls as well as without, and it is more often the professors rather than the students who do so. Professors may not be consciously dishonest or aware of their pedagogical ineptness. However, the lack of clear standards of teaching, the professors' ignorance of pedagogy, and the obligation, which many take too literally, to cover ground prescribed in syllabi produce the same effect as incompetence and dishonesty. The practices of the past are not re-examined, and the challenges that true education might post are ignored. (p.152).

The number of similarities between the Science Council of Canada's Background Study No. 37 **Mathematical Sciences in Canada** and Kline's recent book are quite striking. Anyone concerned about the future of mathematics in our culture would be well advised to consider and act on the positions outlined in these documents.

Beltzner, K., Coleman, A.J. and Edwards, G.D. **Mathematical Sciences in Canada.** Science Council Background Study No. 37. Ottawa: Science Council of Canada, 1976. $6.50. 333pp. Paper. 0660003880.

Kline, Morris. **Mathematical Thought from Ancient to Modern Times.** New York: Oxford University Press, 1972. $45.00. 1238 pp.

**Mathematics in Western Culture.** London: Oxford University Press, 1974 (originally 1953). $5.50. 484pp. Paper.

**Why Johnny Can't Add: The Failure of the New Math.** New York: St. Martin's Press, 1973. $6.95. 173pp. (Paperback edition by Vintage Books, New York, 1974. $2.05. 0394719816).

**Why the Professor Can't Teach: Mathematics and the Dilemma of University Education.** New York: St. Martin's Press, 1977. $12.50. 288pp. 0312878672.

Gardner, Martin and Tappay, Bob. **Aha! Insight Box.** New York/San Francisco: Scientific American/W.H. Freeman, 1978. $135.00 (approximate). (Book: **Aha! Insight**, San Francisco: W.H. Freeman, 1978. $7.00 (approximate). 179pp. Paper. 089450017).

## AHA! INSIGHT BOX

Following on their successful **Paradox Box** Martin Gardner and Bob Tappay have joined forces again to produce the **Aha! Insight Box**. This Box consists of six filmstrips together with six soundtracks on cassette tapes. Each filmstrip presents a number of problems posed by cartoon characters. The problems are grouped into six classes being respectively, Combinatorial, Geometry, Number, Logic, Procedural, and Word. In total, there are some sixty-five problems.

In the introduction guidebook which accompanies the box, Martin Gardner claims that the criterion for selection was that the problems should "seem difficult, and indeed are difficult if you go about trying to solve them in traditional ways. But if you can free your mind from standard problem-solving techniques, you may be receptive to an aha! reaction that leads immediately to a solution" (p.viii).

While the problems are not universally of the highest quality, the majority are interesting and well chosen. (Surely everyone must have heard by now of the surgeon who can't operate on a patient because 'that's my son' although the patient's father was also a victim of the same accident. This would seem to say more about our tendency to sexually stereotype certain roles than our problem-solving ability although, admittedly, the two are related). Without exception the problems are attractively presented. Martin Gardner seems to have a special genius for writing up mathematical ideas.

The filmstrips and audiotapes are of the high quality we have come to expect from **Scientific American.** Boards of Education with substantial media-centre funding should consider purchasing the Box for use in the Intermediate and Senior Divisions. Groups with smaller budgets for the acquisition of materials would find the guidebook an attractive and useful substitute at quite a reasonable price. For individuals with an interest in problem solving, the book is a must.

## Calculator Materials

(For a more extensive coverage of this area see the Review column of the March 1978 number of this journal).

Dr. Gerry Vervoort of Lakehead University and Dale Mason of the Lakehead Board of Education have written **Calculator Activities for the Classroom** in the form of three volumes of ditto-masters.

(Volumes I and II contain 52 activities apiece; Volume III has 60). The books, published by Copp Clark (1977), are supplemented by a 92 page **Teacher's Resource Book** which contains a number of valuable sections including a stimulating and very appropriate short story by Isaac Asimov. The materials, which assume access to a standard form-function calculator, would seem to be mainly appropriate to students in the intermediate division. The activities, which the authors classify into the topics of 'getting to know you', games, estimation, lessons, patterns, problems, and fractions, include many which are original and imaginative. Any mathematics teacher who is looking for suggestions for the effective use of calculators should seriously consider this collection.

A new and highly regarded journal in this field is **Calculators Computers Magazine** (Volume 2, Issue 3, March 1978). Edited by Don Inman, it is published seven times a year by Dymax, P.O. Box 310, Menlo Park, Ca. 94025. The annual subscription fee is $17.00. Approximately one-half of the articles deal with calculator applications at a level which an able secondary school student could understand. A selection of articles from Volume 1 of this journal has recently been published by Addison-Wesley under the title **Calculators Computers: A Sourcebook of Activities** (1978, 216pp.)

## MATH EQUALS: BIOGRAPHIES OF WOMEN MATHEMATICIANS + RELATED ACTIVITIES

Perl, Teri. **Math Equals: Biographies of Women Mathematicians + Related Activities.** Don Mills: Addison-Wesley, 1978. 250pp. Paper. 0201057093.

Over recorded time the number of women mathematicians has been very small and continues to be so today. It is difficult, in some ways, to understand why this should be so. One thing which Ms. Perl succeeds in doing in this excellent book is to make more obvious the enormous social pressures which have existed in the past (and still exist in some form currently?) which made it extremely difficult for a woman to become a mathematician.

The life and work of nine women, from Hypatia (370-415) to Emmy Noether (1882-1935), are outlined in this book. The treatment of the mathematical contributions is well handled. The accompanying illustrations and photographs are appropriate and well executed. A number of fascinating insights into the development of mathematics are revealed in the biographies of these women.

A feature of the book which makes it unique is the very fine collection of 'related activities' which follow each chapter. Many of these activities are such that they could be taken immediately into secondary school curricula. After the materials on Grade Chisholm Young (1868-1944), for instance, there are ten pages of material on polyhedral models and binary numeration.

Students and teachers alike will profit from this first-class book. One might only hope that male mathematicians might find as able an expositor as Ms. Perl.

# FUTURE MEETINGS

**Local**

**Executive Meeting**
First Friday after Labour Day 1978 09 08
**Council Meeting**
First Saturday after Thanksgiving1978 10 14
**Executive Meeting**
First Friday after Nov. 11 1978 11 17
**Council Meeting**
Second Saturday in February 1978 02 10
**Executive Meeting**
First Friday of April 1978 04 06
**Annual Meeting** 1978 05 13
**Executive Meeting**
First Friday after May 24 1979 05 25
**Council Meeting**
Second Saturday after May 24 1979 06 02

**International**

OAME/AOEM 6th Annual, 1978
May 12 - 14 Peterborough
N.C.T.M. 57th Annual, 1979
April 18 - 21 Boston
OAME/AOEM 7th Annual, 1979
May 8 - 11 Toronto
NCTM 58th Annual, 1980
April 16 - 19 Seattle

NCTM 59th Annual, 1981
April 22 - 25 St. Louis



# I DIDN'T KNOW THAT

## TRICKS — GAMES — PUZZLES

**Dr. H. Laurence Ridge, Faculty of Education, University of Toronto**

This issue's column is something of a pot-pourri - something for everybody. The topics have arisen from correspondence and conversation since the March issue. Thanks is due to each of the correspondents for their contribution.

### Rotational Numbers

Mathematics is a hobby for Russell Walker, a retired gentleman who resides in Ravenswood, a community in Lambton County. Rather than keep his age a secret, Russell sent me the information - in puzzle form of course. He was born 66 months after the previous 29th of February. That's right - sixty-six months. So, how old is he now?

Russell also drew my attention to a set of fascinating numbers:

| | |
|---|---|
| 16 digits: | (0) 588 235 294 117 647 |
| 18 digits: | (0) 52 631 578 947 368 421 |
| 22 digits: | (0) 434 782 608 695 652 173 913 |
| 28 digits: | (0) 344 827 586 206 896 551 724 137 931 |

Each of these numbers could be called a "rotational number". To see what is meant by "rotational number", multiply each n-digit number by 2, 3, 4, ... n. (How useful is your hand-held calculator for this chore?)

Now, here are a couple of 'puzzles'[1] for you number buffs:

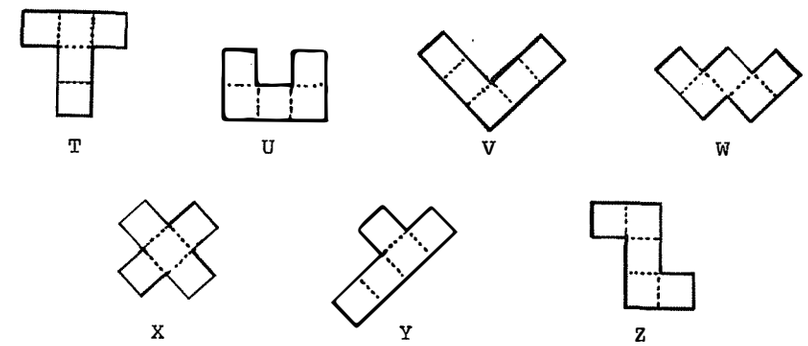
1. What is the smallest (6-digit) "rotational number"?
2. What is the next greatest "rotational number" after the 28 digit case?

It turns out that my explanation and Mr. Walker's differ considerably (at least on the surface). Both approaches will be spelled out in the December issue. As well, your ideas are always most welcome.

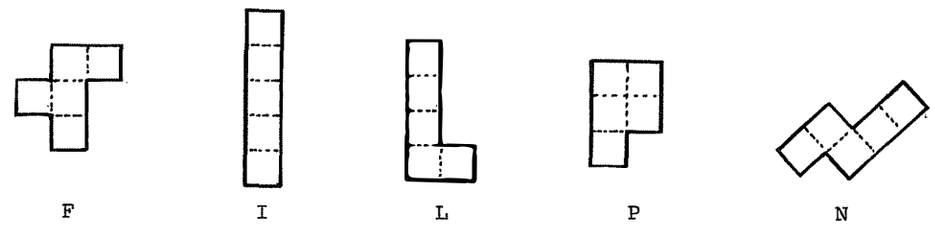
### Pick of the Pentominoes[2]

A most interesting item arrived from George Adams who teaches mathematics at T.L. Kennedy Secondary School in Mississauga. At the beginning of a section on Grade 10 deductive geometry this past year he used pentomino puzzles "as a change of pace from the normal proof techniques and to enhance visual perception".

For the uninitiated, polyominoes (of which pentominoes are a specific case) are configurations of (five) squares arranged so that each square shares a side with at least one of the other (four) squares. The twelve[3] pentominoes are usually identified by letter as follows:




(The last seven letters of the alphabet)

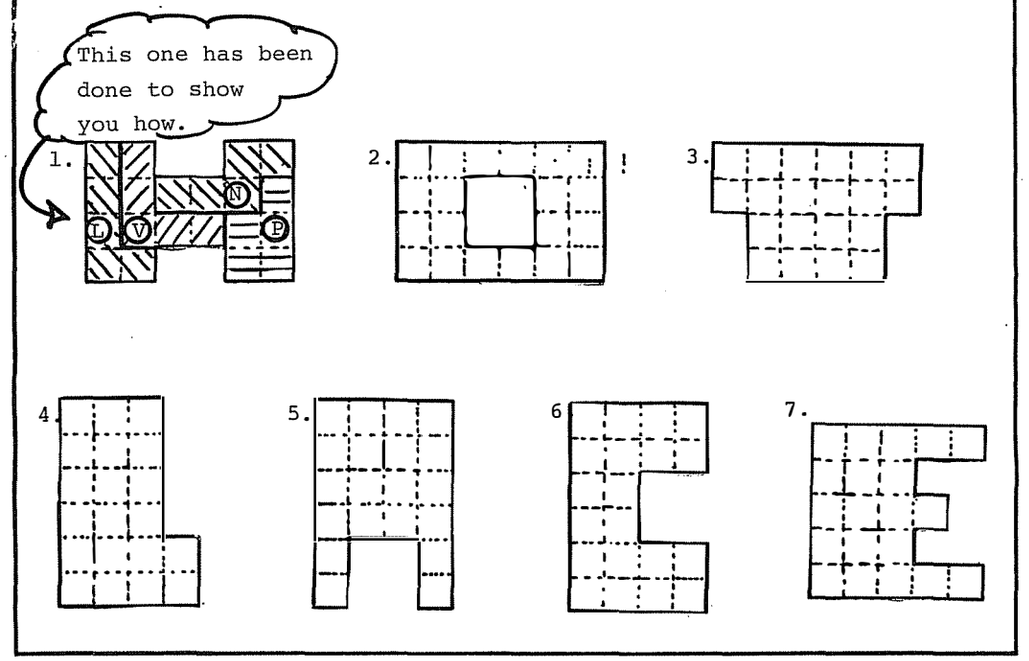



(Key letters of the word FILiPiNo)

George has devised a number of dissection-style exercises which are much more feasible than the standard filling of rectangles (3 by 20, 4 b 15, 5 by 12, and 6 by 10[4] ) using all twelve pentominoes. (Although there are many ways of completing the latter type of puzzle, they are just too difficult for regular full-class use[5] .)

Among the 20-square shapes which can be made from sets of four different pentominoes are letter configurations. Here's a version of one of George's exercises.

**FILL EACH OF THESE 'LETTERS' USING 4 DIFFERENT PENTOMINOES***




*There may be several different solutions for each one. See how many you can find.

George claims that 20-square shapes resembling the letters F, I, S, and U can also be made. Perhaps there are others as well (depending on the 'artistic licence' permitted in interpreting a shape as representative of a given letter). 'La piece de resistance' is George's idea "to spell various 3-letter words using all 12 [pentomino] symbols". Here's a real chance to vie for a place of distinction with the 'calculatorese'[6] fanciers. Let's see what we can do with this challenge - sentences, maybe !?!

**Divisibility Dilemma**

Barry Nix, a mathematics teacher at Athens District High School in Leeds and Grenville County, told me recently of a novel divisibility test for 11 devised by one of his students. This is her method as applied to 6783:

```
 6 7 8 (3)
-   3*
 6 7 (5)
-   5*
 6 2
```

6 2 which is not divisible by 11.

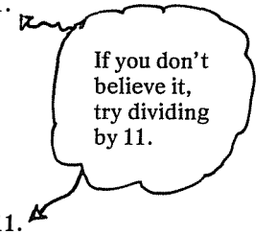
Hence, 6783 is not divisible by 11.

For 42 669:

```
 4 2 6 6 (9)
-       9*
 4 2 5 (7)
-     7*
 4 1 (8)
-   8*
 3 3
```

3 3 which is divisible by 11.

Hence, 42 669 is also divisible by 11.




Barry wondered why the method (always) worked and whether it was exemplary of a more general set of divisibility tests. The answer to both questions is related to an article[7] by David Rosen of Amherst, NY, but before you look it up (or in case you don't have access to it) see what you can come up with on your own.[8]

You're invited to share the wealth of ideas that your students have come up with recently - or long ago for that matter.

**'Match Game' Follow-up**

In his presentation at the O.A.M.E. Annual Conference in Hamilton in April, Julian D'Angela (Hamilton-Wentworth Separate School Board) gave a very nice variation on the MATCH GAME described in this column in the March issue.

The idea of the MATCH GAME was to have participants select any 4-digit number (first and last digits differing by at least 2), reverse (or just reorder) the digits, then subtract the smaller from the greater number. They would then read the difference but omit one of the four digits by saying "Blank". The 'MC' would then provide the missing digit, thereby 'Matching' their result. The process is based on 'casting out nines'.

Instead of rearranging the digits of the selected number and subtracting, Julian had his audience subtract the digital sum from the selected number. Example (using 4 digits):

```
 7 3 2 9
 -   2 1   (since 7+3+2+9 = 21)
-------
 7 3 0 8
```

To avoid the 'zero or nine problem' (discussed in the March issue) he had the folks cross out (equivalent to reading "blank") any non-zero digit in the different then add the remaining digits.
Example (cont'd): 7 ~~3~~ 0 8 --> 7+0+8 = 15
Since the difference and its digital sum must both be multiples of 9 and a number less than 10 (3 in this case) has been deleted from both the difference and its digital sum, the digital sum of the actual difference must be the next greatest multiple of 9 (18 in this case). Hence, in this case, the missing digit would be identified as 18 - 15 or 3.

This variation can be explained very neatly in algebraic form. Any four digit number can be represented by

$1000\,a + 100\,b + 10\,c + d$ with the necessary restrictions

on a, b, c, and d. The digital sum is then represented by

$$a + b + c + d$$

and the difference between the number and its digital root is

$$999\,a + 99\,b + 9\,c \text{ or } 9 \times (111a + 11\,b + c)$$

which is clearly divisible by 9.

**Post Script**

Solutions and comments for all the 'left to the reader' items will appear in the next (December) issue.

Recreationally yours,
H.L.R.

---

1. Call them "problems" if you like but we don't want to infringe on MacKay and Knight's new PROBLEMS section. Welcome aboard Dave and John.
2. The term "pentominoes" is a registered trademark of Solomon W. Golomb.
3. Investigation of all possible configurations (and making of cardboard models for future use) can be useful exercise at various levels (depending on the materials used: coloured cubes, centicubes, unifix cubes in trays, squared paper, etc.). Sets of plastic pentominoes are also available commercially under various trade names.
4. There is no known explicit formula for the number of 'n-ominoes' (polyominoes consisting of n squares).
5. Another approach to easing into pentomino puzzles has been tried successfully by Janet Caldwell of Parkdale Collegiate Institute of Toronto. She used partially completed rectangular patterns.
6. 'Calculatorese' refers to words derived from inverted hand-held calculator displays.

1. Rosen, David A. "An Uncommon Divisibility Test" *Mathematics Teaching*, No. 76, Sept. 1976, pp. 32, 33.

8. Here's a hint (but hints can also serve as distractors depending on your train of thought): the method can be readily adapted to divisibility tests for 7, 13, 17, and 37.




**Editor's Note:** By way of introduction, both authors arrived at Westdale Secondary School, Hamilton but six years ago, Dave from A.C.E. with a B.Sc. from McMaster and John from F.E.U.T. with a B. Math from Waterloo. Both have been active, speaking at conferences and together they have set up an enrichment program at Westdale. Dave chairs the MAGIC (Math Activities and Games in the Classroom) Committee for the Hamilton Board of Education.

**To Submit Problem Solutions**

1. Solutions to one or all the problems are invited. Individual students, small groups, or whole classes should include their name, grade, school, and community.
2. Correct solutions will be published two issues later.

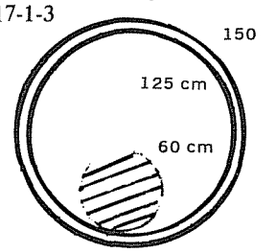
3. Solutions should be submitted to:
   Ontario Mathematics Gazette
   c/o Mr. J. McKnight,
   Westdale Secondary School
   700 Main Street West
   Hamilton, Ontario L8S 3T1
4. If you have an interesting problem you would like to share, please send it along, with a solution, to the above address.

## I UP TO GRADE 10

17-1-1 A hockey team has won 5 out of 8 games played. With 16 games still to be played, how many more games must be won so that the team wins 75% of its season's schedule?

17-1-2 In a survey of year IV students, the numbers studying various Sciences were found to be: Chemistry - 28, Biology - 30, Physics - 42, Chemistry and Biology - 8, Chemistry and Physics - 10, Biology and Physics - 5, all three Sciences - 3. Find the number of students studying exactly one Science.

17-1-3 The diagram shows the cross section of a roller. In addition to the heavy outer casing, it contains a small cylinder which is free to roll around inside. The circumferences of the various parts are marked on the diagram. How many revolutions will the small 60 cm cylinder make if the roller is pushed a distance of 18 m?




17-1-4 If 94 players enter a knockout tennis tournament for a singles championship, how many matches must be played to determine the winner?

**Editors Note:** For a 95 player tournament, how many matches must be played?

## II UP TO GRADE 13

17-1-5 Donald Corleone cashed a $200 cheque at the bank and requested some one-dollar bills, 10 times as many two-dollar bills, and the balance in five-dollar bills. How did the cashier pay him?

17-1-6 Find all real values of k so that the equation $x^3 + x^2 - 4kx - 4k = 0$ has two of its three roots equal.

17-1-7 Three circles are on the same side of a straight line and are tangent to the line. One of the circles has radius 4 and each of the three circles is tangent to the other two. Draw a diagram and then determine the radius of the two equal circles.

17-1-8 Prove, by the methods of analytic geometry, that if 2 medians of a triangle are equal then the triangle is isosceles.

## III ARE YOU READY FOR A CHALLENGE?

17-1-9 Two men sold their herd of x cows at x dollars per head. With the proceeds they bought sheep at $10 each and a single lamb costing less than $10. Each man received the same number of animals but the one receiving the lamb had to be compensated so as to make the division equitable. How much money did he receive from the other man?

# ANSWERS TO BRAIN TEASERS

1. 14 strokes
2. 30 miles.

# O.A.M.E./A.O.E.M. COUNCIL ORGANIZATION 1978-79

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Past President - Williamson, Neil
President - McPhail, David
Pres.-Elect - Norrie, Alex
Director - Higginson, Bill, Faculty of Education, Queen's University, Kingston, K7L 3N6
Director - Pogue, Paul
Director - Sherk, Paul
Sec. Treas. - MacGregor, Morley B.

**Councillors**

Term Expiring

| 1979 | 1980 | 1981 |
|---|---|---|
| Bennie, Jean | Acheson, Roma | Bedard, Jeannette |
| Kyer, Arvon | Crundwell, Mary | Fullerton, Olive |
| Landriault, Pierre | Dottori, Dino | Lessard, Paul |
| Morrow, Lorna | Evans, Douglas | Muller, Eric |
| Parker, Shirley | Jones, Gareth | Smith, Bob |
| Robinson, Shirley | McPhail, Sharon | Topping, Bruce |
| Tuck, Robert | Silbert, Michael | Maskell, Fred |
| Vervoort, Gerry | Ullrich, Peter | |

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AMEO/EOMA - Brule, Paul
COMA - Hubbard, Christing
Golden Section - Marshall, Ray
GVMA - Davidson, Dave
NOMA - Truman, Neil
PRMA - Jeffers, Bill
SLMA - Buskard, Margaret
SEYMA - DiDifelice, Dominic
SWOAME - Peters, Gary
TY - Silcox, Edwin

**Standing Committee Chairpersons**

Projects - Ullrich, Peter
Publications - McPhail, Sharon
Chapters - Robinson, Bob
Nominations - Williamson, Neil

**Ad Hoc Committee Chairpersons**

Constitution - Williamson, Neil
Convention - Jeffers, Bill
Interface - Maskell, Fred
OAME-OECA - Pogue, Paul
Research - Topping, Bruce

# PRESIDENT'S MESSAGE

**D. McPhail**



By the time you read this, the summer of 1978 will almost be or will have been history. I hope that each and every one of you had an enjoyable and relaxing summer pursuing those interests that there is neither the time nor the weather to pursue at other times of the year.

It is indeed both a privilege and a pleasure to be the president of your organization for the coming year, and a busy year it will be.

It is my belief that the coming year will see some changes in the general operation of O.A.M.E. We represent over 1500 mathematics teachers, and perhaps the time has now arrived that this organization becomes the "co-ordinating body" of the **many mathematics factions** across this province. Since the ultimate goal of all the mathematics groups is better programs for the students maybe, just maybe, a totally non political body such as OAME can provide the leadership, the co-ordination and bring together the expertise that is in the field to bring about even better math learning for Ontario students.

This will not be a year, however, where we look only afield for jobs to be done. Many responses and suggestions, received in the spring have proven to be invaluable in giving much direction to the executive and council as we evaluate the internal workings of our own organization with the express goal of making absolutely certain that the Ontario Association of Mathematics Education continues to meet the needs of the membership it serves.

There is much work to be done in the months ahead but with the hard work of the excellent people that you have elected to the executive and council, with the continuing efforts of our ten chapters and with individual submissions from interested members, I am certain that when we look back on this year at the Annual Conference at Trent University in May 1979, that this will prove to be a proud year for O.A.M.E.

# A CHALLENGE FOR CHAPTER PRESIDENTS

It is time for each and every chapter to communicate news, opinions and points of view to the rest of the chapters as well as to your own members - through a newsletter. It is time to consider, over a period of time, an issue. It is time to formulate a position paper on that issue. This article sets forth two basic issues being considered by your provincial council and it needs your help **in writing.**

Recall article II - The Aim of OAME/AOEM. "To promote excellence in mathematics education in the Province of Ontario. The term "excellence" in this context, refers to the contribution that mathematics education can and should make to the growth and intellectual development of all individuals in our Province, with due regard to the needs of our society".

At the spring council meeting 1978 03 11, your elected representatives unanimously approved the following motion, "The Council recommends to the editor of the Gazette that the National Council of Supervisors of Mathematics **Position Paper on Basic Mathematical Skills** be published and that opinions be solicited relevant to the Ontario situation from the chapters.

Herin lies the first issue chapters may well consider during the coming year. Chapter executives might structure an ongoing debate. How useful a newsletter would be! The outcome would be a position paper - your chapter's answer to that popular slogan "Back-to-the-Basics". Here is a chance to help members consider carefully an issue.

To aid you in your discussions, the Gazette will publish provocative articles by authors who have considered different facets of the problem. The more faces of the issue we examine, the better the outcome. Do consider both the NCSWM paper and the article by Ayim and Emerson which follow. Written opinions or articles you feel should be included we will be only too happy to receive. Eventually, your council might well attempt an OAME/AOEM position paper for Ontario.

In the March 1978 issue of the Gazette we printed the NCTM-MAA position paper titled **Recommendations for the Preparation of High School Students for College Mathematics Courses.** It was the opinion of council that for a position paper to be useful to students and teachers it would need to be specific to the Ontario situation. At the last council meeting 1970 06 03 the following motion was passed, "that Alex Morrie prepare a position paper on **Preparation of High School Students for College Mathematics Courses** and to distribute it to any interested group for reaction". Alex has prepared a draft copy and mailed it out for reaction.

Herein lies the second issue chapters might well consider, Alex would be pleased to hear from you.

I believe the strength of OAME/AOEM lies in the chapters. It was stated by President Neil Williamson in his annual report that our nine chapters hosted no less than fourteen major events in 1977/78 with a total attendance of some 1500 members. We have a strong effective leadership course each fall that is highly thought of. Our annual conference continues to be an outstanding event. It is time for all chapters to consider a bulletin or newsletter! COMA does. It serves the two fold purpose of informing members as well as the other chapters across this province. It is good for a group to place their thoughts, experiences, positions in writing. Will not each chapter president make the founding of a chapter newsletter a prime objective for this year?

Imagine, nine chapters, each with a newsletter **and** position papers on the issues your council is wrestling with.

Bob Smith

Already, Associate Editor Frank Ebos has written about the BTB (Back to Basics) issue. Dig out your copy of OMG, September 1976. For our new members, the article makes the following points:

1. **Applications should be basic to any curriculum.** They not only satisfy the question "Why are we taking this stuff?" but may also provide a foundation for topics in next year's curriculum.
2. **Decision about what information is needed to solve a problem is a basic skill.** This skill is probably more useful than the skills advocated by the BTB group. Yet the BTB skills are needed once this information is properly interpreted.
3. **To know the answer is reasonable is a basic skill.** This may be accomplished by providing practice in approximating, estimating as well as decision making.
4. **Teaching students "to sketch" the problem is a basic skill.** Often a start "doing rather than waiting for an inspiration" is what is needed.
5. **Working with, interpreting, and reading statistics is a basic skill.** Statistics permeate too much of our everyday living to be ignored.

Frank admitted his list was neither exhaustive nor complete. His admonition was "Each of us probably have many other recommendations to add, but we do have to make recommendations".

# TEN CHAPTERS OF OAME/AOEM

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Rep. - Edwin Silcox

Chapter secretaries are asked to update the above.

**MATHEMATICAL ESSAY CONTEST**
**for high school students**

The Samuel Beatty Fund has provided money for prizes for a mathematical essay contest for secondary school students. Candidates will prepare a report on one of a list of topics in mathematics not later than February 38, 1979 to qualify for a prize. Further details will be avialable in September. Any interested teacher or student should write to:

Prof. E.J. Barbeau
Secretary, Samuel Beatty Fund Trustees
for more information.

# NATIONAL COUNCIL OF SUPERVISORS OF MATHEMATICS POSITION PAPER ON BASIC MATHEMATICAL SKILLS

**INTRODUCTION**

The currently popular slogan "Back to the Basics" has become a rallying cry of many who perceive a need for certain changes in education. The result is a trend that has gained considerable momentum and has initiated demands for programs and evaluations which emphasize narrowly defined skills.

Mathematics educators find themselves under considerable pressure from boards of education, legislatures, and citizens' groups who are demanding instructional programs which will guarantee acquisition of computational skills. Leaders in mathematics education have expressed a need for clarifying what are the basic skills needed by students who hope to participate successfully in adult society.

The narrow definition of basic skills which equates mathematical competence with computational ability has evolved as a result of several forces:

1. Declining scores on standardized achievement tests and college entrance examinations;
2. Reactions to the results of the National Assessment of Educational Progress;
3. Rising costs of education and increasing demands for accountability;
4. Shifting emphasis in mathematics education from curriculum content to instructional methods and alternatives;
5. Increased awareness of the need to provide remedial and compensatory programs;
6. The widespread publicity given to each of the above by the media.

This widespread publicity, in particular, has generated a call for action from governmental agencies, education organizations, and community groups. In responding to these calls, the National Institute of Education adopted the area of basic skills as a major priority. This resulted in a Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio, in October, 1975.

The National Council of Supervisors of Mathematics (NCSM), during the 1976 Annual Meeting in Atlanta, Georgia, met in a special session to discuss the Euclid Conference Report. More than 100 members participating in that session expressed the need for a unified position on basic mathematical skills which would enable them to provide more effective leadership within their respective school systems, to give adequate rationale and direction in their tasks of implementing basic mathematics programs, and to appropriately expand the definition of basic skills. Hence, by an overwhelming majority, they mandated the NCSM to establish a task force to formulate a position on basic mathematical skills. This statement is the result of that effort.

## RATIONALE FOR THE EXPANDED DEFINITION

There are many reasons why basic skills must include more than computation. The present technological society requires daily use of such skills as estimating, problem solving, interpreting data, organizing data, measuring, predicting, and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills. In recognition of the inadequacy of computation alone, NCSM is going on record as providing both a general list of basic mathematical skills and a clarification of the need for such an expanded definition of basic skills.

Any list of basic skills must include computation. However, the role of computational skills in mathematics must be seen in the light of the contributions they make to one's ability to use mathematics in everyday living. In isolation, computational skills contribute little to one's ability to participate in mainstream society. Combined effectively with the other skill areas, they provide the learner with the basic mathematical ability needed by adults.

## DEFINING BASIC SKILLS

The NCSM views basic mathematical skills as falling under ten vital areas. The ten skill areas are interrelated and many overlap with each other and with other disciplines. All are basic to pupils' development of the ability to reason effectively in varied situations.

This expanded list is presented with the conviction that mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics. The ten components of basic mathematical skills are listed below, but the order of their listing should not be interpreted as indicating either a priority of importance of a sequence for teaching and learning.

Furthermore, as society changes, our ideas about which skills are basic also change. For example, today our students should learn to measure in both the customary and metric systems, but in the future the significance of the customary system will be mostly historical. There will also be increasing emphasis on when and how to use hand-held calculators and other electronic devices in mathematics.

## TEN BASIC SKILL AREAS

### 1. Problem Solving

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.

**2. Applying Mathematics to Everyday Situations**

The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics, and interpret the results in light of the initial situation.

**3. Alertness to the Reasonableness of Results**

Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

**4. Estimation and Approximation**

Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.

**5. Appropriate Computational Skills**

Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single-digit number facts is essential and mental arithmetic is a valuable skill. Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.

Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

**6. Geometry**

Students should learn the geometric concepts they will need to function effectively in the 3-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular. They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills. They also must be able to recognize similarities and differences among objects.

**7. Measurements**

As a minimum skill, students should be able to measure distance, weight, time, capacity, and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

**8. Reading, Interpreting, and Constructing Tables, Charts, and Graphs**

Students should know how to read and draw conclusions from simple tables, maps, charts, and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts, and graphs.

**9. Using Mathematics to Predict**

Students should learn how elementary notions of probability are used to determine the likelihood of future events. They should learn to identify situations where immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make predictions such as election forecasts.

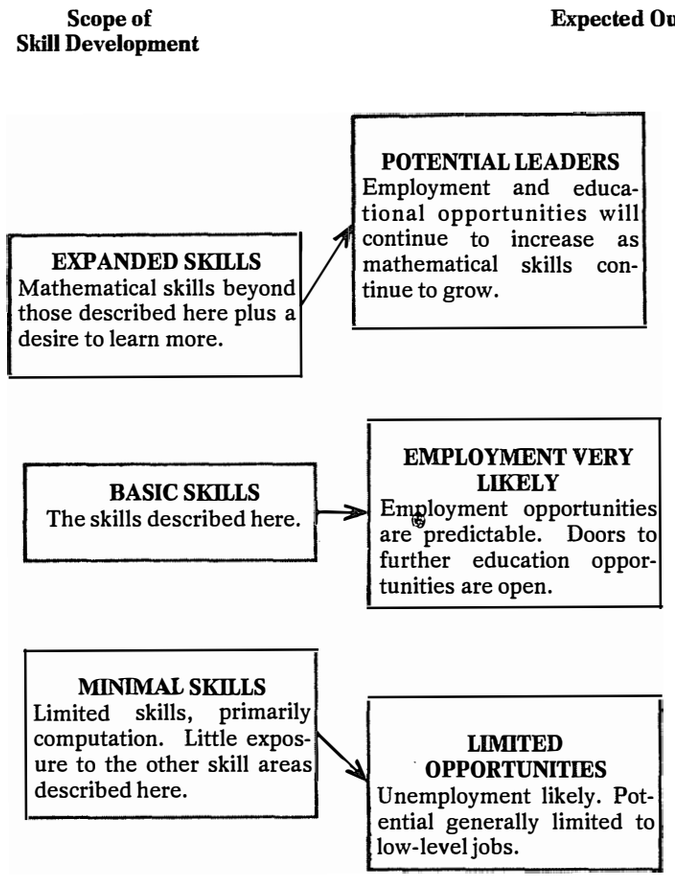
**10. Computer Literacy**

It is important for all citizens to understand what computers can and cannot do. Students should be aware of the many uses of computers in society, such as their use in teaching/learning, financial

transactions, and information storage and retrieval. The "mystique" surrounding computers is disturbing and can put persons with no understanding of computers at a disadvantage. The increasing use of computers by government, industry, and business demands an awareness of computer uses and limitations.

## BASIC SKILLS AND THE STUDENT'S FUTURE

Anyone adopting a definition of basic skills should consider the "door-opening/door-closing" implications of the list. The following diagram illustrates expected outcomes associated with various amounts of skill development.




## MINIMUM ESSENTIALS FOR HIGH-SCHOOL GRADUATION

Today some school boards and state legislatures are starting to mandate mastery of minimum essential skills in reading and mathematics as a requirement for high-school graduation. In the process, they should consider the potential pitfalls of doing this without an appropriate definition of "basic skills". If the mathematics requirements are set inordinately high, then a significant number of students may not be able to graduate. On the other hand, if the mathematics requirements are set too low and mathematical skills are too narrowly defined, the result could be a sterile mathematics program concentrating exclusively on learning of low-level mathematical skills. This position paper neither recommends nor condemns minimal competencies for high-school graduation. However, the ten components of basic skills stated here can serve as guidelines for state and local school systems that are considering the establishment of minimum essential graduation requirements.

## DEVELOPING THE BASIC SKILLS

One individual difference among students is style or way of learning. In offering opportunities to learn the basic skills, options must be provided to meet these varying learning styles. The present "back-to-basics" movement may lead to an emphasis on drill and practice as a way to learn.

Certainly drill and practice is a viable option, but it is only one of many possible ways to bring about learning and to create interest and motivation in students. Learning centers, contracts, tutorial sessions, individual and small-group projects, games, simulations and community-based activities are some of the other options that can provide the opportunity to learn basic skills. Furthermore, to help students fully understand basic mathematical concepts, teachers should utilize the full range of activities and materials available, including objects the students can actually handle.

The learning of basic mathematical skills is a continuing process which extends through all of the years a student is in school. In particular, a tendency to emphasize computation while neglecting the other nine skill areas at the elementary level must be avoided.

## EVALUATING AND REPORTING STUDENT PROGRESS

Any systematic attempt to develop basic skills must necessarily be concerned with evaluating and reporting pupil progress.

In evaluation, test results are used to judge the effectiveness of the instructional process and to make needed adjustments in the curriculum and instruction for the individual student. In general, both educators and the public have accepted and emphasized an overuse of and overconfidence in the results of standardized tests. Standardized tests yield comparisons between students and can provide a rank ordering of individuals, schools, or districts. However, standardized tests have several limitations including the following:

a. Items are not necessarily generated to measure a specific objective or instructional aim.
b. The tests measure only a sample of the content that makes up a program; certain outcomes are not measured at all.

Because they do not supply sufficient information about how much mathematics a student knows, standardized tests are not the best instruments available for reporting individual pupil growth. Other alternatives such as criterion tests or competency tests must be considered. In criterion tests, items are generated which measure the specific objectives of the program and which establish the student's level of mastery of these objectives. Competency tests are designed to determine if the individual has mastered the skills necessary for a certain purpose such as entry into the job market. There is also need for open-ended assessments such as observations, interviews, and manipulative tasks to assess skills which paper and pencil tests do not measure adequately.

Reports of pupil progress will surely be made. But, while standardized tests will probably continue to dominate the testing scene for several years, there is an urgent need to begin reporting pupil progress in other terms, such as criterion tests and competency measures. This will also demand an immediate and extensive program of inservice education to instruct the general public on the meaning and interpretation of such data and to enable teachers to use testing as a vital part of the instructional process.

Large scale testing, whether involving all students or a random sample, can result in interpretations which have great influence on curriculum revisions and development. Test results can indicate, for example, that a particular mathematical topic is being taught at the wrong time in the student's development and that it might better be introduced later or earlier in the curriculum. Or, the results might indicate that students are confused about some topic as a result of inappropriate teaching procedures. In any case, test results should be carefully examined by educators with special skills in the area of curriculum development.

## CONCLUSION

The present paper represents a preliminary attempt by the National Council of Supervisors of Mathematics to clarify and communicate its position on basic mathematical skills. The NCSM

position establishes a framework within which decisions on program planning and implementation can be made. It also sets forth the underlying rationale for identifying and developing basic skills and for evaluating pupils' acquisition of these competencies. The NCSM position underscores the fundamental belief of the National Council of Supervisors of Mathematics that any effective program of basic mathematical skills must be directed not "back" but forward to the essential needs of adults in the present and future.

# BACK TO BASICS VIA THE LANGUAGE OF MATHEMATICS

**Maryann Ayim and Goldwin Emerson, The University of Western Ontario**

One of the problems in getting back to the basics in mathematics is that we are inclined to lose sight of what it was that we considered to be basic in the first place.[1] It is unfortunate that "basics" is frequently confused with rote learning and mindless drill. Surely a more constructive view would emphasize an understanding of mathematical concepts, a facility with the use of everyday mathematics, and the same joy of discovery that children experience when they learn a new word in language study. In returning to the basics, the analogy between learning mathematics and learning language seems appropriate.

This analogy is borne out by contemporary research into the formal structure of language. In the wake of Noam Chomsky's work on grammar,[2] we know that the rules which speakers of a language use in combining words to formulate acceptable sentences reveal a tight logical structure; this structure closely parallels that of the rules governing the combination of mathematical symbols into acceptable mathematical statements. The grammar or syntax of a natural language like English or French may be viewed as an axiomatic system in exactly the sense in which artificial languages such as propositional logic or Euclidean geometry are axiomatic. That is, we can account for why native speakers of a given language will consistently find certain sentence types acceptable and others unacceptable in terms of a set of axioms and rules of inference with which all native speakers of any language seem to operate. Similarly, in accepting a mathematical statement as correct (or rejecting it as incorrect), we will appeal to axioms and rules of inference with which mathematicians operate.

Viewing language in this light permits the drawing out of many interesting similarities. One example is that both natural languages and mathematical systems operate with a finite number of rules which nevertheless make possible the formulation of an infinitely large number and variety of acceptable statements. A second closely related similarity is that certain rules of grammar, like certain rules of mathematics, exhibit the recursive property.

Traditionally, mathematics and language have been viewed as inherently different types of disciplines. Seeking out the similarities would have been the more constructive approach however; this would have facilitated students' seeing these disciplines as part of the whole picture of the human quest for knowledge. Instead, the radical segmentation of the school curriculum, with little effort expended on inter-relating the various disciplines, defeats the attempt to fit either mathematics or language into a total picture of things and maintains the arbitrarily imposed lines of division. If the "back to basics" pandemic could be construed as an argument for the synthesis and unity of subject areas, then it might truly be a move forward with higher educational payoff.

Language provides a useful vehicle for the expression of our thoughts, feelings, and perceptions. It encapsulates concepts and allows us to store them for the future or to rearrange them in meaningful patterns for present needs. It is a useful instrument for bringing fuzzy concepts into sharper focus so that we can see reality more clearly. Mathematical symbols, like words, can be arranged in a variety of patterns to convey different meanings. Yet numbers in isolation are like words taken out of context. They communicate very little to ourselves or others. Moreover in teaching language it is generally true that ideas and meanings precede the printed words we attach to them. It is helpful to keep the same principle in mind when teaching mathematics.

The need to divide primary classes into as many as four separate reading groups is well recognized and commonly practiced. It enables children to comprehend what they read and to acquire a

facility with a vocabulary that is appropriate to their interests and levels of ability. Unfortunately there are comparatively few classes that are similarily divided for the purpose of studying the "language" of mathematics. Too often it is felt that what children lack in ability can be compensated for by drill and repetition. Yet a similar practice in the teaching of language would rightly be considered to be bad pedagogy reminiscent of Dicken's **Hard Times.**[4] In language study, and particularly in reading, wherever repetition occurs the emphasis is upon increased understanding along with an increased facility with the use of words. To paraphrase Isaac Newton the purpose is to make facts meaningful and meaning factual. The raw materials of language and mathematics are words and numerals. The symbols composing such statements as "2 + 2 = 4" carry no meaningful truths until we can provide further information that grows out of our everyday experiences. Consider the following cases. When we say that "2 + 2 = 4" do we mean that two dollars Canadian and two dollars American equals four dollars? Can we assume that by adding two litres of water to two litres of alcohol that we will end up with exactly four litres of liquid? Analogously, does "2 x 2 = 4" entail that four degrees Centigrade is exactly twice as hot as two degrees Centigrade in any absolute sense? Obviously, such mathematical statements need to be qualified in the light of the children's everyday experiences. There is a vast difference between rote "learning" that is mere memorization on one hand and an intuitive facility with the meanings of mathematics on the other. A recognition of this difference is particularly crucial in the developmental stages of preoperational and concrete operational thinking in elementary grades. If we wish mathematics to be meaningful it is important to recognize children's limitations in dealing with abstract concepts.

In summary, "getting back to basics" presents teachers with an obligation to re-think what is essential. It is important to distinguish between mathematics for understanding and mathematical facts taken in isolation. The study of language provides a useful model for mathematics teachers and opens up for students a whole new way of viewing mathematics. (The converse is also true). Seen in this light, the back to basics move may provide the mechanism for moving forward to new insights and deeper understanding.

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1. This concern is expressed in the Position Papers of the National Council of Teachers of Mathematics on Basic Skills and by the National Council of Supervisors of Mathematics on Basic Skills in **Arithmetic Teacher,** Vol. 25, No. 1, Oct. 1977.

2. Noam Chomsky, **Aspects of the Theory of Syntax.** Cambridge, Massachusetts: The M.I.T. Press, 1965.

3. Charles Dickens, **Hard Times.** London: Dent, 1963. See Chapters 1 and 2 "The One Thing Needfull" and "Murdering the Innocents". Here Dickens presents a farcical description of the difficulties inherent in getting back to the basics in education.

4. John Phillips, Jr., **The Origins of Intellect: Piaget's Theory.** San Francisco: Freeman, 1969. See also Robert Sund, **Piaget for Educators.** Columbus: Merrill, 1976.

# CHAPTER ITEMS

**Transformation Geometry ... Something to Think About**

**ASSOCIATION MATHEMATIQUE D'OTTAWA-CARLETON MATHEMATICS ASSOCIATION**

It will be interesting, 25 years hence, to look back on the current "transformation geoemtry" controversy: such things always seem so much simpler in retrospect. In the meantime, however, one must learn to cope with the present, and, to this end, we offer the following quotations, the

selection of which is not intended to give preponderance to any point of view.

W.W. Sawyer, in "Mathematician's Delight": "Why, then, do people dislike geometry? Partly because it is a mystery to them: they do not realize (and are not told) how close it lies to everyday life. Secondly, because mathematics is supposed to be **perfect.** There is nothing in the geometry books about shapes being 'nearly triangles' or 'almost triangles'. This perfection puts people off ... but this perfection comes at the end: **experiment** comes at the beginning".

Howard Eves, in his "Introduction to the History of Mathematics", refers to Euclid's Elements in these terms: "No work, except the Bible, has been more widely used, edited, or studied, and probably no work has exercised a greater influence on scientific thinking".

H.S.M. Coxeter and S.L. Greitzer in "Geometry Revisited": "Until our own century, one of the chief reasons for teaching geometry was that its axiomatic method was considered the best introduction to deductive reasoning. Naturally, the formal method was stressed for effective educational purposes. However, neither ancient nor modern geometers have hesitated to adopt less orthodox methods when it suited them ... One such technique is the use of transformations..."

Simone Weil: "Presque toutes nos actions simples ou savament combiné sont des applications de notions géométriques, l'univers où nous vivons est un tissue de relations géométriques, et la nécessité géométrique est celle-même à laquelle nous sommes soumis, en fait, comme des créatures enfermees dans l'espace et le temps". (tiré de "L'enracinement")

John Bolyai, one of the originators of non-Euclidean geometry: "I have made such wonderful discoveries that I am myself lost in astonishment: out of nothing I have created a new and another world".

# TORONTO - YORK CHAPTER

### TORONTO-YORK CHAPTER

**Editors Note:** From our newest chapter. Congratulations!

Dear Bob:

Just want to report for the fall Gazette that we have established a new O.A.M.E. chapter for North York/Toronto, with a full organization meeting scheduled for November. Temporary president is Walter Howard, Jarvis C.I., (Toronto), I am secretary (representative); current name TORN.Y. although the full meeting may want to change that, possibly to N.Y.T.M.A. Until we have a full membership meeting, there are no definite plans for activities, just hopes.

Ed Silcox,
Sir Sandford Fleming Secondary School

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**Be sure to attend**
**Next Fall's Conference**
in
London
1978 11 04
*Further information and registration forms will be available in September.

# THE OAME - OECA MATHEMATICS DISCIPLINE COMMITTEE

Early in December of 1977 the OAME president, Neil Williamson was approached by OECA regarding the formation of a Mathematics Discipline Committee to advise TV Ontario and suggest programming needs. (Other subject areas were approached in a similar manner).

The executive was consulted and an ad hoc committee was formed to meet with the OECA representatives. The responsibilities of the subject discipline committees were discussed and after further consultation with the executive and council it was decided to proceed.

The objectives of the committee are:

1. To ascertain the curriculum needs of the students and teachers which can be satisfied by television-based learning systems.
2. To prepare lists of topics or topic units ranked according to the priority attached to them by the teachers of the discipline.
3. To examine and judge existing OECA materials to establish whether this material satisfies the identified needs.
4. To assist in the selection of areas to be addressed.
5. To assist in formation of a long range production plan.
6. To assist in the design and execution of learning systems.
7. To assist in the design and execution of utilization and delivery systems.
8. To assist in the dissemination of information to the subject teachers through their professional organization.
9. To assist in the selection of project committee.
10. To assist in the evaluation of projects undertaken to serve the needs of Mathematics in the schools.

Thus far the joint committee has achieved the first two objectives by compiling a list of topics and circulating the questionnaire all members received during May 1978. The committee thanks you for your excellent response.

In the fall, the joint committee will meet to analyse the results of the survey and suggest potential programming priorities.

It is the committee's hope that this project will benefit the teachers of Ontario and the members of OAME by providing - better Mathematics programming in all divisions,
- future foci for OAME conference and committee work based on the survey results.

As the committee has the power to add and recommend future resource people, many OAME members will be involved in determining the specific subject matter to be presented in future OECA programs.

All expenses incurred by this joint committee, including the cost of the survey and all meetings are being paid for by OECA.

We are now awaiting the survey results which should be available in time for the next issue of the Gazette.

Committee members:

Paul Pogue - Chairman
Paul Lessard
Sharon McPhail
Neil Williamson 1977-78
David McPhail 1978-79

**Note:** Any member wishing to assist with the future work of this committee should contact:
Paul Pogue

# THE RESOURCE NEEDS QUESTIONNAIRE

**Arlene Zuckernick, Research Officer, The Ontario Educational Communications Authority,**
**July 1978**

Many of you recently received a questionnaire from the Ontario Association for Mathematics Education and may have been wondering what it was all about.

We owe you an explanation. The survey was conducted by The Ontario Educational Communications Authority in conjunction with the Ontario Association for Mathematics Education.

The OECA is concerned with providing programming to meet the specific curricular needs of Ontario mathematics teachers.

A Mathematics Discipline Committee, under the auspices of the School and Youth Programming Branch, was constituted to provide guidance concerning OECA's mathematics programming at the Intermediate and Senior levels. Its membership consisted of math specialists from various school boards across Ontario (representing the OAME) and the Superintendent of the School and Youth Programming Branch.

The essential first step would be to carry out a province-wide ascertainment study determining the resource needs of mathematics teachers.

The Project Research Section of the Authority undertook this study. The sample consisted of active members of the OAME (1400 individuals across the province), including classroom teachers at all levels and support staff.

The purpose of the study was to determine the areas of need of mathematics teachers (K through 13) and support staff for inclass resources to enhance the teaching/learning process.

A survey questionnaire was jointly designed. Its components were a general needs questionnaire to be completed by all respondents and individual questionnaires specifying resource needs for each of the divisions - Primary/Junior, Intermediate and Senior. The list of potential topics was generated by Committee members.

As you are aware, no mention was made of either TVOntario or television in any of the materials. Reference was made throughout only to "resource needs". It was decided that once needs were established in the field by those in the best position to do so - teachers and support staff - and then those with expertise in instructional TV program planning (in conjunction with Committee members), should make decisions regarding the appropriateness of using the medium for instructing in the individual topics.

A public opinion firm was contracted by Project Research to receive, computerize and do the initial analysis of the data.

Your response was quite gratifying. There were 620 questionnaires mailed back by the specified deadline, indicating a return rate of 44%. The breakdown by division was Primary/Junior 140; Intermediate, 470 and Senior, 406. Of course, the actual number exceeds 620 because some teachers and support staff have responsibilities in two or more divisions.

The findings will be detailed in a future issue of the Gazette. These will specify those resources for which you felt there was a strong need - by grade and by division.

In addition, you expressed preferences about the function such resources should fulfill, their scope within the curriculum and for what learner groups such resources should be developed.

We sincerely thank you for assisting us in determining mathematics resource needs. Any further resources we can provide to you will benefit from your involvement.