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EDITORIAL

New Curriculum Guidelines for the Intermediate Division are needed. I.12 A,B,C,D were the result of the "New Mathematics of the Sixties. They proved unrealistic, yet gave impetus to change in both subject matter and pedagogy. Now, we await solid, well-conceived goals and equally as important, a philosophical basis and rationale for an Intermediate Division Curriculum. But such an event does not happen without much toil and work and consultation. Thus, this editorial and the theme for our next issue.

First an overview of what has gone on thus far.

1. The Minister of Education is committed to a policy of cyclic review. In 1975 **The Formative Years** was published setting out the goals for the Primary and Junior Divisions. Along with it was published the booklet **Education in the Primary and Junior Division**, which provided an extensive philosophical basis and rationale for the programme. It also indicated how the programme expectations might be achieved in an integrated and child-centred framework. These had been accomplished as a result of curriculum review.
2. About the same time, a working draft of a proposed document titled **The Intermediate Years**, and in 1977, sections of the draft copy, **Intermediate Division Mathematics** gave us a philosophical basis and rationale for a Mathematics programme within the context of the adolescent years. The goals for the Intermediate Division are in the latter booklet and were enlarged on in modules called **Outline of topics**. A later version, **Notes for Teachers,** included suggested development and resources as well. Thus, in Primary, Junior and Intermediate Divisions we may have a suggested set of goals and separate from these suggestions about pedagogy.
3. In the March 1975 issue of this journal your editor of the day, Arn Harris stated that the three most important aspects of the (Pogue) report were: (i) a strong commitment to an experimental approach, with emphasis on development of problem solving skills through "real life" examples, (ii) the inclusion of statistics in the core and (iii) continuing emphasis on transformations in geometry. I happen to agree with this statement. So did the writing team, which produced the draft guidelines. In spite of their efforts, perhaps because of the undue haste of the validation process last spring, their work went unrewarded. Little is achieved in a vacuum and a position paper often gives way to better things. The start is often the most difficult. The evident time-line difficulty was alleviated by the following:

April 11, 1978
R.A.L. Thomas
Assistant Deputy Minister

Extension of Time Line For Validation

September 1978 - March 1979	classroom field-testing
March 1979 - April 1979	validation

4. This time-line extension gave rise to a new committee.

Excerpts from a talk given by G. Scroggie at Math '78 — June 2, 1978

"A new committee is being formed . . . . responsible for planning strategies for field testing the guideline."

5. Letter from M.J. McKenna, Chairman, Intermediate Mathematics Validation Committee — October 23, 1978

The preliminary draft of the Intermediate Division Mathematics Guideline was sent to the schools in March of 1978. In order to organize a validation of this draft guideline, a committee including nominees from OTF and subject organizations, and the Ministry of Education has been formed. The membership list is appended at the end of the article.

The committee has three objectives:
(i) to organize a validation process;
(ii) to submit recommendations based upon the input to a reconstituted writing team (the recommendations must be implemented);
(iii) to begin the design of a complete review and redevelopment package to be applied to the senior division mathematics guideline.

The initial task has been completed and will be described in a Ministry memorandum to directors of education and principals of schools. For clarity, the validation process is described herein.

In response to the request for validation, it is hoped that jurisdictions, and/or individual teachers within these jurisdictions, subject organizations and other professional organizations will submit responses. These responses will fall into two categories:

I. **General Suggestions for Change**

These suggestions can be organized using the following criteria;
(a) suggestions can be editorial in nature (e.g. grammar, etc.) or they can be substantive in nature (e.g. change in content, philosophy);
(b) each suggestion must be accompanied by a supporting rationale for it to be of value to the committee;
(c) suggestions should be based upon current classroom experience and current inclusion in local course outlines.

II. **Field Testing**

This involves classroom testing of new topics and can be organized as follows:
(a) consider one or more new sections in the draft guideline for field testing;
(b) organize a summary of the field testing with respect to materials used, time spent, observations from teachers' standpoint, observations from students' standpoint, and recommendations with reasons;
(c) Wherever possible, organize responses across a jurisdiction.

It is recognized that field testing will require resources for classroom planning and use. Notes for Teachers, in draft form, will become available to boards over the course of this school year; these will expand upon the Outline of Topics at each grade level. In addition, the Ministry will prepare a list of other resource materials. It is requested that boards share any material they currently have available by sending a copy to the Intermediate Mathematics Validation Co-ordinator at the following address:

Mr. M.J. McKenna
Intermediate Mathematics Validation Co-ordinator
Ministry of Education, Northeastern Regional Office
P.O. Box 3020
North Bay, Ontario
P1B 8K7

The list will be updated as new materials are added to it.

COMMITTEE

Jacques Lamoureau, MacDonald-Cartier S.S., Sudbury
Marilyn McPhee, Ryerson Senior Public School, Hamilton
Bill Collier, St. Florence Catholic School, Scarborough
Alex Norrie, Mathematics Co-ordinator, Peel County Board
Wyn Bates, Director of Mathematics, Toronto
Fred Burford, Principal, Downsview S.S., Downsview
Jack LeSage, Eastview S.S., Barrie
Sam Pravica, Faculty of Education, The University of Western Ontario
Don Roberts, Beamsville S.S., Beamsville
Wilma Dowling, Waterdown S.S., Waterdown
Maurice Poirier, Ecole Secondaire Garneau, Orleans
Elaine Harvey, Baydridge S.s., Kingston
Peter Weygang, Thomas Aquinas School, Kenora.
Claire Zeller, Morning Star Drive S.S., Mississauga
George Scroggie, Ministry of Education, Toronto
Louise Fleming, Ministry of Education, Toronto
Jim MacLean, Ministry of Education, Toronto
Michael McKenna, Ministry of Education, North Bay (Chairman)

6. Letter from Committee Member Sam Pravica — 1978 11 1

What is being validated

Those directly involved with the field trials will receive
(i) a copy of the Intermediate Division Mathematics Guidelines, 1977.
(ii) teacher's notes for each strand or topic being field tested.
(iii) two questionnaires, Form A and Form B.

Please keep in mind that it is **not** the teacher's notes that are being validated but rather the curriculum guidelines. The teaching notes are support resources in cases where Board developed or teacher developed materials are not available. The committee feels that a fair trial of each topic can only be achieved if teaching notes are available particularly for topics that are unfamiliar to teachers.

A Personal Concern

Each Board through the Director of Education will be responsible for its own field trials. Not all teachers of Intermediate Division Mathematics will take part in the field trials. On one hand the volume of responses would be unmanageable and on the other, statistically, it is not necessary to achieve satisfactory validation. The coordinator of the validation process for each Board will be asked to submit a summary of the responses from the individual questionnaires. In certain cases it is recognized that submissions of individual responses may be appropriate.

I am of the firm belief that the Ministry is committed to the validation process, otherwise I, and I am sure I speak for the other members of the committee, would not subject ourselves to this intensive and time consuming task. Like many of you, I have, in the past known the frustrations of written concerns gone seemingly unheeded. Sitting on the other side of the fence I now understand the reasons.

Subjective decisions are extremely difficult to make. One attempts to solicit opinions from as broad a base as possible. However, ultimately the strength of the decision rests on two major criteria; is the opinion that of the majority and is the opinion supported by a sound rationale?

It is also difficult to make decisions based on a minority response, for, what does a response without a supporting rationale mean, complacency or apathy? The most discouraging aspect of conducting a curriculum validation is the lack of total commitment from the field. Many good ideas have been set aside because of a lack of majority support.

It will not be difficult to pass judgment on opinions based on emotion. It may be true that a point of view has merit, but without a clear concise and rational reason to support it, it merits no more weighting than any other biased opinion.

As a member, I am impressed not only with the membership's mathematics expertise and teaching calibre, but also in the way each member has committed himself to represent the position of those who will be required to implement the final document. It seems that one philosophy prevails; however good the mathematics may be, if it's not "teachable", or "learnable" at a particular level then it should be modified or dropped. The members of the committee are all intent on publishing a document that is sound mathematically, good for the future education of its recipients, teachable and appropriate for the learner at his level of maturity and understanding.

Thank you Sam, for your thoughts. We expect to hear from you again next issue. The communication is useful.

The outline of events given above provides a setting for you, the readers. Intermediate Mathematics Guidelines are needed. I am only too aware that the March issue will soon be upon us - yet the September issue would be too late to address this ongoing concern. Share what you are doing with the rest of us. How suitable is the content of the Grade 9 Level 2 Numerical Methods topics; the Grade 10 Level 4 Geometry; the Grade 7 Algebra? Do you agree with the philosophy of Motion Geometry for the Intermediate Division? What results are you achieving in Data Graphs or Transformations at the Grade 7 Level; in Flow Charts or Dilatations in Grade 9 Level 4; in Problem

Solving in Grade 10 Level 3; etc.?

Good for Dr. Usiskin's article in which he argues **The Case for Transformations in High School Geometry.** This is an issue here in Ontario, in our Intermediate Division Guidelines. If your attempts at measuring & cutting floor tiles have not convinced you of a practical application, perhaps Dr. Usiskin's article or the Interview Brock Rachar had with Dan Moalem from Australia will.

I sometimes wear this button to class.

Bob Smith



LETTERS TO THE EDITOR

Lakehead University,
Thunder Bay, Ontario
October 5, 1978

Dear Prof. Smith,

As my children progress through school I occasionally visit their classrooms. On one 'meet the teacher' night I was glancing at the wall charts in a grade 4 classroom when I saw words that were new to me. In my years of obtaining a Ph.D. degree, and after teaching mathematics for many years at the university level, I had never come across 'minuend', 'subtrahend', 'addends', etcetera.

After some reflection, the meaning of the words became clear, but I wondered how I had managed to progress in mathematics without them. Perhaps arithmetic and mathematics could be made easier for children if they were taught how to use elementary addition, subtraction, multiplication and division without using extraneous words that seem only to complicate the processes.

Another perplexing problem came with a grade 6 problem on order of operations. I could not solve $(2 + 18 \div 6 - 1 \times 3)$ without going to the textbook. In practice parentheses (brackets) are used to indicate the order of operations, so remembering a rule about doing multiplications and divisions first, from left to right, and so on, is worthless.

The point about examples like $(2 + 18 \div 6 - 1 \times 3)$ is that you should **not** write anything like it. Brackets are necessary to show what numbers and operators are linked, but I feel no teacher should spend class time on teaching a rule for such expressions, just say such things are chaotic - that brackets are used to make things clearer.

Incidentally, I still have to concentrate to get numerator and denominator the right way up and often use the terms 'top' and 'bottom'.

Perhaps there are other areas of elementary mathematics where excess of terminology is clouding the child's view of the operations themselves.

J.G. Griffith

Editor's Note: Readers, What are your feelings? Read Sam Pravica's reply.

Professor R. Smith,
Editor, Mathematics Gazette.

A Response to a Letter to the Editor

At last, someone has found the courage to publicly state a concern which I have attempted to raise on a number of occasions with educators but have met with one type of put down or another and, rather than upsetting establishment I have let my frustrations simmer.

For some time now, I have been seriously questioning what we are trying to accomplish in treating most children as though they will ultimately become mathematics majors by having them understand mathematics at the synthesis level ad naseum. It appears to me that we are turning children off mathematics more and more rather than on, which is contrary to the intent of the new approach.

It never ceased to amaze me that we insist on teaching elementary school mathematics to children at a level at which I have consistently seen mathematics majors have an initial difficulty. Further, current research indicates that the majority of elementary school teachers are not operating at Piaget's formal operations level in mathematics. If we agree that Piaget's stages of conceptual growth has some validity does it not seem ludicrous that the mathematics we are attempting to teach children is as difficult for them as it is for those who are attempting to teach it?

We need more exposure to these kinds of concerns. We also need more research to identify the depth as well as the breadth of children's ability to grasp mathematical ideas at all levels . . . .

Yours truly,

Professor S. Pravica,
Department of Mathematics-Science

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Ladies and Gentlemen,

One of our customers is interested to know **The Availability and The Price** for Ontario Mathematics Gazette, Vol 1 up to 12 inclusive. Please quote price and availability!

Yours faithfully,
Exlibris Buckhandelsgesellschaft.

Editor's Note: It seems we are becoming known.

Assumption High School
Burlington, Ont

Dear Professor Smith:

Enclosed a short article which may be of interest to the readers of the Ontario Mathematics Gazette. Title: **Inequalities: Teachers' and Students' Nightmare**

When teaching inequalities at the Grade 9 level (advanced) I slipped, accidentally, the following example in a test: Solve the following inequality: $\frac{9}{-x} < 3$

which generated worthwhile class discussion. Two solutions are possible.

(a) $x < -3$
(b) $x > 0$
values $-3 < x \leq 0$ do not satisfy the inequality.

Conclusion: In an inequality, when the denominator includes a variable it is necessary to define the **domain of the variable.**

O. Pilibbossian

San Diego
April 12, 1978

Editor's Note: As I sat in the audience at my first meeting of N.C.T.M. I wondered how John Egsgard felt as he brought the meeting to order as president of N.C.T.M. His responsibility was a great one. The following is the result of my request to him to reflect & let us know.

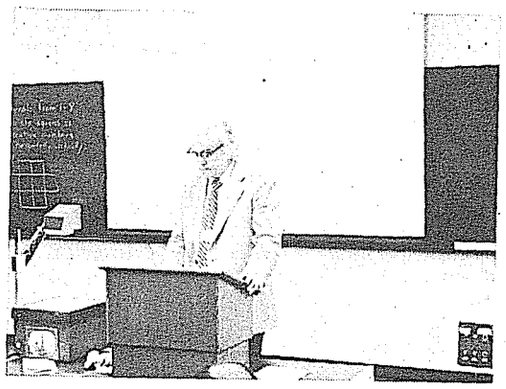
Dear Bob,

May 4, 1978

During the summer I shall write the article that you requested for the gazette regarding my term as President of the NCTM.

Sincerely,

John C. Egsgard,
Past-President, NCTM



WHAT IS IT LIKE TO BE PRESIDENT OF THE NCTM?

What is it like to be President of the National Council of Teachers of Mathematics? Well, first of all it means that someone else reserves your hotel room for you when you go to a meeting. And if the meeting is one of the eight name-of-site meetings or the annual meeting held each year, then the room is a spendid suite If you are a classroom teacher like I am then your students sometimes think that you a "substitute teacher" for them. During my two years as President, year as President-elect and year as Past-President, I will have been away from school on about two hundred and fifty teaching days. Unlike most teachers, my weekend as President began on Wednesday. Unfortunately, my weekends were spent away from my family at committee meetings NCTM conventions, and meetings of other organizations. Last fall from the beginning of September to the end of November I spend every weekend away from Orillia.

The NCTM expends much energy in trying to improve the teaching of mathematics. Its Board of Directors, headed by the President, meets there three times a year for three or four days to make decisions about future trends for the Council and to study recommendations of its ten standing committees. These standing committees are concerned with all aspects of mathematics education - its three journals, the Mathematics Teacher, the Arithmetic Teacher, and the Journal for Research

in Mathematics Education; its 150 other periodicals) its meetings; its relationship with affiliated groups such as OAME; its relationship with other organizations; current mathematics education problems such as the hand-held calculators; and finances. (The annual budget of the NCTM is almost two million dollars).

If the President feels, as I did, that he must know the work of the NCTM in detail then he attends at least one of the two meetings of each committee every year. To decide whether or not the Council is, in reality, sensitive to the needs of its member, he also goes to each of the nine conventions each year. Besides delivering a major address, he makes himself as visible as possible so that he can learn what the members want the NCTM to do.

With interest in committees and members comes correspondence and a secretary. Fortunately, I had a dictating machine so that I easily finished my correspondence nightly by 11 p.m. provided I began by 7:30 p.m. (My excellent secretary was the wife of one of our teachers so that he brought the letters to school with him each day. It certainly impressed the students to see me read and sign ten to twenty letters daily).

Then, of course, being President of NCTM means you go to exciting cities in Canada and the United States such as, Toronto, New York, St. John's, Boston, Victoria, New Orleans, Winnipeg, San Francisco, Missoula, Phoenix, Colorado Springs, etc. etc. "What is _____ like?", the students would ask me upon my return, if they recognized me. "I don't know", I would answer for I only saw what the inside of one of their hotels looked like. Travelling does have some advantages, of course. For example, I was able to sneak in a few days to go skiing at Alta., Sun Valley, Killington and Collingwood.

Newspaper and TV people seemed to be very interested in my opinion about the hand-held calculator, the back to the basics movement, and other current topics in mathematics. I always explained to them that the way to ensure the solution to all problems in mathematics education was to make sure that all mathematics teachers were *good mathematics teachers.* Many times I had the opportunity to tell teachers what I thought was a good mathematics teacher. All of us have our own ideas about this. Fortunately, people wanted to know what my ideas were and I was anxious to tell them. Being able to find an audience to listen to ones' ideas is great! It is sobering too, because it forces one to examine these ideas carefully and to find the means to explain them simply. One of the opportunities that I had along this line that I enjoyed the most was writing an article for each edition of the NCTM Newsletter that comes out four times a year. (Yet, it was disconcerting to learn from meetings with affiliated group presidents at conventions that some NCTM members do not realize that they receive a Newsletter).

The thing with which I am most pleased concerning my term as President of the NCTM is the response I received from the journals and committees as I emphasized that it was extremely important for the Council to keep the needs and interests of the elementary and secondary school teachers in mind in all of its works. In order to help committees realize what these needs were, I appointed as many elementary and secondary school teachers to committees as I could. Unfortunately, many subject oriented organizations in the United States, unlike OAME, have a leadership made up almost entirely of teachers from university and colleges of education.

Finding members for committees was a fascinating experience for me. Even though I would have been able to fill all committee appointments using teachers from Ontario, I could not do so. It was necessary for me to take into consideration the fact that NCTM has members from all over Canada and the United States and from all levels from kindergarten to college. Canada with about 3% of the 80,000 NCTM members should have about 3% of the 86 non-elected members on committees, namely two people. At present there are six non-elected NCTM members on committees, two of these, Dave Alexander and Bob Robinson, are from Ontario. Incidentally, it helps the president greatly to have recommendations regarding committee membership from affiliated groups like OAME.

I did enjoy being president of the National Council of Teachers of Mathematics. It gave me the special opportunity to discuss the teaching of mathematics with teachers from all over the United States and Canada. Unfortunately, it kept me away from my family and my students. I am very glad to be back with both of them again.

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1. Give change for 55 cents in two coins one of which is not a nickel. Answer on pg 64

PLUS + + +
AN INFORMATION COLUMN

Ed Barbeau

The aim of this column is to provide communication among mathematics educators across Canada; to tell you about important events, research, curriculum development and items in the literature of national interest. It is to be inserted in mathematical education journals throughout the country.

Your help is needed in the gathering of information which may be of national interest. Ed has agreed to be the mailbox for Plus. Material should be sent to: *Prof. E.J. Barbeau,*
University College, B201
University of Toronto
Toronto, Ont. M5S 1A1

79.1 The Canadian Mathematics Education Study Group

Twice, in September, 1977 and in June, 1978, a group of mathematics educators, mostly from university mathematics departments and faculties of education were invited to participate in conferences on the state of mathematical education research and teacher training in Canada. The proceedings of the first conference, "Educating teachers of mathematics: the universities' responsibility," published by the Science Council, has appeared recently. This includes the texts of addresses by T.E. Kieren (University of Alberta), C. Gaulin (University of Laval) and A.J. Coleman (Queen's University) as well as the deliberations of the working groups.

At the second conference, besides invited addresses by G. Noelting (Quebec), G. Rising (Buffalo) and I. Weinzweig (Chicago), there were workshops at which discussions begun at the first conference on teaching, training, mathematization and educational research were carried further.

In order to maintain the momentum of discussion and to keep the participants in touch, it was decided to form the **Canadian mathematics education study group**, a loose organization that will function through local, regional and national initiatives.

Further information can be obtained from the organizers, Professor A.J. Coleman (Mathematics Department, Queen's University, Kingston, Ontario K7L 3N6), Professor W. Higginson (Faculty of Education, Queen's University) or Professor D. Wheeler (Mathematics Department, Concordia University, 1455 de Maisonneuve Blvd. W., Montreal, P.Q. H3G 1M8). The address of the Science Council is 150 Kent Street, 7th floor, Ottawa, Ontario K1P 5P4.

79.2 PERMAMA: Inservice teacher training

In 1972, an innovative inservice teacher training programme, "PERfectionnement des MAitres en MAthematique," (PERMAMA for short), was established by Tele-universite du Quebec, a branch of the University of Quebec. Intended for high school mathematics teachers, it is a credit programme leading to a certificate. Participants register for courses in one of 97 local or regional centres. They work through repared modules and meet regularly for discussions directed by a "moniteur-animateur" at the centre. About 60 modules are currently available.

A fuller account is found in the Science Council of Canada report "Educating teachers of mathematics: the universities' responsibility" (pp. 32-36) (see item 79.1) or may be obtained from Professor Claude Gaulin, Faculte des Sciences de l'Education, Universite Laval, Quebec, P.Q. G1K 7P4.

79.3 Survey of educational research in Canada

Dr. D.R. Drost, Department of Curriculum and Instruction, Memorial University, St. John's Nfld. A1B 3X8 is compiling a catalogue of Canadian research related to mathematics education. Please send a list of studies currently in progress or completed in the last five years, including name of

researchers, title of study, grade level and date. These might be graduate theses, personal published reports, both internal and external, or assessment studies. This is the first step of the research subgroup of the Canadian Mathematics Education Study Group towards making knowledge of the work of Canadian researchers more widespread.

I'VE GOT A MILLION - HOW ABOUT YOU?

Ann Lukasevich, Faculty of Education, University of Western Ontario
Irene Gauvin, Fairmont School, London Public School Board

"Nothing should be learnt by heart that has not been thoroughly grasped by the understanding".
Comenius "The Great Didactic."

Understanding is developed without many problems when the teacher introduces the child to small quantities such as 6, 1 dozen, 25 ... For the teacher can easily bring in 6 apples, a dozen eggs, or 25 blocks. But what does the average teacher tend to do when she much teach concepts such as 600, 24 dozen or 250,000? She tends to revert back to abstraction with no manipulative materials, feeling that it would be too difficult to represent such large numbers with concrete material. This is an unwise plan, for large numbers rather than small numbers tend to create the greatest problems for the elementary child. As a result, daily, students present their teachers with answers that are unrealistic and show a lack of understanding of large numbers. It's no wonder when Billy is asked how many groups of 50 are in 874,000 he responds with 50 or 60 which greatly underestimates the true quotient. Or when Jason is asked to multiply 6450 by 372 he obtains the answer 32994. Why does he fail to question the validity of such a product? Is it because he lacks a true understanding of numbers of this size?

Can an 8 year old picture 2000 books, 25,000 cars or 956,000 letters when he has had little experience with numbers of this size in real life? This problem becomes more evident when one begins to question a child about his/her personal possessions.

"How many pennies do you have in your piggy bank?"

"Hundreds, because it is so heavy!" In actual count it may only be fifty. Why do elementary school children respond this way? Probably because their teachers either fail to provide experiences with large numbers or fail to make them aware of how large numbers are utilized in everyday life.

To help overcome this problem there are a number of things the elementary school teacher could and should do.

As early as kindergarten or grade one, students can be taken on number walks where they are asked to look for specific numbers, large and small. Such numbers will be found on houses, stores, street signs and license plates. This information could be recorded on a frequency graph or an experience chart on return to the classroom.

Almost daily primary children eagerly bring in news about themselves which could be recorded in their personal diary or on a class news sheet. Often these items may have a number bias eg Johnny reports that it is his **eighth** birthday and he received a bag of **101** marbles. A more sensitive awareness to numbers and their importance in everyday life can be developed by recording such information in a class Number Newsbook. Attractive illustrations can be added by the students.

As the child's understanding of numbers progresses through the elementary grade he/she meets larger and larger numbers which often appear to be out of his experiental background. This understanding can be furthered through a variety of topics which can be developed at almost any grade level depending on the ability of the child.

Children often hear adults talking about census counts and polls that have been conducted. Why can't children conduct their own polls or census counts on things of interest to them? In the early primary grades when children are talking about themselves they can tally the number of hands in their family, in their classroom, and in their school. In order to deal with even larger numbers, this

can be expanded to fingers, toes, weight, pets, number of hours of sleep, and life span in years, months, days, and minutes depending on the age group. This could lead to pictorial representation, graphs and other forms of mathematical recording.

Field trips at all grade levels have the potential of providing excellent opportunitites to work with large numbers. A trip to the grocery store should be much more than looking around at fruit, vegetables and piles and piles of cans. By the techniques of estimating the contents of one section of a shelf, the students should be able to approximate the total number of canned goods on display on the whole shelf or even the whole store. Containers of various sizes and shapes can lead to interesting experiences estimating and measuring capacity. Through these experiments with volume, the children will soon realize that one cannot accurately estimate the capacity of a container by its shape. They might find that a tall, thin container which appeared larger held as much as a short fat container. A visit to the store's bakery will allow children to see their favourite recipe multiplied by 50, 100 or maybe 150. At home Tim will use 500 ml of flour for chocolate chip cookies whereas the baker will use 50,000 ml of flour. The students may also wish to interview the store manager to find out specific sales information (How many loaves of bread, jugs of milk... are sold in one day, one week, one month and one year?).

Field trips to the lumber yard, the post office, the zoo, the apple orchard and on Ontario farm may also be used to provide interesting mathematical data with large numbers. Visits to these types of places may lead to interesting questions elicited through class discussion:

- How many planks of lumber are needed to build one, 3 bedroom house?
- How many shingles are needed for the roof?
- How many letters are sorted in 1 day, 1 week, 1 month, or 1 year?
- How many parcels are sorted in 1 day, 1 week, 1 month, or 1 year?
- How many bales of hay does Daisy the elephant eat in one week?
- How many people visit the zoo daily and weekly?
- How many cm long is the giraffe's neck?
- How many apples does a tree produce yearly?, in its life span?
- How many apples are needed to fill a bushel?
- How many litres of milk will Bessy produce in 1 day, 1 week, 1 month, 1 year and in her life span?
- How many kg of potatoes are produced on the farm in 1 year?

Besides field trips a lot of numerical information can be gained by looking at the school itself and the children's own homes. The children may count the number of light bulbs in their home. From this information a grand total for the class may be tallied. Other information they might wish to collect about their homes and community might be the number of windows, doors, pets, flowers, trees, bicycles, chairs, books . . . Older children might do a tree count of the school district. This may lead to ecological concerns such a lack of trees within the school district.

Almost every mathematic concept may lead to a fun provoking activity. Large numbers are no exception. Any child will be intrigued by guessing the correct number of jelly beans in a huge jar. A class contest could be set up. Once the contest is over the contents could be replaced with a variety of other materials such as beans, acorns, small blocks, beads, pennies, walnuts. . . Collections can also lead to a lot of fun during the year. The class might start a string collection to see how long a ball of string they can make by tieing various lengths of string together. It might turn out to be over 100 m long. Each m might be marked off in red. The finished ball then could be used as a large tape measure to measure the length of the classroom, the gym, the school yard or even the block. In fact, 2 classes might even make it into a contest. Another alternative for collections are bottle caps. Several classes or the school may get together to see if they can collect 1,000,000. A variety of containers could be used for storage. They may be stored in groups of 100, 500, or 1,000. A weekly or daily tally could be recorded and posted.

for a million bottle caps	
693 659	306 341
WE HAVE	WE NEED

It may not be the same as winning a $1 000 000 lottery, but for young children it's a sure way to make a million.

TEACHER EDUCATION - A BRITISH VIEW

Mrs. E. Allan, Senior Lecturer, Mathematics, Christ Church College, Canterbury, Kent England

I don't feel I have very much to say that is new to you on the subject of our aims in the Mathematics Department in dealing with First School students. You have all been through Curriculum Mathematics classes yourselves. What we're trying to do differs very little, if at all, from what the Mathematics departments of all the colleges in the country have been trying to do for a very long time.

Briefly, our aims are:-

1. to ensure that teachers-in-training are competent themselves in the Mathematical field in relation to the level at which they are going to teach.
2. to develop a coherent scheme of teaching (mainly Arithmetic), to discuss various methods and to point out the pitfalls.

We are never satisfied with our efforts to ensure students' own competence. More and more of our students have their 'O' Levels in Mathematics - which is a sound beginning. Nevertheless, we give them a test on entry and find the most peculiar deficiencies. Some of these arise out of the widely differing 'O' Level syllabuses throughout the country. Roughly those who have followed an S.M.P. or other modern syllabus, have very little competence in elementary Algebra and an almost total ignorance of elementary Geometry, whereas those who have followed a traditional course find it hard to understand why certain modern topics appear on the syllabus at all. Generally, however, we find that students are unable to deal with the manipulation of simple fractions, even of decimal fractions. They find multi-stage problems from 11+ papers extremely difficult and numbers of them have not yet come to terms with the metric system.

This is our raw material, so to speak. It could be argued - though not very cogently - that because they may not be called upon to teach any pupils above first-school level they do not require any breadth or depth of knowledge or skill in Mathematics. We reject this argument for obvious reasons. The teacher must always look ahead to the upper classes to which his pupils proceed and be sure to give them a grounding and a skill, sufficient to cope with the needs of these upper classes. First-school teachers look towards the junior classes and junior teachers look towards the Middle school where every child undertakes scientific experiments and seldom, if ever, meets integral results. Then, as far as possible, the teacher should know why certain topics are being taught and towards what end each lesson is leading. He should also have the necessary background to exercise sound judgment as to what is essential and what mere time-absorbing trivialisation. Teaching for instance, tessellations without extracting the elementary geometry to which they lead, transmutes the Mathematics lesson into an art and craft lesson. Little Mathematics necessarily emerges from folding paper through blots of ink, nor even from finding the symmetry number of a tetrahedron. I was in a class of 13 year olds the other day and the pupils spent two full periods doing nothing more than going up ladders and down snakes ostensibly learning about co-ordinates. I am similarly doubtful of the value of teaching Module Arithmetic in junior or first schools. The topic apears in secondary syllabuses largely as a means of providing laboratory material for the study of structure. It is an open question whether school mathematics should be built on structures. The young teacher needs a background of knowledge against which to rate innovations (especially at the present time), the confidence to discard a great deal and to recognize dilution, trivialisation and waste of time for what they are.

It is unnecessary to say that the emphasis in our First school curriculum courses is on teaching number and the four rules. Without competence in these children are handicapped for life. A great deal of hard work is required of first school teachers in this field. The children learn to count and then to compute. They must acquire skills and these are never acquired without concentration and effort. There is no 'play-way' to numerical or mathematical skills. It is wrong and misleading to represent computation as a game, Arithmetic or Mathematic as something you do when it is easy but abandon when concentrated thought is required. However, the rewards are great. In no other subject is it possible to achieve such complete success and excellence. It is, therefore, the confidence-booster: children ask for more when properly taught. Enthusiasm, coupled with a background of knowledge in the teacher, are the keys to success; the success of seeing his pupils acquire skills which will serve them well, all through their lives.

So our main effort in First school curriculum classes is directed towards the teaching of counting, place value, the four rules, the difficulties and common errors, the need for grading examples, the need to approximate and check. We deal with the teaching of all the measures - a far easier thing to-day, now that metrication has arrived.

A lot of new topics have bene introduced into the curriculum in the last 10-15 years. Some have been laboured and have diverted attention away from basic number work. Very little indeed of truly modern mathematics extends to the first school. What little does, is laboured and often trivialised, because the basic ideas are far beyond first school pupils. "Algebra before Arithmetic" runs one commercial advertisement completely ignoring the fact that algebra is a generalization of arithmetic which can logically only follow Arithmetic and even then, only at a stage when the child is capable of generalization. The single exception to this might be sets, which are basic to human thought in any case. We make quite sure that our students have a sound background in set operations. Graphs - which may be looked upon as traditional - have come down to the first school and we set the students a high standard in the production of these. It is important that they do not produce graphs without axes or scales and that they do not use a continuous curve when only a bar graph is appropriate and so on. So we deal with graphs in detail in their course.

Elementary Geometry seems to have arrived in the Primary schools - under the high sounding name of 'Space and Spatial Relations!' - at almost exactly the same time as it was going out of the Secondary schools as a systematized study! How unfortunate! In our course we have to attempt to make up for this deficiency, so that our students are equipped with sufficient background knowledge to teach the subject properly, sequentially and systematically.

We do not feel that we have 100% success in our efforts. We do the best we can in the time available. Briefly I think we attempt to ensure that the student know the wheat from the chaff, that they concentrate on the substance and do not pursue the shadow in their teaching of Mathematics. Some very long shadows have been cast in Mathematics teaching in primary schools in the last 10-15 years, largely, commercialized shadows. Our aim, along with the Maths departments of all Colleges in the country, and in other countries I've visited in the last few years, is to make sure that we deal with essentials first, and allow nothing whatsoever to "stain the white radiance" of mathematics. We simply wish we'd had more success - but we continue to try. The climate of public opinion has enormously changed in the last year and a half.

FOUR MEASURES vs. TWO MEASURES GIVEN IN AREA/PERIMETER PROBLEMS

A MINI RESEARCH

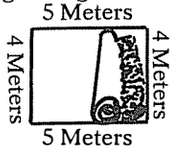
A.V. Buffington, Maryland State Department of Education, P.O. Box 8717, BWI Airport, Baltimore, Maryland. 21240

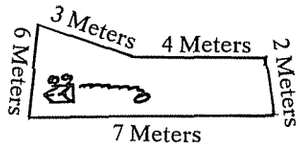
Audrey Buffington is a State Specialist in Mathematics, Baltimore, Maryland. Perhaps this example may serve as a model to help you in your search for answers.

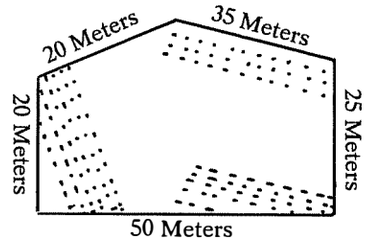
INTRODUCTION

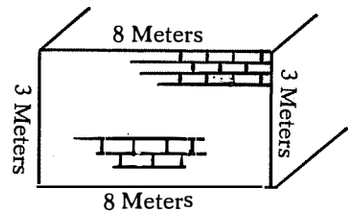
On the Maryland Functional Mathematics Test recently piloted, there were three items written to test the objective dealing with perimeter and area. The average item difficulty for the perimeter problems was 0.69 but for the area problems was 0.27. The analysis of distractor choice for the area problems indicated that an average of 58% of the student chose the distractor which was the perimeter of the rectangular region.

For each item, a picture of the region was drawn -- garden, living room floor, field, park -- and the measure of **each** edge was given. The following are typical examples:









Remembering that in many mathematics textbooks and other print materials, giving the measure of **each** side of a rectangle is usually associated with perimeter and giving the measure of only one of each pair of parallel sides is usually associated with area, the need for research seemed important and timely.

No related research has been found in the literature to date.

Null Hypotheses

Editor's Note: A Null hypothesis is a statement, presented in negative form, concerning results a researcher may expect to find where method A and method B will bring about the same results. The researcher then sets about to demonstrate that the results are in fact different for methods A and B, thereby disproving the negative hypothesis. This method of stating the hypothesis negatively is useful in cases where the results are likely to be different for methods A and B but are nonetheless difficult to predict in advance..

1. There is no significant difference between responses of ninth-grade students on tests of area of rectangular regions where the measure of each of the four sides is given and the responses of ninth-grade students on tests of area of rectangular regions where the measure of only one of each of the pairs of parallel sides is given.
2. There is no significant difference between responses of ninth-grade students on tests of area and perimeter (problems given in area/perimeter sequence) of rectangular regions where the measure of each of the four sides is given and the responses of ninth-grade students on tests of area and perimeter of rectagular regions where the measure of only one of each of the pairs of parallel sides is given.
3. There is no significant difference between responses of ninth-grade students on tests of perimeter and area (problems given in perimeter/area sequence) of rectangular regions where the measure of each of the four sides is given and the responses of ninth-grade students on tests of perimeter and area of rectangular regions where the measure of only one of each of the pairs of parallel sides is given.

Test Instrument Development

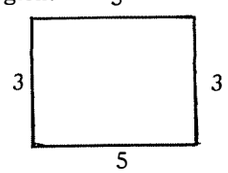
Six mini-tests of six problems each were developed. All measures were intentionally given as whole numbers between 0 and 10 so that, hopefully, computational errors would be avoided. No labels were given either in the problem or in the four multiple-choice responses so that additional cues of square units versus linear units would not be provided. Each test consisted of six items. Examples of the three groups of tests follow.

A **Test I 1 and II 1 were both on area.**

e.g. #1

1. What is the area of this region?

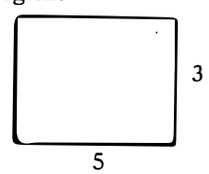
a) 32
b) 16
c) 8
d) 15



e.g. #2

1. What is the area of this region?

a) 32
b) 16
c) 8
d) 15

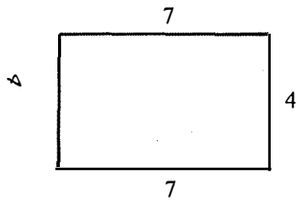


B **Test I 2 and II 2 were in area/perimeter sequence, 3 on area, 3 on perimeter.**

e.g. #1

1. What is the area of this region?

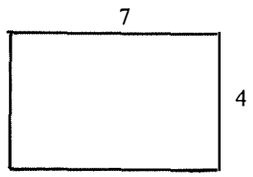
a) 22
b) 28
c) 44
d) 11



e.g. #1

1. What is the area of this region?

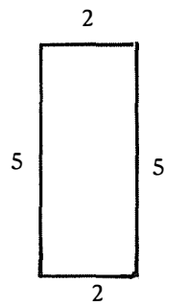
a) 22
b) 28
c) 44
d) 11



e.g. #4

4. What is the perimeter of this figure?

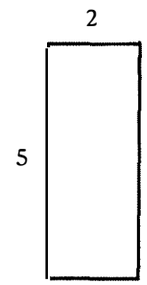
a) 10
b) 7
c) 14
d) 28



e.g. #4

4. What is the perimeter of this figure?

a) 10
b) 7
c) 14
d) 28

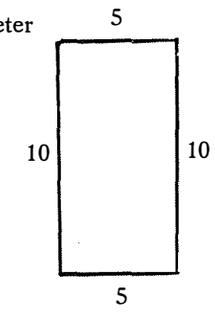


C **Test I 3 and II 3 were in perimeter/area sequence.** Again 3 on perimeter, 3 on area.

e.g. #3

3. What is the perimeter of this figure?

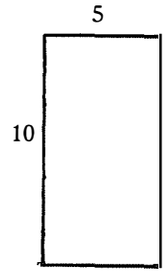
a) 50
b) 30
c) 60
d) 15



e.g. #3

3. What is the perimeter of this figure?

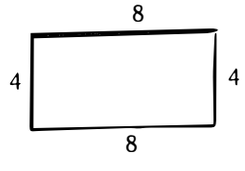
a) 50
b) 30
c) 60
d) 15



e.g. #6

6. What is the area of this region?

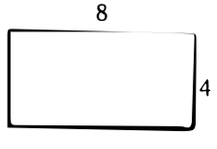
a) 48
b) 32
c) 24
d) 12



e.g. #6

6. What is the area of this region?

a) 48
b) 32
c) 24
d) 12



Test Sample

Copies of the six mini-tests were given to a mathematics supervisor/coordinator in each of the 23 counties and 1 city.

Those who had the time and/or opportunity were asked to have a teacher administer the test to ninth-grade students with no pre-teaching or special introduction.

Where possible, the test was to be administered to students of varying abilities — from those in "remedial" mathematics programs to those in college-bound of "gifted" programs.

Equal numbers of each of the six forms of the test were to be shuffled like a deck of cards and then given to the students in a class in the order they appeared in the "deck."

Research Design

Six different mini-tests were to be randomly given to students of a range of abilities. The tests were to be scored recording five patterns of responses -- all area (bdaacb), all perimeter (abbcbc), area/perimeter (bdacbc), perimeter/area (abbacb), other. Scores on Tests I, 1 will be compared to II,1; Scores on I,2 to II,2; and I,3 to II,3.

The expected number in each cell will be computed by multiplying the probability of a response belonging to the ith row by the probability of a response belonging to the jth column and that product by the sample size n. To further test that rows and columns represent independent classifications, the chi square will be used.

The chi square test of the null hypothesis will be established with the probability level $p = 0.01$.

Tables showing the distribution of scores were compiled. These are available from the author.

Conclusion

Null hypothesis 1, there is no significant diference between responses of ninth-grade students on tests of area of rectangular regions where the measure of each of the four sides is given and the responses of ninth-grade students on tests of area of rectangular regions where the measure of only one of each of the pairs of parallel sides is given, is rejected.

On the two all-area tests there was a significant difference in responses. The greater number of correct responses was associated with those problems where only two measures were given. As the number of correct responses increased, the number of incorrect all-perimeter responses decreased.

Null hypothesis 2, there is no significant difference between responses of ninth-grade students on tests of area and perimeter (problems given in area/perimeter sequence) of rectangular regions where the measure of only one of each of the pairs of parallel sides is given, is rejected.

On the two area-perimeter tests there was a significant difference in responses. There was little change in correct responses, however. The two greatest changes were in the decreased number of all-perimeter responses and the increased number of all-area responses when only two measures were given.

Null hypothesis 3, there is no significant difference between responses of ninth-grade students on tests of perimeter and area (problems given in perimeter/area sequence) of rectangular regions where the measure of each of the four sides is given and the responses of ninth-grade students on tests of perimeter and area of rectangular regions where the measure of only one of each of the pairs of parallel sides is given, is accepted.

On the two perimeter-area tests there was no significant difference in responses. The two greatest changes were in the decreased number of all-perimeter responses and the increased number of "other" responses when only two measures were given.

An interesting aside was that out of 73 students who had 5 correct responses out of 6 possible responses, 30 of these students had an incorrect response to number 6 which required them to find 4 x 8.

Additional research should be done including, perhaps, all-perimeter problems and perimeter problems with plane figures other than rectangles.

This research could be very valuable to developers of text materials and tests in mathematics.

ACTIVITES MATHEMATIQUES, CYCLE MOYEN

1) INITIATION AU PLAN CARTESIEN
2) LES TRANSFORMATIONS DANS LE PLAN
3) LES JEUX MATHEMATIQUES

Rosaline Séguin, Dedictique des enseignants à l'élémentaire, Université d'Ottawa
Jacques Théorêt, Conseiller pédagogique en mathématique, Conseil scolaire d'Ottawa

Note de éditeur: À cause du manque d'espace, nous ne pouvant pas publier tous les articles au complet. Les interesses peuvent s'adresser aux auteurs.

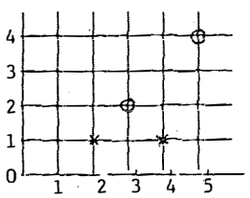
Raison D'Etre de ce Contenu

L'enfant se déplace dans un espace physique, agit et vit dans cet espace. Il faut lui faire vivre des expériences qui lui permettront de mieux observer les éléments de cet espace. Il en dégagera des propriétés, établira des relations, décélera des variables... . Il traduira mathématiquement l'espace dans lequel il évolue et fixera quelques éléments de structure.

Le concept de relation est omniprésent en mathématiques. Il l'est en particulier en géométrie quoique souvent non-explicite. Il est important qu'à l'élémentaire l'enfant découvre, crée, décrive, modifie, reproduise... diverses relations géométriques. Il est également bon qu'il généralise son idée de relation en s'apercevant qu'en géométrie on rencontre des relations qui sont semblables, par leurs propriétés, à des relations observées sur les nombres ou sur d'autres ensembles.

La Régularité dans les Graphiques

Il suffit de savoir compter afin d'apprendre à identifier les points sur une grille: un couple est suffisant pour chaque point. C'est le mathématicien Réné Descartes qui inventa au 17e siècle le système des coordonnées et du même coup trouva une façon de relier les formes géométriques aux équations algébriques. Les enfants de tous les niveaux peuvent bénéficier de l'expérience acquise avec cette importante idée en mathématiques et les activités avec les graphiques raffirmeront les concepts du nombre.



Se les enfants de votre classe ne peuvent localiser un point sur un graphique alors il suffit de jouer au tic-tac-toe. Diviser la classe en deux équipes et demander les de vous indiquer où situer le "x" (pour une équipe) et le "o" pour l'autre. Les enfants saisiront assez rapidement la facon de placer un point sur la grille (2,1) - "x" (4,1).

Ensuite ils (elles) sont prèts(es) à étudier les regularites d'un graphique.

Objectif: L'élève sera capable d'identifier un point d'un graphique par un couple de nombres (une paire ordonnee).

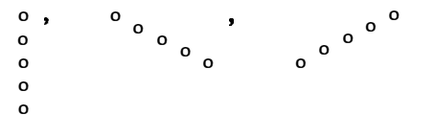
Niveau Approprié: Tout étudiant qui n'est pas familier avec cette notion.

Stratégie d'enseignement:

1. L'enseignant distribue la feuille ci-jointe à chaque étudiant.

2. L'élève indiquera cinq (5) points consécutifs (sans tenir compte de la direction)

e.g., o o o o o ,



3. L'élève indiquera un couple de nombres pour chacun des points de sa droite. (Il sera bon d'encourager l'élève à séparer les chiffres par une virgule, et les situer à l'intérieur des parenthèses.)
4. L'enseignant affichera les travaux terminés au babillard en les regroupant par la direction de chaque droite. A la fin de ce travail l'enseignant étudiera toutes les différentes droites et les couples.

Objectifs: L'élève reviserà l'opération addition en se servant des couples de nombres sur une feuille quadrillée.

Niveau approprié: 1er, 2e, 3e

Stratégie d'enseignement:

1. L'enseignant distribue la feuille ci-jointe à tous les étudiants.
2. L'élève choisit un nombre entre 5 et 12 inclusivement, e.g., 7 - (7,0)(6,1)(5,2)(4,3)(3,4).
3. L'élève inscrit les paires de nombres qui ont comme somme le nombre choisi.
4. L'élève place les paires de nombres sur la feuille quadrillée.

*5. Les élèves comparent leurs graphiques.
*6. Les élèves identifient des couples de nombres qui permettent la construction de formes géométriques connues, e.g., rectangle, triangle, etc.

* - pour le groupe avancé.

Objectif: L'élève revisera l'opération "multiplication" en se servant des couples de nombre sur un graphique.

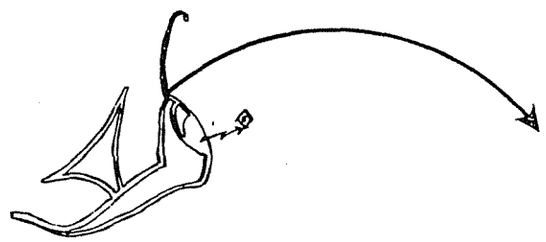
Niveau approprie: 4^e, 5^e, 6^e.

Stratégie d'enseignement:

1. L'enseignant distribue la feuille ci-jointe.
2. L'élève choisit un des nombres suivant 12, 16, 18, 20, 24.
3. L'élève indique tous les couples de facteurs du nombre choisi, y incluant les couples inverses.
4. L'élève situe chaque couple sur la grille.

*5. Les élèves comparent leurs graphiques et discutent des observations.
*6. Les élèves étudient les facteurs de nombres plus grands. Il faut distribuer du papier quadrillé permettant plus de facteurs.
*7. Les élèves peuvent inclure les nombres fractionnaires.

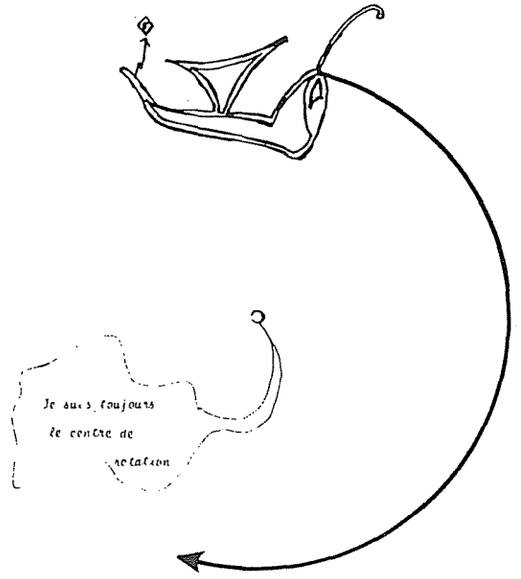
1. Dessine sur ton papier-calque le centre de rotation et le papillon.





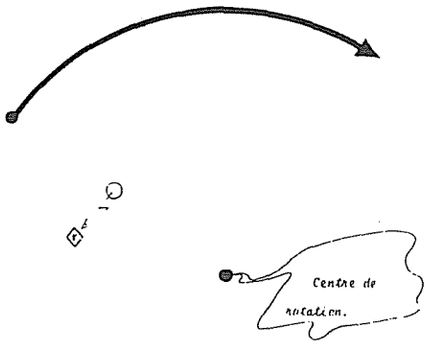
2. Sans déplacer le centre de rotation fais tourner le papillon comme l'indique la flèche et dessine son image.

Le papillon prend gôut aux exploits! Il va effectuer la rotation indiquée par la flèche de rotation et le centre de rotation. Dessine l'image obtenue par sa rotation.

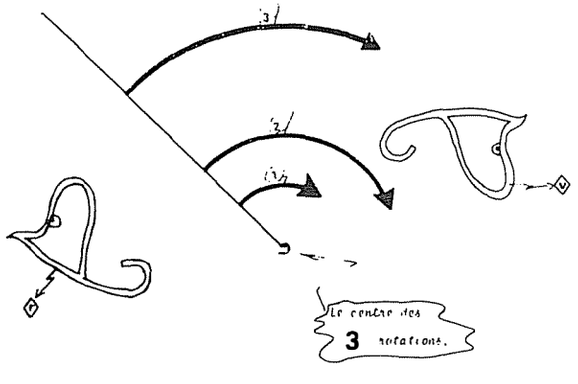


Et voilà ce qui arrive aux papillons qui font de la géométrie!

1. Dessine en vert l'image du point rouge par cette rotation.



2. Dessine la flèche courbée de cette rotation partant du point rouge et se rendant au point vert.



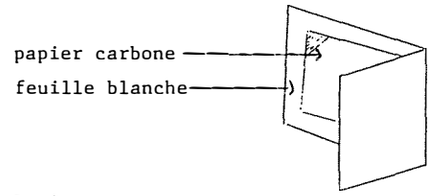
Voici trois rotations. Elles ont le même centre.

Quelle est la rotation qui conduira le Bozo rouge le plus près possible du Bozo noir?

DECOUVERTE DE LA NOTION D'AXE DE SYMETRIE

Activite 1: Première exploration:

Place une feuille de papier carbone sur une feuille blanche. Plie les deux feuilles de sorte que le papier carbone soit à l'intérieur

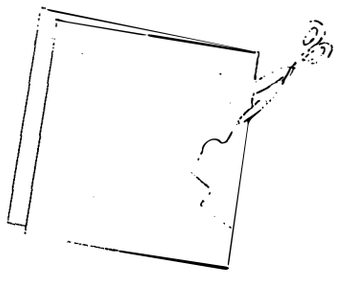


Trace un croquis (un dessin) sur le dessus.
Ouvre ta feuille. Note tes observations.
Réalise d'autres expériences de pliage avec feuille et carbone. Compare tes diverses observations. Discutes-en avec les autres équipiers.

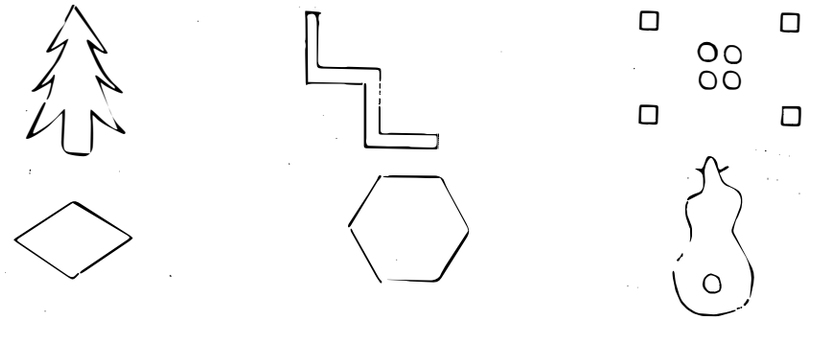


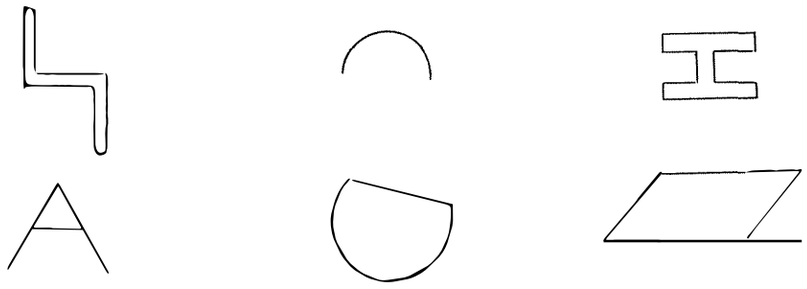
Deuxième exploration:

Plie une feuille en deux. Découpe le long du pli, avec des ciseaux, une figure quelconque. Déplie et observe. Note tes observations. Refais la même activité, mais cette fois, en pliant la feuille en quatre, en huit. Note tes observations. Compare avec les voisins.



Pourrais-tu obtenir les dessins suivants par pliage et ciseaux? Si oui, trace en rouge l'endroit du pli de la feuille.





Troisième exploration: (activité libre)

Fais une tache de peinture, de gouache ou d'encre sur une feuille. Plie la feuille avant que la tache soit sèche. Plie-la de telle sorte que la tache soit à l'intérieur. Déplie et observe. Note tes observations.

Quatrième exploration:

Lis le texte suivant.

Cette fiche aborde l'étude de la symétrie. Observe ce qui se passe quand il y a un axe de symétrie, deux axes de symétrie (parallèles ou non). Une figure est dite symétrique s'il existe un axe de symétrie tel que l'image est la figure elle-même.

As-tu découvert ce que to dois faire pour le lire facilement?

Comment pourrais-tu écrire un texte de la même facon?

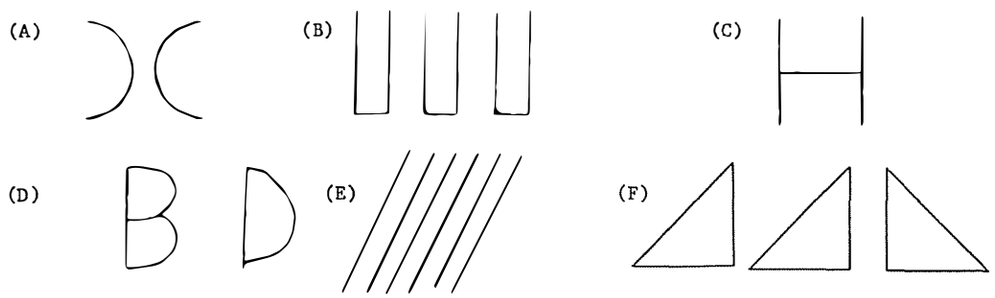
Cinquième exploration:

Avec un miroir transparent, trace au crayon diverses images de la figure suivante. Il suffit de déplacer le miroir.

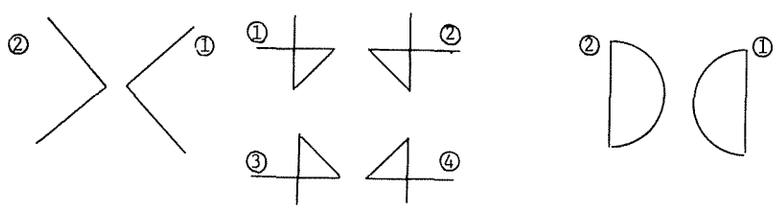
RESUME. Enumère diverses facons d'obtenir des figures "symétriques".

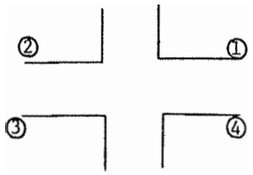
Activite 2: Première exploration:

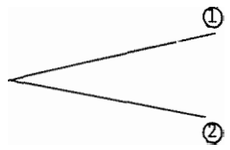
Où faut-il placer le miroir? Où faut-il plier la feuille? Note tes observations. Discutes-in.



Où dois-tu placer le miroir pour que l'image de ② soit ①? De même pour que ③ soit ①, pour que ④ soit ①, pour que ③ soit ② (s'il y a lieu)?

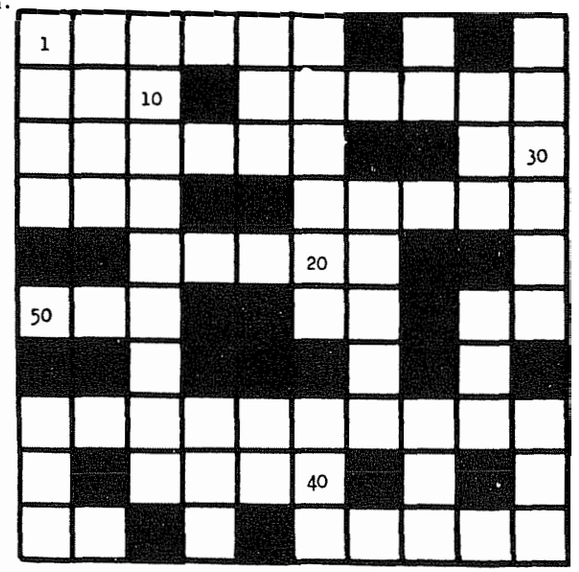






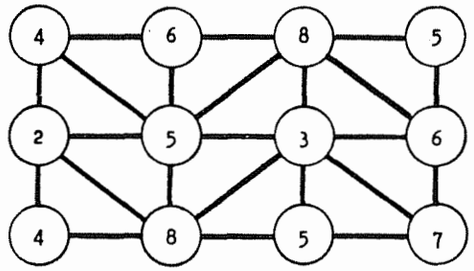
Jeu no 152: Le cinquante

1. Vous partez de 1 dans la case supérieure gauche, Le jeu consiste à trouver le chemin qui mène à la case 50.
2. A mesure que vous avancez horizontalement ou verticalement, vous numérotez les cases.
3. Amusez-vous bien.



Jeu no 157: Un circuit

En passant par six cercles, trouvez un chemin pour que la somme des nombres soit 28.



Jeu no 147: Un carré chiffré

Sachant que B = 8 et D = 6, complétez la grille de sorte que les sommes horizontalles, verticales et diagonales soient égales.

Mini-jeux magiques
Editions du losange Enr.
Rimouski, P.Q.

D	A	D
C	C	C
B	E	B

Mini-Jeux magiques
Editions du losange
Enr., Rimouski, P.Q.

DISCOVERY IN QUADRATICS

J.N. Maycock, Area Superintendent, London Board of Education

Recently, Frank Wohlfort of New York City offered the following method of finite differences to stimulate mathematics students in the area of quadratic functions. He asked each student to create a quadratic function. He then asked them for the values for the function from 0 to 4. With these values he told them the function that they had written. A difference table is the Key.

EXAMPLE #1

x	0	1	2	3	4
y	1	6	17	34	57
		5	11	17	23
			6	6	6

A DIFFERENCE TABLE

The function was $y = 3x^2 + 2x + 1$

EXAMPLE #2

x	0	1	2	3	4
y	-1	0	3	8	15
		1	3	5	7
			2	2	2

The function was $y = x^2 + 0x - 1$, or $y = x^2 - 1$

EXAMPLE #3

x	0	1	2	3	4
y	0	1	4	9	16
		1	3	5	7
			2	2	2

The function was $y = x^2 + 0z + 0$, or $y = x^2$

EXAMPLE #4

x	0	1	2	3	4
y	0	2	4	6	8
	2	2	2	2	
	0	0	0		

→ Constant first difference implies a linear function

↓

The function was $y = 0x^2 + 2x + 0$, or $y = 2x$

By inspection, the students were able to see that the coefficient of the x^2 term is half of the value arrived at in the last line.

To see how the x term and constant term are found (if not clear from the preceding examples), apply the differencing method on the quadratic function $y = ax^2 + bx + c$.

x	0	1	2	3
y	c	a+b+c	4a+2b+c	9a+3b+c
		a+b	3a+b	5a+b
			2a	2a

The coefficient of x, b, can be derived by subtracting half of the last line entry from the first entry in the second line:

$(a + b) - ½ (2a)$
$= a + b - a$
$= b$

The constant, c, is the y - intercept or first entry in the first line.

For a quadratic function $y = ax^2 + bx + c$, with difference table

x	0	1	2	3	4
y	-6	-9	-10	-9	-6
		-3	-1	1	3
			2	2	2

$a = ½ (2)$
$= 1$

$b = -3 - ½ (2)$
$= -4$

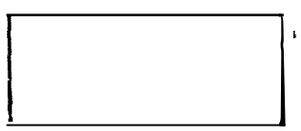
$c = -6$

Hence, the function is $y = x^2 - 4x - 6$.

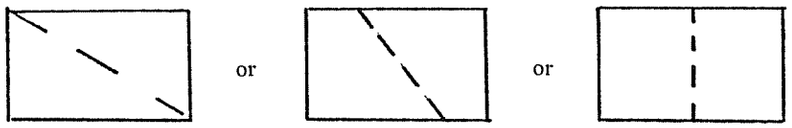
This differencing method can be used to develop quadratic functions in the following way.

Partition the region with 0, 1, 2, 3, 4, 5 lines to produce the maximum number of regions, r.

Case 0

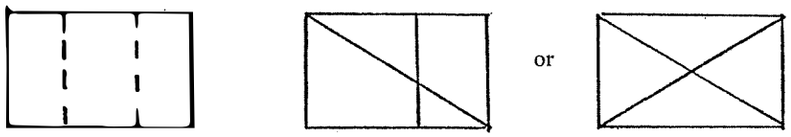


Case 1



Case 2a not allowed

Case 2b



From our experiments, the following table is tabulated.

Table 1. Case 1, Case 2, Case 3 . . .

l	0	1	2	3	4	5	6
r	1	2	4	7	11	?	?

By Trial, student should see that five lines will produce 16 regions and 6 lines will produce 22 regions and so on. A difference table could then be produced as follows;

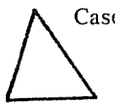
Case 1, Case 2, Case 3 . . .

l	0	1	2	3	4	5
r	1	2	4	7	11	16
		1	2	3	4	5
			1	1	1	1

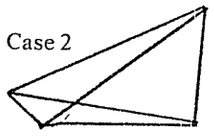
Assuming that the function is quadratic, we find that $r = \frac{1}{2}l^2 + \frac{1}{2}l + 1$.

The function for the number of diagonals produced by other polygonal figures is slightly more difficult, but the process is the same. Partition the polygons by diagonals to produce maximum diagonals, e.g.:

Case 1



Case 2



Case 3

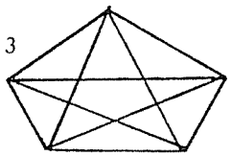


Table 2.

s	3	4	5	6	7	8	(s for sides)
d	0	2	5	9	?	?	(d for diagonals)

However, the necessary values for the production of a function to illustrate the relationship are missing. These can be found by moving the series to the left.

Thus:

s	0	1	2	3	4	5	6
d	0	-1	-1	0	2	5	9
		-1	0	1	2	3	4
			1		1	1	1

Thus the function to illustrate the number of diagonals d in a figure with s sides would be:

Again assuming that the function is quadratic we find that

$$d = \tfrac{1}{2}s^2 - 3/2s + 0 \qquad \text{or} \qquad s = \tfrac{1}{2}s^2 - 3/2s$$

To prove that the function is correct, let d = 4 in the above function.

$$d = \tfrac{1}{2} \times (4)^2 - 3/2(4)$$
$$d = 16/2 - 12/2 = 4/2 = 2$$

To prove that the function is correct for partitioning a figure let $l = 2$.

$$l = 2$$
$$\text{In } r = \tfrac{1}{2}l^2 + \tfrac{1}{2}l + l$$
$$= \tfrac{1}{2}(2)^2 + \tfrac{1}{2}(2) + 1$$
$$= 2 + 1 + 1$$
$$= 4$$

which checks with the value found empirically in Table 1.

SOME THOUGHTS ON MATH ANXIETY

by Peter Hilton

Peter Hilton is a Fellow of the Bottille Seminar and Study Program, Seattle, Washington.
This is the second part of an article which began in the September, 1978 issue.

1. The Constituents of a Sound Mathematics Education

What then should be the conspicuous features of a good mathematics education? Here I revert to some of my thinking at the beginning of the previous section but expand on the ideas introduced there. I take it as obvious that we should be emphasizing the utility of mathematics and that we should be justifying its structural properties as enhancing its utility and simplicity. The mathematics the children learn should be genuinely relevant and should be seen to be relevant by the child; this means that it must be relevant to his (or her) world, and not simply relevant to a world which the adult wishes to impose on the child. I would like to emphasize this last point. I have seen, at so many levels, the forlorn attempt to generate interest in a piece of mathematics A by demonstrating its importance in the solution of some problem B. I can think of nothing more utopian than to suppose that, by exhibiting a connection between A and B, one can generate in the student an interest in both when he starts with an interest in neither.

It follows from what has been said that mathematics must take the child beyond the classroom. It must not be an activity confined to the mathematics lesson. Frequent opportunities must be taken to involve the children in observations and experiments which they can carry out in their own homes or in their own neighbourhoods. Even within the school itself, it is encumbent upon the teachers when teaching subjects which are not explicitly mathematical to demonstrate, wherever appropriate, the importance of genuine mathematical competence as an aid to refined understanding of the topics under study.

These are general propositions; I come now to certain specific proposals relating to curriculum. My one regret here is that I believe I am writing this article for the wrong journal, for I believe that the readers of this journal will regard what I have to say here as being too obvious to be worth saying. I must defend myself by pointing out that so many of the curricula I have studied violate many of the principles embodied in these subsequent proposals.

(i) *Estimation and Approximation: The Use of the Hand Calculator.* I find no justification, as I have said, for instilling useless algorithms. That there is an algorithm is important; the nature of the algorithm is also important; the ability to use the algorithm accurately is today usually of no importance. The availability of the hand calculator has rendered so many algorithms anachronistic. On the other hand, estimation and approximation are of supreme importance in their own right--and take on an added significance in view of the necessity of ensuring that one has

punched the right numbers and the right operations into the calculator. Determining the appropriate degree of accuracy for a particular problem is an essentially human task and therefore absolutely proper to the education of human beings. The algorithms are mechanical tasks and are therefore absolutely proper to machines. We must be guided in our educational philosophy by the aim of achieving the human uses of human beings.

(ii) *Geometry.* To many people, and particularly to many parents and policy makers, elementary mathematics is identified with arithmetic. This is most ironical when one reflects that the sort of arithmetic that many of these people have in mind is mostly out of date (if it ever was useful) and, in its spirit, as antithetical to genuine mathematics as any human activity could be.

The attitude to which I refer leads to a sad neglect of geometry. In the elementary grades, I do not of course advocate any formal geometric reasoning. In those grades there would be an increasing interest on the part of the child in intuitive geometrical concepts and in the relationship between numbers and geometry. As the child matures, so he or she should become increasingly familiar with the three-dimensional world in which he lives. It is a deplorable fact that most of us are entirely unfamiliar with this world. Thus, for many people it is impossible to conceptualize problems which are related to the fact that we live on the surface of the sphere. For example, I have put to undergraduates the question, "Given two cities on the same parallel of latitude, would the shortest route between them follow that parallel?" The answers have been mostly guesswork. I follow the great French mathematician Rene Thom in believing that geometry is the primary way in which we, by means of our thought, convert the three-dimensional world of experience to the one-dimensional world of language. I regard any student deprived of familiarity with geometrical ideas as abused.

(iii) *Decimals.* Decimals come naturally into the curriculum through our monetary system and--thank goodness-today through our use of the metric system of weights and measures. The arithmetic of decimals should present no problems, at least until the multiplication of decimals is involved, and there the model of rectangular area should provide motivation for the appropriate concept. Moreover, the relationship to percentages is immediate and natural. I believe that an important principle here is that decimals must be presented in a way which is quite distinct from fractions--I particularly deplore the hybrid term 'decimal fractions'. Conceptually, decimals and fractions are far apart, and it is only a mathematician's trick to say that they are the same. A decimal starts life as a number and a measure; a fraction starts life as a proportion and remains a proportion for a long time (for a lifetime for most people!). Moreover, it is simply not true that every fraction has a decimal equivalent if, by decimal, we mean a natural *arithmetical* extension of the concept derived from, say, cents and centimetres. The surreptitious introduction of infinite decimals is an example of the authoritarianism which I condemned in the pevious section.

(iv) *Thinking Stories.* I use here the name given in the new mathematics programme *Real Math*[1] to refer to an integral and characteristic part of that programme in which the children are told (or, in later grades, read) stories which contain questions which they have to answer. There is no fixed format. The stories contain much information not relevant to the questions asked; and, in many cases, questions are asked for which adequate information has not yet been provided in the stories. The educational philosophy behind these thinking stories is that children should be encouraged to use their reasoning power. The stories may well have some fantastic elements, but the children recognize these and are by no means fooled by them. Other aspects of the stories may very well be highly realistic. These stories help the child to understand that mathematical problems do not arrive already neatly packaged. The world presents the curious child with questions, and some of these questions can be translated into mathematical terms. There is no algorithm for these translations. The child is thus learning, at the appropriate level and from an early age, how to genuinely apply mathematics. Moreover, he is acquiring understanding of mathematical process through use.

(v) *Qualitative and Quantitative Thinking.* Most of the thinking which adults do is qualitative. For many, quantitative thinking is almost entirely confined to marketing, housekeeping and bank transactions. Thus, numbers tend to be very much underemployed. As a result, much of our thinking is extremely crude. This is very conspicuously true in our use of large numbers where, for many people, the difference beween, say, 1,000 and 1,000,000 is by no means obvious.[2] The student must learn early on that he can obtain a far clearer picture of the world around him by using numbers appropriately. This of course has much to do with questions of estimation and of

appropriate precision for recording information. Experience and practice in such techniques should form an integral part of the elementary mathematics education. By bringing numbers into one's daily life, one acquires a familiarity with them which should destroy their power to induce fear and horror.

Once again, we could extend this list. However the purpose of the list is to exemplify the type of mathematical instruction which will lead to confidence instead of anxiety in the student. We have said nothing about pedagogy, but we recognize there are important principles here too to be put into effect. One such principle we would like to make explicit. The teacher must understand why the mathematics is important and must convey that understanding to the student. This principle implies many things. It makes it plain that the teacher must know more mathematics than he (or she) teaches; he must know applications of the mathematics which are interesting to the student; and he must be able to relate the mathematics to what as gone before in the entirety of the mathematical education of the student. This last point deserves further stress. Too many teachers teach the various parts of mathematics--algebra, arithmetic, geometry, probability, etc.--as if they were entirely separate subjects. They are nothing of the sort; they are all part of mathematics. The fragmentation of mathematical education has its bad efects all the way up, including, of course, the training of the Ph.D. If the students could see mathematics as a single subject in all its fascinating interconnections, they would be far less prone to fear the strange and the unfamiliar, for they would rarely meet it.

2. Dealing with the Current Situation of Mathophobic Adults

In this final brief section, I want to make a few points on the question of how we can ameliorate the current situation. I am dediated to the view that, in contemplating educational malaise, prevention is far better than cure. For this reason the main burden of my thought in this article has been concerned with precollege education. Yet, we owe it to those who have been through the system and find themselves at a disadvantage today to try to do what we can to remedy their situation. I believe that we must make the elimination of math anxiety one of the principal thrusts of continuing education. I think we can learn a lot from what has been done at Wesleyan University, Wellesley College and other institutions where a serious effort has been made to improve the situation.

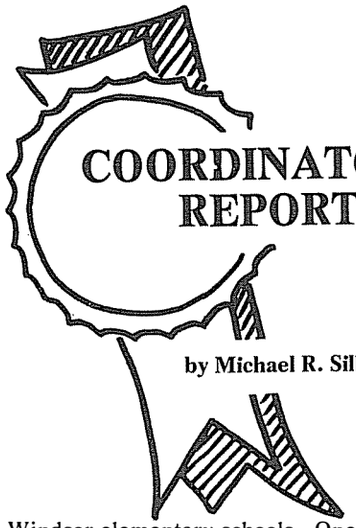
I believe that it may very well be that continuing education well be a major concern of universities in the next few decades. The pattern of education is already undergoing change, so that we are having fewer undergraduates arriving at our colleges direct from high school, while, at the same time, more and more adults are recognizing the need to continue their education. I would like to see the mathematicians of our universities and colleges taking a full resonsibility for dealing with the distressing and vitally important problem of adult math anxiety. I hope that relatively easy options will not be taken. It would be helpful, but still entirely unsatisfactory, if we merely converted adults who were worried that they could not do mathematics into adults who were not worried that they could not do mathematics. We must go further; we must try by every means we know to overcome the reluctance of those adults to involve themselves in mathematical reasoning. For we believe that it is nothing but an accident of their circumstances that these intelligent people are unable to use mathematics. We do not believe that mathematical ability at the level at which we are speaking is a rare gift. All that is rare is the survival of that ability and the appetite to use it in the unfavorable climate of a typically dreary and misleading experience in mathematics education. No self-respecting civilization can accept a situation in which the majority of its citizens are so handicapped. This is a battle which we must win.

Case Western Reserve University
Battelle Seminars and Studies Program

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[1] Open Court Publishing Company, LaSalle, Illinois.

[2] Mitchell Lazarus, writing in *The Common: A Meeting Place for Education in New England,* (December, 1977) refers to a problem involving the fact that 24250 automobiles were produced in the U.S. in 1905 and adds 'The numbers for the year 1905 could just as well have been 24 million cars, rather than 24 thousand, for all the student knows'. I am not sure how to take this remark, but it clearly bears on the point I am making. Lazarus seems to think it invalidates the problem--I think the opposite. He even remarks further, 'And why should anyone care?' I would have no trouble answering that question, especially if I were allowed to assume the truth of his previous comment.



by Michael R. Silbert

"Who are these Math Co-ordinators and what do they do?", I was asked by a teacher after he had read this column in the last issue. He was particularly curious about what they might be doing to improve the quality of mathematics education in Ontario. Please allow me to introduce some of my colleagues from across the province.

* * * * * * * * * *

Dave Skoyles, Co-ordinator of Pure and Applied Science with the Board of Education for the City of Windsor is a relaxed and easy going individual whose responsibilities include Mathematics and Science programs as well as activities related to Metrication. This fall, Dave has been busily involved in testing out a new, locally developed guideline for Junior Division Mathematics in forty-three Windsor elementary schools. One of the new features included in this guideline is a transformational approach to instruction in geometry as suggested in the Formative Years. With teachers providing input to this new program in November and with a rewrite planned for January, a great deal of energy will be put into Junior Division mathematics curriculum this school year.

With the draft of the new Intermediate Division Mathematics guideline under careful study, there is a considerable amount of experimentation with the statistics sections going on in Windsor schools. In addition, as the impact of calculators and computers on the mathematics program is being assessed, Dave has been involved in planning an upcoming dinner meeting and social evening for Windsor mathematics teachers. At this meeting, four faculty members from each of the university of Windsor and the University of Waterloo will be talking about "Calculators and Computers in the Classroom".

As well as carrying out his many administrative duties, Dave still finds the time to teach on occasion. Recently, he was involved in teaching how hand held calculators can be effectively applied in the Mathematics of Investment program in grade thirteen.

"We have more Mathematics of Investment classes than we ever had before", says Dave. "For years we had none. Now we have them in at least three schools, and the calculator has been really the reason that they have started to go as a fourth grade thirteen mathematics course."

* * * * * * * *

Brock Rachar is a soft spoken individual who brings to his job a tremendous love of people. In his capacity as Co-ordinator of Mathematics with the Board of Education for the City of London, he feels that one of his prime jobs is "helping teachers to find the materials that they need in order that they can best help students with special abilities or special needs". As well as providing teachers with individual assistance, Brock provides guidelines that have been developed by committees of teachers for regular classroom programs.

Lab packages on a wide variety of levels and topics are one feature which Brock provides and which he feels are of particular use to teachers. These packages may be ordered by phone or by mail and range from classroom sets of geared clock faces for learning how to tell time to kits on geometry, calculators and computers. Some of these packages are commercially produced and some have been locally developed.

Testing is an area of considerable focus in the Board at this time and some commercially developed criterion referenced tests have been provided to schools. At the present time, work has been proceeding on the development of London Mathematics Tests with tests for grades 4 and 7 already available. Tests for Grades 5 and 6 are also being developed and it is hoped that the grade 5 test will be ready by Christmas with the grade 6 test to follow sometime in the spring. These tests are designed to go along with curriculum guides that have been previously developed and with ones

The puzzle is not too difficult to construct although commercial models are available. A coat-hanger wire for the wire loop, plastic curtain rings for the rings, light welding rod or stiff wire for the straight rod, a strip of wood or metal to separate the straight rods at the bottom, and a piece of broom handle or a wooden block for the handle, is all the material required. The holes in the piece separating the stiff wires are drilled at a distance equal to the diameter of the plastic rings although this distance is not critical. Note that in each case the stiff wire rods pass through the left ring of two adjacent rings then between the wire loop and are finally attached to the right hand ring. There is no limit to the number of rings which may be placed on the wire loop but seven or eight are usually sufficient to provide considerable challenge and to demonstrate the technique required to free the rings.

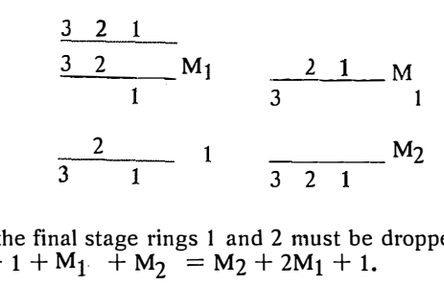
For those who find the puzzle to frustrating, mathematics will come to the rescue. The puzzle itself lends to a solution using a form of binary notation. If the ring on the extreme left, number 7 in the diagram, is on the loop it is assigned the digit 1, and if it is off the loop it is assigned the digit 0. For every other ring, if it is **on** the loop its digit is **opposite** to the digit of the ring to the left of it, and if it is **off** the loop its digit is the **same** as the digit of the ring to the left of it. For example, if there were only four rings in the puzzle the notation 1001 would read, ON, ON, OFF, ON, while 1010 would read ON, ON, ON, ON.

To illustrate how to employ this scheme to determine the proper sequence of moves we will consider a simpler case with a puzzle consisting of only three rings, (numbered 3, 2, 1, from left to right). When the three rings are all on the loop the code reads 101. To determine the sequence of moves required to remove the rings, subtract 1 for each move using the rules of binary arithmetic. Subtracting 1 from 101 yields 100, meaning ON, ON, OFF, and hence the first move is to remove ring 1. Subtracting 1 from 100 yields 011 meaning OFF, ON, OFF, and thus the second move is to remove ring 3. The complete sequence is:

1	0	1		1	0	0		0	1	1		0	1	0	
ON	ON			ON	ON	OFF		OFF	ON	OFF		OFF	ON	ON	
0	0		1		0		0	0							
OFF	OFF		ON		OFF		OFF	OFF							

This same procedure can be applied to determine the sequence of moves required to remove any number of rings. To put the rings back on start with 000 and add 1.

An examination of several particular cases will aid in the development of a recursive formula to determine the number of moves required to remove 'n' rings. In the diagrams the numbers above the line symbolize that the ring is on the loop while the number below indicates that the ring is off the loop. M_k is the number of moves required for all k rings to be dropped (or to be put back on the loop). Consider the case of three rings. The first move requires that ring 1 be dropped which requires M_1 moves. Then ring 3 can be dropped in 1 move. Before ring 2 can be dropped ring 1 must be put back on requiring M_1 moves. To get the puzzle to the final stage rings 1 and 2 must be dropped and this requires M_2 moves. Hence $M_3 = M_1 + 1 + M_1 + M_2 = M_2 + 2M_1 + 1$.

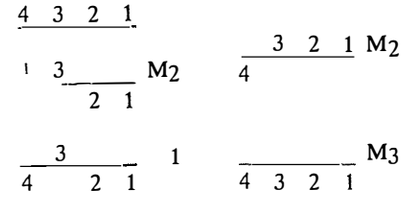


The case for four rings appears in the diagram on the left.

Four rings on the loop.
To drop rings 1 and 2, M_2, moves are needed.

To drop ring 4, one move is required. Before ring 3 can be dropped ring 2 must be on and before ring 2 can be on ring 1 must be on. Hence, it is necessary to return to this position and M_2 moves are needed.

M_3 moves are required to reach the final stage.
Hence $M_4 = M_2 + 1 + M_2 + M_3$ or $M4 = M_3 + 2M_2 + 1$.



The case for 'n' rings follows the same general pattern. n rings on the loop.
To drop the first (n-2) rings M_{n-2} moves are required. One move will then drop ring n. All the rings to the right of ring (n-1) will have to be put back on the loop. The final stage is reached in M_{n-1} moves which are required to drop (n-1) rings. The general formula for 'n' rings is $M_n = M_{n-1} + 2M_{n-2} + 1$.

n (n-1) (n-2) ... 3 2 1

n (n-1) (n-2) ... 3 2 1

n (n-1) M_{n-2}

(n-2) ... 3 2 1

(n-1) 1

n (n-2) ... 3 2 1

(n-1) (n-2) ... 3 2 1 M_{n-2}

n

M_{n-1}

n (n-1) (n-2) ... 3 2 1

Recalling that the two end rings (numbered 2 and 1) can be dropped simultaneously then $M_2 = M_1 = 1$. Substituting in the general formula we have:

$M_3 = M_2 + 2M_1 + 1 = 1 + 2(1) + 1 = 4$ _____ $2^2 - 1$

$M_4 = M_3 + 2M_2 + 1 = 4 + 2(1) + 1 = 7$ _____ $2^3 - 1$

$M_1 = M_4 + 2M_3 + 1 = 7 + 2(4) + 1 = 16$ _____ 2^4

$M_6 = M_5 + 2M_4 + 1 = 16 + 2(7) + 1 = 31$ _____ $2^5 - 1$

$M_7 = M_6 + 2M_5 + 1 = 31 + 2(16) + 1 = 64$ _____ 2^6

$M_1 = M_7 + 2M_6 + 1 = 64 + 2(31) + 1 = 127$ _____ $2^7 - 1$

It might now be conjectured that if n is even then $M_n = 2^{n-1}$, and if n is odd $M_n = 2^{n-1}$ These results may be established by mathematical induction.

Reference: Domoryad A.P., **Mathematical Games and Pasttimes,** Pergamon Press, 1964.
The World of Mathematics, Vol. 4, p. 2424

FUTURE MEETINGS

Local

Council Meeting
Second Saturday in February 1978 02 10
Executive Meeting
First Friday of April 1978 04 06
Annual Meeting 1978 05 13
Executive Meeting
First Friday after May 24 1979 05 25
Council Meeting
Second Saturday after May 24, 1979 06 02

International

Interamerican Conference on Math Education, 1979, Feb. 13-16, Campinas, Sao Paulo State, Brazil
OAME/AOEM 6th Annual, 1979
May 11-13 Peterborough
N.C.T.M. 57th Annual, 1979
April 18-21 Boston
OAME/AOEM 7th Annual, 1979
May 8-11 Toronto
N.C.T.M. 58th Annual, 1980
April 16-19 Seattle
N.C.T.M. 59th Annual, 1981
April 22-25 St. Louis

INTERVIEW

Taped Talks with Brock Rachar

. . . Dan Moalem



Dan Moalem is the noted Australian Mathematics educator who toured the province last April with his down-to-earth approach to transformations. Mr. Moalem's ingenious models of geometrical relationships and his quiet and warm style won him admiration as well as many friends in Ontario. Those of us who had the opportunity to attend some of his sessions were impressed with the calibre of his teaching and were challenged to give his ideas serious consideration.

It was my pleasure to interview Dan at the Annual OAME Conference in Hamilton.

B.R. Dan, Australian School Mathematics has gone through a number of adjustments over the last while. Now we in Ontario are looking at a new curriculum for Grade 7-10 Mathematics..

What do you see as the most important challenge in mathematics for the grade 7 to 10 program?

D.M. The student needs a program which is flexible and gives him or her an adapt ability to face up to change. For this reason it is important to stick with many traditional types of mathematics where competence in mathematical skills are involved; where a kind of well-being in having a facility for handling numerical, algebraic and geometric tools is given. At the same time there should be a part of the course that looks at new ways of solving the old problems.

Students need a broad base of experience so that they are better able to face change when change does come.

The teacher must understand the needs of the student in order to ''tailor make'' the course to fit these needs.

The guidelines are a starting point for the teacher. The teacher then designs the course to meet local needs.

Flexibility and adaptability are key words in my mind if we are to provide programs to meet the needs of the time.

B.R. What place do you see Transformations taking in the total program?

D.M. Transformations would be just one part of Mathematics. It has a role and a very definite role, because it is a very powerful technique of problem solving, of deducing results, discovering new results, and is used continually in research (although people are not aware of that). Transformations could work in tandem with Euclidean, it certainly supplements it and makes it far more effective. It is a means of linking the geometry to the algebra and the number, through graphs and the analytic plane. It is a means of making a student look at the same problem in more ways than one and hence have a large armoury of tools with which to solve problems. For that reason, transformations should have a very important place in our curriculum, but it is not intended to replace Euclidean geometry. A blend of approaches would seem to provide the best program.

B.R. Do you have some hints or final words for us as we embark on our new program?

D.M. No course will last indefinitely. Changes are always coming about where perhaps new and

better techniques are discovered. The essence in meeting change seems to be to hasten slowly. Look at the things in the old program worth keeping, but also make sure that what you are keeping is still useful. Give the new ideas a try before making a hasty judgment about them.

No program should be accepted because it works in the U.S., England or even Australia. Keep an open mind and a flexibility that allows changes to occur as you need them.

Editor's Note: Dan Moalem would be happy to hear from you. His address is:

Dan Moalem
17 Cave Road
South Strathfield, N.S.W.
Australia 2136

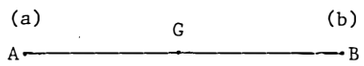
APPLICATIONS OF VECTORS

D. Baker, Glenforest Secondary School

The purpose of this short article is to (1) introduce a physical meaning to the problem of the division of a line segment, and (2) to illustrate how such meanings greatly simplify certain types of problems.

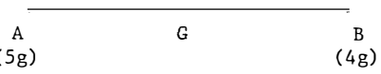
When the study of the division of a line segment is first approached the following physical interpretation should be given.

Consider a rod AB (of negligible mass) connecting two masses of ag and bg. The point of division G of the rod, in the ratio b : a is the center of gravity of the system.



G can be considered to be the point that the total mass (a + b) is acting at. (Displaying this fact requires little or no equipment. A ruler and some chalk brushes on either end do fine.)

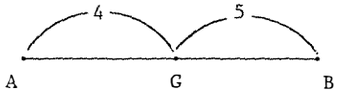
Example 1:



In this situation we can imagine G as a mass of 5 + 4 = 9g acting at the point dividing AB in the ratio 4 : 5.

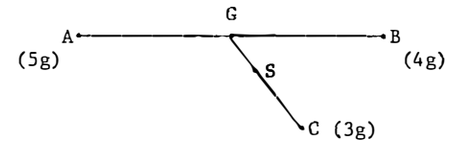
In the study of vectors the result $\vec{OG} = \frac{a}{(a+b)}\vec{OA} + \frac{b}{(a+b)}\vec{OB}$ is proven[1] for the point G that divides AB in the ratio b : a.

Example 2:



In this situation $\vec{OG} = \frac{5}{9}\vec{OA} + \frac{4}{9}\vec{OB}$ or more suggestively $9\,\vec{OG} = 5\,\vec{OA} + 4\,\vec{OB}$. Once again we can interpret G as being the center of mass (9g) of 5g at A and 4g at B. The advantage to this interpretation is the "reason" why the "co-ordinates" 4 and 5 are taken in the reverse order is evident. One would expect the center of mass to be closer to the larger of the two masses.

Example 3: There is no limit to the number of masses that can be considered in such a system.



In this case $\vec{OG} = \frac{5}{9}\vec{OA} + \frac{4}{9}\vec{OB}$ by dealing with the masses at A and B. Similarly $\vec{OS} = \frac{9}{12}\vec{OG} + \frac{3}{12}\vec{OC}$ by dealing with the masses at G and C.

Bringing the two results together we obtain $\vec{OS} = \frac{9}{12}(\frac{5}{9}\vec{OA} + \frac{4}{9}\vec{OB}) + \frac{3}{12}\vec{OC}$ or $\vec{OS} = \frac{5}{12}\vec{OA} + \frac{4}{12}\vec{OB} + \frac{3}{12}\vec{OC}$ or more suggestively.

$$12\,\vec{OS} = 5\,\vec{OA} + 4\,\vec{OB} + 3\,\vec{OC}.$$

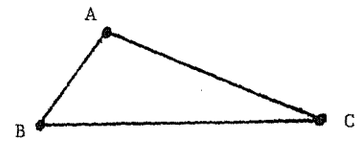
The symmetry of solution continues so that in general, a system of masses $m_1, m_2, m_3, \ldots, m_k$ at points $A_1, A_2, A_3, \ldots, A_k$ has a centroid G such that

$$(m_1 + m_2 + m_3 + \cdots + m_k)\,\vec{OG} = m_1\vec{OA}_1 + m_2\vec{OA}_2 + \cdots + m_k\vec{OA}_k .$$

An illustrative example for this is to take any large wooden polygon.

1) Determine the center of gravity for the polygon by experimental means and by theoretical means.
2) Place weights of various masses on the vertices and determine the balancing point both experimentally and by theoretical means.

<u>Example 4:</u> The proof that the medians of a Triangle intersect at a common point G, that is a point of trisection can be given physical interpretation.



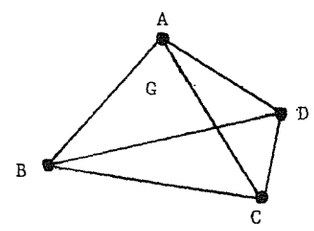
Consider masses of lg at each of the points A, B and C. Then the center of gravity for B and C is the midpoint M. Now considering the masses at A (1g) and at M (2g) the center of gravity G is clearly seen to be the point that divides AM in the ratio 2 : 1.

In problems that involve plane geometry the "reasonableness" of trying to prove a statement or more fundamentally, where the original conjecture came from, can be done by requiring the student to draw some particular cases and measuring. Thus the medians of a triangle trisecting each other could be handled in this way.

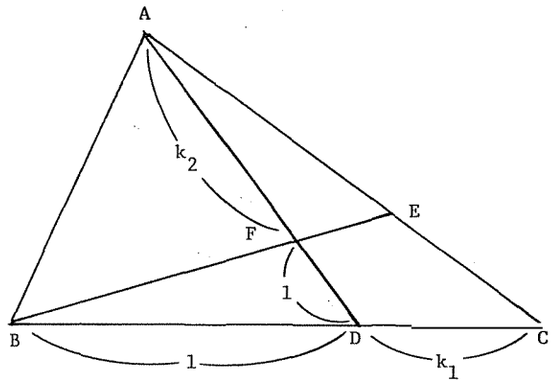
For 3 dimensional figures such a device is not available.

<u>Example 5:</u>[2] Prove that in any tetrahedron, the four line segments which join the vertices to the centroids of the opposite faces are concurrent, and are divided in the ratio 3 : 1 by the point of concurrency.

The student often wonders far more about how one hit upon the ratio 3 : 1 than in actually constructing the proof. In this case one can imagine the masses of lg at each vertex. Thus G, the centroid of face ABC can be considered as a mass of 3g. Dealing with the masses at G and D then leads to the ratio 3 : 1,



The concurrency of the line segments is "natural" in that one expects a system to have only <u>one</u> center of gravity.



<u>Example 6:</u> Applying this "intuitive" procedure to the problem posed by Henry Dernych.[3]

Given: A triangle ABC with D in BC so that AF : FO = k_2 : 1. BF produced meets AC at E.

Required: Express the ratios AE : EC and BF : FE in terms of k_1 and k_2.

Solution: The masses a, b and c are assumed to be placed at vertices A, B and C respectively. Thus,

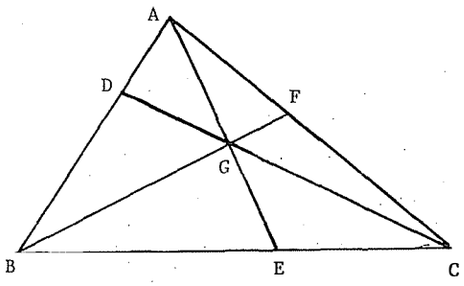
$$\frac{BD}{DC} = \frac{c}{b} = \frac{1}{k_1}. \quad \therefore c = b \text{ and } \frac{AF}{FD} = \frac{b+c}{a} = \frac{k_2}{1}. \quad \therefore a = \frac{b+c}{k_2}.$$

The desired ratio $\frac{AE}{EC} = \frac{c}{a} = \frac{b/k_1}{b + \frac{b}{k_1} / k_2} = \frac{k_2}{k_1 + 1}$ and similarly

$$\frac{BF}{FE} = \frac{a+c}{b} = \frac{b + c/k_2 + k_2 c/k_2}{b} = \frac{k_1 + k_2 + 1}{k_1 k_2}$$

Such problems should be more widely used so that the concepts are investigated fully.

<u>Example 7:</u>[4] Given: BD : DA = a : b
BE : EC = c : d
Find: CF : FA
CG : GD
AG : GE
BG : GF



Solution: Once again we can assume that G is the centroid of three masses, m_1, m_2, and m_3 at A, B and C respectively. Then

$\frac{a}{b} = \frac{m_1}{m_2}$ and $\frac{c}{d} = \frac{m_3}{m_2}$. $\therefore m_1 = \frac{a}{b} m_2, m_3 = \frac{c}{d} m_2$. Thus $\frac{CF}{FA} = \frac{m_1}{m_3} = \frac{\frac{a}{b} m_2}{\frac{c}{d} m_2} = \frac{ad}{bc}$.

Similarly for the other ratios.

<u>Footnotes:</u>
[1]See the Ontario Mathematics Gazette, Dec. 1977, Vector Proofs in Geometry, Bob Alexander, p. 22.
[2]Algebra, Coleman, et al., 1973, p. 143, Qu. 10.
[3]Ontario Mathematics Gazette, March 1978, p. 19.
[4]Ontario Mathematics Gazette, March 1978, Vector Proofs in Geometry, Bob Alexander, Ex. 11.

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2. If butter costs 52 cents a pound, how much butter can be bought for a cent and a quarter?

3. A man sold half his pigs and half a pig more; he then sold half of what remained and half a pig more and had none left. How many had he at first? Answer on pg. 64

AN EXAMPLE WHERE A VECTOR PROOF IS MORE CONCISE

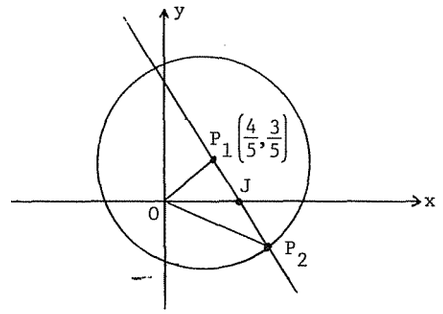
Henry Dernyck, Westgate Collegiate and Vocational Institute, Thunder Bay

Grade 13 Examination in Geometry, 1962

Question #4. A point P_2 is in the fourth quadrant at a distance of 2 units from point $P_1\left(\frac{4}{5}, \frac{3}{5}\right)$ and is on the line with slope $-\frac{4}{3}$ through P_1.

(a) Find the coordinates of P_2.

(b) Calculate the area of the part of the triangle OP_1P_2 that lies in the fourth quadrant.



I. The Conventional Geometric Solution

(a) The equation of the circle with centre $\left(\frac{4}{5}, \frac{3}{5}\right)$ and radius 2 is

$$(x-\tfrac{4}{5})^2 + (y-\tfrac{3}{5})^2 = 4. \qquad (1)$$

The equation of the line through $P_1\left(\frac{4}{5}, \frac{3}{5}\right)$ and having slope $-\frac{4}{3}$ is

$$y - \tfrac{3}{5} = -\tfrac{4}{3}(x-\tfrac{4}{5}) \qquad (2)$$

Substitute (2) in (1) to get $(x-\frac{4}{5})^2 + \frac{16}{9}(x-\frac{4}{5})^2 = 4$

$$\tfrac{25}{9}(x-\tfrac{4}{5})^2 = 4$$

$$\tfrac{5}{3}(x-\tfrac{4}{5}) = \pm 2$$

$$x-\tfrac{4}{5} = \pm\tfrac{6}{5} \quad x = 2 \text{ or } x = -\tfrac{2}{5}$$

Omit $x = -\frac{2}{5}$ as P_2 is in the fourth quadrant.

When $x = 2$, $y - \frac{3}{5} = -\frac{4}{3}(2-\frac{4}{5})$

$$y = \tfrac{3}{5} - \tfrac{4}{3}(\tfrac{6}{5})$$

$$y = \tfrac{3}{5} - \tfrac{8}{5}$$

$$= -1.$$

Coordinates of P_2 are $(2,-1)$.

(b) Let J be the point where P_1P_2 meets the x-axis. In equation 2 put $y = 0$

$$-\tfrac{3}{5} = -\tfrac{4}{3}(x-\tfrac{4}{5})$$

$$\tfrac{9}{20} = x - \tfrac{4}{5}$$

$$x = \tfrac{9}{20} + \tfrac{16}{20}$$

$$= \tfrac{5}{4}.$$

Coordinates of J are $(\frac{5}{4}, 0)$.

Area of $\Delta OJP_2 = \left|\frac{1}{2}(\frac{5}{4})(-1)\right| = \frac{5}{8}$ square units.

II. A Vector Solution

(a) The slope of ℓ is $-\frac{4}{3}$.

The direction vector for ℓ is $\vec{s} = (3,-4)$.

Then $\vec{OP}_2 = \vec{OP}_1 + k\vec{s}$ or

$$\vec{OP}_2 = \left(\frac{4}{5}, \frac{3}{5}\right) + k(3,-4).$$

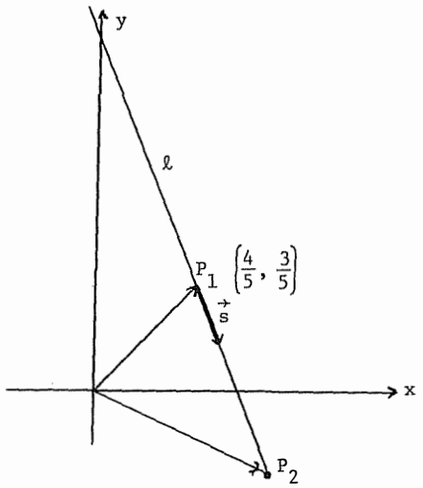
But $|k(3,-4)| = 2$

$5|k| = 2$

$k = \pm\frac{2}{5}$, use $k = \frac{2}{5}$

$$\vec{OP}_2 = \left(\frac{4}{5}, \frac{3}{5}\right) + \frac{2}{5}(3,-4)$$

$\vec{OP}_2 = (2,-1)$. Coordinates of P_2 are $(2,-1)$.



(b) Let J be the point where P_1P_2 meets x-axis. Parametric equations for ℓ are

$x = 2 + 3k$ $\qquad$ $y = -1 - 4k$

Put $y = 0$ $\qquad$ $0 = -1 - 4k$ $\qquad$ $k = -\frac{1}{4}$

When $k = -\frac{1}{4}$, $\qquad$ $x = 2 - \frac{3}{4}$

$= \frac{5}{4}$.

Coordinates of J are $(\frac{5}{4}, 0)$.

Area of $\Delta OJP_2 = \left|\frac{1}{2}(\frac{5}{4})(-1)\right|$

$= \frac{5}{8}$ square units

SWOAME LOGO COMPETITION



Create a logo for SWOAME and win $50.00

*Contest is open to students and teachers.

*Submissions should not be larger than 4 cm x 4 cm and should be done in black india ink on white paper.

*Submit your entry by 1979 01 31 to Irene Gauvin, Fairmont Public School, 1040 Hamilton Road, London, Ontario.

*All entries become the property of the SWOAME council and the prize will be awarded at the discretion of the judges.

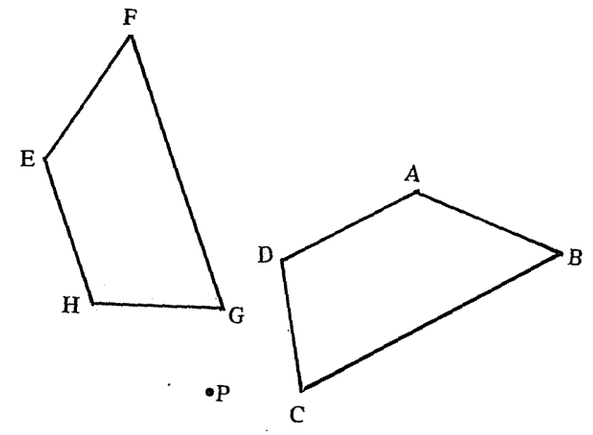
THE CASE FOR TRANSFORMATIONS IN HIGH SCHOOL GEOMETRY

Zalman Usiskin, University of Chicago

Eight years ago, few teachers had heard about transformations except in college linear algebra courses, and certainly not too many teachers were thinking about them in high school geometry, even though there were available materials.[1] Yet today, in an increasing number of texts at elementary, junior high, and high school levels, for both slow and fast learners, transformations are mentioned, given lip-service, given sections, or studied in detail. I have been one of those people who have encouraged the use of transformations both in algebra and in geometry. The purpose of this article is to give one view of the case for transformations. After a brief introduction to transformations, six reasons are given for their inclusion in school curricula.

What are transformations?

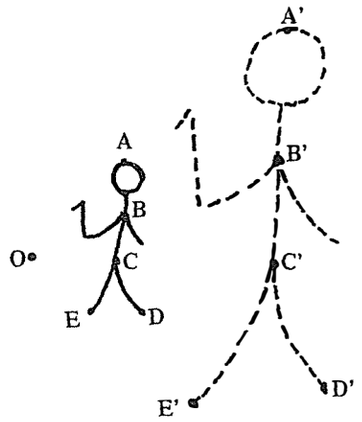
A transformation is a correspondence (often restricted to be 1-1) between the points of one set and the points of a second set. Below is pictured one example of such a correspondence. Point A corresponds to E, B corresponds to F, C corresonds to G, and D corresponds to H. All other points of ABCD correspond to points of EFGH. What do we call this correspondence? It looks like we could have turned ABCD 80° around the point P (stick a needle into P and physically turn the page to verify) and it would fall upon EFGH. So we call this correspondence a **rotation**. In fact, this particular rotation has center P and magnitude 80. We call EFGH the **image** of ABCD under this rotation. ABCD is the **preimage**. We say that the rotation **maps** ABCD onto EFGH.



In the physical model, all points -- not just those on ABCD -- have images. And so we speak of transformations of the plane even though we may focus only on a few points and their images.

Notice that the distance between two preimages (B and D, for example) is equal to the distance between their images (F and H). This is a characteristic of all rotations -- they **preserve** distance. Not all transformations preserve distance. Below is one which does not -- a **size transformation** (also called a dilatation).

In the size transformation, a center and magnitude are chosen. Here we have picked 0 as the center and 2½ as magnitude. For each point A on the preimage, we find its image A' by choosing the point on ray $\overrightarrow{OA}$ which is 2½ times as far from O as A is. That is, $OA' = 2\frac{1}{2}.OA$. This process is repeated with as many points as are needed to determine the image.



The preimage and image probably look similar to the reader. However, the standard definition of "similar" does not cover such complicated figures. Yet many applications of similarity (patterns, scale models and drawings, photographs, magnifying figures) require a general notion of similarity. This gives the first major building block for the case for transformations.

1. Transformations enable one to deal with a much greater variety of figures in geometry.

Here are possible definitions of congruence and similarity as seen from a transformation point of view. Notice that they apply to all figures, not merely segments or angles or triangles.

Definition: Two figures $\mathcal{A}$ and $\mathcal{B}$ are **congruent** if and only if there is a distance-preserving transformation which maps $\mathcal{A}$ onto $\mathcal{B}$.

Definition: Two figures $\mathcal{A}$ and $\mathcal{B}$ are **similar** if and only if there is a distance-multiplying transformation which maps $\mathcal{A}$ onto $\mathcal{B}$.

By "distance-multiplying" we mean that if A and B are any two preimages and A' and B' their images, then A'B'/AB is a constant. This constant is the ratio of similitude. In the above example, this constant ratio is 2½. That is, the size transformation in some sense multiplies distances by distances by 2½.

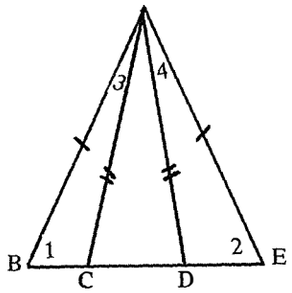
With these definitions, it is **natural** to ask: "When are two triangles congruent?" "When are two polygons similar?" From these questions, the standard content of geometry can be developed. Under traditional courses, it is doubtful that students think of congruence or similarity as being in any way natural. This is a second major point.

2. Transformations bring geometry much closer to the intuition of the child.

The early geometry experiences of the child in grades K-8 lead the child to understand that congruence and similarity depend upon size and shape. From this intuition, the child ought to realize that assembly-line production depends upon the building of congruent parts, that xerox duplication gives us congruent pages, that he or she is congruent to his image in a mirror (at a given point in time), that what you see through a telescope or microscope is similar to what is behind the lens, and so on.

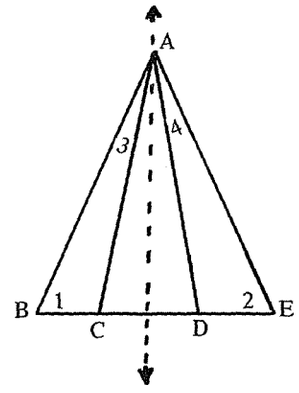
However, standard high school work in geometry divorces these concepts from the intuition of the child by (1) ignoring the very existence of the world it was supposed to model and (2) setting up different definitions of congruence and similarity for each type of figure even though the underlying intuition is the same.

This idea can be illustrated in another way. Below is a figure found in geometry texts. Some information is given, and let us suppose that the child is asked to deduce that $\angle 3 = \angle 4$.



How does the child **know** that $\angle 3 \cong \angle 4$. Certainly not because he proves it. The proof does not convince most children. He knows that $\angle 3 \cong \angle 4$ because in some way the figure is **balanced**. For this reason, the child would balk at being asked to prove $\angle 2 \cong \angle 3$. The figures in geometry texts are almost always symmetric. Symmetry is another important concept which is most easily defined in terms of transformations.

Definition: A figure is symmetric if there is a non-trivial* distance-preserving transformation which maps the figure onto itself. (*The transformation which maps each point onto itself is distance-preserving. [Think of a rotation of 0]. If this transformation were not excluded from the definition as being trivial, then every figure would be symmetric). The figure used above is reflection-symmetric. There is a reflection over the bisector of $\angle CAD$ which maps the figure onto itself. Notice that this reflection maps $\angle 3$ and $\angle 4$. For this reason, they are congruent.



Because transformations bring geometry closer to the child's experience, they can make the development easier. (This is not the case with every development using transformations; some writers insist on making simple mathematics as difficult to learn as they can). Furthermore, the geometry can be made much more visual and physical. This makes the use of transformations particularly appropriate for students who do not easily grasp proof concepts. They can at least understand the material about which they are being asked to make deductions. This is the third major reason for transformations.

3. Transformations make geometry more accessible to the slower student.

It is notable that UICSM decided to use transformations almost exclusively in designing a course in geometry for slow 8th graders.[2] It is also notable that here is a case of "new math" being easier -- it is taught in elementary schools in Europe -- and this goes against the experiences of many of the new curricular materials of the 60's, so that teachers don't believe it until they teach transformations.

Mathematics does not have to be hard to be good.

Any new concept will increase the number of applications available to a student. The increase is even greater with transformations, because they enable **all** figures (not merely polygons and circles) to be considered. Traditional concepts have broader applicability if they have been approached through transformations. Has anyone ever seen an amoeba or paramecium? No! They are too small to be seen. A size transformation image of these protozoa (thus similar to the preimage) is what we can see, and we deduce the dimensions of the preimage from knowing the magnitude (power) of the transformation and the dimensions of the image. Similarly, we have never seen the back of our head but can view its reflection image in mirrors. Other applications of congruence and similarlity have been given above.

Transformations and associated ideas have loads of uses of their own and no book, let alone an article, could include them all.[3] Symmetry is found in art, architecture, wallpaper designs, frieze patterns, ink blots, crystals, flowers and animals, orbits, beehives, sound and other waves. The same ideas that are recreationally applied to analyze Escher drawings are used seriously to study and classify crystals. Properties of reflections are used in billiards, lasers, optics, and to create illusions in magic. Preservation of distance or angle measure is basic to the construction of maps and in the analysis of many physical phenomena. In short:

4. Transformations enable geometry to be more easily applied to the real physical world and vastly increase the number of applications of geometry.

It is not geometry which has so few applications to the real world; it is the standard way of approaching geometry which has so few applications.

The generality with which transformations can be used in geometry makes them much more applicable in later courses. Put another way, there aren't too many triangles in the second-year algebra course, so the standard geometry course cannot easily be applied there. Here are just a few of the applications of transformations which are possible in later high school courses.[4]

a. The congruence of parabolas $y = x^2$ and $y = (x-h)^2 + k$ and $y = -x^2$, for example.
b. Reflections over the line $y = x$ for inverse functions.
c. Symmetry of the conics.
d. Congruence of the graphs of $y = \sin x$ and $y = \cos x$, application to phase shift.
e. The similarity of all parabolas (take a magnifying glass to a thin parabola and it looks wider).
f. Rotating 90° to get slopes of perpendicular lines.
g. Polar coordinates as arising out of a rotation and size transformation.
h. Connections with real number and complex number addition and multiplication.
i. Periodicity of functions as an instance of translation symmetry.

Other applications are related either to matrices or to groups, which arise naturally if one has transformation.

These connections with what used to be exclusively college-level mathematics has forced **every** recent curriculum project for good students to introduce transformations early in their development.[5]

5. Transformations provide assistance (and a strong weapon) for future work in mathematics.

It is not geometry which has no applications in later mathematics, it is the **standard course** in geometry which has so few applications.

Indeed, transformations are examples of functions and hence useful in preparing to study analysis. Knowledge of transformations is necessary in studying groups. The intimate association between transformations and matrices is fundamental in linear algebra. And every geometer must know about transformations because some theorems have no other simple proofs -- except those which use transformations.

There are other concepts which are useful in later mathematics. and coordinates and vectors have been posed as possible tools with which to approach geometry. Thus far, all developments using these ideas available in the U.S. seem to require too many algebra skills for the average student. (The traditional unifying concept, proof, is, of course, not geometric but a logical concept). And these methods do not enable one to easily approach the fundamental concepts of congruence.

symmetry, and similarity and use these ideas to deduce properties of figures. We thus come to the last reason given here for using transformations in geometry.

6. Transformations give a unifying concept to the geometry course which is geometric in nature.

The reader may now wonder if there are any arguments against the use of transformations. One I have often heard is that teachers are not ready to teach transformations. This readiness is an attitudinal problem, not due to insufficient mathematical background. If material is really at the level of the average student, then it can be understood by teachers with no special training. Experience has shown that teachers go through materials more slowly in their first year of teaching transformations than in their second, and this would be expected with any new materials. In short, there is no case against the use of transformations in geometry -- their time has come and they are here to stay.

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1. Usiskin, "Transformations in High School Geometry Before 1970," **The Mathematics Teacher**, 67: 348-360 (October, 1974).
2. University of Illinois Committee on School Mathematics, **Motion Geometry**, Harper and Row, 1969.
3. For many other examples, see Herman Weyl's classic book **Symmetry** (Princeton University Press, 1952).
4. All of these applications can be found in the author's **Advanced Algebra with Transformations and Applications,** Doubleday Canada, Ltd., 1975.
5. For example, see Unified Mathematics, by Fehr, Fey and Hill, published by Addison-Wesley, 1972.

AN INSTRUCTIVE ALGORITHM INVOLVING A NUMBER THEORETIC PROBLEM

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This activity oriented problem is rich in mathematical experiences that are adaptable to students from elementary through high school. The emphasis is on simple arithmetical operations, flow charts, and graphical representation. Another such algorithm has been considered previously by the author (1).

Consider any natural number N > 1. If N is divisible by 3, divide it by 3 and repeat the division by 3 as often as possible. When N is not divisible by 3 and is an odd number, subtract 1 from N and try division by 3 again. Repeat this division if possible. Should N be even and not divisible by 3, multiply N by 2 and add 1. Again try division by 3 and repeat if possible. Division by 3 is attempted at every step. Whenever N is even and division by 3 fails, multiply N by 2 and add 1, but if N is odd and division by 3 fails, subtract 1. The procedure continues until N is reduced to 1.

For N = 14 we tabulate as follows:

Step Number	Procedure	n
0		14
1	M2 + 1	29
2	S1	28
3	M2 + 1	57
4	D3	19
5	S1	18
6	D3	6
7	D3	2
8	M2 + 1	5
9	S1	4
10	M2 + 1	9
11	D3	3
12	D3	1

divide by 3. The natural number n is the number obtained in each step when one of these operations has been performed.

For n = 57 the tabulation becomes:

Step Number	Procedure	n
0		57
1	D3	19
2	S1	18
3	D3	6
4	D3	2
5	M2 + 1	5
6	S1	4
7	M2 + 1	9
8	D3	3
9	D3	1

It will be shown later in this paper that for all natural numbers $N > 1$ the final number in the chain is always 1. The following table gives the number of steps required to complete the chains for natural numbers $2 \leq N \leq 101$. Clearly the number of steps do not tend to increase very rapidly. In this table, the largest number of steps is 24 and this occurs for N = 80 and N = 86. Even for numbers up to 500 the greatest number of steps is 31 and this occurs for N = 494. Incidentally, N = 194 requires 30 steps.

N	Number of Steps	N	Number of Steps	N	Number of Steps	N	Number of Steps
2	5	27	3	52	13	77	17
3	1	28	10	53	14	78	16
4	3	29	11	54	8	79	17
5	4	30	10	55	9	80	24
6	6	31	11	56	21	81	4
7	7	32	18	57	9	82	11
8	14	33	11	58	8	83	12
9	2	34	10	59	9	84	11
10	9	35	11	60	8	85	12
11	10	36	5	61	9	86	24
12	4	37	6	62	16	87	12
13	5	38	18	63	9	88	11
14	12	39	6	64	16	89	12
15	5	40	5	65	17	90	11
16	12	41	6	66	8	91	12
17	13	42	13	67	9	92	11
18	7	43	14	68	16	93	12
19	8	44	13	69	9	94	11
20	7	45	6	70	16	95	12
21	8	46	13	71	17	96	19
22	7	47	14	72	16	97	20
23	8	48	13	73	17	98	19
24	15	49	14	74	16	99	12
25	16	50	13	75	17	100	11
26	15	51	14	76	16	101	12

New chains can be generated by reference to previously determined chains without doing ever step because, eventually, a number less than N is reached. An interesting phenomenon frequentl occurs for certain groups of adjacent numbers. For example, for $87 \leq N \leq 95$ the numbers of ste are, respectively, 12, 11, 12, 11, 12, 11, 12, 11, 12. For $334 \leq N \leq 350$ the numbers of steps are 2 21, 20, ..., 21, 20. Also, by arguing backward, it is easily shown that, whenever three or more ste are needed, the last three values of n are always 9, 3, 1.

I UP TO GRADE 10

17-2-1 A prisoner is given ten green balls, ten red balls, and two identical boxes. He is told that an executioner, while blindfolded, will randomly select one of the two boxes and will then randomly withdraw one ball from the box selected. If the ball drawn is green, the prisoner will be set free; but if the ball drawn is red, the prisoner will die. If the prisoner arranges the balls in the boxes so that he will have the best chance for survival, find the probability of his survival.

17-2-2 A section of railway track 5000 metres long was laid in the desert. Because of the heat during the day the workmen put the track down during the cool of the night and securely fastened each end. In the heat of the following day the section of track expanded by 1 metre in length. If the track bowed upwards, how high would the centre of the track be above the ground level?

17-2-3 If 8888^{8888} is multiplied out, what is the units' digit in the final product?

17-2-4 A traveller sets out to cross a desert. On the first day he covers 1/10 of the journey; on the second day he goes ⅔ of the distance already travelled. He continues on in this manner, alternating the days on which he does 1/10 of the distance still to be done, with days on which he travels ⅔ of the total distance already covered. At the end of the seventh day he finds that another 22.5 kilometers will see the end of his journey. How wide is the desert?

II UP TO GRADE 13

17-2-5 24 players competed in a recent chess tournament. The committee divided them into 2 sections. In each section each player played one game against every other competitor. There were 69 more games in section B than in section A. Mr. Gambit, unbeaten in Section A, scored 5½ points (win = 1 point; draw = ½ point). Determine how many of Mr. Gambit's games were drawn.

17-2-6 A 25 vehicle interterrestial starship fleet took off from rebel headquarters, one ship departing every 5 earth minutes. Each ship travelled at the uniform speed of 4½ intergalactic distance units per earth hour, attaining this velocity instantaneously, and each ship has to journey 12 earth hours before arriving at its destination. The first ship departed at 11 a.m. earth time. Find the total number of intergalactic distance units travelled by all these starships from 11 a.m. to 9 p.m. earth time that night.

17-2-7 If x men working x hours a day for each of x days produce x articles, determine the number of articles produced by y men working y hours a day for each of y days.

17-2-8 In $\triangle ABC$, point D is on BC such that BD:DC = 5:4, and point E is on AC such that AE:EC = 1:2. If AD and BE intersect at point P, then BP:PE = k:4. Find k.

A PROBLEM WITH SOLUTION

W.H. Vander Velde

Problem: Prove (i) $\prod_{k=1}^{m} \sin^2 \frac{\pi k}{2m+1} = \frac{2m+1}{2^{2m}}$

(ii) $\prod_{k=1}^{m} \sin^2 \frac{\pi k}{2m+2} = \frac{m+1}{2^{2m}}$

Submitted by: W. H. van der Velde, Fanshawe College, London, Ontario.

Solution:

LEMMA. The roots of $z^n - 1 = 0$ are given as $z_1, z_2, \ldots, z_n$ with $z_1 = 1$. Then $(1-z_2)(1-z_3)\ldots(1-z_n) = n$.

Proof: $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1)$. Thus $z_2, z_3, \ldots, z_n$ are the roots of $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$. From theory of polynomials, $(z-z_2)(z-z_3)\ldots(z-z_n) = z^{n-1} + z^{n-2} + \cdots + z + 1$. Letting $z = 1$, we get our result.

THEOREM. (i) $\prod_{k=1}^{m} \sin^2 \frac{\pi k}{2m+1} = \frac{2m+1}{2^{2m}}$

(ii) $\prod_{k=1}^{m} \sin^2 \frac{\pi k}{2m+2} = \frac{m+1}{2^{2m}}$

Proof: (i) Let $z_1 = 1$ and $z_{k+1} = \cos k\theta + j \sin k\theta$, $k = 1,2,\ldots, n-1$ be the roots of $z^n - 1 = 0$, where $\theta = \frac{2\pi}{n}$ and n is odd. Then $(z_2 z_n)$, (z_3, z_{n-1}),..., $(z_{\frac{n+1}{2}}, z_{\frac{n+3}{2}})$ form conjugate pairs. Since $(1 - (a+bj))(1 - (a-bj)) = 2 - 2a$ for $a^2 + b^2 = 1$ we have $(1-z_m)(1-z_{n-m+2}) = 2 - 2\,re(z_m) = 2 - 2\cos(m-1)\theta$. From the lemma, $(1-z_2)(1-z_3)\ldots(1-z_n) = n$. $\therefore$ $[(1-z_2)(1-z_n)][(1-z_3)(1-z_{n-1})]\ldots[(1-z_{\frac{n+1}{2}})(1-z_{\frac{n+3}{2}})] = n$. $[2-2\cos\theta][2-2\cos 2\theta]\ldots[2-2\cos\frac{n-1}{2}\theta] = n$.

$2^{\frac{n-1}{2}}(1-\cos\theta)(1-\cos 2\theta)\ldots(1-\cos\frac{n-1}{2}\theta) = n$. $2^{\frac{n-1}{2}}(2\sin^2\frac{\theta}{2})(2\sin^2\frac{2\theta}{2})\ldots(2\sin^2\frac{\frac{n-1}{2}\theta}{2}) = n$. $2^{\frac{n-1}{2}} \cdot 2^{\frac{n-1}{2}} \sin^2\frac{\theta}{2}\sin^2\frac{2\theta}{2}\ldots\sin^2\frac{\frac{n-1}{2}\theta}{2} = n$.

$\sin^2\frac{\theta}{2}\sin^2\frac{2\theta}{2}\ldots\sin^2\frac{\frac{n-1}{2}\theta}{2} = \frac{n}{2^{n-1}}$. But $\theta = \frac{2\pi}{n}$.

$\therefore \sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\ldots\sin^2\frac{\frac{n-1}{2}\pi}{n} = \frac{n}{2^{n-1}}$. $\therefore \prod_{k=1}^{\frac{n-1}{2}} \sin^2\frac{k\pi}{n} = \frac{n}{2^{n-1}}$

Letting $m = \frac{n-1}{2}$, we have our result.

We prove (ii) similarly by letting n be even and remembering that $z_{\frac{n}{2}+1} = -1$.

One example: $\sin^2 20° \sin^2 40° \sin^2 60° \sin^2 80° = \frac{9}{2^8}$.

It may also be that teachers would wish to develop a CSE mode 3 along these lines. Some would certainly not wish to confront all pupils with an examination before leaving; others, conscious of the demands of employers might feel that such an examination would be worthwhile.

It is possible simply to list in syllabus form those specific topics we should cover but some justification for inclusion and exclusion needs to be given.

There is a great deal of concern at the moment about number and in particular the calculation aspects. People do regularly use some simple procedures but the test of regular need must always be made before we decide to include something in our list. The very certain extension of the use of mechanical and electronic calculators of all sorts must also be considered.

No one can really cope without some understanding of our number system. We all need to be able to "read" numbers, to have an understanding of place value and to be able to use our system for counting not only in ones but probably also in twos, threes and tens. Numbers are displayed in many different ways, on circular or linear scales, occasionally on dials based on place value and with a variety of digital presentations, including the computer numerals, which will grow more familiar as calculators are used more widely.

Operations between numbers seldom crop up in the abstract but are usually part of the solution of a problem in money, time or various other measurable quantities. Perhaps the most regular need is to take one two-figure amount from another, the second usually a round number. While in the future calculators may always be at hand it is helpful to be able to perform such simple operations in one's head and with regular practice few people are unable to achieve this skill. Most people can work out change from a pound, or tell how many minutes to the next hour on a digital clock. Probably we shall seek to retain the power to add two two-figure numbers together. Situations do occur, particularly in money where we use numbers of three or more digits but it is and will become even more unusual to operate with them without the help of a calculator. Certainly, there are still people who check the machine addition in the supermarket, which speaks volumes for their lack of trust and willingness to waste time. Of course, one might check that the price of the item is rung up correctly and that every article is rung up only once.

So it seems we need only a low level of addition and subtraction; indeed the test might be whether we can do it in our heads. It might be sensible to re-introduce regular mental arithmetic practice for children for that does largely define the distance we need to go in teaching computation.

We should be able to divide or multiply a two-figure number (perhaps a three-figure one?) by a single figure one - and of course by ten. So the tables are in! The real trouble about the tables is not that the multiplication bonds should not be memorized - they should - but that whenever one is asked "Mustn't they know their tables?" the questioner has the whole situation out of perspective. Tables are not central to mathematics.

There is a basic equipment for handling the computation we have defined. We need to know the number bonds to eighteen and we need to know the forty-five number facts of the multiplication table, relatively few of which are difficult to remember, but those few do need to be worked upon.

Beyond the four basic rules, some very elementary percentage work and some simple ratios (so that one knows the odds!) are probably in, though at a descriptive rather than an operational level. This material may be on the borderline for necessity as we have been seeking to define it.

Rates (of pay for instance) and simple rates of change, like speed need to be understood to some extent, certainly if one is to avoid prosecution, but normally they may be read directly from a scale, without a deep understanding of the units involved.

Reading scales generally seems to be accessible to nearly everyone. A really hard necessity - to tell the time - has until recently involved the rather complex process of two arrows running over a scale with two series of numbers on them. Yet almost no-one fails to learn to tell the time, perhaps because it is a fairly central need.

Cutting across a commonly held view of mathematics as a precise and exact subject, we should greatly encourage estimation, particularly in distance, time, weight, and speed; guessing is

respectable. So also is trial and error, fitting numbers to see if they work, which encourages a valuable facility and familiarity with the size of a number.

Leaving the computational side of number we still have quite a lot of numerical material to handle. Most of this involves some simple knowledge of Cartesion co-ordinates though perhaps that dignifies it a bit. Numbers are often displayed on a rectangular matrix as in a mileage chart, bus fares or a railway timetable, or even a ready reckoner. Practice and familiarity with this layout can be justified, therefore, as important information is frequently presented in this way.

More importantly, graphs are regularly used in the massmedia as persuasive evidence of the virtue of some commercial product or political point of view. These graphs are of the two axis variety but one needs to watch for certain tricks which deliberately mislead the "innumerate" consumer. Other forms of graphical representation, such as pie charts and block diagrams are in regular use. Thus the understanding of elementary statistics and their graphical presentation should be an important part of the basic mathematics programme.

It is rather more difficult to define what geometrical ideas we need. Some have effectively been specified in our section on estimation, where we might expect length and volume (and to a lesser degree area) to become familiar. Problems involving fitting things together - packing a box, planning the furniture in a room - occur quite regularly and the solution of such problems should be encouraged through a great deal of practical work.

A more topological idea - connectivity - is involved in finding one's way about. With the help of a map, bus routes or an underground diagram it should be within nearly everyone's scope to plan a simple journey.

It may be wondered what part the "new maths" plays in our basic scheme. In fact, there is a very extensive area to do with sets and ordering, which modern syllabuses have tended to treat rather formally instead of in everyday contexts. Much suffering and inadequate standards of living derive less from lack of money (though this is often associated) than from people's inability to organize and plan their affairs. One example might suffice here. A mother will usually have to provide a household with food and other materials for a week from payday. It is common, but uneconomic, to buy simply as the need for something becomes apparent. It may not by then be possible to satisfy that need, which indicates how essential is forward planning. Some listing must be done, and presumably in sets. You must know where to buy the various things; it is necessary to estimate cost. Some ordering processes are involved in deciding how and when to make purchases. It is not suggested that we should seek precise and exact lists and a carefully planned route through the shops, but without regimentation one might still hope for something better than total chaos.

Moving into the realm of language, we have another area where we must explore needs. Everyone attempts most days some form of logical discussion; a series of bald statements without logical connections results in a lack of communication with others. Some small skill in argument is necessary to attract respect and one of our tasks must be to develop these argumentative skills. Effectively this means to understand sets, subsets and logical implication, though not using a formal algebra. Perhaps we should work with the English departments in this area and basing the work again on the mass media we should encourage a critical approach to persuasive argument.

Again, in language, we have seen the increasing use of semi-mathematical terms in the media. Words such as fall, rise, more, less, rate of, maximum, minimum, average, trend, increase, often indicate a mathematical idea underlying the words. They have entered common parlance and an acquaintanceship at least with some of the ideas is increasingly necessary.

We are agreed therefore that the mathematical needs of the citizen are considerable and by no means limited to numerical calculation. Let us rid ourselves of false notions about what is useful so that a genuine programme can be offered as basic to all children. Maybe we will then be able to claim that we teach for real life.

ANSWERS TO BRAIN TEASERS

1. A 50c piece and a nickel; the 50c piece is not a nickel.
2. ½ pound of butter for 1c and 25c
3. 3 pigs
4. 64 cats and kittens
5. Bill and Bert will win by 3¾ min.
6. 3 hens

THE OBJECTIVES OF MATHEMATICS EDUCATION

A.J. Coleman

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It would appear rather obvious that there could be only one objective of mathematics education. Clearly, it is to teach mathematics!

Agreed! But this having been said, many questions press upon one. What kind of mathematics? How much? To whom? Why teach any to anyone? Should the programme be the same for all pupils? If not, according to what criteria can pupils be distinguished? After all, in a democracy everyone is equal, so should we not aim to achieve a common universal mediocrity? Is there any percentage in considering objectives in a vacuum? Must we not link with such consideration an evaluation of the practical possibility of achieving them, given the mathematical competency (or incompetency) of the corps of teachers and the strong societal pressures which denigrate intellectual excellence or mental effort? In other words, is there any use discussing "objectives" without considering whether they can be achieved?

As with any issue of basic existential import, once you start thinking about the aims of mathematics education, questions flood into your mind. Clearly, I cannot in a brief article deal with them all or, indeed, adequately with even one of these many issues.

You will recall that fairly recently an OECD study of education in Canada expressed amazement at the extraordinarily high proportion of the GNP which goes into education in a society which seems to have no explicit statement of what it expects the school system to accomplish. Evidently, most other OECD countries have clearer aims for education than has Canada. So perhaps my topic is timely.

Three Objectives

I have read discussions in which as many as twelve aims of mathematics education were distinguished - or, at least, the author thought he could distinguish this number, though my mind was not sufficiently acute to grasp the subtleties of his thought. In any case, I consider such long lists pedantic nonsense and psychologically ineffective. It is best to concentrate on essentials. I fix on three aims which I regard as crucial.

1. The average citizen should be enabled to master the minimum of mathematics needed for ordinary commercial transactions and for understanding the functioning of our society.
2. All people should be given the possibility of enriching themselves culturally and intellectually by extending their basic human capacity for abstract thought.
3. A supply of thoroughly trained mathematical "practitioners" - engineers, economists, research mathematicians - sufficient for the needs of society should be assured.

Let us look at these objectives in order.

Of course, there cannot be a satisfactory dogmatic definition of the precise set of mathematical facts which every Canadian should know. A minimum would surely comprise arithmetic, including percentage; mensuration, including change of units; and simple geometry. However, anyone who has noted the extraordinary rate at which mathematics has increasingly been applied in our society in recent decades would argue that much more than this is needed. The proper functioning of democracy requires a universal ability to interpret, and not be misled by, a wide variety of statistics which various interest groups thrust upon us. Presumably the hand-held computer will be universally available for the generation of students which is now in the schools. They will need to learn how to use it effectively.

No matter how we define the contents of the curriculum required to accomplish the first objective, the material can properly be called **basic.** However, I am not prepared to campaign under the slogan "Back to Basics". We have never done an adequate job in inculcating basic mathematics in the average Canadian, and what might have been sufficient twenty years ago is not enough now

and will be abysmally inadequate in another ten years. So I lift high my banner which reads: "Forward to the New Basics!"

In addition to enabling people to function as citizens or as economic agents in society, mathematics education should contribute to personal and cultural enrichment. David Wheeler has argued correctly that anyone who can use language effectively thereby demonstrates an ability to apprehend structure. But mathematics is the study of abstract structure, so one can argue that to mathematize is to be truly human. If, with Aristotle and St. Thomas Aquinas, we characterize man as the "rational animal", we must recognize that as we increase our ability to reason we enhance our humanness. Mathematical puzzles can be enjoyable and relaxing. But the more mathematics one understands and can use, the easier it is to understand and control the technological environment in which all of us are now immersed. Without such understanding the feeling that one's life is dominated by mysterious unknown forces must be overwhelming. So, to feel at ease with mathematics can enhance one's sense of freedom, as well as opening up the limitless and fascinating literature of mathematics and its applications.

All citizens should know enough mathematics to be able to manoeuvre in our society. All citizens should be given the opportunity to experience the joy of developing their innate capacity to mathematize and exult in this power. But at a more mundane and practical level, Canadian society needs a supply of competent mathematical practitioners. Engineers, physicists, economists, biologists, and social scientists are increasingly making use of more and more sophisticated types of mathematics. They need to be properly trained. Many Canadians are proud that Bell Northern Research Company has a high reputation internationally as one of the few research-oriented companies in Canada which competes effectively for international markets. The Science Council of Canada Background Study, **Mathematical Sciences in Canada**, reports that of 126 professionals in BNR, two had doctorates, 26 had master's and 46 bachelor's degrees in mathematics. It is not a coincidence that the Canadian company with the highest research effectiveness is the one with the highest concentration of mathematically trainer personnel.

If Canada wishes to remain in the forefront of the technological age, as it has in the past, it is desirable, indeed essential, that our programmes of mathematics education encourage gifted students to push their mathematical training forward as fast and as soundly as possible.

Are the objectives achievable?

I expect that most of my readers would agree that the three objectives sound fine. However, doubts might be raised as to whether they are realizable, and questions will be asked about how they are to be achieved.

It is my belief that, at least in Western countries, our past methods of mathematics education have been abysmal. We have barely scratched the surface when it comes to developing and unleashing the power of human beings to mathematize. This is chiefly because for centuries the study of mathematics has been overshadowed by a powerful inhibiting factor which reveals itself in the widespread fear or awe of mathematics. Perhaps, the root source of this fear is the manner in which the rote learning of Euclid was drilled into many successive generations of European, American, and Canadian children. Or perhaps its source is the stance of superiority which the mathematically-gifted, particularly university professors, have often assumed. Whatever the cause, I believe that a fear of mathematics and a feeling that "higher mathematics" (i.e. anything beyond $2 + 2 = 4$!) is accessible only to a gifted elite have been the chief factors in preventing the majority of Canadians from entering with joy and satisfaction into the pleasant field of mathesis.

This conviction was reinforced by my visit to the Soviet Union in April and May of 1977, when I had the opportunity to learn something about the "Kolmogorov reform" and to observe mathematics lessons in two schools at grade 3, 9, and 10 levels. Already in grade 3 (but not in 1 and 2), mathematics is taught by a specialist mathematics teacher. The system is not divided into elementary, junior, and senior schools. The child enters at the age of seven and continues in the same school for 10 years. All pupils study mathematics every year; six periods per week in grades 3 through 8, and five in grades 9 and 10. The textbooks and, in principle, the programme are uniform throughout the USSR. Apparently, the reformed curriculum which has been gradually

6. If three hens lay three eggs in three days, how many such hens will lay 100 eggs in 100 days?

Answer on pg.

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