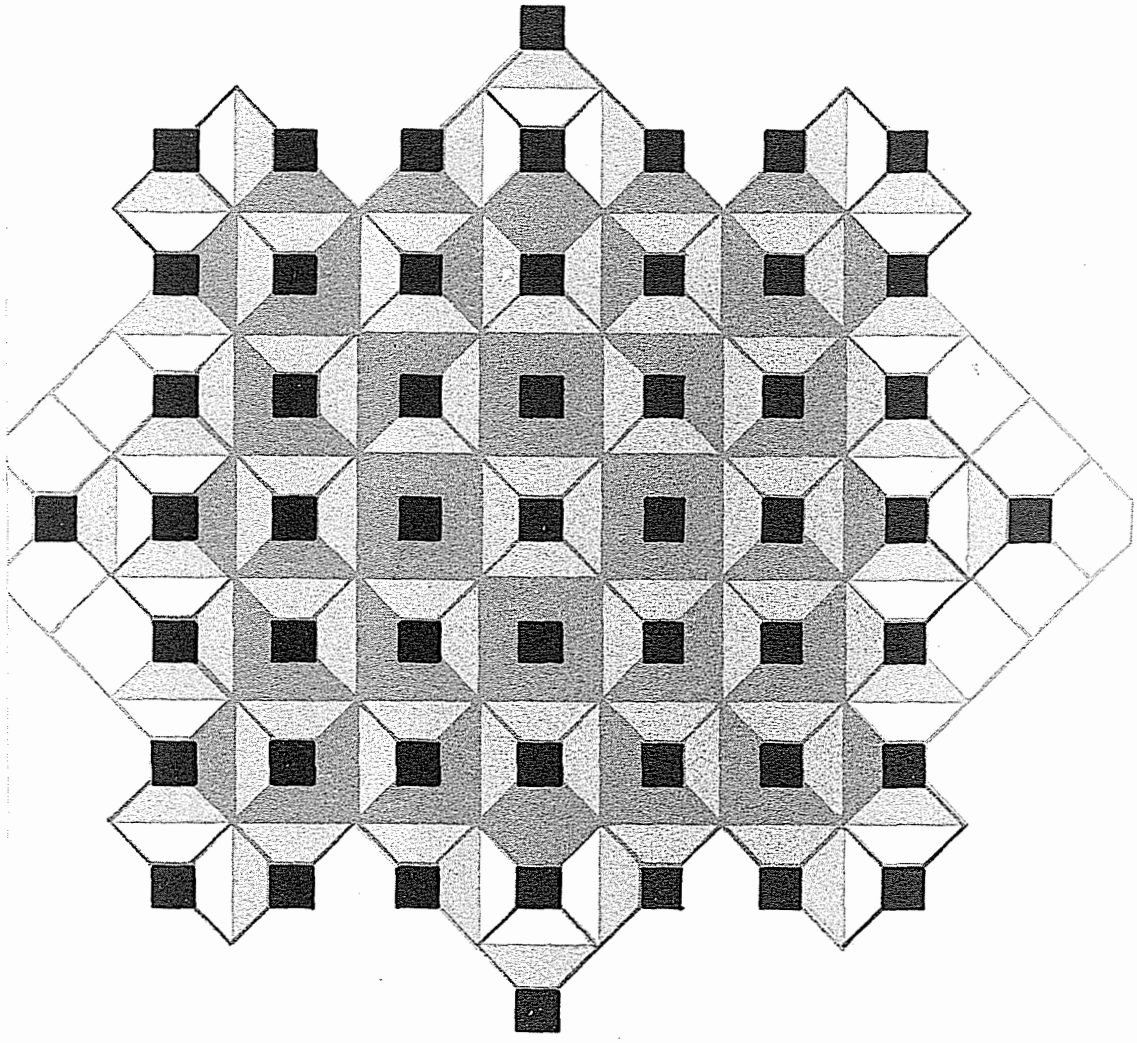
ontario mathematics gazette

VOLUME 18 NUMBER 1 SEPTEMBER, 1979



THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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EDITORIAL

September, 1979

Another summer has come and gone. Once again, the smiling faces of your pupils are before you. May this year be a successful one for both your pupils and you.

Another Federal election has come and gone. A new government has been elected - a Conservative Government. What changes will this bring about? The new Conservative Government in Britain is taking a hard line on spending. Local Education Authorities have already moved to save money by not hiring supply teachers and the drive for comprehensive schools has slowed. Mrs. Thatcher has announced that Grammar Schools still in existence need not change. Mr. Clark has promised a decrease in the Civil Service. Local Boards of Education are struggling with declining enrollments. Yet some young people want to be teachers and are willing to train, to work as occasional teachers, to enter the teaching profession. I say, more power to them and good luck. The joy in teaching is its own reward.

INTERNATIONAL YEAR OF THE CHILD

1979

Année Internationale de l'Enfant

As the Year of the Child comes to a close I suggest that it will have been worthwhile if it has focussed our attention once more on the importance of **observation** and **reflection** about the individual pupil. In recent years, the advent of the behavioural theory and the emphasis on performance competence, on process rather than outcome have brought a renewed interest in direct observation. Now tape recorders and video-tape machines allow a reexamination of a pupils behaviour. Skills developed through systematic observation of pupils and knowledge gained of their behaviour will help develop a genuine awareness and sensitivity to the individuality of pupils. Further, a deep understanding of children's development and of mathematics is necessary.

Perhaps, this fall, when you have a helper in the classroom, i.e. a parent, a student teacher, a visitor; take time for careful observation. Follow this up with a mathematical assessment of the pupil's behaviour, then commit your thoughts to paper. Notice that I am suggesting that you go one step beyond the familiar anecdotal record you may be accustomed to keeping. Perhaps an example will help.

Jason
Jason, age six, was putting number tablets onto a hundred square. There seemed to be about three sets of tablets. He picked up 31, but held it upside down. He held it near the 13 on the hundred square, pondered for a while, then rejected it.

Otherwise, his tactics seemed to be: first find the right row, then find the right column.

He picked up 36.

"Can you read it?" I asked.

"Three and then six," he said. He found the third row, then the sixth column, and put the tablet on top of the 36 on the hundred square. He picked up 45, and hovered around the correct square. Then he realized that there was a 45 tablet there already.

60 and 40 went straight in the correct places. So did 91.

It is all too easy to identify Jason's failures. Presumably his teacher had given him this exercise to reinforce his ideas about the notation for tens and ones. She could have said that Jason did not understand place value, and even had some trouble matching the numerals.

However, let us think positively. Jason's reading of 36 as "three and then six" may not have shown an understanding of tens and ones, but it was a very good operational description of how he found the right place on the hundred square for each tablet. Furthermore, he was able to deal with two attributes at once and in such a way that we could almost say he had a feeling for coordinate geometry.[1]

*As this new year begins make an effort to assess a pupil's work. Tape his dialogue, have it transcribed, write up your assessment, for writing is an excellent way of focussing one's attention. Send it to me. The **December** issue of O.M.G. will focus on children as the International Year of the Child comes to a close. Let us hear what your pupil's are doing. You may be surprised by the child's thinking!

*As suggested in the March editorial, your **March** 1980 issue will have as its theme **Problem Solving.**



*For the **September** 1980 issue, one year hence, we will have Professor Sam Pravica as our guest editor. I know he desires the same whole-hearted support that you have given me.

Addendum

As indicated above, the September 1980 issue of the Mathematics Gazette will have me, Sam Pravica, as the guest editor. I welcome the new challenge and am looking forward to producing an "exceptional" issue.

Although it is still a year away, producing a magazine requires planning ahead. It is a time to focus on a theme, collect materials and generally reinforce relationships with many of my fellow colleagues in education.

As the theme "The Year of the Child" wanes, a new one is emerging with even a stronger emphasis - **Education of the Exceptional Child**. This movement particularly interests me because resources are sparse in the area of mathematics for the exceptional child.

The September 1980 issue then will focus on the theme of exceptionality. What is happening in your board; what is happening in your classroom? Take a moment to gather your thoughts and send me a submission. I will also be calling on the many acquaintances I have made over the years to send me information with regards to special education in their regions. My expectation for the issue will be to give our many readers an overview on what's happening in special education, where resources may be obtained and what courses are available for those interested in upgrading.

Sam Pravica

1. Fielker, David S. "Words, Symbols and Operations" **The Journal**, May 79 Canadian Association for Young Children.

WHAT'S NEW AT THE GAZETTE

The editorship of the journal continues in London, but relies heavily on help from contributers about the province as well as from other countries. As it is the Associate Editors who are largely responsible for ensuring the supply of quality articles, I would draw your attention to the new Associate Editor, Professor Sam Pravica. Sam has been a Mathematics Consultant with the Halton Board of Education and is presently teaching in both the elementary and secondary panels at the Faculty of Education, The University of Western Ontario. He has been a regular contributor, keeping us informed about the work of the Intermediate Mathematics Validation Committee.

We will miss the services of Frank Ebos and Sharon McPhail this year, as both are leaving the editorial board. They have served long and meritoriously. We will miss Bill Higginson for the year as he will be in Great Britain on a well earned sabbatical leave. O.M.G. hopes you enjoy a good year **Bill** and that you return refreshed and renewed ready to begin work once again. We welcome an old friend Dr. Gerry Vervoort, who will be writing the **Book Reviews** column.

The Advisory Board is representative of the mathematics groups we serve and functions not only to advise but also to read articles as they are submitted. Dr. Clive Reis has returned from leave to take over from Dr. Lai Chan. As Alex Norrie is now our president, I have obtained the help of Olive Fullerton, math consultant with the Toronto Board of Education and Wes Keener, math consultant with the London Board of Education to help with this work.

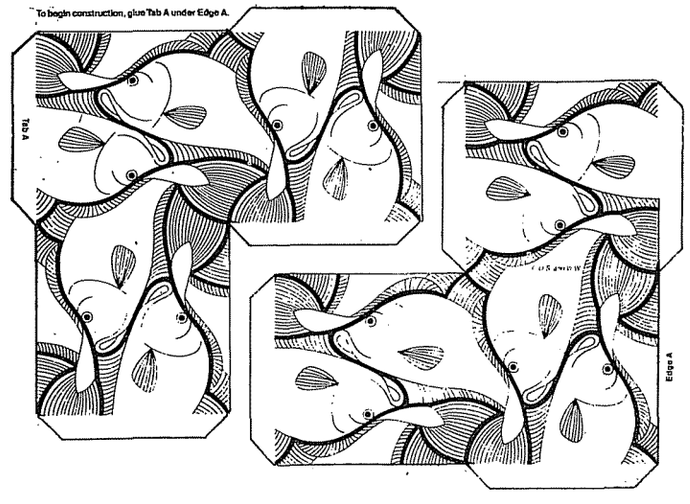
Miss Dorothy Elliot was unable to proof read this past year, but it is hoped she may return this year. Norah Logan and Sam Pravica will continue to help with the proof-reading, a task that is most necessary. Business and mailing arrangements continue in the capable hands of the McGregors.

Do take time to answer the enclosed questionnaire. O.M.G. needs your replies to determine if we are serving you well. Last year two new features were introduced to the Gazette, bringing the total of core items to eight. This year sees another innovation - hopefully a continuing column. Its continuity however will depend upon you. Do read Ed Barbeau's article, "An Application of a Locus." Do fill in the questionnaire.

I cannot finish this column without a mention about the excellent Spring Conference hosted by Bill Jeffers and the members of the Pine Ridge Chapter. We, who have been regular attenders have now completed the round. It has been a pleasure to have visited with mathematics teachers in each area of the province to have enjoyed the student residences in several different Universities, and to have seen the beautiful locations, the variety of architectural styles, and the regions. It has been an education.

Next year's conference brings a change as we move to the Prince Hotel in Toronto. Program planners have most speakers lined up and it promises to be one of the best. Plan now to attend.

Bob Smith



OAME
MAY 8 TO 11
1980
PRINCE HOTEL
TORONTO

LETTERS TO THE EDITOR

Algonquin College of Applied Arts and Technology
College des arts Appliques et de Technologie
21 March 1979

The Editor,
Ontario Mathematics Gazette

Dear Bob,

The last issue of the Gazette has at the bottom of page 47, "... the limit approaches infinity", and also "... one of the solutions approaches infinity as a limit."

Infinity means quite literally not finite, that is, unbounded or limitless. The authors appear to be saying "the limit approaches the limitless".

In my experience, students have difficulty with the symbol . It suggests some sort of number to them. It surely is unfair to students to add to this natural hazard a casual usage of the word infinity in a context which appears to imply that infinity is indeed a number.

Why not use the simple English, "the limit does not exist", and "one of the solutions becomes large without limit"?

I do not believe that common practice is a good enough justification for the creation of a conceptual hazard for students to stumble over.

Sincerely,

(FGB Maskell)

Editors Note: Professors Litwiller and Duncan replied to Fred's letter stating "We appreciate your concern and agree that your comments have merit."

Merivale High School
Ottawa, Ontario
April 18, 1979

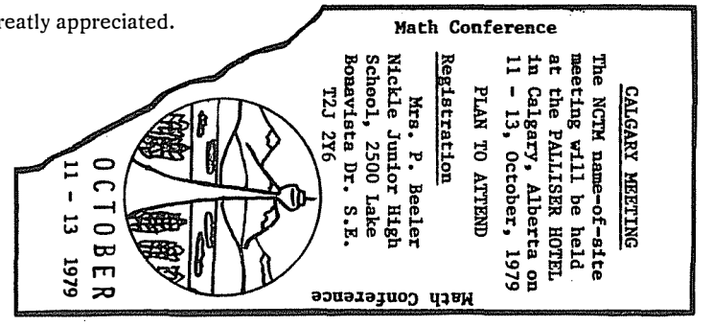
Dear Sir:

Re Article in Ontario Mathematics Gazette on the Rule of 78

I would be interested in receiving all the available information (articles and publications) related to the Rule of 78.

We would like to incorporate this topic into our Mathematics of Investment course at Merivale High School.

Any information would be greatly appreciated.



Bluevale Collegiate Institute
Waterloo, Ontario

Dear Sir:

I would like you to publish the entire article on the Rule of 78 as per your suggestion in an abbreviated article on this same rule in the March 1979 issue of O..G.

If the response to publish this entire article is not sufficient please send me a copy of the article mentioned.

Sincerely,
Anton E. Fieder
Head of Mathematics

Editors Note: Both Mr. Pelton and Mr. Fieder have been sent copies of the entire article as those were the only two requests received, we have decided not to publish Dave Campbell's original article.

State University of New York
Faculty of Educational Studies
March 12, 1979

Dear Bob:

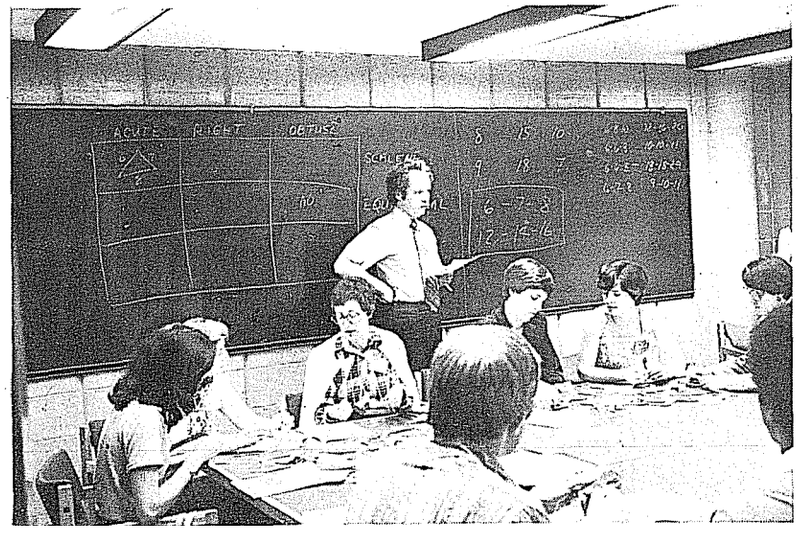
Thank you so much for the copy of the **Gazette**, and for including my orphan. The journal is very attractive (especially in comparison to the somewhat pretentious tone of **Math Teacher** ... my membership on the editorial panel notwithstanding).

Two small editorial points:
1. In the second paragraph, I mean the word to be **causal** not **casual**. Why I chose the former is something I should think some more about.
2. The first footnote on p.37 should have the colleague name be **Walter** not **Walker.**

These aren't worth making a fuss about - perhaps the first will be an inspiration for me to write now in justification to the difference.

Sincerely,
Steve Brown

Editors Note: We would be very pleased to hear from you again. I do believe the problem was one of proof reading. Particularly as I must bring errors not caught to your attention. In March issue, on **P.** 54 3(b) and top of P 55 the $\Rightarrow$ and $\therefore$ symbols have been left out. Also on P. 54 read 2 Even and odd Every odd entry in the table is surrounded by (8 not 9) even numbers (Fig. 3).



Congratulations Pine Ridge Chapter. A Great Conference!

Father Faught Award 1978 awarded to Glebe Collegiate Institute, 212 Glebe Ave., Ottawa



Tom Richford

Ken Dworschak

Andrew Dobrowski

Jonathan

Shelagh Pepper

Christine Hubbard

Norman Cohrs (Head Math Glebe)

A MATTER OF CONSCIENCE

F.B. Maskell, Algonquin College

Ross Honsberger declared in the Preface that his latest book "is written from the viewpoint that all the riches of life are gifts from God and that we should receive them with gladness and thanksgiving and share them to His praise and glory."

J.S. Bach felt the same way about his music; all his compositions are autographed, "In Nomine Jesu".

Such men and women in much of modern society would find themselves obliged to become "otherwise thinkers".

Anatoly Shcharansky is a computer scientist in the USSR whose Jewish consciousness led him into conflict with the political requirements of the Soviet state, and he became an "otherwise thinker". He is now on Amnesty International's list of prisoners of conscience, having been sentenced to a total of 26 years confinement, half of it at hard labour.

In an open society where freedom is taken for granted, one may wonder to what extent such accounts of its repression are accurate. In fact, there is an increasing body of careful documentation and observation becoming available; some of the more recent is listed below.

Leading scientists in this country were among those making representations to the USSR on behalf of Leonid Plyushch, a mathematical engineer who was subsequently released in response to these representations.

Similar moves on behalf of Shcharansky have been initiated, notably by University of Toronto's Professor Israel Halperin, who invited fellow mathematicians to support his appeal to Prime Minister Trudeau to do something, and by McGill University's law Professor Irwin Cotler, who prepared an 800 page legal appeal.

In Ottawa, mathematics teachers in the universities, college, and secondary schools undertook to approach their colleagues who engage in the teaching or practice of mathematics and the related actuarial, computing, and statistical sciences, to give them the opportunity of signing a simple statement to the Soviet Ambassador in support of the legal appeal.

The publicity given by the press of this country to Shcharansky's case places upon us the onus of deciding whether silence on our part may reasonably be interpreted as consent, whether the freedom we enjoy implies an obligation to speak our minds, and whether as members of a profession we accept a responsibility for a matter of conscience.

The Council of the Canadian Mathematical Society saw fit to publish Professor Halperin's letter (now out-of-date since the change of government but appended below). Of course, some members of the Society are active in international mathematical circles.

For members of the OAME and readers of this Journal, the matter may seem more remote. Even so, is it perhaps one which calls for some leadership by the mathematical teaching profession in Ontario?

Mathematical Morsels, M.A.A., 1978 P. VII

The Fame in the Darkness, The Russian Human Rights Struggle, Victor Sparre, Grosvenor; may be obtained from Canterbury House, 760 Somerset Street, Ottawa, K1R 6P9, $5.95.

Prisoners of Conscience in the USSR: Their Treatment and Conditions, Amnesty International, Box 6145, Station J, Ottawa, K2A 1T2, $2.50.

A Chronical of Current Events: Journal of the Human Rights Movement in the USSR, translated into English from the Russian, especially No. 50, November 1978, Amnesty International, $3.45.

To Build A Castle -- My Life as a Dissenter, Bukovsky, Viking; may be obtained from Penguin at any bookstore, $21.95.

Dear Colleagues,

On June 5, 1978 I addressed a request to the Prime Minister of Canada, the Right Honourable Pierre E. Trudeau, that he ask the President of the USSR to release Anatoly Shcharansky with permission to emigrate, and that he ask the President of Uruguay for the same for Professor Jose Luis Massera. My request was supported by letters from many persons, mathematicians and others.

The Prime Minister's reply of August 9, 1978 was courteous and sympathetic but he did not say he would take the action requested. I believe that those of us who are most concerned that this action should be taken, have some obligation to show that a truly significantly large number of Canadians wish that this action should be taken.

Letters have gone and continue to go from doctors, lawyers, dentists, academics, MP's and others. I appeal to every mathematician to write a short note to Mr. Trudeau, referring **to the request of June 5, 1978 by Toronto mathematician Isreal Halperin,** and expressing support for that request.

It is important that a copy of each such letter be sent to me, and that this be indicated on the letter to Mr. Trudeau.

Thank you.

Israel Halperin

University College
University of Toronto
Toronto, Canada
M5S 1A1
22 May 1979

Dear Professor Smith,

Here is a list of the winners for this year's Beatty Prize essay context, which I would be grateful for you to publish along with the announcement of next year's contest.

John Del Grande and I were wondering whether you would be interested in running some of the top essays, suitably revised, in the Gazette. This would provide an extra bit of encouragement for the winners and their teachers. What do you think?

With best regards,
Ed Barbeau

Editors Note: I agree and will run one or two of the essays in the December issue. What could be more fitting?

THE SAMUEL BEATTY PRIZE MATHEMATICS ESSAY CONTEST

Six cash prizes and three honorable mentions have been awarded in the first Samuel Beatty Prize Mathematics Essay Contest, sponsored by the Samuel Beatty Fund established in 1952 by the Mathematics and Physics graduates of the University of Toronto. Here is the list:

First prize ($200) to Doug Bonn, Grade 13, Northern C.I. & V.S. Sarnia, Ont. Topic: "Egyptian fractions"
Second prize ($100) to Lai Lane Luey, Grade 12, York Mills C.I., North York Topic: "Sir Isaac Newton"
Third prize ($80) to Paola di Paolo, Grade 13, North Toronto C.I., Toronto Topic: "Game Theory"
Fourth prize ($60) to Kevin Jardine, Grade 12, West Humber C.I., Etobicoke Topic: "Surface theory"
First junior prize ($40) to John Chew, Grade 10, University of Toronto Schools Topic: "Area of a quadrilateral"
Second junior prize ($20) to Michael Tanglao, Grade 7, St. Martin's School, Smithville, Ont. Topic: "Sir Isaac Newton"
Honorable mention: Peter King, T.A. Blakelock H.S., Oakville Topic: "Sir Isaac Newton"
Brigitte Duchesne, Branksome Hall, Toronto Topic: "The use of computers in medicine"
Michele Brouwers, Oakridge S.S., London Topic: "Topology"

The contest will be repeated in the academic year, 1979-80, under similar rules to the first contest. Further information can be obtained from E.J. Barbeau, Secretary, Samuel Beatty Fund, Department of Mathematics, University of Toronto, Toronto, Ontario M5S 1A1.



Social Time at Pine Ridge was great also

VALIDATION - THE FINAL STAGE IS SET

Sam Pravica, Faculty of Education, The University of Western Ontario

The modus operandi of a committee is generally determined by administrative bodies attempting to fulfill a perceived need. Often the need as identified by administration is perceived by those at the receiving end as shortsighted or opposite in view. The Intermediate Mathematics Validation Committee was established to prevent such a polarization. For a committee to achieve success, both groups must feel a sense of satisfaction with the end product. In order to achieve a satisfactory end product the needs and concerns of those required to implement the product must be obtained. But what happens to the mandate of the committee if the response from the field provides little or no feedback upon which major decisions rely? It necessarily must change, otherwise the committee cannot fulfill any function.

The number of completed questionnaires returned to the Validation Committee from the field, was to say the least, disappointing. Of the small number of boards or schools responding, most of the feedback relative to common concerns on specific topics, showed, you guessed it, approximately a third were neutral, a third responded positively, and a third negatively. When puzzling about the large majority that did not respond, the lyrics by Randy Vanwarmer, "You left me just when I needed you most", kept repeating like a broken record in my mind.

It was the chairman's foresight in selecting a membership that represented a broad education spectrum and his extremely capable leadership that enabled the committee to fulfill its mandate.

The committee met for one week in June. The task - to review the responses from the field and begin formulating recommendations. The meeting ended on a seemingly divergent theme. It appeared that the members overreacted to the realization that the revisions would be largely a committee responsibility and consequently attempted to incorporate the philosophy of the groups they represented as well as their own into the document. Nerves became frayed. The deadline for completion loomed frustratingly near. With only one week left in July to complete our task, little seemed to have been accomplished. In retrospect, I believe that this reaction was a result of the unexpected "birth of twins", the lack of feedback from the field and, the sudden added dimension to the decision making process placed on our shoulders.

But in fact, a great deal was accomplished in June. Members returned in July knowing that they had to come to grips with issues as a unified group and not as a fragmented one. Major decisions in fact had been achieved. What remained was to organize the package such that the decisions made were philosophically and conceptually consistent.

The task was completed on schedule. A number of changes were recommended and in my estimation the committee has put together the best possible guideline given the constraints of available time and information feedback.

What are some of my personal reservations? Firstly, will teachers recognize the document for what it is - a guideline? The information between the covers suggest a content for curriculum and program development at local levels. It cannot provide any more than that. The educational needs for students in Ontario are quite different from community to community. This became increasingly clear as the committee tried to come to grips with identifying a common core for the province. We were unable to define the meaning of "core" to everyone's satisfaction. Try it with a group of teachers from **different** schools. It's an interesting exercise. What should be core and what optional? For some students the suggested core program may be more than they can comfortably handle, for others a core program alone could stifle creative minds. For these reasons local boards must make the final curriculum decisions. It is often said the Ministry officials are too far removed from the classroom to make realistic judgements. This then, is all the more reasons why boards must be given the autonomy and accountability to develop programs to meet local needs. A Provincial Curriculum Guideline must necessarily suggest a degree of prescriptiveness, yet be flexible to accommodate for a variety of situations.

Secondly, content without suggested processes and pedagogy is subject to misinterpretation. For example, the interpretation given to the statement, "use properties of arithmetic operation such as commutative, associative ..." or the statement "volumes of rectangular solids and capacities of containers developed through activities, estimation and calculation", can vary a great deal from teacher to teacher. What is meant by "activities", "real world problems", etc? Precise definitions of these words are as impossible to define as the word "understand". There are varying degrees of interpretations given to these terms which are directly related to the user's

experience, background knowledge, and interests. Although it would be desireable to incorporate these processes in the document, two reasons dictate against it. First, the syllabus would become unmanageable in size. Secondly, it is contradictory to suggest a uniform approach and still maintain the philosophy of a flexible document. Teaching strategies are as individual as learning styles. The responsibility for identifying suitable strategies must rest with local boards. The teacher's responsibility is to take these suggested processes and translate them into classroom practice - dictated by student and community needs.

Thirdly, left for individual boards to interpret the guideline, how will consistency be maintained in its implementation throughout Ontario? This will largely depend on whether or not boards of education perceive this as a problem and how much attention they wish to give to inservice education. Much of the initial frustrations associated with interpreting new curriculum material can be minimized if local curriculum development groups seek assistance. Some of the committee members can be freed to provide inservice for curriculum committees. As I mentioned in the March issue, members of the mathematics departments of the faculties of education will be available on a consultative basis to assist in the development of a curriculum for boards requiring such expertise.

In addition, the Ontario Mathematics Coordinators' Association has produced (presently under revision) a very informative booklet, "A Rationale for Elementary and Secondary School Mathematics". It was an effective resource for the Validation Committee during the decision making process and will be a valuable resource for the upcoming review of the Senior Division Mathematics Guidelines. This booklet can be obtained on request. I will be happy to respond to letters of inquiry. Drop me or Bob Smith a line.

I also anticipate that the O.A.M.E. conferences, regional and annual, will include in their themes activities associated with the implementation of the revised guideline. The question of implementation need not be as imposing as it initially appears. Human and material resources are available. Time and money will be the most difficult barriers to overcome.

Finally, criticisms of the guidelines are anticipated. That's the price one pays for attempting to optimally satisfy a large group of individuals ... you just can't please everyone. The committee expects it and welcomes it because that is how future guidelines are improved. From my vantage point, the lack of response from the field suggests that the majority of teachers were reasonably content with the initial syllabus. It is highly likely that the majority of teachers accept the realities of teaching, that regardless of what is on paper, ultimately it is their professional judgement that translates content and philosophy into classroom practice.

The guideline will be in your hands shortly after this issue of the Gazette is distributed. Keep in mind that it is just that, a guideline developed to meet the needs of a variety of students. As is true for any document either in business or in education, inservice accompanying its distribution is extremely important and in this case is the responsibility of local boards of education. Insist on it. Get clarification if you sense a feeling of vagueness. Assistance is available. Use it. If you feel you have a concern, put it on paper and send it to the Gazette. What better media to air your views?

We were given a challenge, we responded ... Senior Division, here we come!

THE UNSTABLE ARITHMETIC CURRICULUM: SOME PERSPECTIVES AND PROPOSALS FOR ELEMENTARY SCHOOL TEACHERS

Dr. Roland F. Gray, Professor of Education, University of British Columbia

Editors Note: Dr. Gray has brought Brownell's Meaning Theory of the Fifties and the Modern Math Movement of the Sixties nicely into perspective. He sees them as complimentary and urges teachers to employ this view in teaching arithmetic.

Two important curriculum reform movements have influenced the teaching of elementary school mathematics in this century. The earlier movement, variously expressed, but most generally known as the Meaning Theory of teaching arithmetic, influenced the literature of arithmetic teaching from 1935 until after World War II. The later movement, commonly referred to as the Modern Mathematics Movement, exploded into the literature as a part of the curriculum reform

movement of the 1960's. While it was not always recognized, the two movements were interrelated and complementary. Had they been adequately seen as such, the arithmetic curriculum might have remained more stable, more vital, and better able to withstand the current "Back-to-the-Basics" pressures than seems presently to be the case.

The Meaning Theory emerged as a result of the research and writing of numerous scholars following lines of investigation suggested by William Brownell in 1935. In the tenth yearbook of the **National Council of Teachers of Mathematics**, of that year, Brownell described three theories of arithmetic instruction which he labelled the Drill Theory, The Incidental Learning Theory, and the Meaning Theory. In his discussion of the Meaning Theory, Brownell stated that children should be taught arithmetic in a meaningful way in order that they might understand the arithmetic processes they were called upon to perform. In large measure, Brownell's position could be seen as a revolt against the then prevailing rote drill procedures where understanding was expected to emerge from repeated practice. (3)

During the 1930's and 1940's, Brownell, his students and other investigators, produced a body of research literature that seemed to lend support to the views of the meaning theorists. A general summary of the findings of these studies would include the following statements of points of view: (1, 5, 8, 10, 14, 15, 21, 22, 27, 28)

a. Development of understanding of general principles, patterns, relationships through concrete learning experiences followed by practice on facts and algorithms seemed to result in less time being required for drill, better retention of facts over time, and superior transfer of principles to new and untaught procedures. (14, 15, 21, 22)
b. Premature drill could serve to inhibit development of maturity of understanding later. (27:187-202)
c. The basis for teaching arithmetic meaningfully was said to lie in the mathematics of arithmetic. This concept called for understanding place value, the zero principle and the structure of the numeration system before practicing basic facts and algorithms. In learning subtraction, for example, it was thought, and to some extent demonstrated, that if children understood how numbers in a base ten numeration system could be re-named by re-grouping as in the case, $174=100+70+4=100+60+14$, then less time would need to be spent on practicing the subtraction algorithm. (5, 27:253-170, 28: 71-89)
d. Basic facts, instead of being learned separately, should be presented in patterns or groups such as, $6 + 2 = 8$, $2 + 6 = 8$, $8 - 2 = 6$, and $8 - 6 = 2$. Also in learning multiplication, such patterns should be sought as that which occurs when 9 is a factor, **viz.,** that the sum of the digits of the products, except for zero, always equals 9; the digit in ones' column decreases by one and in tens' column increases by one. Children should be urged to look for and discover such patterns for themselves because of expected retention and transfer benefits. (15,22,28:71-89)

The meaning theorists had a strong impact on thinking about the teaching of arithmetic, but perhaps less influence on actual classroom procedures. Many of this group of researchers and writers were primarily concerned with the psychology of learning. (1, 3, 14). They stressed the importance of learning general principles of mathematics and stated that meaningful and understandable arithmetic rested on the mathematics of arithmetic, but they were not always precisely clear about what mathematics would render arithmetic meaningful. The theory, in this sense was never fully complete and, although texts were written to reflect the Meaning Theory, the teaching of arithmetic in many classrooms, continued to be characterized by social applications or rote drill procedures. (28:71-89).

By the mid 1950's, as a result of changes in university level mathematics curricula, professional and academic mathematicians undertook a study of the mathematics preparation of elementary and secondary school students. Judging from new school mathematics curricula that were to emerge, these scholars seemed concerned with what they perceived to be a deficiency in mathematics understanding, a lack of acquaintance with some of the less traditional mathematics such as non-Euclidean geometry, group theory and probability, and an inadequate independence and ingenuity in approaching problem-solving situations. The existing school mathematics curricula, they believed, failed to give students insight into the underlying structure of mathematics and tended to produce too few students adequately prepared to succeed in a study of higher level mathematics. (12:15-23). They urged a new approach to the teaching of school mathematics and by the early 1960's new programs of study for elementary and secondary schools began to appear.

These new programs, produced with the active participation of college and university level mathematicians, came to be known generally to teachers as the "new math". Such seemingly new topics as geometry and set language were introduced into the primary grades and non-base ten arithmetic, modular mathematics, probability and more rigorous and extensive number theory,

were introduced into the intermediate grades. The commutative, associative, distributive, identity and inverse properties were employed as a vehicle for promoting understanding of some of the structure of mathematics and great stress appeared to be placed on precision of language used in describing mathematics. Logic, mathematical models and sentences were used in some programs as an approach to problem solving and in others considerable time was devoted to developing understanding of relations, functions and operations in a more or less systematic manner. (2, 4, 6, 9, 16, 17, 19:17-27, 20). In the main, the "new math" programs were designed to produce students who could both do arithmetic and understand arithmetic as a branch of mathematics - an aim not inconsistent with that of the earlier Meaning Theory.

The modern mathematics movement had in reality, to a great extent, carried on where the Meaning Theory left off, by providing a more precise delineation of the particular principles, properties, relationships and generalizations upon which a meaningful program of arithmetic could be built. Unfortunately, this connection between the two movements was not always made explicit, and, in classrooms, the new curricula were sometimes seen and taught as separate and distinct from anything that had gone before. The langauge employed exemplified by set theory, geometry and the field properties frequently seemed arcane, esoteric, or simply unintelligible to some teachers. In many cases, inadequate opportunity was provided for teachers in large numbers to become acquainted with the underlying mathematics of the new programs and to see its relationship to the teaching of ordinary arithmetic. (24,25). To many, it appears, arithmetic seemed to have disappeared from the elementary curriculum and had been replaced by something called elementary school mathematics. In too many cases, seemingly, the teaching of basic arithmetic seems to have received inadequate attention. Teachers', often bewildered, perhaps, and unprepared to handle the new programs in the manner intended, seemed to fall back on what they knew best--teaching by drill procedures, but now by drilling on language and bits and pieces of mathematics that failed to add up to an integrated study of arithmetic. This phenomenon has been described by one writer as a "New Formalism." (23)

The outcome was almost predictable. Some writers alleged that students in considerable numbers failed to attain the desired outcomes of the "new math," and also, in many cases, seemed less well prepared in the traditional arithmetic areas of basic facts and algorithms. (12, 13, 18) By the early 1970's public criticism of the "new math" began to be voiced, and by mid-decade the "new math" was being branded a failure. (7, 12, 13, 18, 24) School districts, state and provincial departments of education, under public pressure, mounted a "back to the basics" movement. In arithmetic classes this seemed to mean a return to rote memorization of basic facts and algorithms and more systematic and rigorous testing of children's memorizations. At the present time, it would seem that, in some situations, the arithmetic curriculum has gone full circle and is about where it was shortly before 1920.

Teachers of elementary school mathematics are now faced with a forked road situation. On the one hand, they can accept current "back to the basics" pressures and return to rote procedures for teaching arithmetic. On the other hand, they can, once again, come to grips with the problem of developing a more meaningful approach to arithmetic, but this time by integrating the contributions of both the Meaning Theory and the Modern Mathematics Movement. The former will be relatively easy to do. The latter, however, will be more difficult. It will require much time and thought by each teacher, but, if the accumulated literature of the past is correct, such an effort is more likely to produce the outcomes so long sought in mathematics education. Despite current public pressures, it is this second approach that needs urgently to be followed.

By way of a single and rather simple example of how a teacher might analyze the task of teaching arithmetic and incorporate the contributions of the two movements discussed above to achieve an understanding of arithmetic as mathematics, let us look to the problem of teaching multiplication. There appear to be about six steps in the procedure a teacher might follow. At the outset, the procedure should be independent of textbooks and other pre-packaged materials, except as these sources may be of help to the teacher in understanding the process. The procedure is discussed in some detail below and the reader might find it helpful to follow the development with paper and pencil.

First, the teacher must ask what has to be learned. The answer is that children must undertsand the operation, remember the 100 basic facts, and be able to use the multiplication algorithm to find answers to multiplication combinations with factors greater than 10, e.g., 16 x 12.

Second, the teacher must ask, what do children need to know in order to be successful in learning multiplication. The answer to this question is more complex. In sum, children need to know most of the concepts and skills antecedent to multiplication. These include number concept, the structure of the base-ten numeration system including the process of re-naming, the addition

operation, the commutative and associative properties for addition, and memorization of many, but not necessarily all, of the addition facts. (Some addition facts can be memorized while learning multiplication).

Third, the teacher must ask how to find out whether or not the antecedent skills and concepts are present. The best way to proceed may well be to ask appropriate questions individually while children are working at arithmetic. (8, 27:5-47)

Fourth, the teacher must ask what to do with those who have not yet acquired the antecedent skills and concepts. The answer to this is to decide on two groups of children. The first will consist of those who are lacking skills and concepts that can not be acquired in learning multiplication. The second will consist of those lacking skills and concepts that can easily be acquired in learning multiplication. To the first group must be taught the antecedent concepts and skills before proceeding to multiplication, otherwise they will most certainly fail in multiplication. It is of no consequence that some of the unlearned skills and concepts should have been learned last year or the year before, but it is important that these be taught and learned now.

Fifth, the multiplication operation should be presented initially as a shortcut for addition. Thus 6 x 7 is presented as $7 + 7 + 7 + 7 + 7 + 7 = 42$. It will readily be appreciated by most children that remembering the answer to such examples as 6 x 7 is easier than adding six sevens. This is an adequate initial approach to multiplication. More sophisticated conceptualizations can await added maturity.

Sixth, the teacher should analyze the table of multiplication facts and decide on the most economical way to proceed to introduce early those concepts and principles that will render children independent in finding answers in multiplication situations. The teacher should make an examination of the basic facts, reproduced below, and then determine how many facts must be memorized, how many can be remembered easily by recognizing patterns, and how many can be found quickly by employing the commutative and distributive properties. The challenge is to see how easily this can be done. Teachers might play a game with themselves by asking how many **really** have to be memorized, and then go through a process of eliminating those that can be learned without memorization. They may be surprised at the outcome.

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	35
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

It will be seen quickly by observation that many facts don't have to be memorized at all. Consider for example:

a. The zero facts. Early in the century the zero facts were considered by some to be the hardest to remember. However, by employing the concept of zero as the name of the empty set from the "new math," then by showing a number of empty sets it can be easily seen that the name of a collection of empty sets if still zero. Hence $1 \times 0 = 0$, $2 \times 0 = 0$, $3 \times 0 = 0$, and so on. Therefore, the zero facts really don't need to be memorized. There are 19 zero facts that can now be eliminated.
b. Similarly, the ones facts can quickly be learned by applying the identity property, again from the "new math". One is the identity for multiplication. Simply put, it means that one times anything is anything, so that one group of 7 is 7 and likewise $1 \times 6 = 6$, $1 \times 5 = 5$, and so on. Therefore, the ones facts don't need to be memorized. There are now 17 additional facts eliminated.

c. The two facts and five facts are easily acquired by addition or skip counting. In fact, many children will have already acquired them by game-like counting by twos or fives. Thus, 7 x 2 is simply a matter of counting by two until seven of them have been counted. Fingers can be put to good use for keeping count and 7 x 2 becomes 2, 4, 6, 8, 10, 12, 14. Similarly, with the fives. Twenty-eight additional facts can thereby be eliminated.
d. The nine facts have often been thought to be the most difficult to recall. In reality, the nine facts may be the easiest to remember--at least easiest to figure out quickly. When 9 is a factor the digits of the answer (the product) always sum to 9, except for zero, and the first digit (in 10's place) is always one less than the multiplier. Given the combination 7 x 9 the digit in the 10's column must be 6 (one less than 7) and the second digit must be 3, the number which when added to 6 gives 9. Thus 7 x 9 = 63. This pattern was reported in part by Thiele in 1935. (22) Even more facts, not already eliminated above, do not need to be memorized.
e. At this point, a total of 75 facts have been eliminated from rote memory procedures, which leaves 25 not previously eliminated. However, many of these can also be eliminated. Consider 4 x 3 and 3 x 4. These are two of the 25 combinations not yet eliminated, but by applying the commutative property it is readily seen that these two combinations do not need to be learned separately. It is clear that 4 x 3 and 3 x 4 equal the same number - 12. By applying the commutative property similarly to the 25 remaining facts it will be seen that 10 more facts can be eliminated. This leaves a total of 15 facts that need special attention--much fewer than the original 100 facts.
f. One of the 15 facts remaining is 8 x 7 (or 7 x 8). By applying two more properties from the "modern math" literature, facts like this one can be aquired without memorization. Consider the principle that any number can be named many ways and apply it to 7, then 7 can be named as 3 + 4, 4 + 3, 5 + 2, etc. By choosing 5 + 2, writing the combination 8 x 7 as 8 x (5 + 2) and applying the distributive property, 8 x 7 now becomes 8 x 5 and 8 x 2--both easy combinations already acquired by skip counting by 2 and 5. Thus 8 x 7 is 8 x 5 + 8 x 2 or 40 + 16 = 56. The key here, of course, is to rename a number such that easy and already known combinations are involved. By applying the distributive property in this way to the remaining facts, then it could be argued that none of the facts has to be memorized by rote. Also, the transfer power of learning to use the distributive property to find answers to unknown combinations is important. If a child can find the answer to 8 x 7 by approaching it as 8 x 5 + 8 x 2, then nothing new is involved in finding the answer to 16 x 12, which becomes 16 x 10 + 16 x 2. The distributive property is the basis of the multiplication algorithm and explains such hitherto mysteries as why the second partial product is moved to the left and why the two partial products are added. Learning to apply the distributive property early may render children independent later in more complex situations.

It may be the case that teachers will see this approach as too simplistic. But when it is considered that some three to four years are spent memorizing the multiplication combinations with considerably less than complete success, then it is worth trying a different approach. Certainly some rote drill may still be required, but it should be employed after development of insights into the patterns, properties and principles discussed above. A frequent error in teaching multiplication lies in starting with rote drill instead of ending with it.

A further comment seems necessary at this point. While children may well develop an ability to solve multiplication examples through following the foregoing procedures, it will still not guarantee automatic recall of all 100 facts. What teachers need to add to this procedure is a lively repertoire of problems, games, puzzles, etc., which provide opportunities for children to **remember** the combinations by using them in a wide variety of challenging activities. For help, teachers should turn to the extensive literature available on enrichment. games and laboratory activities.

In closing it seems appropriate to state that by applying a similar analysis and procedure to other aspects of teaching arithmetic in the elementary schools, teachers themselves could assure that arithmetic could become meaningfully taught. Children might, then, acquire understanding of those principles, properties, and generalizations that apply to nearly all mathematics, and thereby produce a larger number of students better prepared to study more mathematics. This is an objective long sought by mathematics educators, and is still worthy of pursuit.

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The track of Billy's electric train is 300 inches in circumference while Jack's is 150 inches around and his train runs two thirds as fast as Billy's. If the two trains simultaneously leave the stations which are side by side, how often will they arrive there at the same time? Answer on pg. 40

EULER'S RELATION IN PLANE LANGUAGE

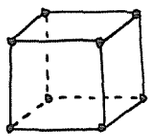
James E. Riley, Western Michigan Unviersity

Reprinted from Mathematics in Michigan Volume XVII #1

Many elementary teachers are aware of Euler's relation for simple polyhedrons (solid figures), V + F = E + 2, where V is the number of vertices, F the number of faces, and E the number of edges. The cube (figure 1), for example, has 8 vertices, 6 faces, and 12 edges.

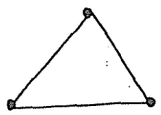
$$V + F = E + 2$$
$$8 + 6 = 12 + 2$$

Figure 1



There is analogous relationship for plane figures (sometimes called graphs) which is R + V = L + 2, where R is the number of regions the plane is partitioned into by the figure, V the number of vertices, and L the number of lines. For example, the triangle (figure 2) partitions the plane into two regions (inside and outside) and has three lines and three vertices.

Figure 2



This relationship can be used to motivate later elementary children in discovery and drill in number relations. One could start by showing the children a number of figures and pointing out the regions, vertices, and lines. The children could then be given a series of figures and asked to tabulate the regions, vertices, and lines. Try this yourself with the following graphs (figure 3) and complete Table 1.

Figure 3

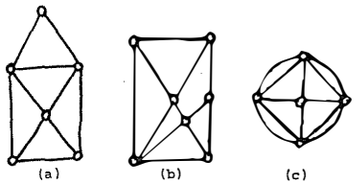


Table 1

Graph	R	V	L
a		6	
b			
c			

After the children have completed their tables ask if any among them can identify a pattern. They may or may not discover Euler's relation. The fourth graders at Woodward School, Kalamazoo, Michigan, where we tried this had no difficulty with it. If they had we would have given a few hints.

When the children discover the relation ask them if they think it will always work. Ask them to make up their own graphs to test the relationship. Ask them to try and make a graph for which the relationship does not hold. Why don't you, the reader, get a piece of scrap paper and try it now.

As you and your students are convinced that the relationship is always true you may consider the following question. Given any triple (R, V, L) for which R + V = L + 2, can you construct a graph

showing the relationship. Try it yourself using the triples (a) (3, 5, 6) and (b) (7,4,9). Make up some triples of your own and then construct the graphs.

There is one final extension. The relationship holds for any graph in the plane — even those that do not partition the plane into two or more regions. For example, the graph below (figure 4) has one region, 13 vertices, and 12 lines.

$$R + V = L + 2$$
$$1 + 13 = 12 + 2$$

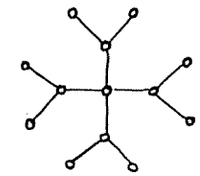


Figure 4

Your assignment for tomorrow is to verify Euler's relation for the following graphs (figure 5).

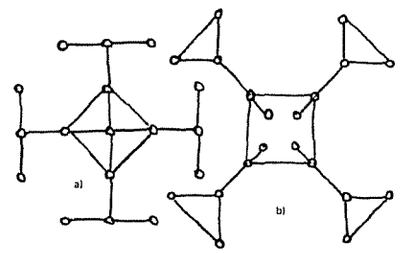


Figure 5

Reference

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A PSYCHOLOGY OF CHILDREN'S ARITHMETIC, WITH IMPLICATIONS FOR EDUCATION

Herbert Ginsburg, University of Maryland Baltimore County

This paper proposes some psychological principles which are firmly established by recent research, and which can offer broad guidelines for educational practice. The principles do not themselves dictate educational practice; instead they provide a general framework within which educators can develop sound approaches to classroom activities. Also, it should be emphasized at the outset that the guidelines cover only a very limited range of mathematics. They focus on elementary arithmetic, on counting, on the understanding of a few fundamental principles concerning whole numbers, and obviously offer little direct application to other aspects of mathematical activity of interest to educators. At the same time, since arithmetic is fundamental to early mathematics education, these principles speak to a central concern of mathematics educators.

In recent years, there has been an enormous increase in developmental psychology research into young children's mathematical thinking. Much of this research was of course stimulated by

Piaget, and focuses on traditional Piagetian problems, like the conservation of number. Piaget finds, as everyone knows, that the child below the age of about 6 or 7 years fails to recognize that two sets remain equivalent in number even when one of them is changed so as to look different from the other. This finding has been replicated many times, in a variety of cultures, and is probably as certain as any result in psychology. While Piaget uses the conservation result to draw implications concerning preoperational and concrete thought, we use it here to ullstrate our first certain proposition.

1. **In some key respects, children's thought is qualitatively different from the adult's.** There is no need to enter into debate concerning the adequacy or details of Piaget's theory. At the minimum, his results demonstrate that in some key respects young children are very different from us. This is the starting point from which we must begin, and if this is the only way in which we use Piaget's work, then we have learned a good deal from him.

Usually those accepting Piaget's contention that children's thought is sometimes distinctive limit themselves to a focus on young children's immature or deficient thought. A main lesson often taken from Piaget is that young children **cannot** conserve, or **cannot** think flexibly or **cannot** engage in hypothetico-deductive thought. While there is something to this, recent research stresses the opposite point, which is our second proposition.

2. **Without the benefit of schooling, young children possess basic mathematical "intuitions."** Research shows that before attending school or in cultures without schools, virtually all children possess some basic mathematical perceptions, techniques, ideas, or procedures--all of which I lump together in the deliberately vague term "intuitions."[1] For example, when presented with two randomly arrayed sets of dots, as in figure 1, children as young as 3 years of age can consistently point to which of the two sets "has more." African children in the Ivory Coast,



Figure 1

members of illiterate cultures, can solve such tasks at about the same level as North American children (Posner, 1978). This is done without the benefit of counting, and seems to depent on the perception of physical attributes like area covered. This "perception of more" is obviously not foolproof. It breaks down with large numbers and with small differences between sets, and with particular misleading arrangements of the objects (as when the less numerous set covers a larger area than the more numerous). At the same time, the perception of more is a fundamental mathematical "intuition" possessed by virtually all children. And there are many other intuitions of this type.

Not only do young children begin with intuitions, they also develop a "practical arithmetic."

3. **Without the benefit of schooling, young children develop a practical arithmetic, involving elementary addition and subtraction**. Before the onset of schooling in Western societies, and in societies which offer no formal education, children at the ages of 3 or 4 begin to develop a "practical arithmetic" which usually involves the addition and subtraction of concrete objects (Hebbeler, 1978). The young child's society makes available to him a cultural legacy of great value, namely a system of counting. In many societies throughout the world, even "primitive" societies, counting is a well-developed system, passed down through the generations; the child does not have to invent it for himself. In the absence of schooling or before its onset, he assimilates counting from adults and then spontaneously applies it to basic problems in addition and

[1] Much of the basic developmental psychology research into children's mathematical thinking is summarized in two books. One is by Gelman, R. and Gallistel, R. Cambridge Mass; Harvard University Press, 1978. Another book is my own, **Children's Arithmetic: The Learning Process**. N.Y.: D. Van Nostrand Co., 1977.

subtraction. At first, he can solve such problems only when concrete objects are involved. But by the age of 5 or 6, he can work with imaginary objects as in ordinary story problems like, "how much are 5 apples and 3 apples?" By the age of 5 or 6, his experience in calculation leads him to memorize certain basic number facts. Soon after that, he can short-cut the process of calculation by the use of various mental tricks, usually involving redistribution, as when he solves the problem 9 + 5 by converting it into a simpler form, 10 + 4. By adulthood, individuals in illiterate societies can engage in impressive feats of mental calculation, and even exhibit some explicit understanding of certain arithmetic principles involved (Petitto, 1979). Cross-cultural research provides us with the perspective to realize that a good deal of sensible, powerful, and even explicitly understood arithmetic can be accomplished in the complete absence of schooling.

Provided with intuitions which may be a biological contribution, and with a practical arithmetic which is mainly a cultural legacy, the young child usually enters school at about the age of 5 or 6 years. There the child encounters a formal, written, codified system of arithmetic, a very clear and specific cultural legacy of great potential value.

4. **Children assimilate what is taught in school into what they already know.** Before entrance to school, children already know a good deal about arithmetic, at least at an implicit level. This informal knowledge provides them with a framework with which to assimilate, to incorporate, or to interpret the codified school arithmetic. The assimilation takes many forms. From the present point of view, one of the most important types of assimilation is the "invented strategy." Children do not simply calculate written problems in the manner taught, via the standard algorithm. Instead, children invent a combination of algorithmic procedures and practical arithmetic (Houlihan and Ginsburg, unpublished). Children use various tricks, they improvise to solve problems in a manner congruous with their already existing cognitive structures. For example, to solve the problem 6 x 8, one child spontaneously converted it into 4 x 6, whose product he remembered, and then added together two 24s. He cleverly transformed the given problem into a sequence of problems that for him were easier--that is problems that he could assimilate into already existing cognitive structures. This kind of cleverness, of exploitation of already existing knowledge, of economical shortcuts, of "cheating"--all of these are at the heart of invented strategies.

Of course, children do not always succeed at their arithmetic. Often they make mistakes.

5. **Children's mistakes are usually the systematic results of organized rules**. Mistakes are not the outcome of some kind of random process. Nor is it very helpful to think of mistakes in terms of vague notions of "low intelligence" or "low mathematical aptitude". Instead it is useful to consider mistakes as the outcome of determinate cognitive processes. In the case of column addition, for example, it is usually possible to specify the mistake generating rules with both precision and ease. Addition errors usually occur because children employ particular types of alignment procedures, or "carrying" procedures, etc. There is nothing particularly mysterious about the errors or the reasons for them.

6. **Often there is a gap between academic and informal knowledge**. We have seen that children come to school with already developed intuitions and a working practical arithmetic. In school, children attempt to assimilate academic arithmetic into already existing structures. But the attempt is not always successful. Often children fail to understand basic mathematical principles; often they develop faulty calculational procedures which systematically lead to error. Furthermore, many children find it so difficult to assimilate academic mathematics that they see it as a topic existing by itself, without any relation to anything real, or to the child's own informal procedures. Children like this can seriously believe that the exact same problem can yield one answer when calculated "in the head" and another when calculated on paper, the way it is taught in school. In short, there exists a wide gap between the child's reasonably effective informal knowledge and his "book learning," which in the popular imagination is "academic," in the worse sense of the word, namely trivial, useless, senseless.

In summary, these are fundamental principles with which it is hard to quarrel: children may think in different ways from adults; children begin with mathematical intuitions and a counting-based practical arithmetic; they interpret academic knowledge in terms of what they already know, and develop "invented strategies"; their errors are the result of systematic procedures; and often there is a wide gap between their reasonably sensible and effective informal mathematics and their faulty and poorly understood school arithmetic.

These principles can provide an effective framework for early mathematics education, particularly in arithmetic. First, however, a preamble: the principles talk mainly about the nature of children's knowledge; they do not say very much about the nature of learning or techniques for teaching. The existing research does not point to one specific method for teaching arithmetic, like programmed

instruction, or the discovery method, or mathematical games, etc. Personally, I believe on the basis of my observations of children in school that a wide variety of techniques--those mentioned above, textbooks, and just about anything else imaginable--can be effective under specific circumstances with different children. There are many ways to learn; many ways to teach. Consequently, the suggestions to be described below may be implemented in a variety of ways. A second preliminary point is that the suggestions may be applied on a number of different levels. They may be used to design entirely new curricula; they may be used to choose among already available curricula and techniques; or they may be used to develop materials to supplement already existing curricula. First, however, a suggestion concerning testing.

1. **Assessment of children must be flexible**. Existing standard tests fail to measure many of the phenomena--like invented strategies--that seem central to understanding children's arithmetic. Perhaps better standard tests might be developed in the future. To some extent this may be possible, but the attempt is not likely to be very useful. Children's mathematical thinking is too complex to be adequately measured by standard tests of the ordinary type. In cognitive psychology, there is a trend toward the use of more complex assessment procedures; in the study of mathematical thinking in particular, many investigators are beginning to use a version of the Piagetian clinical interview procedure. Such flexibility of testing seems essential, and indeed may become scientifically orthodox in the next several years. In any event, informal testing and interviewing seems to be as necessary for the teacher as it is for the psychologist. Both are researchers requiring an accurate portrait of the child's thought. The research literature already contains many hints on how such flexible testing can be done; but much more needs to be accomplished.

2. **Don't teach readiness activities, Piagetian tasks, or the new mathematics**. Sometimes in the early grades, teachers engage in various sorts of readiness activities--like discriminating between circles of various sizes or attempting to teach the word "more"--of dubious relevance to early arithmetic. The psychological research shows that preschool children are already engaged in genuine mathematical activities from the age of 2 or 3 (at least). They do not need to be made ready to engage in arithmetic; they are already ready. In fact, they already know many of the things which readiness activities try to teach them. As far as Piagetian training is concerned, it seems quite beside the point for at least two reasons. One is that children spontaneously develop Piagetian notions quite on their own, without the benefit of schooling. Therefore, for schools to spend time on the training of Piagetian notions is a waste. A second is that is it not clear how the Piagetian notions relate to school arithmetic. It is quite possible that a child may learn a good deal of school arithmetic without understanding basic Piagetian notions. In any event, the more direct approach is to teach arithmetic itself and forget about the Piagetian notions which will take care of themselves. It is also ineffective to teach the "new math." Today almost everyone recognizes that the new math was a mistake since it ignored the psychological nature of the child. For the young child, abstract ideas conerning sets and logic do not make a grat deal of sense. To achieve such abstract ideas, the educator must paradoxically avoid them at the outset and instead engage the child in appently non-abstract activities like counting.

3. **A good part of an arithmetic curriculum can be based on counting**. Counting-based practical arithmetic is "natural" for the young child. He develops it outside of school, and it is reasonably effective. It seems to me--a psychologist, not a mathematics educator or mathematician--that a good deal of elementary arithmetic can be interpreted in terms of counting. It is important to recognize, however, that such counting-based arithmetic need not involve concrete objects in the usual sense. Children often enjoy playing with numbers themselves; sometimes nothing more concrete is required. Addition can be taught as counting forward from a given number on the number line; subtraction can be interpreted as counting "backward" on the number line. Multiplication can be interpreted as repeated and constant additions or "jumps" on the number line. The addition facts can be interpreted as the patterned results of systematic counting. Thus, if we know what $2 + 4$ is by counting, it is easy to figure out what $2 + 5$ is, and to see the pattern resulting from successive increments of 1. Similarly, I even think that it is possible to teach column addition through counting-type procedures. Indeed, research has shown that young children often carry out a kind of mental column addition in a spontaneous fashion (Houlihan and Ginsburg, unpublished). Thus, asked to add $23 + 34$ mentally, young children often regroup the numbers into $20 + 30 + 3 + 4$, since the individual additions involved are easier to perform mentally than the original ones. This regrouping implicitly exploits the place value approach to addition. A little elaboration of the spontaneous procedure should suffice to make the principles explicit. In brief, counting procedures, including finger-counting, may prove extremely useful as the basis for a good deal of arithmetic. Models for this type of approach already exist, for example in the recent work of Hasler Whitney, Tom O'Brien and others.

4. **Relate the existing curriculum to what children already know**. At every step, the teacher must attempt to find in children's experience and competence some analogue for curricular material.

The trick for teaching is to locate the analogue and make explicit the relationship. Thus, if the textbook adopts a union of sets approach to addition, the teacher may attempt to clarify the matter by drawing the parallel with the child's concrete acts of putting things together. Many children have the idea that school mathematics relates to nothing whatsoever. Given this attitude, they have considerable difficulty in checking their work. The explicit linking of academic procedures with informal ones may help to overcome these difficulties. Thus, children should be taught how to check the results of column addition by making informal estimates via rounding or by counting on their fingers. Children desperately need techniques which can allow them to determine whether their mathematical work makes any sense whatsoever, and perhaps one way of doing this is by providing a link with their ordinary experience.

5. **Encourage children's invented procedures**. Instead of using standard algorithms, children develop various "tricks" for solving standard problems. These tricks involve shortcuts and drawing on available knowledge. Such invented strategies should be encouraged and exploited. They should be encouraged because they involve some degree of creativity and may involve some interesting arithmetic principles. Indeed, these principles can be exploited as the basis for instruction. Thus, when children use a mental addition technique of regrouping, the teacher can abstract from the child's activities a number of principles which are worthy of conscious deliberation.

These then are some principles which may guide educational practice. While they are few in number and while we may have expected more of the results of psychological research, the principles are quite fundamental and can make a basic contribution. They seem to be particularly useful in the case of children experiencing difficulties in the learning of arithmetic. Even in cases where children are labelled as "learning disabled," it is often of great benefit to engage in flexible assessment, to base a good deal of instruction on counting-related activities, to attempt to reduce the gap between academic knowledge and the child's information knowledge, and to encourage and exploit various "invented procedures"

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Coordinators Corner

As one of the original three math coordinators - John Del Grande, Norm Sharp, Win Bates, and since I am retiring at the end of June, I am pleased to make somc comments on the Ontario Math Coordinators. Over the years it has been a pleasure to belong to the association and take part in their discussions and deliberations. Always there has been a pleasant balance between good comradeship and serious attention to issues under consideration. From the beginning a spirit of sharing, honest appraisal and carefully arrived at decisions was the established policy. Curriculum outlines, support materials, testing materials, enrichment booklets, etc. developed in any one centre are shared with

all others. It is not surprising that such a method of operation now involving 14 highly qualified experienced persons vitally interested in mathematics has become an important group in the field of elementary and secondary mathematics. The influence is vastly increased by the range of expertise that each coordinator has through association with Math Heads, teachers and Administrators. A most fortunate organization is created from which a major contribution is understandably expected.

The K-13 rationale referred to in the March issue of the Gazette is the latest work of the coordinators. At present, it comprises a carefully developed position paper for K-8 and 9-13 advanced level mathematics. The final document will include a similar consideration of all levels. This will then provide a sound basis for curriculum building in mathematics.

At a time when so many questions and concerns are raised with respect to the goals of mathematics and methods of implementing them, the guidance of the coordinators is vital.

Editors Note: As the Ontario Mathematics Coordinators' Association brief to the Intermediate Mathematics Validation Committee is now a public document, I should like to quote three small sections from it which I believe are significant. Hopefully, all readers will have the opportunity to read a draft copy soon. "Mathematics, as presented in the rationale, is viewed as a useful tool both in an individual's problem-solving arsenal and in communicating both qualitative and quantitative ideas about the problem-solving process in particular, and relational and spatial ideas in general. Mathematics both complements and enriches the general language development of students. Teachers of mathematics must share, with all teachers, the responsibility of maintaining good standards of both oral and written communication."[1]

"The content of this rationale has been organized into five strands: NUMERATION, ALGEBRA, GEOMETRY, MEASUREMENT and INFORMATION PROCESSING. While the first four strands reflect a generally accepted curriculum organization, the last strand is new in its philosophical approach. This is in keeping with the spirit of the rationale for the NCSM's expanded definition of basic skills which states:

> The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills."[2]

"A century ago, many Canadians were unprepared for the coming of the twentieth century because they could neither read nor write. Literacy for the general populace was therefore an inescapable goal for the education system of our young nation. We are now faced with the vision of a vastly different society in the twenty-first century -- one in which the computer data base will replace the printed word as the medium of storage for much of man's accumulated knowledge.

Today our nation's educational systems are faced with the challenge of providing a revised and expanded set of basic skills including a 'new literacy' for a rapidly changing environment. It is our sincere hope that this rationale will provide a sound basis for the development of mathematics curricula as we enter the second last decade of the twentieth century." [3]

AN APPLICATION OF A LOCUS

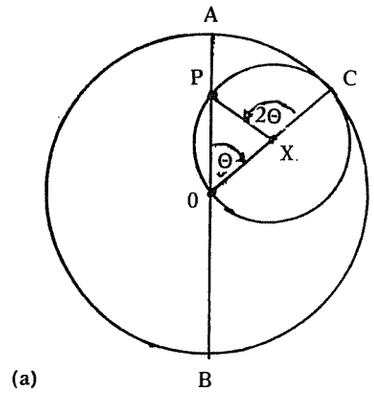
Ed Barbeau, University College, University of Toronto

The National Museum of Science and Technology in Ottawa has a display of miniature models of steam engines. One of these illustrates the so called hypocycloidal engine desinged in 1802 by Matthew Murray of Leeds, England. It transforms the linear motion of a piston into the circular motion of a driving wheel.

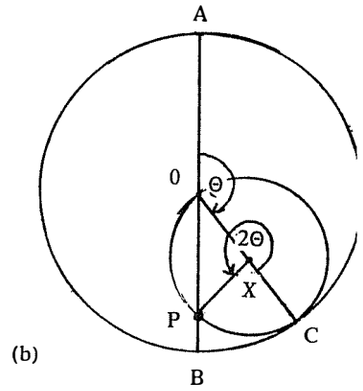
The operation of this engine depends on the fact that the locus of a point on the circumference of a circle, which rolls without slipping on the inside of a fixed circle of double the diameter, is a straight line. An alternative way of looking at this is pictures in Figure 1. Let AB be a fixed

1. O.M.C.A. **A Rationale for Elementary & Secondary School Mathematics** p. 2, Draft No. 1, 1979 05 10
2. Ibid, p. 3
3. Ibid, p. 5.

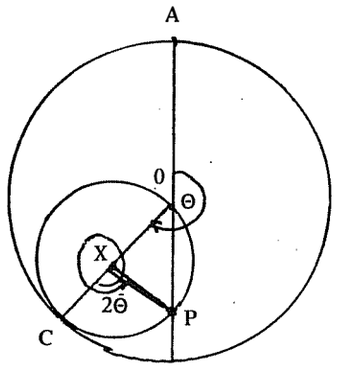
diameter and 0 the centre of the larger circle. Suppose the smaller circle touches the larger at C. Since the normals through C of both circles coincide and since the diameter of the larger is twice that of the smaller, the other end of the diameter of the small circle through C must be 0. Let P be the point generally distince from 0 at which the small circle intersects AB. The assertion is then that the point P is a **fixed point** on the smaller circle.



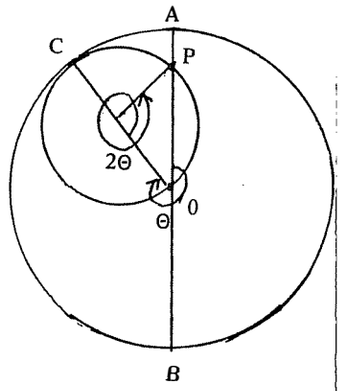
(a)



(b)



(c)



(d)

Figure 1

Since there is no slipping, this assertion can be established by showing that the arc AC is equal to the arc PC. But this is quite straightforward. Let X be the centre of the smaller circle and r be its radius. Suppose angle A0C (= angle P0C) is equal to Θ. Then angle CXP must be equal to 2Θ. (In Figure 1(b), note that angle CXP (reflex) = 360° - angle CXP (obtuse or acute) = 360° - 2 angle C0P = 2 (180° - angle C0P) = 2Θ. Likewise, the size of angle CXP can be checked in all the cases). Therefore, arc AC = (2r)Θ= r(2Θ) = arc PC.

A further observation can be made. Let x be the directed distance of P above 0. Then $x = 2r \sin \Theta$. If P is executing simple harmonic motion with period u about 0, then $x(t) = 2r \sin {}^{2\pi}/_{u}\, t$, where t represents time elapsing since P coincides with 0. Then $\Theta = {}^{2\pi}/_{u}\, t$, so that the point X is moving at a constant speed in a circle about 0.

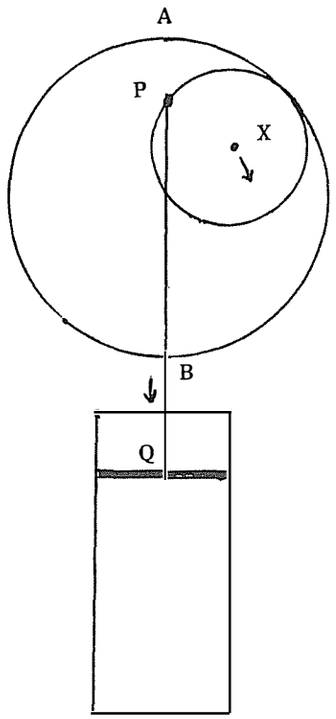


Figure 2

Figure 2 is a scheme of the engine. AB is a diameter of a fixed metal circular ring, with its inner circumference toothed. C is the centre of a movable ring of half the diameter, the teeth on whose outer circumference will mesh with the teeth of the fixed ring. A point P on the circumference of the samller ring is attached to the end of a piston rod PQ. As the piston moves up and down in simple harmonic motion, P moves vertically between A and B, thus driving the point X at the centre of the smaller circle around in a circle.

Once I discussed this locus with a group of high school students. None of them guessed that a fixed point on the smaller circle moved on a straight path, and there was considerable skepticism that this was so. Fortunately, at hand were two movie projector reels, one having twice the diameter of the other, so that a demonstration on the spot was possible.

Editors Note: As there is to be a renewed emphasis placed upon application, O.M.G. requests that you consider sending an interesting application that one might appear in each issue. Ed's article has set a good standard.

A PROJECT TO IMPROVE TEACHING PRACTICES

N. MacEwan, L. Shapiro, G. Drouillard, Herman Secondary School, Windsor

Editors Note: In the September issue, one year ago, O.M.G. called for all of us to be concerned with evaluation. We are glad to print this article, which may prove to be controversial. Do write.

INTRODUCTION

This project started in September 1975, and involved the Mathematics Department of W.F. Herman Secondary School in Windsor. Herman has an enrolment of about 1700 students in a traditional 3-term-year with three sets of examinations. There are about ten full-time Math Teachers in the Department.

PURPOSE OF PROJECT

(i) To define a STANDARD of performance for students, against which teachers could measure.
(ii) To remove "VARIABLENESS" of existing standards demanded by teachers, to ensure fair treatment of students.
(iii) To help teachers evaluate their course-design, teaching, testing, standards, --- with a view to self-improvement.
(iv) To help the Department Head monitor the program offered in the school, with a view to responding to apparent weaknesses.

METHODS USED

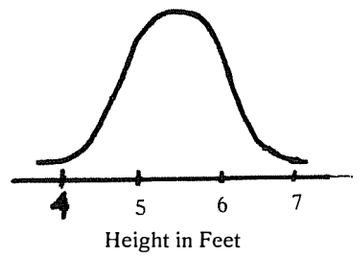
As a result of several Department meetings, where the above ideas were discussed, the teachers arrived at consensus on what a reasonable "standard" was.

This they defined in terms of student performance by means of a "PERFORMANCE HISTOGRAM". This histogram became the STANDARD for the Department.

Ratinale for Definition of STANDARD

When dealing with a large group, the distribution of certain characteristics of the group follow a pattern. As examples:

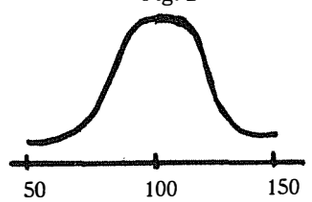
Fig. 1



Performance Levels

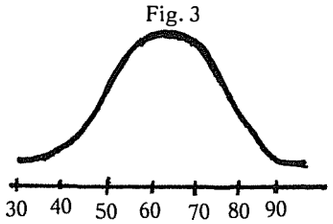
(i) If you were to measure the heights of a large number of randomly selected adults, the distribution would appear as in Fig. 1. There would be very few who were less than 4 feet tall or more than 7 feet tall. Most would be in the 5 to 6 feet category.

Fig. 2



(ii) A second example might be the measures of the I.Q.'s of each member of the group (Figure 2). Few would be retarded, and few would be genius. Most would be near the 100 level.

Fig. 3



(iii) A third example might be the life-expectancy of the group (Figure 3)

These familiar "bell-shaped" curves are called "normal distributions", and most mathematics teachers have studied these curves in University.

Based on the ideas behind the above examples, it would seem reasonable to assume that this pattern would apply to the 1600-1700 students taking Mathematics at our school.

The characteristic in which we were interested was

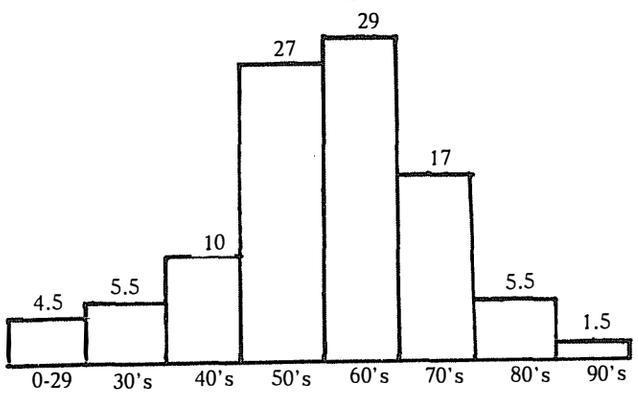
"student performance in mathematics"

Few should perform at the 10% level, and few should perform at the 90% level, with the bulk of the students in the 60+ - 70% range.

Based on this observation, and the mathematical theory behind "normal" distributions, we defined the ideal mathematics-performance distribution or

THE STANDARD FOR THE MATHEMATICS DEPARTMENT

Fig. 4



Past records had indicated that a reasonable Department pass-rate goals would be 80% for each term (i.e. 80% of the students should get at least 50% on the performance criteria established by each teacher in the classroom and the common examinations). Thus, Figure 4 became the STANDARD for the Department.

By dividing the performance levels into deciles, we could now establish what percentage of the students should perform in the 50's, 60's, 70's, and so on.

These percentages are based on the theory behind "normal distribution", and are predictable based on the agreed-upon pass-rate of 80%. (For simplicity, note that the first 3 deciles (0-9), 10-19, 20-29) are grouped into one column, as very few students are expected to perform at less than 30%)

To illustrate:

If our courses are properly designed, and our teaching and testing patterns were proper, we should expect to have 29% of the students performing in the 60's, 17% in the 70's, and so on.

But this at the Department (or school) level!

What of **each** teacher?

Each teacher teaches at least 150 students, divided between the General and Advanced Program, in at least two different grade-levels.

Again, based on theory, we believe these smaller groups are still large enough to have the STANDARD applied to each of them as well!

Thus the STANDARD for the Department became the STANDARD FOR EACH TEACHER!!

Data Collection System

Next we designed a system to collect the data at the end of each term, and organized a team to

analyse the data. A report * was produced creating histograms as indicated in the following table.

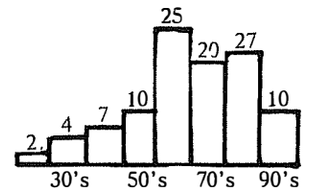
	By Teacher	By Course	By Level (General)	By Level Advanced)	By Dept.
Class (Mark) Performance		✓	✓	✓	✓
Exam (Mark) Performance.	✓	✓	✓	✓	✓
Final (Mark) Performance	✓	✓	✓	✓	✓

Note that the Final Mark is made up of 50% Class Marks and 50% Examination Marks.

Analysis of Results

The next problem to be solved was "How do we analyse the histogram report?"

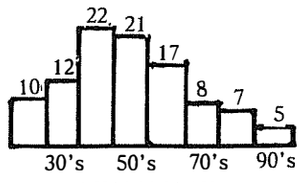
Each teacher was able to compare his/her students against the STANDARD, against the Department, against students taking the same courses, etc., by comparing the shapes of the histograms involved. To illustrate:



CLASS MARK

Consider the Class Mark, Exam Mark, Final Mark Histograms for Teacher A and his 150 + students.

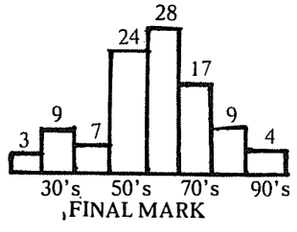
Note that the Class Mark Histogram is skewed (pushed) to the right. This could be caused by setting class-tests that were too easy, or by having an unusually gifted group of students.



EXAM MARK

However, note the Examination Mark Histogram is skewed to the left - perhaps caused by examinations that were generally too difficult. Or maybe the suggested easier class testing did not properly prepare these students for the examination.

*For a sample contact the authors.



FINAL MARK

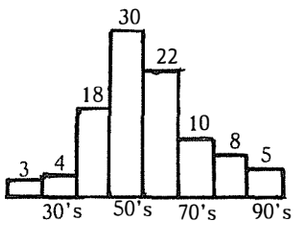
Notice the Final Mark Histogram is almost ideal. But has this teacher done as good a job as the Final Mark Histogram would indicate?

These visual comparisons were very beneficial to the teachers.

But we needed something a bit more concrete than just this visual comparison of Histograms.

Thus the DEVIATION SCORES and CONSISTENCY SCORES!!

The Deviation Score Defined (D-Score)



An example will best describe this idea. Suppose Teacher B has the following Class-Mark Histogram for his/her students. Remember, numbers at the top of each decile reflect percents. i.e. 30% of the students scored in the 50's.

Compare these results to the STANDARD

Class Marks for Teacher B

	0-29	**30's**	**40's**	**50's**	**60's**	**70's**	**80's**	**90's**
Teacher B	3	4	18	30	22	10	8	5
STANDARD	4.5	5.5	10	27	29	17	5.5	1.5
Difference	1.5	1.5	8	3	7	7	2.5	3.5

The Class Mark DEVIATION = the sum of the differences
= 34 (in this case)

So a set of Deviation scores could be calculated for each teacher, (class mark, exam mark, final mark), each course, each level (General and Advanced) and the Department. We had Deviations ranging from 36 to 70, with the median about 50. Deviations less than 55 were considered acceptable, and higher scores caused close scrutiny by teachers.

Using the range of deviation Scores, each teacher could compare himself against the rest of the teachers in the Department.

The Consistency Score (C-Score)

This score was created to measure how consistent the students' scores were from class work performance to examination performance.

Ideally, if the examinations, testing and teaching were proper, each teacher's class-mark-performance-histogram should be congruent to his examinations-performance-histogram.

In practice of course, they are not!

To measure the consistency between the two, we just added the differences between corresponding percentile scores.

For example:

For Teacher A

	0-29	30's	40's	50's	60's	70's	80's	90's
Class Mark Distribution	2	4	7	10	25	20	22	10
Exam Mark Distribution	10	12	22	21	17	8	7	3
The Differences	8	8	15	11	8	12	15	7

Consistency Score C = 8 + 8 + 15 + 11 + 8 + 12 + 15 + 7
= 84

It's difficult to interpret the C-score by itself -- it can be generally agreed that the ideal C-score should be zero based on the congruence assumption. But what constitutes a "good" C-score?

So far, the only way we have used the C-score was to compare C-scores within the Department. For example, the C-scores for the 9 teachers for one term were as follows:
56, 24, 33, 36, 30, 38, 16, 66, 51
As a guide we calculated the C-scores for the General Level (28), for the Advanced Level (23) and for the whole Department (25).

This gave each teacher a chance to compare his C-score against the Department C-score. Clearly a score of 16 was the best among the teachers compared to C = 25 (Department). Also, a score of 66 would cause that teacher to examine more closely what was happening in his classes.

GENERAL OBSERVATIONS

1. At the Department Level

The first time we did the analysis, we noted that the Department distribution was skewed to the right. We interpreted this as a weakness in our testing program and thus each teacher was encouraged to give more challenging problems to the students.

2. At the Course Level

The histograms clearly showed:
(i) a weakness in the transition from grade 10 to grade 11; (The grade 10 histogram was pushed to the right and the grade 11 to the left. We are now trying to toughen the grade 10 course and ease up a bit on the grade 11)
(ii) that our expectation from General Level students is too high, and we're trying to correct that.

3. At the Teacher Level

The histograms gave the teachers an opportunity to make the various comparisons mentioned earlier.

The C and D-scores gave encouragement to those who scored well relative to the Department, and provided incentive to others whose scores were not as good.

The Department set as a goal, a reduction in the next set of D and C-scores -- a reasonable measureable goal. This should help to reduce the variableness, one of the goals of the project.

The Department Head could also use the C and D-scores to indicate what teachers needed closer supervision and direction.

4. At the Examination Level

The D and C-scores on examinations could give a measure of the effectiveness of the examinations, when compared with the range mentioned earlier. So far, we haven't had time to do this.

5. At the Class-Mark Level

As above (at the Examination Level)

The Big Change

As a result of this project, the most significant change in teacher behaviour took place at the testing level. Without exception, all teachers became more careful about the degree of difficulty of tests they gave their students.

Further changes took place with teachers at the high end of the C and C-scores range. They were reduced in several cases quite dramatically, as a result of changed testing patterns - a healthy sign to reduce the variableness of standards.

The Teacher Reaction to the Analysis;

We have been blessed with a staff that is open to try things that are new, and require more effort on their part. The Head of the Department surveyed the teachers with the following results.

1. Without exception, all teachers wanted the analysis to continue in future, as it did help them evaluate what they were doing.
2. None considered the analysis a threat to them, but one did admit it created some pressure to do a better job. They were concerned about how the information was to be used.
3. All teachers liked the idea of being able to make comparisons, to see how their students were doing, and to see how they (the teachers) were doing relative to the rest of the Department.

4. All teachers accepted the fact that student performance and C and D-scores were not absolutes in evaluating the effectiveness of a teacher.
5. In discussing the range of C and D-scores, most teachers preferred to have the Department range and average D and C-scores published rather than have names assigned to scores. The Head was very much in favour of this.
6. As mentioned above, the most significant change in teacher behaviour took place in their testing patterns and procedures.
7. Many of the teachers agreed that, in their opinion, this type of analysis might benefit other Departments within a school.
8. The collection of this type of information could be quite meaningful to teachers, when viewed over a 2 to 3 year period, as trends might start to develop in that time. This gives the teachers a more accurate picture of what's happening in his classes, and would encourage him to react in a positive way.

Impact on the Windsor System

The project-team made a presentation to the Secondary School Principals on this project, and what the benefits are. They (the principals) were overwhelming in their approval of the theme and spirit of the project. As a result, Windsor has changed the existing computer reporting system to reflect the decile format in line with the project's recommendations.

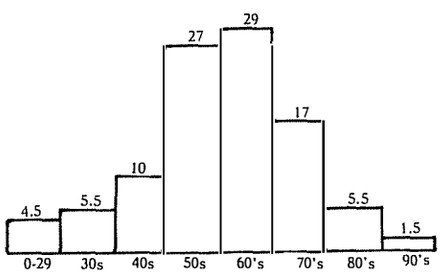
The University of Windsor Gets Involved

The project team made a presentation on the above concept to a group of Essex County Secondary Mathematics teachers at a seminar sponsored by the University of Windsor.

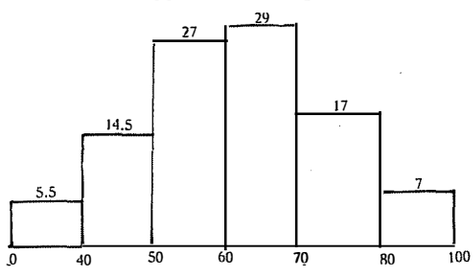
In the audience was C.S. Wong, Ph.D., a Statistics professor who was intrigued by what we were doing.

As a result, Dr. Wong enlisted one of his graduate students in Statistics, F.Y. Li to undertake a Computer Analysis and 10,000 class simulation using our basic assumptions. A 60 page report (partially funded by the NRC) was prepared. Statistically there were a few bugs in the concept, but the concept was fundamentally statistically sound. The report's major recommendation we are adopting this year is as follows:

Original Model



Revised Model
(suggested by Computer Analysis)



The final version of this 60 page report will probably be completed this school year. Based on this better model, our data collection system will have to change.

CONCLUSIONS:

With the support we've received so far, we can forsee the day when the computer will do **all** the analysis, draw the histograms and calculate the D and C-scores for other Departments that want this type of analysis as well as our own. As more opinion is brought to bear on this theme, the idea will be further refined and improved, as it was by Dr. Wong and Miss Li.

As a result of this broader exposure, the setting of realistic standards and the improvements in testing teacher self-evaluatioln which we are now experiencing, can only improve the quality of teaching and the educational program of our greatest resource -- our young people.

PROJECT TEAM
N. MacEwan, Head of Math
L. Shapiro, Assistant Head
G. Drouillard, Assistant Head

INTERVIEW

Taped Talks with Bob Smith
. . . David Fielker



David Fielker is the Director of the Abbey Wood Mathematics Centre, Inner London Education Authority and editor of **Mathematics Teaching**. It was my pleasure to meet David in November, 1975 at his centre and to renew acquaintances in Leicester Square on Tuesday, June 12, 1979. It was a typical English spring day with some sun, then overcast and even some rain by the end of the afternoon. We had a bite of lunch together at a nearby pub, then wandered into the square to find a quiet bench for the interview. It was not all that quiet! A musical group entertained at the top near the theatres, a negro sax player was being filmed nearby and children fed the pigeons while the tourists wandered across the square. However, these did not distract us.

R.S. First off, tell us about Edith Biggs.

D.F. She is spending most of her time completing her doctorate, which I gather is almost finished. But she can be enticed out occasionally. In fact, she was with me last March, helping out on a course at Abbey Wood and as lively as ever; as full of ideas.

R.S. It is fifteen years since Nuffield and Edith Biggs first made their contribution. What effect has their "hands on" approach had in Britain?

D.F. Well, I suppose it depends very much on what you put your hands on. Edith has always been very keen on the environmental aspects of mathematics and Nuffield took this up as well. Now, there were some very good things about this approach. In a way it was applied mathematics. And it got children out of their desks; they worked with a friend; they went out and measured things, and Mathematics became an active subject.

R.S. The pencil and paper were used to record findings only, then.

D.F. Well, a problem existed in that it was very easy to measure everything in sight but not so easy to do something with the measurements when the children returned to the classroom.

The other phrase that was bandied about so much was "Discovery Approach". People thought that you went out into the environment and you discovered mathematics. Well, this is absolute rubbish. I don't think that either Edith or Nuffield ever said this (that Mathematics was discovered). Teachers eventually found that their pupils were not discovering mathematics in the environment. So I feel that practical methods have suffered somewhat of a setback. In recent years teachers have given up environmental mathematics because it is not delivering the mathematical goods.

The trouble is that the teacher has to play a part in the discovery process; to ask the right question, and to pinpoint what the children are discovering. The biggest difficulty in teaching is knowing what question to ask and to ask it at the right time, no matter what method you use. Teachers in general are not experienced enough to know this, in mathematics.

R.S. So your work at the Centre is to help teachers learn more mathematics and to change their attitudes toward the discipline. Do the teachers come for help with strategies or in fact to learn more mathematics?

D.F. I think they come wanting anything to help them teach mathematics in a better way and that is certainly what I try to do. For most courses, I do not have a set plan to follow. For instance, annually we run a full-time six-week course. On the first morning, there is a bit of work to do so

that we might have something to talk about. A time table schedule is posted but not filled in. It is blank. I suggest they fill it in with the sorts of things they feel they need, as they go along. A lot of time is spent in discussion, looking at the literature, making manipulative materials, much of this without my help. They usually ask to visit schools. There are two basic things I do insist on, however. Either my colleague or I will teach children and follow this up with discussion.

Also, each teacher must go around to a neighborhood school and teach three children for an hour each week. This is something they have not had much chance to do, since in your own classroom you seldom have the opportunity to sit down for an hour with only three children. You never really have the chance to see how a child gets to work on a problem, to find out how he thinks. Then in order to focus the teachers' attention on what is going on, we ask them to write out their observations on a ditto, which is circulated and discussed.

R.S. What a good idea!

D.F. Well, writing is a good way of focussing one's attention.

R.S. Do you have a feeling sometimes that your sessions are becoming repetitious?

D.F. Well, although teachers do have the chance to make things for their class, to take away lessons for Monday morning as well as a wealth of ideas, what I am really trying to do is to get teachers to think about what they are doing. I am not so interested in **curriculum** development. I am more interested in developing **teachers** as people.

R.S. Surely you are hopeful that they will be people thinking about mathematics.

D.F. Well, fair enough. Yes. They will have to learn more mathematics... and none of them know enough. They also must meet new ideas about teaching. I find this works best when I teach children myself. Not that I expect them to imitate my way, but I hope that they might become aware of other ideas about teaching. After the first time they observe me teach, they have commented "Well, you didn't tell them anything," or "You didn't tell them when they were right or wrong", "You didn't praise them", "You didn't correct them." Now, I am able to point out that, in fact, children can correct themselves when they are wrong; children can make judgements about other children's work. It is good for them to learn to rely on their own ideas rather than to become too dependent upon the teacher.

Generally, I would like to challenge teachers to think about what they are doing.

R.S. Let me change the subject. Your interest has been geometry for the primary school up to age eleven and in fact this has been the topic of several articles you have written. What are your concerns?

D.F. Well, my original concern was that nobody did enough of it in the primary school. When I first asked teachers what geometry they did they talked about things like area and length, things which were really to do with measurement. This is partly due to the fact that mathematics for most primary teachers has to do with arithmetic. The feeling they have is that you can not do any geometry unless you have some numbers attached to it. This lack of concern for the teachinf of spatial relationships - geometry without number, then, was a second concern that became evident. Where there was non-metric geometry being taught I found it lacked sequence and purpose. There was much fragmentation. More often than not tangrams, curve stitching, or tesellating were done on a Friday afternoon when the children were too tired to do anything more taxing mathematically. Very often symmetry was done at Christmas time when it provided stars or angels or christmas trees for decorations.

R.S. Teachers do see a development in number; from cardinal and ordinal numbers, through operations and algorithms, to the various sets of numbers and relationships. Skills and techniques are much more in evidence. Why cannot this be developed in geometry?

D.F. Teachers have certain obsessions about geometry apart from the above mentioned gimmicks. One is the obsession about shape, and unfortunately, it is not even concepts about shape, rather it is the recognition of shapes only. But, the triangle is always equilateral and always the same way up. Another obsession is feeling that you must do three dimensional work before two dimensional. Certainly we should do both, and there are all sorts of interrelationships we could explore, including perspective drawings. But as far as I know there is no evidence per se that children should do three dimensional work first. It is one of those myths around in education. In the first place, the three dimensional world is really quite a complicated one. In the second place, although

children live in a three dimensional world there is some evidence, I believe that they view it as two dimensional. In face you never know when they use a name whether they mean the two dimensional phrase, e.g. a triangle, or if it is their label for the three dimensional shape - a triangular prism.

R.S. What have you been doing to help primary teachers structure the study of space?

D.F. If algorithms are a large part of the study of number, and the part teachers feel quite comfortable with, then I would say that the advantage of geometry is that there are no algorithms. There are no procedures for doing certain tasks and this frees teachers to look at geometry in so many other ways. However it also requires that math educators outline some of the other ways of imposing some sort of order on it so that teachers will know answers to the sort of question that keeps popping up: "When do I teach this?" Unfortunately, the teaching strategies are not easy because the concepts are, on the whole, mathematical and psychological and cannot be described briefly. The best thing I can do is refer you to the recent article I wrote for the February issue of **Educational Studies in Mathematics** titled "Strategies For Teaching Geometry To Younger Children".

R.S. Yes, I have read it. It is an excellent article. Unfortunately, it requires a teacher to go off on his own and construct a syllabus. Indeed, that is what you have been attempting to help the teachers do but I am afraid it would be a little foreign to most teachers.

D.F. Well, that may well be true. I suppose you have to weigh up your teacher and decide if you are going to give them a syllabus to work from based on this idea or encourage them to go it alone. I would prefer to get teachers to want to do it themselves.

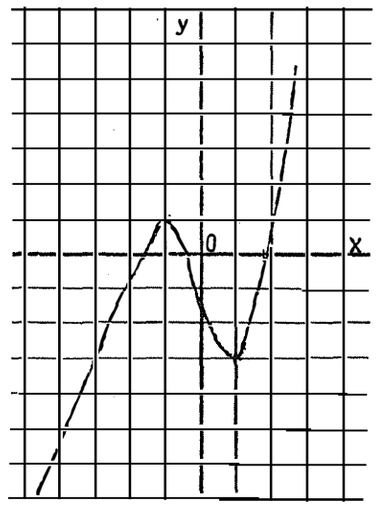
R.S. David, it has been great talking with you. I do hope you will be able to bring your views personally to Ontario and in the near future.

D.F. I would enjoy that very much. I have been to British Columbia and worked with Jim McLean in Illinois, so next year how would it be that I come to London, Ontario?

SOLVING A CUBIC EQUATION

J. Norman C. Sharp, Co-ordinator of Mathematics, Etobicoke

For many years teachers and their students have been aware of different methods for solving a cubic equation. In most cases we described how it would be done but the arithmetic involved was so time-consuming that the equation really wasn't solved. The hand-held calculator has now removed th ened for time-consuming calculation and we can now obtain a result to whatever degree of accuracy we desire. Here are four methods used for solving a particular cubic equation (chosen because it lends itself to trigonometry).



Method 1: By Trigonometry

Solve: $x^3 - 3x - 1 = 0$

From $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

we get $\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0$ (1)

Let $x = y\cos\theta$ where $y > 0$

$y^3\cos^3\theta - 3y\cos\theta - 1 = 0$

or $\cos^3\theta - \frac{3}{y^2}\cos\theta - \frac{1}{y^3} = 0$ (2)

From (1) and (2), $\frac{3}{4} = \frac{3}{y^2}$ and $\frac{1}{4}\cos 3\theta = \frac{1}{y^3}$

$\therefore$ $y = 2$ and $\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$$\theta = \frac{\pi}{9},\ \frac{5\pi}{9},\ \frac{7\pi}{9}$$

$$x = 2\cos\frac{\pi}{9},\ 2\cos\frac{5\pi}{9},\ 2\cos\frac{7\pi}{9}$$

$$= 1.87938,\ -0.34729,\ -1.53209.$$

Method 2: By Isolating the Roots Through Graphing

Use $y = x^3 - 3x - 1$

then $y' = 3x^2 - 3$

when $y' = 0$ $x = \pm 1$ and $y = -3$ or 1

maximum point is $(-1,1)$

minimum point is $(1,-3)$

x	1	2	a root lies between 1 and 2
y	-3	1	and closer to 2

x	1.87	1.9	a root lies between 1.8 and 1.9
y	-0.568	0.159	and is closer to 1.9

x	1.87	a root lies between 1.87 and 1.88
y	-0.071	and is closer to 1.88

x	1.879	1.880	a root lies between 1.879 and
y	-0.002925	0.00467	1.880 and is closer to 1.879

x	1.8793	1.8794	a root lies between 1.8793 and
y	-0.0006	0.0001	1.8794 and is closer to 1.8794

One root is 1.8794 correct to 4 decimal places; other roots can be similarly found.

Method 3: By Using the Tangent to the Curve

$y = x^3 - 3x - 1$

$y' = 3x^2 - 3$

x	2	1.889	1.8795
y	1	0.073	0.00087
y'	9	7.704	7.598

The equation of the tangent at (2,1) is $\frac{y-1}{x-2} = 9$.

The tanget cuts the x-axis at (1.889,0).

The equation of the tangent at (1.889, 0.073) is $\frac{y-0.073}{x-1.889} = 7.704$.

The tangent cuts the x-axis at (1.8795,0).

The equation of the tangent at (1.8795, 0.00087) is $\frac{y-0.00087}{x-1.8795} = 7.598$.

The tangent cuts the x-axis at (1.87939,0).

A root of the equation is 1.87939.

x	-0.5	-0.3333	-0.34966
y	0.375	-0.0370	+0.00622
y'	-2.25	-2.667	-2.6332

The equation of the tangent at (-0.5, 0.375) is $\frac{y-0.375}{x+0.5} = -2.25$.

The tangent cuts the x-axis at (-0.3333,0).

The equation of the tangent at (-0.333, 0.0370) is $\frac{y-0.0370}{x+0.3333} = -2.667$.

The tangent cuts the x-axis at (-0.3497,0).

The equation of the tangent at (-0.34966, 0.00622) is $\frac{y - 0.00622}{x + 0.34966} = -2.6332$.

The tangent cuts the x-axis at (-0.34729,0).

The root of the equation is -0.34729.

The third root which lies between -2 and -1 can similarly be found.

A good discussion can develop to determine which method arrives at the desired result faster. It is also interesting to discuss how quickly the numbers converge.

Method 4: By Iteration

The equation $x^3 - 3x - 1 = 0$ can be replaced by the pair $y = x^3$ and $y = 3x + 1$ and solved simultaneously.

By iteration this involves using $y = 3x + 1$ and $x = \sqrt[3]{y}$. By the graph it is known that a value of x is close to 2. Let $x = 2$ then $y = 3x + 1$
$= 7$
then $x = \sqrt[3]{7}$ and $y = 3(1.9123) + 1$
$= 1.91293 \quad = 6.73879$
Continue the iterative process and successive values of x and y are:
$x = 1.88884 \quad y = 6.66651$
$x = 1.88206 \quad y = 6.64617$
$x = 1.88014 \quad y = 6.64042$
$x = 1.87960 \quad y = 6.63880$
$x = 1.87945 \quad y = 6.63834$
$x = 1.87940$

Observe that the pairs of values of x and y are approaching the point of intersection of the two curves $h = x^3$ and $y = 3x + 1$. The value of x can then be determined to any degree of accuracy required. A caution to observe is that many calculators will not accept negative numbers in the y or $\sqrt[x]{y}$ keys.

THE PROBLEM OF CALCULATING

B.L.R. Shawyer, Department of Mathematics, The University of Western Ontario

It has been known from antiquity that the ratio of the circumference of a circle to the diameter is constant and the determination of that value has invoked man's ingenuity for thousands of years.

The Babylonians had a very pragmatic approach. They simply constructed (or drew) a circle and measured the diameter and the circumference. Now, the Babylonians used a 60-ry system of numbers (we use a decimal or 10-ry system). It was a cumbersome system requiring many symbols. Thus, when they invented circular measure, it was natural to use 60. They knew that an equilateral triangle had all its angles equal, and that six such triangles fitted into a circle forming a regular hexagon. 60 "degrees" was chosen as the measure of the angle in an equilateral triangle, so that 360 degrees gives a complete revolution. Since their base was 60, each degree was divided into 60 "minutes". An ancient Babylonian tablet states that the ratio of the perimeter of the regular hexagon to the circumference of the circle is 57 "degrees" and 36 "minutes".

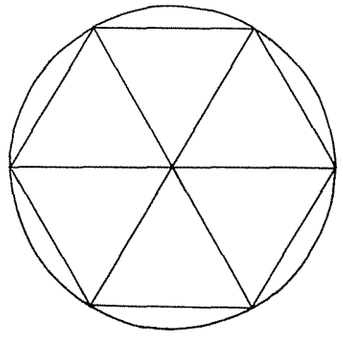


FIGURE 1

In modern notation, this is $\frac{3}{\pi} = \frac{57}{60} + \frac{36}{(60)^2}$ or $\pi = 3 + \frac{7}{60} + \frac{30}{(60)^2} = 3\frac{1}{8}$.

Now a better approximation is $\pi \doteq 3 + \frac{8}{60} + \frac{29}{(60)^2} + \frac{44}{(60)^3} = 3.14159259$,

so that $3 + \frac{8}{60} + \frac{30}{(60)^2} = 3\frac{17}{120}$ is a good approximation and this suggests

$\frac{3}{\pi} = \frac{57}{60} + \frac{18}{(60)^2}$ as a better Babylonian value with $\pi = 3.1413612566$.

Errors in the measurement could arise from non-mathematical items such as the physical properties of the rope used for measuring (thickness and tendency to stretch).

The Egyptians also had a practical approach. The Scribe Ahmes, about 1650 B.C. copied an ancient Egyptian book on "Accurate Reckoning". In it is stated that a circular field with diameter 9 units is equal in area to a square field with side 8 units or

$$\pi(\tfrac{9}{2})^2 = 8^2 \qquad \text{given} \qquad \pi = (\tfrac{16}{9})^2 = 3.16049\ldots .$$

Ahmes also indicates how the ancient Egyptians decided on this value. He considers an (irregular) octagon formed from a square of sides 9 units as shown in Figure 2. He notes that the area of the octagon ABCDEFGH is about the same as that of the circle inscribed in the square. The area of the octagon is $5 \times 3^2 + 4 \times \frac{1}{2} \times 3^2 = 63$ and $63 \doteq 64 = 8^2$ so that the circle of diameter 9 has the same area as a square of side 8. There are two approximations involved which do not quite balance out the errors.

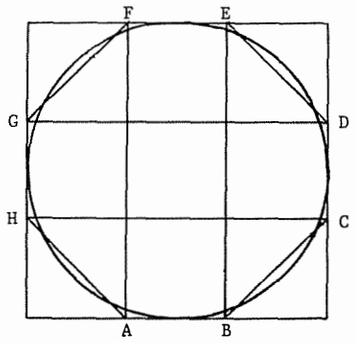


FIGURE 2

The Greeks, and in particular Archimedes applied a more systematic approach by discovering methods of calculating the areas of certain inscribed and escribed polygons for a circle. He started with a hexagon and successively doubled the number of sides until he reached a polygon with 96 sides. And this he did with no trigonometry or decimal notation. Archimedes obtained the result $3\frac{10}{71} < \pi < 3\frac{1}{7}$ or $3.14084\ldots < \pi < 3.142857$.

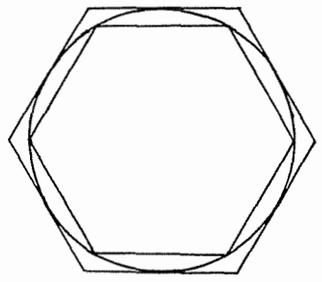


FIGURE 3

However Science was soon to fall into the dark ages and investigation in Europe ceased for almost two thousand years. With the Renaissance, investigation started again and Archimedes' method was used

Date	Name	Accuracy (# of decimal places correct)	# of sides of polygon
1573	Otho	6	7
1579	Vieta	9	6×2^{16}
1593	Romanus	15	2^{30}
1596	vanGeuden	20	15×2^{35}
1610	van Geuden	32	2^{62}
1621	Snell	34	2^{30}*
1630	Grienbergen	39	

*Improved Method

TABLE 1

with polygons with larger and larger numbers of sides.

Unfortunately, the accuracy of the method behaves like the logarithm of the number of sides in the polygon and places a severe limitation on the practical use of the method. Eventually in 1654, investigation ceased.

It was fortunate for the π calculators that the Calculus was invented in the 1660's and shortly afterwards, Gregory in 1671 discovered the series for the inverse tangent.

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{giving} \qquad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This and other results about inverse tangents gave new life to the investigations and once again results poured forth. It must be remembered that all the calculations shown in Table 2 (with the exception of Ferguson's) were made without mechanical aids to arithmetic. Ferguson did have a mechanical calculator.

Date	Name	Accuracy (# of decimal places correct)
1699	Shard	71
1706	Machin	100
1719	DeLagny	112
1789	Vega	126
1794	Vega	136
1841	Rutherford	152
1844	Dase	200
1847	Clausen	248
1853	Rutherford	440
1853	Shanks	527
1946	Ferguson	620

TABLE 2

But then the machine took over and Table 3 shows some recent progress. The last result I know of is ten years old and doubtless it is out of date.

Date	Name	Accuracy (# of decimal places correct)
1947	Ferguson	808
1949	Eniac	2,037
1959	IBM (Paris)	16,167
1961	IBM (N.Y.)	100,000
1966	IBM (Paris)	250,000
1967	CDC (Paris)	500,000

TABLE 3

Using Chebyshev means for π we obtain

$$m_1 = 4$$
$$m_2 = 3.111\ 111\ 111$$
$$m_3 = 3.121\ 568\ 628$$
$$m_4 = 3.141\ 895\ 142$$
$$m_5 = 3.141\ 525\ 680$$
$$m_6 = 3.141\ 598\ 561$$
$$m_7 = 3.141\ 591\ 927$$

m_8 = 3.141 592 745

m_9 = 3.141 592 642

m_{10} = 3.141 592 655

TABLE 5

We shall assume now that you have been able to calculate π with as much acuracy as you desire ($\pi \doteqdot$ 3.14159 26535 89793 23846 26433 83279 50288).

Now we should ask the question that was as relevant to the ancient Babylonian as it is today.

"How accurately do you need to know π?"

The radius of the universe is at present estimated to be about 10^{10} light years or about 10^{24} meters. How many decimal places of π do you need to know in order to get a good estimate of the circumference of the universe? An answer lies in Table 6.

AGE	ERROR IN π	ERROR IN CIRCUMFERENCE OF UNIVERSE
Babylon	10^{-2}	10^{22} Meters
Greek	10^{-4}	10^{20} Meters
Polygon	10^{-40}	10^{-16} Meters = 10^{-6} A°
Hand	10^{-500}	10^{-486} Meters
Machine	$10^{-500,000}$	$10^{-499,986}$ Meters

TABLE 6

Let us return now to Gregory in 1671 and the newly discovered result

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots .$$

Let us suppose that Gregory decided to use this to calculate π. Series which alternate like Gregory's series have the property that the sum of the first n terms differs in size from the actual sum by no more than the size of the next term.

$$\text{If } S = \sum_{k=0}^{\infty} (-1)^k a_k \text{ with } a_k \geq a_{k+1} \geq 0 \quad \text{then } \left| S - \sum_{k=0}^{n} (-1)^k a_k \right| \leq a_{n+1} .$$

So, for an error of less than 10^{-7}, we need to add up 10^7 terms.

How long will this take you —

by hand remember 10^7 secs $\doteqdot$ 2778 hours $\doteqdot$ 118 days

1 term per minute @ 8 hours per day $\doteqdot$ 57 years.

This is a little too long, so we will transport Gregory to the present day and give him the benefit of a modern computer which can perform arithmetic operations at a rate of about 10^6 per second

13 decimal place accuracy needs

10^{14} terms at 10^6 per second gives

10^8 seconds $\doteqdot$ 3.17 years.

Cost @ $2,000,000 per year !

And Gregory would want at least 10^{40} terms to "beat" Grienbergen's result.

10^{40} terms at 10^6 per second gives

10^{34} seconds $\doteqdot 3 \times 10^{26}$ years !!

And to beat CDC (Paris)

$10^{500,000}$ terms at 10^6 per second gives
$10^{499,994}$ seconds $\doteq 3 \times 10^{499,986}$ years

both of which are somewhat longer than present estimates for the age of the universe (approximately 10^{10} years!).

Clearly, Gregory is not going to have much success in pursuing this solution to the problem of calculating π and even waiting for advances in computing technology is unlikely to help. Thus another attack must be found. Here are two possibilities.

I Use Gregory's series for other values of x which make the series converge very rapidly. For example use

$$\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239},$$

or $$\frac{\pi}{4} = 22 \tan^{-1}\frac{1}{28} + 2 \tan^{-1}\frac{1}{443} - 5 \tan^{-1}\frac{1}{1393} - 10 \tan^{-1}\frac{1}{11018},$$

and this is what people like Shanks and Rutherford did.

II Use Gregory's series for $\frac{\pi}{4}$ and transform it to make it converge very rapidly. As an illustration of this type of process, think of the use of running averages to eliminate the effect of seasonal variations in say, unemployment statistics.

A simple way is just to average things

$$\frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Most of you are aware of "weighted" averages often used in calculating grades

$$\frac{p_1a_1 + p_2a_2 + \cdots + p_na_n}{p_1 + p_2 + \cdots + p_n}$$

where p_k is the "weight" given to the "mark" a_k.

Why do we go on?
Perhaps, because it is there?

Finally, for the really practically minded who want an approximate value better than most people remember and which is correct to 14 decimal places so that, when required you can calculate the circumference of the universe with an error of less than 100 meters, or the circumference of the earth with an error of less than 10^{-6} cm! It is given in the form of a mneumonic where the number of letters in the word represent the digit

e.g., DIGIT → 5.

HOW I WANT A DRINK
ALCOHOLIC OF COURSE
AFTER THE HEAVY LECTURES
INVOLVING DECIMAL FRACTIONS

Literature

1. A History of π, P. Beckmann, St. Martin's Press, New York, 3rd ed., 1974.
2. Mathematical Recreations and Essays, W. W. Rouse Ball and H. S. M. Coxeter, University of Toronto Press, Toronto, 12th ed., 1974.

Anyone interested in receiving a copy of the complete article, please contact Professor B. L. R. Shawyer, Department of Mathematics, The University of Western Ontario, London, Ontario, N6A 5B9.

Answer from pg. 16 Jack's does 4 laps on his while Billy's does 3 laps on his.

CALCULATING NATURAL LOGS BY TAKING SQUARE ROOTS

Norman M. Rice, Department of Mathematics and Statistics, Queen's University

The purpose of this note is to make better known the little known formula

$$\ln a = \lim_{k\to\infty} 2^k(\sqrt{\cdots\sqrt{a}} - 1) \qquad (1)$$

The formula enables a student using a pocket calculator with a square root button to calculate good approximations to the natural logarithm of a number by finding k successive square roots of the number, subtracting 1, and doubling the result k times. The approximation should get better as bigger k's are used, but students may find it interesting to experiment with different k's as round-off errors become important if k is too big.

Formula (1) is obtained from the following formula by taking $2^k = m$:

$$\ln a = \lim_{m\to\infty} m(\sqrt[m]{a} - 1)\,. \qquad (2)$$

It is shown below how formula (2) can be obtained from the fairly well known formula.

$$e^b = \lim_{m\to\infty} \left(1 + \frac{b}{m}\right)^m \qquad (3)$$

which in turn follows from the well known formula $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ by writing

$$e^b = \left[\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right]^b$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{nb}$$

$$= \lim_{m\to\infty} \left(1 + \frac{b}{m}\right)^m, \quad \text{where } m = nb\,.$$

We note in passing that formula (3) provides a method for approximately calculating e^b using a simple hand calculator.

To see how formula (2) follows from formula (3) let $b = \ln a$, so $a = e^b$, and note that for large m formula (3) says that $a \approx \left(1 + \frac{\ln a}{m}\right)^m$, so $\sqrt[m]{a} \approx 1 + \frac{\ln a}{m}$, so $\ln a \approx m(\sqrt[m]{a}-1)$ as required.

An alternate proof of formula (2), using the fact that $\frac{d}{dx} a^x = \ln a \cdot a^x$, is the following. Let $f(x) = a^x$. Then, since $f'(x) = \ln a \cdot a^x$ and $a^0 = 1$, we have

$$\ln a = f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{a^h - 1}{h}$$

$$= \lim_{m\to\infty} m\left(a^{\frac{1}{m}} - 1\right), \quad \text{where } m = \frac{1}{h}\,.$$

LA SERIE DE MACLAURIN, LE SINUS, ET LA CALCULATRICE A PRIX MODIQUE

Maurice Poirier, Ecole Secondaire Garneau, Orleans, Ontario

Les calculatrices électroniques deviennent de plus en plus accessibles de nos jours, grâce aux progrès, techniques et à la production en grande quantite. Cependant, pour des raisons financières, on ne peut s'attendre á ce que tous nos étudiants en possedent une qui affiche toutes les fonctions disponibles sur les calculatrices de grand prix, en particulier, les fonctions trigonométriques.

Dans un article paru dans la revue **Popular Science**, D. Huff donne l'algorithme suivant pour trouver la valeur approximative du sinus d'un angle avec une calculatrice sans touches trigonométriques:

> On divise le nombre de degrés par 57.296 et on élève ce résultat au carré. Après avoir retranché, 10, on élève encore au carré. Ensuite, on ajoute 20 et on multiplie par le nombre de degrés. Enfin, on divise par 6875.5

Une vérification rapide de ce calcul donne $\sin 30° = 0.500$ et $\sin 60° = 0.866$, au millième près. De fait, nous allons démontrer que cette méthode produit des valeurs approximatives, exactes au dix-millième, pourvu que l'argument soil inférieur à 45°; cette restriction ne nuit pas, en pratique, grâce aux formules de réduction trigonométriques.

Algébriquement, le procédé se traduit comme suit: soit d le nombre de degres; alors

$$\sin d \doteq \{[(\frac{d}{57.296})^2 - 10]^2 + 20\} \frac{d}{6875.5}$$

On reconnaît facilement que $57.296 \doteq \frac{180}{\pi}$, et un peu moins facilement que $6875.5 \doteq 120 \times 57.296 \doteq 120 \times \frac{180}{\pi}$.

L'approximation devient alors:

$$\sin d \doteq \{[(\frac{\pi d}{180})^2 - 10]^2 + 20\} \frac{\pi d}{180} \times \frac{1}{120}.$$

Avec $x = \frac{\pi d}{180}$ radians, on a, d'une facon équivalente,

$$\begin{aligned} \sin x &\doteq \{[(x)^2 - 10]^2 + 20\} \frac{x}{120} \\ &= \{x^4 - 20x^2 + 120\} \frac{x}{120} \\ &= \frac{x^5}{120} - \frac{x^3}{6} + x \end{aligned}$$

$$\text{Ou, } \sin x \doteq x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Or, cette expression est formée des trois premiers termes de la série de Maclaurin pour sin x:

$$\begin{aligned} \sin x &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \end{aligned}$$

Cette série converge rapidement dans l'intervale $[0, \pi/4]$, et le calcul decrit ci-dessus est donc très exact. En fait, la différence entre sin x et $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ est inférieure a $\frac{x^7}{7!} \leq (\frac{\pi}{4})^7 \times \frac{1}{7!} < 0.00004$ dans cet intervalle.

Le tableau suivant illustre la précision de la méthode.

θ	sin θ	
	valeur exacte à 6 décimales près	valeur obtenue par le calcul donné
10	0.173648	0.173648
20	0.342020	0.342020
30	0.500000	0.500002
40	0.642788	0.642803
45	0.707107	0.707143

Pour des valeurs de l'argument supérieures à 45°, on recourt aux formules de réduction. Par exemple, pour obtenir le sinus de 61° au dix-millième près, on se sert des formules: $\sin \theta = \cos (90° - \theta)$ et $\cos \theta = \sqrt{1 - \sin^2 \theta}$, $0 \leq \theta \leq 45°$ pour former $\sin \theta = \sqrt{1 - \sin^2 (90° - \theta)}$. En calculant, successivement, sin 29° (avec le calcul proposé), $\sin^2 29°$, $1 - \sin^2 29°$, et $\sqrt{1 - \sin^2 29°}$, on obtient, avec sa calculatrice: $\sin 61° = 0.874619$ (la vraie valeur est $\sin 61° = 0.8746197...$). Notons que le calcul de $\sin 61°$ directement à partir du procédé donne 0.874923, soit une erreur supérieure à 0.0003.

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Ressources:

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Huff, D.: rubrique Calcu-letter dans **Popular Science**; janvier 1979.
Kopchenova, N.V. et I.A. Maron: **Computational Mathematics**; MIR, Moscou, 1975.

THE PROBABILITIES OF THE NUMBERS OF SUCCESSES IN A SERIES OF INDEPENDENT TRIALS

David R. Duncan and Bonnie H. Litwiller, Professors of Mathematics, University of Northern Iowa, Cedar Falls, Iowa

Editors Note: How fortunate we are to have had Professors Duncan and Litwiller as regular contributors. Their articles are always topical, interesting and fun. Seldom does an article contain reference to a former article from the same journal dated 6 years previous.

A variation of the famous "Birthday Problem" is the following: Suppose that a card is selected at random from each of n standard decks of playing cards. What is the probability that at least two of the cards drawn are exactly the same (same suit and same face value). The authors investigated this problem in. "Birthdays, Cards, and Matches: An Experience in Probability," **Ontario Mathematics Gazette**, Vol. 11, No. 4, September 1973, pp. 279-282. Some noteworthy results were:

1. If cards are drawn from at least nine decks, the probability of there being at least one match is greater than ½.

2. If 13 decks are involved the probability of at least one match is .805. This result is interesting because most students assume the results of at least one match when 13 decks are used is 13/52 = ¼ = .25.

Subsequent to this investigation the authors have demonstrated this activity with many classes and other groups. The standard practice has been to distribute 13 decks of cards and to ask the audience for an estimate of the probability of at least one match if 13 cards are drawn from the 13 decks. The estimated probabilities are usually very small. The activity of drawing 13 cards from 13 decks is then performed 10 times. In each case the 13 "deck-holders" compare drawn cards to see whether or not a match has occurred. For each of the ten performances a tally is kept as to whether or not any matches occurred. The theoretical probability of there being a match is then computed, discussed and compared with the actual outcomes.

After many such presentations, it occurred to the authors that almost without exception either 7, 8, or 9 of 10 drawings contained a match. We wondered what the probability was of this close correspondence between the theoretical and actual number of occurrences. The problem is simple to resolve and gives rise to an interesting table.

Let an independent trial be performed 10 consecutive-times. Let p be the probability of success on a given trial and q be the probability of failure ($p + q = 1$). From the familiar binomial theorem, the probabilities of 0, 1, 2, ..., 10 successes are computed and displayed in Table \$.

Table 1

Successes	Probability
10	p^{10}
9	$10p^9q$
8	$45p^8q^2$
7	$120p^7q^3$
6	$210p^4q^6$
5	$252p^5q^5$
4	$210p^4q^6$
3	$120p^3q^7$
2	$45p^2q^8$
1	$10pq^9$
0	q^{10}

Table II reports the probabilities (three decimal places) of Table I for specific values of p.

p	0	1	2	3	4	5	6	7	8	9	10
1.00	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	1.000
.90	.000	.000	.000	.000	.000	.001	.011	.057	.194	.387	.349
.80	.000	.000	.000	.001	.006	.026	.088	.201	.302	.268	.107
.70	.000	.000	.001	.009	.037	.103	.200	.267	.233	.121	.028
.60	.000	.002	.011	.042	.111	.201	.251	.215	.121	.040	.006
.50	.001	.010	.044	.117	.205	.246	.205	.117	.044	.010	.001
.30	.028	.121	.233	.267	.200	.103	.037	.009	.001	.000	.000
.20	.107	.268	.302	.201	.088	.026	.006	.001	.000	.000	.000
.10	.349	.387	.194	.057	.011	.001	.000	.000	.000	.000	.000
.00	1.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000

)bservations:

1. In each row the most likely number of successes is the same as the expected number of successes (10p). For instance, if p = .70, the most likely number of successes is 7; there will be 7 successes 26.7% of the time. The expected number of successes 10(.7) is also 7.
2. The table possesses a "reverse symmetry". For example, the 11 probabilities listed in the 'p = .60" row are the same as those listed in the "p = .40" row but in reverse order.
3. The probabilities of the most likely number of successes decreases as p decreases from 1 to).5. For example, when p = 1, the probability ot 10 successes is 1 as well. If p = .90, the probability of 9 successes is .387 while of p = .80, the probability of 8 successes is .302. This pattern continues until p = .5 and then reverses because of the reverse symmetry of the table.
4. For p = .60, .70, .80 and .90 the probability of getting more than the expected number of successes is greater than the probability of getting fewer than the expected number of successes. If p = .80, the probability of 8 successes is .302; the probability of more than 8 successes (9 or 10), is .268 + .107 = .375. The probability of fewer than 8 successes (o, 1, 2, ... 7) is .001 + .006 + 026 + .088 + .201 = .322. In other words in our card activity where p≈.80, the actual number of draws with matches is more likely to exceed 8 than to fall short of 8. (This impresses our classes which thought that matches would be very unlikely).
5. If p = .80, the probability of 7, 8, or 9 matches is .771. It is therefore not unusual that these three results should seem to turn up so frequently.

VECTOR-MATRIX REPRESENTATIONS OF CONICS AND REDUCTION TO STANDARD FORMS

G.R. Magee, Professor Emeritus, The University of Western Ontario

Editors Note: It is most fitting to hear again from Dr. Magee.

INTRODUCTION

The integration of linear algebra and geometry has provided efficient and general methods, based on matrix theory, to reduce quadratic forms to so-called standard or "canonical" forms. In euclidean spaces of two and three dimensions, quadratic forms are associated with conics and quadric surfaces, respectively. Canonical forms of equations of these configurations are of great importance to geometers, not only because a classification scheme is thereby provided but also because many geometrical properties of these configurations can be determined by inspection of the identifying equations.

To understand and appreciate the reduction methods provided by linear algebra requires the development of a considerable body of theory. We propose to avoid this path and to return, with some modifications, to an older and more direct approach which is available in a real euclidean plane E_2. The use of matrix theory will be limited, not only in the dimensions of the matrices, but also to simple rules of operation such as the "row by column" rule of multiplication of two matrices, and the recognition of the transpose and the inverse (if the latter exists) of a matrix.

PART I

We start with a general polynomial equation of degree two in the variables x and y:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{1}$$

Relative to equation (1) it will be assumed that (i) x,y are real variables, (ii) a,b,c,f, g,h represent real numbers, (iii) a,b,h are not simultaneously zero. If a single rectangular cartesian frame of reference is established in E_2, then each point P in E_2 has a unique coordinate vector X = (x,y), known as a row vector. The transpose of vector X is a column vector, indicated by

$$X^t = \begin{pmatrix} x \\ y \end{pmatrix} .$$

Note the possible interpretation that X is a 1×2 matrix, that is, one row and two columns, whereas X^t may be regarded as a 2×1 matrix which has two rows and one column. If we introduce a 2×2 matrix A and a 2×1 matrix B where

$$A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}, \qquad B = \begin{pmatrix} g \\ f \end{pmatrix},$$

we may write equation (1) in a more compact form, namely,

$$XAX^t + 2XB + c = 0 . \tag{2}$$

For example, the equation

$$5x^2 - 4xy + 8y^2 + 18x - 36y + 9 = 0$$

is given by (2) in which

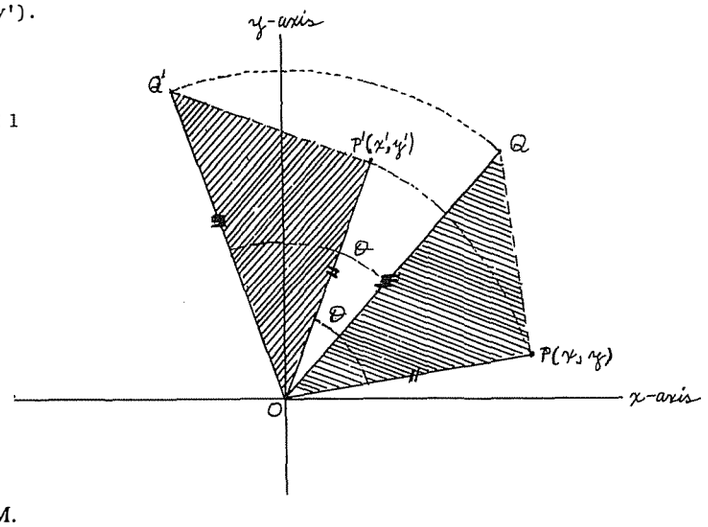
$$A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}, \qquad B = \begin{pmatrix} 9 \\ -18 \end{pmatrix}, \qquad c = 9.$$

DEFINITION: A non-empty set of points in E_2 whose coordinate vectors $X = (x,y)$ satisfy an equation of the form (2) is called a (point) conic.

A reference frame for the coordinate vectors may be arbitrarily selected in E_2 and may have little or no geometric relationship to the conic. A conic is regarded as "proper", or non-degenerate, if the left member of (1) is not factorable as the product of two linear functions of x and y with real coefficients. In the contrary case, the conic consists of two lines (distinct or otherwise) and it is said to be a degenerate conic.

We seek now a mapping which associates with each point P in E_2 a unique point P' in the same plane. We require that the mapping be "distance preserving", that is, if P is mapped onto P' and another point Q is mapped onto Q', the distance PQ is equal to the distance P'Q', whatever the points P and Q. One way to accomplish this is to choose an arbitrary point O in E_2 and rotate points in E_2 about O through a specified angle θ. The centre of rotation is usually chosen as the origin of the reference frame, and the usual convention is adopted concerning the direction of rotation when θ is a positive angle. This kind of mapping is a point transformation in which a point with a non-null coordinate vector (x,y) is mapped onto a unique "image" point which has a coordinate vector (x',y'). Symbolically, $(x,y) \rightarrow (x',y')$.

DIAGRAM 1



A study of Diagram 1 will show that the mapping $(x,y) \to (x',y')$ is given algebraically by the equations

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta, \end{cases} \tag{3}$$

or, in vector-matrix form, by

$$X'^t = RX^t, \tag{4}$$

where $X' = (x',y')$, $X'^t = \begin{pmatrix} x' \\ y' \end{pmatrix}$, and the "rotation" matrix R is given by

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad -\pi \le \theta \le \pi.$$

A mapping, described by (3) or, equivalently, by (4) may be called a rotation of the plane E_2. Note that the "inverse" mapping $(x',y') \to (x,y)$ is represented by

$$\begin{cases} x = x'\cos(-\theta) - y'\sin(-\theta) = x'\cos\theta + y'\sin\theta \\ y = x'\sin(-\theta) + y'\cos(-\theta) = -x'\sin\theta + y'\cos\theta, \end{cases} \tag{5}$$

that is,

$$X^t = R^{-1}X'^t, \tag{6}$$

where

$$R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

As the notation suggests, R^{-1} is the inverse of the matrix R.

EXAMPLE: A rotation of the plane through the angle $\theta = \arctan 2$ is given by

$$\begin{cases} x' = \frac{1}{\sqrt{5}}x - \frac{2}{\sqrt{5}}y = \frac{1}{\sqrt{5}}(x - 2y) \\ y' = \frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = \frac{1}{\sqrt{5}}(2x + y) \end{cases}$$

or, in vector-matrix form, by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For later use, we note that the mapping (5) may be written in row-vector form and in this form it is represented by

$$X = X'R. \tag{7}$$

It should be observed that (7) is obtainable from (6) by taking the transpose of both sides of equation (6).

PART II

We now consider a conic C in E_2, whose equation in vector-matrix form is

$$XAX^t + 2XB + c = 0. \tag{2}$$

Recall that C is a non-empty set of points in E_2 whose coordinate vectors $X = (x,y)$ satisfy (2). Under a rotation of the plane E_2, C is mapped onto a set C' whose equation may be obtained by substituting for X and X^t from (7) and (6), respectively. Following this procedure, we obtain

$$X'RAR^{-1}X'^t + 2X'RB + c = 0. \tag{8}$$

The mapping equations (6) and (7) are linear and homogeneous in the variables x',y', and therefore the image C' of C is a conic which may be represented by an equation of the form

(9) $$a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c = 0.$$

It is convenient to let

$$A' = RAR^{-1}, \quad B' = RB,$$

and we then write (8) as

$$X'A'X'^t + 2X'B' + c = 0. \quad \text{[Cf. equation (2)]}$$

The transpose of the 2×2 matrix A' is given by

$$A'^t = (RAR^{-1})^t = R^{-1^t} A^t R^t = RAR^{-1}.$$

Observe that $A'^t = A'$ and, accordingly, A' is a <u>symmetric</u> matrix; its elements are the coefficients a', h', b' in equation (9). Hence

$$A' = \begin{pmatrix} a' & h' \\ h' & b' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

which yields

(10) $$\begin{cases} a' = a\cos^2\theta - 2h\sin\theta\cos\theta + b\sin^2\theta \\ h' = (a-b)\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta) \\ b' = a\sin^2\theta + 2h\sin\theta\cos\theta + b\cos^2\theta. \end{cases}$$

Moreover

$$B' = \begin{pmatrix} g' \\ f' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix},$$

and consequently

(11) $$\begin{cases} g' = g\cos\theta - f\sin\theta \\ f' = g\sin\theta + f\cos\theta. \end{cases}$$

It should be noted that $a' + b' = a + b$ and $a'b' - h'^2 = ab - h^2$. The expressions $a + b$ and $ab - h^2$ are <u>invariants</u> of the conic C under any rotation transformation of C. (Invariants of a configuration under any mapping have to do with intrinsic geometrical properties of the configuration.)

Our actual target is the expression for h'. We wish to impose the condition $h' = 0$. First, if $\cos\theta = 0$ in (10) and (11), we observe that $a' = b$, $h' = -h$, $b' = a$, $g' = \mp f$, $f' = \pm g$ and there has been no reduction in the form of equation (1) of C. Consequently, we may assume $\cos\theta \neq 0$ and find that $h' = 0$ implies

(12) $$h\tan^2\theta - (a-b)\tan\theta - h = 0,$$

which yields

(13) $$\tan\theta = \frac{a - b \pm \sqrt{(a-b)^2 + 4h^2}}{2h}, \quad h \neq 0.$$

Values of $\tan\theta$ which satisfy (13) are real and have a product equal to negative unity. Therefore, two rotation angles are available in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, and one of these is positive. A natural choice for a useful rotation angle is the <u>positive accute</u> value of θ. Corresponding to this angle there exists a rotation transformation (3) which will map the set of points represented by (1) onto the set of points whose coordinates satisfy

(14) $$a'x'^2 + b'y'^2 + 2g'x' + 2f'y' + c = 0.$$

In a numerical example, values (in some order) for a', b' in (14) may be obtained from the relations

$$a' + b' = a + b, \quad a'b' = ab - h^2.$$

However, it is advisable to compute all the coefficients a', b', g', f' by substituting for

sin θ and cos θ in the product matrices RAR^{-1} and RB.

EXAMPLE 1: Consider the conic

$$C = \{x,y \mid 5x^2 - 4xy + 8y^2 + 18x - 36y + 9 = 0\}.$$

A rotation angle is determined by

$$2\tan^2\theta - 3\tan\theta - 2 = 0. \quad \text{[Cf. equation (12)]}$$

Hence $\tan\theta = 2$ and $\tan\theta = -\frac{1}{2}$; we choose $\theta = \arctan 2$. The corresponding rotation matrix is

$$R = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Therefore

$$A' = RAR^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix},$$

and

$$B' = RB = \begin{pmatrix} g' \\ f' \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 9 \\ -18 \end{pmatrix} = \begin{pmatrix} 9\sqrt{5} \\ 0 \end{pmatrix}.$$

The rotation of the plane E_2 through $\theta = \arctan 2$ maps the conic C onto the conic C', which is given by

$$9x'^2 + 4y'^2 + 18\sqrt{5}\,x + 9 = 0,$$

in which

$$\begin{cases} x' = \frac{1}{\sqrt{5}}(x - 2y) \\ y' = \frac{1}{\sqrt{5}}(2x + y). \end{cases}$$

By a simple algebraic device ("completing squares") we find that C' in example 1 is represented by the following equation:

$$9(x' + \sqrt{5})^2 + 4y'^2 = 36.$$

The substitution of $x'' = x' + \sqrt{5}$, $y'' = y'$ represents a second type of mapping of E_2 onto itself—known as a translation T of E_2. In this numerical example, T moves any point in the plane through a distance parallel to the x-axis equal to $\sqrt{5}$ units. In particular, T maps the conic C' onto the conic C'' which is given by

$$9x''^2 + 4y''^2 = 36.$$

Thus the image of the given conic C under the product of two transformations (rotation followed by translation) is the congruent conic C'' whose equation, in canonical form, is

$$\frac{x''^2}{4} + \frac{y''^2}{9} = 1.$$

The mapping equations of the product transformation are

$$\begin{cases} x'' = x' + \sqrt{5} = \frac{1}{\sqrt{5}}(x - 2y + 5) \\ y'' = y' \qquad\quad = \frac{1}{\sqrt{5}}(2x + y). \end{cases}$$

The ellipse C'' has axes of symmetry (principal axes) which are along the coordinate axes, and so we perceive that the major axis of the given conic C lies on the line $x - 2y + 5 = 0$ while the minor axis of C is along the line $2x + y = 0$. The centre of C corresponds to the centre of C'', and it is therefore the point (-1,2). The eccentricity of C (being equal to the ecoentricity of C'') is $\frac{1}{3}\sqrt{5}$. Additional information concerning C, such as location of

focal points, equations of directrices, etc., can be obtained by using the mapping equations of the product transformation. To illustrate, note that the foci of C" are the points $(0,\sqrt{5})$ and $(0, -\sqrt{5})$; on substituting these values for x" and y" in the mapping equations we find that the focal points of the conic C are the points $x = 1$, $y = 3$ and $x = -3$, $y = 1$.

The following additional examples are recommended:

EXAMPLE 2: A hyperbola is represented by

$$C = \{x,y \mid 7x^2 - 60xy + 32y^2 - 106x + 68y - 37 = 0\}.$$

Employing mapping techniques, show that the asymptotes of C are the lines

$$7x - 4y - 1 = 0,$$
$$x - 8y - 15 = 0.$$

EXAMPLE 3: Find a mapping of the plane which will reduce

$$6x^2 - 5xy - 6y^2 - 46x - 9y + 60 = 0$$

to the form

$$x^2 - y^2 = 0.$$

EXAMPLE 4: A parabola is represented by $XAX^t + 2XB + c = 0$ when

$$A = \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \quad c = 21.$$

Using mapping processes, show that its axis of symmetry is the line $2x + 3y + 1 = 0$ and the tangent at the vertex is $3x - 2y - 10 = 0$.

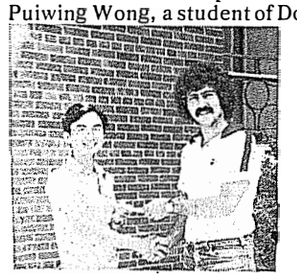
JOHN G. McCARTNEY (1939-1977) ANNUAL MATHEMATICS SCHOLARSHIP: $400

Born in Glasgow, Scotland, Jack emigrated to Canada with his family when he was a child. He attended Secondary School and Teacher's College in Peterborough and was a graduate of Queen's University.

After teaching in the Kingston and Stoney Creek area, Jack joined Science Research Associates (Canada) Limited in 1967 as an editor of educational programs and subsequently became Vice-President of Development. During his ten years with SRA, Jack made a significant contribution to publishing innovative learning materials, particularly in the field of mathematics. When he died in 1977, the educational programs he had instigated were being used in four countries around the world.

In recognition of Jack's contribution to education, this scholarship in the amount of $400 will be awarded annually to an outstanding Ontario Grade 13 mathematics scholar. Details and application forms are available from the Ontario Association of Mathematics Educators, who were to administer the scholarship.

OAME/AOEM takes pleasure in announcing the first winner of the John McCartney Scholarship in Puiwing Wong, a student of Dominic DeFelice at Overlea Secondary School, Don Mills.



This year, Puiwing Wong was awarded a $6000. Descartes Scholarship in recognition for his outstanding mathematical ability. His accomplishments included: 100% on the 1978 Euclid exam, 2nd and 5th in the 1977 and 1978 Olympiad exams, 4th in the 1979 Descartes exam. Member of the winning Euclid team 1978 and Descartes team 1979. A worthy recipient of the FIRST JOHN McCARTNEY MATHEMATICS SCHOLARSHIP.



I DIDN'T KNOW THAT

TRICKS — GAMES — PUZZLES

Dr. H. Laurence Ridge, Faculty of Education, University of Toronto

Here it is a new season - and we do have a new feature! One of the most popular topics under the banner of recreational mathematics is that of **figurate numbers**, a fascinating combination of pattern in number and shape. The best known members of the family are **square numbers** but even more fundamental are **triangular number**. Much of the interest in triangular numbers lies in the surprising frequency with which they occur in unexpected places.

This issue will introduce triangular numbers through a puzzle motif. Subsequent issues will include at least one occurrence or application of these numbers. Readers are cordially invited to share any triangular number experiences through this column.

Before moving into the triangular number feature, we'll keep variety alive with 'oldie but goldie' puzzle from Russell Walker, then a follow-up to Julian D'Angela's Single Pile Nim from the December '78 issue.

Your number's up

Here's a favourite puzzle which many of you may have seen and perhaps even used. Try putting a set of cards like these on a bulletin board in your classroom.

YOUR NUMBER'S UP - ON THESE CARDS.
PICK A NUMBER FROM 1 TO 63.
SHOW ME WHAT CARDS IT'S ON AND I'll
TELL YOU THE NUMBER!

1	17	33	49	2	18	34	50	4	20	36	52
3	19	35	51	3	19	35	51	5	21	37	53
5	21	37	53	6	22	38	54	6	22	38	54
7	23	39	55	7	23	39	55	7	23	39	55
9	25	41	57	10	26	42	58	12	28	44	60
11	27	43	59	11	27	43	59	13	29	45	61
13	29	45	61	14	30	46	62	14	30	46	62
15	31	47	63	15	31	47	63	15	31	47	63
8	24	40	56	16	24	48	56	32	40	48	56
9	25	41	57	17	25	49	57	33	41	49	57
10	26	42	58	18	26	50	58	34	42	50	58
11	27	43	59	19	27	51	59	35	43	51	59
12	28	44	60	20	28	52	60	36	44	52	60
13	29	45	61	21	29	53	61	37	45	53	61
14	30	46	62	22	30	54	62	38	46	54	62
15	31	47	63	23	31	55	63	39	47	55	63

The fact that 37 occurs only on the cards beginning 1, 4, and 32 should give enough of a clue as to how to determine the number from the cards named.

To figure out what numbers should appear on each card Russell Walker of Ravenswood uses the folowing system.

1-3-5-7-9-11-13-15-17-19-21-23-25-27-29-31-33-35-37-39-41-43-45-47-49-51-53-55-57-59-61-63.

2,3,--6,7,--10,11--14,15--18,19--22,23--26,27--30,31--34,35--38,39--42,43--46,47--50,51--54,55--58,59--62,63.

4,5,6,7,----12,13,14,15,----20,21,22,23,----28,29,30,31,----36,37,38,39,----44,45,46,47,----52,53,54,55,----60,61,62,63.

8,9,10,11,12,13,14,15,--------24,25,26,27,28,29,30,31,--------40,41,42,43,44,45,46,47,--------56,57,58,59,60,61,62,63.

16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,----------48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63.

32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63.

Russell then goes on to say, "The method of arranging the numbers is quite simple and easy to remember, but why it works in the elimination process, I have been unable to come up with a mathematical solution." Now there's a challenge for those of you and your students who aren't familiar with an explanation. One key is that the leading numbers on the cards are successive powers of 2. Let's pitch in and give Russell a hand with this one!

Single Pile Nim

Finding a winning strategy for this game can involve a classic example of working backwards from the answer (sometimes called 'analysis'), a most important problem solving technique.

In Julian D'Angel's version two players, X and Y, play the game by starting at 25 and alternatively subtracting 1, 2, or 3 until one player wins by arriving at zero. Let's start with a winning situation for one of the players, say X. By working back through all the possible moves, perhaps we'll be able to see how X could always win; that is, find a winning strategy for X.

X wins (reaches zero) if Y leaves 1, 2, or 3
(since X would then subtract 1, 2, or 3 respectively to arrive at zero),
but Y is forced to leave 1, 2, or 3 if X leaves 4.
X can leave 4 if Y leaves 5, 6, or 7
But Y is forced to leave 5, 6, or 7 if X leaves 8.
X can leave 8 if Y leaves 9, 10, or 11
but Y is forced to leave 9, 10, or 11 if X leaves 12.
•
•
•
X can leave 20 if Y leaves 21, 22, or 23
but Y is forced to leave 21, 22, or 23 if X leaves 24.
X can leave 24 by **going first** and subtracting 1 from 25.

Conclusion:

So long as only 1, 2, or 3 can be subtracted, the first player (usually called 'A') also has a winning strategy in games which start at 26 or 27. The same strategy applies in games which start at 29, 30 or 31; 33, 34, or 35; or at any number which is **not** a multiple of 4. The strategy is:
subtract 1, 2, or 3 to reach a multiple of 4, then. on subsequent moves, subtract (4 minus opponent B's move); that is, the complement of 4.

The second player. 'B' has a winning strategy for any similar game (subtracting 1, 2, or 3) which starts at any multiple of 4. The strategy is:
subtract (4 minus opponent A's move)

For algebraically inclined readers, generalized Single Pile Nim would consist of alternating subtractions of 1, 2, 3, ..., k from a starting point n with the winner being the first to reach a target number t. Using a process similar to that in the specific case,

X wins (reaches t) if Y leaves $t + 1, t + 2, \ldots, t + k$ but Y is forced to leave $t + 1, t + 2, \ldots, t + k$ if X leaves $t + (k + 1)$.

X can leave $t + k + 1$ if Y leaves $(t + k + 1) + 1, (t + k + 1) + 2, \ldots, (t + k + 1) + k$
but Y is forced to leave $(t + k + 1) + 1, (t + k + 1) + 2, \ldots, (t + k + 1) + k$
if X leaves $(t + k + 1) + k + 1$; i.e. $t + 1(k + 1)$

•
•
•

X can leave $t + m(k + 1)$ if Y leaves $t + m(k + 1) + 1, t + m(k + 1 + 2, \ldots, t + m(k + 1) + k$, but Y is forced to leave $t + m(k + 1) + 1, t + m(k + 2) + 2, \ldots, t + m(k + 1) + k$ if X leaves $t + (m + 1)(k + 1)$.
X can leave $t + (m + 1)(k + 1)$ by going first and subtracting 1, 2, 3, ..., or k from n so long as n is **not** equal to t plus a multiple of $k + 1$: that is $n \neq t + r(k + 1)$ for some natural number r.

Conclusion:

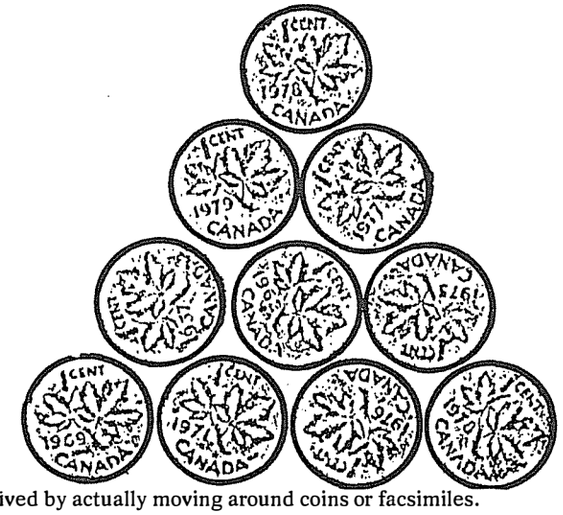
A (the first player) has a winning strategy when $n \neq t + r(k + 1)$. The strategy is:
subtract 1, 2, 3, ..., or k to reach $t + r(k + 1)$, then, on subsequent moves, subtract ($k + 1$ minus B's move), that is, the complement of $k + 1$.

B (the second player) has a winning strategy when $n = t + r(k + 1)$. The strategy is:
subtract ($k + 1$ - A's move).

Triangular numbers* unlimited

To paraphrase 'Donald in Mathmagicland' - "You find triangular numbers in the darndest places'. For starters though, here is a puzzle whose form suggests that the solution should involve triangular numbers.

Move as few coins as possible to reverse the triangular array of coins.



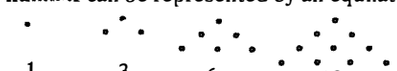
A solution is best derived by actually moving around coins or facsimiles.

Some solutions:

Strategy 1: Think of reversing the configuration by reflecting it in the line shown in Fig. 1. All but 3 of the coins reflect onto (the positions of) other coins or are invarient; that is, reflect on to themselves.

---

* **Triangular number** can be represented by an equilateral triangular configuration of dots.



1 3 6 10

$T = 1, \quad T = 3, \quad T = 6, \quad T = 10, \ldots T = n(n + 1)/2$

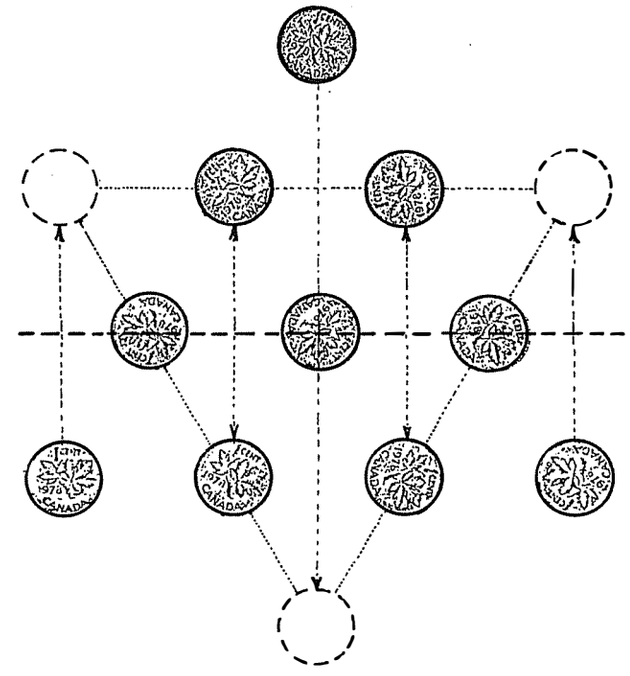


Fig. 1

Strategy 2: Think of reversing the configuration by rotating it 60° in either direction with respect to (the coin at) the centroid of the triangle as shown in Fig. 2. Again, all but 3 of the coins are rotated onto (the positions of) other coins or are invariant.

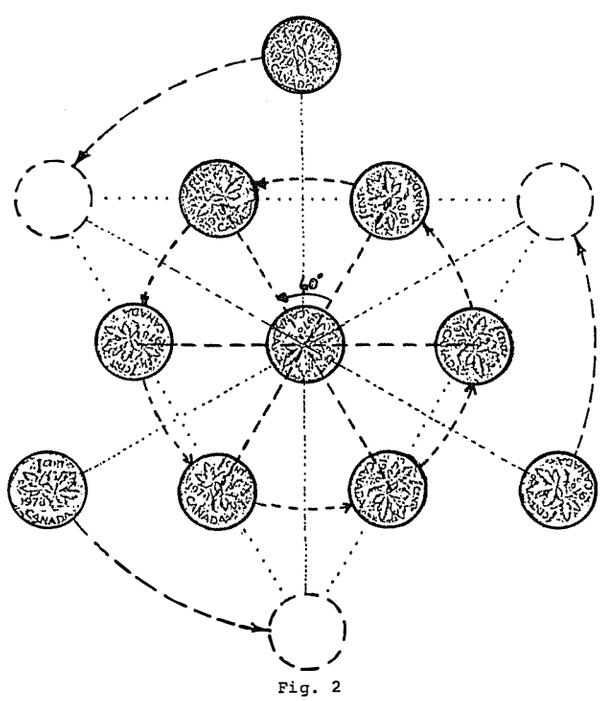


Fig. 2

Extending the puzzle.

One of the most important aspects of problem solving is to extend or generalize the situation beyond the specific case for which a solution has just been found. In this case an appropriate generalization would be:

What is the minimum number of coins which can be moved in order to reverse any equilateral triangular configuration of coins?

Investigation of this situation makes for a good exercise in pattern recognition and application.

Consideration 1:

Which of the 'reflection' or 'rotation' strategies will be more fruitful in the general case? Is there another strategy which is better than either one of the preceding strategies?

Consideration 2:

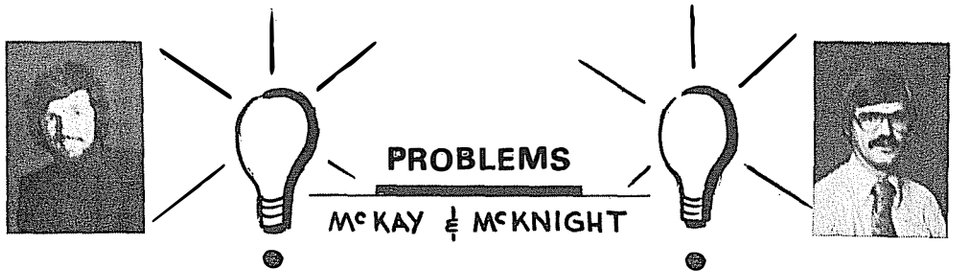
The minimum number of moves can be expressed in terms of triangular numbers in various ways. How many can you find?

Consideration 3:

The minimum condition can be seen inductively by patterns and can also be proved deductively.

Responses to the generalized puzzle will appear in the next issue along with other triangular number pursuits. Let's hear from you on one or both counts.

Recreationally yours,
H.L.R.



McKay & McKnight

The following problems are compiled for use in the classroom. They could be given to individuals, small groups or entire classes as an enrichment activity or they could be used as an interesting way for bright students to fill some extra class time.

We have changed the format slgithly from last year in that we are not asking for solutions to be mailed in to be published later. Instead, we are publishing brief solutions to the problems in the same issue.

If you have an interesting problem you would like to share, please sent it, along with a solution to:
Ontario Mathematics Gazette
c/o Mr. J. McKnight

UP TO GRADE 10

1

Members of a local teen-age club disagreed about the way a certain outing was mananged and 15 girls withdrew. This left two boys for each girl. The boys were unhappy about the new set up and 45 moved out, leaving only one boy for each five girls. Work out how many girls there were in the club at the time of the outing.

2

On the average, over a period of years, how frequently does Friday the thirteenth occur?

3

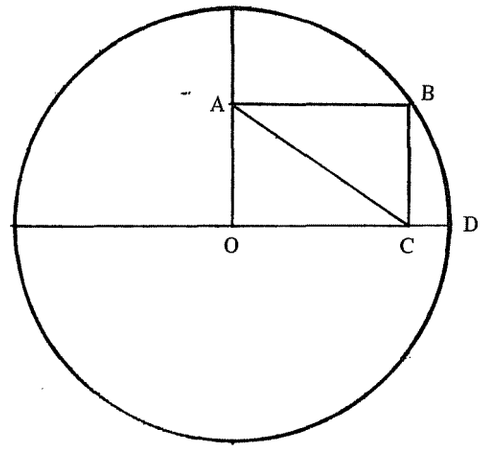
The positive integers 1, 2, 3, ..., 25 have been arranged very carefully into the table below:

11	17	25	19	16
24	10	13	15	3
12	5	14	2	18
23	4	1	8	22
6	20	7	21	9

In 120 different ways a set of 5 numbers from this table can be chosen so that a number is taken from each row and from each column. In each set of 5 there occurs a minimum number. Find the largest number which occurs as one of these minima.

4

In the diagram below, O is the center of the circle and AB is parallel to OD. AO and BC are perpendicular to OD. OC is 5 units long and CD is 1 unit long. Find the length of AC.



UP TO GRADE 13

5

A circle is inscribed in a right-angled triangle ABC in which c is the hypotenuse. Prove that the diameter of the circle is $(a + b + c)$.

6

Prove the identity $(1 + \tan A + \operatorname{Cot} A)^2 = \dfrac{(\sin A \bullet \operatorname{Cos} A + 1)^2}{\sin^2 A \bullet \cos^2 A}$

7

Solve $(x + 1)(x + 3)(x + 5)(x + 7) = 9$

8

A clock hangs on the wall of a railway station. The wall is 71 ft. 9 in. long and 10 ft. 4 in. high. If the hands of the clock were pointing in opposite directions, and were parallel to one of the diagonals of the wall, what was the time?

III ARE YOU READY FOR A CHALLENGE?

9

A cattle dealer had 5 droves of animals consisting of oxen, pigs and sheep, with the same number of animals in each drove. He sold them all to 8 dealers. Each dealer bought the same number of animals, paying $17 per ox, $4 per pig and $2 per sheep, and the dealer received $301 in all. What was the greatest number of animals the dealer could have had and how many of each kind were there?

BOOK REVIEWS



Dr. G. Vervoort
Lakehead University

Starting with this issue an attempt is being made to have more of the reviews written by those who teach at the level for which the book was intended.

If you are one of those who would like to contribute by reviewing an assigned text. please drop a note with some information on your background, current position, and address to G. Vervoort, Lakehead University, Thunder Bay, Ont. P7B 5E1. In particular some reviewers of elementary school texts and materials are needed.

Elementary Schools

Mathematics and the 10-year-old, Ward, Murray. Schools Council Working Paper 61. Evan/-Methuen Educational, London 1979. 4.25. 168 p. ISBN 1 423 50600 5
Reviewed by G. Vervoort, Lakehead University, Thunder Bay, Ont. This paperback is part of a British primary school evaluation study designed to provide teachers with information on the strengths and deficiencies in mathematics of the current crop of pupils. to describe teacher expectations and student reactions, and so on. No attempt was made to compare methodologies, though it does provide some indication of what skills and processes are in need of more attention. The study involved some 2300 children, their teachers and headmasters.

It appears that British schools spend somewhat more time on mathematics than Canadian schools, an average of 60 minutes a day, though individual schools varied from a mean of 30 minutes to 90 minutes. For this, and a long list of other reasons, the results are not necessarily valid for Canada, but it does provide us with an indication what to be on guard against.

While many results are what one would expect, there are a number of surprises in the overall picture. For instance, consider the results to test question 13:

0	6	2	9	9

What is the next number this meter will show?

Write it in here | | | | | |

Q.13 Results

Performance	Girls %	Boys %	All children %
Right	34	47	41
Wrong	57	46	51
Not attempted	9	7	8

Children's performance 70/91
Teachers' rating of importance 8/45

When compared to other test items, this one ranked 70th among 91 questions. Yet in the rating of importance, the teachers ranked in 8th among 45. (parallel tests were used).

Equally disconcerting are the results to question 39:
The value of the 4 in the number 347 is 40.
The value of the 4 in the number 481 is _____. .

Q.39 Results

Performance	Girls %	Boys %	All children %
Right	44	50	47
Wrong	34	29	32
Not attempted	22	21	22

Children's performance 57/91

Another intriguing concern is the gap in performance between the sexes with regard to specific test items, in general the girls out-peforming the boys is straightforward computation, the boys doing better in word problems. One can only speculate on the cause of these differences.

All in all an interesting report to stimulate the thoughtful mathematics educator.

Secondary School:

An Introduction to Number Theory. Harold M. Stark, 1978. MIT Press: 28 Carleton St. Cambridge, Mass. Paperback. $10.75 (Can.) Reviewed by Henry M. Derynck, Westgate Collegiate and Vocational Institute.

Professor Stark begins his books with an interesting 15 page introductory chapter. Some of the topics included here are the prime number theorem, Fermat's last theorem, Polya's conjecture, and Euclid's proof that there are infinitely many primes.

Chapter 2 deals with the Euclidean algorithm and the fundamental theorem of arithmetic. The author also includes sections on multiplicative functions and on Diophantine equations. Plenty of good exercises are provided.

Congruences are covered in chapter 3. Professor Stark begins with a good review and even includes an amusing application. From the fundamental properties of congruences he progresses into complete residue systems, linear congruent equations, Chinese remainder theorem, reduced residue systems, Euler's ϕ-function, and Fermat's theorem.

Chapter 4 is on magic squares. Diabolic and Symmetric Squares and a historical account on magic squares are also included. Diophantine equations are again discussed in chapter 5, this time using congruences for solving the equations. Other topics covered are Pythagorean triples and Fermat's Method of Descent.

Rational and irrational numbers are explained next. The chapter starts with a review of terminating and non-terminating, repeating and non-repeating decimals and from there progresses into algebraic and transcendental numbers.

Chapter 7, on continued fractions, is the longest one in the book, covering 75 pages. Continued fractions are discussed from a geometric point of view. This approach differs considerably from that of C.D. Olds, L.W. Singer Company. The chapter concludes with the Fermat-Pell equation

and the continued fraction expansion of d where d is a positive integer which is not a perfect square. An abundance of exercises follows this section.

The last chapter, chapter 8, deals with quadratic fields. The author progresses from fairly elementary topics to divisibility and factorization into primes. Also covered are unique factorization, Euclidean domains and applications of quadratic fields to Diophantine equations.

Excellent appendices are provided for review of 2-dimensional vectors, 2 by 2 matrices, and fields. A bibliography and answers to selected exercises are also included.

Though this book is probably intended as a text for first or second year university students, it makes an excellent source of high grade enrichment material for the gifted student. For that purpose and for one's own stimulation it is a worthwhile addition to a secondary mathematics teacher's library shelf.

BOOK REVIEWS

University:

Remarks on the Foundations of Mathematics, Ludwig Wittgenstein; revised edition, edited by G.H. von Wright, R. Rhees, and G.E.M. Anscombe; M.I.T. Press, 28 Carleton St., Cambridge, Mass., 1978; 444 pp. Reviewed by John E. McPeck, Faculty of Education, University of Western Ontario.

Bertrand Russell once said that, as a boy, he believed along with others that "two plus two equaled four", but he always thought it strange that we should be able to say so with such certainty. This apprehension might or might not have been the inspiration for his monumental **Principia Mathematica**, but the search for the foundations of mathematics will never be the same as a result. Following Russell, Ludwig Wittgenstein continues this search in **Remarks on the Founations of Mathematics**, first published as a collection in 1956, but revised here with a new chapter (Part VI) which has never been published before.

Although Wittgenstein is similarly interested in the foundations of mathematics, as was Russell, the resemblance between them stops there: both in the conclusions they reached, and in their methods of presenting their results. Where Russell's reduction of mathematics to logic was formal, systematic, and mathematically elegant, Wittgenstein's thoughts on the foundations of mathematics are meandering, fragmatic (resembling Heraclitean fragments), and written exclusively in prose. Like Russell's boyhood apprehensions about certainty in mathematics, Wittgenstein's major preoccupation in this work is to examine the source of logical compulsion, the "must" in mathematical conviction. Though his thoughts keep returning to this question, he discusses at some length Russell's logic, Godel's theorem, Cantor's diagonal procedure, Dedekind's cuts, and the nature of proof and contradiction.

Where Russell was content to rest mathematical certainty on logical certainty, Wittgenstein provides reasons to take pause over logical certainty itself. From whence does **it** come? To make a very involved story short, Wittgenstein's answer to this question is that logic itself is essentially **normative**, issuing from certain human decisions to "follow rules" which we create. There is nothing, however, in the nature of things which compel them to obey logic. Our felt need to yield to logical and/or mathematical demonstration stems from our sense of (conceptual) propriety for "following rules", which, in turn, comes from our desire to play certain "language-games". Herein lies the source of the "must" in all mathematical conviction and logical compulsion. Wittgenstein argues therefore, that "The mathematician is an inventor, not a discoverer." (p.99) In particular, the mathematician invents new games with new rules in the progress of mathematics. In this vein, two of Wittgenstein's fragments are illustrative:

45. You want to say that every new proof alters the concept of proof in one way or another. But then by what principle is something recognized as a new proof? Or rather there is certainly no 'principle' here.
62. The introduction of a new rule of inference can be conceived as a transition to a new language-game. I can imagine one in which for example one person pronounces: '$p \supset q$', another 'p' and a third draws the conclusion.

From an historical point of view, it is interesting to note that W.V.O. Quine's famous paper "Two Dogmas of Empricism" succeeded in convincing most contemporary philosophers of the fundamentally normative nature of analytic propositions (**e.g.**, logic and mathematics). But I think it accurate to say that Quine's position on this question was fully anticipated by Wittgenstein some ten years earlier, and to be found here in the **Remarks on the Foundations of Mathematics**.

This book however, is about the philosophical foundations of mathematics, and is therefore not a book within mathematics. For this reason the book could not be recommended as a text or learning-aid for the teaching of mathematics, even on a university level. Moreover, the fragmentary nature of Wittgenstein's presentation makes the overall argument difficult to follow even for those familiar with his method in the **Investigations**. But these fragments do make it possible to pick out some Wittgensteinian insights on each of the discrete subjects he treats. And teachers of mathematics who are interested in mathematical theory will find many of these fragments provocative. And for those keen on teaching mathematics through the use of games and puzzles, they will find solace in Wittgenstein's view that mathematics **is** nothing more than games and puzzles -- important as those games may be at times.

SOLUTIONS TO PROBLEM SET 18.1

18.1.1 If there are x boys and y girls then 2 equations can be formed:

$$x = 2(y - 15) \qquad 1$$

$$5(x - 45) = y - 15 \qquad 2$$

Solving yields $y = 40$. $\therefore$ there were 40 girls.

Note: This problem can also be solved in one variable.

18.1.2 The thirteenth of the month occurs every $\frac{365\ 1/4}{12}$ days.

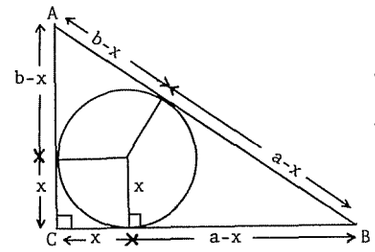
Each day of the week is equally likely.

$\therefore$ Friday the thirteenth occurs every $7 \times \frac{365\ 1/4}{12} = 213$ days.

18.1.3 15

18.1.4 Since AOCB is a rectangle then AC = OB = OD = 6.

18.1.5



$$c = a - x + b - x$$
$$= a + b - 2x$$
$$2x = a + b - c$$

$\therefore$ the diameter is 2x

$\therefore$ the diameter is a + b - c.

18.1.6 $LS = (1 + \tan A + \cot A)^2$

$$= \left(1 + \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)^2$$

$$= \frac{(\cos A \sin A + \sin^2 A + \cos^2 A)^2}{\sin^2 A \cos^2 A}$$

$$RS = \frac{(\sin A \cos A + 1)^2}{\sin^2 A \cos^2 A}$$

18.1.7 4 consecutive odd integers adding to 9 are -3, -1, 1, 3.

If $x + 7 = 3$ then $x = -4$ is a solution.

Now $(x + 1)(x + 3)(x + 5)(x + 7) = 9$ becomes

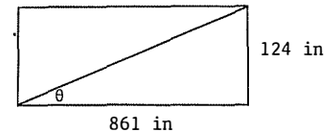
$x^4 + 16x^3 + 86x^2 + 176x + 96 = 0$.

Using polynomial division this becomes $(x + 4)(x^3 + 12x^2 + 38x + 24) = 0$.

Now using the factor theorem yields $(x + 4)(x + 4)(x^2 + 8x + 6) = 0$.

The solutions are $x = -4, -4 + \sqrt{10}, -4 - \sqrt{10}$.

18.1.8



$$\cot \theta = \frac{861}{124} = 6.9435$$

$$\theta \approx 8\tfrac{1}{5}°$$

$8\frac{1}{5}°$ translates to $\frac{8\frac{1}{5} \times 60}{360} = 1\frac{11}{30}$ min.

$\therefore$ the time is 2:43 and 38 seconds, or 9:16 and 22 seconds

18.1.9 Let x represent the number of oxen.

Let y represent the number of pigs.

Let z represent the number of sheep.

$\therefore 17x + 4y + 2z = 301$ and $5 \mid (x+y+z)$ and $8 \mid (x+y+z)$

$\therefore x + y + z = 40$ or 80 or 120

Let x = 1 $\quad 4y + 2z = 284$, $\quad y + z = 119 \Rightarrow y = 23, \quad z = 96$

if x = 3 $\quad 4y + 2z = 250$, $\quad y + z = 117 \Rightarrow y = 8, \quad z = 109$

$\therefore$ he had 120 animals with 3 oxen, 8 pigs and 109 sheep, or 1 ox, 23 pigs and 96 sheep.

O.A.M.E. A.O.E.M. COUNCIL ORGANIZATION 1979-1980

Executive

Past President - McPhail, David
President - Norrie, Alex
Pres. Elect - Morrow, Lorna
Director - Pogue, Paul
Director - Sherk, Paul
Director - Ullrich, Peter

Councillors

Term Expiring

1980	**1981**	**1982**
Acheson, Roma	Bedard, Jeanette	Conelly, Ralph
Crundwell, Mary	Fullerton, Olive	DeFelice, Dominic
Dottori, Dino	Lessard, Paul	Kyer, Avron
Evans, Douglas	Muller, Eric	Sr. Laframboise
Jones, Gareth	Smith, Robert	Neufeld, Rudy
McPhail, Sharon	Topping, Bruce	Robinson, Robert
Silbert, Michael	Maskell, Fred	Tuck, Robert
		Vervoort, Gerry

Chapter Representatives

C.O.M.A. - Armstrong, Bruce
Golden Section - Marshall, Roy
G.V.M.A. - Davidson, Dave
N.O.M.A. - Buhr, Ken
Q.S.L.M.A. - Buskard, Margaret
S.E.Y.M.A. - Walsh, Paul
S.W.O.A.M.E. - Palmer, Barbara
T.O.R.N.Y. - Feldson, Philip
P.R.M.A. - Jeffers, Bill



PRESIDENT'S MESSAGE

A.L. Norrie

A noted mathematics educator once stated that he based his work on an old Latin principle which stated "primum non nocere". Translated this meant "first do no harm". This principle would appear to be a very useful one for me to keep in mind as I approach the Presidency of this unique organization.

First, let me do no harm to the excellent work done by David McPhail over the past year and secondly let me do no harm in my year as President.

I must warn you, however, that while not wishing to harm I nevertheless wish to convey the message that I am very concerned about mathematics education in this Province. The role of our organization as we approach the complexities of present issues is also one of my concerns. We cannot afford to spend a year rearranging the deck chairs on the Titanic.

At least six major issues are dominating mathematics education in this Province at the present time. These are issues that cross all levels of education and they are not unique to mathematics.

I would charge **all** members of O.A.M.E. to help the Executive and Council to find ways and means of dealing with:

#1 The question of "CURRICULUM DEVELOPMENT". At present, the Intermediate and Senior Divisions are a particular challenge. However, the fragile links between Divisions and on into University are an important part of any development. Within O.A.M.E. a truly fine interface between elementary, secondary and post-secondary education can and should be developed.

#2 The question of "POLITICS". This issue continues to be crucial as the debate between the Ministry and School Boards continues. The incessant reorganization within the Ministry itself is barely comprehensible. Many individuals and powerful interest groups are also adding their voice to issues in education. O.A.M.E. needs to rise above politics and find positive ways of representing our membership for the benefit of mathematics students at all levels.

#3 The question of "ACCOUNTABILITY". Do we really know what the existing situation is? Are we teaching the correct programs? Are we meeting the needs of our students?

#4 The question of "EVALUATION". History may come back to haunt us unless we can generate the proper ways of finding out what is really happening. O.A.M.E. members would do this Province a great service by answering the simple question; "How much evaluation is enough?" It would be dangerous to leave the topic of evaluation to the whim of the politicians or academics.

#5 The question of "PROBLEM SOLVING". What are O.A.M.E. members doing to achieve this major aim of all mathematics education?

#6 The question of "RESEARCH IN EDUCATION". What kind is appropriate? Who should be involved? How shall the results be used? These are not easy questions and mathematics educators have long since worked out adequate strategies for dodging the issues.

Let us first list a few strategies with examples and then suggest how O.A.M.E. might change a negative strategy to a positive one:

(a) Find a scape-goat. Mathematics teachers can blame administrators; administrators can blame teachers; both can blame the parents and students. Everyone can blame the state of society.
Positive strategy - identify and carry out responsibilities at all levels.

(b) Profess not to have the answer. This lets you out of the need to have any answer.
Positive strategy - O.A.M.E. can actively search for answers.

(c) Say that we must not move too rapidly. This avoids the necessity of getting started.
Positive strategy - O.A.M.E. can capitalize on solutions now available.

(d) For every proposal set up a counter-proposal.
Positive strategy - Try new proposals and develop a rationale based on experience.

(e) Rationalize the status quo with minor improvements.
Positive strategy - O.A.M.E. can recognize the status quo as a starting point for future development.

(f) Point out that some of the greatest minds have struggled with this problem, implying that it does us credit to have even thought of it.
Positive strategy - O.A.M.E. can get involved in finding solutions for problems based upon practical experience.

(g) Put off recommendations until every related problem has been definitely settled and answered by scientific research.
Positive strategy - take a calculated risk and base recommendations upon the needs of students.

(h) Retreat into general objectives on which everyone can agree but which suggest little content and no changes in the present program.
Positive strategy - O.A.M.E. can specify important objectives for new program direction.

(i) Promote mass confusion by using as much educational jargon as possible.
Positive strategy - O.A.M.E. must clarify its position as much as possible.

(j) Explain and clarify over and over again what you have already said.
Positive strategy - determine new and divergent strategies for clarifying program objectives.

Let us not be overwhelmed by the issues. O.A.M.E. can continue to support Ontario mathematics which still is among the best in the world.

Alexander L. Norrie
President, O.A.M.E.

TO THE CHAPTERS



If your deliberations on **Basic Mathematical Skills** is continuing, I offer the following N.C.T.M. Annotated Bibliography. As I have received no written opinion from either a chapter or an individual member, the annotated bibliography will be my last reference for the basics debate. I feel I have fulfilled the council's request, "that opinions be solicited relevant to the Ontario situation." Perhaps we as individuals and as chapter members would rather let our opinions be expressed by others. The Ontario Mathematics Coordinators' Association has adopted the N.C.S.M. position paper on Basic Mathematical Skills and stated this in its brief to the Intermediate Mathematics Validation Committee. Perhaps we fully endorse the N.C.S.M. statement. We have not said so, however.

I am happy to hear that the Publications Sub-Committee is going to attempt better communications with the chapters during the coming year. This should help make Chapter News more interesting. I still believe a chapter newsletter or bulletin is a great help. Does your chapter have one yet?

Your new president has placed clearly before us issues in the form of six questions. Will your chapter be considering them during the year ahead? This is your forum. Do not be afraid to write us expressing either your personal opinion or the opinion of your chapter.

Bob Smith

ANNOTATED BIBLIOGRAPHY ON BASIC MATHEMATICAL SKILLS

1. Bell, Max S. "What Does 'Everyman' Really Need from School Mathematics?" **MATHEMATICS TEACHER** 67 (March 1974): 196-202.
   One person's view of what people should accomplish through their experiences in school mathematics.
2. Michigan Council of Teachers of Mathematic (MCTM)/NACOME Study Group. **REPORT OF THE NCTM/NACOME STUDY GROUP.** 1976.
   The NCTM/NACOME Study Group reviewed the NACOME report to sort out implications for improving the development of present practice and for improving research in mathematics education in Michigan. These implications focus on goals, basic skills, and classroom practice.
3. National Advisory Committee on Mathematical Education (NACOME). **OVERVIEW AND ANALYSIS OF SCHOOL MATHEMATICS GRADES K-12.** 1975. NCTM, 1906 Association Drive, Reston, VA 22091.
   A comprehensive overview and analysis of the current status of mathematics education K-12--its objectives, current and innovative practices, and attainments. Special recommendations on the needs and opportunities for improvement in school-level mathematics education are also made.
4. Taylor, Ross. "What to Do about Basic Skills in Mathematics." **TODAY'S EDUCATION 66** (March/April 1977):32-33.
   A discussion of the opportunity to turn the back-to-the-basics thrust forward to the basics and offer students the background in basic mathematical skills they will need to meet the challenges they will face.
5. The Committee on Basic Mathematical Competencies and Skills. "Mathematical Competencies and Skills Essential for Enlightened Citizens." **MATHEMATICS TEACHER** 65 (November 1972):671-76.
   A report containing a list of basic mathematical competencies, skills, and attitudes essential room for enlightened citizenship in contemporary society.
6. **THE NIE CONFERENCE ON BASIC MATHEMATICAL SKILLS AND LEARNING, VOLUME I: CONTRIBUTED POSITION PAPERS.** 1975. ERIC Document Reproduction Service, Box 190, Arlington, VA 22210. (ED 125 908)
   A collection of short papers by the participants of a three-day conference on basic mathematical skills and learning.
7. **THE NIE CONFERENCE ON BASIC MATHEMATICAL SKILLS AND LEARNING, VOLUME II: REPORTS ON THE WORKING GROUP.** 1975. ERIC Document Reproduction Service, Box 190, Arlington, VA 22210 (ED 125 909)
   A description of the organization of the conference, summarizing remarks by the cochairmen of the conference, and reports of the working group.

TWELFTH ANNUAL DESCARTES DINNER

the 1979 Medallists

Edwin Baumgart, Kitchener

James Boyd, Kingston

Congratulations to this illustrious group and in particular to the 1979 Descartes Medallists:
Ed Baumgart, Kitchener
Jim Boyd, Kingston

The Descartes Medal is given by the Faculty of Mathematics Waterloo, to outstanding mathematics edocators throughout Ontario. Each year, the previous winners and spouces meet to receive the new members and enjoy the evening. Each new member receives a medal mounted, which contains the following message.

THE SOCIETY OF DESCARTES MEDALLISTS

1968: Rodger Allan, Frank Asbury, Lloyd Auckland, Wilf Cunnington, John Del Grande, John Egsgard, Jim Gardner, Jim Fencott, Jean Leppard, **John McKnight

1969: Jim Dean, **Jean McRobert, Norman Sharp

1970: Henry Dearborn, Jerry Lowden, Marion McKinley

1971: Earl Marcy, Harold Minielly, Jack LeSage, Gladys Tunks

1972: Don Fox, Martha Gartrell, Doug Henderson

1973: Bob Behan, Gene Durrant, Dave Palmer

1974: Don Attridge, Fred Miller, Joan Routledge, Geoff Steel

1975: Duff Butterill, Bob Jackson

1976: Ann Harris, Paul Sherk

1977: Don Kaye, Brock Rachar

1978: Roma Acheson, Bob McNeil

1979: Ed Baumgart, Jim Boyd

FUTURE MEETINGS

O.A.M.E./A.O.E.M.
Executive - Sept. 14
Council - Oct. 13
Executive - Nov. 16
Council - Feb. 9
Executive - April 4
Local
N.O.M.A. Sept. 28-29
at Cambrian College, Sudbury

International
N.C.T.M. Calgary Meeting, 1979
October 11-13 Calgary
A.A.M.T. 8th Biennial, 1980
January 14-18 Canberra
O.A.M.E./A.O.E.M. 7th Annual, 1980
May 8-11 Toronto
N.C.T.M. 58th Annual, 1980
April 16-19 Seattle