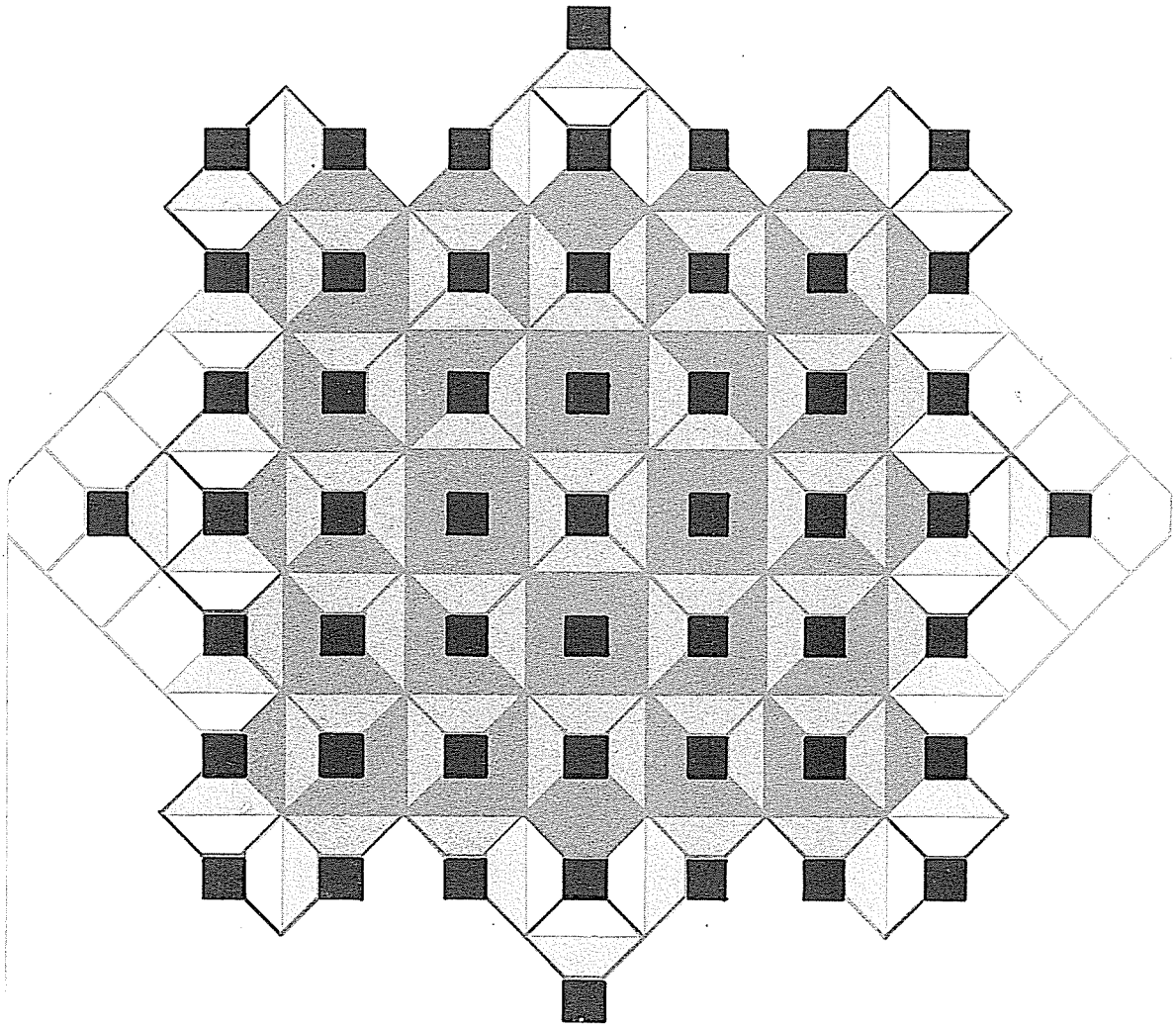
ontario mathematics gazette

VOLUME 18 **NUMBER 2** **DECEMBER, 1979**



THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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INTERNATIONAL YEAR OF THE CHILD

1979

Année Internationale de l'Enfant

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EDITORIAL

December, 1979

In 1971-72 I was a member of the Ontario Mathematics Commission, one of the two founding groups of O.A.M.E At that time I was concerned that the new organization would continue two projects that O.M.C. was involved in. The first was a publication. A grant from the Beatty Foundation gave assistance in establishing the journal and Dr. Coleman's determination that it continue resulted in the journal you are reading. The second concern was that the Leadership Seminars, Dr. John Girhiny of Hamilton and I were involved in, would continue. It has not only continued, it has grown and it has prospered. Along with President Alex, I congratulate the co-chairmen on the fine program presented last month. From the written evaluations summarized below, it is evident the participants found the 3 day seminar worthwhile.

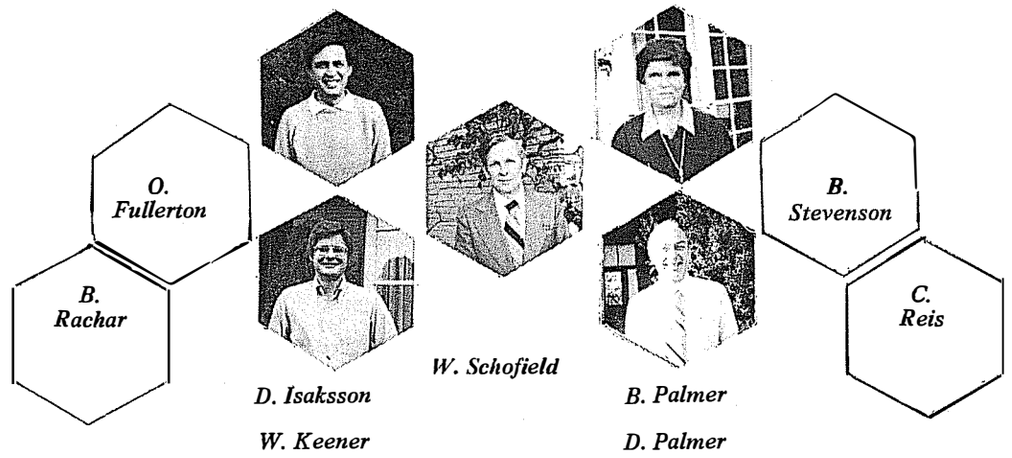
EVALUATION
MATHEMATICS LEADERSHIP SEMINAR

	Excellent	Good	Total	Total	Fair	Poor
1. General Organization	65%	35%	100	0	0%	0%
2. Seminar Topics	42%	49%	91	9	7%	2%
3. Seminar Leaders	42%	53%	95	5	5%	0%
4. Ideas for improving math prograı	45%	42%	89	13	13%	0%
5. Improving leadership	33%	51%	84	16	16%	0%
6. Participation	27%	50%	77	23	18%	5%

This Leadership Seminar has been a project of OAME/EAOM now for the past six years. At the conference, my informal sampling of opinions found the participants even more enthusiastic than the above results would indicate. I did ask those I talked with, to write me a letter, after reflection, on their experience, and indicate if they indeed were helped in providing leadership. I encourage any of you, who have attended a Leadership Seminar in the past, to write. We need more than an immediate evaluation to be sure that this project is meeting its objective.

Bob Lowitt

THE ADVISORY BOARD



FROM THE PRESIDENT



Once again OAME sponsored a leadership seminar at the Ascot Hotel on October 18. David McPhail (Past-President) and Morley MacGregor coordinated one of the finest programs ever presented to leadership participants. OAME is grateful for their hard work and support.

I was privileged to be a small part of the seminar program and at that time I shared my thoughts about "OAME and Leadership". My comments seemed appropriate to be included as an editorial for all OAME members.

As we move into the 1980's it is clear that the leadership role in mathematics education is shifting from a type of centrally controlled Provincial leadership to a more participatory kind of leadership. The new leadership is evident by the increasing involvement, of OAME, OMCA and specific members of some, in the mathematical decision making throughout the Province.

How does OAME exhibit leadership as an organization? Consider the fact that our organization actively supports nine chapters across the Province and this support is reciprocal. Our direct input into Provincial curriculum projects has increased as we continue to be an influence at the Intermediate Division levels and we will shortly be influential at the Senior levels. The publications (Gazette and Abacus) of OAME are of the highest quality and are valued not only in Ontario but across North America. We consistently provide excellent professional development for our members by such events as the leadership seminar and our annual conference. Chapter activities also provide good sources of involvement.

What of the individual member as a leader? Chances are multi-fold: become involved with a Chapter, be a council member, volunteer to be a speaker, or workshop leader and share your views by writing for our publications. You should know the organization's position and involvement in mathematic's education. Then get out and work to make our position stronger.

Leaders within our organization are easily recognizable because they: recognize their responsibility at all levels, actively search for sound educational answers, try new proposals and develop new rationales based upon experience, get involved in finding solutions for problems, specify important objectives for new program directions and recognize a need for future oriented development.

"OAME is not led, OAME leads".

Alex Norrie,
O.A.M.E. President A.E.O.M.

LETTERS TO THE EDITOR

Waterloo, Ontario
October 2, 1979

The Editor,
Ontario Mathematics Gazette

Dear Sir:

This is with regard to the letter of FGB Maskell in the September issue of the Gazette.

Occam's Razor has many forms. One is "Do not multiply opportunities for misunderstanding unnecessarily." Mr. Maskell's comment is well taken from this point of view. His simple English paraphrase is aimed specifically at the problems involved in treating infinity as if it were a number, and avoids the problem most successfully.

One might argue that the student has to come to grips with such language as "the limit is infinity" sooner or later, and that hence the question is essentially one of timing. But there is a more serious flaw in "... the limit approaches infinity" which he also avoids, but without calling attention to

this. This flaw has nothing to do with infinity - it appears also in "the limit approaches 2". It is a good example of an error in the use of ordinary language which can play havoc with a beginner's attempts at understanding. What has happened is that the approacher has become confused with the approachee: this is an actual error, even though one sees it in many books (not the better ones) and hears it in many lectures (likewise). It is not removable by appealing to standard correct mathematical usage. In "the limit of x^2 as x approaches 4 is 16", it is x^2 which does the approaching - not the limit, 16. The limit does not approach the limit - it **is** the limit.

I am especially interested in such examples, as I regularly encounter freshmen who are confused by this very point.

Sincerely yours,
R.A. Staal

Editors Note: F.G.B. Maskell is "in complete agreement" with the above comment.

1979-09-19

Dear Bob:

I have read and reread the article on evaluation procedures by **N.** MacEwan and associates in the September issue of the Gazette.

Mr. MacEwan and the members of his department are to be commended for identifying the problems associated with assigning marks to students, for having the dedication to do something about it, and for having the courage and forthrightness to tell the rest of the professional community what they have done. I would wager that a similar exercise would prove beneficial to the proper functioning of the mathematics department of every secondary school in Ontario.

However, my disagreement with some of the assumptions stated is so profound that I have attempted to articulate my feelings in the enclosed article. I leave to your judgment whether there is the need or the space to print it.

D. Henderson,
Master Teacher of Mathematics

Editors Note: I respect your opinion Doug, and have printed your reply. Where do you readers stand on these two different views regarding evaluation procedures?

University of Waterloo
Waterloo, Ontario
September 20, 1979

The Editor
Ontario Mathematics Gazette

Dear Sir:

This is to comment on the interesting article by Dr. Gray, "The Unstable Arithmetic Curriculum ... Elementary School Teachers" in the September Issue of the Gazette. The perspective gained by covering the period from the thirties to the present is most valuable, and the bibliography is especially well selected for this purpose.

A feature of the Modern Mathematics Movement which should be emphasized is that it was not just concerned with promoting understanding of mathematical structure, but (at least in most of the literature) with a **particular** approach to this goal. The problems which arose may not have been due to the goal, or to the teachers' preparation, but to this approach.

A rather typical example was the use of the terminology of the "empty set", even though everything that needed to be said could be said in the language of everyday discourse. One of the advantages of everyday language is that the language itself fades into the background and what it says becomes primary: in addition, whatever indeterminacy it may have is something we are used to, and reasonably capable of handling. When using special terminology, one must be doubly careful to **use it properly**. It may not be necessary to explain fully just why it is being used properly in a particular case, but errors should be avoided none the less. On page 14, under "a. The zero

facts'' we have what has become fairly standard practice in school texts in ''the concept of zero as the name of the empty set'' and ''the name of a collection of empty sets is still zero''. Now, the M.M.M. created confusion because it made a fuss over many things which it proceeded either to ignore or treat carelessly, and this is a good example. In the treatment of the ordinal numbers one sometimes begins with the null set playing the role of zero. But generally, it is very dangerous to encourage the habit of confusing the null set with the number zero, especially in dealing with the arithmetic of cardinal numbers. By using zero as the **name of** the empty set, one provides an ''out'', but the plot thickens at this point. By ''zero'' here does one mean the word with four letters or does one mean the cardinal number zero? Is zero the concept or is ''zero'' the concept? Furthermore, a collection of empty sets (by the way, there is just one empty set, according to standard terminology) is not the empty set, and hence its name should not be zero (or ''zero''?). The answer given to such musings is usually that we are making too much of a fuss in our criticisms, and can only confuse the students. But surely the shoe is on the other foot. Those who waken sleeping dogs in the first place must be prepared for the consequences. (If metaphors occur in separate sentences, are they actually mixed?) Not only does the more-or-less standard M.M.M. treatment of these matters have these hazards, but the entire issue can be avoided by the use of ordinary English, **and just as rigorously** (even when the errors outlined above are eliminated). To explain why $3 \times 0 = 0$, it is enough to say something such as ''If John, Peter and Tom have each caught no fish, then together they haven't caught any fish.''

This treatment goes to the heart of the arithmetical truth without mentioning the null set or talking about naming. **What the M.M.M. approach tries to do is teach the M.M.M. approach** to arithmetic, and this, unfortunately turns out to be harder than the arithmetic itself. (It is rarely done correctly).

On page 15, under ''f.'' there is further confusion due to the apparent desire to **talk about** naming, which in turn I suspect is due to the confusion between **naming** and **describing**. In ordinary discourse, names are to a large extent non-descriptive. ''John Jones'' may indicate that the person named is a child of Mr. Jones, but ''John'' and ''Jones'' by themselves have no descriptive properties. The phrase ''by choosing $5 + 2$ (as a name of 7)'' is bound to be confusing because of this as well as the use of ''choosing'' which suggests freedom on the part of the chooser. The usefulness of ''$5 + 2$'' has nothing to do with naming: it has to do with a **property** of 7. The reference to naming is but an exotic and artificial device for using this property. Even if we replace ''name'' by ''describe'', thereby getting a far sounder treatment, much unnecessary awkwardness remains.

It is now generally agreed that meaning and structure are both essential to understanding. It would, however, seem to be a worthwhile project to replace the more or less standard M.M.M. treatment of the latter with something more likely to achieve its goals. What seems to be needed is a full appreciation of the logic (in the broad sense) involved together with simple, direct exposition in the vernacular. Many teachers provide just this, but our ''system'' doesn't.

Yours truly, Ralph Staal
Waterloo



January: Children need love, especially when they do not deserve it.
- Harold S. Hulbert



October 23, 1979

Mr. J. LeSage
Barrie, Ont
L4M 1T4

Dear Jack,

Just a short note to formally thank you for representing Ontario Association for Mathematics Education on the recent Intermediate Validation Committee.

I was there with you and I know just how good a job you were able to do. OAME members can rest assured that the guideline will be better because you were present.

The Executive, Council and Membership of OAME thank you for all your efforts.

Sincerely,
Alexander L. Norrie
President, OAME

AN OPEN LETTER TO O.A.M.E.

A Mathematical Community in Canada

Time For Some Affiliations

The Science Council of Canada's report, Mathematical Sciences in Canada, 1975 observes that in Canada, "There are so many centres of mathematical activity, and there is so little communication between people in different centres!" (p.63)

The Constitution of the Canadian Mathematical Society (CMS) includes in its objects, "to assist in improving the teaching of mathematics in Canadian universities, colleges, and schools," and "to encourage and assist in the development of mathematics and mathematics education."

The Constitution of the Ontario Association for Mathematics Education has for its aim "to promote excellence in mathematics education in the Province in Ontario." There is no constitutional restriction to the school system; in fact, an explanatory note implies the reverse.

In 1979, "part of the programme of the CMS Winter Meeting will be devoted to matters of interest to secondary school teachers as well as university mathematicians. ... Peter Lancaster will discuss the role of the CMS in promoting the teaching of mathematics and problem-solving ..." (From the CMS's October Notes News & Comments).

I hope this is the beginning of a recognition by post-secondary teachers of mathematics that the foundations of what goes on mathematically in the colleges and universities is laid during the course of some twelve years in the country's school systems, and that the universities must give some attention to these foundations, and be prepared to propose changes in specifications.

In 1979, the OAME formally affiliated with the National Council of Teachers of Mathematics - an organization based in USA. I am not opposed to this move; I am in favour of more communication and cooperation across the border. However, the move does highlight the dichotomy among the Canadian mathematical communities: university - research - CMS; and school - maths education - NCTM.

I hope it does not mean that the OAME - the country's leading provincial mathematical organization, at least in numerical terms - will shed what little interest it has in post-secondary education, and hole up in the school system.

There are serious research needs in mathematics education. Little is known about what precisely is mislearned in mathematics, nor precisely how such mislearning occurs. Throwing blame about is to my mind entirely fruitless. A joint approach involving both sides of the interface is needed - not just educational faculty, but mathematical faculty as well.

Perhaps it is opportune to consider the creation of a national mathematics coordinating council, which would provide a forum for representatives of organizations of professionals who engage in the practice as well as in the teaching of mathematics.

Such a body could provide a powerful incentive for the development of a mathematical community and the release of mathematical resources for the improvement of mathematics education. A proposal for a national council was put forward in 1975 in Mathematical Sciences in Canada (pp. 175 and 182).

I would like to see such a Council address itself inter alia to the Canadian policy and practice of excluding mathematics from Canadian culture and from the responsibilities of the Canada Council.

But first, let there be dialogue between the national organization and the provincial organizations. An important new initiative under the editorship of E.J. Barbean is a new publication "to tell you about important events, research, curriculum development and items in the literature of national interest" for insertion "in mathematical educational journals across the country." (From the CMS's October Notes News & Comments). [which we carry]

As a member of both the OAME and the CMS, I hope that an increasing number of university and college teachers of mathematics will participate in and be made welcome by provincial mathematics organizations, that room will be made in the national organization for mathematics specialists in the country's schools, that initiative will be taken in 1980 to bring about affiliation between the CMS and the OAME.

To this end, I am submitting this observation for publication in both the Ontario Mathematics Gazette, and the CMS's Notes, News & Comments. That I have made no reference to other provincial organizations does not mean that I hold them unimportant, but only that I know my own organization and feel keenly the incongruity of an affiliation which ignores our own national organization.

FGB Maskell
Algonquin College
22 October 1979

UNITED NATIONS' DECLARATION OF THE RIGHTS OF THE CHILD

THE RIGHT

To affection, love and understanding.
To adequate nutrition and medical care.
To free education.
To full opportunity for play and recreation.
To a name and nationality.
To special care, if handicapped.
To be among the first to receive relief in times of disaster.
To learn to be a useful member of society and to develop individual abilities.
To be brought up in a spirit of peace and universal brotherhood.
To enjoy these rights, regardless of race, colour, sex, religion, national, or social origin.

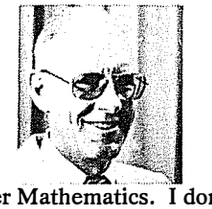


INTERNATIONAL YEAR OF THE CHILD



CHILDREN'S TALK

R.S. Smith, Faculty of Education,
The University of Western Ontario



Math is the Most. Mostly Math. The Now Math. Monster Mathematics. I don't get it. How do you do this? Who's got the answer?

Grade 4 children using an acetate cm^2 grid to find the area of a convex pentagon (a house end figure). "Oh boy! Halves are a problem. Do you count half ones? I can't do it because the line comes across." Then said, "To heck with the halves."

Before and After Math. Mathematics Revisited. Forever Mathematics. My, Aren't These Hard! I Love Math. What Question will we do next?

Using the same acetate grid but given a rectangle with dimensions 4 cm wide and 8 cm long, a grade 4 child said, "I counted eight across the bottom and four up and I multiplied because it is easier than going 1, 2, 3, 4, up and down." Children from the same Grade when asked to draw a figure whose area is twice that of (b) a square centimetre, made squares $3cm^2$, $4cm^2$ or $8cm^2$ in area. Mark recognized his error when questioned further saying, "I made a boo-boo, because I made the lines twice as long. I did like 1, 2, 3, 4 (counting the sides) times 2 equals 8. ??? Paula (explaining her answer - a figure of area $3cm^2$ "One square there ... to make it twice as big, I make two other squares. "Tester, How many have you drawn?" Paula, "Three". Tester, "Is this twice a big?" Paula, "Yes."

Magnetic Math. Merry Math. Man From Math. Mathmanship.
Only God Could Do This. That's hard. Golly - What Next. It's a challenge!

Grade 6 pupils when asked to show area concept with the convex pentagon suggested, "Oh, it is the distance around." A couple of pupils said "Well, it's - we multiply the distance across the bottom times the height of the house - Oh, no, I can't do that because it is not a rectangle." A few children suggested it was the inside of the plane figure, but I was not sure whether they were talking about volume. So I showed them a litre container. They then said, "Well, volume is inside there, and area is inside this flat shape," pointing to the inside of the house-end."

Basic Math. Math Phobia. Abstract Math. Into The Math Room, Again.
Math is the pits! Why is Math so difficult? There is no way.

Sometimes the written answers tell us a lot about children's misunderstandings. When asked to write down the name of the unit used to measure area, I got the following efforts: 1^{2} cm, 1^{sq} cm, $1\ cm^{4}$, 1 sq. cm. and $1^{\mathfrak{cl}}$ cm. When asked to write down the name of the unit used to measure volume, I got the following efforts: 1 cubic cm, 1 qu. cm. 1^{3} cm. and 1 cu. cm.

Math Media. Math is the Message. The Now Math. The Liberated Math.
It's not fair. That wasn't covered in class. We didn't do THAT.

Dr. H. Ginsburg in his book "Children's Arithmetic: the learning process" states that incorrect answers are seldom due to guessing, to low intelligence, or to low mathematical aptitude. A close look shows that they derive from systematic procedures having sensible origins.

"Sherry, a third grader, was the child asked to add 52 + 123 + 4. She got 17, an apparently absurd answer - so absurd that it must have been the result of an unthinking guess. Was it? The interviewer asked "how did you get it?" Sherry replied; "First I added 5 and 2 and then I added 1 and 4 and the 2 and 3." In other words, Sherry simply added the individual digits without regard to place value.

In doing subtraction, many children use the following misguided but systematic procedure. Asked to do: 21
$\underline{-\ 5}$
Bob came up with the answer 24. He had subtracted 1 from 5 to get 4 and then simply brought

down the 2 (or subtracted 0 from it)."[1]

The children's informal methods of computing often rely on counting. Standard tests are of little value for diagnosing their methods. Worst of all, they fail to reveal children's strengths. The tests say nothing about how instruction should proceed. Listening to the child's talk may well reveal the wrong strategies, which then can be tackled. Anyone wishing to teach must pay attention to children, must observe them going through the learning process.

A LESS VISIBLE DEATH



Norman M. Globe, Secretary General, Canadian Teachers' Federation

Every year, 15½ million children, aged four and under, die. Of these deaths, 15 million occur in the "developing countries". Half of them are due, in part at least, to hunger or malnutrition.

No such dreadful statistics haunt the Canadian scene. Does that mean that, in our concern for the children of other countries, we can afford to ignore the less visible problems of our own young people?

To do so would be a betrayal of the purposes of the International Year of the Child.

Some of our children will grow up to be functionally illiterate. The chances of this are greatest for those whose parents are among the 5 million of today's adults who, in Canada, are in or close to that category themselves. Their children stand a good chance of inheriting their birthright of poverty, powerlessness, marginal employment, dependence on welfare, chronic ill health.

Many of our children will grow up with little comprehension of the fundamentals of healthy relationships. Many more will never find a basis for the self-esteem that everyone needs. Some will be criminals. Many will inflict cruelty on their own children. Very many will be alcoholic or otherwise drug-addicted. Some will kill themselves.

Some of our children will tread the well-worn paths reserved for members of their own ethnic group, through low expectations to low performance, and out into low-status, low-income jobs.

Many of our children will be denied the enrichment of the spirit that comes from contact with the arts. Many will leave school unprepared for employment, deficient both in their skills and in their understanding of the world. Many will be quite unprepared for the exercise of democratic rights, for the responsibilities of citizenship, or even for the task of managing their own personal finances.

Very many will never know what fields of endeavour might have been opened to them, or what interests they might have pursued that would have given more substance to their lives.

How much the schools could ever do about such things, if teachers had the help they needed to create appropriate programs, the chance to implement them, and the time to get to know their students' needs, we do not know. We may never know more than we do now — we may never be allowed to even try as much as we are trying now, if the political mood which seems to be growing in Canada continues to develop. It is a mood of indifference to the needs of the disadvantaged, a rejection of the compassion and concern for equity that characterized the last two decades.

The education ministers of the OECD countries talked about that at a meeting last October. "Much of the debate about education today," said their chairman, Dr. Pais of the Netherlands, "is

1. Ginsburg, H. **Children's Arithmetic: the learning process.** p. 108.

politically motivated ... a matter of the aspirations and capacities which education generates among various social groups: access to the tools which will allow them to improve their status in society." "The challenge," said Mr. Egeland of Norway, "is to build up the strength of the weaker groups ... to fight for a fully accepted equal status in society."

What do *our* politicians say?

What can non-politicians do?

Nurses in Ontario were concerned that their workload was becoming too heavy to permit proper care of their patients. Three arbitration boards, backed in December 1978 by the Supreme Court of Ontario, have upheld the inclusion of the issue of professional responsibility in collective bargaining, in the form of the negotiated right to refuse a workload that jeopardizes the welfare of patients, and to appeal to a professional committee for a ruling. Is there a lesson there for teacher-negotiators?

There are many ways of dying. To be denied a needed education is to suffer, at the least, a small quiet death of the intellect, the spirit, the personality, the death of hope. Physical starvation threatens few of our children, but the Year of the Child challenges the political leaders, the negotiators and the practitioners of education to be aware of the spiritual and psychic starvation, the denial of opportunity, the confirmation of disadvantage and inequality that *do* threaten and which must be fought, in the name of our children's lives.

SIR ISAAK NEWTON

Michael Tangloo

One day while I was watching television I saw this commercial about Sir Isaak Newton. He was getting hit on the head by an apple and discovering the theory of gravity.

I had known about gravity for quite some time, but I didn't know about how it came to be discovered. Being of a curious nature I decided to check and see if this story about Newton was true.

I borrowed many books from the library on his life and works. I soon found out it didn't take this great mathematician a hit on the head in order for him to become curious as to why things fell to the ground. All Newton had to do was watch the apples fall to the ground and immediately he started thinking.

After conducting a series of experiments, he soon concluded that a force called gravity pulled the apples to the earth.

Childhood

After reading further I ascertained that Newton lived from 1642 until 1727 and in his lifetime he had made a long list of important discoveries. Many of these discoveries affected the lives of numerous people.

Newton was born on Christmas Day in Woolsthorpe, England. His father had died a few months earlier. Newton's mother thought it best that Isaak should stay with his grandmother.

At twelve years of age he was sent to Mr. Stoke's Grammar School by

his grandmother. He did very well in school and even became the leading student. Though he was capable of doing more complex work, he did just enough so that he would stay the leading student in his class.

Short-lived was his grammar school life because his stepfather died in 1656 and his mother needed him to help on the farm. Newton was soon withdrawn from school.

Those two years of school had given him just enough knowledge of mathematics and other various sciences so that he would hunger for more. No matter what, Newton was always trying to find ways to acquire knowledge. His uncle sympathized with him and obtained for him admission to Trinity College in Cambridge as a sub-sizar on June 5, 1661. This was only done after Newton's mother had consented.

Sub-sizar to Scientific Genius

This change in his life was probably the most important of his whole career. It started him on his long list of major scientific achievements.

He advanced to a sizar on July 8, 1661. On April 28, 1664 he was elected a scholar. During the month of January in 1665 he received his Bachelor of Arts degree. While being a student he also became an authority on most of the worthwhile existing mathematical works. After graduation his main years began. Newton himself acknowledged them as his best years in the following quote which was written in May, 1666. "I had entrance into the inverse method of fluxions, and in the same year I began to think of gravity extending to the orb of the moon.... All this was in the two years 1665 and 1666, for in those years I was in the prime of my age." ("100 Great Lives", p. 37, P. I)

In 1665 he discovered the binomial theorem. A small expanse of time later, he discovered the principles of integral calculus. This is the method for calculating the area of curves and the volume of solids. From this Newton went on to discover the Universal Law of Gravitation, invent the first reflecting telescope, write the "Principia" and also publish many other scientific papers. I have classified most of his discoveries under related categories

Contributions in Mathematics

In my opinion, no other mathematician of the seventeenth century

gave more of himself than Isaak Newton.

The binomial theorem was one of Newton's earliest contributions. Newton also introduced the concept of infinite series by the binomial theorem.

The fundamental principles of calculus were discovered by Newton when he learned that integrations could be performed easier by inverting the process of differentiation. It was possible for him to learn this because he had studied the methods of his teacher Isaak Barrow, and also the works of Fermat and Wallis. Newton's first publication on the subject of calculus was a short scholium in the "Philosiphiae Naturalis Principia Mathematica" which was published in 1687.

This paper had little immediate effect on the scientific public because it was hard for skilled mathematicians of that day to comprehend this paper. Only a few of Newton's contemporaries understood it, none of whom would explain it to the general scientific public.

Contributions in Physics

Newton's work on the laws of mechanics started men thinking about the theories of aerodynamics. Newton reached the formula simply by deduction. He formulated that pressure on a flat plate was proportional to the product of the density of the air, the area of the plate and the square of the plate facing the airstream. His work didn't go very far into aerodynamics.

The properties of the force of gravitation can be examined by astronomical observations. Newton probably used one of his reflecting telescopes to gather this information. He proved that the planets stay in their orbits round the sun, and the moons in their orbits round the planets because of the gravitational pull of each celestial body. His work on celestial mechanics such as this was published in 1687 in "Principia".

Newton wrote about the three laws of motion which talk about the principles of mechanics. This became known as a sort of follow-up to Galileo's work in the early seventeenth century.

Newtons formulation of the universal law of gravitation stated that any two particles of mass in the universe attract each other with a force that varies directly with the product of the mass and inversely as the square of the distance between them. We now know that Newton's laws of gravitation and motion were approximations of the true laws of the motion of celestial bodies.

At one point in his life Newton started experimenting with sound. He tried to find the speed of sound. Using his full intellect Newton reasoned that the velocity of sound in air should be equal to the velocity acquired by a body following under gravity through a distance equal to half the height of the atmosphere, as being of uniform density equal to that of the surface of the earth. Though Newton knew that the speed of sound was greater than this he suggested it might be due to the size of the air particles which could not be calculated.

While Newton was a student he took interest in astronomy. He made his own telescopes and he would grind his own lenses himself. Newton started experimenting with various lenses and taking down notes on what he observed. These notes would soon become his theory of light and colour. On August 1665, bubonic plague broke out in England, interrupting his experiment. Trinity College was shut down and the students were sent home. Newton packed his telescopes and notebooks and went home to Woodsthorpe.

Although he made his own lenses he could never correct the blurry fringes of colour that outlined the images he saw in his telescope. He soon deduced that it was not an imperfection in the lens that was causing it, but that it was caused by light itself. Having this in mind Newton began a series of experiments. He soon became convinced that pure sunlight was composed of a rainbow of colours. Newton learned this by using a prism to separate a beam of sunlight into a spectrum of colours and then passed this spectrum through a second prism. Once this was done he noticed that the light was again pure white sunlight.

In this experiment he showed that all visible colours are contained in sunlight and that white light could be reproduced by combining lights of many colours. After learning this Newton set out to correct the fringes of colour in his telescope. Today these fringes of colour are known as chromatic aberrations and can be corrected by using a colourless lens. Newton did not know this so he used a curved mirror instead of the usual lense to bring distant stars into focus. By doing so he eliminated the aberrations. He invented this in 1668 and it soon became known as the reflecting telescope.

When these discoveries became known to the Royal Society, Newton was immediately made a candidate for the society. This was a great honour because the Royal Society was a group of leading mathematicians and scientists and they had not regarded him as a part of the scientific set.

Newton's Senior Years

During the years of 1692 through 1694, Newton was taken ill. He suffered from insomnia and also a severe nervous disorder. This disorder was so severe that people rumoured he was insane. They even went so far as to say he had been put in a sanitorium. His friends who really knew what had happened to him secured him the job of Warden of the Mint in 1695 after his recovery. In 1699 he became Master of the Mint and after this he was elected an associate of the French Academy of Sciences.

He was again elected to the Parliament for the university in 1701. This, and his other duties took up most of his time and there was little time left for scientific study. When he was appointed Master of the Mint, he had to find somebody to take his duties as Lucasian professor. In 1701 he resigned from the professorship. He was a thorough Master of the Mint until his death.

The Royal Society elected Newton to be their president every year for twenty five years beginning in 1703.

During the year of 1705 Queen Anne came to visit Cambridge as a guest. While visiting, she bestowed the honour of knighthood upon Newton.

In the early part of 1727 Newton became ill once again. On March 20, 1727, his life ended. Eight days later Newton was buried at Westminster Abbey. Four years later a monument was erected in his memory.

In the Future

In the future as in the past, there will continue to be great men such as Newton. Newton was born after Galileo had made his mark in the world of science and just before Einstein made his. Great men such as these have come into our midst without us having noticed them. They were treated at most the same as artists. These special artists and their works were only recognized after their deaths.

If this world was to have a handfull of Isaak Newtons it would mean, perhaps, a solution to many of the world's major problems. Newton might have solved the energy problem had he been alive today. Newton wouldn't stop there, he would continue to solve other problems.

Newton made so many discoveries in the various areas of science that nobody would be able to list them in a mere two thousand words.

"Nature and nature's laws lay hid in night; God said, 'Let Newton be' and all was light." (100 Great Lives, p. 41, P. 6).

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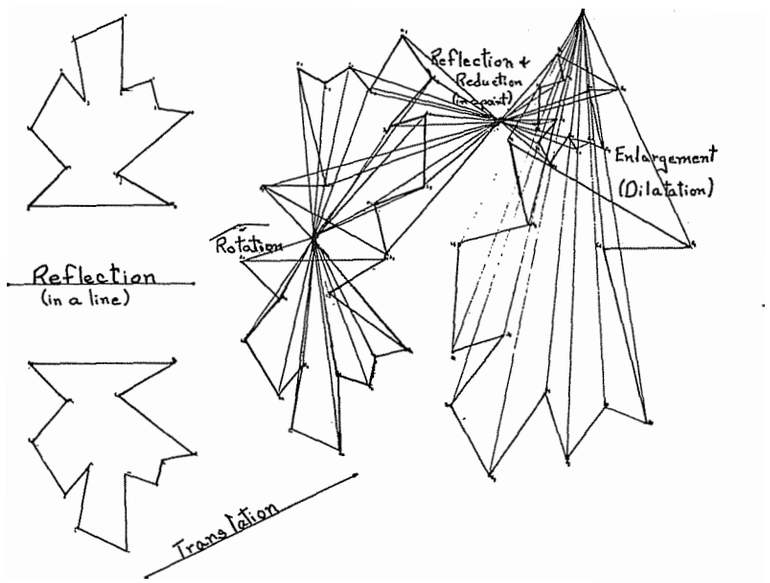
Michael Tanglao, 12 years,
Grade 7, St. Martin's School.

Editors Note: This essay was a winner in the Samuel Beatty Prize Mathematics Essay Contest.

THE MAPLE LEAF FOREVER

Jonathan Batty

While a student in Charlie Wright's Grade 8 class at Islington Public School in Etobicoke last year, young Jonathan Batty created this patriotically motivated transformational collage from a maple leaf motif. Jonathan is to be congratulated on his artistic design, precision in geometric construction, and obvious appreciation of the character of geometric transformations.





by Duff Butterill

At our meeting of September 25th, one topic caught the attention of all of us - the reform and updating of Computer Curriculum Services. These are the services related to the teaching of curriculum as opposed to administrative serving. There are a surprising number of such services.

Take note of the following:

Computer Educational Services

Administrative

Finance
- accounts receivable
- budget development
- purchasing
- fund raising
- fixed assets accounting

Facilities
- scheduling & utilization
- food services
- supplies inventory
- maintenance schedules
- vehicle schedules

Library
- acquisitions
- cataloguing
- circulation control
- serials/periodicals
- bibliographic search

Student
- admissions
- timetabling
- student records
- census
- student health

Curriculum Services for Teachers and Students

Computer Assisted Instruction
- test scoring and analysis
- diagnosis and analysis
- tutorial/dialogue
- literacy/concepts
- program training

Teacher/Student Computing
- computer science
- statistical analysis
- non-numeric analysis
- simulation
- information retrieval

The following is a small view of a large picture, of people, places, and things, associated with computer services; a picture which almost daily becomes larger and more complex.

First, a new type of master teacher has come into being. His title may be that of coordinator, supervisor, consultant, or something else. He is the resident expert on computer software and hardware. He knows their capabilities and costs. He is producing programs useful to teachers and students. Men such as Dave Didur, Hamilton Board of Education who have headed up the change from using a central batch service to using minicomputers; Dale Mills, Lambton County Board of Education, who wrote an excellent article on "The Feasibility of Microcomputers in the Classroom"; Maurice Poirier, Carleton Board of Education, who is investigating the feasibility of Computer Assisted Instruction (French) as part of the Ontario Assessment Instrument Pool of which Don Fox is chairman.

There are advisors who are willing to talk and give workshops about computer services in our schools. Profesor Wes. Graham, University of Waterloo, is an advocate of structured programming and of minicomputers in high schools. He is very convincing. Professor A.E. Lott, Faculty of Education, The University of Western Ontario, gives workshops and makes an interesting case for the use of microcomputers in the classroom. William Olivier, O.I.S.E., is doing research of the use of the microcomputer in education. Jim McLean, Ministry of Education, has held a computer awareness day for Ministry Officials. As a result, an inventory of computer facilities in Boards across the Province will be taken, their plans for the future considered, and a revised grant structure devised. Finally, if you keep your tongue in your cheek, you can always get plenty of advice from the people who man the computer sales outlets.

Places in the computer news are Hamilton, Peel, Etobicoke, Renfrew County, North York and Scarborough. These Boards have bought minicomputers and are contemplating the buying of more. They are also investigating the educational uses of microcomputers. They use structured programming in WATBOL and WATFIV-S, and some form of BASIC. They all teach computer usage to mathematics and business students of grades X, XI, and XII. Several have computer clubs designed to attract grade IX students and encourage the use of the computer by grade XII students.

The pressing problem about computers is that of choosing in what form computer services can best be provided. Computers range in levels of sophistication from the hand-held calculator (2 to 3 digit price range) to micros (4 digit price range) to minis (5 digit price range) to central processing computers (very expensive). Computer technology is advancing very rapidly. To commit oneself now or to wait and see can be equally wrong. A careful assessment of priorities and a plan of phasing in of the more desireable services seems most logical. This phasing in must maintain a nice balance between centralized control and leadership and the innovative work of turned-on teachers in individual schools.

You can see that a board which goes after all the administrative services will require a rather sophisticated system, certainly a central computer with plenty of storage capacity. Terminals having access to the computer and interconnected with each other will be in every school and in the central office. Nor have I touched upon the part to be played by the telephone system, television, and the cable system. The possibilities of such a set-up are mind boggling. It is difficult even to imagine what the future will bring by way of electronics. No wonder the co-ordinators were interested. I'm sure you all are for "computer Literacy" which will probably be a watch-word in education during the 1980's.

Editors Note: In part the computer has made feasible **The Ontario Assessment Instrument Pool.** First, a bit of background, then more about its purpose. In July 1976, the Ministry of Education formed the Research and Evaluation Branch. The Ministry needs the capacity to evaluate provincial programs. On a different level, school boards need to evaluate local policies and programs. At the centre of the educational process, students and teachers need to know how successful they have been in reaching their goals. By the spring of 1977, a Work Group on Evaluation and Reporting made several recommendations which favoured a curriculum-oriented approach to student testing. Concurrently, Ministry officials were collecting information about evaluation practices, both in Canada and abroad. The "state-of-the-art" was examined.

As a result of these two efforts, several convictions emerged.

1. "The primary purpose of evaluation is to make judgments that will help to improve the quality of education, while the primary purpose of assessment is to provide data on which the evaluative judgments can be based ...
2. Any assessment program must include not only the methods of gathering data but also the techniques for analyzing and communicating the results ...
3. The assessment instruments currently available are not good enough ...
4. The expertise that is available must be harnessed and put at the disposal of all those who are engaged in any kind of evaluative process.[1] "

Thus, the Ontario Assessment Instrument Pool was initiated. It will consist of the usual achievement-oriented test items **and** instruments which focus on the affective domain. Since not all learning outcomes can be assessed by written tests, the Pool will contain other types of assessment materials:

- methods for conducting consistent classroom observations and structured interviews;
- methods and reference materials to act as standards in assessing student growth in areas such as listening, speaking, writing, music and art;
- methods for assessing student growth in affective areas such as attitudes to others, self-image, moral development and self-discipline.[2]

A single Technical Coordinating Group will review contracts let to ensure both quality and compatibility with the concept of O.A.I.P. Some of the major contracts already let and of interest to us as mathematics educators are:

Completed.

1. Measuring Achievement at the Primary and Junior Levels. (M.W. Wahlstrom, O.I.S.E.)

Quelques causes d'erreurs en calcul

David F. Robitaille, Faculte de l'Education, Universite de la Columbie Britannique

En pratique, il y a dans chaque classe au niveau élémentaire au moins un élève qui éprouve des difficultés modérées ou sévères avec le calcul en mathématique. Une telle situation représente une source d'inquiétude pour les professeurs de mathématique mais elle n'est pas surprenante quand on considère le nombre et le degré de complexité des algorithmes arithmétiques que les élèves doivent apprendre et se rappeler.

Plusieurs éléments contribuent aux difficultés que les élèves éprouvent avec les algorithmes d'arithmétique. Parmi ces éléments on distingue ceux qui sont soit intrinsèques soit extrinsèques à l'élève. Chaque étudiant qui éprouve des difficultés en calcul a son bagage particulier de forces et de faiblesses psychologiques, physiologiques, et intellectuelles lesquelles font de ses difficultés un cas très personnel et spécifique. Pourtant, il est possible, en généralisant les difficultés de nombreux étudiants, d'identifier quelques grandes catégories de causes d'erreurs en calcul et d'employer celles-ci pour réaliser des programmes de perfectionnement. Trois de ces catégories seront discutées ici.

Défauts de connaissance

Les mathématiques ont un ordre hiérarchique qui est très évident. Le progrès d'un élève peut être ralenti ou complètement arrêté à cause d'un défaut de maîtrise des habilités et concepts préalables. L'exemple dans l'Illustration Numéro 1 démontre bien ce point. L'élève a obtenu la mauvaise réponse à cause d'une erreur en soustraction (312 - 235 = 97) et non à cause d'un erreur en division. Il est peu probable q'un tel élève puisse utiliser correctement l'algorithme de division avant d'avoir maîtrisé l'algorithme de soustracttion.

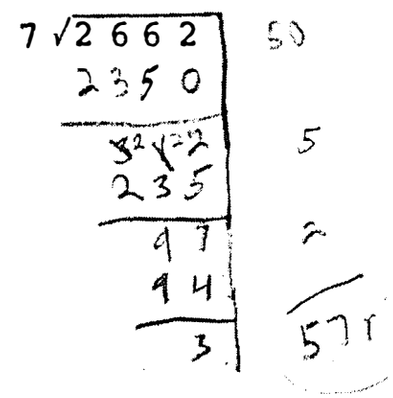


Figure 1

L'exemple le plus fréquent de défaut de connaissance est un défaut de maîtrise des tables[1] des quatres opérations avec les nombres entiers. Dans un sens les tables font partie du vocabulaire minimum de mathématique. Si les enfants veulent maîtriser les habilités de calcul, il est évident qu'ils doivent avoir un rappel immédiat de ces tables. Les élèves

[1]Les tables pour addition (multiplication) incluent les combinaisons ou chaque nombre est moins de 10. De cette façon, 9+9=18 et 8x6=48 sont membres des tables mais 12x4=48 et 11+2=13 ne le sont pas. Les tables pour soustraction (division) sont les combinaisons qui sont des transformations algébriques de leurs contreparties en addition (multiplication). De cette façon, 13-8=5 est membre de la table parce que 8+5=13 en est. D'ailleurs 56/4 = 14 n'est past membre de la table pour division parce que 4x14=56 n'en est pas un.

SOME SOURCES OF STUDENTS' ERRORS IN COMPUTATION

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In virtually every elementary school classroom there is at least one student who is experiencing moderate to severe difficulties with the computational skills of arithmetic. That such a situation exists is a continuing cause of concern to teachers of mathematics, but it is not a surprising state of affairs when one considers the number and the degree of complexity of the various computational algorithms which children are expected to learn and remember.

Many factors contribute to the difficulties that students have with computational algorithms. Some of these are intrinsic to the students, while others are more the result of external influences. All students who are experiencing difficulty in computation have their own particular sets of emotional, psychological, physiological, and intellectual strengths and weaknesses which make their difficulties in arithmetic personal and specific. However, it is possible, by generalizing from many students' individual dificulties, to identify a few broad categories of error sources and to use these to develop suggestions for carrying out remedial work. Three such categories of causes of errors will be discussed here.

Gaps in knowledge

Mathematics as a subject is highly structured and hierarchical. A student's progress in arithmetic may be slowed or completely halted because of a lack of mastery of prerequisite skills and concepts. The example shown in Figure 1 illustrates this point well. The student has obtained an incorrect quotient because of an error in subtraction (312 - 235 = 97) not division. It seems unlikely that such a student will ever be successful in using

See Figure #1

the division algorithm until he has mastered the process of subtraction.

The most frequent instance of a gap in knowledge is lack of mastery of basic facts[1] for the four basic operations with whole numbers. In a sense, basic facts are part of the "sight vocabulary" of mathematics. If children are to master computational skills, then they must have immediate recall of basic facts. Students who have to count out basic facts on their fingers or use some other laborious procedure will never be efficient at using computational algorithms.

The fact that mastery of basic facts is necessary does not mean that attaining such mastery is easy. Every teacher of mathematics has struggled, sometimes in vain, to develop such mastery in a student. At the same time, we know that this is a task which must be attempted and pursued constantly. Moreover, we know that students' degree of mastery of basic facts improves with regular practice, particularly when such practice consists of speeded drills.

1. Basic facts for addition (multiplication) are those combinations where both addends (factors) are whole numbers less than 10. Thus, 9 + 9 and 8 x 6 are basic facts, but 12 x 4 = 48 and 11 + 2 = 13 are not. Basic facts for subtraction (division) are those combinations which are algebraic transformations of their addition (multiplication) counterparts. Thus, 13 - 8 = 5 is a basic fact because 8 + 5 = 13 is a basic fact; however 56/4 = 14 is not a basic fact because 4 x 14 = 56 is not.

qui doivent les compter sur leurs doigts ou qui utilisent quelque autre processus pénible ne seront jamais capables d'appliquer les algorithmes de calcul d'une manière efficace.

Dire que les élèves doivent maîtriser les tables n'est pas dire que d'atteindre ce but est facile. Chaque professeur de mathématique a travaillé, parfois en vain, à développer cette maîtrise chez un de ses élèves. De même, nous savons que si cela est une tâche qui est absolument nécessaire, elle est rarement achevée.

Les connaissances inexactes

L'expérience clinique démontre que des élèves développent fréquemment leurs propres algorithmes de calcul. Pour une diversité de raisons, les informations transmises à l'élève par le professeur deviennent boulversées, sont mal apprises ou ne sont apprises qu'en partie. Le résultat d'un tel apprentissage est souvent le développement chez l'élève d'un algorithme idiosyncratique répétée jusqu'à ce qu'il devienne automatique et avec lequel l'élève devient fréquemment très compétent.

Quelques exemples des algorithmes idiosyncratiques sont presentés dans les Illustrations numéros 2 à 6. Avant de lire plus loin vous êtes invité à essayer d'identifier la connaissance inexacte illustreé par chaque exemple.

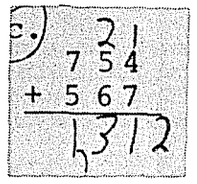


Figure 2

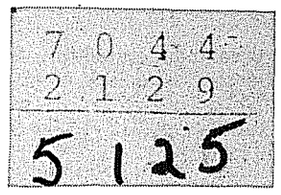


Figure 3

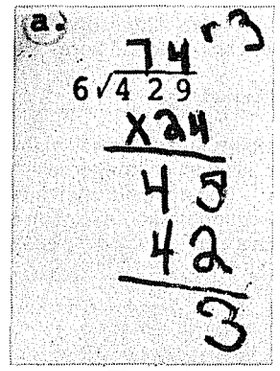


Figure 4

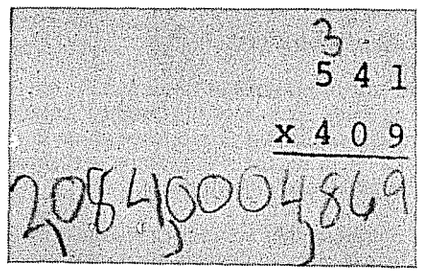


Figure 5

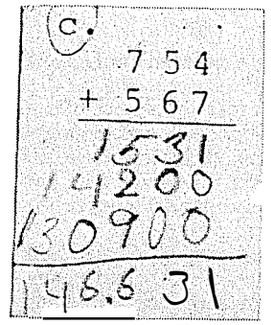


Figure 6

Illustration 2: Cet élève semble avoir appris à additioner en commençant par la gauche au lieu de par la droite. Son travail se poursuit de la manière suivante, "7 + 5 = 12. Je pose 1 et je retiens 2: ...".

Illustration 3: Cet exemple démontre ce qui est sans doute l'algorithme de soustraction inexact le plus répondu. L'élève a calculé la différence absolue entre chaque paire de chiffres: "4 - 9 = 5. 4 - 2 = 2. 0 - 1 n'est pas possible, alors 1 - 0 = 1 ...".

Incorrect Learnings

Clinical experience has shown that students frequently develop their own computational algorithms. For whatever reason, the information transmitted to the student by the teacher becomes scrambled, is mislearned, or some portion of it is not learned. The result of such learning frequently is the development of an idiosyncratic algorithm which the student practices until it becomes automatic and with which the student may attain a high degree of proficiency.

Several examples of such idiosyncratic algorithms are shown below. Before reading farther, try to identify the incorrect learning or learnings illustrated in each case.

See Figures 2-6

Figure 2: The student appears to have learned to add from left-to-right, rather than the reverse. The work proceeds as follows, "7 + 5 = 12. Put down 1 and carry 2; ...." Figure 3: This example illustrates what is perhaps the most prevalent incorrect subtraction algorithm. The student has calculated the absolute difference of the pair of numbers in each place: "9 from 4 I cannot do, so 4 from 9 is 5. 2 from 4 is 2. 1 from 0 I cannot do, so 0 from 1 is 1. ..."

That this algorithm is learned by so many students instead of the one taught by most teachers in Canada is undoubtedly due, at least in part, to the fact that this algorithm is easier to apply than the standard one. More importantly, students using this algorithm obtain correct answers whenever no regrouping (borrowing) is required, thereby obtaining positive reinforcement to continue doing subtraction exercises in the same manner.

Figure 4: As in the example in Figure 2, the basic difficulty here is one of orientation: the pupil, using the traditional division algorithm, has worked from right-to-left rather than from left-to-right as required for this algorithm. It is interesting to speculate about the possible source of such an error on the part of the student. One possibility is that the student has decided that, since the algorithms for the other three basic operations on whole numbers require right-to-left processing, it seems reasonable for the same rule to apply to division. Another possible cause of this mislearning may be a confusion of the traditional and the Greenwood algorithms for division. Certainly, many of the difficulties that children today have with long division are directly attributable to the fact that mathematics educators have been unable to agree among themselves about the best way for children to learn how to do long division.

Figure 5: This example illustrates more than one difficulty, but the most important and most obvious one has to do with how the student has recorded the three partial products. Each digit in the multiplier has been used in turn to find the partial products, but these have been written beside each other rather than in the usual way.

Figure 6: Here we have a striking example of algorithm confusion: the student has used some of the steps from the multiplication algorithm to find the sum of these two numbers. First, 7 is added to each digit in 754. The second partial sum is obtained by writing a zero and then adding 6 to each digit in 754. Finally, the sum of the three partial sums is correctly obtained using the usual addition algorithm. It seems clear from the example that the student uses this algorithm only in cases where there are two addends.

Each of the foregoing examples illustrates the result of incorrect learning by students. In each of these cases, the result of such learning has been the development of an idiosyncratic algorithm which the student has learned to use consistently and proficiently. While it is true that examples as striking and clear-cut as some of these are not to be found everyday in every classroom, these illustrations help to bring home the fact that students do at times learn something other than what is intended by the teacher, and that they learn such material very well.

Level of Abstraction

Frequently, perhaps too frequently, teachers of mathematics teach at a level that is too abstract for their students. While the students may need to be working with concrete materials in order to

Que cet algorithme soit appris par tant de nos élèves au lieu de celui qui est le standard est sans doute dû au fait qu'il est plus facile à appliquer que l'algorithme usuel. De plus, les élèves qui utilisent cet algorithme obtiennent des réponses justes quand il n'est pas nécessaire d'emprunter. Leur succès avec de tels exercices leur donnent de l'appui pour continuer à soustraire de la même manière.

Illustration 4: La difficulté fondamentale est ici, comme elle était dan l'Illustration 2, l'orientation. L'élève, en utilisant l'algorithme traditionnel de division a travaillé de droite à gauche au lieu de gauche à droite comme il est requis. Il est intéressant de spéculer sur les causes qui auraient pu produire une telle erreur chez l'étudiant. Il est possible que l'élève a décidé que, tandis que les algorithmes usuels pour les trois autres opérations sur les nombres entiers demandent qu'on procède de droite à gauche, il est raisonnable d'appliquer la même règle dans le cas de la division. Une autre possibilité peut être une confusion entre les deux algorithmes de division enseignés dans les écoles contemporaines. Certes, plusieurs parmi les difficultés que les élèves éprouvent avec l'opération de division sont directement dûes au fait que nous, les professeurs de maths, ne sommes pas du même avis en ce qui concerne le meilleur algorithme de division à enseigner dans les écoles élémentaires.

Illustration 5: Cet exemple démontre plus d'une difficulté, mais la plus importante et la plus évidente concerne le processus utilisé par l'élève pour enregistrer ses produits partiels. Les trois chiffres du multiplicateur ont été utilisés un par un afin de calculer les produits partiels, mais ceux-là ont été écrits l'un à côté de l'autre au lieu de la façon correcte.

Illustration 6: Ici nous voyons un exemple frappant de confusion d'algorithme: l'élève a utilisé quelques étapes de l'algorithme usuel de multiplication pour trouver la somme de ces deux nombres. D'abord, il a additionné 7 à chaque chiffre de 754. La deuxième somme partielle est obtenu en écrivant 0 et puis en additionnant 6 à chaque chiffre de 754. Ensuite, il a écrit deux zéros et il a additionné 5 à chaque chiffre de 754. Finalement, la somme de ces trois sommes partielles a été obtenue correctement en utilisant l'algorithme usuel. De cet exemple il semble que l'élève utiliserait cet algorithme exclusivement dans les cas où il n'y a que deux nombres à additionner.

Tous ces exemples illustrent le résultat de connaissances inexactes, chez nos étudiants. Dans chaque cas, le résultant était le développement par l'étudiant d'un algorithme idiosyncratique qu'il a appris à appliquer régulièrement et avec compétence. Tandis qu'il est sans doute le cas que des exemples aussi clairs ne se trouvent pas tous les jours dans chaque classe, il est vrai que ces exemples aident à démontrer le fait que nos élèves apprennent, en certains cas, des choses autres que ceux voulues par nous les professeurs et qu'ils apprennent ce matériel très bien. Il s'ensuit qu'il est souvent très difficile de défaire ce qu'ils ont mal appris et de leur faire apprendre le processus correct.

Niveau d'abstraction

Fréquemment, peut être trop fréquemment, les professeurs de maths enseignent à un niveau qui est trop abstrait pour leurs élèves. Au moment où l'élève a besoin de travailler avec des matériaux concrets, le professeur présente la leçon d'une manière semi-concrète ou abstraite. Une telle absence de congruence entre le niveau visé par l'enseignant et le niveau auquel apprend l'élève peut mener à un défaut de compréhension du matériel présenté.

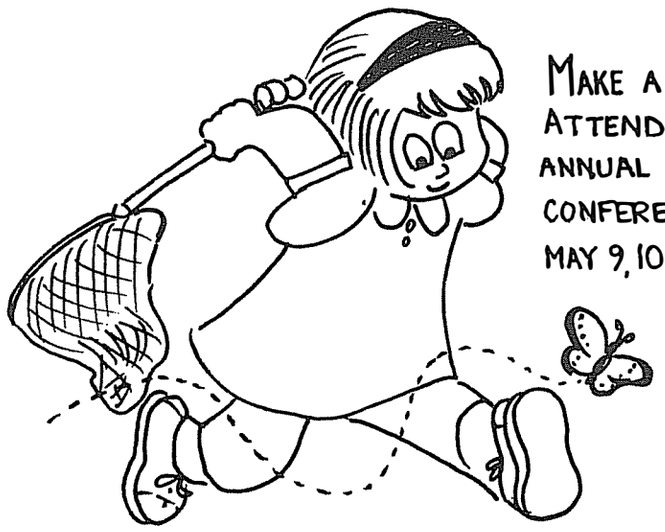
Peut être le meilleur exemple de l'importance du niveau d'abstraction de l'enseignement

clarify a concept, the teacher may be presenting the material at a semi-concrete or an abstract level. Such a lack of congruence between the teacher's teaching level and the student's learning level may result in a student's failing to understand the material being presented, and hence, failing to learn that content correctly.

A good illustration of the importance of the level of abstraction at which teaching takes place concerns place value. In the early grades, when children learn about one and two-digit numbers, teachers typically have their students work with counters and manipulative materials of various kinds in order to clarify the important underlying concepts. Thus, in a lesson dealing with the numbers 20 to 29, each child might be asked to count out two groups of ten counters and seven more to show the number 27. Such "handson" experience is of the utmost importance to the successful learning of concepts.

In the intermediate grades, on the other hand, much of the teaching of mathematics takes place at a more abstract level. Instead of having the children use materials, the teacher may give a demonstration or the children may be referred to an illustration in their textbook. As far as place value is concerned, part of the reason for this shift to a more abstract teaching mode may be due to the size of the numbers involved: representing four - and five-digit numbers concretely is not as easy as representing one- and two-digit numbers concretely.

The truth of the matter is that, regardless of any inconvenience to the teacher, children need to have concrete experiences on which to base their understandings of concepts. It would also appear to be the case that less capable students require more of such experiences than others, and that students receiving remedial assistance require even more. What all children require is a well-designed series of experiences, beginning with the concrete and progressing gradually to the abstract. Given such experiences, children will have a firm foundation upon which to build their understanding of abstract concepts such as place value.



April: To be a child
is to know the fun of living.
To have a child
is to know the beauty of life.
- Alan Doan.

concerne les valeurs positionnelles des nombres. Dans les premières années de l'école, quand les élèves apprennent les nombres à un ou à deux chiffres, les professeurs utilisent d'habitude des matériaux concrets de toutes sortes afin de clarifier les principes fondamonteux. Donc, durant une leçon qui a pour but d'enseigner les nombres de 20 à 29, chaque élève pourra avoir à compter deux groupes de dix objets et puis sept autres objets afin de démontrer le nombre 27. On sait que ces expériences avec les objets concrets sont vraiment importantes pour assurer que les élèves apprennent les concepts.

Dans les classes intermédaires de l'école primaire la situation n'est pas la même. A ce niveau, une grande partie de l'enseignement des mathématiques se déroule d'une façon beaucoup plus abstraite. Au lieu de s'assurer que les élèves se servent de matériaux concrets, le professeur leur donnera peut être une démonstration, ou les élèves examineront une illustration dans leur manuel. Dans le cas des valeurs positionelles des nombres, une raison possible pour ce changement de mode d'enseignement peut être la grandeur des nombres concernés: représenter des nombres à quatre ou à cinq chiffres avec des objets concrets n'est pas aussi facile que de représenter des nombres à un ou à deux chiffres.

En réalité, malgré tout inconvénient causé au professeur, on sait que les élèves ont besoin d'expérience avec les matériaux concrets sur lesquelles ils peuvent établir leur compréhension. De plus, il apparaîtra que les élèves moins capables ont plus besoin de telles expériences que les autres, et que les élèves qui reçoivent de l'aide en ont encore plus besoin. Tous les élèves ont besoin d'une série d'expériences qui procède graduellement à l'abstrait. Avec ces expériences, les élèves auront une base ferme sur laquelle ils pourront construire leur édifice de compréhension des mathématiques.

ONE OF THE BASICS, A HUMANE APPROACH TO STUDENT EVALUATION

Doug Henderson, Master Teacher of Mathematics, Sudbury Board of Education



Editors Note: This is a reply to the article in the September issue by N. MacEwan titled **A Project to Improve Teaching Practices.**

My reaction to the ideas put forward in the article on student evaluation in the September issue of the Gazette probably has a lot to do with a point of view developed in my first years of teaching mathematics. I belong to a generation of teachers who, at the present time can likely be divided into three distinct groups, those who do use Grecian Formula, those who should use Grecian Formula, and those who do not even use a comb.

In short, I began teaching in that golden age, a brief period of history which might be described as after Sputnik and before Hall Dennis. In those days the program was sensible and consistent, the student uniformly diligent and industrious and the skies were not cloudy all day. One common characteristic of teachers at that time was that they genuinely believed (erroneously as we were later informed) that everything that they did, teaching, testing, discipline, everything, was designed with only one goal, to help students to learn. Consideration of new proposals or changes in practice would be based solely on their probable effect on students' learning.

Now in the article in question two assumptions are stated explicitly:

(i) that assigned marks for a reasonably large group of students taking the same course (such as grade IX advanced) should closely fit a normal distribution.

(ii) that marks on class tests should be congruent to marks on term examinations.

I wish now to argue against the truth of these two assumptions and then to put forward some suggestions in their place.

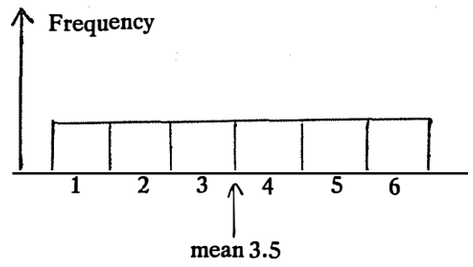
Student Marks and the Normal Distribution

(i) **The Origin of the Normal Distribution**

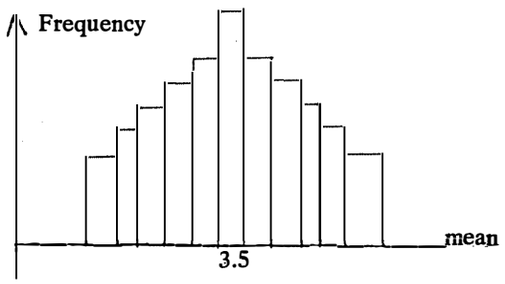
The normal distribution, like most things mathematical, is an abstract formula and may fit some concrete situations well and others not at all. This abstract formula does not fit any concrete situation **exactly.**

Its use in practice arises mostly from the central limit theorem, which states that, regardless of the original distribution from which we take samples, the distribution of means of samples of size n **approaches** the normal distribution as n increases.

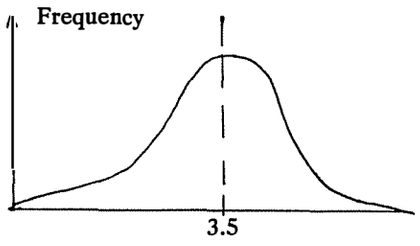
For example if you roll one fair die the distribution of outcomes is uniform.



If you roll two fair dice and average the total showing, the distribution will immediately begin to become taller in the middle, symmetric and centred on the mean of the original distribution.



If 100 dice were rolled and the mean scores taken, the resulting distribution would be scarcely discernible from the normal curve.



Some Questions and Answers

Q. When should we use the normal distribution to approximate a concrete situation?
A. (i) when we are considering a distribution of sample means with sample size sufficiently large

OR

(ii) when we are able to make objective measurements of some variable quantity and to show inductively that the distribution is close to the normal model.

Q. Why is it that such things as shoe sizes, clothing sizes and heights of adult males closely approximately the normal curve?

A. Perhaps it is because each of us is, in some sense, the mean of a sample of size 2^n taken from the human gene pool where n is the number of generations since Adam. The other answer, more scientific perhaps, is that this is what we observe to be true.

Q. If it can be shown through observation that shoe sizes and heights vary according to the normal distribution, does it not seem reasonable to assume that characteristics less easily measured such as intelligence or musical or mathematical ability also follow the normal distribution?

A. To me that seems not necessarily so. Even if we grant the assumption that math ability is normally distributed, it does not follow that actual achievement will act in the same way. Achievement is the result of aptitude, ambition, time spent and the quality of teaching. To assume that achievement will be distributed in the same way as aptitude is to believe that all students work equally hard and this is obviously false.

Q. If the assumption that intelligence is normally distributed is open to question, then why do I.Q.'s so closely fit the model?

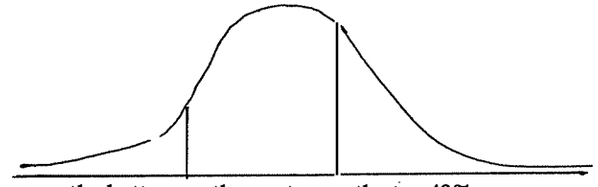
A. Now an I.Q. test is designed to measure differences between persons. Therefore, they do not include any test items which will be correctly answered by almost everyone or by almost nobody. As a matter of fact the ideal test item is one which will be correctly answered 50% of the time. Thus, the scores on an I.Q. test will be close to a binomial distribution ($p \doteqdot 0.5$), which for a large number of test items may be approximated with the normal. In addition to this, many commercially available tests put the scores through a series of transformations to **make** the scores fit the normal.

To Summarize

(i) the assumption that mathematical ability is normally distributed throughout the population may be seriously questioned.
(ii) Aptitude and actual achievement are not necessarily distributed in the same way.
(iii) Scores on commercially prepared tests fit the normal distribution partly because of the nature of the tests and partly because of transformations performed on the raw results.

What Has This To Do With The Marks In Grade IX Basic?

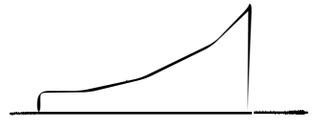
Let us assume for the moment that mathematical aptitude **is** distributed normally throughout the population and that the marks assigned to any group of students should reflect this. Now as the population of students enters grade IX, they are streamed into three groups:



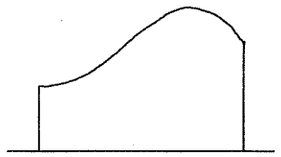
Now the streaming process is far from perfect but it seems to me that it is also far from imperfect. Let us assume for the moment that students choose the "right" level at which to study mathematics.

This would imply that

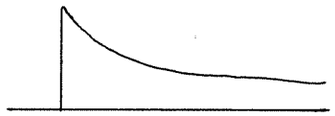
(i) the distribution of marks in the basic course should be like this. The fact that the marks do not usually fit this model is more an indication of poor work habits and background than of poor ability. Perhaps when we submit a set of marks for a basic level class which approximates the normal distribution we should be criticized for failing to motivate our students rather than congratulated for the nice fit.



(ii) at the general level the students would be scarcely distinguishable in ability and hence in achievement. Is this not now the case in most mark distributions for the general level? The few students who score significantly higher than the group do so because of better work habits and attitudes, not because of superior ability. Those who fail are those we have been unable to coax or coerce into expending some effort.



(iii) marks for advanced classes would be distributed this way. Once again discrepancies between this model and what we actually experience can be more easily explained by differences in attitude rather than differences in ability. Also a 0 - 100 scale may not be adequate to indicate progress being made by students who are very able and also willing to work hard.



To Summarize

(i) Either the distribution of math ability in the whole population fits the normal model or it does not.
  (a) If it does not, then why should math achievement fit the model?
  (b) If it does, then math achievement in special groups such as the basic, general and advanced levels should definitely not have their marks distributed in the same way as they would be in the whole population.

(ii) Which ever of (a) or (b) is true, it is erroneous and even dangerous to demand that marks for a particular course should be made to fit a preconceived pattern.

Class Tests and Examinations

A period test, covering a unit of work or less has two main purposes:
  (a) to inform the student of how well he is learning the current material and to indicate which ideas require further study and which skills further practice.
  (b) to inform the teacher which students need extra help, which ideas require further teaching and whether the members of the class are ready for the next topic.

An examination, covering several units, performs these two functions, as well as two more,
  (c) to allow both the teacher and student to see the extent to which the interrelationships between the ideas and skills learned in individual units have been recognized and mastered.
  (d) to provide the student with an opportunity to demonstrate that she has profitted from the information obtained through period tests and mastered certain skills which were inadequate part way through the term.

To me a period test in mathematics is a measure of learning before you even expect it to be complete. Its prime purpose is to give information to the student and teacher on how he is progressing towards his goal, not whether he has reached it.

To demand that term marks and exam marks should be congruent is equivalent to demanding that a runner's placing part way through a race must be the same as his placing at the finish. Would a history teacher expect to place the same mark on the first draft of an essay as on the finished product? Should a calculus student who failed a test on limits in September have that failure to learn the first time held against her for the rest of the year, even though in November she could now pass that same test with ease?

If students are learning, the results on examinations should be **higher** than those on term tests. The fact that almost every school experiences the opposite effect reveals that there is something gravely wrong with schools, students and teaching.

Some General Principles

It is always easy to criticize the work and ideas of someone else and much more difficult to try to

articulate some proposals of your own. Probably nobody has the ultimate solution to the problem of student evaluation and I feel a bit like the politician who said "I don't have a solution to unemployment myself but, if I am elected, and someone comes up with a solution, I promise to support it." But here goes.

I The Principle Which Overrides All Others

Testing takes place to promote student learning.

It accomplishes this function by providing feedback to students on their accomplishments and deficiencies and to assist teachers in deciding what to do next. Since the teacher is the one person most sensitive to the needs of his students, he must be free to decide whether a particular group would profit best from many short tests or a few long tests, an easy, confidence building test or a challenging difficult test. In making these decisions the teacher may be guided, but must not be constrained, by rules made at the school or Board level.

II The Guidance Function

Test and examination results serve to help the student make **decisions regarding the careers in which he has the potential for success**, but **this is a secondary rather than a primary function.**

There are two critical points in our system at which career decisions become highly important, at the end of elementary school and in grades XII and XIII. It would be immoral to encourage, through the marks assigned, a student to attempt to enter a profession for which that student does not possess the required aptitude in mathematics. It would be equally immoral, in the name of maintaining standards, to discourage a student from pursuing a career in which he has a reasonable chance of success.

The guidance function must be administered differently with different groups of students; for example advanced level students may require different kinds of information about their potential than do general level students.

Providing guidance for students seems to me to be the only valid argument for norm-referencing, Board-wide or provincial tests.

III The Ideal

The ideal mark distribution would indicate that every student has learned what he set out to learn, and learned it well.

Accepting and even institutionalizing failure rates of 15% to 25% should upset the conscience of every teacher. When test results indicate that a student has not learned what he set out to learn, the remedy would lie in a combination of adjustments to the following variables:

(i) Course content

Perhaps we are setting unattainable goals. Goals for students must be set high enough to require their best efforts but not so high that they cannot be met. Close communication at the school, the system and provincial levels are required so that teachers may determine just what is the best work which may be obtained from pupils in various ability groups.

(ii) Teaching Approaches

Perhaps we could change the mark distribution by finding a better way to teach per cent, or operations with integers, or deductive geometry.

(iii) Student Placement

If students are obtaining failure marks because they are "in the wrong course", we should make greater efforts to avoid this situation, not simply punish them for the failures of the system.

(iv) The Tests

We must consider the possibility that the testing procedures do not allow the students to display what they really have learned. Every teacher can cite examples of graduates and even dropouts from the general and basic programs who have become eminently successful after leaving school when their marks would have forecast mediocrity or even failure. It seems to me that they were learning many things which our evaluation procedures failed to reveal.

(v) The Student

It may be that it is the student's own attitude which must be altered. We must not avoid our responsibility to encourage students to learn by burying it amongst approved failure rates.

IV **The Validity of a Test**

My own working definition has always been that **"a test is valid if, for the most part, it reinforces my own preconceived opinions".**

While working with a class, asking and answering questions, checking homework, we develop a feeling for what the students collectively and individually can do.

The past ten years have seen a growing undermining and erosion of teachers' faith in their own judgement and competence. It is time to reassert ourselves, tempering our confidence with principle number III.

V **Forgiveness**

In mathematics it is possible to be born again.

The grading system must be such that being poorly motivated and even obnoxious in one term is not hung about the student's neck like an albatross for the rest of his career. At the other extreme we should not allow a student to coast after receiving high marks for part of the year.

At the present time rigid formulas combining term and examination marks to reach a report card mark are telling some students that there is no use working and others that there is no need.

VI **When in doubt, pass him**

Many persons who now hold responsible and productive positions in our communities, perhaps even in our schools, have been "pushed through" by the humane application of promotion standards.

In Conclusion, At Last

Teachers are not gods, nor angels, nor even saints. We do not even know all that there is to know about our subject area and how to teach it, let alone how to unerringly read the motives, discern the potential and forecast the future of our students. Yet we must reaffirm the conviction that of all the groups in society, we are the ones who do these things best.

We have every reason, based on our training, our intelligence and experience, to believe that the decisions we make in the interests of our students are the right decisions. Let us resist attempts by those who, in these matters, are not our equals to change what we know to be right for political, quasi-theoretical, or any other reasons.

That same training, intelligence and experience should give us the humility to continue to strive for improvement in ourselves. We need to know even more about mathematics and its teaching, even more about the psychology of learning and of young people, even more of the successes and problems of our colleagues in our own community and throughout Ontario.

Never should we attempt to hide our shortcomings behind medians, standard deviations, correlation coefficients and the like; always we must strive to make our own contacts with students occasions for their genuine growth as human persons.

INTERVIEW

Taped Talks with Bob Smith

. . . Dietmar Küchemann



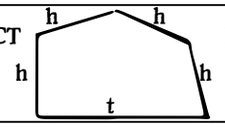
Dietmar Küchemann has been a part of the mathematics team of the "Concepts in Secondary Mathematics and Science" project (C.S.M.S.) located at Chelsea College, University of London, since its beginnings in 1974. It was my pleasure to meet him first in January 1976 in London, and to renew acquaintances this past spring. The project finished in September, 1979. A book is being written containing a chapter about each of the ten studies that were undertaken. Dietmar did the studies on Algebra, and Reflections and Rotations.

R.S. What did you discover about children's (median age 15.3) understanding of numerical variables?

D.K. The majority of children had extreme difficulties using letters in an algebraic way. They gave a large number of different meanings to unknowns as represented by letters. The blanket term 'variable', is commonly used by teachers to denote any and all letters in generalized arithmetic or algebra, but we found six different ways in which children used or interpreted the letters.

R.S. How would you want the children to think about a variable?

D.K. A variable, as its name implies, suggests that you are concerned with something whose values are changing, and if you follow that through, to relating the changes in one variable to the changes in another, the idea becomes extremely complex. In fact our research has shown that very few children, maybe five percent of 15 year olds are able to grasp the concept. Collis identified several ways in which children interpret letters in generalized arithmetic. We have developed his ideas further to give the following six categories describing the different ways letters can be used.

Category	Example	Explanation
Letter Evaluated	$a + 5 = 8$ $a = ?$	Here the letter can be evaluated immediately.
Letter IGNORED	$a + b = 43$ $a + b + 2 = ?$	2nd equation differs by +2 a + b ignored
Letter as OBJECT	 P=	h and t can be regarded as names or labels rather than numbers.
Letter as SPECIFIC UNKNOWN	 There are n sides all of length 2. p=	n stands for an unknown number which cannot be evaluated.
Letter as GENERALIZED NUMBER	$c + d + 10$ $c < d$ $c + ?$	c represents a set of numbers rather than just one value
Letter as VARIABLE	Which is larger, $2n$ or $n + 2$?	A second order relationship needs to be found between $2n$ and $n + 2$ as n varies.

R.S. Let us look at the hardest category first - letter as **VARIABLE**. What were some of the answers you obtained to the question "which is larger 2n or n + 2?"

D.K. We asked children for an explanation for their choice. Obviously we were hoping that children would see that the relative sizes of the expressions were dependent on whether n was greater than 2, in which case 2n is larger. Very few children looked at the problem in this way. Usually they said that 2n was larger "because it's multiply". Most did not even suspect that there was more to it. Very few children gave a numerical example to demonstrate their answer. Some children gave the rather nice answer that 2n was larger because n + 2 was smaller.

R.S. As we do not have time to look at each category, what do you mean by letter as **SPECIFIC UNKNOWN?**

D.K. We used this label where children were willing to use a letter as representing an unknown, but where their view was limited to the idea that the letter had a unique value, even though they may not have known what the value was. The above example of an incomplete figure of n sides all of length 2, the perimeter to be found, high-lighted it. A lot of children were reluctant to give an answer involving n when they didn't know its value. Many counted the number of sides drawn, knew their lengths were 2 and gave the answer 2 x 17, or 34, or they added a few more sides to close the figure in their mind.

R.S. They needed a definite answer.

D.K. Yes, and this relates to another idea put forth by Collis. He speaks of the child's willingness to accept an answer which in a sense in unresolved or incomplete; a willingness to give an answer like 2n when we don't know what n is. Collis calls the willingness to cope with an expression of this type "Acceptance of Lack of Closure" and regards it as an important indicator of formal operational thought. We found that with the children we tested who were in the 13, 14, 15 year old range, less than half of each age-group seemed able to use the letter consistently in this way. Another example of this was the following question: "4 added to n can be written n + 4. Add 4 onto each of these"

8	n + 5	3n"
.....	.....	.....

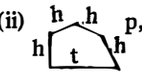
For the 3rd part of the question - 3n, 31% of the children gave the answer 7n and a further 16% gave the answer 7, ignoring n entirely. The strategy used here might be called "association" (3 + 4 = 7), which is adequate for the 2nd part where 68% correctly gave the answer n + 9.

R.S. A letter can be regarded not as an unknown number but as an object and it is often introduced to children in this way when they first meet 3a + 4b and are told "a stands for apples, b for bananas." On your test paper, when did children treat the letter as **OBJECT?**

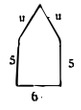
D.K. For example, in question 9, where children were asked to state the perimeter of various figures on which letters were used to denote the lengths of the sides.

(i) p,= 

In parts (i) and (ii) the letters can successfully be treated and simply listed or collected together. — 94% got it correct

(ii) 

p,= Here many children gave answers like p. = 4h, t or p. = hhht — 68% got it correct

(iii) 

Children gave answers of the type; p. = 2u + 2.5 + 1.6 — 67% got it correct

In the above examples the letter is seen as something quite tangible, not as something unknown and mysterious. This however, brings conflict later when the last thing we want is for the letter to be thought of as a number. For example, consider the following question: "Cabbages cost 8 pence

each and turnips cost 6 pence each. If c stands for the number of cabbages bought, and t stands for the number of turnips bought, what does 8c + 6t stand for?'' Most children thought 8c + 6t meant 8 cabbages and 6 turnips. Only 1% gave a clearly correct answer, and most children then wrote 14 for the total number of vegetables instead of c + t (4% correct).

R.S. ***From my own experience with the following question asked of student teachers: ''Write as a part of a dollar - .25c _____'' Usual answer $.25. I wonder if both age groups make the error because of programming, i.e., we read was we expect to read.***

R.S. Well, you have found some of the problems. What can teachers do to help children with Algebra?

D.K. Unfortunately, I'don't have the answer. This is something we want to research much more fully. One approach I should like to try is to first use their primitive strategy, ''a stands for apples'' because children can at least cope with it and then at a judicious moment deliberately set up conflict by introducing situations where their strategy no longer works, to see whether this might encourage children to restructure their thinking. One of the strongest impressions we have from the present research, is that children have a very strong tendency to use their own methods, which very often have little to do with what the children have been taught, and which are very persistent. It seems to me that we ought therefore to acknowledge the existence of these methods and possibly even try to use them rather than ignore the way children think.

R.S. Geoffrey Matthews obtained the research grants from C.S.M.S. Is he still with you at Chelsea College to help find new grant money?

D.K. Unfortunately, he has retired. But I think we may get some money, though it may not keep all of us here at C.S.M.S. However, at least some of the work we have done will be carried on.

R.S. Good luck, Bon chance.



May: There are only two lasting bequests we can hope to give our children. One of these is roots; the other, wings.
- Hodding Carter.

INTERMEDIATE MATHEMATICS: WHAT DO WE KNOW AND WHERE DO WE GO?

A Report on Domain Referenced Tests of the Intermediate Evaluation Project



Background of the Intermediate Tests

In the fall of 1975, there was a change of direction in the educational winds in Ontario. When I, newly appointed as Coordinator of Field Development in OISE, talked to school board administrators about their needs, again and again I was told that there was a real shortage of tests that would establish just what students, individually and in groups, were achieving in relation to Ontario's educational objectives.

The Intermediate Evaluation Project, financed by OISE and participating boards, was born with the following objectives:

1) To provide information to students, parents and teachers concerning the achievement of the individual;
2) to provide general information on the achievement of students overall, within a particular system, and within a particular school; and
3) to help school systems change their programs in the light of specific information about strengths and weaknesses in the achievement of their students.

As a researcher, I was also interested in obtaining some rare information on achievement in Ontario, particularly with respect to factors that appear to be related to achievement. Nevertheless, the project was essentially a field service rather than a research activity. The background information was a bonus.

Two tests developed as part of Phase I (for preliminary administration in May 1977) were in mathematics and problem solving. We translated major objectives from the official guidelines and circulated them to teachers whose pupils would be involved in the testing for validation and comment. The ratings obtained from teachers helped modify our objectives and helped determine the weight that should be allocated to the testing of the various objectives. We then collapsed the objectives in mathematics into five domains which became sub-tests in: computation; measurement; problems; understanding; and algebra/geometry. Comparable sub-tests in problem solving (as revised for 1978) were: logic, relations and experimental.

All tests were scored on a 0 to 240 scale. Test items were variously weighted so that those basic items judged by the researcher to fall in the "must know" category constituted 110 points; those in the "should know" composed the next 40 points, and those in the "nice to know" category fell in the final 90 points. With an allowance for clerical and measurement error, we judged that the vast majority of Ontario pupils **could** be brought to the 110 level with appropriate teaching (assuming that the teachers shared the stated objectives of the tests, and did not give higher priority to other non-stated objectives). Students who obtained a score of 150 could, in our view, be seen as having a degree of mastery such that they would be unlikely to meet significant problems during the secondary years. The high honors level (over 200) constitutes a measure of excellence.

Revised mathematics and problem solving tests, based essentially on our 1977 and 1978 Ontario tests, were field tested in June 1978 with a national sample. A sample population was constructed for the mathematics and problem solving test groups according to the following criteria:

i) A mean I.Q. of 100 with an approximately normal distribution;
ii) equal distribution by sex;
iii) appropriate distribution by geographical region (of Anglophone Canada);
iv) appropriate distribution between urban and rural Canada.

The first criterion was met almost exactly, the last three were approximated. National norms have been prepared from those samples and the tests are available from OISE's Publications Sales.

Results of the Testing

In 1977, the math test was taken by 4268 Anglophone students from 14 boards, problem solving by 2828 from 11. In 1978, the figures were 4756 from 27, and 1542 from 15 respectively. The boards

are widely represented geographically but there was a bias towards rural western and northern Ontario and away from urban and suburban Ontario. Most significant is the lack of involvement of the boards from the Toronto region, perhaps because the boards are larger and the accountability question has been less pressing until very recently. I believe the overall results are probably reasonably representative of the province as a whole. We collected I.Q. scores for most students in the 1977 sample and the mean I.Q. was just over 100. We would expect a province-wide sample to perform at approximately the same level as our population, or perhaps slightly below that level. Most students taking the test were from eighth grade. For most purposes, we have not separated the eighth grade responses as the small number of seventh and ninth grade students have little or no effect on the overall data.

TABLE 1

Mathematics and Problem Solving Results 1977 and 1978

	Math Gr. 8 1977	Math Gr. 8 1978	Problem Solving Gr. 8 1977	Problem Solving Gr. 8 1978
Mean	125	125	111	99
Standard deviation	39	39	34	34
Reliability	.80	.80	.70	.64
% Competent	66	62	50	38
% High honors	3	3	.3	.1

Table 1 summarizes the results (grade 8 only). No comparisons can be made meaningfully between the two years. The populations are different and the tests themselves are not exactly equivalent. As the statistics suggest, the 1978 problem solving test was more difficult than the 1977 test.

The reader should note the reliability coefficients, which indicate a standard error higher than we would wish (17 in mathematics and 19 in problem solving). We deliberately sacrificed some reliability in order to attain the greater validity obtained by including a wide range of items testing a wide range of objectives, some of which are not closely related. If we had been constructing norm referenced tests (as distinct from our domain referenced tests) we would have eliminated all aberrant items irrespective of their idiosyncratic quality and relevance to the guidelines, retaining only those items that are closely interrelated. The consequence is that our reliability coefficients are much lower in tests like problem solving (.68) and Canadian studies (.68), where we are testing quite disparate objectives, than they are in tests like health (.87) and science (.87) where the objectives, and the items, are more similar. Thus a norm referenced test in mathematics will give the teacher a more precise score for a given student than will ours, but the score will reflect a much more limited range of skills and abilities.

The results in mathematics (1978) were somewhat disappointing. One might conclude that Ontario has sacrificed accuracy in mathematical computation and gained in return only a cursory understanding of mathematical principles. While 85% were able to add three 4 digit numbers only 45% could divide a five digit number by a three digit number to 2 decimal places. On the other hand, as many as 61% could apply the idea of the set to a simple case and 71% showed understanding of simple number statements.

It is perhaps in the area of everyday, "low" mathematics that the results were most disappointing. Only 56% could calculate correctly simple percentage questions such as 5% of 500. One-third were unable to calculate change correctly (e.g. to pay an exact fare on a bus for a family). A simple problem requiring the calculation of discount and sales tax was answered correctly by only 26%. Only 40% could answer a simple time, distance, speed question.

In problem solving, the results are much harded to describe. Logical problems formed a large part of the test. Logic is arguably a part of most subjects in school--certainly in mathematics, geography, language and science--but is it ever systematically taught? I have my doubts. Our experimental sub-test emphasized the scientific method--particularly the need to control variables and change one independent variable while holding the others constant, and the need for a random experimental design. Once again, the results of our science test (which support the results of our problem solving test) suggest that the scientific method is one of the least learned aspects of science in the elementary years. Our third sub-test looked at relationships--the kind of problem solving particularly important in geography, when studying the interplay of factors that influence

climate, for example. Our generalized finding that a disappointingly high proportion of students is unable to tackle problem solving in any systematic manner can be ascribed to the fact that problem solving is not much taught. Experience in some schools suggests it can be taught. On the other hand, many teachers may prefer to believe that the results in problem solving simply reflect the fact that the questions were unrealistic for young people of thirteen or fourteen.

I shall give one example of the dilemma. One of our "must know" (i.e. competence) questions demanded an understanding of intersecting sets. No mathematical calculations were required but written statements had to be interpreted in terms of a diagrammatic representation. Only 23% answered correctly. Is this too difficult for 77% of the population, or is it just not systematically taught?

What Does the Research Tell us About the Factors Related to Achievement?

There is not space here to describe all the research that has been carried out on the background characteristics related to mathematics achievement, and, as one would expect, many of the findings are insignificant and inconsequential. The reader may well say, "Of course" to the few that are consequential. Research demands that we painstakingly separate what we think and expect from what we know. I shall try to summarize briefly the major findings.

Essentially, the background research tries to answer two questions:
1) How are the various tests and sub-tests interrelated?
2) Which home and school factors contribute to (or are related with achievement in mathematics?

The first question is important for several reasons, most notably because it provides evidence of concurrent validity for the tests. In other words, it helps us understand just what the tests are measuring. If the tests are valid, there are certain kinds of relationships we would expect, and some we would not expect. In problem solving for example, it is important to see if the test is anything more than a test of reading. The second question is important because it provides a preliminary basis for determining decisions that will lead to improved competence in various skills with a minimum cost to the public.

Tests and Sub-Tests

TABLE 2

Problem Solving Correlation Matrix (based on a small sample of 1978 data only)

	Prob.Sol.	Log.	Exp.	Rel.	Math.	Reading	Lang.	Sci.	I.Q.
Problem Solving	-	.81	.71	.84	.58	.53	.58	.51	.61
Logic			.38	.47	.47	.44	.48	.42	.51
Experimental				.43	.42	.41	.45	.37	.28
Relation-ships					.47	.43	.46	.41	.58

An examination of Table 2 does not suggest an unusually strong relationship between problem solving and either reading or I.Q. The other intermediate tests have relationships with I.Q. at least as high (mathematics .67, science .59, language .72, and reading .78). The sub-tests, as one would expect, have less strong relationships with the other variables as they are less reliable. However, it is noteworthy that the sub-tests are correlated as strongly with the other tests as they are with the other problem solving sub-tests, suggesting that there is a fair amount of independence in the three separate sub-test skills. It is clear that problem solving, as defined in our tests, is somewhat independent of mathematics, the correlation coefficient being no greater than that with language.



June: If there is anything that we wish to change in the child, we should first examine it and see whether it is not something that could better be changed in ourselves.
- Carl Jung.



TABLE 3

Mathematics and Problem Solving (Grade 8) Correlation Matrix)

	Math.	1 Comp.	2 Understanding	3 Prob.	4 Alg/Geo.	5 Meas.	Prob.Sol.	1 Log.	2 D.M.	3 Corr.	4 Exp.	I.Q.
Mathematics		.87	.75	.69	.63	.51	.63	.56	.38	.48	.30	.68
Computation			.50	.46	.45	.33	.53	.46	.31	.41	.26	.58
Understanding				.45	.41	.24	.51	.46	.29	.37	.25	.53
Problems					.35	.28	.50	.45	.31	.38	.21	.52
Algebra/Geometry						.22	.38	.37	.20	.25	.22	.40
Measurement							.31	.25	.21	.28	.10	.30
Problem Solving								.83	.59	.78	.56	.54
Logic									.35	.45	.31	.50
Decision Making										.31	.25	.34
Correlational											.28	.37
Experimental												.21
I.Q.												

Table 3 provides some concurrent validity for the tests. Most of the relationships make sense. Too much emphasis should not be given to the variation in the relationships between the sub-tests and the total tests. Some sub-tests carried more weight than others and were also more reliable. Hence, the difference between the relationship between mathematics and computation (.87) and between mathematics and measurement (.51) could be mainly or entirely a statistical artifact, i.e. computation is considered more important than measurement by the test makers. However, that would not explain so satisfactorily the variations among sub-tests. The logic sub-test, for example, correlates remarkably well with mathematics and most of the mathematical sub-tests (with the understandable exception of measurement). I am tempted to conclude that a much more formal attempt to teach logic would pay dividends in mathematics (as well as, in all probability, some other subjects).

The sceptic may still have lingering doubts as to whether the problem solving ability is really anything different from intelligence. Interesting support for my belief that problem solving ability is something more than intelligence (which is not to deny the strong relationship between intelligence and problem solving ability, in the same way as there is a strong relationship between intelligence and all other cognitive abilities) comes from our moral reasoning test, developed for administration in 1978. The correlation coefficient for moral reasoning and I.Q. prove to be -.07, whereas its relationship with problem solving was .18.

Background Factors

In 1977, we obtained measures of the following "background" factors: I.Q., social status, attendance, coverage, and attitudes. I have placed the word "background" in quotation marks, because the term may be inadvertently misleading. Probably occupational status and I.Q. can be considered antecedent, background factors. Thus, in the case of the other variables, we have to consider very carefully as to whether there is any implication of cause and effect from a relationship. For example, do favorable attitudes produce good performance in mathematics, or vice versa, or does the home that produces the one ipso facto produce the other? There are three logical explanations for the same relationship, and the truth may well lie in a combination of the three. More recently, under a Ministry of Education contract, I examined instructional time as an additional background factor. The reader should be aware that the following very brief summary does scant justice to the complexity of the topic.

I.Q.

As expected, this is the single most powerful and stable predictor for all academic achievement, and one over which I believe schools have almost no influence. About 40% of the variance in mathematics achievement is explained by measured intelligence, a factor that is itself probably determined about 70% by inheritance, approximately 25% by home cultural conditions, and perhaps about 5% by other factors (such as the pre-natal and post-natal physical environment). Some will argue that I.Q. is a product of schooling, and therefore not a predictor. Suffice it to say

that I have assumed otherwise, that there is considerable evidence on the inheritability of I.Q. (an absolute minimum of 50%), and that we found huge inter-school variations in achievement given the same I.Q. In other words, if schools really produce I.Q. (for which I do not think there is any real evidence within a reasonably homogeneous culture such as ours) then why do schools with the same I.Q. distribution produce such different results in mathematical achievement?

Occupational Status

We collected occupational status by asking schools to code the occupation of each participant's father (or mother where there was no father). Rather surprisingly, the correlation between occupation and mathematics achievement was only .15 (.25 in the case of problem solving). From studies in other countries, we would have expected a coefficient between .4 and .5. Perhaps there was measurement error. Another possibility is that social status is much less important in rural Ontario (and remember that our population was biassed in that direction) than in the urban and suburban regions. A last (and pleasant) possibility is that there is less bias by social background in Ontario schools than in schools in the USA and western Europe. Even more remarkable is our finding that occupational status virtually disappears as a factor if I.Q. is controlled. The Beta factor for occupational status (independent of I.Q.) is .01, whereas the Beta factor for I.Q. (independent of occupational status) if .67. Once more, this tends to suggest that intelligence, and inherited intelligence at that, is the most important factor in contributing to mathematics achievement.

Attendance

We collected a coded measure of attendance from September until January of the same school year in which the tests were to be written the following May. The relationship between attendance and mathematics is .10 (between problem solving and attendance .17). Independently of I.Q., attendance disappears almost completely. This confirms the common sense observation of teachers that "good" students will achieve well even if they miss school but the "poor" students will achieve badly irrespective of whether they attend or not.

Coverage

We asked teachers (in the November before the May testing) to indicate the importance they assigned to various objectives. We then tried to determine whether this coverage factor helped explain achievement. At both the school and individual levels overall emphasis on the total spectrum of objectives proved to be of no value, probably because all teachers emphasized or claimed to emphasize our objectives (on which there was strong consensus) approximately equally. In the case of problem solving (which I believe is not taught systematically), coverage also proved to be unrelated to achievement. However, if we compare achievement at the school level with coverage at the school level in the specific domains, we do find, in the cases of two domains, correlation coefficients of .25 and .21. No relationship was found in the case of any of the problem solving sub-tests. In other words, it is just possible that teachers' stated emphasis on certain aspects (e.g. metric measurement and algebra) may contribute to greater achievement in those areas.

Attitudes

We measured five different sets of attitudes encouraged by school programs and compiled a total attitudes score. We found no relationship between the mean level of attitudes in a school and performance in mathematics in the school, although, as expected, there was a relationship ($r = .22$) at the individual level. In other words, schools with more favorable sets of attitudes do not produce better performance in mathematics, but individuals with favorable attitudes tend to perform better than individuals with less favorable attitudes. There is no inconsistency in these two findings, which are consistent with the literature. For example, one explanation might be that students coming from homes that help to develop favorable attitudes tend to do slightly better than students coming from homes where less favorable attitudes are developed. On the other hand, if a school improves attitudes overall, that has no effect on academic performance.

Instructional Time

In 1979, under a Ministry of Education contract, a study was carried out on the relationship between instructional time and achievement in a number of subjects. Mathematics proved to be one of the subjects most sensitive to change in instructional time at the eighth grade level. Using the school as unit of analysis, the relationship between allocated time and achievement was found

to be significant in the following cases: French (r = .60); geography (r = .49); and mathematics (r = .35). No significant relationship was found for language, reading, written English, or history. In science, we found a negative relationship (r = -.4). We examined more closely a small sample from 1977 where I.Q. scores were available, and the findings in English and mathematics were the same with I.Q. controlled. Incidentally, we also found a very great variation in allocated time in all subjects. In mathematics, for example, the minutes per day ranged from 20 to 80 with 60 and 30 being quite common. As a matter of interest, the range of allocated time was just as great in those subject areas where there was no relationship between time and achievement.

A Few Conclusions and Suggestions

The Intermediate Evaluation Project was essentially a service project--providing an evaluation program to schools in a wide variety of subject areas. In terms of that objective it was successful. If we examine achievement in mathematics overall, intermediate students do not seem to be remarkably accomplished in basic computation and measurement--the new emphasis on every day "low" mathematics may therefore be a salutary one. On the other hand, one might reasonably question excursions into informal or transformational geometry, at least until more basic and useful skills have been mastered to a greater degree. The "new mathematics" has taken some hold, but we would be naive indeed if we believed that a greater understanding of mathematics leads ineluctably to greater computational skills. As for improving mathematics scores, instructional time would appear to be one factor at the school level that could contribute to an improved performance, and perhaps least harm would be done if that additional time were taken from science. What is perhaps most remarkable is the small amount of achievement testing carried out in many boards in Ontario. It is not at all unusual for parents and students, and often teachers, to have virtually no idea about the achievement level or competencies. The message from south of the border is becoming clearer and louder. If educators do not step in and provide systematic information to students, parents, and the public at large it is likely to be imposed from above.

Note:

Some of the research on which this report is based has not been published before. The **Reports on the Intermediate Evaluation Project** (1977 and 1978) are available as a pair from the Office of Field Development at $2.50 post-paid. The Ministry report, **Instruction Time and Academic Achievement**, by Mark Holmes, assisted by Richard Wolfe, is to be made available by the Ministry of Education on microfiche. Information on the Intermediate Tests is available from the Office of Field Development, 252 Bloor Street West, Toronto, Ontario, M5S 1V6. Sales of tests, handbooks, answer keys, etc. are made through Publications Sales at the same address.



July: Know you, what it is to be a child?... it is to be so little that the elves can reach to whisper in your ears, it is to turn pumpkins into coaches, and mice into horses, lowness into loftiness, and nothing into everything, for each child has its fairy godmother in its soul.

- Francis Thompson.

A STATISTICAL ACTIVITY USING DATA CARDS

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The writer used specially designed data cards to give multiple choice quizzes to his classes. The method was reported in the Ontario Mathematica Gazette, March 1975, vol. 13, No. 3., titled "AN INEXPENSIVE METHOD OF RAPIDLY SCORING MULTIPLE CHOICE TESTS".

A fringe benefit that has resulted from the use of this scoring method is a hands-on assignment for students in an introductory statistics class. An example follows:

The cards from a quiz in any subject can be used. In our example, the subject was "EFFECTIVE SUPERVISION". There were thirty multiple choice questions valued at one mark each. The results were given in the table below.

MARK	FREQUENCY	MARK	FREQUENCY
1		16	7
2		17	3
3		18	3
4		19	1
5	1	20	2
6		21	2
7		22	1
8		23	
9	1	24	
10	2	25	
11	4	26	
12	3	27	
13	3	28	
14	9	29	
15	4	30	

$$n = \overline{46}$$

The mean and standard deviation of the marks could be determined on a pocket calculator capable of doing statistics by keying in each mark. However, an easier and more interesting method for a statistics assignment is to sort the cards in piles by mark. With a little experience this can be done very quickly. Next, count the number of cards in each pile. This is the frequency of each data item (or group if desired).

Using the formula.

$$\sigma = \sqrt{\frac{\Sigma x^2 - \frac{(\Sigma x)^2}{N}}{N}}$$

where σ is the standard deviation of the whole population

Σx is the sum of all the marks

Σx^2 is the sum of the squares of all the marks

N is the number of the data items.

On a calculator that has an algebraic operating system (AOS) we get:

$\Sigma x = 5 + 9 + 10 \times 2 + 11 \times 4 + 12 \times 3 + 13 \times 3 + 14 \times 9 + 15 \times 4 + 16 \times 7 + 17 \times 3 + 18 \times 3 + 19 + 20 \times 2 + 21 \times 2 + 22 = 679$

$\Sigma x^2 = 5^2 + 9^2 + 10^2 \times 2 + 11^2 \times 4 + 12^2 \times 3 + 13^2 \times 3 + 14^2 \times 9 + 15^2 \times 4 + 16^2 \times 7 + 17^2 \times 3 + 18^2 \times 3 + 19^2 + 20^2 \times 2 + 21^2 \times 2 + 22^2 = 10551$

$N = 46.$

If one is using a calculator with statistical capability (e.g., Texas Instrument SR-51 II) $\Sigma\chi = 679$ can be put in memory 1; $\Sigma\chi^2 = 10551$ can be put in memory 2 and $N = 46$ put in memory 3.

Now the mean, variance, and standard deviation can be found readily by keying in.

If the calculator you are using has not got statistical capability, do not give up. A slight alteration in method of keying-in will help you. For instance, the use of brackets or memory plus may be used, e.g.,

$\Sigma\chi = 5 + 9 + (10 \times 2) + (11 \times 4) + (12 \times 3)$ etc. $= 679$

$\Sigma\chi^2 = 5^2 + 9^2 + (10^2 \times 2) + (11^2 \times 4) + (12^2 \times 3)$ etc. $= 10551.$

These values are then inserted in the formula:

$$\sigma = \sqrt{\frac{\Sigma\chi^2 - \frac{\{\Sigma\chi\}^2}{N}}{N}}$$

$$= \sqrt{\frac{10551 - \frac{(679)}{46}}{46}}$$

$$= 3.4$$

the mean, $\bar{\chi} = \frac{\Sigma\chi}{N} = \frac{679}{46} = 14.8.$

Other statistical related activities are possible. For instance, from the cards, the students could determine the questions that were too easy and the ones that had most wrong answers by doing a simple tally. This will also indicate to the teacher which questions should be deleted as too simple or too difficult.

Another activity that is useful is to make a frequency distribution chart of the marks. This can also be revealing to the teacher. For instance, a frequency polygon of the above test results would indicate that the test was very difficult for the class and that there was likely no cribbing!

Still another activity is to show how sampling works in a real life case. If there are a large number of cards involved, then take at random samples of different sizes, e.g. 1%, 5%, 10%, etc. Have the students do a standard deviation on the samples using the formula:

Sample standard deviation $S = \sqrt{\frac{\Sigma\chi^2 = \frac{\{\Sigma\chi\}^2}{n}}{n-1}}$

where $\Sigma\chi$ = sum of marks in sample

$\Sigma\chi^2$ = sum of marks squared in sample

n = sample size.

The student can then compare inferences from the samples through use of one, two or three deviations and compare them with the whole population. If done properly, the idea of sampling and inference comes into focus.

The innovative instructor will find other useful activities such as the central limit theorem. The use of the cards allows the teacher to tie in sampling with the whole population with a minimum of effort and cost and it is a hands-on activity.

Author's Note: Since data cards are used in most educational institutions, we have a hands-on activity at no additional cost.

A second item that I would like to bring to the attention of teachers is the very high cost of scoring multiple choice tests on a computer. Some business firms estimate that each line of print-out from a computer costs the firm one dollar. One might disagree with this cost - but the fact remains that it is high. The equipment for hand scoring is about fifty dollars. Possibly, re-reading or re-printing the article "An Inexpensive Method of Rapidly Scoring Multiple Choice Tests" could save some money.

AN APPROACH TO SOME NON-TRIVIAL INEQUALITIES

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Generally in algebraic solutions to inequalities there is a tendency to rely on what is often called the "case" method to solve inequations. The problem $x^2 - 5x + 6 < 0$ is considered as $(x - 3)(x - 2) < 0$, and then the two cases:

(1) $x - 3 < 0$ and $x - 2 > 0$

or, (2) $x - 3 > 0$ and $x - 2 < 0$ are treated.

The solution being $\{x \in \mathbb{R} \mid 2 < x < 3\}$ from case (1), as case (2) is vacuous.

In moving from $x^2 - 5x + 6 < 0$ to $x^3 - 3x^2 - 4x + 12 < 0$, the same technique would apply, but the number of cases would increase to four. It becomes clear, however, that the problem

$$\frac{x^3 - 3x^2 - 4x + 12}{x^2 - 1} < 0$$

could entail checking up to sixteen cases. At this point the mathematical mind decides to view the problem in a somewhat different manner.

First, we know that corresponding to each inequation $f(x) \leq 0$ or $f(x) \geq 0$, there could be associated a relation $y = f(x)$ for which we are asking the question: "For what values of x is the graph of $y = f(x)$ above, or below the x-axis?".

Secondly, it is known that the only points at which the graph of a relation crosses the axis are at intercepts or at points of discontinuity. Figure 1 illustrates an intercept crossing, and Figure 2 illustrates a discontinuity crossing.

FIGURE 1

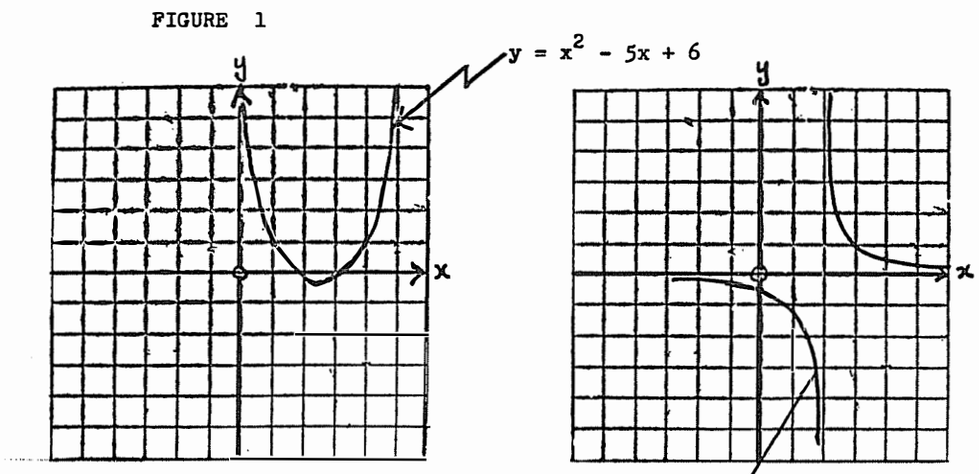


FIGURE 2

With these observations together with the remarks of Hille and Salas [1], I formulated some years ago an algorithm to deal with multiple case problems, which seems to reduce the work and maximize the understanding. I make no pretence of originality in this pursuit,

but write it only because I know of no clear reference. Moreover, my colleagues have urged that the idea be shared.

Step One: The streamlining of notation by use of standard interval notation is taught first. That is, by (a,b) we shall mean $\{x \in \mathbb{R} \mid a < x < b\}$, and similarly $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, $[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$, etc. ... to include $(a,b]$, $(-\infty,a)$, $(-\infty,a]$, (a,∞), $[a,\infty)$, and $(-\infty,\infty) = \mathbb{R}$ (see [2]).

Step Two: Discuss the need of a simpler approach and the idea of the functional geometric interpretation of an inequality as expressed in the preceding prologue. Include at this point some demonstration of the voluminous checking involved in a problem such as $\frac{x^2 - 9}{x^2 - 4} > 0$.

Case I: $x^2 - 9 > 0$ and $x^2 - 4 > 0$

subcase i) $x - 3 > 0$ and $x + 3 > 0$ and $x - 2 > 0$ and $x + 2 > 0$

ii) $x - 3 < 0$ and $x + 3 < 0$ and $x - 2 > 0$ and $x + 2 > 0$

iii) $x - 3 > 0$ and $x + 3 > 0$ and $x - 2 < 0$ and $x + 2 < 0$

iv) $x - 3 < 0$ and $x + 3 < 0$ and $x - 2 < 0$ and $x + 2 < 0$

Case II: $x^2 - 9 < 0$ and $x^2 - 4 < 0$ has subcases

i) to iv) dual to Case I.

Step Three: The Algorithm ---

(1) Find the intercepts by solving $f(x) = 0$. $a_1, a_2, \ldots, a_k$ are intercepts.

(2) Determine the discontinuities if any. (If the concept of discontinuity is premature, use "the values where f(x) is undefined".) Call these values $b_1, b_2, \ldots, b_m$.

(3) Order the $k + m$ values from (1) and (2) from smallest to largest giving an ordered set of numbers $c_1, c_2, \ldots, c_{k+m}$.

(4) Use the values from (3) to exhaust the real line, $\mathbb{R}$, with intervals.

(5) Check a point from each interval, and the endpoint, if it is an intercept.

(6) The union of the satisfying intervals will be the solution set (S.S.).

Example 1: $\frac{x^2 - 9}{x^2 - 4} > 0$

re (1): $\frac{x^2 - 9}{x^2 - 4} = 0$ iff $x^2 - 9 = 0$, and $x^2 - 9 = 0$ iff $x = 3$, or $x = -3$. Hence, $a_1 = 3$ and $a_2 = -3$ with reference to the algorithm.

re (2): $\frac{x^2 - 9}{x^2 - 4}$ is discontinuous (undefined) iff $x^2 - 4 = 0$, and $x^2 - 4 = 0$ iff $x = 2$, or $x = -2$. That is, $b_1 = 2$, and $b_2 = -2$ with reference to the algorithm.

re (3): $c_1 = -3$, $c_2 = -2$, $c_3 = 2$, $c_4 = 3$ is the required ordering.

re (4): $(-\infty,-3)(-3,-2)(-2,2)(2,3)(3,\infty)$ are the intervals, and since we are interested in the strict inequality, there will be no need to check intercepts.

re (5): $-4 \in (-\infty,-3)$ and $\frac{(-4)^2 - 9}{(-4)^2 - 4} = \frac{16 - 9}{16 - 4} > 0$, and hence the interval is satisfactory.

$-\frac{5}{2} \in (-3,-2)$ and $\frac{(-\frac{5}{2})^2 - 9}{(-\frac{5}{2})^2 - 4} = \frac{\frac{24}{4} - 9}{\frac{25}{4} - 4} = \frac{\frac{-11}{4}}{\frac{9}{4}} < 0$, and thus the interval $(-3,-2)$ is unsatisfactory. $0 \in (-2,2)$ and $\frac{0^2 - 9}{0^2 - 4} > 0$, and hence $(-2,2)$ is satisfactory.

$\frac{5}{2} \in (2,3)$ and $\frac{(\frac{5}{2})^2 - 9}{(\frac{5}{2})^2 - 4} < 0$ makes $(2,3)$ unsatisfactory. Finally, $4 \in (3,\infty)$ and

$\frac{4^2 - 9}{4^2 - 4} > 0$ making $(3,\infty)$ satisfactory.

re (6): The solution set is $(-\infty,-3) \cup (-2,2) \cup (3,\infty)$.

Of course, once understood, this technique does not require all of the foregoing detail. Example 2 will illustrate this fact.

Example 2: $f(x) = \frac{x^2 - 3x^2 - 4x + 12}{x^2 - 1} \leq 0$.

re (1): $\frac{x^3 - 3x^2 - 4x + 12}{x^2 - 1} = 0$ iff $(x + 2)(x - 3)(x - 2) = 0$, hence -2, 2, and 3 are the intercepts.

re (2): $\frac{x^3 - 3x^2 - 4x + 12}{x^2 - 1}$ is discontinuous when $x^2 - 1 = 0$, hence -1 and 1 are discontinuities.

re (3): -2, -1, 1, 2, 3.

re (4): $(-\infty,-2)(-2,-1)(-1,1)(1,2)(2,3)(3,\infty)$.

re (5): $-3 \in (-\infty,-2) f(-3) < 0$, and $f(-2) = 0$, therefore, $(-\infty,-2] \subset$ S.S.

$-\frac{3}{2} \in (-2,-1)$ and $f(-\frac{3}{2}) > 0$

$0 \in (-1,1)$ and $f(0) < 0$ therefore $(-1,1) \subset$ S.S.

$\frac{3}{2} \in (1,2)$ and $f(\frac{3}{2}) > 0$

$\frac{5}{2} \in (2,3)$ and $f(\frac{5}{2}) < 0$, with $f(2) = f(3) = 0$ therefore, $[2.3] \subset$ S.S.

$4 \in (3,\infty)$ and $f(4) > 0$.

(This checking is extremely easy with a calculator, and a knowledge of nested forms for polynomials.)

re (6): S.S. $= (-\infty,-2] \cup (-1,1) \cup [2,3]$.

At this stage it might appear that the intervals will alternate from left to right in a "flip-flop" fashion, but that will only be the case where there are no duplicate factors. In the presence of duplicate factors all checking is essential.

In conclusion, it is interesting to use the technique on expressions involving absolute value, as the only wrinkle which occurs is solving two equations for intercepts. More particularly the first stage of $|2x + 1| < |x - 5|$ involves solving $2x + 1 = x - 5$ and $2x + 1 = -x + 5$; the balance of the solution is unaltered. Also, observe that if the expression is continuous everywhere, then stage two of the solution is vacuous.

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[1] Hille, Einar and Saturnino Salas, First Year Calculus, Waltham, Mass.: Blaisdell Publishing Company, 1968.

[2] Protter, Murray H. and Charles B. Morrey, Jr., Calculus with Analytic Geometry: A First Course, Reading, Mass.: Addison-Wesley Publishing Company, 1970.

Editor's Note: So that readers will not accuse Claire of sloppy notation, his article in Vol. 17, No. 3, March 1979 contained a typing error on line 6, page 14 "L^2 10 =" should read "L_{10}^2 " in the article titled "Do Ya Wanna See a Card Trick?"



August: You know children are growing up when they start asking questions that have answers.
- John J. Plomp.



THE USE OF TRIGONOMETRY TO SOLVE EQUATIONS

OR

HOW FAST WOULD YOU HAVE TO GO TO BE HALF AS TALL AS YOU ARE NOW?

T.J. Racey, Bayridge Secondary School, Kingston

"It was his (Edward Kasner) theory that the way to interest children in mathematics was to begin with difficult concepts and gradually work up to the multiplication tables."
The Mathematical Magpie pxvi

The gap between mathematics and a physical idea connected to it is sometimes larger than we would like in the current school curriculum. I offer the following as one example of the many ways that this fragmentation can be lessened to help create a smooth transition between the two areas.

Many senior students in our high schools have heard of some of the exotic areas of physics, such as Einstein's special theory of relativity, where an object moving quickly does not age as fast as an object standing still, where the size of an object gets smaller as it travels faster, and where the 'mass' of an object gets larger as it travels faster, with respect to us. The exciting elements of this concept can be used as an example between the physical world and the mathematical techniques of predicting this world. The example illustrated offers another use of the Pythagorean theorem and has a particular relevance to some good trigonometric exercises, and illustrates that trigonometry is sometimes a more efficient way of solving equations in a given problem than the standard algebraic techniques.

The equations of Einstein's special relativity can be given to the students. Using two frames of reference, a frame S (where the object is examined while standing still) and a frame M (where the object is examined while moving with respect to S), the equations describing the relationship between the object when it is standing still and when it is moving are given in Table 1.

TABLE 1: The Equations of Einstein's Special Relativity

$$L_M = L_S \sqrt{1 - \frac{v^2}{c^2}} \qquad (1), \text{ where L is length}$$

$$M_M = \frac{M_S}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (2), \text{ where M is mass}$$

$$T_M = \frac{T_S}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (3), \text{ where T is time}$$

In each equation: v is the velocity of the moving object and c is the velocity of the speed of light (3×10^8 m/s).

At this point students can often be encouraged to find how fast a student must travel to be half as tall as he or she is now. This provides a good algebraic review and leads the studnet into possible reasons for finding an alternative technique to solve the equations. Students should always be encouraged to guess at the speed first. Rearranging equation (1) gives

$$L_M = L_S \sqrt{1 - \frac{v^2}{c^2}}$$

$$\frac{1}{2} L_S = L_S \sqrt{1 - \frac{v^2}{c^2}}$$

$$\frac{1}{4} = 1 - \frac{v^2}{c^2}$$

$$\frac{V^2}{c^2} = \frac{3}{4}$$

$$V = \sqrt{\frac{3}{2}}\, c$$

$$V = 2.6 \times 10^8 \text{ m/s}$$

To find the equivalent values of M and T at this velocity would also be long algebraic solutions for most students. Likewise, the general solution would also appear difficult to a student. This can be a good exercise to encourage students in algebraic manipulation and will contrast later to the trigonometric solution.

$$L_M = L_S \sqrt{1 - \frac{v^2}{c^2}}$$

$$\frac{L_M}{L_S} = \sqrt{1 - \frac{v^2}{c^2}}$$

$$\left(\frac{L_M}{L_S}\right)^2 = 1 - \frac{v^2}{c^2}$$

$$\frac{V^2}{c^2} = 1 - \left(\frac{L_M}{L_S}\right)^2$$

$$V = \sqrt{1 - \left(\frac{L_M}{L_S}\right)^2}\; c$$

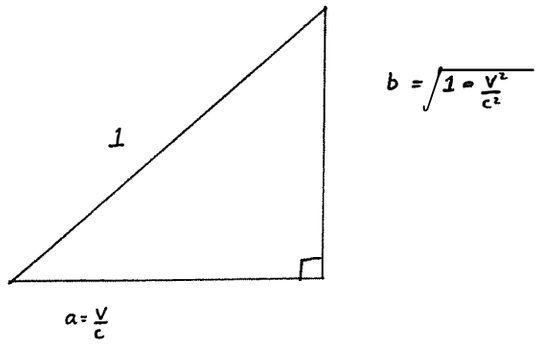
This is where the 'new' technique of a trigonometric approach can be introduced. In equation (1) consider only the radical

$$\sqrt{1 - \frac{v^2}{c^2}}$$

Using substitution, let $a = \frac{v}{c}$ and $b = \sqrt{1 - \frac{v^2}{c^2}}$, this gives $b = \sqrt{1 - a^2}$. Squaring, one gets

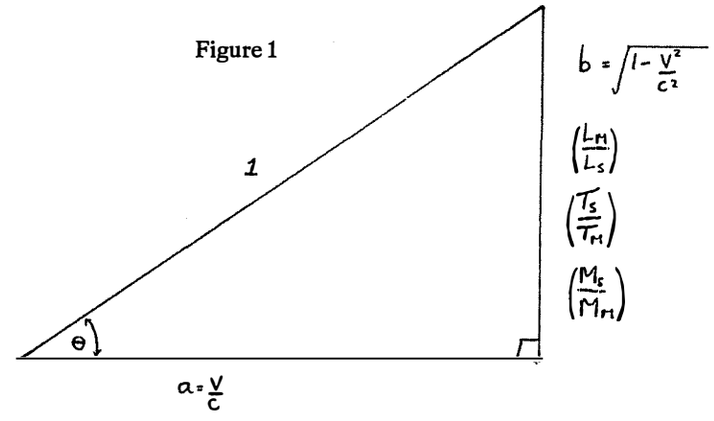
$$b^2 = 1 - a^2 \quad \text{or} \quad a^2 + b^2 = 1 \qquad (4)$$

Students would recognize this as the Pythagorean equation and therefore could draw the triangle shown below.



Relating the equations of Table 1 to a single triangle determined by the velocity of the moving object allows the triangle below to be created. This triangle is equivalent in form to equation (4).

Figure 1



Representing the equations in a cartesian form allows the student to quickly observe that standard trigonometric ratios can now be used to connect the side labelled 'a', which is just the relative velocity of the object V/c, with the relative physical measurement of the object, i.e. its time, length or mass. Trigonometric tables or calculators can now be used to manipulate the ratios to allow calculations of the equations to be carried out quite easily.

Example: How fast would you have to go to be half as tall as you are now?

From Figure 1 $\dfrac{\left(\frac{L_M}{L_S}\right)}{1} = \sin\theta$; where $\left(\frac{L_M}{L_S}\right) = \frac{1}{2}$

therefore $\sin\theta = \frac{1}{2}$

which from trigonometric tables gives $\theta = 30°$.

Now from Figure 1 $\cos\theta = \frac{V}{c}$

this gives $\frac{V}{c} = \cos 30°$ or $\frac{V}{c} = .87$.

This gives the result $V = 2.6 \times 10^8$ m/s.

This type of calculation is now easily repeated for the rest of Table 1. The results are all equivalent to the algebraic solution. Once the trigonometric concept for solving equations of this type has been introduced to the student then trigonometric ratios become an efficient technique for making predictions from these real world equations.

The student can now be motivated to use this trigonometric technique to answer questions such as
- how fast do you have to go to live 10 times as long as your stationary friend?
- the escape velocity from Earth is about 11 km/s. How much does your mass increase in a rocket shot? (You may need to use calculator tables here)

Motivational questions of this sort are endless and they all relate to the physical world around the student. This technique now leads into such concepts as congruent triangles (same velocity), similar triangles and trigonometric identieis $(\sin^2\theta + \cos^2\theta = 1)$. The equivalence of geometric and algebraic problem solving techniques is also demonstrated for the student. The student is lead into a real physical world, where the power of mathematical predictions illustrate its beauty and strangeness.

September: A boy's will is the wind's will,/And the thoughts of youth are long, long thoughts.
- Henry Wadsworth Longfellow.

INTRODUCING THE CALCULUS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

T.J. Griffiths, A.B. Lucas Secondary School



In the initial study of Calculus, one of the topics for which it is difficult to find a good introduction is the differentiation of exponential and logarithmic functions. The following is a development of this topic which I have found to be reasonably satisfying both for myself and for my grade 13 students. It should be noted however, that this is not meant to be a rigorous, dotted i's, crossed t's proof, but more of a reasonable justification.

First consider functions whose rate of change, or slope, is proportional to the value of the function. Using envelope techniques reasonable graphs can be drawn from this definition, although a little sliding of the position of the tangents is needed to obtain a smooth curve. Examples which can be used in this initial preparatory study include population growth, bacteria growth, radioactive decay, bank interest and depreciation. In all of these the rate of change of the sample is proportional to the size of the sample. Functions with this characteristic are called exponential functions, and have equations of the form $f(x) = ax$

In Calculus notation, if f(x) is such a function, then the rate of change being proportional to the function gives $f'(x) = k \cdot f(x)$.

For simplicity let k = 1. Now assume that such a function does indeed exist. Then

$$f'(x) = f(x). \qquad (1)$$

Therefore $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x)$. Consider this equation without the limit, i.e.,

$\frac{f(x+h) - f(x)}{h} = f(x)$.

$$\therefore \quad f(x+h) - f(x) = h \cdot f(x)$$

$$\therefore \quad f(x+h) = h \cdot f(x) + f(x)$$

$$\therefore \quad f(x+h) = f(x) \cdot (h + 1)$$

$$\therefore \quad \frac{f(x+h)}{f(x)} = (1+h). \qquad (2)$$

Since we started this with the concept of an exponential function it would be reasonable to let $f(x) = a^x$. In fact, this is the only function with the property that $\frac{f(x+h)}{f(x)}$ is independent of x. So $\frac{f(x+h)}{f(x)} = \frac{a^{x+h}}{a^x} = \frac{a^x \cdot a^h}{a^x} = a^h$. So from equation (2) we have

$$a^h = (1+h). \quad \therefore \; a = (1+h)^{1/h}. \qquad (3)$$

Now reintroduce the limit and let $h \to 0$. It is more convenient to replace h by 1/n. As $h \to 0$ then $n \to \infty$ and $(1+h)^{1/h} = (1+\frac{1}{n})^n$, which is simpler to evaluate. Evaluating $(1+\frac{1}{n})^n$ for increasingly larger values of n.

n	$(1+\frac{1}{n})^n$
1	2
2	2.25
5	2.4883
10	2.5937
100	2.7048
1000	2.7169
10 000	2.7181
100 000	2.7182

This sequence would seem to be increasing to a limit. Now note that $(1+\frac{1}{n}) \to 1$.

$$\therefore \quad (1+\frac{1}{n})^{n+1} = (1+\frac{1}{n}) \cdot (1+\frac{1}{n})^{n} > (1+\frac{1}{n})^{n}.$$

Evaluating $(1+\frac{1}{n})^{n+1}$ for increasingly large values of n

n	$(1+\frac{1}{n})^{n+1}$
1	4
2	3.375
5	2.9860
10	2.8531
100	2.7318
1000	2.7196
10 000	2.7184
100 000	2.7182

This sequence would seem to be decreasing to a limit. Therefore we have two sequences, one for $(1+\frac{1}{n})^{n+1}$, which is decreasing, but every term of which is greater than those from the sequence generated by $(1+\frac{1}{n})^{n}$, which is increasing. Also $(1+\frac{1}{n})^{n+1} = (1+\frac{1}{n}) \cdot (1+\frac{1}{n})^{n}$. But as $n \to \infty$, $(1+\frac{1}{n}) \to 1$. Therefore $(1+\frac{1}{n})^{n+1}$ and $(1+\frac{1}{n})^{n}$ must approach the same limit. Therefore the limit for both of them must be sandwiched between them. This limit is defined as "e", and it can be seen from the tables that $e \underline{\sim} 2.7182$, the value of a in equation (3). So $f(x) = e^x$. We have also from equation (1) that $\frac{d}{dx} e^x = e^x$. From this $\int e^x dx = e^x + c$. The Inverse of the function $y = e^x$ is $x = e^y$. By the definition of logarithm this can be written as $y = \log_e x$. For convenience we define this as ln x, and it is called the natural logarithm of x.

$$\therefore \quad y = \ln x$$

$$\therefore \quad y^y = x$$

$$\therefore \quad \frac{d}{dx} e^y = \frac{d}{dx} x$$

$$\therefore \quad \frac{d}{dy} e^y \cdot \frac{dy}{dx} = 1 \qquad \text{by the chain rule}$$

$$\therefore \quad e^y \cdot \frac{dy}{dx} = 1 \qquad \text{since } e^y \text{ is its own derivative}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{e^y}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x} \qquad \text{since } e^y = x$$

$$\therefore \quad \frac{d}{dx} \ln x = \frac{1}{x} \qquad \text{since } y = \ln x$$

or $\int \frac{1}{x} dx = \ln x + c.$

The studies of $\frac{d}{dx} e^{f(x)}$, $\frac{d}{dx} \ln f(x)$, $\int f'(x) \ e^{f(x)} dx$, and $\int \frac{f'(x)}{f(x)} dx$ follow directly by applying various previously established rules.



October: 'Tis education forms the common minds/Just as the twig is bent, the tree's inclined.
- Alexander Pope.



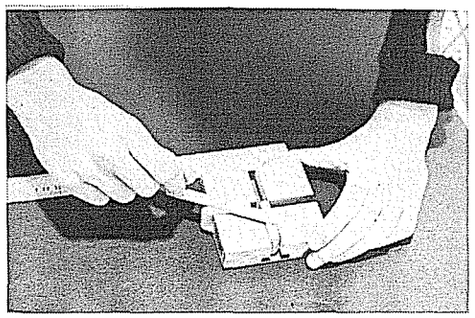
AN EQUATION FOR THE DO-NOTHING GRINDER

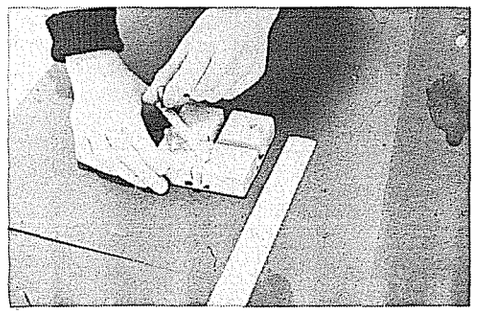


AN APPLICATION FROM A X-MAS PRESENT

Maurice Poirier, Ecole Secondaire Garneau, Orleans

As a joke, my father-in-law gave me a "Do-Nothing Grinder" (see fig. 1) for Christmas, suggesting that we mathematics teachers have an easy time of it, and such a machine could keep us busy during "all (our) spare time".





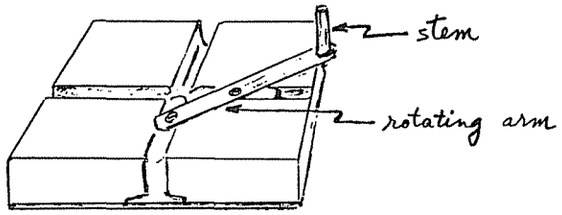


fig. 1

I was intrigued by its motion, and wondered if such a linkage produced a known elementary curve. A bit of exploration produced the following:

Let the two perpendicular grooves be the x- and y-axes, respectively, of the Cartesian plane, and let P(x,y) be a point on the locus traced by the stem of the "Do-Nothing Grinder". Label the points as in figure 2.

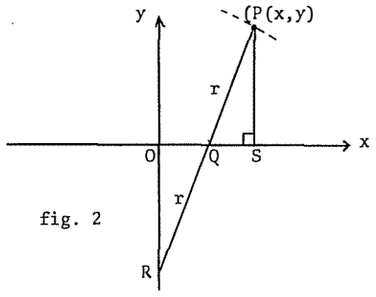


fig. 2

The structure of the grinder is such that PQ = QR = r, say. Constructing PS⊥ x-axis gives us quite easily that $\Delta OQR \sim \Delta SQP$, and, consequently, that $OQ = QS = \frac{x}{2}$. Considering ΔSQP, by the Pythagorean theorem, we get:

$$QS^2 + SP^2 = PQ^2$$

or, $$\left(\frac{x}{2}\right)^2 + y^2 = r^2$$

$$\therefore \quad \frac{x^2}{4r^2} + \frac{y^2}{r^2} = 1 \text{ is the required equation.}$$

This is easily recognized as an ellipse, with foci on the x-axis, having its major axis = 4r, and its minor axis = 2r. This locus, of course, is that traced by the stem

of the "Do-Nothing Grinder". What about <u>other</u> points T on the rotating arm?

Do they all produce ellipses?

There are 4 other cases to consider (see fig. 3):

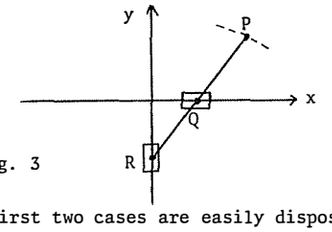


fig. 3

(i) the point T is at Q on the arm;

(ii) the point T is at R on the arm;

(iii) the point T lies between P and Q on the arm;

(iv) the point T lies between Q and R on the arm.

The first two cases are easily disposed of, by noting that Q "slides" back and forth on the x-axis, while R does the same on the y-axis, i.e., cases (i) and (ii) produce line segments concurrent with the two axes.

Consider case (iii): Let T(x,y) be in PQ (see fig. 4).

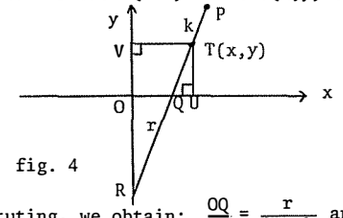


fig. 4

Draw TU⊥ x-axis, and TV⊥ y-axis. Let QT = r - k, r > k > 0.

Since $\Delta OQR \sim \Delta VTR \sim \Delta UQT$, we have:

$\frac{OQ}{VT} = \frac{RQ}{RT}$ and $\frac{OR}{UT} = \frac{RQ}{TQ}$.

Substituting, we obtain: $\frac{OQ}{x} = \frac{r}{2r - k}$ and $\frac{OR}{y} = \frac{r}{r - k}$ or $OQ = \frac{xr}{2r - k}$ and $OR = \frac{yr}{r - k}$.

Considering ΔOQR, again, Pythagoras gives us: $OQ^2 + OR^2 = RQ^2$ or $\left(\frac{xr}{2r - k}\right)^2 + \left(\frac{yr}{r - k}\right)^2 = r^2$.

Dividing through by r^2, we get: $\frac{x^2}{(2r - k)^2} + \frac{y^2}{(r - k)^2} = 1$, an ellipse, foci on the x-axis, with major axis = 2(2r - k) and minor axis = 2(r - k).

Case (iv) is handled in much the same way; letting T(x,y) be in QR (see fig. 5) gives us the equation: $\frac{x^2}{(r - k)^2} + \frac{y^2}{k^2} = 1$, an ellipse, with major and minor axes measuring 2(r - k) and 2k, respectively. In this case, however, depending on the relative magnitudes of r and k, we obtain:

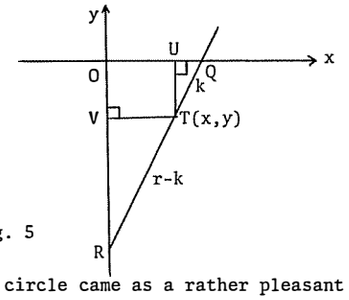


fig. 5

a) an ellipse with foci on the x-axis if r - k > k, i.e., if r > 2k (the point is closer to the "x-screw" of the grinder);

b) an ellipse with foci on the y-axis if r - k < k, i.e., if r < 2k (the point is closer to the "y-screw" of the grinder);

c) a <u>circle</u> if r - k = k, i.e., if r = 2k (the point is half-way between the two screws).

The circle came as a rather pleasant surprise, at the end of this analysis. Other interesting questions to consider while "grinding away" with the linkage: where are the foci in each case? Are the ellipses confocal? Where is the directrix? What is the eccentricity? What happens if a point is chosen on an "extended arm"?

These and other questions kept me busy and absorbed most of the Christmas evening, while the children scurried around, playing with <u>their</u> toys. When I thanked my father-in-law for the "wonderful" gift, he replied, half-jokingly, with thinly veiled disgust: "That's the trouble with you mathematicians, you can't take a joke!"

Addendum: If you are interested in obtaining your own "Do-Nothing Grinder", try your local variety or joke shop: they usually carry such items. Some readers may have recognized the "Grinder" as a variant of the elliptical blackboard compass. The construction of such an apparatus would be an interesting project for your students.

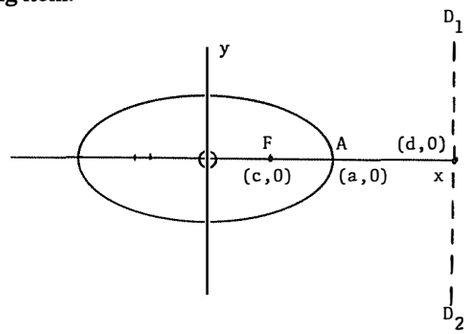
Editors Note: Thanks to B. Palmer for the following item.

Here is a simple way to remember the formulas for eccentricity and for the equation of the directrix for both the ellipse and the hyperbola.

a is the mean proportional between c and d.

$$\frac{c}{a} = \frac{a}{d} = e .$$

Combining these 3 quantities in pairs will quickly give the desired formula.



"Comme le directeur, Gloster Ridley, dans le roman Leaven of Malice, par Robertson Davies, je ne peux pas excuser les erreurs dans O.M.G.--je peux seulement en accepter la responsibilité. Après qu'on l'a fait remarquer à Gloster, c'était facile de se rendre compte qu'aucune date comme le 31 novembre n'existe dans notre système, ni en fait, qu'un faire-part du marriage n'aurait été imprimé portant cette date. Dans l'édition de septembre de O.M.G.' l'article par Maurice Poirier est incomplet. Il y a un paragraphe qui manque. Il devrait venir juste avant la série de MacLaurin pour cos x.

"De même, pour obtenir le cosinus ou la tangente d'un angle donné, on n'a qu'à se servir d'une formule de réduction qui utilise le sinus. Il suit, de la définition des fonctions inverses, que la cosécante, la sécante et la cotangente se calculent aussi facilement.

Les analystes mordus pourraient développer un calcul semblable pour le cosinus, dont la série de MacLaurin est:"

A LIMITATION ON MATHEMATICAL INDUCTION AT AN ELEMENTARY LEVEL

J. Rubinstein, Northview Heights S.S., North York Board of Education

Mathematical Induction is a powerful tool for attempting to solve the following problem: given a statement P_n involving the natural number n, is P true **for all** natural numbers n? For instance, let P_n be the statement:

$$1 + 2 + 3 + \bullet\bullet\bullet\bullet\bullet + n = n(n + 1)/2, n \in N,$$

which can be readily hypothesized from the pattern $1 = 1 \times 2/2$, $1 + 2 = 2 \times 3/2$, $1 + 2 + 3 = 3 \times 4/2$, etc., and then proven true, by Mathematical Induction, for all $n \in N$.

Mathematical Induction is applied as follows:

a) verify that P_n is true for $n = 1$.
b) assume that P_n is true for $n = k$ (say), $K \in N$
c) prove that P_n is true for $n = k + 1$.

Completion of (a), (b), and (c) proves that P_n is true **for all** $n \in N$. Step (c) is usually the most difficult one for high school students, and in some cases even for mathematicians. A notorious example of this kind is the recently solved Four-Colour problem[1] . In such cases, a natural doubt arises: can **all** appropriate statements be proven, at least in principle, by Mathematical Induction? Although we cannot fully answer this question, this paper presents a simple class of counter-

examples to prove that "Elementary" Mathematical Induction proofs (to be defined below) are not possible for a certain class of statements.

Let $\{t_1, t_2, \ldots, t_i, \ldots\}$ be an infinite sequence of real numbers and let

$$S_n = \sum_{i=1}^{n} t_i .$$

Suppose that we have reasons to hypothesize that for a given function f(n),

$$S_n \geq f(n), \quad n \in N, \tag{1}$$

i.e., that for some values of n, $S_n > f(n)$, while for others $S_n = f(n)$. Suppose further that

$$S_1 > f_1, \tag{2}$$

and that $S_n = f(n)$ for <u>at least</u> one value of n, say p, i.e., that

$$S_p = f(p) \text{ for some } p \in N,\ p > 1. \tag{3}$$

Notice that (2) and (3) are consistent with (1).

The usual way of proving (1) by Mathematical Induction involves assuming $S_k \geq f(k)$, $k \in N$. Therefore $S_{k+1} \geq f(k) + t_{k+1}$, and then proving that $f(k) + t_{k+1} \geq f(k+1)$. Therefore $S_{k+1} \geq f(k+1)$. This process, which we shall presently develop in more detail, shall from now on be called <u>Elementary Mathematical Induction</u>, or EMI for short. Notice that EMI is a particular method for proving the truth of a special type of proposition.

We can then prove the following:

<u>Theorem:</u> (1), subject to (2) and (3), cannot be proven true, for all $n \in N$, by "Elementary" Mathematical Induction.

<u>Proof</u>: The proof is by "reductio ad absurdum", i.e., we will show that the negation of the theorem leads to a contradiction.

If (1) could be proven true for all n, by EMI, in the proof we would

(a) verify that (1) is true for n = 1, due to (2)

(b) assume that $S_k \geq f(k)$ (4)

(c) add t_{k+1} to both sides of (4), to obtain

$$S_k + t_{k+1} \geq f(k) + t_{k+1} \therefore S_{k+1} \geq f(k) + t_{k+1} \tag{5}$$

and then prove that $f(k) + t_{k+1} \geq f(k+1)$; (6)

from (5) and (6) $S_{k+1} \geq f(k+1)$,

i.e., (1) is true for n = k + 1. This would complete the proof that (1) is true <u>for all</u> $n \in N$.

But then, we could also prove, by EMI, that

$$S_n > f(n), \quad n \in N \tag{7}$$

is true <u>for all</u> $n \in N$. In the proof we would

(a) verify that (7) is true for n = 1, due to (2)

(b) assume that $S_k > f(k)$ (8)

(c) add t_{k+1} to both sides of (8) to obtain

$$S_{k+1} > f(k) + t_{k+1}; \tag{9}$$

since the assumed proof of (6) would still be valid, we can use (6) in conjunction with (9) to obtain $S_{k+1} > f(k+1)$. This would complete the proof that (7) is true <u>for all</u> $n \in N$. Since this contradicts (3), we have proven that the negation of the theorem, i.e., the assumption that (1) <u>can</u> be proven by EMI, is false. Therefore, the theorem is true. #

Notice the qualifier "Elementary" in the statement of the theorem cannot be easily removed. It is conceivable that we could prove the <u>implication</u>

$$[S_k \geq f(k)] \Rightarrow [S_{k+1} \geq f(k+1)],$$

even if (6), on which our theorem depends, were false. However, such a proof, if possible, is likely to be more difficult than the "Elementary" one, and even beyond the capabilities of most high school students.

We can build an example for which (1), (2) and (3) hold. Let $t_i = i$. Therefore $S_n = \sum_{i=1}^{n} i = \frac{1}{2}(n^2 + n)$, $n \in N$. The graph of S_n as a function of n is shown in Fig. 1 as dots on the parabola $y = \frac{1}{2}(x^2 + x)$, $x \in R$, for x = 1,2,3, etc.

We now build a function f(n) such that $S_4 = f(4)$; $S_n > f(n)$, $n \neq 4$, $n \in N$. Let $g(x) = \frac{1}{2}(x^2 + x)$, $x \in R$. The line tangent to the parabola $y = g(x)$ at the point A(4, g(4)) is clearly below the parabola for all $x \neq 4$; since g(4) = 10 and $g'(4) = \frac{9}{2}$, the equation of this line is $y = \frac{9}{2}(x - 4) + 10 = \frac{9}{2}n - 8$ and we have

$$\frac{1}{2}(x^2 + x) \geq \frac{9}{2}x - 8, \quad x \in R \quad \text{(Fig. 1)}$$

$$\therefore \quad \frac{1}{2}(n^2 + n) \geq \frac{9}{2}n - 8, \quad n \in N$$

$$\therefore \quad S_n \geq \frac{9}{2}n - 8, \quad n \in N; \qquad (10)$$

(10) is a special case of (1), with $f(n) = \frac{9}{2}n - 8$, and for which (2) and (3) apply:

$$S_1 = 1 > \frac{9}{2} \times 1 - 8 = f(1)$$

$$S_4 = 1 + 2 + 3 + 4 = \frac{9}{2} \times 4 - 8 = f(4) \quad \therefore \quad p = 4.$$

Then according to our theorem, (10) cannot be proven by EMI. Notice that here S_n stands for $1 + 2 + \cdots + n$, and that we purposely ignore in this example the fact that S_n can be evaluated in closed form, i.e., $S_n = \frac{1}{2}(n^2 + n)$.

It is instructive to see what happens if we <u>do</u> try to prove (10) by EMI:

(a) $S_1 > \frac{9}{2} \times 1 - 8$ $\therefore$ (10) is true for n = 1

(b) assume $S_k \geq \frac{9}{2}k - 8$

(c) $\therefore S_{k+1} \geq \frac{9}{2}k - 8 + (k + 1)$ $\therefore$ $S_{k+1} \geq \frac{11}{2}k - 7$,

we now need to prove that

$$\frac{11}{2}k - 7 \geq f(k + 1)$$

i.e.,

$$\frac{11}{2}k - 7 \geq \frac{1}{2}(k + 1) - 8,$$

but this is true only for $k \geq \frac{7}{2}$, i.e., for $k \geq 4$! $\therefore$ (10) cannot be proven by EMI. Incidentally, the statement $S_n > \frac{9}{2}n - 9$ (we have replaced f(n) by f(n) - 1) cannot be proven by EMI either.

Clearly, the example can be amended by using a simple modification of EMI, namely, starting the induction process at n = 4; however, amending in this manner would not be possible if (3) were true for an infinite sequence of p's. A trivial example is given by $S_n = \sum_{i=1}^{n} (-1)^{i-1} \geq 0$; attempts to prove it by EMI by several in our Math. Department have resulted in proofs that, implicitly or explicitly, use the fact that $\sum_{i=1}^{n} (-1)^{i-1} = 0$ or 1. If there is a greatest p, p_m, for which (3) holds, it must be known in order to amend the EMI proof; in our example, p_m was obtained in a systematic way, but this may not be always the case in more complex proofs.

The theorem remains valid if, in (1), $\geq$ is replaced by $\leq$, and/or $t_i > 0$ and $S_n = \sum^{n} t_i$ is replaced by $B_n = \prod^{n} t_i = t_1 t_2 \ldots t_n$. In fact, a question of the latter nature in the 1978 Descartes Competition motivated the present paper.

This completes the class of problems for which the theorem is now proven. Inclusion of additional types of problems into the class is not, however, precluded.

A word of caution: if the reader teaches mathematical induction he is aware of the persistent difficulties encountered by many students in apprehending both the concept and the techniques; we have often heard the complaint that "it can't be done!" Obviously, such students should **not** be exposed to the ideas presented in this paper, since it would only compound their doubts. Only a **good** senior class, or selected students, would benefit. This benefit would be twofold:

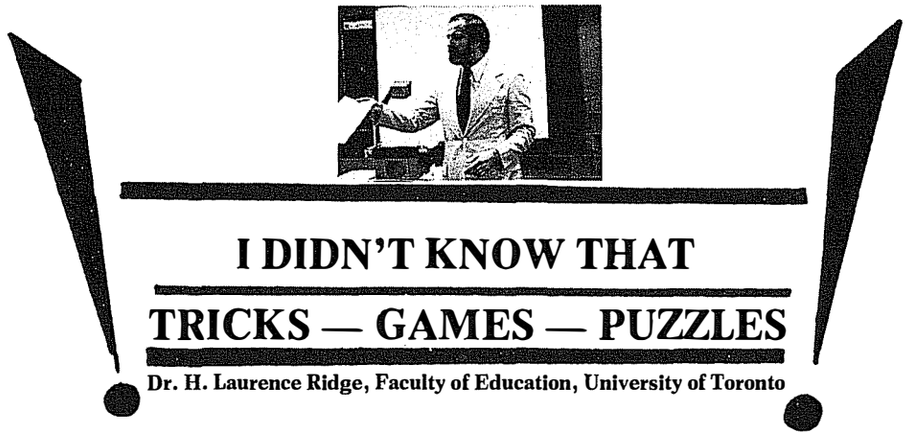
a) by providing the students with a better insight into the process of Mathematical Induction.
b) by exposing them to an application of quasi-formal logics, which is often neglected after they have been taught Euclidean geometry.

On the other hand, the **teacher** has now a good reason to try every proposed Mathematical Induction assignment by him/herself first, just in case ...

A quick survey of available literature[2,3,4] has failed to reveal prior discussion of the topic presented in this paper.

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1 K. Appel and W. Haken, Sc. American, Vol. 237, No. 4, pages 108-121, October 1977. See also Sc. American Vol. 235, No. 4, page 57, October 1976.
2 Albert A. Blank, Mathematical Induction, in **Enrichment Mathematics for High School**, page 118, NCTM, Washington, D.C., 1963.
3 I.S. Sominskii, **The Method of Mathematical Induction**, Translation by L. Lange and E.E. Enochs, D.C. Heath, Boston, 1963.
4 Bevan K. Youse, **Mathematical Induction**, Prentice-Hall, Englewood Cliffs, N.J., 1964.



Dr. H. Laurence Ridge, Faculty of Education, University of Toronto

This issue's column contains solutions to "Your Number's Up" and the "Coin Reversal Puzzle". It also features a "Missing Address Puzzle" (which has become popular from the elementary school level on up), then closes with a related puzzle from recent correspondence.

The secret of "Your Number's Up"

The entries on the cards set out in the September issue are determined according to the binary numeral for each number. Order the cards by means of the upper left entry as shown in Figure 1.

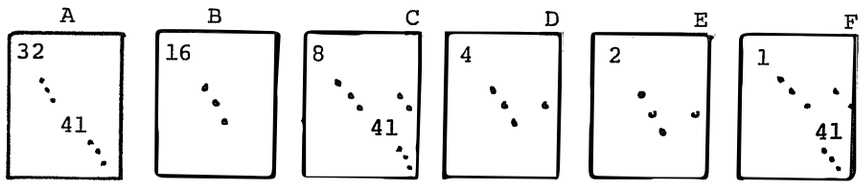


Figure 1

Each of the cards is associated with a power of 2:32, 16, 8, 4, 2, or 1. 41 occurs on cards A, C, and F because

$$\begin{aligned} 41 &= 32 + 8 + 1 \\ &= 2^5 + 2^3 + 2^0 \\ &= 101001_{\text{two}} \end{aligned}$$

If a number, n, is found on cards B, D, E, and F (and only those cards), then

$$\begin{aligned} n &= 16 + 4 + 2 + 1 \\ &= 23 \end{aligned}$$

Also

$$\begin{aligned} 23 &= 2^4 + 2^2 + 2^1 + 2^0 \\ &= 10111_{\text{two}} \end{aligned}$$

In general, the numbers on card F all have 1 in the 2^0 column in binary notation. You can see that all these numbers are odd. (Multiples of 2 must end in 0 in a base two system just as multiples of ten end in 0 in the denary numeration system). Each successive card contains only those numbers which have '1' in the corresponding column of their binary form.

Trevor Rice, in his **Mathematical Puzzles** and **Games** (published in Australia by the Batsford Publishing Company), claims that

> [t]he trick is made much more impressive by colouring the cards and memorizing which colour represents which [power of two]. The operator can then stand well out of reading distance and still be able to say which number was chosen. The numbers on the cards can then be mixed up, adding greater mystery to the trick (p. 57).

The Coin Reversal Puzzle - solved and generalized

The basic ten coin configuration in the September issue could be reversed by moving three of the coins as shown in Figures 2 and 3.

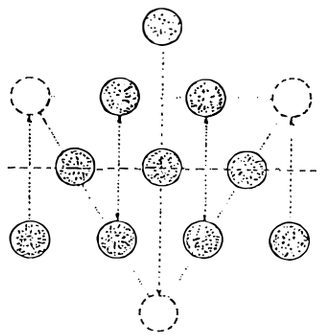


Figure 2: Reflection Strategy

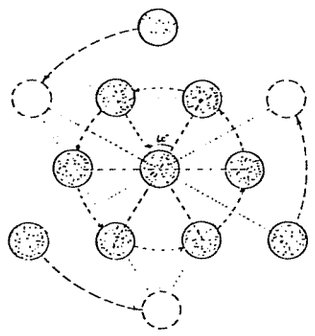
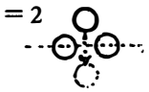


Figure 3: Rotation Strategy

To generalize the puzzle you were left with three considerations.

1. **Which of a reflection or rotation strategy is the more useful?**

Consider a few specific cases identified by the number of rows in the configuration.

n = 2  Reflection shows a single-coin move. (The position of each of the other two is invariant under the reflection).

 Rotation about the centroid of the (implied) triangle does the job but requires that all three coins be moved.

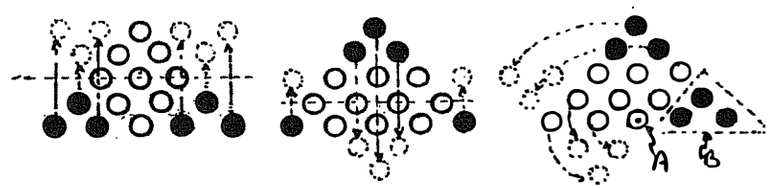
n = 3  Reflection makes reversal possible by moving only two coins.



Rotation of 60° about coin A requires only a two-coin move but the centre of rotation is no longer the centroid. Coins B and C would be moved.

n = 4 has been dealt with. Both approaches require a three coin move.

n = 5 The reflection at the left below involves a six-coin move. The reflection centre below involves only a five-coin move. Just any old reflecting line will not guarantee a minimum number of moves.



A rotation of 60° about coin A would do the job (as for n=3) but it would require a six-coin move as shown at the right above. In practice, coin cluster B would be moved to the lower position.

It appears that reflection in an appropriate row of coins consistantly produces the least number of moves.

2. How can the minimum number of moves be expressed in terms of triangular numbers?

a) Here is a chart of results.

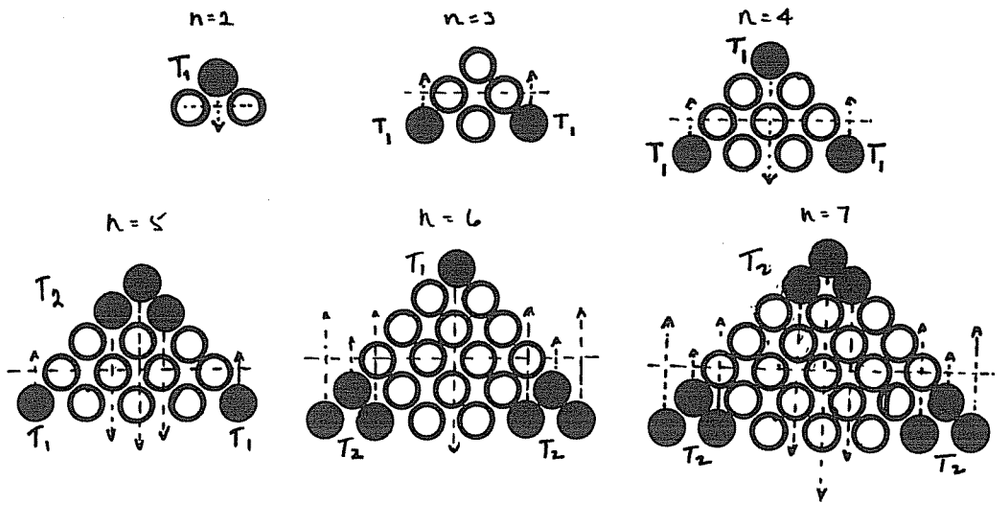
n	1	2	3	4	5	6	7	8	9	10
M_n	0	1	2	3	5	7	9	12	15	18
T_n	1	3	6	10	15	21	28	36	45	55

where n is the number of rows in the triangular configuration,
M_n is the minimum number of moves to reverse the configuration, and
T_n is the nth triangular number.

It appears that M_n can be found by dividing the corresponding triangular number by 3 and dropping the remainder. This relation can be expressed as
$M_n = [T_n/3]$ where [ ] means "greatest integer less than or equal to".

b) The relation of M_n to triangular numbers can be made more specific by noting the form of the number n.
For $n = 3k + 1$ for some integer k (ie, $n \equiv 1 \pmod 3$),
$M_n = (T_n - 1)/3$.
For $n \neq 3k + 1$ for some integer k (ie, $n \equiv 0$ or $n \equiv 2 \pmod 3$),
$M_n = T_n - 3$.

c) The following diagrams show the triangular number components of each M_n.



The following chart summarizes the results.

n	1	2	3	4	5	6	7	8	9	10
T_n	1	3	6	10	15	21	28	36	45	55
M_n	0	1	2	3	5	7	9	12	15	18
Triangular components of Mn	1(1)	2(1) + 2(0)	3(1) + 1(0)	1(3) + 2(1)	2(3) + 1(1)	3(3)	1(6) + 2(3)	2(6) + 1(3)	3(6)	

The chart can be generalized as follows:

$M_{3n+1} = 3T_n$ as in the case of $M_7 = M_{3(2)+1}$
$= 3T_2 = 3(3) = 9$

$M_{3n+2} = 2T_n + T_{n+1}$ as in the case of $M_8 = M_{3(2)+2}$
$= 2T_2 + T_3$
$= 2(3) + 6$
$= 12$

$M_{3n+3} = T_n + 2T_{n+1}$ as in the case of $M_9 = M_{3(2)+3}$
$= T_2 + 2T_3$
$= 3 + 2(6)$
$= 15$

These results can be consolidated as shown in part d).

d) Further observation of the triangular number components in the chart in part c) leads to the following relation.

If $n - 1 = 3m + p$ where $p = 1, 2,$ or 3
then $M_n = pT_{m+1} + (3 - p)T_m$

For example, when $n = 7$

$n - 1 = 6$
$= 3(1) + 3$

which means that

$m = 1$ and $p = 3$.

It follows that

$$M_7 = 3T_2 + (3 - 3)T_1$$
$$= 3T_2$$
$$= 3(3)$$
$$= 9$$

For another version of the above result, consider the form of n rather than the form of n-1.

If $n = 3m + p$ where $p = 0, 1,$ or 2

then $M_n = (p + 2)T_m + (1 - p)T_{m-1}$

For example, when $n = 7$

$7 = 3(2) + 1$

which means that

$m = 2$ and $p = 1$.

It follows that

$$M_7 = (1+2)T_2 + (1 - 1)\,T_1$$
$$= 3T_2$$ as before.

3. **Can the minimum conditions concluded inductively be proved deductively?**

Yes. The expressions for the minimum number of moves can be derived by applying a zero derivative condition. Such a solution is somewhat heavy for this column but is available on request.

What makes this puzzle rather attractive is the variety of levels on which it can be attacked and generalized.

Missing Address Puzzles

These are usually puzzles which involve number properties and logic in the guise of lost, missing, or forgotten addresses. The first example has been going the rounds of the conference circuit for the past couple of years, notably at the 1978 O.A.M.E. Annual Conference in Hamilton. There are many versions but the essentials are as follows.

Pat and Mike meet on the street.

"Pat", says Mike, "how's that fine family of yours?"

"Very well, thanks" replies Pat. "How old are your three children now?"

"It's hard to believe" explains Mike, "but the product of their ages is 36 and the sum of their ages is one more than the address on that white building across the street."

Later on, Pat calls Mike to say, "I still can't figure out your children's ages. Tell me something more."

"OK" agrees Mike, "the oldest girl takes zither lessons."

"That's better" exclaims Pat, "now I know their ages."

How old are Mike's children?

Solution.

This puzzle really is a classic because of it's clever red-herrings. (You might want to try solving it before reading further). The fact that the sum of the ages is related to an address is immaterial. What is important is that **Pat knows what the sum is** even though the reader doesn't. The other (double) distractor is the 'daughter' and 'zither lesson' reference. Neither the sex of the child nor the activity is of any importance. The operational word in that clue is 'oldest'. Its significance becomes clear when the possible triples of ages are considered.

Solution of the puzzle calls for careful analysis of 36 into all possible triples of factors. This can be done systematically by exhaustion starting with the smallest possible factors.

If 1 is the smallest factor then the other two must be factors of 36.

1, 1, 36 (a second marriage?)
1, 2, 18
1, 3, 12
1, 4, 9
1, 6, 6 (twins)

If 2 is the smallest factor then the other two must be factors of 18 (No factors of 1 will appear in this list).

2, 2, 9
2, 3, 6

If 3 is the smallest factor, the other two must be factors of 12. (No 1's or 2's will appear in this list).

3, 3, 4 (twins)

4, 6, 12, 18, and 36 are all covered in the previous triples.

Pat knows what the sum of the ages has to be, yet he has to call Mike for more information. This means that two of the triples must have the same sum. Indeed, only two of the triples do have the same sum:

$$1 + 6 + 6 = 13 \text{ and } 2 + 2 + 9 = 13.$$

Pat needs to know whether the twins are younger or older than the other child. Mike's reference to 'oldest' (daughter) identifies the ages as 2, 2, and 9 years.

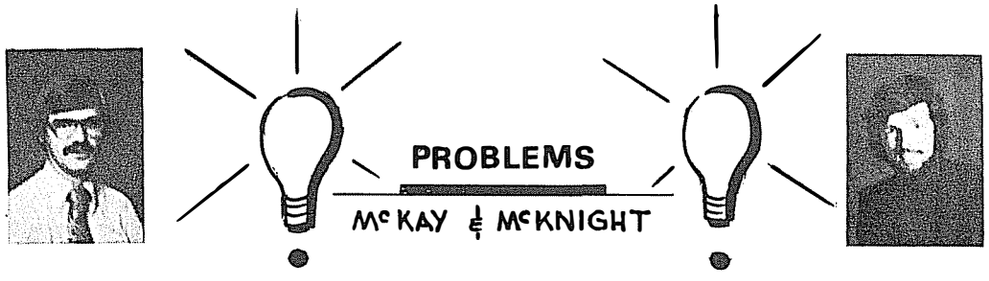
If you or your students found that one interesting here is another address-related puzzle courtesy of Rhode Island College in Providence, RI.

ALICE'S RESTAURANT

One day, Alice met one of her old beaus, Bob. "I heard that you own a restaurant now, and, as I recall, the address on President Avenue is somewhere between 13 and 1300. But don't tell me the address, just answer my questions and see if I can guess. First, is the address over 500?" Now Alice did not consider Bob to be one of her favorite beaus, so she wanted to forestall any visit he might plan; thus she lied. Bob, the undaunted mathematician, then asked whether the address was a perfect square. Again, Alice lied. Then, when asked whether the address was a perfect cube, Alice told the truth. "I need one more question," Bob said. "Is the second digit a one?" When he heard her answer, he thought he had the address, but alas he was, of course, wrong. He did not realize that Alice had been "toying with the truth", as it were. The reader, however, should be able to tell what the correct address is, no matter what Alice answered to question 4.

I'll give you Rod's solution and mine in the next issue. In the meantime, try it for yourself and on your classes.

Recreationally yours,
H.L.R.



I UP TO GRADE 10

18.2.1

At a dance there are 50 men and 38 women.

a) How many different couples could appear on the dance floor?
b) How many couples could appear if there are three men such that two certain women each refuse to dance with any in that set of three men?

18.2.2

If the sum of two numbers is 8 and the product of these two numbers is 10, find the sum of the squares of these numbers.

18.2.3

The manager, the accountant, the teller and the auditor at a local bank are Mr. Smith, Mr. Brown, Mr. Jones and Mr. Foster, but I can never remember who is who. I do know that:
1) Mr. Brown is taller than the auditor or the teller.
2) The manager lunches alone.
3) Mr. Jones plays bridge with Mr. Smith
4) The tallest of the four plays basketball.
5) Mr. Foster lunches with the auditor and the teller.
6) Mr. Smith is older than the auditor.
7) Mr. Brown plays no sports.
Determine which job each man performs?

18.2.4

In the local Sunday School picnic, men are asked to pay 50 cents for refreshments, women are asked to pay 30 cents and children only 1 cent. At the last picnic, the total attendance was 100. If everyone paid the correct change, and the total receipts were exactly $10, how many men, women and children attended?

II UP TO GRADE 13

18.2.5

In any triangle ABC
a) prove $\frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2} = \frac{\tan B}{\tan C}$
b) If a:b:c = 4:5:6, find tan A:tan B.
c) If tan B = 2tan C, prove $a^2 + 3c^2 = 3b^2$.

18.2.6

Four High School Soccer teams each played one game against each of the others. The scoring is in the following table:

	Goals For	Goals Against	Points
MACDONALD	13	17	4
LAURIER	17	13	3
CLARK	17	13	3
WESTVIEW	13	17	2

Two points were scored for a win and one for a tie. Each game produced the same number of goals but no two matches produced the same score. Of their 13 goals, Westview scored two against Clark. What was the result of the match between Westview and Laurier?

18.2.7

I have two little bags, of which the contents are identical. Each has in it four blue marbles, four red ones, and four yellow ones. I close my eyes and remove from Bag No. 1 enough marbles (but only just enough) to ensure that my selection includes two marbles at least of any one colour, and one marble at least of either of the other colours. These marbles I transfer to Bag No. 2. Now (again closing my eyes), I transfer from Bag No. 2 to Bag No. 1 enough marbles to ensure that, in Bag No. 1, there will at least be three marbles of each of the three colours. How many marbles will be left in Bag No. 2?

18.2.8

Prove, using the methods of Analytic Geometry, that the diagonals of a rhombus are perpendicular.

ARE YOU READY FOR A CHALLENGE?

18.2.9

It is 52 kilometres by road from Hamilton to Toronto. At 10 a.m. Peter Brown left Hamilton and cycled at a uniform pace, without stopping, to Toronto and back again. Some time later, Bill Storey left Toronto and drove his car at a uniform speed to Hamilton and back again. Storey passed Brown, on his outward journey, 15 kilometres from Toronto. He passed him again, on his return journey, 11 kilometres from Toronto. Storey was back in Toronto at 4:40 p.m. What time was it when Brown arrived back in Hamilton?

ONE PROBLEM FROM CRUX MATHEMATICORUM
Propose par Leo Sauve, College Algonquin

Determiner le couple (p,q) sachant p et q (a) sont **les** racines de l'equation $x^2 + px + q = 0$;
(b) sont chacun racine de l'equation $x^2 + px + q = 0$.
Find the numbers p and q if it is known that (a) they are **the** roots of the equation $x^2 + pz + q = 0$
(b) each is a root of the equation $x^2 + px + q = 0$.
Solution is found in Vol. 2, No. 1, January 1976*

*Schools which do not subscribe to CRUX MATH may obtain monthly issues for $1 each, and bound volumes 1 & 2 (1975, 1976), 3 (1977), 4 (1978), 5 (1979), each for $10 by writing to CRUX MATHEMATICORUM, Algonquin College, 200 Lees Avenue, Ottawa, Ontario, K1S 0C5.

BOOK REVIEWS

Dr. G. Vervoort, Lakehead University



If you see a child without a smile give him one of yours

1979 International Year of the Child

How long since you checked the mathematics collection in the school library? Or does your Dewey Decimal System skip the 510's entirely? If your school is typical, the mathematics shelf will either be bare or filled with surplus classroom texts from the fifties. Take a look at the due dates on the cards in the book pockets. Is the most recent stamp for Jan. 12, 1962?

Now, during the year of the child, have a critical go at your collection. How child centred is the mathematics section? How enjoyable? Never mind how profound and worthwhile the content. If it is not being read, it may as well not be there!

Elementary and intermediate teachers in need of suitable titles should look at **-MATHEMATICS LIBRARY - Elementary and Junior High school** by C.E. Hardgrove and H.F. Miller, published by N.C.T.M. This is a bibliography of some 650 mathematics books for the library. Thumbnail descriptions and grade levels from K - 9 for each book are included. For references to more advanced materials for the high school level, see the companion bibliography **The High School Mathematics Library**, also published by N.C.T.M.

Elementary Schools

Because this is the year of the child, we asked students in a local elementary school to write their own reviews. The following is based on their combined efforts.

A WRINKLE IN TIME. Madeline L'Engle. Scholastic Book Services. Toronto 1973. Paperback 256 p. Review compiled from book reports by grade 6 students of St. Thomas Aquinas School, Thunder Bay, Ont.

A Wrinkle in Time is about two children, Meg and Calvin, and Charles who is a psychic and five years old. They must travel many miles through space to save Meg's and Charlie's scientist father who is held captive by IT at the CENTRAL Central Intelligence Centre. The distances are so large, even at the speed of light that they take shortcuts whenever possible by wrinkling time/space and travelling in the 5th dimension. With the help of the spirits of 3 dead stars - Mrs. Whatsit, Mrs. Who, and Mrs. Which - they save their father but lose Charles to the control of IT. After fleeing to another planet, Meg returns alone to the Black Planet, defeats IT and saves her brother by her love.

There is not much mathematics in this book but it is fun and made me think about other dimensions.

Intermediate and Secondary Schools

THE MATHEMATICAL MAGPIE. Clifton Fadiman, 1962. Simon and Schuster: Rockefeller Center, 630 Fifth Avenue, New York 20, New York. Paperback. 300 pages. Reviewed by Mary Stapleton, Teacher, E.Q. Jennings School, Thunder Bay, Ont.

Clifton Fadimon does not consider himself a mathematician, but rather one who enjoys dabbling in mathematical fantasies. In **The Mathematical Magpie**, his intention is, ''to give the reader, by showing him what imaginative writers, humorists, and journalists have done with the subject, tantalizing sense of the wonder, the oddity, and the fantastical fun that pervades the most basic of the sciences''. In my view, he succeeds admirably.

The book is a sequel to **Fantasia Mathematica** which was written four years previous and with the same aim. The two books are similar, yet Fadiman has in only a few cases called upon the same authors. (Arthur C. Clarke, Lewis Carroll, H. Nearing, Jr. and Miles J. Breuer, M.D.).

The book is comprised of five sections. ''A Set of Imaginaries'' includes ten science fiction stories by such well known authors as Arthur C. Clarke and Issac Asimov. They deal with such topics as computers, basic arithmetic, the fourth dimension and topological problems.

''Comic Sections'' tell twelve amusing stories about such things as word problems, parallel lines and the mathematics of golf.

''Irregular Figures'' includes seven stories each of which concerns a rather strange principal character involved in a variety of topics from elementary arithmetic through probability and on to topological considerations.

''Simple Harmonic Motions'' is the shortest section comprised of a cartoon and two musical pieces, one about the Pythagorean Theorem called ''The Square of the Hypotenuse'' and the other topological in nature.

''Dividends and Remainders'' is a large collection of Apothems, Anecdotes, Nursery Rhymes, Poems, Cartoons and Limericks treating the whole gamut of mathematical ideas.

Fadiman introduces each story, poem, song, etc. with a few remarks about the author and the essence of the work. This is a valuable asset especially to the less experienced mathematics reader. The book is a great bed side book for one with mathematical curiosity rather than mathematical genius. It promises to entertain and amuse a wide audience. Some of the stories about computers or the fourth dimension as well as the songs and poems, are totally suitable for grade seven and eight. One of the stories that I have used in Grade 7 with great success is ''The Appendix and the Spectacles''. The story begins with a greedy and cantankerous banker calling in the note of a struggling medical student. Of course, the student can't continue his studies and has to settle for being a professor of mathematics. Fifteen years later the banker has an attack of acute appendicitis but is unwilling to have an operation. This is brought to the attention of the professor who has a solution. He will use his ''fourth dimension elevator'' which he has recently invented to remove the appendix. This is done for the cost of $2000 and all are happy. There is no scar, no incision, no apparent operation. However, later the banker becomes ill and upon being x-rayed, it is learned that a pair of spectacles is in his very large abdomen. He is outraged and tries to sue, but to no avail as there is no evidence of any operation having taken place. In desperation and still not wanting an operation, the banker agrees to finance poor students to the tune of $50,000 if the

professor will use his "fourth dimension elevator" again to remove the spectacles. The professor does the job and all ends well.

A teacher with a little imagination could do wonders with the material provided in the book to help math students enjoy the subject better. Some of the stories could provide a new insight into mathematics and science for students who have been bored with the subjects. Senior high school students could well appreciate the book in its entirety. Their background should enable them to understand and enjoy most topics. Perhaps it would wet their appetite for more mathematics. At the very least it should make some mathematical concepts more "human" for them.

The only objection one might raise is that two stories poke fun at women and their mathematical sense. "Needed: Feminine Math" and "The Wonderful World of Figures" portray women as less than capable in doing mathematical figuring. Perhaps we must remember that the book was written in 1962 rather than 1979:

Certainly this book deserves a place in every intermediate and high school library for the enjoyment of students and staff alike.

Mathematics for Technicians. Volume I. Frank Tabberer, Boston: Butterworths U.S.A. 19 Cummings Park, Woburn, Mass. 01801. 1978. 177 pp. Paperback. 0-408-00326-X. $6.95 (U.S.) Reviewed by William C. Farmer, Fort William Collegiate

Organization: Pages are well laid out with good shading to emphasize important parts. This reviewer found it somewhat disconcerting to have the chapters labelled as letters, though once one gets used to it, it is quite acceptable. There is no index. However, there is a one page table of contents. Moreover, the content of each chapter is outlined at the start of that chapter. A summary at the end of each chapter helps to bring out important ideas. Occasionally one questions why some important content is deleted such as inverse trig ratios in chapter G. The summary of chapter F is weak.

Content: The vocabulary differs from that used in the North American schools e.g. "A whole number is an integer and may be positive or negative." Empty terms which most teachers are trying to avoid occur repeatedly, e.g. cancelling and no explanation is included why one may cancel. Often the statements are open to misinterpretation. In summarizing the rules of signs for addition and subtraction it reads "like signs combine to a positive result, unlike signs combine to a negative result". The struggling student may interpret this to mean that the sum of 2 negative numbers is positive. What the author refers to is subtracting negative numbers. The worked examples are sufficient and with enough detail. I question the inclusion of some topics such as the slide rule in chapter B and introducing and applying the concept of limit to explain accuracy and validity.

Exercises: Throughout the text, exercises which relate directly to a topic just covered follow immediately after the presentation. At the end of each chapter there is comprehensive problem section using the concepts and skills of that chapter. Overall the exercises are just adequate. One wishes there was more practice with the basic ideas and more practical applications.

Evaluation: As a teacher I would not purchase this book as a classroom text. It may be somewhat useful as a source of problems on the teacher's library shelves.

Nonparametric Probability Density Estimation. R.A. Tapia and J.R. Thompson. Baltimore. John Hopkins University Press. (Available in Canada from Burns & MacEachern Ltd., 62 Railside Rd., Don Mills, Ont. M3A 1A6) 1978. 176 pp. ISBN 0-8018-2031-6. Reviewed by W.R. Allaway, Lakehead University, Thunder Bay, Ont.

Most teachers have at some time or another adjusted final marks so as to conform to some given curve. The curve that is typically used is the famous bell shape curve of the graph of the Normal Distribution. In this book the authors make the case that the normal distribution is not always the most suitable. A number of alternatives is suggested.

The text is written for the researcher who is confronted with a non-standard probability density (which is common) and who realizes that fact (which is uncommon). For a complete understanding

and appreciation of this work one should have a good undergraduate background in statistics. The text is not suitable for high school students. Even high school teachers with a flair for probability and statistics will find most of the proofs hard slugging. Better to save one's money for another text.

SOLUTIONS

SOLUTIONS TO PROBLEM SET 18.2

18.2.1 (a) $50 \times 38 = 1900$ couples; (b) $47 + 47 + 50 \times 36 = 1894$ couples

18.2.2 $x + y = 8$ $\quad (x+y)^2 = x^2 + 3xy + y^2$

$xy = 10$ $\quad x^2 + y^2 = (x+y)^2 - 2xy$

$= (8)^2 - 2(10)$

$= 44$ is the sum of the squares

18.2.3

	Audit	Acc	Man	Tell
SMITH	x	x	x	✓
BROWN	x	x	✓	x
FOSTER	x	✓	x	x
JONES	✓	x	x	x

Jones is the auditor,
Smith is the teller,
Brown is the manager, and
Foster is the accoutant.

18.2.4 Let x represent the number of men

Let y represent the number of women

Let z represent the number of children

$\therefore x + y + z = 100$ and $50x + 30y + z = 1000$

The number of children is divisible by 10 and must be at least 70 so that the money collected is under $10.

Try 80 children $\quad x + y = 20$

$50x + 30y = 920 \quad x = 16, \quad y = 4$

$\therefore$ there are 16 men, 4 women and 80 children.

18.2.5 (a) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ $\qquad \dfrac{c}{\sin C} = \dfrac{b}{\sin B}$

$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$ $\qquad \dfrac{\sin B}{\sin C} = \dfrac{b}{c}$

$$\frac{\tan B}{\tan C} = \frac{\frac{\sin B}{\cos B}}{\frac{\sin C}{\cos C}} = \frac{\sin B}{\sin C} \cdot \frac{\cos C}{\cos B} = \frac{b}{c} \cdot \frac{\frac{a^2+b^2-c^2}{2ab}}{\frac{a^2+c^2-b^2}{2ac}}$$

$$\frac{\tan B}{\tan C} = \frac{a^2+b^2-c^2}{a^2+c^2-b^2}$$

(b) from (a) $\dfrac{\tan A}{\tan B} = \dfrac{c^2 + a^2 - b^2}{c^2 + b^2 - a^2} = \dfrac{36 + 16 - 25}{36 + 25 - 16} = \dfrac{3}{5}$

$\therefore \tan A : \tan B = 3 : 5$

(c) $\tan B = 2 \tan c \Rightarrow \dfrac{\tan B}{\tan C} = 2$

from (a) $\dfrac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2} = 2$

$a^2 + b^2 - c^2 = 2a^2 + 2c^2 - 2b^2$

$\therefore a^2 + 3c^2 = 3b^2$

18.2.6 The only possible scores were 10-0, 9-1, 8-2, 7-3, 6-4, 5-5

CLARK vs. WESTVIEW 8-2

LAURIER vs. CLARK 5-5 tie
$\therefore$ MACDONALD vs. CLARK 6-4
MACDONALD vs. WESTVIEW 0-10
MACDONALD vs. LAURIER 7-3

and the score for WESTVIEW vs. LAURIER is 1 to 9.

18.2.7 Step 1, remove 5 from Bag No. 1

Marbles left

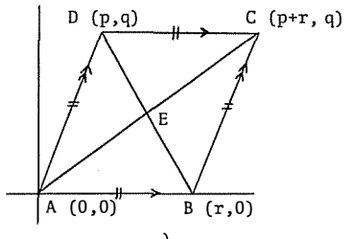
	a	b	c
Bag 1	0	3	4
	1	2	4
	1	3	3
	2	2	3

	a	b	c
Bag 2	8	5	4
	7	6	4
	7	5	5
	6	6	5

Looking at the second possibility, 12 must be removed to meet the requirements

$\therefore$ 5 marbles will be left in Bag No. 2.

18.2.8



Given: Rhombus ABCD

Prove: $mDB \cdot mAC = -1$

Proof: $AD = DC$

$\sqrt{p^2 + q^2} = r$

$p^2 + q^2 = r^2 \quad \therefore q^2 = r^2 - p^2$

$mDB = \frac{q}{p - r}$ and $mAC = \frac{q}{p + r}$ $\Rightarrow \quad mDB \cdot mAC = \frac{q}{p - r} \cdot \frac{q}{p + r} = \frac{q^2}{p^2 - r^2} = \frac{r^2 - p^2}{p^2 - r^2} = -1.$

Q.E.D.

18.2.9 Peter covers 26 km while Bill covered 78 km so Bill's speed is 3 times Peter's speed.
When Bill left Toronto, Peter was 20 km (15 + 5) distant;
Hence, if Peter's speed is x kph:

$\frac{32}{x} + \frac{104}{3x} = \frac{20}{3}$ (i.e., (time on bike) + (time in car) = 6:40)

$x = 10$ kph.

$\therefore$ Peter Brown was back in Hamilton by 8:24 p.m.

PLUS + + +
AN INFORMATION COLUMN

Ed Barbeau

The aim of this column is to provide communication among mathematics educators across Canada; to tell you about important events, research, curriculum development and items in the literature of national interest. It is to be inserted in mathematical education journals throughout the country.

Your help is needed in the gathering of information which may be of national interest. Ed has agreed to be the mailbox for Plus. Material should be sent to: *Prof. E.J. Barbeau,*
University College, B201
University of Toronto
Toronto, Ont. M5S 1A1

79.4 Canadian Mathematical Society Winter Meeting (Ottawa, Dec. 14-16, 1979)

Part of the programme of the C.M.S. Winter meeting will be devoted to matters of interest to secondary school teachers as well as university mathematicians. A keynote speaker will be Professor Peter Lancaster of the University of Calgary who will discuss the role of C.M.S. in promoting the teaching of mathematics and problem-solving in the applied sense in secondary schools. Professor Ed Williams of Memorial Univeristy will chair a session of six 15-minute talks on contests and other resources. Professor Ed Barbeau of the University of Toronto will chair a panel,

including a teacher and a professor, on the mathematician's role in assisting schools. All those events are planned for Saturday, December 15, 1979 in Ottawa. For further details, write Professor D.A. Dawson, C.M.S. Winter Meeting, Department of Mathematics, Carleton University, Ottawa, Ontario K1S 5B6.

79.5 Samuel Beatty Mathematics Essay Contest

The Beatty Essay Contest is open to Canadian elementary and secondary school students. Sponsored by the Samuel Beatty Fund, established by University of Toronto graduates of mathematics and physics, the first contest was held in 1978-79 with a first prize of $200. going to Doug Bonn, a grade 13 student of Norther C.I. & V.S. in Sarnia, Ontario. Topics and rules for the second contest are obtainable from E.J. Barbeau, Secretary, Samuel Beatty Fund, University College, B201, University of Toronto, Toronto, Ontario M5S 1A1. Deadline for submissions: March 31, 1979.

79.4 4th International Conference on Mathematical Education (Berkeley, August 10-16, 1980).

ICME will be held at Berkeley, California from August 10 to 16, 1980. The scientific programme examines problems in education at all levels and for every type of learner. Topics of emphasis: universal primary education, research, technology, applications, profession of teaching, relationship of languages and mathematics. For further information write ICME IV, Mathematics Department, University of California, Berkeley, BA. 94720 U.S.A. Specify your language (English, Espanol, Francaise) and level of interest (elementary, secondary, college, teacher training, supervision, university).

79.7 Canadian Mathematics Education Study Group

In June, 1979, the C.M.E.S.G. met for the third time at Kingston, Ontario--to discuss preservice and inservice teacher education, mathematization and research in hand. The group has become more formally organized with an advisory committee and a secretary, Professor Joel Hillel, Department of Mathematics, Concordia University--S.G.W., Montreal, Quebec H3G 1M8, to whom requests for a report on the proceddings should be sent.

CHAPTER ITEMS

CHAIRMAN'S MESSAGE

A number of my close mathematics associates happen to enjoy the past-time of 'angling' in their leisure time. There is a saying from this special breed that goes like this ... "angling may be said to be so like the mathematics, that it can never be fully learnt" (from the Compleat Angler). For many of us therein lies the beauty of the 'Queen of the Sciences' but we never forget that it is this very nature of mathematics that often makes its study a frustrating experience for many others.

The study of mathematics, like so many things, is a matter of perspective. When you finally reach the summit after many trying, even exhausting, moments you are usually recompensed for your troubles. For the mathematically inclined, it is usually discovered that although your solution is correct, it has many implications that entices you to tread on new ground (for them, at least). But it is this that so frustrates those less enamoured with the discipline of mathematics. ...

This year COMA will be dealing with many items of concern including the mathematics examining board, grade XIII committee, new directions in primary-junior level, new (?) senior guidelines, implementing the intermediate guidelines and the list goes on. ...

We are all privileged to have Maurice Poirier, a much respected mathematics educator, as our editor. Moe's leadership skills in teaching and curriculum are well known and I thank him for adding the responsibility of this newsletter to his already full schedule. (Ed. note: He made me publish this!)

You are invited to our first program on November 5. Details are available through your constituency representative. Support your provincial mathematics organization (O.A.M.E.) and its local chapter (COMA). Perhaps you could share this newsletter with a colleague. I trust the ideas in this issue will stimulate discussion.

Bruce E. Armstrong

.....PEOPLE, PLACES ..... OU SONT-ILS? .....

Peter Ulrich, well-known Ottawa area mathematics teacher, is in his second year as Mathematics Consultant for the Ottawa Board of Education a post filled by Weldon McLeod for the previous four years (Weldon is now back in the classroom at Vincent Massey Public School).

Jacques Theoret et de retour au conseil scolaire d'Ottawa, comme conseiller pedagogique en mathematiques au secondaire, a mi-temps. Jacques consacre l'autre moitie de son temps aux activites des ''equipes volantes'' du Ministere de l'education.

Always a ''mover'', Bruce Topping is now admin. head at Earl of March Secondary in Kanata.

Christine Hubbard spent the summer in Africa, and promised she would be reporting her experiences in this publication some time this year.

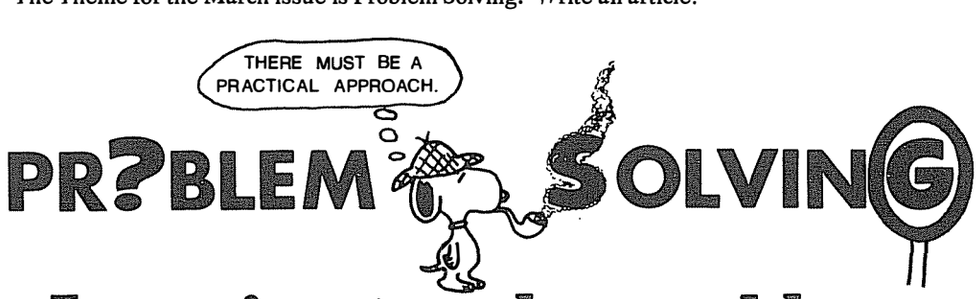
Francois Dupont (faculte d'education) est en conge sabbatique cette annee: il poursuit des etudes en andragogie (...? demandez-lui ce que c'est...)

Fall Conferences Held:

Golden Section	Oct. 2 in St. Catherines Nov. 9 in Hamilton
G.V.M.A.	Nov. 1 in Waterloo
N.O.M.A.	Sept. 28-29 in Sudbury
Pine Ridge	Nov. 8 in Peterborough
Q.S.L.M.A.	Nov. 8 in Kingston
S.E.Y.M.A.	Oct. 29 in East York
T.O.R.N.Y.	Nov. 21

The contest winner was Michelle Zimmer of London.

The Theme for the March issue is Problem Solving. Write an article.



Learning to solve problems is the principal reason for studying Mathematics.

The theme for the September issue is **''Education of the Exceptional Child''.**

November: The sublimest song to be heard on earth is the lisping of the human soul on the lips of children.
- Victor Hugo.

A REPORT FROM OAME/AOEM RESEARCH COMMITTEE TO COUNCIL

13 October 1979

1 The Committee noted that its effort is and will be handicapped by lack of membership and hopes that some Council members will consider whether Council might be better served by their membership on this Committee than on one of the others.

EFFECT OF XIII CALCULUS

2 There is some question on the effect of the Grade XIII calculus course on the subsequent performance and dropout rate in university calculus. Consequently, the Mathematics Department of Lakehead University has initiated a study to obtain statistical information on such effects. Initial information has been collected to be correlated with performance and dropout rate at the end of the year.

3 The Committee decided to search the literature for relevant information on the matter. Depending on the outcome of the search and of the investigation at Lakehead, a questionnaire for circulation to mathematics departments of universities and colleges may be prepared.

ADEQUACY OF PREPARATORY COURSES

4 Experience at Brock University and at Algonquin College suggests doubt about the adequacy of preparatory courses for students wanting to upgrade their mathematics to improve their chances of success in mathematically dependent academic programmes.

5 Investigation of the subject has been initiated at the University of Western Ontario.

6 The Committee decided to find out more about the matter.

EDUCATIONAL POLICY

7 Research is related to educational policy, and it is reasonable to suppose that the Council of a professional provincial body such as OAME should have some competence in putting forward proposals concerning mathematical educational policy as well as proposals for mathematical educational research.

8 The Committee invites members of O.A.M.E./A.O.E.M. to forward concerns which might imply the need for educational research to one or other of the following:
David Handley Paul Sherk Fred Maskell

IMPRESSIONS FROM THE GAZETTE QUESTIONNAIRE

Rudy V. Neufeld
Sir Frederick Banting S.S.
London, Ont.

Thank you for returning the questionnaire which was enclosed with the September issue of the Gazette. Let me briefly give my overall impressions from the responses received. It is gratifying to note that reactions were received from the various panels.

People seem to be using the Gazette for reference material and for 'Monday Morning' classroom material. We were presently surprised to see that quite a number of people are reprinting from the Gazette. It should be noted that, as of the last issue, O.A.ME./A.E.O.M has been printed at the bottom of each page. The organization gets some credit in the reprints.

Gazette readers are interested in O.A.M.E. news. The last part of the journal (meeting dates, council organization, chapter news) is very well received. Chapter News could be improved if chapter members and representatives would notify Bob Smith of upcoming events well in advance of the submission date.

The Gazette is a good journal but it could be improved. We need your suggestions for first rate authors. We need your help in proofreading articles. We need to know the Chapter news. Please contribute to your professional journal.

Editors Note: The majority of persons responding want to see more author's pictures. "It would be nice to see who is behind each article." "It livens it up." So if you are sending us an article, include a photograph to be printed. "More puzzles of a mature nature to be used for Grade 13 courses," was another common comment.

As a result of the above two requests, you will find more pictures per issue and a Gr. 13 contest type problem in each issue. Fred Maskell has agreed to pick one of the best from Crux Mathematicorum and it will be added to the problems section.

It was good to hear your feelings about O.M.G. "Well done. Keep at It. It is nice to see that there are concerned professionals who are trying to improve mathematics education. I look forward to receiving the Gazette. I have found more than one article per issue for usage. I'm up to p.52 in December issue and already have 9 articles marked for future reference. It is necessary to maintain an effective Canadian Math journal which is capable of stimulating independent development of theory and practice which can stand on an equal par with other national journals. It provides an excellent source of Professional Development Material. O.M.G. has something for all levels. I do feel the technical quality could be improved. My last issue was a trapezoidol shape. Print is too small. Improve the proofreading".

For writers

Topics suggested for future articles were:

Transformations and Vector Geometry Approaches.
Transformation Geometry - the 80 - 81 range.
The Emphasis of Euclidean Geometry (Grade 10,11).
A Rationale for Teaching Motion Geometry.
G Level Mathematics.
Strategies to Help Parents Understand Children and Their Mathematics.
The Future of Mathematics and the Hand-Calculator.
Integrating Calculators and Computers into the Curriculum.
The Problem Solving Aspect of So-called Recreational Mathematics.
Problem Solving - Grades 4-8.
Microcomputers and Software in Mathematics.
Applications of Mathematics Studied at the Intermediate Level.
How to Run a Successful Mathematic Contest Team.
Slow Learner.
The Place of Observation in Primary Education.

Take up your pen. Choose a topic. Write an article.

MEMBERSHIP NOTICE

The history of OAME is a story of growth and development. Our achievements are many and our involvement in mathematics education can only be praised. Unfortunately, the cost of achievement and involvement is being inflated daily.

Your Executive, after considerable study and discussion have concluded that OAME annual fees must be increased to $15, as of May 1, 1980.

Many factors have contributed to put our Association into a position where a fee increase is unavoidable. For clarification, some of the major issues are listed here:

(i) In a time span from 1973 to 1980 with an inflation rate of 8% annually, your fee should be $17.
(ii) Postage on the Gazette has now increased to 39c.
(iii) Each issue of the Abacus cost about $250 to produce and 15c per member to mail.
(iv) OAME now sends member refunds to 9 chapter organizations.
(v) Members now receive certificates, seals, and membership cards.

The Executive believes that each and every member of OAME will support this fee increase and continue to receive the benefits of a unique professional association.

President
Alex Norrie

AT THE LAST MINUTE

Kingston,
Ontario Nov. 2/79

Dear Prof. Smith,

I had the fortunate experience of attending the Eighth Annual OAME Elementary Leadership Conference held recently in Toronto. Dave McPhail certainly presented a super program.

The seminars entitled "Monday Morning Specials" were interesting, presenting masses of materials and fresh ideas. Intended for participants to take leadership roles by passing the information along to colleagues, these seminars covered a variety of teaching levels, areas of math, and methodology. Unfortunately, they were not long enough. Presenters, all leaders in their own right, did a superb job.

Guest speakers were both interesting, controversial, and well-versed in their particular fields. Thanks need be extended to these professionals as well as to the company representatives who oprovided a large display of new and varied materials pertaining to this field.
Every aspect of the conference was obviously well-organized and a tribute to the high standards established by the O.A.M.E. Ample time was also set aside for the mixing of participants. I gained many new friends and numerous ideas for my classroom.

Congratulations are extended especially to Mona and Morley McGregor for their special efforts with registration, accommodation, meals, etc.

May I close in saying that I highly recommend this conference, held yearly the third week of October, to all math leaders and hope that all Boards of Education could send representatives. You can bet that I'll be there next year if at all possible.

Sincerely,
Russell R. Davies



December: Not believe in Santa Claus? You might as well not believe in fairies... No Santa Claus! Thank God, he lives, and he lives forever.
- Francis Pheraellus Church.

