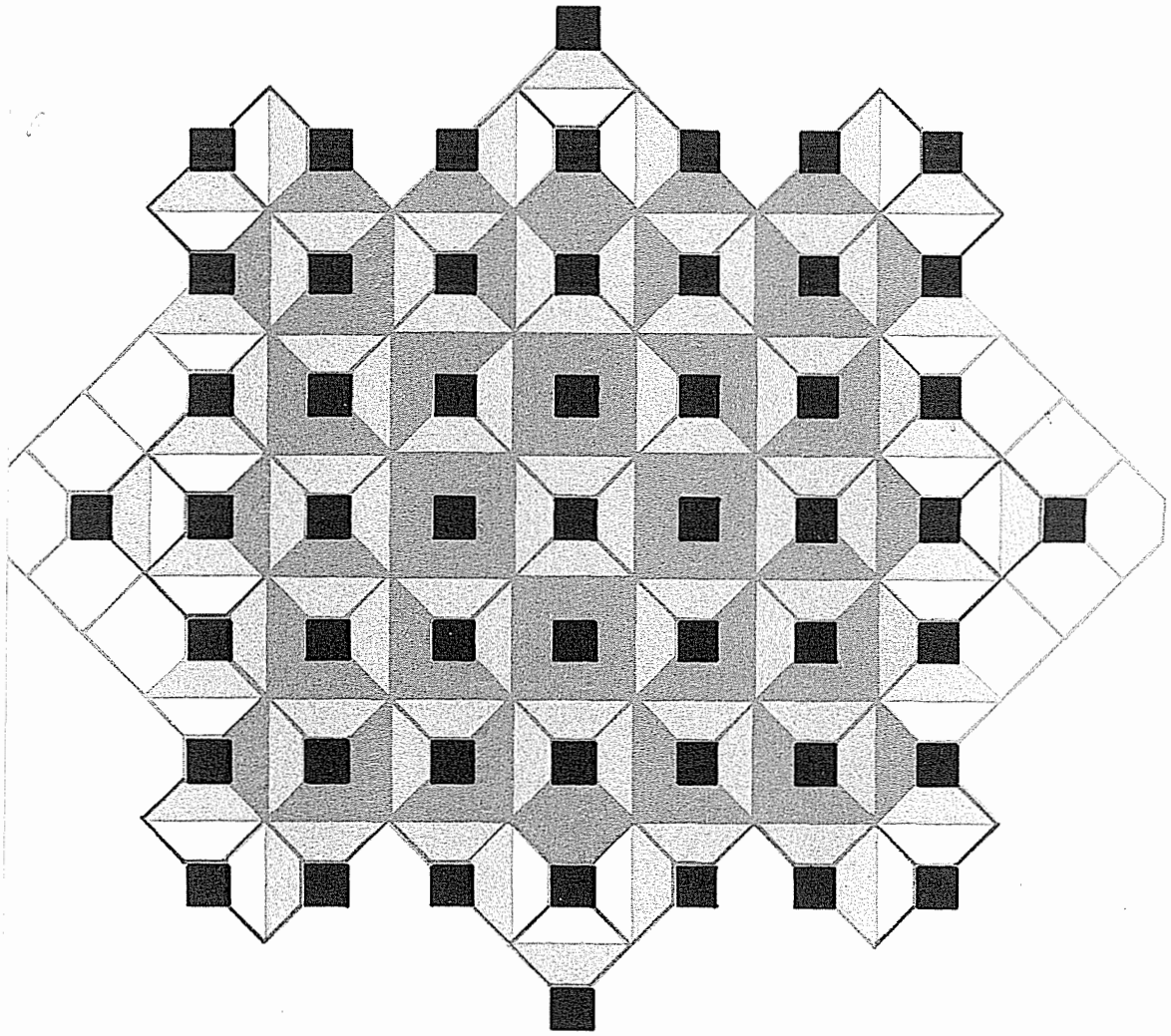
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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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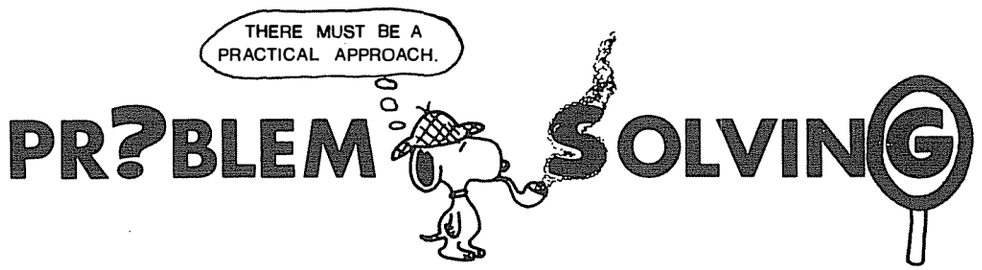
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CONTENTS

Editorial 2
Letters to the Editor 4
At What Price Validation? *by Sam Pravica* 9
Some Alternate Paths to the Goal of Problem Solving *by Angela Walsh* 10
You Can Teach Problem Solving *by John F. LeBlanc* 15
Problem-Solving in the Primary Grades *by Ann Lukasevich* 21
Coordinator's Corner *by Brock Rachar* 24
A New Look into Problem Solving *by Chester D. Carlow* 27
The Language of Verbal Problems in Mathematics *by Marcel Danesi* 28
Some Ideas for Teaching "Problem Solving" *by Rudy Neufeld* 33
Problem Solving *by Charles D. Gallant* 36
Interview - Taped Talks with Bob Smith 39
Algebra - Imagery - Problem Solving *by Dave Alexander* 42
Canadian Mortgages *by Richard D. Wink* 45
Games Bring Life to Math *by Mariano Rodrigues* 46
Ressources pedagogiques en mathematiques en francais *par Cleo Beaudry* 49
Some Problems from Grade 10 Mathematics *by Prof. E.J. Barbeau* 51
Sea and Air Navigation *by David R. Campbell* 53
I Didn't Know That Tricks - Games - Puzzles *by Dr. H. Laurence Ridge* 60
Problems *by McKay & McKnight* 65
Book Reviews *by Dr. S. Vervont* 66
Plus + + + An Information Column *by Ed Barbeau* 68
Chapter Affiliations 69
Chapter Items 70
Future Meetings 72

Learning to solve problems is the principal reason for studying Mathematics.

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EDITORIAL

Today is Saturday, February 16th. It is the first day of the Chinese New Year, the Year of the Monkey - 4678. As I write this editorial, I am trying to make sense of the issues inherent in the theme - Problem Solving and to make sense of the issues confronting all Canadians, for Monday is election day. By the time you read this, the decision of the Canadian people will be known. I have just read that (it is a Chinese custom) you must not use knife or scissors on this first day of the New Year. The consequence for doing so might be the loss of good luck for the coming year. I must be careful not to cut and paste as I work today for I would suggest that we need all the luck that is possible. We are going to need not only persuasive, determined, able leaders to enable Canadians to set achievable goals and to do everything possible to achieve them, but also the laws which will wisely deal with the problems of energy, economic development and unity.

Problem solving needs not only alternative strategies but also teachers to present effective programs. No one doubts the key position of Problem Solving in the school curriculum. Your response has required that I set the March 1981 issue aside for Problem Solving Part II. Let us hear your reactions to the articles or send an article which adds to the debate.

James and James in their **Mathematical Dictionary** define a problem as "a question for solution; a matter for examination." For a problem statement to be a problem **for some particular learner**, the problem must be unfamiliar in the sense that the learner has no "standard procedure." If the learner has a standard procedure, the so-called problem becomes merely an exercise. Perhaps the purpose of the typical text-book problem is to reinforce the pupil's understanding of a concept or to use a skill learned earlier. Helping the child to acquire a set of processes and to know when to apply these, is a far more complicated and less well defined task than teaching the other facets of the mathematics program. A higher level of cognitive development is also required. A variety of strategies are suggested in this issue.

Creative problems do not fit directly into specific mathematic topics. These are often found in the corners of textbook pages or as Mind Benders. They require originality, intuition, conjecture - the things the mathematician uses all the time except when in the classroom. Some good examples follow. Try some. Actual, not vicarious participation, is an indispensable part of your experience. Do have a pencil in hand while reading!

Most problem solvers, young or old, experience a sense of achievement when they discover a successful attack on a problem which has resisted their initial efforts. Many people in fact are fascinated with mathematical problems and enjoy puzzling over them. These problems are regular features in newspapers and journals such as "Fun With Figures" in the **Globe and Mail** and the "Mathematical Games" section in **Scientific American**. For some of our students there is an aspect of shear enjoyment in solving problems. Consider a Math Club for your school and do encourage your students to enter the many Math Contests that are now available.[1]

Recent studies have shown that students and graduates have serious deficiencies in their problem solving abilities. This may be due in part to the fact that problem-solving plays only a minor part in mathematics instruction. Yet, as Chester Carlow's article suggests, problem solving has many features in common with systematic inquiry and decision making in other fields. Certainly, such attributes as inquisitiveness and persistence must be nurtured over long periods of time and in more than one subject area.

I have raised but a few of the issues inherent in this theme. I look forward to examining a few others in the editorial one year hence.

Bob Smith

1. "Other Math Competitions," **Ontario Mathematics Gazette**. Vol. 17 No. 3, March 1979, page 50.

Across The Editor's Desk

The following items are worth noting:

1. Manuscripts are invited by NCTM for the "How to ..." series. Three topics have been selected for top priority:

- How to Deal with Reading Difficulties.
- How to Cope with Math Anxiety.
- How to use Games Effectively.

2. The editorial committee of the 1982 NCTM yearbook on "Mathematics for Grades 5 - 9" is requesting articles for possible inclusion.

3. The Fourth International Congress Mathematical Education is being held at Berkeley, California, August 10-16, 1980. Plenary addresses are:
Hans Freudenthal, The Netherlands. Topic: Major Problems of Mathematics Education.
Hua Loo-King, The People's Republic of China. Topic: Applications of Mathematics and Mathematics Education.
Seymour Pappert, The United States. Topic: Technology and Mathematics Education.
Hermine Sinclair, Switzerland. Topic: Language and Mathematics Learning.
For detailed information, write to:
ICME IV Registration
The Haley Corporation

4. The Diagnostic and Prescriptive Mathematics Conference is being held at the Hyatt Regency Hotel, Vancouver, April 13 - 15. Plenary addresses are:
Vincent Glennon - Neuropsychology and the Instructional Psychology of Mathematics.
Marcel Kinsbourne M.D. - The Brain Basis of Developing Mathematical Skills.
Richard Skemp - Mathematics as an Activity of Intelligence.
Gerald Rising - Foundations of Diagnosis: A Critique.
For detailed information write to:
Registrations,
The University of British Columbia,
Centre for Continuing Education

5. The 58th Annual Meeting of NCTM is being held at The Seattle Centre, Seattle, Washington, April 16 - 19, 1980. The theme is The Mathematics Curriculum of the 1980's.

6. The 18th Annual Summer Workshop of AMTNYS will be held at Niagara University, August 10 -14, 1980. Workshop themes will be developed around six strands: Art, Microcomputers, Life Skills, Problem Solving Enrichment Mathematics and New Curricula.
For detailed information write to:
Sr. John Francis Gilman,
1980 AMTNYS Summer Workshop
Department of Mathematics
Niagara University, N.Y. 14109 U.S.A.

7. The 1st Annual one-week non-credit short courses to be given at the University of Western Ontario, July 14 - 18 and 21 - 25, 1980. David S. Fielker Director of Abbey Woods Math Centre, London, England, editor of Mathematics Teaching and Jim MacLean, Education Officer, Ministry of Education, Toronto will tailor the week's investigations to topics of concern to the participants. Children will be available for demonstration teaching and interaction with participating teachers. Open to teachers in the Pr. Jr. and Int. Divisions.
For detailed information write to:
The Co-ordinator,
Prof. R.S. Smith, Faculty of Education,
c/o Elborn College, U.W.O.

8. The Twelfth Canadian Mathematical Olympiad will be written on Wednesday, April 30, 1980 at schools throughout Canada. Copies of nomination forms can be obtained from:
The Olympiad Committee,
Department of Mathematics and Statistics,
Memorial University of Newfoundland,

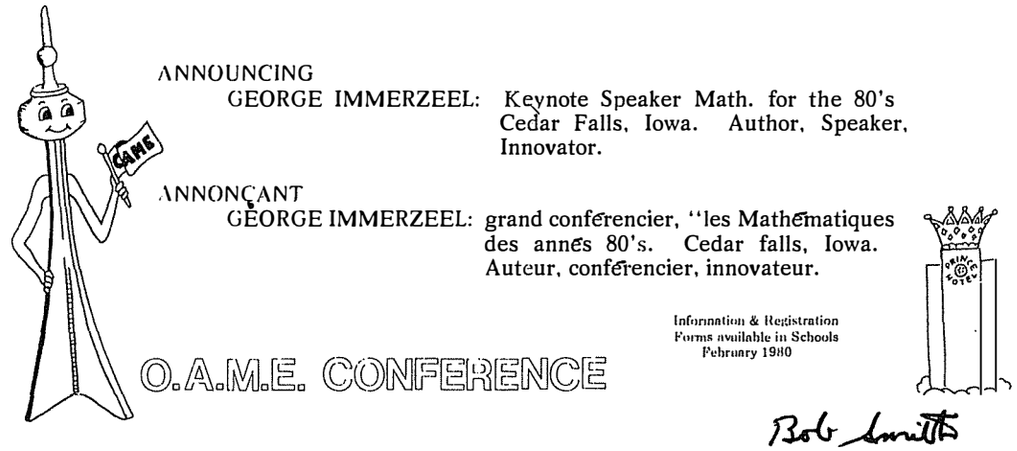
St. John's, Newfoundland A1B 3X6
The Ontario chairman is Prof. K.D. Fryer, Department of Mathematics, University of Waterloo, Ontario, N2L 3G1 and the Ontario quota is 50. Joe Poirer recommends the following two publications as a "must" for competitors and coaches.

a) The Canadian Mathematical Olympiads
Editors: E.J. Barbeau, W. Moser - $2.50

b) 1001 Problems in High School Mathematics
Book I Problems 1 - 100, Solutions 1 - 50
Book II Problems 51 - 200, Solutions 51 - 150
Book III Problems 151 - 300, Solutions 151 - 250
Cost $1.80 each, including handling charges.
Please address to:
Canadian Mathematical Society,
3421 Drummond Street, Suite 15
Montreal, P.Q. H3G 1X7, Canada

You will note a new format for many of the articles. Please let us know your reaction. Is the split column more readable?

10. Last but not least let me remind you of our own conference. The Seventh Annual Conference of OAME/AOEM will be held at the Prince Hotel, Toronto, May 9 - 11, 1980. Lorna Morrow and Neil Williamson have a great program for us to enjoy and the committee has assured me that it will be an outstanding convention. I am not sure about the jogging facilities, but the Prince Hotel has other enticements. Do join us there.



LETTERS TO THE EDITOR

The University of British Columbia
December 5, 1979

Dear Bob:

I have only a few comments to make (about to the letter you enclosed written by Ralph Staal). I don't think a lengthy debate in your column would be worthwhile. My chief intent in my article "The Unstable Arithmetic Curriculum . . ." was to state the appropriate principles, properties and generalizations in the simplest possible language in order to reduce what I have considered to be a comunication barrier set up between mathematicians and elementary teachers. In doing this, one runs the risk of over simplifying to the point of producing conceptual imprecision, if not error. I take Professor Staal's criticism of my discussion of the null set, zero and zero products to be a case

in point. Professor Staal's criticism is valid and his suggestion is an improvement over my exposition of the matter. I am grateful for his comments.

My response to his second criticism relating to the notion of "renaming" numbers is somewhat less enthusiastic. I doubt that increased precision or rigor of expression will really make the process any more clear and may perhaps only serve to increase communication barriers. In a sense our abstract numerical symbols are ways of "naming" - representing - concrete quantities. Also by using the notion of the identity symbol one could argue that one can "name" or represent a quantity several ways. In any case I hold the general process to be an important tool to be learned early in a child's mathematical experience and perhaps is best learned in the simplest language possible. As long as no serious conceptual or logical damage is done, the precision and rigor can develop later. I therefore do not take Professor Staal's comments about "re-naming" seven quite so seriously, though I concede the essential correctness of his argument.

I trust this is adequate to amplify my position on the matters raised.

Very truly yours,

Roland F. Gray,
Professor,
Faculty of Education.

Centre for Science Education
January 10, 1980.

Dear Bob:

Thank you for your letter. Been madly busy (naturally) since we've at last got the book for teachers finished - its called "Children's Understanding of Mathematics, 11-16" to be published by John Murray in the summer of 1980. The test papers (all but two) are with the National Foundation for Educational Research and due out this year. If that is not enough, we have almost finished a research monograph describing the work in very much greater detail, with lots of data!

I have a new project 'Strategies and Errors' starting January 1st but unfortunately Dietmar is not employed on it, the funding body wanted somebody with a more "psychological" background and appointed Lesley Booth who used to work here at CSMS with us.

Happy New Year,

Yours Sincerely,
DR. K. HART

The Editor,

Pickering, Ontario
January 14, 1980

Dear Bob:

Keep up the great work The Gazette. The last issue was the best yet.

Thanks.

Neil Williamson

University of Toronto
January 31, 1980

Dear Bob:

I think the O.M.G. has had some stimulating articles over the past few years - keep up the good work!

Dave Alexander

W.T. Herman Secondary School
January 4, 1980

Dear Sir:

This is in reference to "A Project to Improve Teaching Practices" written by Mr. N. MacEwan and associates. I am one of the associates who worked on this project.

I am not taking offence to the article written by Mr. Doug Henderson, but rather I am attempting to clarify some of the ideas that were in our approach.

1. We do not force our marks to coincide with "our model" at all. The "model" is merely used for comparison purposes.

2. The "model" was not constructed solely from theoretical mathematics but arose from our own experience of test results of the past decade.

3. We are not suggesting therefore that everyone use "our model". You might have "your model" or "no model". We are suggesting, however that it is important to examine your test results and use them for the same reasons that Mr. Henderson outlined. For us, the "model" enabled us to do just that.

4. We are not expecting test results to be the same as exam results. However, we frown upon results of tests and exams that differ greatly.

To conclude I would like to say that "our model" and "our assumptions" are not the main purposes of the project. Rather, we are attempting to get teachers to examine their teaching and testing procedures for the ultimate benefit of both the students and teachers.

Yours Sincerely,
L. Shapiro
Math Department
Herman S.S.

Algonquin College
Rideau Campus, Ottawa, Ontario
January 29, 1980

Dear Bob:

I agree entirely with Doug Henderson on page 4 of the December 1979 issue when he observes that "Mr. MacEwan and members of his department are to be commended for identifying the problems associated with assigning marks to students, for having the dedication to do something about it, and for having the courage and forthrightness to tell the rest of the professional community what they have done."

Like Doug, I too have fundamental reservations. However, without a starting point such as has been provided, dialogue is difficult. So I am grateful that opportunity has been provided in the columns of the Gazette for discussion on this important but troublesome matter.

Doug has pointed out on page 28 that many commercially prepared tests are designed to approximate the normal distribution, presumably for good reasons.

It seems to me that MacEwan et al also designed their tests to help them approximate the normal distribution on the assumption that what their tests measure is normally distributed. Doug has questioned the validity of this assumption. I believe we should question its desirability, particularly when it is associated in the conventional way with the percentage scale.

Without necessarily implying that such tests should never be used, one should at least consider the implications of customarily using them. With a pass mark and a median somewhere near the middle of the percentage scale, it is evident that in a general sort of way a considerable proportion of students achieve about half or less of what it is possible to achieve.

What happens next? Is there an improvement in achievement in (say) simple arithmetic and algebra at the primary and intermediate levels as students progress through secondary levels?

I do not know the answer to this question. I do know as a post-secondary teacher that by far the most troublesome type of problems that students encounter in pre-calculus, calculus, statistics, and mathematics applications is unreliability in simple arithmetic and algebra - including addition and subtraction with or without pocket calculators!

All too often, students with high hopes for successful post-secondary mathematically dependent careers are simply unable to cope with the simple arithmetic and algebraic skills which are foundations for a host of mathematics - dependent courses as well as mathematics courses.

Ought not there to come a time when students should score 80, 85, 90% on tests of basic skills?

Is it unreasonable to expect that sooner or later students might become sufficiently reliable to be able to say, "My answer is right, and I know it is right"?

If in fact simple basic skills can be identified which are pre-requisite for further progress in mathematics and mathematically dependent courses, then it would seem that tests of such skills with a high threshhold are more appropriate than those designed to fit a normal distribution - at least for students choosing courses dependent upon elementary mathematics.

Sincerely,
(FGB Maskell)

Stratford Northwestern Secondary School
Stratford, Ontario
January 4, 1980

The
Ontario Mathematics Gazette.

Dear Sir:

I have read the article on evaluation procedures by N. MacEwan in the September issue of the GAZETTE and also the reply by D. Henderson in the December issue.

I also had much disagreement with the views held by Mr. MacEwan and I am almost totally in agreement with Mr. Henderson.

I would disagree with Mr. Henderson only on his claim that student results on examinations should be higher than those on term tests. It should be remembered than an examination covers materials

for a period of three or four months, or more; whereas term tests cover material for a much briefer period of time.

For many students, expecially those in the general level, the amount of recall is inversely proportional to the length of time since a topic was covered.

Ron Marcy,
Head of Mathematics

Gordon Graydon Memorial Secondary School
January 22, 1980

The Editor
Ontario Mathematics Gazette

Dear Sir:

This is with regard to the article ''The Use of Trigonometry to Solve Equations'' by T.J. Racey. I would like to point out that his third equation in table 1. should read

$$Tm = Ts\sqrt{1 - \frac{v^2}{c^2}}$$

Sincerely yours,
J.V. Mussen
Asst. Head of Math

Faculty of Education
Althouse College
January 4, 1980

Letter to the Editor

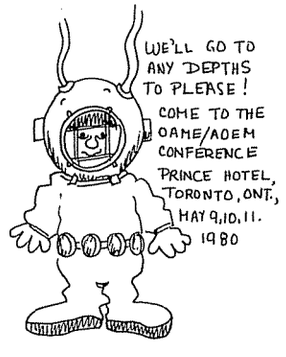
Dear Bob:

I would like to quell rumours and confirm that Mike McKenna has resigned from the Ministry to occupy the position of Superintendent of Curriculum with the Sault Ste. Marie Board of Education.

His move has left me with feelings of ambivalence, happy for him in his new venture, yet saddened by the loss of his dedicated leadership in mathematics education. I am sure that I speak for those who have worked with Mike in saying that his unselfish efforts in working to improve mathematics education in the province will be sorely missed.

The Sault Ste. Marie Board of Education has made a wise selection and is very fortunate in acquiring an individual of Mike's caliber.

Sincerely,
Sam Pravica





AT WHAT PRICE VALIDATION?

Sam Pravica
Faculty of Education
The University of Western Ontario



Editor's Note: Sam Pravica is the guest editor for September 1980. The theme is "Education of the Exceptional Child".

Periods of time pass all too quickly. Without memory time would have no meaning. It is memory that enables us to reshape the future by acting upon past mistakes and taking steps to minimize their recurrence.

The production of the Intermediate Mathematics guideline was, to say the least, not without its problems. The question worth considering is, was the cost spent in human time and energy, and outlay of money, justified? This of course is a debatable issue; largely a matter of one's perspective. What have we learned as a result of the validation process and at what price?

What price working with professionals? The make up of the validation committee was designed to provide representation and thus input from all the major educational organizations in Ontario that would be directly or indirectly affected by the implementation of the Intermediate Mathematics guideline. The members came, as I did, rather uncertain of their eventual role; cautious, not willing to show their hand until the game plan was revealed. Friendships grew quickly. Points of view were exchanged - heated at times, but respected. A mandate had to be fulfilled and all contributed beyond initial expectations. It is because of individual professionals such as these that Mathematics is alive and doing well in Ontario. Their impact on Mathematics education will be felt regardless of their future roles in education. The price? . . . the best possible Mathematics curriculum guideline under the constraints of time, money and terms of reference given.

What price producing and printing? It depends on how one views cost. Cost is a relative issue; relative to a perceived value. Value and cost, in many ways, form an inverse relation. The guideline may have a great deal of intrinsic value. However, if one uses it only to decorate a shelf, its production cost is too high at any price.

There are in existence today literally millions of pages of mathematics resources gathering dust in board of education storage facilities. The cost in man-hours to produce them, and the cost of paper and printing, is beyond imagination. Yet there they sit. Why are they not being used? The answer is simple. Without a proper implementation program, concepts written on these pages are obscure. The words become subject to translation and misconception.

Teachers are constantly looking for new ideas. Many of the new ideas never reach the implementation stage because when asked of curriculum writers "Are you aware of my priorities?" or "Are you aware of what my tradeoffs might be if I introduced your materials?" the questions are largely ignored. If this reflects reality, then is the production cost justifiable? In terms of getting value for dollars spent, is the Ministry justified in its cutbacks in human resources and a proper implementation program? The price? . . . paying lip service to the curriculum guideline.

What price turning a deaf ear? The guideline is in the final stage of editing and printing. The work of the validation committee has been completed for some time. Yet, I am nagged with the concern that even given a proper implementation program, it may not be possible to implement the guideline fully. My fear is that given the realities of the classroom, the expectations related to implementing the core program are too high.

It is suggested that the core program should represent about 80% of the teacher's total mathematics program for the year. Upon reflection, however, this seems overly ambitious. Take a pencil and paper and estimate the number of days that would represent a realistic review, testing and evaluation program. You will discover that this activity alone requires from 15% to 20% of the teacher's program. This is absurb - cut back you say? Realistically - impossible! These are administrative requirements for the purpose of eliciting marks for reporting and promotion considerations. Pressures for accountability and mastery learning will, in fact, increase the time devoted to formative as well as summative evaluation, rather than reduce it. If this is classroom reality, then teachers have little choice but to spend their time covering core topic requirements

and very little time, if any, with the optional topics. Unfortunately, this is just the beginning. Curriculum writers often fail to take into account lost days due to professional development, field days or athletic events, school closings due to inclement weather, field trips (an important requirement for real world experiences), religious exercises in the separate school system, etc. As Peter Trueman, a noted newscaster, would say, "That's not news, but that too is reality." Should I menion the number of programs that teachers are being overwhelmed with, which demand simultaneous implementation . . . ?

Students should be given the opportunity to experience real situations to reinforce skills learned. Ideally the teacher should provide the experiences to allow the natural process of learning to take place through experimentation, analysis and allowing students sufficient time to make generalizations. The problem is that such a strategy is a time consuming process. The price? . . . Concepts are memorized and not fully understood.

What price apathy? The process of reviewing the Senior Division Mathematics guidelines will soon be under way. Mike McKenna, Chairman of the validation committee, has drafted a sound curriculum development plan. The plan is very complete, taking into account the problems encountered with the production and validation of the intermediate document. If adopted, many groups of educators will be asked to provide input relative to the strengths and weaknesses of the existing syllabus. Majority responses could play an important role in the types of revisions that would be implemented. What would be the consequences if the Ministry did not adopt a realistic curriculum review model, and the response from the field generated a meager feedback similar to the input received for the validation of the intermediate mathematics guideline? The price? . . . Lack of clear directions in the curriculum development process resulting in continued dissatisfaction in curriculum content and the loss of strong leadership.

SOME ALTERNATE PATHS TO THE GOAL OF PROBLEM SOLVING



by
Angela Walsh, Mathematics Co-ordinator
London & Middlesex County R.C.S.S. Bd. of Ed.

Suggest Title be **Some Alternate Paths to the Goal of Problem-Solving.**

Whether or not problem-solving can be taught is open to debate. It depends, I would say, on the intellectual stage of development of the learner, on one's definition of a problem, and on one's perception of the art of problem-solving. According to Stephen Krulik, a problem exists if:.

1. The individual has a clear goal of which he is aware; in other words there is some personal involvement.

2. The individual's path toward this goal is blocked. His usual, habitual responses do not remove the block; he experiences a sense of frustration.

3. The individual must deliberate carefully; he must **think** about how to remove the block.[1]

George Polya, the author of **How to Solve It** suggests a problem solving definition in the following crisp way "attaining an aim which was not immediately attainable." This definition immediately excludes the practice exercises of word problems designed to give the pupil an opportunity to apply a certain computational skill or to translate a written situation into an appropriate mathematical sentence. Rarely do these permit the possibility of alternate paths to a goal. Their solution depends almost entirely on simple recall, rather than upon deliberation as Kulik suggests.

The most common approach to problem-solving includes these tips:

1. Read the problem.
2. Find out what is given.

3. Find out what is asked for.
4. Decide what process to use.
5. Estimate the final answer.
6. Find the answer.
7. Check it.

While this straight jacket approach does offer a step-by-step pattern, it leaves no room for flexibility or creativity. If this is so, what then are the alternatives? May I recommend the following:

1. **Restating the Problem**

Research has shown that errors in problem-solving frequently result from an incorrect reading of the problem. So, paraphrasing a problem clarifies given information and the question to be answered, and identifies hidden assumptions that may make the difference between success and failure. The following nursery rhyme illustrates my point.

As I was going to St. Ives.
I met a man with seven wives,
Each wife had seven sacks,
Each sack had seven cats,
Each cat had seven kits,
Kits, cats, sacks, wives -
How many were there going to St. Ives? [2]

In their initial reading of a problem, some students may miss the essential question and consequently the solution. Being able to restate a problem in a different way, then, is one skill that students need to practise if they are going to become more problem-wise.

2. **Brainstorming**

If students are to demonstrate creativity in problem-solving, they must develop flexible thinking habits. Brainstorming will help to do this. Maier calls it "generating alternatives". According to this psychologist, cited by Frank K. Lester in **The Arithmetic Teacher**, November, 1977,

> Finding a final solution to a problem involves two stages, idea-getting and idea-evaluating. Idea-getting is concerned with generating alternatives. Idea-evaluating is concerned with selecting the best alternatives. A common difficulty for a problem-solver is the tendency to evaluate and select an alternative before the best one has been generated.[3]

In brainstorming, all suggestions for the possible solution of the given problem are accepted without judgment or evaluation, and are recorded and discussed until the most feasible alternative emerges. This technique promotes divergent thinking which comes into play in the two sample problems to follow.

(a) Try to fit nine matches together to form five triangles.[4]
(b) A farmer has chickens and horses. These animals together have 50 heads and 140 legs. How many chickens and how many horses does the farmer have? [5]

3. **Developing Patterns**

One successful technique for problem-solving uses the development of patterns as its basis. But before students can develop patterns they must first demonstrate their ability to **identify** and **define** patterns like the one that exists in the following simple item, 1, 3, 6, 10, _____, _____, _____, more sophisticated and varied examples of which frequently appear in current textbooks. Experience with identification of patterns gives the student one more technique with which to experiment when he attempts to solve problems like the Le Blanc handshake problem.

> There are 8 people at a party. If each person shakes hands with the other guests, how many handshakes will there be? [6]

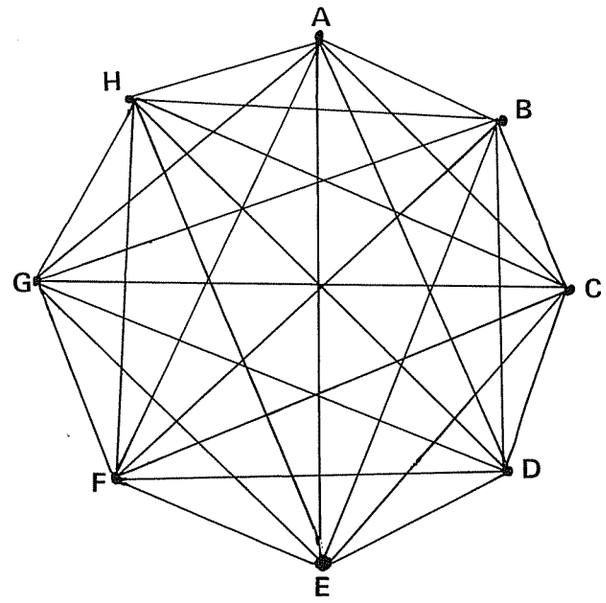
In this particular situation, to record the results of an actual student demonstration would permit the learner to **see** and to **predict** from the emerging pattern the number of handshakes there would be.

People	1	2	3	-	-	-	-	-
Handshakes	0	1	-	-	-	-	-	-

The above information can also be displayed in technique, number 4, favoured and promoted particularly by Floyd G. Robinson in his celebrated article "Can You Draw Me A Picture?".[7]

4. **Drawing Diagrams**

According to Robinson, the superior problem-solver also tends to be a superior diagrammer. "It is not that he always draws diagrams," says Robinson, "but that he can do so when the situation requires it."[8] If he is going to be able to respresent data by diagramming the information, he needs practice so that he can add this skill to his growing repertoire of problem-solving techniques. If, for some students, the completed diagram (as it appears on the following page) which represents the solution to the handshake problem proves to be confusing with so many lines, they should be encouraged to practise and use the next related technique.



5. **Making Organized Lists and Tables**

Also associated with the pattern development technique is that of organizing data into lists or tables from which generalizations can more easily be made. Using the handshake problem once again, the information in that problem can be represented as follows:

	Jim	Jane	John	Joan	Bob	Beth	Bill	Barb	
Shakes hands with	Jane	John	Joan	Bob	Beth	Bill	Barb		
	John	Joan	Bob	Beth	Bill	Barb			
	Joan	Bob	Beth	Bill	Barb				
	Bob	Beth	Bill	Borb					
	Beth	Bill	Barb						
	Bill	Barb							
	Barb								
	7	+ 6	+ 5	+4	+ 3	+ 2	+ 1		=28

6. **Making Models**

This technique helps the student to sort out and clarify information that deals with logical reasoning like, for example, the horse race puzzler which requires that the problem-solver sort out the given clues to determine which horse goes in which stall. In this type of problem that contains information about relative positions of objects, making a model would, perhaps, involve labelling slips of paper and manipulating the slips according to the conditions of the problem, until a solution has been found.

7. **Deduction**

Since students will encounter problems, the solutions for which will not respond to Searching for Patterns, Drawing Diagrams or Making Organized Lists and Tables, they should be encouraged to practise and to add the deductive technique to their repertoire of problem-solving skills. This technique breaks down the problem as a whole into its various parts, considers each part by itself or in combination with each other to achieve subsolutions and then fits these subsolutions together to develop the overall solution.[10] Polya refers to this technique as "Decomposing and Recombining" in his book **How To Solve It**. The features of the deduction technique are inherent in the following example:

> Mike is paid for writing numbers on pages of a book. Since different pages require different numbers of digits, Mike is paid for writing each digit. In his last book, he wrote 642 digits. How many pages were in the book? [11]

8. **Games**

The strategies required in playing games correlate well with the preceding suggestions for improving problem-solving skills. This has been shown by Stephen Krulik in a test conducted with grade nine students using four strategy games - Tic Tac Toe, Hex, Nim, and Saturation, and reported in **School Science and Mathematics**. The pilot teacher felt that games developed interest in problem-solving, enhanced students' analytic powers and enriched her own mathematics teaching by offering something with wide appeal to the students.[12]

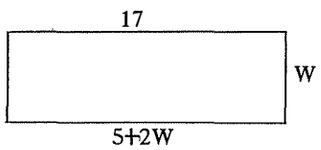
While there is no guarantee that any of the above suggesions will improve a student's ability to solve a problem, they will provide him with alternate paths to that goal, and, at the same time, put the emphasis where it belongs - on the process. Sufficient experience with these alternatives and with Polya's well-known line of analytic questioning should enable a student to **approach** a problem more confidently.

But he still has to **solve** the problem! Let us examine two methods championed by different authors. Consider the following problem and its solution:

> The length of a rectangle is 5 more than twice the width. Find the width if the length is 17 .[13]

9. **Translational Method**

If we use the translational method advocated by Maurice E. Dahmus, the pupil translates each phrase **in the order** that it appears in the problem. The solution would be as follows:



$L = 5 + 2 \times W$ $17 = 5 + 2W$ $5 + 2W = 17$
$W = ?$ $12 = 2W$ $2W = 12$
$L = 17$ $6 = W$ $W = 6$

10. **Step Method**

In the step method the student applies Polya's questioning technique and acts accordingly.

1. Read the question.
2. Find out what is asked and choose a variable to represent the unknown.
3. Examine all other data and how they relate to the unknown.
4. Write a number sentence expressing this given relationship.
5. Solve the equation.
6. Check your work.[14]

To solve the problem using Polya's method would result in the following solution.

Let x represent the width.

$$2x+5 = 17$$
$$2x = 12$$
$$x = 6$$

Dahmus disapproves of this method found in current textbooks because the emphasis is on the variable rather than on the facts, and the data are not used in the order in which they appear in the problem.

Although I can see some strength in Dahmus' approach, I lean towards Polya's. For me, it accommodates alternate paths to the goal. Like Rubin, I believe that

> If we can arrange the curriculum so that the student repetitively engages in as many kinds of thinking as possible, we will in all probability: (a) provide him with a basis for recognizing opportunities for transferring his learning; (b) expand his capacity for responding in several ways to a simple phenomenon; (c) improve his ability to select an effective way of thinking about a task; and (d) provide him with cognitive alternatives he might otherwise have ignored.[15]

FOOTNOTES

[1] Stephen Krulik, "Problem Solving: Some Considerations," **"The Arithmetic Teacher"**, Volume 25, Number 3, 1973, p. 51.

[2] Marguerite de Angeli, **Book of Nursery and Mother Goose Rhymes** (Doubleday and Co., Inc., 1954, p. 113.)

[3] Norman R.F. Maier, "Problem Solving and Creativity in Individuals and Groups", 1970, cited in Frank K. Lester, "Ideas About Problem Solving: A Look at Some Psychological Research," **The Arithmetic Teacher**, Volume 25, Number 2, 1977, p. 14.

[4] Carole Greenes et al., **Successful Problem Solving Techniques**, Creative Pulication, 1977, p. 60.

[5] **More Topics in Mathematics, Thirtieth Yearbook**, National Council Teachers of Mathematics, 1979, p. 494.

[6] John F. LeBlanc, "You Can Teach Problem Solving", **The Arithmetic Teacher**, Volume 25, No. 2, 1977, pp. 16-20.

[7] Floyd G. Robinson, "Can You Draw Me A Picture?" The Educational Courier, Volume 44, Number 2, 1974, pp. 26-29.

[8] Ibid., p. 28.

[9] LeBlanc, p. 19.

[10] Greenes, p. 31.

[11] Ibid. p. 31.

[12] Stephen Krulik, "Mathematics Class Strategy Games = Problem Solving" **School Science and Mathematics**, Volume 75, Number 3, 1976, p. 224.

[13] Maurice E. Dahmus, "How to Teach Verbal Problems," **School Science & Mathematics**, Volume 70, 1970, p. 134.

[14] Lloyd R. Richardson, "The Role of Strategies for Teaching Pupils to Solve Verbal Problems," **The Arithmetic Teacher**, Volume 22, Number 5, 1975, p. 415.

[15] L.J. Rubin, "Creativity and the Curriculum", **Phi Delta Kappan**, June 1963, pp. 438-440.

YOU CAN TEACH PROBLEM SOLVING

By John F. LeBlanc
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Teaching problem solving is a problem, but like most other problems it can be solved. And like other teaching situations, there is no formula that can be automatically applied to guarantee success in teaching problem solving. There are, however, some guidelines and hints that can go a long way in aiding a teacher in problem-solving instruction.

Before these guidelines and hints are explored, a few general comments might be helpful in placing the focus of this article in context. There are three recognized aspects of teaching mathematics in the school, **computational proficiency, concept attainment**, and **problem solving**. (Some mathematics educators state that problem solving is the goal of school mathematics). Each aspect deserves appropriate emphasis and each requires a different instructional approach. The instructional approach to be used depends on the goals to be attained. **In teaching problem solving the instructional goal should be to help children learn procedures for solving problems**. These procedures are easy to state and recognize, but they are often quite elusive when teaching. It seems so much easier to help children acquire a computation skill or to help them understand a concept than to help them acquire skill with problem solving procedures. Among the reasons for the difficulty in teaching/learning problem-solving procedures is that there is no specific content as in computational skills or in concept attainment. In problem solving, an individually acquired set of processes is brought to bear on a situation that confronts the individual. Helping the child acquire and apply these processes is a more complicated and less well-defined task than helping a child learn a computational skill or understand a concept.

What procedures are inherent in problem solving? There are four that are stated in various ways by a number of authors:

1. Understanding the problem.
2. Planning to solve the problem.
3. Solving the problem.
4. Reviewing the problem and the solution.

How can a teacher help children acquire and use the four procedures? In the discussion that follows, each of the four procedures will be illustrated using problems appropriate for elementary school children. These illustrations will be accompanied by some guidelines and instructional moves that can be used to help children in their problem solving. Two different word problems will be used in the discussion. The first is a typical textbook problem; the second problem can be called a "process" problem.

TYPICAL TEXTBOOK PROBLEM

The basic purpose of the typical textbook problem is to reinforce children's understanding of a concept or to use a skill learned earlier by presenting a "real-world" situation that embodies that concept or skill. **Using these typical textbook problems to teach problem-solving procedures has the shortcoming** that for many children the solution, or at least the plan for solving the given problem, is so apparent that no problem exists. However, a teacher can emphasize the procedures for the children.

> A shopkeeper sold 485 bottles of cola in one day. The cola is packaged in cartons holding 6 bottles. How many cartons of cola were sold?

1. ***Understanding the problem***

In helping children understand the problem a teacher might use questions that focus the child's attention on the information or conditions given in the problem.

"John, how many bottles were sold?"
"Nina, how are the cola bottles packaged?"
"Felix, how many bottles in a carton?"
"What does a carton look like?"
"Does everyone always buy a full carton? Does that make any difference in our problem?"
"If one carton were sold, how many bottles of cola would be sold?"

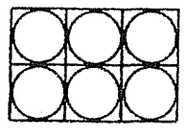
Finally, a teacher might ask, "What does the problem ask for? Can you tell it in your own words?" Until a thorough discussion of the information and conditions takes place, this last question might make little sense. Advice such as "Think" and "Read the problem again" does not help the child without the accompanying questions that focus on understanding the problem.

By the foregoing sequence of questions the teacher has helped the children learn to help themselves understand a problem. **Teachers should call on children to do the question asking and to formulate questions for their classmates**. This experience will help them grow in their ability to focus on important information.

2. ***Planning to solve the problem***

In the discussion of this procedure a teacher should be aware of and look for different strategies that might be used in solving the problem. Initially, the teacher may have to suggest possible strategies for the children, but after they have some experience using various strategies children will choose their own. **Children should be encouraged to suggest strategies**. Throughout the planning-to-solve-the-problem step, the teacher should **emphasize the type of strategy the child has suggested**.

Teacher: Kids, how might we plan to solve this?
First student: Well, we could get 485 bottles of chips and put them into groups of six.
Teacher: Good suggestion. In other words, we could actually **conduct an experiment**.
Second student: It helps me to draw something like a carton so I can think about six bottles in each.



Teacher: I like that strategy. **Making a drawing or a diagram** is a good way to visualize the problem.
Third student: Well, it takes six bottles for one carton and so for ten cartons sixty bottles would be used. I would add sixties until I got to 485 or near it.
Teacher: Good thinking! **Using patterns is an excellent way of solving some problems.** You could even put your numbers into a **table.** (Fig. 1)

Fig. 1

Cartons	10	20	30	···
Bottles	60	120	180	···

Fourth student: I would divide 485 by 6 to find the number of cartons.
Teacher: Good. Sometimes **direct computation is a good strategy to try.**

One of the shortcomings of the standard textbook problems (as was mentioned earlier) is that they are often **designed to provide computational practice rather than to teach problem solving**. This places an additional instructional burden on the teacher - the teacher must create a problem-solving situation by suggesting or trying to elicit other strategies.

For some students, **complete understanding of the problem may come at the "planning to solve the problem" stage**.

Strategies may be grouped into *general* and *helping*. A general strategy may be thought of as an overall plan designed to solve the problem. A helping strategy may be thought of as an intermediate step used to carry out the general strategy. Some **general strategies** are **trial and error, organized listing, simplification, searching for a pattern, experimentation, deduction, computation,** and **working backwards.** Some **helping strategies** are **diagrams, tables, graphs, lists,** and **equations.**

3. *Solving the problem*

In this step the plan selected in step 2 is carried out. The teacher should encourage the children to **solve the problem using the variety of plans elicited earlier**. The importance of this step is not how efficiently the problem can be solved. Rather, **this step should be viewed as the step that carries out the plan**. For some children the plan selected will not lend itself to solution (e.g., diagramming cartons of 6 until 485 bottles are drawn). **If the plan does not work, the problem solver should return to step 2 to devise another plan.**

The teacher's role in promoting this step should be advisory in that he or she should ask how the solution process fits the selected plan. At this stage in the problem-solving process temptations afflict teachers. First there is an impulse to overemphasize the solving-the-problem step, thereby minimizing the other steps. Secondly, **there is a strong tendency to press so hard for computational accuracy that problem solving is associated in the minds of children only** with onerous computation.

4. *Reviewing the problem and its solution*

In developing problem-solving procedures in children there is **no step more important than this fourth step.** There are two aspects of this step; one is **looking back over the steps taken and the other is extending the problem situation to create variations or an entirely new problem.** The former aspect should never be omitted in any problem-solving instruction.

"Tell us how you though about solving it".
"Show us at the board how you worked it out."
"What strategy did you use to get your answer?"
"Does this solution fit the question in the problem?"
"Does that answer make sense to you?"

All the preceding comments are asking the children to introspect, to look at **their own thinking, and to describe it.** This activity is perhaps the most important part of the instruction procedure. It benefits the child doing the introspection, and it benefits the other children who hear how someone else thinks about the problem.

The problem can be extended by the teacher asking additional questions.

"Suppose the shopkeeper puts four cartons into each case. How many cases would he fill?"
"What if there were 600 bottles; could we solve the problem using the same strategies?"
"What profit did the shopkeeper make if he made four cents (four and three-tenths cents!) on each bottle?"

The typical textbook problem is not an ideal vehicle for teaching problem-solving procedures since **most of the textbook problems, by design, can be solved by direct application of an arithmetic algorithm** (addition, division, percent, averaging, and so on). Furthermore, **most of the typical textbook problems follow a section of skill or concept development and are related to that skill or concept**. In such a situation it takes little "problem-solving" ability to guess an appropriate strategy. On the other hand, it is important to use every opportunity possible to teach problem-solving procedures, even using the standard textbook problem.

PROCESS PROBLEM

There is another type of problem, a "process' problem, which is so called because it lends itself to exemplifying the procedures inherent in problem solving. Process problems are found in several textbooks in the corners of some pages. **While these problems are presently often viewed as enrichment or as recreational, they are fast becoming recognized as an important vehicle for teaching problem solving. In the near future they may become part of the mainstream elementary mathematics curriculum**. Not only can process problems be solved using a variety of strategies, but they **also require very little formal mathematics**. The following example is a process problem.

There are 8 people at at party. If each person shakes hands with the other guests, how many handshakes will there be?

1. *Understanding the problem*

As in the previous example, a teacher can help the children understand the problem by selected questions. Again, the children should be encouraged to ask the questions themselves.

"How many guests are at the party?"
"If Jim shakes hands with Carole, should Carole later shake hands with Jim?"
"If you were at the party, how many people would you shake hands with?"
"Now who can give the problem in his or her own words?"

2. *Planning to solve the problem*

One of the features of the process problem is that it lends itself to many different strategies for solution. The following are some of the strategies that children came up with in regular **fourth - and fifth - grade classrooms**.

First student: Let's act it out by using eight kids. Come on, let's get in a circle. (Experiment)
Second student: I'm going to draw it out by putting eight dots in a circle and using lines to show handshakes. (Diagram)
Third student: I'm going to give them names and make a list of the people each shakes hands with. (Organized listing)
Fourth student: I wonder if there is a pattern. Let's see, if there are two people, there is one handshake. If there are three people there are three handshakes . . .(Using tables to find a pattern) (See Fig. 2)

Fig. 2

People	2	3	4	5	...
Handshakes	1	3	6		...

Fifth student: Well, if each person shakes hands with the others, then eight people shake hands with seven others so there are eight times seven handshakes. But . . .(Deduction)

These five responses come from children after they and their teacher had solved several problems together. Initially, but not always, the **teacher many have to supply some of the strategies**. The teacher will also have to identify and label the strategies used by the children so they will learn to use the labels in communicating to others.

3. *Solving the problem*

The children usually do not have any difficulty solving the problem once a strategy has been selected. Even if the strategy is inappropriate, the insight acquired by following the inappropriate strategy usually helps the child to select another strategy.

In the classroom setting, solution of the problems brings enough questions from other children to forestall errors like the one apparent in the fifth child's deductive thinking, not taking into account that his method would have each person shaking hands twice with everyone else. A demonstration by eight children in a circle would soon show him the fallacy in his reasoning.

4. *Reviewing the problem and its solution*

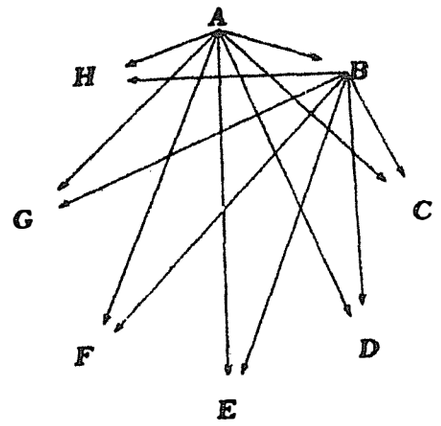
After the children have solved the problem, the discussion time should focus on having the children introspect.

Teacher (to first student): You and your group used an experiment. Will you show the class how you solved the problem?
First student: Jean shakes hands with seven people; Paul shakes hands with six; Mary shakes hands with five; Phyllis with four; Peter with three; Levi with two, and Juan with one.
$7+6+5+4+3+2+1 = 28$ handshakes

Teacher (to first student): Using an *experiment* is a good way to solve a problem sometimes. Did everyone follow the solution?

Second student: Hey, that's the same as our drawing. There are seven lines from A, six from B, five from C, four from D, three from E, two from F and one from G. We only drew lines from A and B, but we could see the pattern. (Fig. 3). We get twenty-eight handshakes, also.

Fig 3



Teacher (to second student): A *diagram* is one of the most helpful ways of displaying a problem, and often it leads directly to a solution. Here you used the **start of a diagram and noted a pattern**. I agree that a completed diagram would have been confusing with so many lines.

Third student: We come up with twenty-eight handshakes too, but we like our method best. (Fig. 4)

Fig. 4

	Jim	Jane	John	Joan	Bob	Beth	Bill	Barb
Shakes hands with	Jane	John	Joan	Bob	Beth	Bill	Barb	
	John	Joan	Bob	Beth	Bill	Barb		
	Joan	Bob	Beth	Bill	Barb			
	Bob	Beth	Bill	Barb				
	Beth	Bill	Barb					
	Bill	Barb						
	Barb							
	7	+6	+5	+4	+3	+2	+1	=28

Teacher (to third student): Your solution uses an **organized listing** strategy. Do you see that, here, every handshake is listed in an orderly fashion?

Fourth student: I think we have discovered something. When we used a table to find a pattern we discovered that could tell the number of handshakes for other parties too. You see, with each person the number of handshakes is increased by a number one less than the new number of people. That makes sense because if a new guest came and he was the 7th person he would have to shake hands with the other six. (Fig. 5)

Fig. 5

People	2	3	4	5	6	7	8	9	...
Hand-shakes	1	3	6	10	15	21	28	36	...
New Hand-shakes	2	3	4	5	6	7	8		...

Teacher (to fourth student): That's nice. Can anyone in class tell me how many handshakes there would be for ten people? For twenty people? Can we find **a general rule** for any number of people?

Fifth student: What's wrong with my reasoning? I get fifty-six handshakes. Since each person must shake hands with seven others, there must be eight times seven handshakes.

Teacher: Who can tell what might be incorrect about that reasoning? It sounds good to me.

And so the discussion continues. At some point in the problem-solving instruction the **teacher should present a list of strategies**. Such a list serves several purposes, including helping children consider a variety of attacks on the problem.

The teacher should extend the problem by asking children questions similar to the following:

How many handshakes would there be in your family?
How many handshakes would there be in our classroom?
How many handshakes would there be if all the boys shook hands just with the girls in the class?

By providing extensions of the problems for children to work on in school or at home, the teacher is giving the children a chance to use their own strategies or those of another, or to find a generalized solution.

In choosing process problems for problem-solving instruction, care should be taken to **choose problems** that **lend themselves to solution using a variety of strategies, reguire little formal mathematics and are interesting to the children**. Process problems have an appeal to children; children enjoy solving them. At the same time it is important to use the standard textbook problem to teach problem-solving procedures. In other words, use every opportunity to teach problem solving, especially those that arise from classroom, school, or community events. Although this article has not specifically addressed the issue of the **role of the affect in problem solving**, attention to this aspect is important. **Helping children become motivated and willing to solve problems is an important prerequisity that a teacher must consider.**

The **"problem of the week"** is one of the specific techniques that have been used successfully by classroom teachers in teaching problem solving. A problem is placed on the bulletin board once a week. A discussion of the problem follows, as previously described, emphasizing "understanding the problem." The children then work on the problem until a preassigned day. Often the children involve their parents or friends outside school in looking for strategies that might be different or unique. At the designated time, the various solutions are discussed, as described in the earlier examples. This activity, together with an emphasis on problem-solving procedures when standard textbook problems are used, can provide the basis for a sound curriculum in problem solving.

Teaching problem solving may be a problem, but tackling the challenge of this important facet of elementary school mathematics can be enjoyable and rewarding. The strongest motivation for successful instruction in problem solving is success - seeing and hearing children eagerly attack and solve problems. □



PROBLEM-SOLVING IN THE PRIMARY GRADES

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Problem-solving is an ability people need throughout life. Since this skill is so important to daily living, it has been a primary stated goal of mathematics instruction since the turn of the century. Although this skill appears to be highly prized by society and educators alike, recent studies have shown that students and graduates have serious deficiencies in their problem-solving abilities. This may be due in part to the fact that problem-solving plays only a minor role in mathematics instruction in some elementary classrooms.

Primary teachers can do much to change the present situation by making their pupils better problem solvers. This however, will not be an easy task! They will have to use a larger number of effective teaching strategies and problem-solving activities in their mathematics programs. They will have to see that their students come into contact with a large number of problems in a variety of ways and situations. For learning to solve problems, like learning to skate or catch a ball, is done vy **doing it**. Greater use should be made of such specific prodedures as writing the number question or mathematical sentence, having pupils designate the process to be used, using a tape recorder to aid poor readers, and presenting students with many varied word problems. Some other useful strategies to follow are noted here:

1. PRESENT ORAL PROBLEMS OFTEN

Beginning with the kindergarten program and continuing into the higher primary grades, oral problems should be asked frequently since the ability to understand and apply mathematical concepts frequently exceeds the young child's ability in reading and writing. Short stories about numbers of things should be given orally by the teacher and should be made up and given orally by the children. This procedure requires the young child to analyze the important information in the problem quickly. It is also the manner in which most out of school problems occur. In addition to teacher-presented problems in the classroom, tape recorded problems can also be used for small group and individual study in the listening centre. Such recordings also aid children with reading difficulties.

2. DRAMATIZE AND DIAGRAM PROBLEMS AND PROBLEM SITUATIONS

As soon as young children know how to count i.e., to find the cardinal number of a set, they can act out and diagram problems and problem situations. In this way, the problem situation becomes much more meaningful as the children pretend to be whatever objects are mentioned in the problem (i.e., Six bears were in the zoo. A circus man bought one of the bears. How many bears are left in the zoo?0. It is also a good idea to use counters or small toys to represent problem situations. In dealing with the problem above, the children can pretend that they are bears or that their counters are bears. This technique can be used effectively all through kindergarten and early grade one. As the children get older, the technique of diagramming the problem can be introduced as well. For if the child is to solve the problem situation, he must first be able to visualize the problem situation in some way. A child who can best visualize a problem by manipulating concrete objects, should be encouraged to do so. Similarly, a child who can best visualize a problem by sketching, diagramming, or simply doodling should be allowed to do so. As patterns of solution are realized, children should be encouraged to proceed to more advanced methods of solution. Remember, that cutouts or small models may also be manipulated on flannel boards, magnetic boards, or on overhead projectors.

3. USE PUPIL MADE PROBLEMS

At all elementary levels, remember to use pupil-made problems. One of the best ways to help a child learn to solve problems is to have him make up problems for himself and his classmates, to solve. These might be placed in a MATH PUZZLER box in the mathematics

centre. The one main requirement is that the child should be able to solve the problem he makes up himself. These pupil-made problems should be both oral and written. It is also a good idea to have pupils formulate problems for given conditions.

4. HAVE PUPILS ESTIMATE THEIR ANSWERS

The habit of estimating answers should begin in first grade at the latest. Even with very simple problems children may be asked such questions as, "Should the answer be more than 10? How can you tell?" As the level of maturity rises, the children should be encouraged to make judgements such as: "It has to be at least 20. It can't possibly be as much as 100." The child who continually asks himself, "About how much should this answer be?" will avoid mistakes which would make his answer ridiculous. That is, he needs to say to himself, "Does this solution make sense in the problem situation?" This practice of estimating a reasonable answer before performing the necessary computation will establish a criteria for plausible responses and help to discourage wild guesses. The estimate can establish the limits within which to work the problem and also serve as a check on the final solution.

5. USE PROBLEMS WITHOUT NUMBERS

Problems without numbers are valuable in that they focus attention on the process which might be used to solve the problem if the numbers were given, "Mike had several toy cars. He got some more for his birthday. How many cars did Mike have then?" The children should be able to see that this is an adding situation. After this has been decided by the class, the teacher should have different pupils supply different pairs of numbers so that the problems can be solved. They should then find the answers in each case and should note how they found it. They should be brought to the conclusion that the problem is an addition problem regardless of the numbers used. Eventually they will learn that it's not the numbers nor the specifics of the story line, but the relationship among the numbers, which determines the process or processes to be used in solving problems.

6. HAVE PUPILS WORK TOGETHER

It is a good idea to arrange for two or three students to work together at times as they analyze the information in the problem and write the related mathematical sentences and the solution.

7. HAVE PUPILS WRITE NUMBER questions or mathematical sentences

The development of number sentence aids is a clear and logical attack on the problem to be solved. Pupils should be encouraged to develop this technique.

8. TRAIN PUPILS TO CHECK AND VERIFY THEIR WORK

All teachers should help children to learn how to check and verify their work. She should teach children to check each solution in the problem situation, and then put the answer in the appropriate place in a restatement of the problem to see whether it fits. The teacher should also take time to discuss any problem-solving errors found in the children's work.

9. TEACH THE READING OF MATHEMATICS

Giving conscious attention to the teaching of math vocabulary and the material of the kind found in math textbooks at the higher primary grade level will help young children read work problems more effectively. A child who cannot read a problem and know what the problem tells him and what it asks cannot possibly solve the problem. Hunt a word puzzles and cross-number puzzles are an excellent way to reinforce the meaning of new concepts and vocabulary. In planning to teach reading of this type, remember that in solving a new problem, children must be able to: (a) read the words and symbols, (b) attack literal meaning, (c) analyze relationships, (d) set up the related math equation, (e) compute and (f) interpret the answer. Teachers may help them do this by having them preview the vocabulary words, mathematical terms and symbols before doing the problems. It is also advantageous to have them read the word problems orally and then rephrase them in their own words. In this way, the teacher will be able to determine whether or not the problems were really understood by the children. The purpose at hand is to be sure that each child understands correctly on his own terms what the problem says. Once a child is able to **read** a problem - to comprehend what it **tells** and what it **asks** - he is off to a good start in solving a problem.

10. PROVIDE A DIFFERENTIAL PROGRAM

In preparing lessons and materials, the teacher must remember to develop materials of multilevel difficulty. This will enhance their educational value by allowing the slower child to work at a level at which he is comfortable, and by challenging the gifted

child. This implies that different children should be given different sets of problems to solve according to their abilities, a technique not usually associated with the traditional use of word problems in mathematics classes.

11. PROVIDE VARIED ACTIVITIES TO INCREASE PROBLEM-SOLVING SKILLS

Primary teachers need to present problems and problem situations to children in a variety of ways, not just in written form. Some activities which work well with primary children are noted here:

a. Teach number verses that stress addition and subtraction. These could be dramatized by the children, or presented on a felt board or magnetic board by the teacher.

i.e. ADD ONE MORE
This is one cat.
Now two cats I see.
Another cat will make it three.
I add another
Then there are four.
Mother cat joins them
To make one more.
Now let's count them:
1, 2, 3, 4, 5.

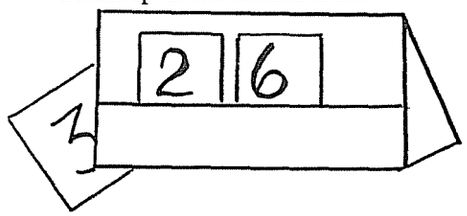
b. Collect large pictures suitable for a number picture file. These could be used by the teacher and pupils to make up oral problems. Well chosen small pictures could be pasted on small file cards. These could be placed in the math center. Interested students could write problems on the cards to challenge their fellow pupils.

c. Play the card game PROBLEM OLD MAID. In this case, the pairs are problems and answers. Seasonal problems add a strong motivational factor to this game.

d. Catalog shopping appeals to young children. Take advantage of this natural interest. Ordinary department store catalogs are useful, especially the Christmas ones! They are usually smaller and much easier for the young child to handle. Children might be asked to do such things as:
- List the members in your family. Choose one present for each member. Remember that you **only have thirty dollars** to spend! List the individual prices and the total price to prove that you did not go over thirty dollars.
- Look in the clothing section. Find two things that you would like to buy. Then find out the total cost of the two things.
- You have five dollars to spend on your best friend. Find something less than five dollars that you think he would like. Tell how much money you will have left after you pay for the surprise gift.

e. Egg cartons make an excellent device for problem solving. Mark answers in the bottom of each section, and then make problem cards in the shape of eggs. The children are to put the egg shapes in the section containing the correct answer.

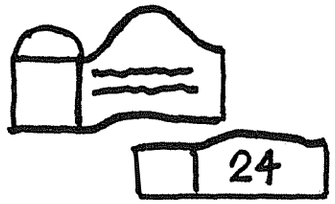
f. Oral problem pockets are a good way to handle oral problem work. Children are given a sequence of numbers, 0 to 9, and an individual pocket stand made from construction paper. When working with larger numbers they may be given two of each. The teacher then gives an oral problem and the children place their answers in the pocket and hold it up for the teacher to see.



g. Greeting card problems are fun for young children. Christmas cards, Easter cards, and Valentines work well. A card is placed over the verse and a problem added. The problems could relate to the season or the picture on the card.



h. Putting together problem puzzles in relation to a class theme is fun. For example, during a farm unit problems could be written on a barn shape with the answer near the bottom. The barns are then cut into two parts. The children put the barns together by matching the problem with the correct answer. The problems should be about farm animals and farm activities.



i. Banking problems can be solved using make believe banks made from margarine containers and play coins. A slot is cut into the lid of the container. Addition problems are given (Bill saved ten cents on Monday and seven cents on Tuesday. How much money did he save?) and the children drop the correct coins into their banks. They then record on paper the amount of money they think they have in the bank. As soon as this has been done, they open their banks and count the money to see if they are correct.

j. Set up a wise thinkers problem in one corner of a bulletin board. It might have the title, PROBLEMS FOR WISE PEOPLE or SOLVE THE MYSTERY PROBLEM. These problems could be changed weekly or daily.



k. Dominoe sets could be made and used where children match problems with answers.

In conclusion, may I add, recent research on the teaching of mathematics has shown increased mathematics achievement in those classes where teachers are enthusiastic about mathematics!

> "Arithmetic should not mean to a child only tables learnt by heart and rules of thumb; if it does then intelligent children will be bored, and advanced work made impossible."

R.K. & M.I.R. Polkinghorne in **Fundamental Number Teaching**



COORDINATOR'S CORNER

by Brock Rachar

One of my hobbies is old books. As you might have guessed old math books are included.

So called 'New Math' has had both its supporters and its detractors. There are swings of the pendulum from old formalisms to new ones; from less structured approaches to very structured ones. 'Old Math' is touted as having been the "real thing" by some or as being entirely inadequate by others. I think, as with most things, it depends upon your perspective.

In the text with the title page shown here, John Herbert Snagster, Esquire, Mathematics Master of the Normal School for Upper Canada, put together a compendium of arithmetic suitable for his time.

If some of his approaches seem rather

rule-bound, we must remember that clerks in an age devoid of the calculator needed 'fast and dirty' ways of getting answers.

Here is a typical question using the "new" decimal currency of the 1860's.

"What is the interest of $1 for 8 months at 6 per cent?"

The approved solution was, "8 ÷ 2 = 4 cents". The rule that was taught and used was, "Divide the number of months by 2 and call the quotient cents."

Another question along the same line:

"What is the interest of $1 for 26 days at 6%?"

Solution: "18 ÷ 6 = 3 mills = $0.003"

Rule: "Divide the number of days by 6 and call the quotient mills."

We can point out the misuse of the equal signs and wonder at teaching such rules. But it got the correct answer and in a context where all interest rates were legally 6 per cent per annum, unless otherwise, the learning of the rules made some sense.

LOVELL'S SERIES OF SCHOOL BOOKS.

ELEMENTARY

ARITHMETIC,

IN DECIMAL CURRENCY;

DESIGNED FOR THE USE OF

CANADIAN SCHOOLS.

BY

JOHN HERBERT SANGSTER, ESQ.,

MATHEMATICAL MASTER AND LECTURER IN CHEMISTRY AND NATURAL PHILOSOPHY IN THE NORMAL SCHOOL FOR UPPER CANADA.

THIRD EDITION.

MONTREAL:
PRINTED AND PUBLISHED BY JOHN LOVELL;
AND SOLD BY R. & A. MILLER.
TORONTO:
R. & A. MILLER, 62 KING STREET EAST.
1863.

Perhaps we would not be getting the reactions that keep cropping up over metrication if people were using the older and more complete tables of the 1860's as seen here.

LONG MEASURE.

TABLE.

12 lines (l.)	make 1 inch,	marked	in.
12 inches	" 1 foot,	"	ft.
3 feet	" 1 yard,	"	yd.
5½ yards	" 1 rod, pole, or perch,	"	rd. or pe.
40 rods or perches	" 1 furlong,	"	fur.
8 furlongs	" 1 mile,	"	m.
3 miles	" 1 league,	"	lea.

NOTE.—The degree, or 360th part of the circumference of the earth, is about 69½ miles.

4 inches	make	1 hand (used in measuring horses).
18 inches	"	1 cubit.
3 feet	"	1 pace.
6 feet	"	1 fathom.
120 fathoms	"	1 cable-length.

SQUARE OR LAND MEASURE.

TABLE.

144 square inches	make 1 square foot,	marked	sq. ft.
9 square feet	" 1 square yard,	"	sq. yd.
30¼ square yards	" 1 square rod,	"	sq. rd.
40 square rods	" 1 rood,	"	r.
4 roods	" 1 acre,	"	a.
640 acres	" 1 square mile,	"	sq. m.

NOTE.—Square Measure is used in measuring surfaces, as, for example, in estimating the work of painters, plasterers, pavers, etc.; also in measuring land.

In measuring land, Gunter's chain is used. It is divided into 100 links.

7.92 inches	make 1 link,	marked	l.
100 links or 4 rods	" 1 chain,	"	c.
80 chains	" 1 mile,	"	m.
100000 square links	" 1 sq. chain,	"	sq. c.
10 square chains	" 1 acre,	"	a.

CUBIC OR SOLID MEASURE.

TABLE.

1728 cubic inches (cub. in.) make 1 cubic foot, marked cub. ft.
27 cubic feet make 1 cubic yard, marked cub. yd.
40 cubic feet of round timber } make 1 ton, marked
50 cubic feet of hewn timber } ton.
128 cubic feet of firewood make 1 cord, marked c.

A pile of cord-wood 4 feet high, 4 feet wide, and 8 feet long, contains 128 cubic feet, or 1 cord. One foot in length of such a pile is called a *cord-foot;* it is equal to 16 solid feet, and is consequently equivalent to the eighth part of a cord.

CLOTH MEASURE.

TABLE.

2¼ inches (in.)	make 1 nail,	marked	na.
4 nails	" 1 quarter,	"	qr.
3 quarters	" 1 Flemish ell,	"	Fl. e.
4 quarters	" 1 yard,	"	yd.
5 quarters	" 1 English ell,	"	E. e.
6 quarters	" 1 French ell,	"	F. e.

NOTE.—The Scotch ell contains 4 quarters 1¼ inch.

DRY MEASURE.

TABLE.

2 pints (pt.)	make 1 quart,	marked	qt.
4 quarts	" 1 gallon,	"	gal.
2 gallons	" 1 peck,	"	pk.
4 pecks	" 1 bushel,	"	bu.
36 bushels	" 1 chaldron,	"	ch.

NOTE.—The measure is used in buying and selling vegetables, fruits, grains, &c.

LIQUID MEASURE.

TABLE.

4 gills (gill)	make 1 pint,	marked	pt.
2 pints	" 1 quart,	"	qt.
4 quarts	" 1 gallon,	"	gal.
31½ gallons	" 1 barrel,	"	bar.
2 barrels	" 1 hogshead,	"	hhd.
2 hogsheads	" 1 pipe,	"	pi.
2 pipes	" 1 tun,	"	tun.

Having looked through this book used by our grandparents, it is not so hard to understand why it has been difficult to shake the "rule without reason" approach to teaching mathematics, so prevalent a few years ago and which still persists today in some quarters. It is still seen especially in teaching topics such as long division or the subtraction of mixed numbers.

This past week, Channel 10, TV London did a news short from the Math Centre. The slide-rule that old workhorse of pre-calculator days was given, for all intents and purposes, a decent burial. Perhaps you will want to hang onto your old slide-rule, but don't count on its value increasing at the same rate as some other collectables these days.

As noted in previous Co-ordinator's Corners, the co-ordinator's role is an interesting and varied one. The opportunities to view what is going on in education from a broader view than is possible in one school or one classroom is a reward in itself.

The valued associations with stimulating teachers of all levels and with many points of view make up for what could be a lonely position.

Things happen through the efforts of people working together. The co-ordinator can facilitate their happening by bringing together those people with similar interests and concerns and complimentary abilities. This is why so many local Boards have been able to produce excellent detailed guidelines and support materials for the implimentation of curriculum suggested by the Ministry Guidelines. This has also made possible the creation of local testing instruments geared to the material actually taught and learned in the classrooms.

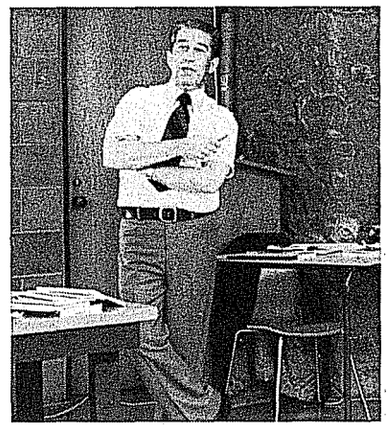
There are many issues and problems which concern co-ordinators and teachers alike. Partial solutions to old concerns and tantalizing new concerns will occupy us all for the foreseeable future. In conclusion are listed a few of these issues for which we are striving to find answers.

- What wil be the role of the microcomputer in the Math classes of the late 1980's?

- What can we do to build greater trust between the teachers of the elementary and seconedary panels? Or between the teachers of university and college and their secondary colleagues? Each group has much to offer, and indeed to teach each other. Perhaps the old adage of 'walking a moon in the other's mocassins' is one answer.

- What can the universities do about providing credit courses for elementary math teachers without math backgrounds? Can we enhance the teaching of this subject (dear to our hearts) at a level of great importance for our students, by encouraging these teachers to get involved, rather than scaring them off with ill-founded arguments about academic standards?

- How can we convince others and ourselves to look at topics and approaches with fresh eyes and not continually just teach the way we were taught? The old way may still be good or even best, but, as with the old book, it is not good just because it is old.



Next Question Please

A NEW LOOK INTO PROBLEM SOLVING

Chester D. Carlow, O.I.S.E.



It is widely recognized that genuine problem solving, at any level, involves the synthesis of a number of skills, and represents the ultimate mathematical activity for that level. It is also recognized that problem solving in mathematics has many features in common with systematic inquiry and decision making in other fields.

In September, 1980, I will begin a one-year leave to do a systematic study of what is being done and what might be done in problem solving education at the secondary level.

Some of the major questions to be investigated are listed below:

Is there an identifiable core of skills, concepts, generalizations, strategies and mental attitudes common to all successful problem solving? What additional skills, strategies, etc., are required for different broad classes of problems? Can all the components of problem solving be taught directly, or can some, such as the mental attitude of persistence, only be nurtured over long periods? How is this best done?

To what extent have growth schema for the core components of problem solving been developed? What evaluation instruments have been developed for measuring progress on these components? Where are the gaps?

To what extent do students with different learning styles tend to use different strategies to solve the same kinds of problems? Are "reflective" learners inevitably better problem solvers that "safety-oriented" or "impulsive" learners? If so, what might be done to improve the problem solving abilities of the latter? In particular, would a learning environment which is designed to raise the conceptual level of a "level B" student to "level C" tend to have a direct influence upon the problem solving ability of that student? [1])

What have been some of the more successful sequences of problems in secondary mathematics programs? What are their general characteristics? To what extent can recent theories of learning and motivaion provide insight about why the successful ones yield superior outcomes, and what might be done to improve the sequences further for all or some students?

What are the legitimate roles of algorithmic and non-algorithmic problems in a secondary mathematics program? To what extent, and under what conditions, is it advisable to teach algorithmic problems according to "type". What are the implications for greater emphasis upon non-algorithmic problems in the present Ministry guidelines for intermediate mathematics education in Ontario?

By what principles and procedures might one strengthen the problem solving component of existing secondary mathematics programs? To what extent can it be organically integrated, rather than simply added on? How might it be done, if at all, with the time normally available to classroom teachers?

How are problems solving strategies in mathematics related to strategies in such fields as scientific inquiry or practical decision making? To what extent would mathematics problem solving, as traditionally perceived, be strengthened by generalizing the concept of problem solving. [2]

During the study leave I expect to look at what has been going on in Canada, the United States, England, Israel, and perhaps some of Europe. I would appreciate from the reader suggestions about interesting sources of information or other important areas of investigation.

(1) Learning Styles and Teaching Strategies, David E. Hunt, Behavioral and Social Science Teacher, Vol. 2 (1), 1974.

(2) See George Polya, **Mathematical Discovery**, (New York, John Wiley and Sons) 1962 as an example of traditional mathematics problem solving.

THE LANGUAGE OF VERBAL PROBLEMS IN MATHEMATICS



Marcel Danesi
University of Toronto

1. Introduction

In a recent article on the meaning and scope of recreational mathematics, Trigg observes that: ''Some rather dry sounding problems can be recast into very nice recreational settings'' [22, p. 18]. This statement incapsulates the very essence ot recreational mathematics.

A frequently used setting or medium in puzzles is the linguistic one, and the resulting problem is usually called ''verbal'' . In such problems, language is a dress for the ever elusive variables **x** and **y** when they occur in the mathematical statement known as an equation. The language, or ''grammar'', or verbal problems is an environmental decor for an underlying mathematical relationship or concept. Verbal problems are therefore interesting not only to the inveterate problem solver, but to linguists and mathematical educators as well. Those problems which use the ambiguous and enigmatic properties of the linguistic medium in which they are cast are of particular interest because in most cases a successful solution requires a correct interpretation of their language. To quote Kasner and Newman: 'It is common experience that often the most formidable algebraic equations are easier to solve than problems formulated in words. Such problems must first be translated into symbols, and the symbols placed into proper equations before the problems can be solved'' [13, p. 158].

That mathematics and language are closely related is a widely held opinion. Psycholinguistic experiments by Luria [14] in Russia have in fact shown that language use and problem solving are intrinsically related. As Beard observes, these experiments demonstrate that language ''is a means of storing experience and, if this is sufficiently general in nature, of facilitating future problemsolving [2, p. 168]. Morgan and King explain problem solving in a similar manner: ''Problem solving consists of manipulating the symbols of language and thought in an attempt to reach a particular goal'' [16, p. 197].

Rapoport, among others, defines mathematics as a symbolic language, and states that its function is: ''(1) to say things tersely and precisely (so that the clumsy grammar and the vague semantics of ordinary language do not get in the way of logical thinking) and (2) to provide an apparatus for making complex deductions'' [20, p. 277]. In reference to the verbal statement of mathematical problems, Rapoport observes that 'mathematical language cleans up the ''pollution'' introduced into reasoning by ordinary language' [20, p. 298]. It is probably this reason which led the great American linguist Leonard Bloomfield to say that mathematics is the ''ideal use of language'' [4, p. 29].

The purpose of this paper is to offer some observations on the nature of the linguistic medium of verbal problems. Such a description may, of course, be of use as a heuristic aid in actual problem solving, since it attempts to provide a ''linguistic'' perspective of the problem solving process.

2. The Grammar of Verbal Problems

A grammar is simply the set of operational rules of a language (see, e.g., [15, p. 55]. Grammatical rules are characterized by the following features: (1) the objects, symbols or signs of a language are its words (or, as linguists call them, lexemes and morphemes); (2) these objects are concatenated or structured into phrases, clauses and sentences (i.e., they are arranged syntactically); (3) the immediate output of the structuring or ordering of words is a thought unit known as a sentence; (4) each sentence has meaning (or, in linguistic terms, a semantic configuration).

Thus, the rules of grammar allow a speaker to generate well-formed and meaningful sentences. Modern linguistics distinguishes between two basic sentence types: kernel and non-kernel. Kernel sentences are simple (declarative) and unambiguous ones, whereas non-kernel ones are complex structures which may be derived from kernels by formal algorithmic processes. This convenient model of language design is known as transformational grammar, and its chief exponent is Noam Chomsky ([5]), [6]).

The central idea in solving verbal problems is to set up an equation or equations (linear, quadratic, differential, etc.) corresponding to the verbal statement of the problem. Therefore, if an equation is the final output and goal of the linguistic deciphering process, then it may be referred to as a mathematical "sentence", and in an analogous fashion a grammar of equations may set up as follows. The symbols or objects of equations (the variables or the unknowns) correspond to the objects of ordinary language, namely words. Words in natural language have, of course, a precise meaning or finite range of meanings, whereas the letters or symbols of equations may have an infinite range of meanings (i.e., numerical interpretations). The symbols in mathematical equations are either alphabet characters (**a, b, c,** etc.) or other typical mathematical symbols (e.g., **dx/dy**, P(**n, r**), etc). Furthermore, the various operations found in equations correspond to the syntactic rules of language. Thus, $+$, $=$, , etc. corresponds to certain syntactic or semantic operations (e.g., compounding, relativization, etc.). Finally, the underlying meaning of an equation, which is its numerical solution set, corresponds to the meaning of a sentence in ordinary language.

The process of arriving at a solution, or of setting up equations, consists basically in translating ordinary sentences into mathematical ones. As Polya notes,[1] the term **translation** is appropriate because the problem solving process in verbal problems may be compared to ordinary "translation from one language into another" [19, p. 174]. Polya illustrates what he means with the following problem in plane analytic geometry:

(2.1) Being given the equation of a straight line and the coordinates of a point, find the point which is symmetrical to the given point with respect to the given straight line [19, pp. 176-177].

The solution of this problem consists in setting up grammatical isomorphisms as suggested in the foregoing discussion; i.e., in translating linguistic into mathematical grammar:

(1) **Symbol translation:** In the language of analytic geometry, the "coordinates of a point" may be translated as (**a, b**) If this is the "given" point, then the coordinates of the required point are, say, (**p, q**). The "equation of a straight line" translates into $y = mx + n$.

(2) **Syntactic operations:** The geometrical condition expressed verbally in the problem consists of two components which are both implied by the word "symmetrical". Thus, the two points are so related that: (a) the line joining them is perpendicular to the given line; and (b) the midpoint of the line joining them lies on the given line.

(3) **Output:** These syntactic conditions give the following two equations:

$$\frac{q - b}{p - a} = -\frac{1}{m}$$

(the perpendicularity condition)

$$\frac{b + q}{2} = m\frac{a + p}{2}, \text{ i.e.,}$$

(the midpoint condition)

(4) **Meaning:** This problem has a numerical solution when the coordinates of the given point (**a, b**), as well as the slope **m** and the numerical constant **n**, are replaced with numbers.

This procedure, or translation process, is characteristic of most verbal problems. As Polya states: "To set up equations means to express in mathematical symbols a condition that is stated in words; it is translation from ordinary language into the language of mathematical formulas. The difficulties which we may have in setting up equations are difficulties of translation" [19, p. 174].

3. Literal Translations

A "literal translation" is a term that may be used to describe the simplest form of verbal problem, namely the one in which the language offers few or no difficulties to the problem solver. Difficulties usually arise when the meaning of words, phrases or sentences are vague, ambiguous or elliptical. Therefore, before discussing literal translations, the type of linguistic mechanisms involved in producing so-called "ambiguous" sentences may be briefly described as follows.

Every language contains two levels: a "deep structure" and a "surface structure". The deep structure contains all the syntactic and semantic information required in decoding a sentence. At this level, each sentence has one and only one unambiguous semantic interpretation. Ambiguity occurs (by means of transformational filters) at the surface level where a sentence may have more than one interpretation. The following examples illustrate the difference:

(3.1) "Old men and women receive that journal".

(3.2) "Mary's murder amazed us" [i, p. 28].

In example (3.1), the meaning is ambiguous because the sentence may have two underlying or deep structure forms: (1) "Old men and old women receive that journal"; (2) "Old men and women (who are not necessarily old) receive that journal". The problem of interpreting the meaning of the sequence "old men and women" in the surface structure stems from the fact that the adjective may have two distributional possibilities in English (x = old; y = men; z = women): (1) $xy + xz$, which may be written in the surface structure as $x(y + z)$; (2) $xy + z$, which has an identical form in the surface structure. The substitution of $x(y + z)$ for $xy + xz$ collides with the sequence $xy + z$, and this is the source of confusion. Similarly, in example (3.2), there are two deep structure meanings: (1) "That Mary murdered someone amazed us"; (2) "That Mary was murdered by someone amazed us". To describe the technical apparatus used in relating deep to surface structures is beyond the scope of this paper. It should, however, be evident than an apparently innocuous sentence may have several meanings, and that the difference in interpretation may be due to discrepant deep and surface structure sequences.

A translation of verbal problems may now be viewed as an interpretation of the surface structure of the problem, expressed in ordinary language, in terms of its underlying deep structure, which consists of an unambiguous mathematical statement. Obviously, problems in setting up equations may arise when the surface structure of the problem is ambiguous.

In literal translations, the deep mathematical structure is isomorphic to its linguistic surface structure. The following examples may be used to illustrate literal translations:

(3.3) "Jack is twice as old as John. In four years the sum of the ages of Jack and John will be 20 years. How old is each at the present time?" [21, p. 27].

(3.4) "How many quarts of a 30% alcohol solution should be added to 8 quarts of a 40% solution to make a new solution which is 38% alcohol?" [3, p. 135].

(3.5) "From a point **A** on level ground, the angles of elevation of the top **D** and bottom **B** of the flagpole situated on the top of a hill are measured as 47°54' and 39°45'. Find the height of the hill if the height of the flagpole is 115.5 ft." [1, p. 30].

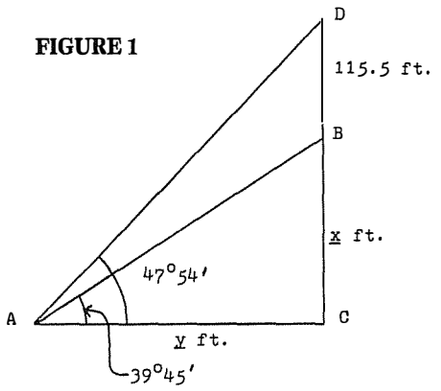
In example (3.3), the symbol x may be used to substitute John's age, and since "Jack is twice as old as John", Jack's age is $2x$. It is interesting to note that, although x and $2x$ stand for "age", mathematical objects normally replace entire phrases, clauses or sentences, as well as single words. It is therefore more correct to say that they substitute the semantic concept implied by the various morpho-syntactic classes (i.e., words, phrases, etc.). Difficulties in substitution emerge when it is not clear if a symbol may be used to substitute a word or some other structure. Accordingly, it is helpful to segment the verbal problem into conceptual areas and to cover all the areas of the "semantic map" with appropriate mathematical objects or symbols. The statement in (3.3) that "in four years the sum of the ages of Jack and John will be 20 years" implies three syntactic operations: (1) "in four years" is equivalent to adding 4 to x and $2x$; (2) "the sum of the ages of Jack and John" indicates the addition of the derived expressions $x + 4$ and $2x + 4$; (3) "will be 20 years" signals an equivalence relationship. The result is the equation $x + 4 + 2x + 4 = 20$. By substituting different letters for the ages (x for John's age and y for Jack's), it is easily seen that the verbal problem may also be translated into two linear equations: (1) $y = 2x$; (2) $y + 4 + x = 20$.

In example (3.4), the need for segmentation into conceptual areas arises more than n (3.3) because some of the concepts are implied rather that stated overtly. If x stands for the quantity of 30% solution to be transferred, then it is clear that this quantity contains $.3x$ quarts of alcohol. Similarly, the 40% solution contains .4(8) or 3.2 quarts of alcohol. The third conceptual area is the amount of alcohol in the new solution. The total number of quarts of liquid present in the solution is $x + 8$, and the number of quarts of alcohol of this new 38% solution is $.38(x + 8)$. The equivalence relationship derives from the axiom that the whole is equal to the sum of its parts. So, the translation of this verbal problem is the linear equation $.3x + 3.2 = .38(x + 8)$.

In example (3.4), the deep structure consists of a trigonometric relationship. Since the bottom of the flagpole may be extended (conceptually) to meet the horizontal through **A** in **C**, and **A** may be joined to **D** and **B**, a new

conceptual area emerges from the resulting diagram - the right triangle (see FIGURE 1). If **x** is the length of **BC** in ft. and **y** of **AC** in ft., the following two syntactic operations apply: (1) $y = (115.5 + x) \cot 47°54'$; (2) $y = x \cot 39°45'$. From these two relations, the following equivalence is established: $(115.5 + x) \cot 47°54' = x \cot 39°45'$. In this problem, as in most mensuration problems, a diagram is an important heuristic device because it is a visual representation or map of all the conceptual areas present in the deep structure.

FIGURE 1



4. Non-Literal Translations

Non-literal translations are verbal problems which are expressed in ambiguous or complex language so that the deep structure is not easily discoverable. The linguistic setting often obfuscates the underlying mathematical relationship; i.e., the path connecting the surface to the deep structure is laden with linguistic obstacles. Indeed, there may not even be a direct connecting path, but one which leads to the deep structure in a labyrinthic manner. The following examples may be used to illustrate such problems:

(4.1) "If a brick balances evenly with three-quarters of a pound and three-quarters of a brick, what is the weight of the whole brick?" [17, p. 6].

(4.2) "If Jack were two years younger than Jill would be if Jill were two years older than half as old as Jack would be if Jack were two years younger than twice as old as Jill would be if Jill were twice as old as Jack is, he would be ten years older than he is now. All of this makes Jack's age quite clear - or does it?" ['11, p. 58].

In (4.1), the compound statement "three quarters of a pound and three-quarters of a brick" may lead to the erroneous conclusion that the brick weighs ¾ + ¾ pounds. However, if the conceptual area "weight of the whole brick" is covered by **x** lb., then the deep structure relationship is seen to be **x** = ¾ + ¾ (**x**). The operation of equivalence translates the statement "balances evenly" and ¾ (**x**). The operation of equivalence translates the statement "balances evenly", and ¾ (**x**) translates "three-quarters of a brick" (i.e., of the **weight** of a brick).

Example (4.2) illustrates the complexity that may result from clause embedding - the process of connecting clauses together into a complex sentence. The mathematical equivalent of embedding is bracketing. The translation technique used in such problems may be referred to as "inverse" translation. The embedded clauses are converted to mathematical expressions by inverting their order of occurence. Thus, the last embedded clause contains the most deeply embedded mathematical expression. By working in reverse, each clause may be easily translated in terms of the preceding mathematical expression. Thus, if **x** is Jack's age now, then the problem may be translated as follows:

Embedded Clause	Mathematical Trans.
"If Jill were twice as old as Jack is"	$2x$
"If Jack were two years younger than twice as old as Jill would be"	$2(2x) - 2$
"If Jill were two years older than half as old as Jack would be"	$\frac{1}{2}[2(2x) - 2] + 2$
"If Jack were two years younger than Jill would be"	$\frac{1}{2}[2(2x) - 2] + 2 - 2$

The final output of this inverse procedure is equivalent to $(x + 10)$ because if Jack were $\frac{1}{2}[2(2x) - 2] + 2 - 2$ years of age, then he "would be ten years older than he is now." The translation of the problem is $\frac{1}{2}[2(2x) - 2] + 2 - 2 = x + 10$.

5. Pseudo Translations

A third type, which is only marginally interesting, may be referred to as a pseudo translation. This type resembles an ordinary verbal problem, but it contains either a play on words or deliberately deceptive statements. As Churchman points out, language is used in such problems with a "malicious intent" to deceive the problem solver [7, p. 151]. A few examples may be used to illustrate pseudo translations:

(5.1) "How many cubic inches of dirt are there in a hole that is one foot deep,

two feet wide, and six feet long?'' [12, p. 16].

(5.2) ''An express leaves London for Brighton at the same time as a slow train leaves Brighton for London. The express travels at a rate of 50 miles per hour, the slow train at the rate of 30 miles per hour. Which is farther from London when they meet?'' [18, p. 23].

There are two general characteristics which are normally found in puzzles of this type: (1) a syntactic environment consisting of irrelevant data (usually in the form of numbers) which directs the problem solver's attention to itself; (2) a ''stylistic'' imitation of ordinary verbal problems. Thus, in (5.1), the meaning of the word **hole** indicates that the dirt has already been taken out, and therefore it contains no dirt. Example (5.2), which simulates the style of an algebraic rate problem, may be solved correctly only if the meaning of the verb **meet** is considered. Obviously, when the two trains **meet**, neither one is farther from London.

6. Conclusion

Even a cursory look at the history of mathematics will show that problems expressed in linguistic form have always intrigued mathematicians. Such problems have a pedagogical value as well. In fact, the use of puzzles to teach mathematics is becoming a common practice. As Gardner states: ''Mathematics has never been a dreary topic, although too often it has been taught in the dreariest possible way. There is no better way to relieve the tedium than by injecting recreational topics into a course'' [8, p. ix].

The classificatory schema of verbal problems proposed in this paper (i.e., literal, non-literal and pseudo translations) is based on the linguistic properties of these problems, and may have a practical value since it can be used as a heuristic tool in problem solving. Two other classificatory schemas are of course possible: (1) a purely mathematical one which lists verbal problems according to their deep structure (e.g., linear equations, quadratic equations, etc.); (2) a ''stylistic'' one which classifies them according to recurrent themes (age problems, rate problems, etc.). The schema adopted here reflects more accurately the kind of relationship present between the linguistic form of the problem and its mathematical form.

In conclusion, it may be pointed out that investigations into the relationship between language, mathematics and thought are becoming more frequent. Language and mathematics are closely linked human artifacts which have allowed us to systematize and classify our experience. Linguists have always investigated the possibility of applying mathematical models to the study of language. The reason for this is that mathematics possesses language-like attributes, which, as Hockett points out, may not always be recognized by mathematicians: ''This ignorance on the part of the average mathematician has no bearing on the quality of his mathematics; but ultimately the language-like nature of mathematics is of basic importance, since it is the most critical clue to an understanding of the place of mathematics in the physical universe of which mathematics, mathematicians, language, linguists, and all of us are a part'' [10, p. 6].

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SOME IDEAS FOR TEACHING "PROBLEM SOLVING"

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Banting S.S.
London



Until recently, we, the educators, have assumed that expertise in problem solving developed somehow incidentally as a result of solving many different problems. Problem solving skills can be taught. In our teaching of mathematics and other disciplines we need to give more attention to the development of problem solving techniques and skills.

In this presentation we will consider some of the skills and strategies that are basic to problem solving. A problem is presented along with the discussion of the skill involved.

SKILL #1 READ, REREAD AND UNDERSTAND THE QUESTION

Restating the question may separate the important information from the 'filler'. It may help to act out the problem.

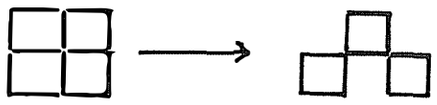
sample problem: Coach Joe Dunkshot of Dribble High had 12 basketball players. All but 7 fouled out of the game. How many remained in the game?

SKILL #2: BRAINSTORMING

Brainstorming consists of recording immediate conclusions, related facts, and alternatives.

sample problem: 12 fencing units are shown in the diagram. Move only 3 units to form 3 squares of the same size. Use all of the fencing units.

solution: brainstorm - have 12 widths
- want 3 units
- 1 square has 4 units
- therefore 3 squares with **no common sides** would require 12 units.



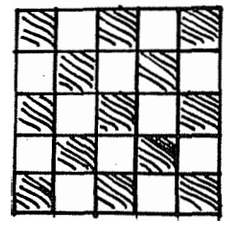
SKILL #3: USE A MODEL

Solve a familiar related problem and use it as a model for solving at least part of the more difficult question. The model may involve a diagram.

sample problem: In a certain classroom the 25 seats are arranged in a square of 5 rows with 5 seats in each row. The teacher announces that each student is to change his seat by going either to the seat in front

of him, the one behind him, the one to his right or the one to his left. Can this order be carried out?

solution: Consider the 5 by 5 checkerboard as a model.



- black moves to white
- white moves to black
- 25 squares - 13 black and 12 white
- It's impossible!!! (beautiful)

SKILL #4: BREAK FUNCTIONAL FIXEDNESS

Become aware of what psychologists call 'functional fixedness' or 'the mind set'. We often appraoch a problem with a one-track mind. We have difficulty in using devices in unaccustomed ways.

sample problem:
(x-a) (x-b) (x-c) ..... (x-z) = ????

solution:
The initial reaction may be to use Pascal's Triangle. Please look again.

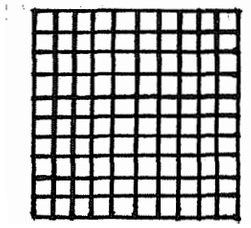
sample problem:
A man and his dog set out for a destination 6 miles away. The man walked at 2 mph and the dog at 5 mph. The dog arrived first and immediately turned back toward the man. Reaching the man, the dog turned back toward the destination and so on until both arrived. How fard did the dog walk?

solution:
Would you add up all the dog distances walked (series)? This would likely be the initial reaction. But read the question again, brainstorm. Break the mind set!

SKILL #5 IDENTIFY CASES-FLOW CHART

This may lead to a deductive or indirect proof. Work backwards.

sample problem:
How many squares are there in the following 10 by 10 grid?



solution:
Try to consider different cases.
n(1x1) = 10^2 = 100
n(2x2) = 9^2 = 81
n(3x3) = 8^2 = 64
etc.

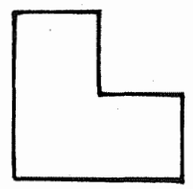
After investigating a few cases a pattern develops.
number = 100 + 81 + 64 + ... 4 + 1 = 385

SKILL #6: MAKE A GUESS; USE THAT AHA! ABILITY

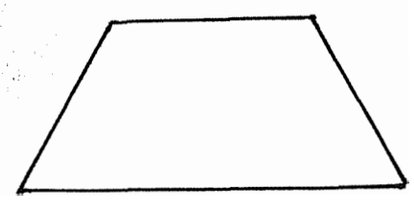
Guessing is important from a motivational point of view. If the problem solver makes a guess, at least he is committed to the problem which he has started.

Sudden hunches are called AHA!! reactions. They seem to come out of the blue. They may be proven by indirect proof or other methods. Even if the hunch turns out to be incorrect some additional information may have been obtained.

sample problem:
Ransom is a surveyor who specializes in dividing curious shaped lots into congruent parts. On one occasion he was asked to cut this lot into 4 identical regions. How?



Ransom's next job was to cut this piece of land into 4 congruent parts. There exist 2 solutions.



SKILL #7: ARRANGING DATA

sample problem: (The following example or versions there of has become popular since the 1978 O.A.M.E. conference)

The host at a party turned to a guest and said, "I have three daughters and I will tell you how old they are. The product of their ages is 72. The sum of their ages is my house number. How old is each?

The guest reached to the door, looked at the house number, and informed the host that he needed more information. The host then added, "The oldest likes strawberry pudding." The guest then announced the ages of the three girls. What are the ages of the three daughters? (all ages are integers)

solution: Initially this question is rather confusing. Don't expect to see the answer immediately. Many students will read a problem quickly and if they don't see an immediate way to solve it, will feel that they could never solve the problem. Students must realize that they are **supposed** to have a problem.

The key to this problem is to start. Perhaps start by listing data. List triplets with a product of 72.

(1, 1, 72) (1, 2, 36) (1, 3, 24) (1, 4, 18) (1, 6, 12) (1, 8, 9) (2, 2, 18) (2, 3, 12) (2, 4, 9) (2, 6, 6) (3, 4, 6) (3, 3, 8)

It may be helpful to find the sums of numbers in each triplet. The house number and ages follow.

SKILL #8: WALK AWAY

What do you do when you have tried everything and nothing works? My advice is to simply walk away, go for a run, go to bed, anything to get away from the problem. Frequently, this period of time allows misleading information to become less significant.

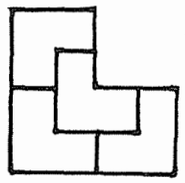
SKILL #9: LOOKING BACK

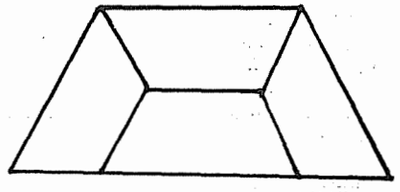
A good problem solver is someone who has learned to ask himself the right questions. After solving a problem ask yourself some useful questions.

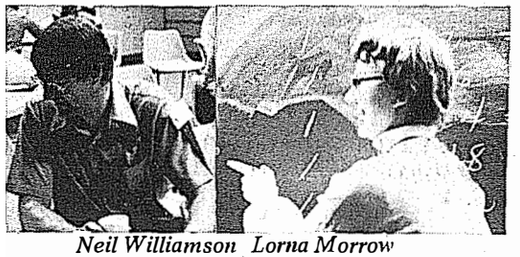
What was the decisive point?
What was the main difficulty?
What could I have done better?

IN SUMMARY, at least one part of mathematics education should be devoted to the analysis of problems. This is in contrast with the almost total previous preoccupation of mathematics education with basic skills themselves to the exclusion of an analytical approach to problem solving which selects the appropriate skills. It is important to note that our **knowledge** about any subject consists of both **information** (i.e. basic skills) and of **know-how**. Know-how is the ability to solve problems - not merely routine problems but problems requiring independence, judgement, originality, and creativity.

some solutions for problems stated under Skill #6.







Neil Williamson Lorna Morrow
Co-chairman Programme Committee
OAME Conference

PROBLEM SOLVING

Charles D. Gallant
St. Francis Xavier University, N.S.

. . . a teacher of Mathematics has a great opportunity. If he fails his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

Polya.

A ski instructor does not get his students heavily engaged in the history of skiing, the makeup of snow crystals, or the inner dynamics of woods, epoxies, and glass resins.
THE CRAFT OF TEACHING -- K.E. Eble.

Content and process goals in Mathematics instruction can be arranged as follows:

1. Knowledge: Facts and skills
2. Understanding: Concepts and principles.
3. Problem Solving: Strategies of choice.

Prior to 1950, knowledge goals dominated in Mathematics Education. The revolution in school mathematics, although addressing both goals #2 and #3, resulted, primarily, in an exaggeration of understanding goals. We set out to be better missionaries and ended up theologians.

Everyone agrees that Problem Solving is important -- the ultimate goal. But more than this, I believe that *knowledge and understanding are important only because they make problem solving possible and justified only if they make problem solving possible.*

I find it interesting to draw the analogy between the teaching of French and the teaching of Mathematics. Certainly, the goals stated above are essentially: Vocabulary, syntax, and conversation. Problem solving IS the conversation of Mathematics. We're still teaching grammar/

How can we teach problem solving? I'm convinced there's an answer to this!! And, we need not wait for another revolution, more research, or improved technology.

Problem solvers exist. Problem solving can be learned -- no doubt, to a different degree in different individuals -- but, again, problem solving can and must be learned. To teach problem solving means to arrange conditions so that problem solving can be learned more easily.

How, then, is problem solving learned? I believe the answer lies primarily in the strategies of choice. Students, in the main, have sufficient knowledge -- they have sufficient skills. *The difficulty is that they have no competence in making choices.* What is the relevant concept; the appropriate algorithm? Students ask -- Do I add or subtract? Is this a rate problem? Should I use logs here? Is this the type where we use ratio? If you agree this is the difficulty, let's address it.

It appears to me that we're presenting students with too many exercises and not enough problems. *What students need is much experience -- in the classroom and at home -- with well-chosen problems, whereby they will eventually learn which strategic choices to make.*

This paper will deal with the Strategies of Choice -- or, Initiating Heuristics. Perhaps a note on semantics is appropriate here, since, surely, the general topic of problem solving has been addressed many times before under differing labels, e.g., adventurous thinking, divergent thinking, lateral thinking, intuition, induction, know-how, open search, process, investigations, cognition, Socratics, active involvement, mathematizing, and the pedagogy of situations. Before addressing the strategies of choice, however, I'd like to emphasize what we're omitting.

Consider the following MOVES in Problem Solving. This is an organized 5-point plan for teaching problem solving.

MOVES In Teaching Problem Solving

1. MOTIVATION AND ATTITUDE. Provide an atmosphere conducive to problem solving.
2. COGNITIVE KNOWLEDGE. Make students explicitly aware of the **initiating** and **reflective** heuristics of problem solving. Identify the characteristics of the problem SOLVER and utilize research findings in problem solving.
3. PRACTICE. Provide opportunity for practice, especially in quizzes, homework, occasional math labs, independent study projects, and as a classroom substitute for drill.

4. METHODOLOGY. Use problem situations to introduce new concepts and/or principles.
5. IMITATION. Become a problem solver yourself. Allow students TO SEE YOU SOLVE A PROBLEM. Join with students in group approach to problem solving. COLLECT PROBLEMS of historical significance or of practical application value.

The main topic of this paper, as indicated earlier, is then only a subset of Moves #2 and #3.

So, there's much being left out. Many inconsistencies to resolve. For example, we say that problem solving is important. Yet, our attitude conveys that it's not important, or at least that problem solving is very difficult. We recall from our readings in Education the genetic principle of Kline and the dynamic principles of Dienes and yet mathematics continues to be so obviously isolated from problem solving -- from real-life applications. We isolate the right triangle from the carpenter's square; we isolate the linear equation from the so-familiar real-life situations which it models. We begin class with: "Today we will consider plotting parabolas using symmetry," or "Let V be a vector bundle."

Strategies of Choice / Initiating Heuristics

1. Identify WANTED, GIVEN, and NEEDED information.
2. Select appropriate notation.
3. Make a drawing, figure or graph.
4. Restate the problem.
5. Write an open sentence.
6. Draw from a cognitive background.
7. Construct a table and search for pattern.
8. Guess and check.
9. Systematize -- a strategy, e.g., elimination, cases, etc.
10. Make a simpler problem.
11. Construct a physical model.
12. Work backwards.

NB: Numbers 1-5 ARE TRANSLATION SKILLS!

The following examples of problems with what one might call "a pedagogical purpose," i.e., the solution is intended to practice some appropriate initiating heuristic. Hence, these problems and others of that type could be used to provide experience in the selection of problem solving strategies.

Examples of Problems

1. MARTIN GARDNER -- DICE PROBLEM
Three dice are thrown. Someone takes the number on one of these dice, multiples it by two and adds five. This result is multiplied by five, and the second number on a dice added to this result. This answer is multiplied by 10 and the third number added. The person working the trick (*who has not seen the dice*) looks at the result of these operations.

By subtracting 250 he can find the three numbers on the dice. Explain WHY this works!!

STRATEGY: 1. Select appropriate notation.

2. If you pick 50 telephone numbers and add the last two digits, what sum will turn up most often -- 18, 12, 9, or 0?

STRATEGY: 1. Construct a table.
2. Systematize.

3. Given a segment of length $a + d$, where a is the edge and d the diagonal of a square, construct the square using Euclidean tools, only.

STRATEGY: 1. *Work backwards.*

4. At what latitude is the distance around the earth half of what is is at the equator?

STRATEGY: 1. *Construct a model.*

Examples of Problems

5. One day, while hiking in the royal forest, Sir Bumble became lost. He came upon a marker which indicated he was in the exact centre of the forest. He knew that the forest area was a square 60 meters on a side. Not knowing how to get out, he started walking in the following pattern:

5 meters south, then 10 meters west, then 15 meters north, 20 meters east, 25 meters south, and so on.

How far did Sir Bumble walk before he finally got out of the forest and on which side did he exit?

STRATEGY: 1. *Make a figure.*

6. A man and his grandson have the same birthday. Is it possible that on SIX consecutive birthdays the man's age is an integral multiple of the grandson's age?

STRATEGY: 1. *Guess and check.*

7. What is the maximum number of questions required in order to determine my telephone number if each question is answered "Yes" or "No"?

STRATEGY: 1. Make a simpler problem.

STRATEGY: 1. *Make a simpler problem.*
2. *Table and pattern.*

Examples of Problems

8. A drawer contains 4 blue socks and 8 red socks. How many socks must be taken out (*the room is dark*!) to:
- ensure one of each colour?
- Ensure a pair?
- ensure a red pair?
- ensure a pair of blue socks?

STRATEGY: 1. *Systematize.*
2. *Guess and check.*

9. An alumnus wishes to leave St. Francis Xavier University sufficient money to ensure that a prize of $100 can be awarded each year at graduation, forever. What sum should he donate, assuming 9% interest?

STRATEGY: 1. *Draw from a cognitive background.*
2. *Guess and check.*

10. If a-1 and a+1 are twin primes greater than 10, then, $a^3 - 4a$ is divisible by 120.

STRATEGY: 1. *Restate the problem.*
2. *Subgoals (simpler problem).*

Examples of Problems

11. Beginning with the statement Numbered 1 (#1), arrange these in a logical order. (Each statement should be on a small separate sheet.)

$\sec B \cdot \csc B$

$\frac{\sec^2 B}{\tan B}$ (#1) *****

$\tan B + \frac{1}{\tan B}$

$\sec B \left(\sec B \left(\frac{1}{\tan B}\right)\right)$

$\sec B \left(\frac{1}{\cos B} \cdot \frac{\cos B}{\sin B}\right)$

$\sec B \left(\frac{\sec B}{\tan B}\right)$

$\tan B + \cot B$

$\frac{\tan^2 B}{\tan B} + \frac{1}{\tan B}$

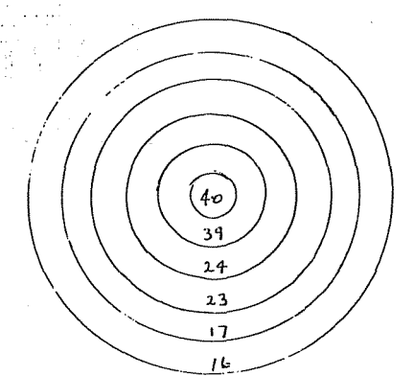
$\frac{\tan^2 B + 1}{\tan B}$

$\sec B \left(\frac{1}{\sin B}\right)$

STRATEGY: 1. *Systematize.*

Examples of Problems

12. What is the minimum number of darts which must be tossed (assuming accuracy) in order to get a score of exactly 100?



STRATEGY: 1. *Guess and check.*
2. *Systematize.*

In conclusion, then, this paper proposes that problem solving is, indeed, a learnable and a teachable object. The recommendation that is made is essentially that teachers direct attention to problem solving specifically by developing student ability to make strategic choices.

These recommendations will lead nowhere unless teachers are willing to break with current practice. We continue to be caught in the clutches of Connectionism -- teaching according to a logical sequence, an hierarchy, a taxonomy, which places problem solving at the end. The result, of course, is that problem solving is often not addressed at all.

Editor's Note: Answers to the problems will be printed in the September issue.

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INTERVIEW

Taped Talks with Bob Smith



Dr. Ed. Summers

Doctor Ed. Summers is at the University of Western Ontario as a visiting scholar with the Faculty of Education.. This means he is on leave from his home base, the University of British Columbia and is now involved with our faculty in seminar work, research, consultation and other faculty projects as well as having time for visiting schools, travel and library work. This interview grew out of discussions with him and a series of seminars on information retrieval given to faculty.

R.S.: Just what is ERIC?

E.G.S.: The acronym stands for the Educational Resources Information Center. As the name implies, ERIC is an educational data base and contains information of value to teachers, administrators, parents, counsellors, librarians, researchers, students, Boards of Education, Ministries of Education -- literally anyone with an interest in education. ERIC's mission is to identify and make available the significant literature produced in the field of education.

R.S.: How is this accomplished?

E.G.S.: ERIC consists of two cooperating units (1) 16 clearing houses, each working with the literature from a specific subject area, and (2) Central ERIC. Other elements of the public and private sector also get involved in printing and making ERIC computer tapes and products available. The clearing houses monitor their subject specialties and locate and slicit documents. These are then indexed, abstracted, catalogued with appropriate bibliographic information, and sent to Central ERIC. There the information is prepared for computer storage and indexed and magnetic tapes are produced on a monthly basis. The information is then made available in two monthly publications, **Resources in Education** (RIE) and **Current Index to Journals in Education** (CIJE).

R.S.: What type of materials can one expect to find in these two indexes?

E.G.S.: CIJE was begun in 1969 and announces pertinant educational articles from over 800 publishing journals on a monthly basis. Some are searched selectively and some cover-to-cover by the clearing house whose scope covers the area of the journal. RIE was first published in 1966. It's task is a bit more difficult. It announces a wide range of documents, largely unpublished, on a monthly basis. This includes curriculum guides, government reports, unpublished research reports, School Board publications, reviews, bibliographies, conference papers, evaluation studies, and the like. There are now close to 400,000 education documents which have appeared in these two publications and material is being added by the clearing houses at the rate of approximately 2,500 documents a month.

R.S.: That's a lot of information on education! I understand that RIE and CIJE **announce** documents and provide abstracts and bibliographic data. But, that does not totally meet the needs of users. Are full-text copies of the materials also available?

E.G.S.: Yes Bob, full-text materials are also available. This is one of the significant accomplishments of the ERIC system. Unless there are copyright restrictions by the authors, over ninety percent of the material in RIE is available from a private source (ERIC Document Reproduction Service) in microfiche or hard copy form. Complete information for ordering documents is contained in RIE. Normally, users wishing to obtain copies of the journal articles announced in CIJE would access libraries and other sources of journal collections. A recent ERIC innovation has been the addition of a commercial reprint service (University Micofilm International) which makes available full-text copy for approximately fifty-five percent of the articles indexed. This is a good supplement to traditional library collections. Full information for ordering reprints can be found in CIJE.

R.S.: How much Canadian content could one expect to find in ERIC?

E.G.S.: It is difficult to state an exact figure, but a rather significant amount of Canadian material is announced by ERIC. The system is U.S. in its orientation of course. No systematic coverage is made of Canadian materials although some of the journals are now regularly indexed. However, many of the clearinghouses are aware of Canadian materials and input them. In addition, many School Boards, Faculties and Ministries of education, and individuals in Canada are very much aware of ERIC and regularly submit materials. Retrieval of specific Canadian content has been made easier in recent years by the addition of descriptors reflecting Canadian interests.

R.S.: How does one best go about searching the ERIC data base?

E.G.S.: One of the most productive techniques is to obtain recent issues of RIE and CIJE and do a hard search on the topic of interest. This is accomplished using the subject indexes. Many times this type of search locates sufficient information to satisfy the users needs. If one wishes to cover an extensive time span and the search is more complex, a computer search can be run using one of the available commercial services now operating (DIALOG, BRS, ORBIT). Descriptors are selected from the 6,000 plus terms listed in the **Thesaurus of ERIC Descriptors** and formed into a search statement using a Boolean approach to indicate inclusion, exclusion, and intersection of content. Many Faculties of Education, academic libraries, Ministries of Education and some School Boards now regularly access ERIC through on-line commercial retrieval services. Costs can run anywhere from $20.00 and up for an individual search. Hard copy is usually available but with a two to three week turn-around time. At UBC we have developed a different approach. Faculty of Education students and faculty make extensive use of ERIC. We subscribe to the ERIC computer tapes and mount them in the UBC computing centre. Thus, with our in-house approach, we can search on-line directly or batch a number of requests. The most practical approach is to batch requests for economy and we run searches about every two or three days. We also provide user seminars to introduce them to the ERIC system and the searching of computer data bases. I have also developed a course on information systems and educational research and development which covers aspects of development and use of machine readable bibliographic data bases. Our approach provides more than just a search service. In an adademic environment, we want to help users become proficient searchers and not have to depend on an intermediary for searching. In addition, we want to educate users about the information explosion and the emergence of machine readable data bases (such as ERIC) and how to cope with new information tools.

R.S.: Does ERIC do anything in addition to locating, organizing, and making information available?

E.G.S.: I'm glad you asked that Bob. Many people are aware of ERIC's information retieval mission. However, few realize the role it plays in another important area, i.e., the analysis and synthesis of information. This is accomplished through the efforts of the 16 clearinghouses, Central ERIC and the many professional associations in education. Analysis and synthesis efforts try to review the information being produced and determine the state-of-the-art on practices and issues in education.

R.S.: How does one become aware of analysis and synthesis publications?

E.G.S.: Publications are often announced in RIE when they are completed. Professional journals also carry information about such publications. A good way to find out about these and other ERIC products is to write to the clearinghouse of interest and ask for information that is available.

R.S.: It sounds like ERIC contains a lot of very useful information. Is there any comparable system that handles just Canadian documents?

E.G.S.: No computerized information system is now operating to organize and retrieve on a national basis related to Canadian education, although I have long felt one should be created. the ONTERIS system is somewhat comparable but is limited to documents from one province. It locates research and other materials, produces an abstract publication, and also makes a good percentage of the material available in microfiche and hardcopy form. It is a well developed, in fact an excellent system and reflects the state-of-the-art in developing computerized information retrieval services. As the ONTERIS file expands computerized searching also becomes possible through use of its PRECIS index language. The Canadian Education Association now makes some of the **Canadian Education Index** materials available in microfiche format and **Canadiana,** the national bibliography produced by the National Library, also announces educational materials of various kinds. These two are in varying

stages of computerization, mostly in production of their indexes, although they are not really bona fide machine readable bibliographic data bases at present. I've often thought that with some creative planning and additional funding, input to ONTERIS could be expanded and in a short time a national educational information system could be built. This system would collect information from across all provinces and serve all educators in Canada. Education is a keystone in the national identity of a nation and by creating a national, bilingual system much could be done in focusing the national identity of Canada through its educational system. It's something to consider. We should avoid relying on other countries to collect and make available our educational information.

R.S.: With all the material now being stored and made available, what can a journal like the Gazette do to aid readers in accessing these systems?

E.G.S.: One thing I would suggest Bob is that the journal create a special column to run each issue announcing some of the more pertinent materials available which would be of interest to math educators. You could call it **Math Bits,** or **Math Info On-Line,** or some such thing. The idea would be to have one or two authors monitor systems like ERIC, select useful information, and describe it for the readers. Occasionally, a whole column could be devoted to information available on a single topic. At times several pieces of information could be synthesized and the implications presented for reaction. Many different formats and approaches are possible. This type of column is presently being carried in many educational journals and is an effective way of using such systems and sharing information. Finally, such a column could also serve the valuable function of keeping readers up-to-date on the new developments in information systems as we continue to grapple with the explosion of information.

R.S.: What is meant by the term **information explosion?** Is it really an **explosion** or **flood** of information that is causing concern?

E.G.S.: I think what we have is a flood rather than an explosion of information. Explosion has been the **in** term because of its dramatic effect. Since the 1940's we have experienced a dramatic increase in the information available in all areas. It is suggested that the increase is exponential rather than arithmetical. This is the root of the problem. For example, research libraries now expect to double their holdings every ten years. In 1969 the UBC library was just under 1 million volumes, in 1979 the holdings had increased to 2 million, and it is expected that we will reach 3 million before the next six years. Libraries have been well aware of this problem and others are becoming increasingly atuned to it. Many of the newly developed computerized information retrieval systems have emerged as a response to the need to find better ways of organizing and retrieving information.

R.S.: Do you see the information explosion continuing?

E.G.S.: Definitely, although I think we are seeing a bit of a plateau effect at present. We have to also ask whether there is a concomitant increase in **knowledge** as well as **information**. We are flooded with information but this does not necessarily suggest that knowledge is increasing all that fast. It's interesting to note that it has been suggested that if you really want to estimate the significant paper's being published in an area take the square root of the total number in print. So the amount of significant material ends up being rather small. Retrieval systems do aid in bringing quantities of information to our attention so we can be more efficient in our evaluation and sifting to get at the really useful materials. western society is founded on inquiry. Problem solving and the resultant production of new information and knowledge is literally our raison d'etre and I don't see any marked shifts in this philosophy in the forseeable future. Unless we conduct our intellectual business in markedly different fashion, I see no decrease in sight in the amount of information produced. Computerized bibliographic data bases, of which there are now over 500 available, will continue to play a significant role in society. It has been estimated by 1990 we'll spend 85% of our time in bibliographic research accessing, directly or through intermediaries, some form of computerized retrieval system. Such systems can take the drudgery and inefficiency out of research and development activities and allow us to get in with the intellectual task of making sense of what is being produced and creating and exploring new paradigms in the Khunian sense of the term paradigm.

R.S.: Thank you Ed. I do hope our readers will try ERIC. I believe O.M.G. should carry a regular column. Who will be the first to volunteer?

SOME USEFUL INFORMATION

Macmillan Information
866 Third Avenue
New York, New York 10022
Published the 7th edition of **The Thesaurus of ERIC Descriptors**

Oryx Press
3930 East Camelback Road
Phoenix, Arizona
Publishes CISE

Superintendent of Documents
U.S. Government Printing Office
Washington, D.C. 20402
Publishes RIE

ERIC Document Reproduction Service
P.O. Box 190
Arlington, Virginia 22210
Sells microfiche and hardcopy of ERIC documents (RIE)

University Microfilms International
UMI Article Reprint Department
300 North Zeeb Road
Ann Arbor, Michigan 48106
Sells reprints of CISE articles

National Institute of Education
Office of Dissemination and Resources
Washington, D.C. 20208
Central ERIC

ERIC Clearing House for Science
Mathematics and Environmental Education
Room 310
1200 Chambers Road
Columbus, Ohio 43312
The most relevant ERIC Clearing House for O.M.G. readers

ALGEBRA - IMAGERY - PROBLEM SOLVING



Dave Alexander, Faculty of Education, University of Toronto

How do beginning students of algebra solve algebraic problems? One way of finding out is to ask them. Roma, Joe, Danny, Sally and Kit are pseudonyms for Grade 9 students interviewed by the author over their Grade 9 year. In reading the extracts involved you may imagine the interviewer biting his tongue many times to prevent changing from "observer" to "teacher."

Interviewer: V = *lwh*. Express *w* in terms of variables.
Roma: All 3 of them? . . . um, I am not sure but I would say that (writes $w = lh/V$).
Interviewer: How did you work that out?
Roma: Well, you have to keep these multiplying together [points to *lh*] as they are and you bring this [points to *V*] to the other side. You take it under . . . I am not sure, I am not doing it by any mathematical thing. I am trying to guess it.
Interviewer: Do you know what this formula would represent?
Roma: Volume.
Interviewer: Mm hm, volume.
Roma: Equals length times width times height.
Interviewer: Now, if you are trying to find the width?
Roma: Oh, well then, I am right, wait. Oh, you could take the volume divided by length times width.
Interviewer: Length times height?
Roma: Yes, and you get width.
Interviewer: Yes, is that what you've got?
Roma: No, I've got it backwards.
Interviewer: Mm hm.
Roma: But, oh! O.K. But I, um just do that from thinking about it, thinking of a box.

Roma has solved an algebraic problem, but is Roma thinking algebraically? Her attempt to solve the problem by symbolic means is simply guessing, not "any mathematical thing." She is successful when "thinking about a BOX" (probably something like a shoe box) but can't relate the answer when obtained this way to an algebraic process.

Interviewer: Now can you make this, what you have done, agree with the answer that you got by thinking about the box?
Roma: Do you mean write it?
Interviewer: Well, you tried to work it out here, by doing the same thing as you have done with equations.
Roma: I already, maybe it would be because you have to . . . take . . . take these down [points to *l* and *h*], but I don't know why, I just don't know why!

Janice Flake in the Gazette of March 1979 points out the importance of the development of mental imagery in the mastery of elementary mathematics (for both children and prospective teachers). This same theme is referred to by Stuart Plunkett in Mathematics Teaching, March 1979.

> "But it is important how we help children to deal with abstractions. Too often number work is symbol manipulation where children do not fully understand the symbols, and do not have anything to which they can refer."

The importance of the imagery employed by a learner in mathematics is not limited to arithmetic. Dietmar Kucheman in the interview recorded for us by Bob Smith (Ontario Mathematics Gazette, December 1979) points out that 15 year olds assign a large number of different meanings to letters used in an algebraic way.

Roma: $a + a = 2a$ because there are two a's
$a + a + b = 2a + b$ because you cant't add an a and b.
Interviewer: Why can't you add an a and a b?
Roma: Because it is like apples and oranges, you cannot add them.

If we are to follow Kucheman's suggestion

> "we ought to acknowledge the existence of these methods and possibly even try to use them rather than ignore the way children think."

then we should start to collect information on the types of imagery students use in their initial exposure to algebra.

In doing so, we should not be so simplistic as to consider that the imagery used will remain static for an individual student. For example, consider the way Roma deals with the algebraic problem only one month after "thinking about a box."

Interviewer: $V = lwh$. Express h in terms of other variables.
Roma: . . . O.K. . . . So if **V = lwh** then l would be equal to V over wh, I think.
Interviewer: Mm hm. But I was aking for h there.

Roma: Oh, h. Well, it would be the same thing then (writes $h = V/lw$).
Interviewer: How did you work that out?
Roma: Well, I did it with numbers I used, V, I said, was 8. l and w were 2. And 8 is 2x2x2, and so if you have 8 divided by 2x2 that's 4. You get 2. I don't know whether it would work out in algebra.

Once again, although Roma succeeds in solving the problem by employing imagery (this time numbers, rather than a box) she still fails to relate the arithmetic processes to the algebraic processes. In fact, she soon demonstrates her fundamental method for solving such problems successfully is still her "box."

Interviewer: $P = 21 + 2w$. Express w in terms of P and 1.
Roma: . . . Express w . . . It would be P, I think it would be P over 21. Oh! But they're the same thing. That is 8 and that's, 1 is 2 and w is 2 so it's 4 + 4. And 8 divided by 4 is equal to 2. I was very. Guessing, and I don't know any mathematical way to do it.
Interviewer: Could you try some other numbers and see what happens?
Roma: O.K. You could say P is 16 (writes 16 = 2.4 + 2.4). That works too.
Interviewer: Mm hm.
Roma: Yeh! No it doesn't! It doesn't work! It would have to be, for this one it would have to be P divided by 1. Wait a second! This, I guess, is 20 (writes 20 = 2.6 + 2.4). That doesn't work at all!
Interviewer: Do you know what that represents?
Roma: Well, that could be length and width.
Interviewer: So what could the P be?
Roma: Perimeter, oh! . . . so you want width. Oh, you just . . . Oh, you just say (writes $w = P - 21$), no that would give you $2w$. Oh, you could say, I think you could say, (writes $w = P - 21/2$). That's easy when you know what it means.

Of course, some students can handle numerical imagery with confidence.
Interviewer: (reading) $V = lwh$. Express h in terms of the other variables.
Joe: Oh. O.K. um . . . you'd have a volume . . divided by length times width (writes $v \div lh$).
Interviewer: Is equal to the --?
Joe: Is equal to the height (writes $= h$). If you put that into numbers, let's say the volume is 21 and the length is 7 and the width is 3, you'd have, and the height is 1 (writes V = lwh
21 7 3 1
If you multiply the 7 and 3 and the 1 together, you'd get 21. So if you divide the 21 by 7 and 3 you get 1 and that is the same number as the height (writes V ÷ lw = h
21 7 x 3 = 1
Interviewer: Suppose I said that h was equal to length times width divided by v, could you show me that's wrong?
Joe: . . . No it wouldn't be wrong with the numbers I am using. But maybe if I use something else. O.K. we will say that length is 2, width is 4, height is 8. So it looks like the volume should be 64 (writes $l = 2$, $w = 4$, $h = 8$, $V = 64$). So you have length times width would be 8, divided by 64 (writes $lw \div V$, $8 \div 64$). 64 can't really go into 8. Well it can. You'd get and that is not the same as your height which is 8.
Despite this exchange we get the impression

that Joe thinks of the variables as **objects** to be manipulated not really numbers.

Interviewer: How did you figure out the *V* divided by the l times w?
Roma: I don't know, I think I guessed a lot of it. But it seems to make sense that if you multiply things together to make something, that when you divide them, divide any of those things, into what you've got as your total, that you should get the answer of the others. What's left.

Perhaps the real power of problem solving in algebra comes from being able to switch the imagery used as the complexity of the problem faced varies. Danny can employ the imagery of numbers.

Danny: *a* + *a would be 2a.*
Interviewer: How do you know?
Danny: Well, multiplication is really addition. So 2 x 3 is 2 groups of 3, or 3 + 3, so that *a* + *a* would be *2a.*

But sometimes he thinks of letters as "things" that are "put on" the numbers.

Danny: *2a* + *3a* is *a* + *a* + *a*, which is five *a*'s which would be *5a*.
Interviewer: Do you always think that way when you are doing that sort of question?
Danny: Well, not really, I just take the *a*'s for granted. I know that I can add those (the '2' and '3') together and so I can take the '2' and the '3' and I add those together and make '5' and put the *a* on.
When faced with another problem he employs numerical imagery with good insight.

Danny: (x^{-1}) (x^{1}) Well it is the same thing when you are multiplying by the same base, then you can add the exponents so you've got (writes x^{-1+1}). They cancel out each other so its x^0 and any number to the zero, or any number or unknown to the zero as an exponent, is always 1 except for 0.
Interviewer: How do you know that?
Danny: Well it is just a law, and let's see . . . 10 squared is a hundred, and 10 to the 1 is ten, and 10 to the cube is one thousand. (writes

10^3 - 1000
10^2 - 100
10^1 - 10

It is easy to see that every time the exponent goes down by one . . . um, let's see, the answer you end up with goes down . . . How can I phrase this? . . . goes down to one tenth of what it was, so what the exponent goes from 3 to 2, 1000 goes to 100 and 100 is one tenth of 1000. So when you go down to 10^0 then one tenth of 10 is 1. (writes 10^0 --1), and (writes $10^{1/10}$) and it goes all the way down that way.

One sometimes wonders if the school experience usually stresses the manipulation of symbols rather than the use of imagery, destroys the natural insights that the imagery supples. Or is it simply that we move on to another set of problems and are satisfied with surface solutions to the old problems. Whatever the reasons Danny responds differently to this problem a few weeks later.

Interviewer: Mm hm and what is x^{-1}equal to?
Danny: $x^{-1}$is x, wait a sec . . . it is (writes $x^{-1}x \div 10x$), I think, mmmm. Oh no! (makes pencil scratches across written work). Now wait a sec. that won't . . . x divided by . . . Oh! One over x (writes $\frac{1}{x}$). That's right, it's one over x.
Interviewer: How did you figure that out?
Danny: Well, um, um, well, it's just that whenever, like negative numbers, it's a fraction. When you get positive it's x times x times x, but when it's negative, like negative, um. I don't know how to explain this. It just seems that I have been taught that. It's just that that's the way it works out I guess.

Not only has he lost his inspiration, he has become complacent about a mechanical grasp of the concept and has no interest in recalling his more insightful view.

Interviewer: That's interesting because the last time you explained it very well.
Danny: Really?

Problem solving, in algebra, or anything else, is more than the application of a procedure. What exactly its components are we don't know. We do know that problems cannot be solved without thoroughly understanding what the problem is about. Is understand "what something is about" really having some image we can manipulate in our minds? Should the emphasis of our teaching be on the transmission of appropriate images rather than procedures or abstractions?

Interviewer: If you had to describe what algebra was to somebody that had never taken it, what would you say?
Sally: I think it's working with variables, these are, I am pretty sure they are called variables, aren't they? And there are all sorts of rules that you have to remember.
Roma: We never knew what it was, we just sort of learned it. When I think of algebra I think of using x's and y's, that's all I know.
Kit: There is only one work for algebra and that is "yuk!"

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CANADIAN MORTGAGES

Richard D. Wink, Adam Scott C.V.I., Peterborough

Mortgages and the cost of home ownership have found their way into several mathematics courses including grade 13 functions, grade 12 level and some of the basic math programmes. The topic is a highly motivating one for students - especially when they discover the astonishing amount of interest paid on a home mortgage. An amortisation schedule usually proves an eye opener when the details of the very few payments are studied. The effect of varying the interest rate and the period of amortisation can also be examined and hopefully the student can be persuaded that a 15 or 20 year mortgage is a very much better prospect than a 35 year one. The payments are very little greater but the reduction in interest is very significant.

This article describes how common Canadian mortgages are calculated and a BASIC computer programme listing is included. The equations can easily be handled by a programmable calculator and a Texas Instruments SR 52 listing may be obtained from the magazine reference at the end of this article. The BASIC programme described here enables one to enter the compounding period and three of the four remaining variables, whereupon the fourth is calculated. Usually the principal of the debt, interest rate and number of years of amortisation will be input and the monthly payment will be output. However, the problem can be turned around and if a hypothetical couple can afford so much a month for their home mortgage, the maximum size of their debt can be determined. The unknown is always entered as a 0.

A mortgage quoted with an annual interest rate i and compounded c times a year has a monthly interest factor i' given by

$$i' = (1 + i/c)^{c/12} - 1$$

where interest rates are in decimal rather that percentage form. The monthly installment payment (equivalent to Plani) is

$$PMT = Pi'/(1-(1 + i')^{-12t})$$

where P is the principal and t is the time in years over which the mortgage is amortised. This time should not be confused with the

```
NOTE: N$ IS AN INTEGER VARIABLE WHICH
      CAN BE REPLACED WITH THE REAL
      VARIABLE N. SIMILARLY FOR C%.


10 PRINT"THIS PROGRAMME COMPUTES"
20 PRINT"MORTGAGE DETAILS INPUT"
30 PRINT"ANY 4 VARIABLES INCLUDING"
40 PRINT"COMPOUNDING PERIOD. PUT A 0"
41 PRINT"OPPOSITE THE UNKNOWN"
50 PRINT"ENTER PRINCIPAL $"
60 INPUT P
70 PRINT"ENTER ANNUAL INTEREST RATE %"
80 INPUTI
90 PRINT"ENTER NUMBER OF YEARS OVER"
100 PRINT"WHICH LOAN IS AMORTISED"
110 INPUT N%
111 N%=N%*12
120 PRINT"ENTER MONTHLY PAYMENT $"
130 INPUT PT
140 PRINT"ENTER COMPOUNDING PERIOD"
150 PRINT"MONTHLY=M, QUARTERLY=Q,"
160 PRINT"SEMI-ANNUALLY=S, ANNUALLY=A"
170 INPUT C$
171 REM C% = #COMPOUNDINGS 1,2,4,12
180 IF C$ ="M"THEN C%=12
190 IF C$ ="Q"THEN C%=4
200 IF C$ ="S" THEN C%=2
210 IF C$ ="A" THENC%=1
211 REM FNA ROUNDS TO 2 PLACES
212 DEF FNA(X) = INT((X+.005)*100)/100
280 IF I = 0 GOTO 420
281 REM CALCULATION OF DECIMAL MONTHLY RATE
290 R=(1+I/100/C%)↑(C%/12)-1
300 IF P=0 GOTO360
310 IF N%=0 GOTO330
315 REM FNA ROUNDS TO 2 PLACES
320 DEF FNA(X) = INT((X+.005)*100)/100
321 REM CALCULATION OF MONTHLY PAYMENT
330 PT=FNA(P*R/(1-1/(1+R)↑N%))
340 PRINT"MONTHLY PAYMENT = $";PT
350 GOTO 480
351 REM CALCULATION OF PRINCIPAL
360 P=FNA(PT*(1-1/(1+R)↑N%)/R)
370 PRINT"PRINCIPAL = $";P
380 GOTO 480
381 REM CALCULATION OF YEARS OF AMORTISATION
390 N=FNA(LOG(PT/(PT-P*R))/LOG(1+R)/12)
400 PRINT"NO. OF YEARS OF AMORTISATION";N
401 N%=N*12
410 GOTO 480
411 REMCALCULATION OF RATE BY ITERATION
420 R=.1
430 R2=PT*(1-1/(1+R)↑N%)/P
440 IF ABS(R2-R)<.00000001 GOTO470
450 R=(R+R2)/2
460 GOTO 430
470 PRINT"ANNUAL INTEREST RATE =";
471 PRINTFNA(((1+R)↑(12/C%)-1)*100*C%);"%"
480 PRINT"TOTAL INTEREST = $";FNA(PT*N%-P)
490 PRINT"DO YOU REQUIRE AMORTISATION TABLE? Y OR N"
500 INPUT D$
510 IF D$ = "N" GOTO 610
520 PRINT"PAYMENT INTEREST DECREMENT OUTSTANDING"
530 PRINT"PERIOD          PRINCIPAL.PRINCIPAL"
539 REM OP=OUTSTANDING PRINCIPAL
540 OP = P
549 REM K = LINE OF AMORTISATION TABLE
550 FOR K= 1 TO N%
559 REM IN = INTEREST PAID THAT MONTH
560 IN =FNA(OP*R)
569 REM DP = DECREMENT OF PRINCIPAL
570 DP=FNA(PT-IN)
580 OP=FNA(OP-DP)
590 PRINTTAB(4);K;TAB(9);IN;TAB(18);DP;TAB(28);OP
600 NEXT K
610 END
```

term over which the interest rate is fixed (often five years). The amortisation period is the time it takes to pay off the debt totally.
The interest paid over the full period of the loan is simple the total paid less the principal or

$$\text{TOT INT} = \text{PMT} \times 12t - P.$$

It would be a simple matter to derive an expression for outstanding balance at any time, but it would give results in conflict with "professional" printouts because it would neglect the cent round-off factor which occurs each time a payment is made. Consequently the amortisation table must be produced in an iterative fashion. If OP (n-1) stands for the debt just prior to the nth payment, then the interest payable at that time is I(n) where

$$I(n) = OP(n-1)\, i'$$

The reduction of the debt DP (n) is the regular monthly payment less interest.

$$DP(n) = PMT - I(n).$$

The outstanding debt after payment has been made is OP (n) where

$$OP(n) = OP(n-1) - DP(n).$$

The BASIC programme is self-prompting and a "list" and two sample "RUNS" are shown. For a debt of $40,000 paid off over 20 years the monthly payment = $425.81 and the total interest = $62194.4, for the same debt paid off over 35 years the monthly payment = $389.57 but total interest = $123619.4.

Depressing isn't it?

Reference: Easy Mortgage Math. Electronics Today "Softspot" September 1978, Vol 2, #8.

GAMES BRING LIFE TO MATH

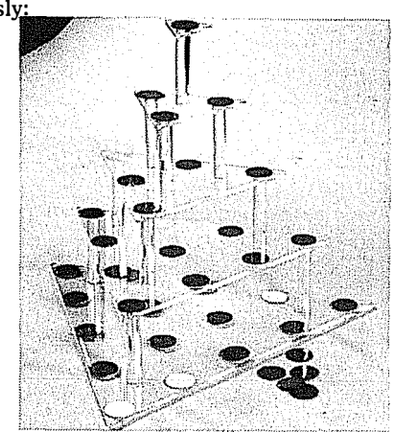
Mariano Rodrigues
Rhode Island College

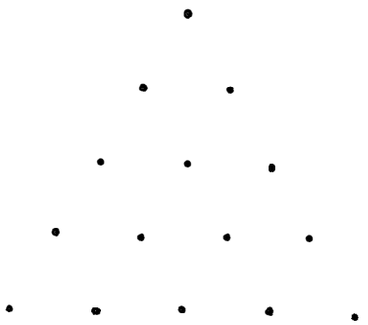


Although some still question the use of games in the classroom, this writer believes that using games can serve serveral purposes including

a. the acquisition of language and techniques of problem solving, particularly with respect to patterns helpful in solving math problems and underlying the methodology of math; and
b. the sharing of an experience which says to the participants, "Math is alive!"

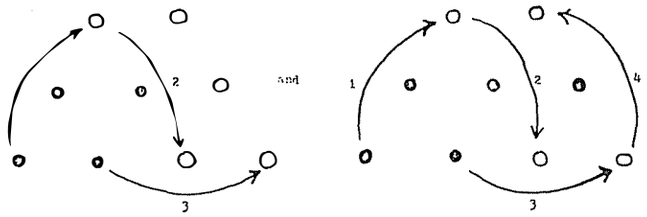
At one time, I had the occasion to visit a colleague's class. A triangular solitaire game had been brought in for study, and I joined in by prodding them in several directions. The game, which is well known, is a two-dimensional board game in which holes are arranged in a triangular array thusly:





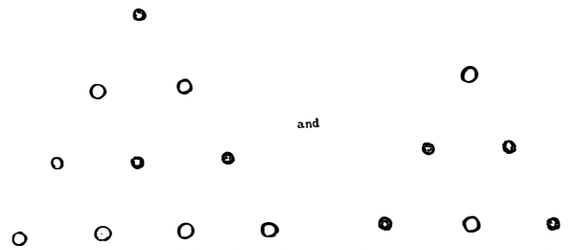
Games of this type are available with five, six, or even seven rows. Pegs are placed in all but one of the holes, usually one of the vertices, and the object of the game is to remove all but one of the pegs by jumping and removing the jumped peg. Jumps are made along a straight line and are similar to the king's jump move in checkers.

After a period of no success, I suggested that, to analyze the game further, we could study the potential end-states of the problem. We then attempted to work backwards. In so doing, students were beginning to recognize arrangements, simpler than the original, which were easily solved; but they were also learning to identify unsolvable arrangements. For example, solvable arrangements include



and

while some unsolvable arrangements include



and

Having achieved no success trying to work back to the initial state, we returned to working forward. Many of the students now encountered patterns which they had learned to recognize when they were "playing around" earlier with the simpler end-game set-ups. A solution came rather quickly. Once its replicability was established (we related here to the concept of "proof"), the question I asked was whether that was the end of it all?

When we solve a problem, do we then simply go on to other problems, or do we delve further? Frequently, I asked, could we create a new game which used the old one, in its entirety, as a substructure?

Students readily suggested adding more rows to the triangle. When I suggested that the plane produced a mind-set of its own, I elicited the suggestion that the game could be developed in three dimensions. Still, the question was "How?" But building a five-tiered triangular pyramid was easily demonstrated using flimsy cut-outs.

One student said that she had always wanted to build some three-dimensional geometric models, and she was in need of a "project" for an ensuing math fair. Indeed, two weeks later, she brought her full-scale model to the fair. Since it was a new and unsolved problem, it generated a good deal of interest at the fair, but none could successfully get the puzzle down to one piece. Solutions were later developed by her teacher and myself.

Once we were sure that solutions existed, we proceded to a patent lawyer. TRI-O is now registered with U.S. Pat. No. 4082283 c 1976 Javiro. Problems are indeed not dead when they have been solved!

The new game consists of five triangular tiers of differing sizes. The top triangle has one dot on it; the next tier is a triangle with two dots on each side; the next tier has three rows of dots, etc. Jumps can be performed on any of the faces of the resulting tetrahedron, on any of the levels, or within triangular cross-sections. There are three moves that can, as a result, be made **through** the object, adding still further challenge to the player. We have also devised two-person games which can be played with the apparatus. Many have been immersed in the game for hours.

All who participated admit that the activities leading up to the game were exciting and informative. We had studied geometric patterns for solvability, and we had **explored**, using patterns of mathematical reasoning. We had posed a more generalized problem, using a solved problem as a special case in guiding us to the generalization, and we had all come to realize that solving puzzles is not a mindless, pointless activity in which nothing constructive can be learned; indeed, we had all learned something about mathematical methodology and problem-solving.





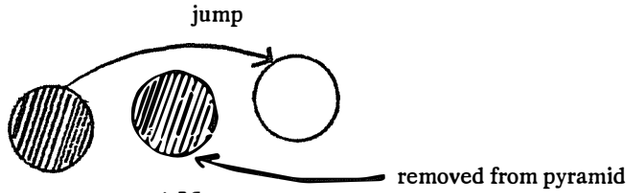
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U.S. Pat N. 4082283

Tri-O builds on an old mountain game with which you may be familiar. This new game is more challenging and more exciting than the old; by lifting the game into three dimensions, we have brought the game to new heights of challenge and versatility. The game involves five triangles of differing sizes arranged to form a pyramid. Each triangle is marked with dots.

The game comes equipped with colored chips. To start the game, place a colored chip on each dot except the one at the peak. The object of the game, when played solitaire, is to remove all but one of the chips. You remove a chip by jumping it, chechers style.

In order to jump, you need three mutually adjacent dots alined in a straight line. When one of the end dots is vacant and the other two are not, the middle piece may be jumped. The jumping piece moves to the vacant dot, and the jumped piece is removed from the pyramid.



A Move
Shaded circle indicates occupied position

The jump can be performed on any of the horizontal levels, along any edge, on any face, or in any triangular cross section. In the case that a three dimensional move is made, it should involve three adjacent levels. Therefore, a jump cannot be made from the first level to the fifth level.

RULES OF THE GAME

Solitaire 1 This is the game mentioned above. Start with all of the dots occupied except the one at the top. Through permissible jumps, remove all but one piece from the game.

Solitaire 2 Start with any other dot unoccupied and proceed as in Solitaire 1. Be careful, though; there are some positions which, if left unoccupied, will not yield a solution.

Solitaire 3 Before starting the game, pick a level on which you want to **end**. Start with the top position unoccupied (or any other of your choice) and end on the selected level.

You can go on inventing your own variations of the solitaire game.

Two-Person Games

Game 1 Starting with the top unoccupied, proceed by alternating turns (in all two person games,

select a "first" player by any of the usual means). No multiple jumping is allowed, and a person must jump if he is able. The last to make a move wins. Note that in this and other two-person games, the game may end with several pieces left in the pyramid.

Game 2 As in game 1, but the last to move loses.

Game 3 Vary the initial unoccupied position.

Game 4 Allow multiple jumps.

Game 5 Form other variations, including the possibility of three persons, team play, or any combination of these and other two-persons games.

Should any of the pieces come apart, simply glue them back together. Simply keep in mind that this object is made of plexiglass, and an appropriate adhesive should be used.

Solutions to the solitaire game may be obtained by sending 25 cents and a self-addressed, stamped envelope to JAVIRO, 100 Friendly Road, Cranston, R.I. 02910.

Insert between . . . "Many have been immersed in the game for hours," and "all who participated admit . . . (that is, insert just before the last paragraph in the paper).

Since the three dimensionality significantly raised the level of complexity of the game, it also enhanced the level of two-person games which could be played on the apparatus. For example, starting with the top unoccupied, each person removes a single piece on his turn (the first being selected by flip of a coin). A person must jump if there is a jump available. The last to move wins. Note that, as in other possible two-person games, the game may end with several pieces left in the pyramid; in fact, success in two-person games depends on one's ability to control the number of pieces which must be left. One can vary the game still further by playing that the one who moves last loses. In both variations, of course, the players are playing a very complex version of Nim.

Ressources pedagogiques en mathematiques en francais

Cléo Beaudry

Le Centre franco-ontarien de ressources pédagogiques est un organisme à but non lucratif avec charte provinciale et régi par un Bureau de Gouverneurs. Cet organisme assure un échange de ressources didactiques entre conseils scolaires de l'Ontario, et développe en plus du matériel pédagogique suivant les besoins identifiés. Le secteur d'activités touche tous les niveaux de l'enseignement en français, de la maternelle à la 13e année.

A) Historique du centre

Depuis longtemps, les éducateurs franco-ontariens déploraient la pénurie de programmes d'études complets et d'aides didactiques ainsi que l'absence totale d'échange en matière de curriculum, entre les conseils scolaires ayant sous leur juridiction des classes françaises.

Ces constatations et cette lacune se devaient d'être comblées afin que les éducateurs franco-ontariens aient meilleure idée de ce qui se faisait ailleurs en développement de curriculum. De cette façon, on pourrait viser davantage à obtenir une qualité uniforme d'enseignement à travers la province.

Le Centre franco-ontarien de ressources pédagogiques ouvrit ses portes le 7 janvier 1974 dans le cadre d'un projet d'initiatives locales. Par la suite, c'est le gouvernement provincial de l'Ontario par l'entremise de son ministère de l'Education qui en assuma la respons-

abilité financière majeure.

Le Centre est sous la régie d'un Bureau de gouverneurs auquel siègent des représentants du ministère de l'Education ainsi que des éducateurs-cadre de diverses parties de la province.

B. Documents relatifs à l'enseignement des mathématiques

Tous ces documents sont disponibles au Centre franco-ontarien de ressources pédagogiques.

TITRE	DESCRIPTION	CYCLE
Le système métrique	Guide général - les diverses mesures.	moyen
Jouons avec les nombres	Jeux mathématiques et exercices de répétition.	primaire et moyen
Si, Si Simple	Système métrique: longueur, volumes, masse, température.	primaire et moyen
Mathématiques	Plan curriculaire avec buts et objectifs précis. Nombre et opération, géométrie, mesures.	primaire, moyen et intermédiaire (7e, 8e)
Cahiers d'activités sur les mesures SI	Mesures métriques avec exercices variés et pratiques.	primaire, moyen et intermédiaire (7e, 8e)
Mathématiques - éducation spéciale	Exercices et jeux mathématiques sur les nombres et les opérations.	primaire
Mathématiques	Tests diagnostiques.	primaire
Viens compter	Chiffres et nombres. Education spéciale. Série d'exercices.	moyen
Mathématiques	Guide pédagogique pour utilisation avec le manuel La mathématique à l'élémentaire, niveau 1. Série de 10 cahiers.	1re année
Mathématiques	Arithmétique, géométrie, mesures métriques. Tests diagnostiques.	5e, 6e, 7e, 8e.
Carnet métrique	Mesures linéaires, de masses, de volumes, de température et de temps.	primaire
Cartes d'activités métriques	Série d'exercices et de stratégies d'enseignement sur le système métrique. Sur cartons-couleurs.	primaire et moyen
Cahier d'exercices	Nombres. Pourcentage. Rapports et proportions. Peut servir au recyclage.	9e et 10e
Mathématiques au préparatoire	Arithmétique, mesures métriques, géométrie. Education spéciale.	10e
Mathématiques	Nombres, géométrie, algèbre, statistiques. Nombreux exercices.	9e
Mathématiques	Puissances et racines. Relations et graphiques. Géométrie. Rapports et proportions. Organisation financière. Vecteurs. Matrices.	10e
Pourquoi et comment maîtriser le mètre?	Système métrique: mesures linéaires, de masses, de températures, d'aires et de volumes. Nombreux exercices.	intermédiaire
Cahier sur la statistique	Données. Echantillonnage. Graphiques. Moyenne. Médiane. Fréquences. Techniques de calcul.	9e, 10e

TITRE	DESCRIPTION	CYCLE
Mathématiques pour classes spéciales	Les quatre opérations. Jeux de parcours. Codes à déchiffrer.	moyen
Statistiques	Données. Table de fréquence. Echantillon. Moyenne. Médiane et mode. Feuilles d'activités pour l'élève et notes pour l'enseignant.	9e

C) Renseignements supplémentaires sur le Centre franco-ontarien

Toute personne intéressée à connaître davantage le Centre franco-ontarien de ressources pédagogiques est priée de communiquer avec la directrice-générale à l'adresse suivante:

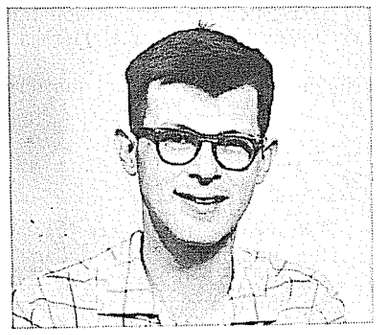
Madame Gisèle Lalonde,
Centre franco-ontarien de ressources pédagogiques,
339, rue Wilbrod,
Ottawa, Ontario. K1N 6M4
(613) 238-7957

Un catalogue avec la description et les prix de tous les documents est distribué gratuitement sur demande.

Les responsables du Centre se feront un plaisir de vous recevoir dans la salle de montre au 339, rue Wilbrod, ou d'organiser des ateliers et des expositions dans votre milieu.

SOME PROBLEMS FROM GRADE 10 MATHEMATICS

Prof. E.J. Barbeau, University College, University of Toronto



1. A bill of $13.75 was paid in half dollars and quarters, and the number of half dollars exceeded the number of quarters by 5. How many were there of each?
Approaches: (1) as a two-variable problem
Let x be the number of quarters and y be the number of half dollars. Then $x + 5 = y$ and $25x + 50y = 1375$.
Variant: a half dollar is equivalent to two quarters and $13.75 to 55 quarters, so the equations become $x + 5 = y$ and $x + 2y = 55$.

(2) as a single variable problem
Let x be the number of quarters. Then $25x + 50(x + 5) = 1375$.

(3) by arithmetic reasoning
The five "excess" half dollars are worth $2.50. Taking $2.50 away from $13.75 leaves $11.25, which is made up of equal numbers of half dollars and quarters. Take each half dollar with a quarter to get a 75c package. Since $11.25 contains $1125 + 75 = 15$ of these packages, there must be 15 quarters and 20 half dollars.

(4) by guesswork
Clearly there are no more than 28 half dollars (this is $14). But there must be more than 15 (which makes up less than half of $13.75), since there are more half dollars than quarters. The number of

quarters is odd, so the number of half dollars must be even. Halfway between 15 and 28 is about 22 for the number of half dollars, so this is not a bad first guess. However, this uses up $11 in half dollars, and we only have "room" for 11 quarters, far too few. So cut down on the number of half dollars. Let's try 20 half dollars. Hurray, it works!

(5) by pattern

If we have only one quarter, there must be 27 half dollars, to make up $13.75. Let us cut down the number of half dollars. Each half dollar we take away must be replaced by 2 quarters, so the pairs (number of quarters, number of half dollars) must be (1, 27), (3, 26), (5, 25), (7, 24). In each successive pair, the difference between the numbers of half dollars and quarters is reduced by 3. We start off with a difference of 26; we want a difference of 5. 26-5 = 21 which is 3x7. So the right pair is the seventh after the first one, i.e. with $1+2 \times 7 = 15$ quarters and $27 - 7 = 20$ half dollars.

(6) by false position

We know that there are 5 more half dollars than quarters, so let us look at various amounts made up with this constraint. Since 6 half dollars make up $3, it is convenient to increase the numbers of half dollars by 6 each time. Here is a table:

No. of half dollars	No. of quarters	Amount
6	1	$3.25
12	7	$7.75
18	13	$12.25
24	19	$16.75

We observe that every time we increase the number of half dollars by 6, the amount goes up by $4.50. Evidently, the amount increases steadily with the number of half dollars for intermediate numbers of half dollars (notion of an increasing function intervenes here - how much should it be made explicit?). It seems that the answer should be somewhere between 18 and 24. We want the amount to be $13.75, which is $1.50 more than $12.25. Thus, the required amount is ⅓ of the way from $12.25 to $16.75. So, the right number of half dollars should be ⅓ of the way between 18 and 24; that is, it should be 20. (King Tut might have learned to do the problem this way).

Variant: What happens if we change some of the numbers in the problem? Will it continue to make sense? Suppose we have 6 more half dollars than quarters? 20 more half dollars than quarters? For what numbers k will the problem make sense for k more half dollars than quarters? Can k be negative? How flexible can we be in changing the amount, if we keep to $k = 5$? if we allow other values of k?

Implicit ideas in situation:

(a) function - notice how in some of our approaches we see how one quantity varies with respect to another

(b) interpolation, extrapolation - approaches (5) and (6) amount to performing a linear extrapolation or interpolation.

(c) ratio & proportion - in approach (6), the excess of half dollars over 6 is proportional to the excess of the amount over $3.25

(d) linear variation (related to (a), (b), (c))

(e) structural properties of the real numbers, in particular divisibility properties (whether or not one number is exactly divisible by the other) and ordering properties (adding equal amounts to an inequality preserves the inequality - see table in (6) with reference to numbers of half dollars and quarters). Without being too trite about this, one should get the sense of increasing familiarity with the structure of the ordinary number system. This is done by experience, not by burdening the pupil with formalism and jargon.

2. Divide $6400 among A, B and C, so that B will receive $120 more than A and C! $160 more than A.

Biographical note: Stephen Leacock has described the characters and fortunes of the "full-blooded blustering fellow, of energetic temperament, hotheaded and strong-willed", a "man of great physical strength and phenomenal endurance", but warns that "a mistake in the working of a sum may keep him digging a fortnight without sleep". B is "quiet, easy-going", while "poor C is an undersized, frail man, with a plaintive face" whose health has been broken by constant walking, digging and pumping. The problem under discussion therefore seems somewhat uncharacteristic, for A is clearly the underdog, while C is best off (unless of course he is put into a higher tax bracket).

Approaches: (1) as a three-variable problem; (2) as a single variable problem; (3) by arithmetic reasoning (take the $6400; give B $120, C $160, and split the remainder

three ways); (4) by guesswork (the amounts are roughly equal, so each gets about one third, with A's amount definitely on the shy side. $6300 is a little less than $6400 and obviously divisible by 3, so start by giving A $2100. Then B will get $2220 and C $2260. But this is $180 too much. So knock $60 off each one's share to give A only $2040, B $2120 and C $2160. So we guess and adjust until right); :5; by false position (try giving A $500, $1000, $1500, $2000, $2500, for example, with B $120 more each time, C $160 more for each time. See what the total amount is in each case. Now calibrate until the total is $6400).

Variants: Can any or all the figures of the problem be changed at will, or are there constraints which must be observed for the problem to make sense? How do these constraints reflect the properties of the number system, and the uses to which numbers are put in practice?

3. Of two sums of money, the second is $500 more than the first. The first is invested at 2% and the second at 3%. If the total yearly interest is $225, find the two sums.

   Economics note: This problem evidently has been kicking around for some time. Bring it up to date, but keep in mind any constraints on the numbers you pick.

Approaches; The standard approach is to use straight algebra. If you try to guess, observe first that $255 is between 2% and 3% of the total amount (why? this approach might give a practical feel for inequalities), so the total amount is between $255 ÷ 0.03 = $8500 and $255 ÷ 0.02 = $12,250. So the sum is roughly divided into two parts and the average rate of interest is about 2½%. So try $255 ÷ 0.025 = $10,200 as the total amount, with part half of $255 over this, the other part half of $255 under this. How close are we?

Another approach is to take $500 of the total amount invested at 3% and half of the remainder at 5%; this will give $255 interest. So half the remainder is (255 - (500 x 0.03)) ÷ .05 = 4800 dollars.

SEA AND AIR NAVIGATION

David R. Campbell, Faculty of Education
University of Western Ontario





One of the major aims of the new (1977) Intermediate Division Guidelines is "to develop in students an appreciation of the place of Mathematics and its widespread application in our culture". The growth of Mathematics throughout the ages has been spurred on by the needs of the navigator making his way across the trackless oceans. Today, a major application of Mathematics is in sea and air navigation, and yet little is done in the schools with this important and interesting topic. The series of lesson aids and activities which follow could be used with minor modifications in any grade from 7 to 10. The aim is to give the student a better understanding of maps, distance and direction and some appreciation of the application of Mathematics in sea and air navigation. The sequence consists of five activity lessons as follows:

1. Boxing the Compass, 2. Bearing and Direction, 3. Latitude and Longitude, 4. Distance and Direction, 5. Dead Reckoning.

The lessons are set up for maximum participation both in the presentation and in individual activity. Extra examples can easily be devised by the teacher if it is felt they are necessary.

This sequence fulfils many of the criteria for topics for the Intermediate Division as outlined in the "Pogue" report and is particularly pertinent in the following areas:

1. It is a "real world" application of Mathematics.
2. It gives practice in (and/or introduces) worthwhile Mathematics - particularly angle, scale,

distance, speed, time, measurement and accurate drawing.
3. It involves solving real problems.
4. It is interdisciplinary - especially with Social Studies.
5. It is activity oriented.
6. It can (and should) be extended to more complicated problems using velocity vectors in Grades 10-13.

Extra materials required are not expensive and are interesting and useful for the students. 360° protractors marked in degrees in a clockwise direction are essential and are available in most book stores for about 40 cents each. Air navigation maps of scale 1:500,000 are recommended and are currently one dollar each. One per pair would be plenty and can be used several times. For maps, write to the Canada Map Office, 615 Booth Street, Ottawa, Ontario, K1A 0E9 and they will send a brochure of maps available and a price list. If economy is paramount, one map posted on the wall would be sufficient, with the teacher making "plotting charts" as shown in the lesson on Dead Reckoning.

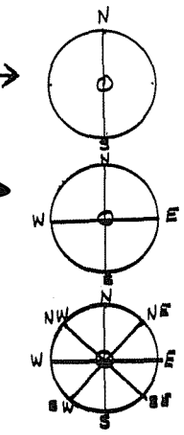
The author would be pleased to answer questions and to hear comments from any teacher who uses or wishes to use these exercises.

BOXING THE COMPASS

Aims: 1. To become acquainted with compass directions.
2. To learn the 32 compass points.
3. To learn to give directions by the 360° compass.

Materials: Ruler, Pair of compasses, 360° protractor.

Procedure:
1. On a piece of paper, draw a circle with a radius 8 cm and Centre 0.
2. Draw a vertical diameter of this circle.
3. Put N. (North) at the top and S (South) at the bottom.
3. Draw a perpendicular to this diameter at 0.
Produce in both directions to meet the circle at
E (East) on the right and W (West) on the left.
4. We now have the 4 cardinal points of the compass,
North, South, East, West.
5. Now Bisect angles NOE, SOE, SOW, NOW, and produce
each bisector to cut the circle. Mark the directions
NE (North East), SE(South East), SW(South West), NW(North West)
These are the 4 quadrantal points of the compass.
We now have 8 compass points.
6. Now, bisect each angle again to produce 8 more compass points.
In order, clockwise from north, these 8 new points are:
NNE, ENE, ESE, SSE, SSW, WSW, WNW, NNW.
Mark each on your circle.
7. Now, bisect each angle again to produce 16 more compas points.
In order clockwise from North, these points in the First
quadrant are: N by E; NE by N; NE by E; E by N.
8. Mark these on your chart.
9. Name the remaining compass points.
[Start with N, NE, E, SE, S, SW, W, or NW (whichever is closest), then add **by** N, S, E or W].
10. Calculate the number of degrees between each adjacent compass direction and check with your protractor.



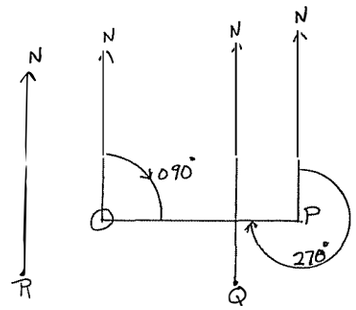
11. Modern ships and aircraft use the 360° degree compass.
Directions are given by the number of degrees from North.
12. Put your 360° protractor on your direction diagram and note that North is 000°, East is 090°, South is 180°, West is 270°.
13. Now **calculate exactly** the number of degrees corresponding to each compass point and put the answers on your diagram. Check with your protractor.
e.g. NE 045, NNE 022 ½, N by E 011 ¼
14. Which system of directions do you consider better for modern conditions?
15. How does the compass help the navigator? (on sea, land or air)?

16. What is meant by steering by a star?

BEARING AND DIRECTION

Aims: 1. To understand and be able to plot a bearing.
2. To understand and calculate reciprocal bearings.
3. To learn how to locate a position by bearings.

Materials: Rulers, 360° protractor, graph paper.

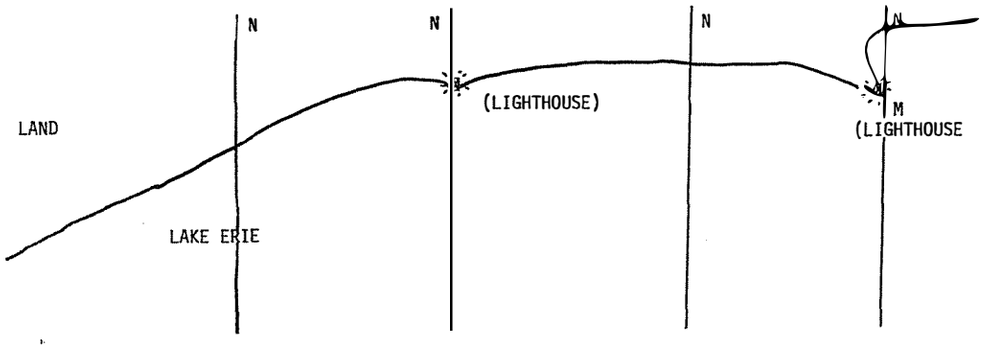


1. The BEARING of a point is its DIRECTION from the Observer. e.g. the BEARING of point P from Observer 0 is 090.
2. a) Bearing of Q from 0 is _____
   b) Bearing of R from 0 is _____
   c) Bearing of P from R is _____
3. The Bearing of P from 0 is 090°, but the Bearing of 0 from P is 270°. These are called RECIPROCAL BEARINGS.
4. By completing the table below, see if you can find a relationship between RECIPROCAL BEARINGS.

	BEARING	RECIPROCAL BEARING
a)	P from 0 090	0 from P 270
b)	Q from 0	0 from Q
c)	R from 0	0 from R
d)	P from R	R from P
	Relationship	

5. a) If the bearing of A from B is 240°, what is the bearing of B from A? _____
   b) If the bearing of C from D is 020, what is the bearing of D from C? _____

6. A ship cruising on LAKE ERIE finds the bearing of Lighthouse L to be 330° and of Lighthouse M to be 060°. Locate the position of the ship on the map below.



LATITUDE AND LONGITUDE

Aims: 1. To examine the method of locating a position on the GLOBE.
2. To appreciate the problems of map making.
3. To examine parallels and meridians on a map.
4. To be able to find latitude and longitude on a map.

Materials: Globe, Windsor-Toronto Air Navigation Maps, 360° protractors, rulers.

Procedure:

1. Examine the GLOBE, locate the parallels of latitude and note their numbering system.
2. Locate the MERIDIANS and note their numbering system. Note how the meridians come together at the poles.
3. Find the approximate LATITUDE and LONGITUDE of the following:
   a) TORONTO _____ b) RIO DE JANEIRO _____
   c) MOSCOW _____ d) SYDNEY (Australia) _____
4. What are the problems of representing the earth on a flat surface?
5. Examine your Air Navigation Map, Windsor-Toronto NW 42/83
   a) Locate the parallels of Latitude. Which ones are on this map? _____
   b) Measure the distance between the 43rd and 44th parallet.
   i) at the left of the map _____cm ii) at the right of the map _____cm
   iii) are they parallel? _____
   e] How many divisions are there between the parallels of latitude?
   These divisions are called minutes. 60 minutes = 1 degree
   f) What is the LATITUDE OF i) AJAX? _____ ii) PORT STANLEY _____
6. a) Locate the meridians. Which ones are on this map? _____
   b) In which direction are the numbers increasing? _____
   c) Measure the distance between the 79th and 80th meridian.
   i) at the top of the map _____cm ii) at the bottom of the map _____cm
   iii) Are the meridians parallel? _____
   d) How many divisions are there for each degree _____? minutes = 1°
   e) What is the longitude of a) LUCKNOW _____ b) AURORA _____
7. Find the town nearest each of the Following locations (Latitude and Longitude)
   a) 43° 45' N 81° 15' W _____b) 42° 32' N.79° 10' W _____
   c) 42° 38' N 81° 28' W _____d) 43° 10' N 80° 54' W _____
8. Find the LATITUDE and LONGITUDE of each of the following:
   a) CLINTON _____ b) AJAX _____
   c) RIDGETOWN _____ d) PORT DOVER _____
9. Back to the GLOBE. The earth turns once in 24 hours.
   a) How many minutes does it take to turn 1° _____
   b) How many minutes does it take to turn 15° _____
   c) How many minutes does it take to turn 4° _____
   d) What is the difference in SUN TIME between a point on the East of your map (79°W) and the West of your map (83°W) _____

DISTANCE AND DIRECTION ON A MAP

Aims: **1. To become acquainted with map scale.**
2. To be able to find the distance between points in 3 different units.
3. To become acquainted with the use of speed, distance and direction in navigation.

Materials: Map - Windsor - Toronto NW 42/83, Ruler, 360° Protractor.

Procedure:

1. Examine the scale at the bottom of the map.
2. What does SCALE 1:500,000 mean?
3. Name the three units that could be used to measure distance on this map.
   1. _____ 2. _____ 3. _____
4. Mark off 50 nautical miles on the edge of a piece of paper. Place the edge of the paper along a meridian and compare with the latitude scale. 50 nautical miles = _____ minutes of latitude.
5. Find the shortest distance from RIDGETOWN to PORT STANLEY in all 3 units, (use the edge of a sheet of paper to transfer the distance to the scale).
   Distance = _____km or _____Nautical Miles or _____statute Miles
6. If you were in an aeroplane going 180 km per hr, how long would it take you to go from Ridgetown to Port Stanley?
7. To find the bearing (Direction) of Port Stanley from Ridgetown, draw a line from Ridgetown to Port Stanley. Measure the angle this line makes with a meridian, i.e. with North. Bearing is

_____ degrees.

8. What is the bearing of Ridgetown from Port Stanley _____?
9. For each of the following pairs, find the distance, direction and time.

	A	B	B in Kilometres	B from A	Speed	A to B
a)	RIDGETOWN	PORT STANLEY	60 km	062°	180 km/h	20 min.
b)	RIDGETOWN	SOMBRA			150 km/h	
c)	SOMBRA	CLINTON			240 km/h	
d)	42° 33' N 80° 02'W	42° 03' N 79° 10' W			300 km/h	

10. (This question has been done on the plotting chart provided).

A ship leaves PORT STANLEY at 1000 hrs. and steers in a direction 188° at a speed of 12 knots (Nautical Miles per hour).

a) Where would the navigator expect to be at 1300 hrs.? _____N_____W
b) To determine the true position, the navigator takes two bearings. The lighthouse at LONG-POINT (42° 10' N 80° 02' W) bears 044° at 1300 hrs. and the lighthouse at ERIE (42° 10' N 80° 07' W) bears 115° at 1300 hrs.
c) Find the actual position of the ship at 1300 hrs. _____N_____W
d) Why do you think the direction and speed of the ship were not what was expected?

DEAD RECKONING

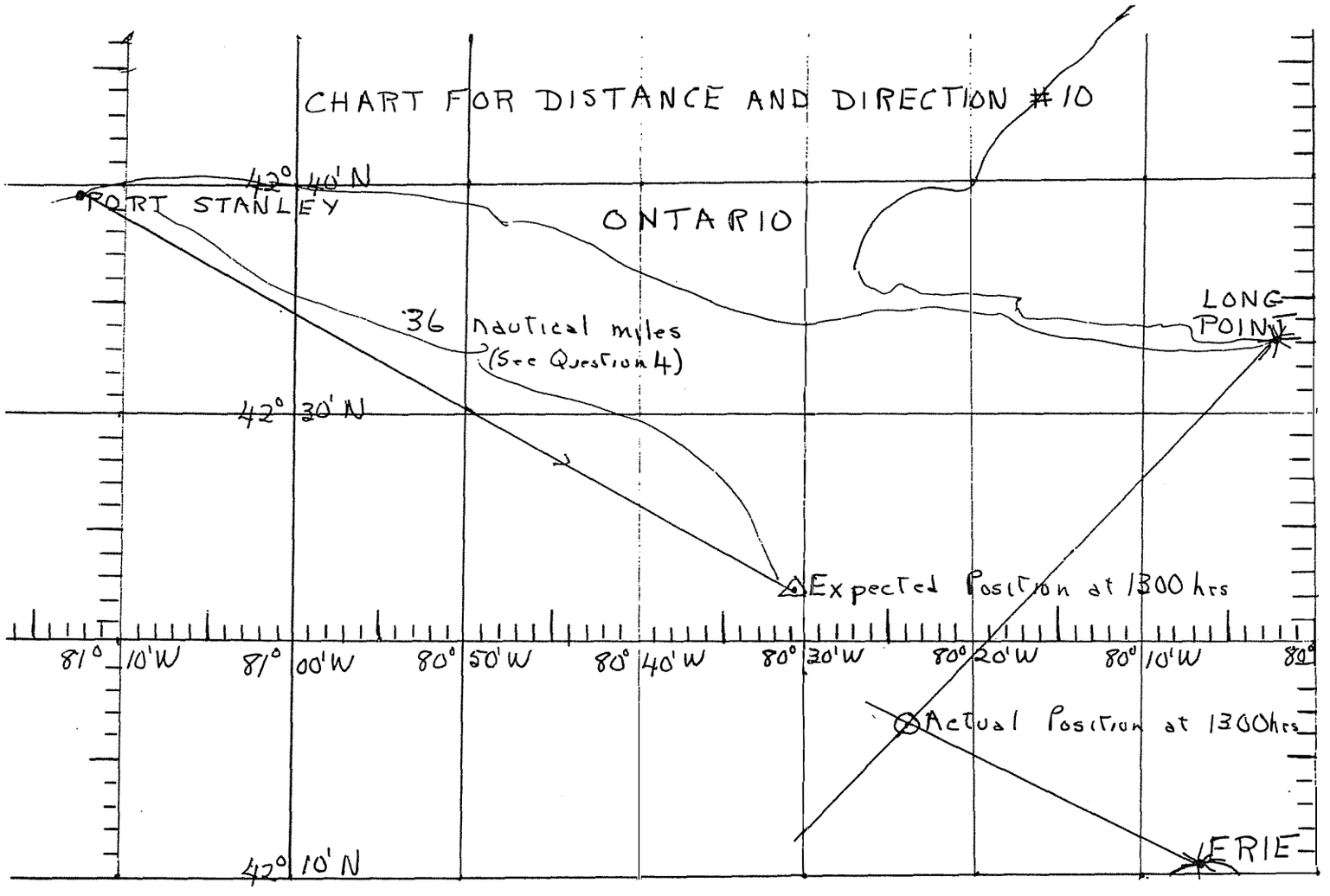
Aims: 1. To learn how a navigator can determine his position by dead reckoning.
2. To be able to find "air position" on a map.
3. To understand the effect of wind on an aircraft.

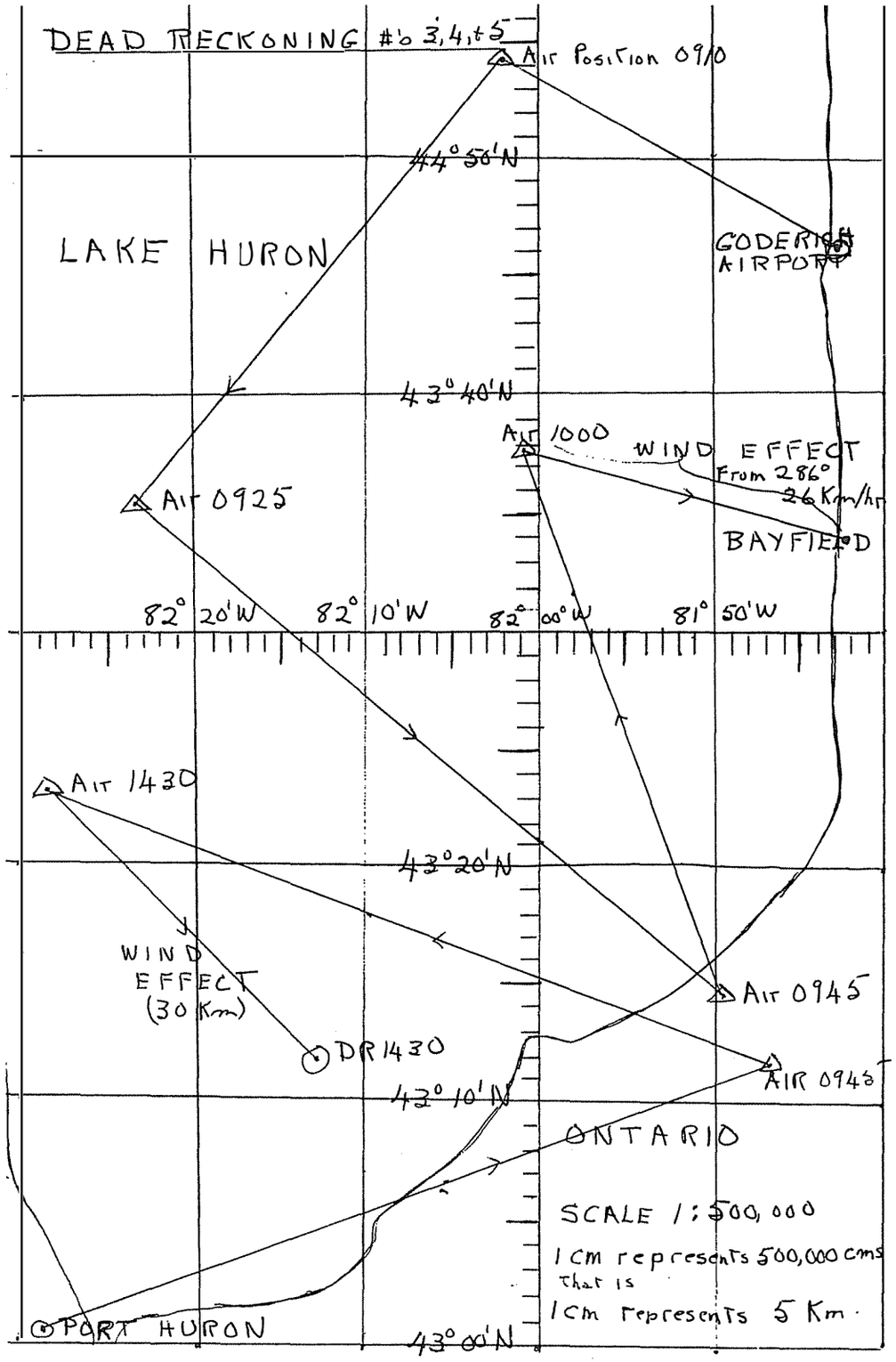
Materials: Windsor-Toronto Air Nav. Map; 360° Proctractor, Ruler

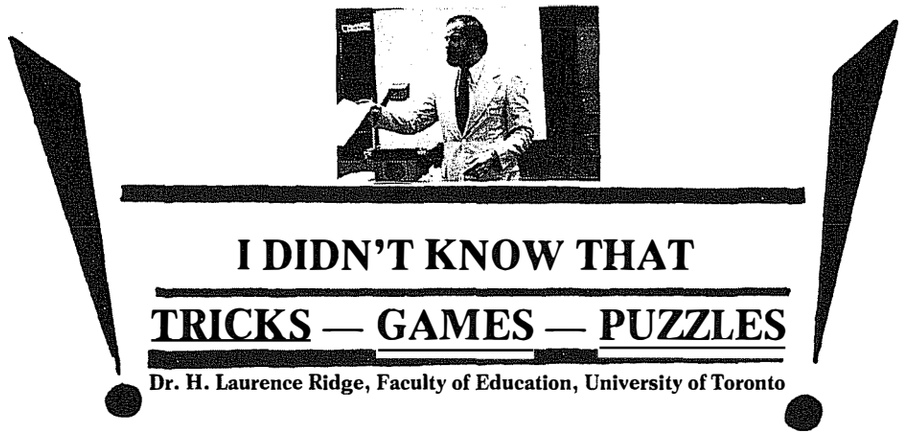
1. The process of determining the position of an aircraft on a map from a knowledge of direction, speed and wind is called DEAD RECKONING (Dead - derived from DEDUCED).
2. The direction an aircraft is pointed is called its COURSE or HEADING. The speed of the aircraft through the air is called its AIR SPEED.
Questions 3, 4, and 5 are done on the plotting sheet provided.
3. An aircraft leaves GODERICH AIRPORT (43° 46' N 81° 43' W) at 0900 hrs on a COURSE (HEADING, DIRECTION) of 300°. Its AIRSPEED is 180 km/hr.
a) Find the air position (position if there were NO WIND) at 0910 hrs. _____
b) At 0910, the aircraft changes its course to 220°.
Find the air position at 0925 (same airspeed) _____
c) At 0925, the course is changed to 130°
Find the air position at 0945 hrs _____
d) At 0945, the course is changed to 340°
Find the air position at 1000 hrs _____

You have just completed an AIR PLOT which gives the AIR POSITION.

4. At 1000 hrs, the Navigator notes that he is directly over BAYFIELD village, 43° 33' N, 81° 42' W.
a) Why is the aircraft not at its AIR position?
_____
b) Assuming the wind was constant for the whole trip,
i) What was the speed of the wind? _____
ii) What direction was the wind from? _____
_____
5. An Aircraft leaves PORT HURON **AIRPORT** 43° 01' N 82° 28' W at 1400 hrs. on a course of 070° with an air speed of 240 km/hr.
a) Find its Air Position at 1415 hrs. _____
At 1415, the course is changed to 290°.
b) Find its Air Position at 1430 hrs. _____
c) If there was a North-West Wind (blowing from 315°) with a speed of 60 km/h find the D.R. (Deaad Reckoning) position of the aircraft at 1430 (Lat. & Long).


CHART FOR DISTANCE AND DIRECTION #10
42° 40' N
PORT STANLEY
ONTARIO
LONG POINT
36 nautical miles
(See Question 4)
42° 30' N
Expected Position at 1300 hrs
81° 10' W
81° 00' W
80° 50' W
80° 40' W
80° 30' W
80° 20' W
80° 10' W
Actual Position at 1300 hrs
ERIE
42° 10' N


DEAD RECKONING #'s 3, 4, & 5
Air Position 0910
44° 50'N
LAKE HURON
GODERICH AIRPORT
43° 40'N
Air 1000
WIND EFFECT
From 286°
26 Km/hr
Air 0925
BAYFIELD
82° 20'W
82° 10'W
82° 00'W
81° 50'W
Air 1430
43° 20'N
WIND EFFECT (30 Km)
Air 0945
DR 1430
Air 0948
43° 10'N
ONTARIO
SCALE 1:500,000
1 cm represents 500,000 cms
that is
1 cm represents 5 Km.
PORT HURON
43° 00'N



I DIDNT KNOW THAT

In keeping with the theme of problem solving, this issue's column is devoted to two problems with a recreational flavour.

The first one is a follow up to December's "Alice's Restaurant" puzzle from 'Rod' Rodriguez., The second is an intriguing handshake puzzle which has been enthusiastically received by several audiences over the last couple of years.

ALICE'S RESTAURANT

To refresh your memory, here is a reprint of the situation.

> One day, Alice met one of her old beaus, Bob. "I heard that you own a restaurant now, an, as I recall, the address on President Avenue is somewhere between 13 and 1300. But don't tell me the address, just answer my questions and see if I can guess. First, is the address over 500?" Now Alice did not consider Bob to be one of her favorite beaus, so she wanted to forestall any visit he might plan; thus she lied. Bob, the undaunted mathematician, then asked whether the address was a perfect square. Again Alice lied. Then, when asked whether the address was a perfect cube, Alice told the truth. "I need one more question," Bob said. "Is the second digit a one?" When he heard her answer, he thought he had the address, but alas he was, of course, wrong. He did not realize that Alice had been "toying with the truth," as it were. The reader, however, should be able to tell what the correct address is, no matter what Alice answered to question 4.

Solution (by 'Rod' Rodriguez - problem proposer)

It should be noted, before we start, that the fourth question is itself information in the problem. That is to say, Bob must have thought that the question's answer would help. Therefore, in his perception, he was trying to choose from amony possibilities, one of which had a second digit of one. This information is relevant. The answer she gave to the question was therefore relevant to Bob. Since we don't know what she gave, it must be irrelevant to us.

Let XXX (X=Y or X=N) represent answer sequences for the questions in order. Hence YNNY would mean that the answers he heard were yes, no, no, yes, in that order. Similarly, let xxx (x=y or x=n) represent the truth. Hence the truth is represented with small letters, and the answers he heard were in caps. We know that the truth must sufficiently narrow the number of pigeonholes so that **we** can decide the address. That is, the truth must narrow down ultimately to one pigeonhole. Thus, we rule out xnnx. For if the truth to the middle two questions was "no" in both instances, the first and fourth questions are insufficient to cut down the number of pigeonholes. If, on the other hand, the truth were xynx, then Bob would have heard XNNX, and **he** would not have been able to decide, even with questions one and four.

* See Rod's "Games Bring Math to Life" article on page 46.

Therefore, the truth is either xnyx or xyyx, and, correspondingly, Bob heard XYYX or XNYX.

However, Bob could not have heard XYYX. For, if he had heard XYYX, he would not have had to ask the fourth question. For $13 \leq n \leq 1300$, there are only two numbers which are both perfect squares and perfect cubes: 64 and 729. Bob would therefore have chosen one of these on the basis of que· tion one's answer. But Bob still indicated a need to ask a fourth question.

Therefore, Bob heard XNYX, and the truth is xyyx; and as a result the address is either 64 or 729. We must now figure out what HE MUST HAVE HEARD in answer to question one, so that we can judge the truth.

Case 1 Suppose Bob heard NNYX. Then either he heard NNYY or NNYN.

a. NNYN. This sequence limits his choices to 27, 64, 125, or 343. Since this is insufficient for Bob to draw a conclusion, he could not have heard NNYN.

b. NNYY. In this case, he would conclude that the address is 216. Our conclusion is that the truth is yyyn or yyyy. But in either case, we are led to the conclusion that the correct address is 729.

Case 2 Suppose Bob heard YNYX. Then either it was YNYY or YNYN.

a. If he heard YNYY, then he would conclude that the correct answer is 512. The truth would be nyyx, and we conclude that the correct address is 64.

b. If he heard YNYN, then he concludes that the address is 1000. But the truth would be nyyx, so that the correct address would be 64.

Case 1 should be rejected for, in that case, the answer to the fourth question is not irrelevant; one of the possible answers le ds one to"no conclusion," and the reader is supposed to know, in any case, that the truth is **independent** of the answer to question four. Case two fits all of the information, then, since the "truth" of 64 is reached, independent of the answers to the other questions.

Solution (by 'yours truly')

The first lie puts the number into the wrong (greater or lesser than) '500 category'. The second lie puts the number into the wrong 'square category'. The cube response was correct. Let's see where Bob would be at on the basis of **his** information.

13<Squares<1300			**13<Cubes<1300**
16	225	729	27
25	289	784	64
36	324	841	125
49	361	900	216
64	400	961	343
81	441	1024	512
100	484	1089	729
121	529	1156	
144	576	1225	
169	625	1296	
196	676		

If all Bob needed to know was whether the second digit was 1 then he must have narrowed the possibilities to **cubes over 500 which were not squares**; that is 512 or 1000. All of the other situations either do not require the question or involve more than one possibility if the answer is no.

The true number then is a square under 500 which is also a cube. Hence, the address is 64 President Ave.

A GET-ACQUAINTED 'PROBLEM'

Now here is the handshake puzzle☆ which was given such big billing at the outset. It came to me from Bob Cherniak who gives a course in Recreational Mathematics at York University.

Four married couples meet for dinner.
There is **some** shaking of hands.
When the handshaking is finished, one husband asks **everyone else** how many times they shook hands.
Everyone gives a different answer.
Assuming no husband and wife shake hands and no two people shake hands more than once . . .
How many times did the questioner's wife shake hands?

One attraction of this puzzle is the intriguing aspect of the apparent lack of data. Another aspect which makes it most appropriate for classroom discussion is the wide range of problem solving strategies which lend themselves.

At the outset, people usually twig to the different answers as being 0, 1, 2, 3, 4, 5, 6 but see little else as to how to proceed. To charify the situation, acting it out or invoking an EXPERIMENTAL STRATEGY can be a most enlightening initial activity. Should volunteers of the required gender not be available, appropriate head gear can be provided for authenticity and a touch of humour.

Arrange eight people in four couples. If they simply go about shaking some hands more or less randomly the desired results just don't happen. Then it becomes a matter of assuming more definite roles in order to put some structure into the activity.

It becomes equally confusing to try to assign numbers of shakes to each person because of having to try to figure out who shakes with whom. It comes down to determining if there is any person in the group all of whose respondents are known. The only such person is "6". Someone volunteers to be "6" then actually shakes with everyone but his or her spouse. Now everyone else, except "6"'s spouse has shaken at least once. Hence "6"'s spouse is the only candidate for "0".

Now someone else volunteers to be "5" and proceeds to shake 4 more times (since "5" has already shaken with "6"). The only candidate for "1" is now "5"'s spouse. Everyone else has shaken at least twice.

Now someone else volunteers to be "4" and proceeds to shake 2 more times. Only "4" s spouse is a candidate for "2".

This procedure leaves only **one couple** yet to volunteer. Both of them are automatically "3" since they have both their hands shaken by each of "6", "5", and "4".

Out of all this, which husband must have been the questioner? In order for all the answere to have been different, "3" had to be the questioner. His wife also is "3".

Q.E.D.

Such an experimental procedure leads to a DIAGRAMMATIC STRATEGY (which could also have been invoked initially).

Figure 1 is really just a pictorial representation of the EXPERIMENTAL STRATEGY described above. The reasoning is identical. Numerals in brackets indicate the number of handshakes while the circled numbers indicate the order of the steps in the procedure. Clearly, H¼ (husband 4) is the "questioner". His wife (W¼) is seen to shake hands three times.

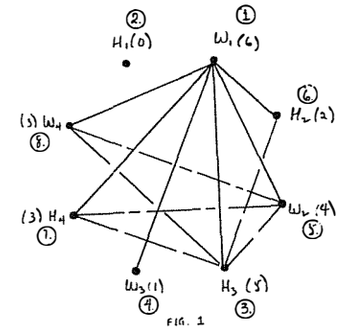


FIG. 1

* This puzzle should not be confused with the very popular handshakes-amoung-eight-people-problem which involves everyone shaking hands with everyone else. The latter problem is the basis of the excellent article by John LeBlanc, "You Can Teach Problem Solving" in the November '77 Arithmetic Teacher.

My original solution consisted of an ORGANIZED LIST of who shook hands with whom.

PERSON (named by number of shakes)	SHAKES WITH (persons named by number of shakes)	
6*	5, 4, 3, 2, 1, Q (questioner)	
5	6, 4, 3, 2, Q	(can't be 1 because 1 has already shaken once with 6)
4	6, 5, 3, Q	(can't be 2 because 2 has already shaken with 5 and 6)
3	6, 5, 4	on the basis of the previous lists
2	6, 5	
1	6	
0		
Q	6, 5, 4	on the basis of the previous lists

This time though we have to figure out who the couples are. There is only one person with whom '6' didn't shake hands. This is '0', who must be '6''s spouse. It follows by a similar argument that '5' and '1', '4' and '2', and '3' and Q are couples. Hence the result regarding the questioner's wife follows.

A fourth and elegant solution uses a TABULAR STRATEGY related to the previous solution. In the table, operation ☆ means "shakes hands with". P shook hands once, P½ twice, etc. In the table 1 means they shook, 0 means they didn't.

*	P_0	P_1	P_2	P_3	P_4	P_5	P_6	Q
P_0	0	0	0	0	0	0	0	0
P_1	0	0	0	0	0	0	1	0
P_2	0	0	0				1	
P_3	0	0		0			1	
P_4	0	0			0		1	
P_5	0	0				0	1	
P_6	0	1	1	1	1	1	0	1
Q	0	0					1	0

* To clarify this representation, you could use P^{6}_{3} for 'person 6'.

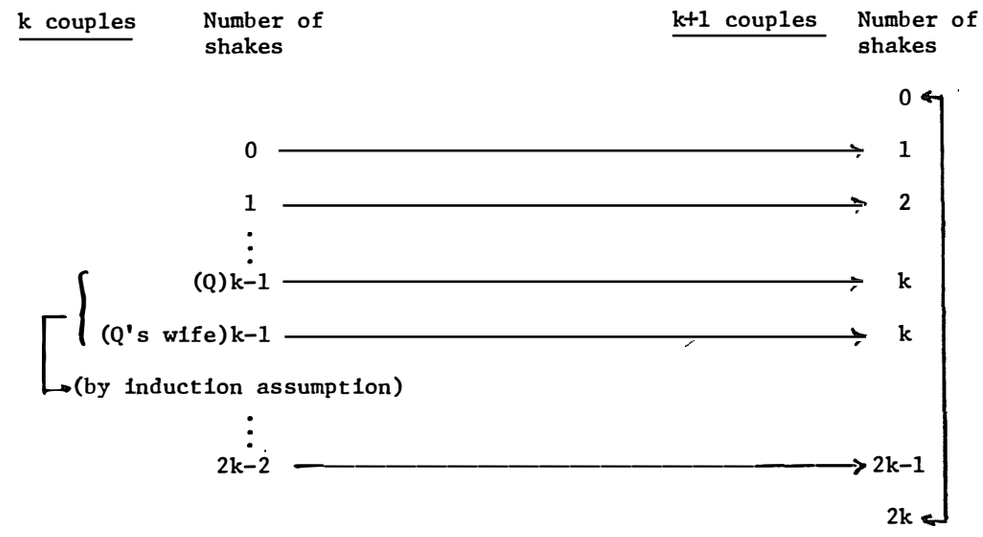
STEP 1: Complete row P_0 and column P_0.
STEP 2: 0 along main diagonal since no one shakes with themselves.
STEP 3: Row P_6 and column P_6.*
STEP 4: Note that row P_1 and column P_1 now have their only 1 entries complete. Fill in the remaining zeroes.

The remaining steps and the determination of the couples is left to the reader.
The previous solution was suggested by the now famous 'Rod' during a presentation at last year's NCTM Annual Meeting in Boston. The use of the 0, 1 notation suggests that function could be defined on the pairs (P_n , P_{n+1}) and that a set-theoretic solution might be possible dealing with any number of couples.

One of Rod's colleagues at Rhode Island Collage has produced such a higher level generalization. While the details are somewhat heavy for this column, I would be pleased to send copies to readers on request.

Of course, the four couple situation can be generalized to n couples on a more modest basis. Students might be encouraged to pose the problem for fewer or more than four couples, conjecture, then check the correct number. It soon becomes evident that the questioner's wife shakes hands n-1 times.

To establish this result by mathematical induction is a little tricky because couple k+1 must be introduced into the situation already existing for the k couples. You can't just start from ground zero with k+1 couples. The easiest way is to have one member of the new couple shake with everyone and the spouse with no one. This will still get a different answer from everyone to the questioner's question as shown below.

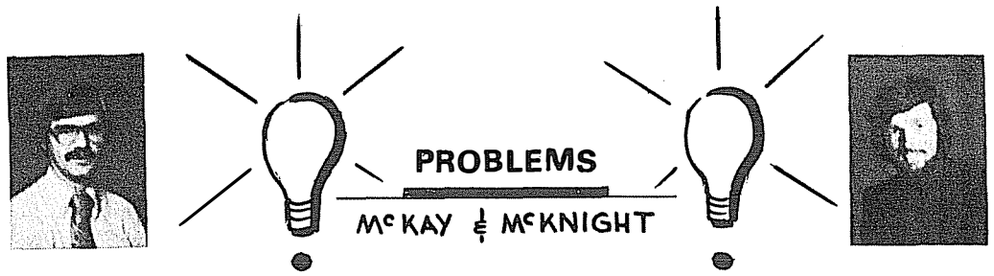


Hence, the questioner's wife shakes k hands in a k+1 couple situation which completes the induction since she clearly shakes hands only once in a 2 couple situation. This application gives a somewhat different wrinkle to mathematical induction than you meet in summing series. Why not try it with your senior classes or math club?

Recreationally yours,

H.L.R.

* This strategy has an advantage over the organized list because each row generates a column which, in term, helps generate the remaining rows.



I. UP TO GRADE 10

18.3.1. A clock strikes the hours (e.g. four times at four o'clock) and "one" each half hour. If you woke up at night and heard the clock strike "one", what is the longest time you would have to lie awake to be sure of the time?

18.3.2. In grade 9, 160 students are enrolled in Mathematics, 175 in English and 60 in French. No student is permitted to take more than two of the three subjects, but every student is required to take at least one. With 350 grade 9 students, it is known that no student taking French is taking either of the other two subjects. How many students are taking both Mathematics and English?

18.3.3 Dianne goes to school cycling at 12 k.p.h. and she is 10 minutes late. Next day she goes at 15 k.p.h. and reaches school 10 minutes early. Find the distance of the school from her house. At what speed should she cycle to reach school precisely on time?

18.3.4 In ABC, show that twice the angle contained by the bisector of A and the perpendicular from A to BC is equal to the difference of angles B and C.

II. UP TO GRADE 13

18.3.5 There are ten bags, each containing ten weights, all of which look identical. In nine of the bags each weight is 16 grams, but in one of the bags the weights are actually 17 grams each. How is it possible, in a single weighing on an accurate weighing scale, to determine which bag contains the 17 gram weights?

18.3.6 Two trains start at 7 a.m., one from A going to B and the other from B going to A. The first train makes the trip in 8 hours and the second in 12 hours. At what hour of the day will the two trains pass each other?

18.3.7 What two whole numbers, neither containing any zeros, will multiply together to equal exactly 1,000,000,000?

18.3.8 Prove, using the methods of Analytic Geometry, that a triangle is right angled if the square on the hypotenuse equals the sum of the squares on the other two sides.

III. ARE YOU READY FOR A CHALLENGE?

18.3.9 An old invoice showed that 72 turkeys had been purchased for $*67.9* the first and last digits were illegible. What were they?

ONE PROBLEM FROM CRUX MATHEMATICORUM

Propose par Fred Maskell, College Algonquin

A young lady went to the store to purchase four items. In computing her bill, the nervous clerk multiplied the four amounts together and announced that the bill was $6.75. Since the young lady had added the four amounts mentally and obtained the same total, she paid her bill and left.

Assuming that the prices for each item are distinct, what are the individual prices?

Solution in Vol. 4, No. 6, June-July 1978.

Schools which do not subscribe to CRUX MATH may obtain monthly issues for $1 each, and bound volumes 1 & 2 (1975, 1976), 3 (1977), 4 (1978), 5 (1979), each for $10 by writing to CRUX MATHEMATICORUM, Algonquin College, 200 Lees Avenue, Ottawa, Ontario, K1S 0C5.



BOOK REVIEWS

Dr. S. Vervont, Lakehead University
Elementary & Secondary Schools

OVERCOMING MATH ANXIETY. Sheila Tobias, 1978. New York: Norton, $10.95, U.S. 278 pp.
Reviewed by R.S. Smith, Faculty of Education, University of Western Ontario.

One of the most interesting spin-offs from the women's movement has been the identification of a phenomenon called Math Anxiety. For Sheila Tobias it came suddenly while reading a survey report. She was struck by the following: "With no mathematics - 92 percent of the females entering freshman year at Berkeley would be excluded from ten out of twelve colleges and twenty-two out of forty-four majors. Instead, they would be restricted to the 'feminine fields'. All other options were closed even before these women arrived at Berkeley". She decided to do something about the problem. This book is the result. It is worthwhile reading.

Some of her statements obviously apply to either sex. Some you might want to challenge.

> "Most people leave school as failures at math, or at least feeling like failures. Some students are not even given a chance to fail.. Identified early as non-college material, they are steered away from pre-college mathematics . . .
>
> There is evidence that a knowledge of pre-college algebra and geometry will make the difference between a low score and a high score on most standardized tests for civil service, federal service and many industrial occupations, not to mention the armed services."

After examining this phenomenon called Math Anxiety from several different angles (some you probably have already pondered) and tackling her own particular nemesis - calculus - she comes to a set of suggestions in the final chapter. These many not be a surefire cure she admits, yet are better than nothing and may stop us from making decisions as a result of math phobia.

I cannot finish this review without reiterating the truism that "states each of us must take the initiative in our own learning" and I would add, in overcoming math anxiety.

Finally, a quote from W.W. Sawyer comes to mind:

> "In discovering something for ourselves, we have a sense of freedom and conquest. In memorizing something that another person tells us and that we do not understand, we are slaves."

How true this is in Mathematics!

VACATIONAL SCHOOLS

Mathematics For Technicians. Volume I. Frank Tabberer, Boston: Butterworths U.S.A. 19 Cummings Park, Woburn, Mass. 01801. 1978. 177 pp. Paperback. 0-408-00326-X. $6.95 (U.S.)
Reviewed by William C. Farmer, Fort William Collegiate

Organization: Pages are well laid out with good shading to emphasize important parts. This

reviewer found it somewhat disconcerting to have the chapters labelled as letters, though once one gets used to it, it is quite acceptable. There is no index. However, there is a one page table of contents. Moreover, the content of each chapter is outlined at the start of that chapter. A summary at the end of each chapter helps to bring out important ideas. Occasionally one questions why some important content is deleted such as inverse trig ratios in chapter G. The summary of chapter F is weak.

Content: The vocabulary differs from that used in the North American schools e.g. "A whole num ber is an integer and may be positive or negative." Empty terms which most teachers are trying to avoid occur repeatedly, e.g. cancelling and no explanation is included why one may cancel. Often the statements are open to misinterpretation. In summarizing the rules of signs for addition and subtraction it reads "like signs combine to a positive result, unlike signs combine to a negative result". The struggling student may interpret this to mean that the sum of 2 negative numbers is positive. What the author refers to is subtracting negative numbers.

The worked examples are sufficient and with enough detail.

I question the inclusion of some topics such as the slide rule in chapter B and introducing and applying the concept of limit to explain accuracy and validity.

Exercises: Throughout the text, exercises which relate directly to a topic just covered follow immediately after the presentation. At the end of each chapter there is comprehensive problem section using the concepts and skills of that chapter. Overall the exercises are just adequate. One wishes there was more practice with the basic ideas and more practical applications.

Evaluation: As a teacher I would not purchase this book as a classroom text. It may be somewhat useful as a source of problems on the teacher's library shelves.

SOLUTIONS TO PROBLEM SET 18.3

18.3.1 Because 12:30, 1:00, 1:30 all have one chime he must stay awake one hour.

18.3.2 Venn diagram solution

or

350		175
-60	taking French	160
290	taking Math., English or both.	335 total for both

$\therefore$ there are 335 - 290 = 45 taking Math and English.

18.3.3 Let x represent time it takes to get to school on time.

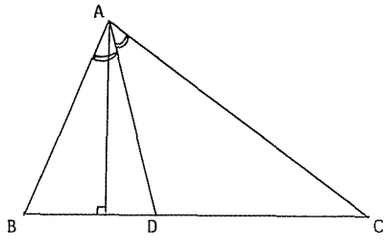
$d = st \quad \therefore 12(x + \frac{1}{6}) = 15(x - \frac{1}{6})$ (distance equal)

$x = \frac{3}{2}$

$\therefore$ distance is $12(\frac{3}{2} + \frac{1}{6}) = 20$ km.

speed $= \frac{d}{t} = \frac{20}{3/2} = 13\frac{1}{3}$ k.p.h.

18.3.4



$x + \angle DAC + \angle C = 90°$ EATT

$\angle DAB - x + \angle B = 90°$ EATT

$\therefore x + \angle DAC + \angle C = \angle DAB - x + \angle B$

since $\angle DAC = \angle DAB$

$2x = \angle B - \angle C$

18.3.5 Take one weight from bag one, two from bag two, etc.

18.3.6

	T	D	S
A to B	x	xy	y
B to A	x	$\frac{2}{3}$xy	$\frac{2}{3}$y

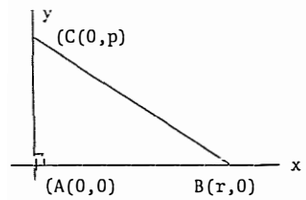
Let x be time until trains meet.
Total distance covered by the two trains when they meet is $xy + \frac{2}{3}xy$. But total distance covered is just the distance from A to B (i.e., S = y when t = 8. $\therefore$ d = st = 8y). $\therefore$ $\frac{5}{3}xy = 8y$

$$x = \frac{24}{5} = 4\frac{4}{5}$$

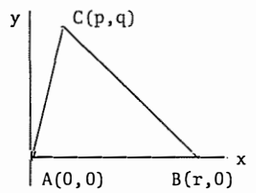
$\therefore$ the trains met at 11:48.

18.3.7 $1{,}000{,}000{,}000 = 10^9 = (2 \times 5)^9 = 2^9 \times 5^9 = 512 \times 1953125$.

18.3.8



Given: ΔABC is right angled
Prove: $CB^2 = CA^2 + AB^2$
Proof: $CB^2 = (\sqrt{r^2 + p^2})^2 = r^2 + p^2$
$CA^2 + AB^2 = p^2 + r^2$
$\Rightarrow CB^2 = CA^2 + AB^2$



Given: $CB^2 = CA^2 + AB^2$
Prove: $\angle CAB = 90°$
Proof: $CB^2 = CA^2 + AB^2$
$(r - p)^2 + q^2 = p^2 + q^2 + r^2$
$r^2 - 2pr + p^2 + q^2 = p^2 + q^2 + r^2$
$2pr = 0$
$\therefore$ p = 0 or r = 0

But $r \neq 0$ $\therefore$ p = 0 $\therefore$ C is the point (0,q) $\therefore$ CA is on the y-axis and AB is on the x-axis. $\therefore$ $\angle CAB = 90°$.

18.3.9 Since 72 is divisible by 8, the amount paid must be divisible by 8. One thousand and higher powers of 10 are always divisible by 8, hence only 79* must be considered to make the entire cost divisible by 8. Thus the last digit must be 2. The number of turkeys is divisible by 9 and so the cost must also be divisible by 9. Thus the sum of the digits must be divisible by 9. Hence the first digit must be 3. The total cost was $367.92.



PLUS + + +
AN INFORMATION COLUMN

Ed Barbeau

As an aid to communication among mathematics educators across Canada, this short magazine is to be inserted in mathematical education journals throughout the country. We hope to tell you about important events, research, curriculm development and items in the literature of national interest. For this, we need **your** help in sending information.

Material (in either English or French) should be sent to:
E.J. Barbeau, Editor
University College, B201
University of Toronto
Toronto, Ontario M5S 1A1

80.1 **Newsletter on Problem Solving**
In June, 1979, the Franklin Institute Press began to produce a monthly newsletter entitled **Problem**

Solving. According to Managing Editor, Julia S. Hough, it will keep the reader "up-to-date on material presented at conferences, as well as books and papers available on the subject". It will include reports on programmes to develop methods of teaching problem solving and on research conducted in university and industry. Write to The Franklin Institute Press, 20th and Race Streets (Box 2266), Philadelphia, PA 19103.

80.2 **The Second International Mathematics Study**

In 1964, the International Association for the Evaluation of Educational Achievement (IEA) sponsored a study of the mathematics, achievement, interests and attitudes of students aged 13, 16 and 18 in each of twelve countries. A second study is now under way. There are three components: curriculum analysis (intended objectives and methodologies for mathematics teaching and learning), classroom process (instructional practices analyzed and compared), student processes (attitudes and achievement in the light of curricular emphasis and classroom practices). Two sessions at ICME IV in Berkeley, California in August 1980 will have progress reports:

(1) curriculum analysis (A.I. Weinzweig, H. Steiner), (2) classroom process, with the results of a pilot study of teacher questionnaires in seven countries (K.J. Travers). International reports are scheduled to appear in December 1982.

Several countries have national committees associated with the study. Canada is not one of these, although the following have been active in pilot testing and other developmental activities: David Bale (Regina U.), John DelGrande (North York B.E.), Lars Jansson (Manitoba U.), Thomas Kieren (Alberta U.), Ronald Ragsdale (O.I.S.E.), David Robitaille (U.B.C.), Howard Russell (O.I.S.E.).

Literature available:

IEA Activities: description of IEA and projects, participating national institutions; from Dr. T. Neville Postlethwaite, Dept. of Comparative Education, University of Hamburg, Sedanstrasse 19, 2000 Hamburg 13, Federal Republic of Germany.

2. **The Second International Mathematics Study:** purposes and plans for study; from Mr. Roy W. Phillipps, Chairman, Mathematics Project Council, Private Bag, Dept. of Education, Wellington, New Zealand.

3. **Second Study of Mathematics, Bulletin No. 3:** purposes and designs, sampling plan, analyses, IEA papers relevant to study, timetable, personnel; from Dr. Kenneth J. Travers, Chairman, International Mathematics Committee, 341 Armory, University of Illinois, Urbana, IL 61801 U.S.A.

80.3 **Master of Science in Teaching at Toronto**

The Mathematics Department and Faculty of Education at the University of Toronto have embarked on a part-time joint graduate programme for experienced teachers. It will include four courses, two jointly given by the two divisions (problem solving, mathematical modelling), one given in the Faculty of Education and one given in the Department of Mathematics. For further information, write Professor D. Alexander, Mathematics Department, Room 373, Faculty of Education, University of Toronto, Toronto M5S 1A1.

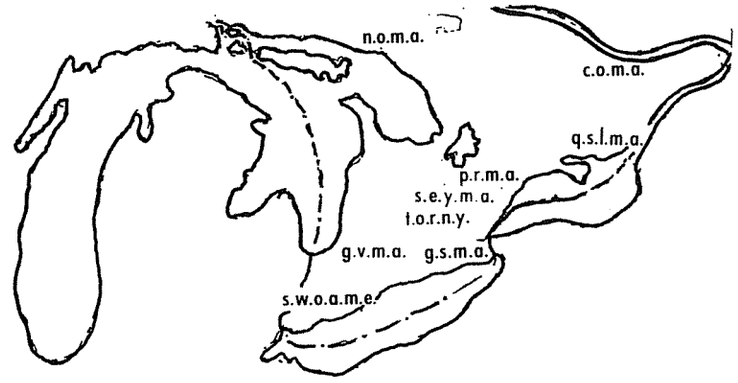
80.4 **Concourse des Jeunes Mathematiciens** (du niveau secondaire)

Pour des renseignements, ecrire a Department de Mathematiques, Universite d'Ottawa, Ottawa, Ontario K1N 9B4.

CHAPTER AFFILIATIONS

C.O.M.A. - **CARLETON OTTAWA MATH ASSOC.**
G.S.M.A. - **GOLDEN SECTION MATH ASSOC.**
G.V.M.A. - **GRAND VALLEY MATH ASSOC.**
N.O.M.A. - **NORTHERN ONTARIO MATH ASSOC.**
P.R.M.A. - **PINE RIDGE MATH ASSOC.**
Q.S.L.M.A. - **QUINTE ST. LAWRENCE MATH ASSOC.**
S.E.Y.M.A. - **SCARBOROUGH EAST YORK MATH ASSOC.**
S.W.O.A.M.E. - **SOUTH WESTERN ONTARIO ASSOC. FOR MATH EDUCATION**
T.O.R.N.Y. - **TORONTO NORTH YORK MATH ASSOC.**

PROVINCE OF ONTARIO



CHAPTER ITEMS

That was the decade that was . . . the 70's

The new math became the old math . . . The old math became the good old math . . . Geometry spread her wings and was "Transformed" . . . Calculators replaced slide-rules . . . Micro-computers made their appearance, while Chisanboppers were counting on their fingers . . . The foundations were laid for a metric Canada . . . Two important documents were developed: The Formative Years (1972), The Intermediate Mathematics Guidelines (in your hands by June 1980, by latest estimates) . . . NCTM had its first Canadian president (John Egsgard) . . . OAME was created from the OATM and the OMC . . . Closer to home: COMA was founded (in 1973, a "prime" year, numerically speaking); Eureka (later Crux Mathematicorum) appeared, under the editorship of Leo Sauve; a Mathematics Examining Board was proposed and discussed at length (the 80's will tell of its birth); the OATIM launched its Math Challenge . . . For these and many other activities from which all of us have benefited, a sincere thank you to all those involved.

Here come the 80's

Crystal-gazing, one sees: a new set of senior mathematics guidelines (by 1985, at the earliest) . . . Arguments over calculators will go the way of arguments over ball point pens (remember the 50's?) . . . New arguments: computers - where will they fit? To what extent will they modify curriculum? (don't "ostrich" on that one) how will they affect methods of teaching? . . . And what of transformations, you say? The crystal ball says that's not an issue: the most radical changes have been made, and the natural follow-up involves only minor modifications of the senior content . . . A real issue? Statistics: to what depth and to what breadth? . . . Problem solving: can it be taught?

Want to find out **more**? Want to have **your** say in the eighties? OAME is **your** organization, and COMA is **your** local chapter. Support COMA and make yourself heard.

$$70+10 = ?$$

Qu'est-ce que cette nouvelle decennie va nour apporter? Les annees 70 terminees, on se pose la question: "Avon-nous progresse, en tant qu'educateurs francophones?" S'il faut croire a l'enthousiasme de nos membres, quoique relativement peu nombreux, ce fut une decennie d'epanouissement: plusieurs ont pris conscience de leur valeur et ont fail preuve de leur initiative au niveau de leur ecole, de leur conseil, voire meme de la province.

Les annees 80? Ce sera une periode d'etablissement. Continuons a nous faire valoir, et faisons partie integrale de l'avancement de l'enseignement des mathematiques, en supportant notre association locale AMOC.

- HUMBUG: Help Undo Metrification, Bug Your MP.

That's the name of the organization as well as the message Sally Hayes brought to Ottawa last week.

Hayes, a Peterborough resident, is spearheading a fight to make the metric system "voluntary" and she's optimistic she can succeed.

Not only is she adamant that eight years after metric conversion began it is not too late to stop it, she is buoyed by last week's announcement of a one-year delay on the metric conversion of weights and measures.

She contended no government agency should be able to promote a metric system over the Imperial system of weights and measures and says, left to their own devices, Canadian businessmen would "find a unit which fits and adopt that unit."

She underlines in all of the discussions about metric, the government has talked and acted as if there is only one metric system. There are "several", she says, and the one Canada is in the process of adopting - the Système Internationale - is used by less than five per cent of the world. As well, she points out Britain has abandonned its metric board and the United States, our major trading partner, is making little effort to "go metric". (Article appearing in the Citizen, November 29, 1979) . . . Oh, well. The name was just right for the Christmas season, anyway!

The Grand Valley

Mathematics Association



In Memoriam

ROMA ACHESON

Every member of the Ontario Association for Mathematics Education was saddened by the news of the untimely death of Mrs. Lorne (Roma) Acheson on November 24, 1979. She was in her 54th year and had spent her entire 25 year teaching career in Grey County. Roma had been in failing health for about two years but courageously carried on her teaching duties before losing her battle with cancer.

The University of Waterloo recognized her contribution to the study of mathematics by awarding her a Descartes Medal in 1978, an honour reserved for only a very few.

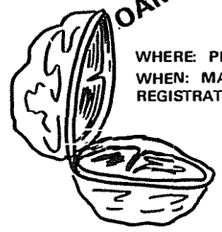
She was a founding member of the Grand Valley Mathematics Association and served on the Council of O.A.M.E. for seven years, six as a Counsellor and one as Director.

Peter Gower sent us this description of Roma: "Never satisfied with either her own skills in the classroom, nor of those of her students, whe worked hard herself and expected the same from those she taught. It was her belief that your mind grew only when it was challenged, only when you demanded more of yourself than you thought possible. And she lived by that code."

A valued member of our profession has been lost. She will be missed by us all.

OAME/AOEM-TORONTO CONFERENCE

IN A NUTSHELL!



WHERE: PRINCE HOTEL, TORONTO
WHEN: MAY 9, 10, 11, 1980
REGISTRATION: $25 for EARLY BIRD MEMBERS
$27 for MEMBERS
$32 for NON-MEMBERS
SOCIAL EVENTS: $4 for FRIDAY NIGHT RECEPTION
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INFORMATION: OAME/AOEM-TORONTO,
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TORONTO, ONT. M4C 2V3
TELEPHONE: 416-465-4631

O.A.M.E. / E.A.O.M.

CONFERENCE, '80

FUTURE MEETINGS

R.C.D.P.M.
April 13-15, 1980
Vancouver, B.C.

N.C.T.M. Annual
April 16-19
Seattle, Washington

4th I.C.M.E.
Aug. 10-16, 1980
Berkeley, California

O.A.M.E./A.O.E.M.
Executive - 1980 04 11
Annual - 1980 05 8-11
Council - 1980 06

O.A.M.E./A.O.E.M. Leadership
Oct. 22-25, 1980 Toronto
N.C.T.M. Oct. 1980, Regina
O.A.M.E./A.O.E.M. Annual
May 1981, Sarnia
N.C.T.M. April 22-25, St. Louis
N.C.T.M. April 14-17, Toronto

***MEMBERSHIP IN ONTARIO ASSOCIATION FOR TEACHERS OF MATHEMATICS**

In the December issue your president outlined the new membership rates. The Association encourages everyone to renew by August 1, 1980 so that the proper number of Gazettes and Abacus can be printed. The membership year in O.A.M.E. is May 1 to April 30. Since this membership year is a change, we will for this year **only**, supply new members who joined after September 20, 1979 with a September Abacus and Gazette. This was indicated either verbally or along with your certificate when you joined. It is our desire that by 1981 all members will be on the new membership year. If you are planning on attending the annual meeting we trust you will join when you register or at the convention. Perhaps if you can't attend the annual meeting you will use the form below, and forward your membership fees of $15.00.

Immediately after the Annual Meeting, renewal notices will be sent to all members who have not yet renewed.

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

MEMBERSHIP APPLICATION

Dr. ☐ Mrs. ☐
Mr. ☐ Miss ☐ _____
(Please Print)

☐ Same address as before or new address as below.

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Chapter Affiliation (if possible _____

Please enclose Cheque or Money Order for - $15.00 _____ payable to O.A.M.E.