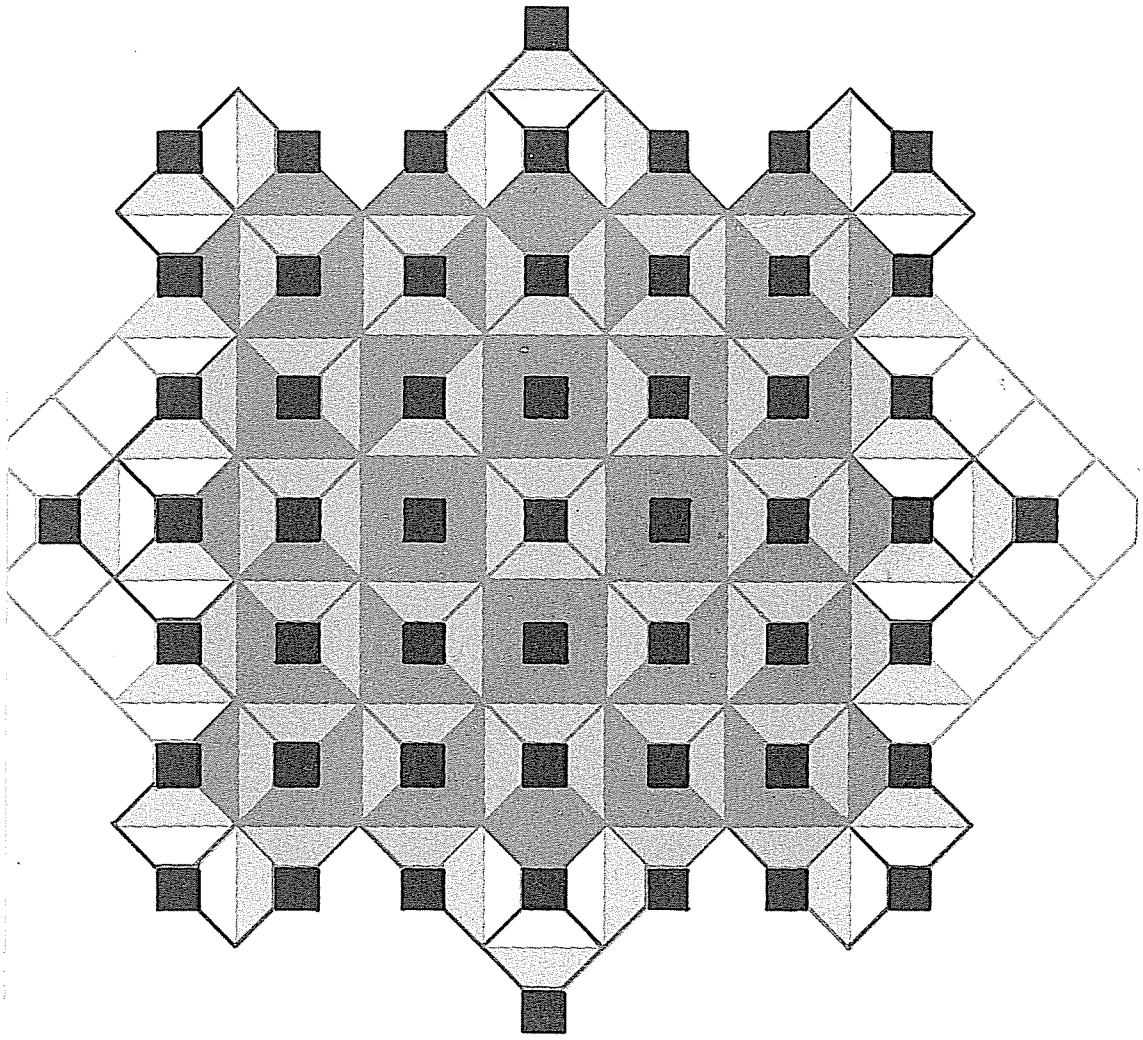
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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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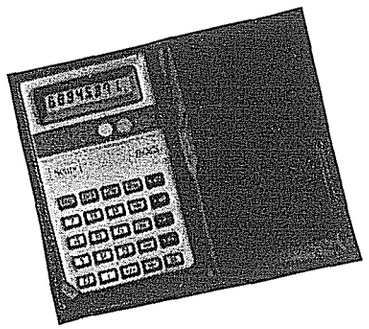
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A MINI THEME ON CALCULATORS

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EDITORIAL

In the September issue, Sam Pravica used the analogy of a football game to express his feelings as guest editor. I'll use the analogy of a marathon -- you run along at a reasonable pace, knowing there's a finish line out there somewhere, and then -- Wham! The Wall! It's two weeks before the deadline and there are about a thousand details to be taken care of -- not the least of which is the preparation of this editorial. But, you do finsih -- exhausted, completely out of steam, but feeling good about the entire race.

I hope you'll enjoy reading this issue as much as I've enjoyed seeing it develop and evolve into its final form. You'll see a couple of things that are different. For example, this issue has a "mini-theme" on calculators and computers. What's a mini-theme? Simply 7 or 8 articles in an issue devoted to a particular topic. Clearly the topic of calculators and computers would easily warrant a whole issue (or more), but by presenting it as a mini-theme other equally interesting, but non-topical articles can be included. Another difference that you might see more frequently in the future, is the inclusion of an excerpt from a manuscript (e.g. the Sovchik article), when printing the whole article is not feasible. Many manuscripts come across the editor's desk which contain some interesting ideas, but are simply too long for inclusion in their entirety. By excerpting the parts that might be of general interest and referring those readers with a specific interest in the topic to the whole manuscript, more material can be included in each issue. With talk being prevalent of cutting the O.M.G. down by 20 pages, this alternative will need to be explored even more fully in the future. Finally, I "fiddled around" with the arrangement of articles. The first articles are all related to the mini-theme, the second "batch" consist of articles **mainly** of interest to teachers in the primary/-junior grades, and the last "batch" **mainly** of interest to the intermediate/senior division teachers, followed by the regular features, etc.

As guest editors, both Sam Pravica and myself would appreciate hearing any reader comments regarding our issues. Your "regular" editor, Bob Smith, will again be at the helm guiding the March, '81 issue on Problem-Solving. He'd love to receive your submissions on the topic.

Now, to get you "in the mood" for our mini-theme, I've collated some of the references on calculators from previous Calculator Information Center bulletins and a quick summary of Marilyn Suydam's State of the Art Review on Calculators.

I've appreciated the opportunity to put this issue together -- happy reading.

Ralph Connelly

LETTERS TO THE EDITOR

University of Guelph
College of Physical Science
Department of Physics
September 2, 1980

Dear Mr. Smith:

It was brought to my attention by acquaintances that Mr. Munsen had made a "correction" to the time equation in the article that I had submitted to your journal.

This "correction" is inappropriate. The stationary and moving frames, as defined in the original article (**18;2**, 1979 p. 46) were consistent in all three equations as stated. Mr. Munsen's equation is inconsistent with the reference frames as defined in equations (1), (2) and (3), and might only be appropriate with a redefinition of those frames. His comment serves only to mislead readers who would not recognize that his equation requires that his reference frame transformation would also have to be applied to the length and mass equations to leave the article consistent.

It seems appropriate, therefore, that the editor of the journal should point out to the general reader that Mr. Munsen's "correction" is inconsistent with the rest of the article.

Yours truly,

Thomas J. Racey

South Strothfield
Australia, N.S.W.
October 15, 1980

Dear Bob:

Thank you for your letter of 28th August. It took over a month to get through. I have enclosed some test questions and their solutions from an all in (integrated) approach. In addition, I have summarised some conclusions I have made on the teaching of this subject around the world. I did receive some mail regarding the interview and surprisingly, some were from Europe and Asia. Apparently your journal gets around! Most of the letters were seeking further information. Time has not been sufficient to do justice to all replies but I did supply considerable information.

Thank you for writing to me. I am anxious to promote geometry in the schools and if there is any medium through which I can communicate these ideas, I am only too happy to do so.

With best wishes,

Dan Moalem

Editor's Note:
We look forward to the article. As it is some eleven closely written pages, we may well have to carry it in two parts, part 1 in March and part 2 in September. The original interview with Dan Moalem was carried in Vol. 17, No. 2, Dec. 1978.

SOME SELECTED REFERENCES ON ACTIVITIES AND RESEARCH RELATED TO CALCULATOR USE

ACTIVITIES FOR STUDENTS, K-13

Bitter, Gary. Count on the Calculator! **Teacher** 96: 67-68ff; February 1979. Use of calculators for elementary students is recommended, so that they may concentrate on the sense of a problem. Activities on place value, estimation and operations are suggested.

Bitter, Gary and Mikesell, Jerald. **Activities Handbook for Teaching With the Hand-Held Calculator.** Boston: Allyn & Bacon, 1980. Activities, grouped by mathematical topic, are given for exploring calculator applications in the elementary and junior high curriculum.

Burt, Bruce C. (editor). Calculators: **Readings from the Arithmetic Teacher and the Mathematics Teacher.** Reston, Virginia: National Council of Teachers of Mathematics, 1979. Articles from the two journals are reproduced.

Duea, Joan; Immerzeel, George; Ockenga, Earl; and Tarr, John. Problem Solving Using the Calculator. In **Problem Solving in School Mathematics.** 1980 NCTM Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1980. Pp. 117-126. Problems for use with calculators are provided; modifying textbook materials and developing a file are suggested.

Fisher, Bill. Calculator Games: Combining Skills and Problem Solving. **Arithmetic Teacher** 27: 40-41; December 1979. Several calculator games involving computational practice are presented.

Meyer, Phillis I. When You Use a Calculator You Have to Think! **Arithmetic Teacher** 27: 18-21; January 1980. Ways in which a fourth-grade class used calculators are described.

Sackson, Sid. **Calculate!** New York: Pantheon Books, 1979. Activities for several age levels are included in this book.

Shields, Joseph J. Mini-Calculators and Problem Solving. **School Science and Mathematics 80:** 211-217; March 1980. How the calculator can facilitate the development of problem solving skills is discussed.

Woodward, Ernest and Hamel, Thomas. Calculator Lessons Involving Population, Inflation and Energy. **Mathematics Teacher** 72: 450-457; September, 1979. Two lessons showing how a calculator can be used to help students discover the "rule of 72" and use it to investigate problems are presented.

Yvon, Bernard R., and Downing, Davis R. **Math Explorations with the Simple Calculator.** Portland, Maine: J. Weston Welch, 1978. This text, appropriate for elementary and junior high school students, presents games, puzzles, problems and exercises.

Zakariya, Norma; McClung, Margo; and Winner, Alice-Ann. The Calculator in the Classroom. **Arithmetic Teacher** 27: 12-16; March, 1980. Three lessons using calculators in grades 3, 4 and 5 are outlined.

RESEARCH REPORTS

Reys, Robert E.; Bestgen, Barbara J.; Rybolt, James F.; and Wyatt, J. Wendell. Hand Calculators: What's Happening in Schools Today? **Arithmetic Teacher** 27: 38-43; February 1980. See also: Wyatt, J. Wendell; Rybolt, James F.; Reys, Robert E.; and Bestgen, Barbara J. The Status of the Hand-Held Calculator in School -- Implications for Parents, Teachers and Administrators. **Phi Delta Kappan** 61: 217-218; November 1979. Results from interviews with 194 elementary and secondary teachers in Missouri are presented. Eighty-four percent said that calculators should be available to children in school. Eighty percent felt that children should master the four operations before using calculators. Almost two-thirds indicated the need for in-service training.

Priorities in School Mathematics (PRISM). Final Report April 1980. ERIC: SE 030 577, 030 578. A variety of items on this survey, conducted for the National Council of Teachers of Mathematics, probed reactions to the use of calculators across major content areas by samples of nine populations. Support varied depending on the topic and the sample, with supervisors and teacher educators most favorable toward calculator use.

MISCELLANEOUS CONCERNS

Werner, Marijane. The Hand-Held Calculator and Its Impact on Mathematics Curricula. **School Science and Mathematics** 80: 29-36; January 1980. Suggestions from reports on calculators are presented, including conference recommendations.

R.D.C.

SUMMARY OF MARILYN SUYDAM'S THIRD ANNUAL STATE OF THE ART REVIEW ON USING CALCULATORS IN PRE-COLLEGE EDUCATION

Development of Instructional Materials

Materials which integrate the use of calculators to teach mathematical ideas are still comparatively scarce. Most of the published articles, however, do present ideas for using calculators to promote learning on specific topics, including work with operations, functions, exponents, polynomials, square roots, and problem solving. There appears to be a decrease in the number of books focused solely on games, and an increase in the number of books which could be used to supplement ongoing instruction.

Two compilations of materials may prove useful to teachers. One is a collection of articles from the **Arithmetic Teacher** and the **Mathematics Teacher** (Burt, 1979); the other is a categorized listing of references on calculators (Suydam, 1979). As has been true ever since calculators appeared in schools, however, there is a continuing need for materials which develop mathematical ideas.

Concluding Comment

While support from some groups for the use of calculators in schools is low, it is nevertheless changing as people accept the existence of calculators in their lives and in their children's lives. Concern continues to revolve around the issue of when the calculator should be used in relation to instruction on basic facts and algorithms: there is fear that paper-and-pencil computational skills will be lost and achievement scores will decline, despite the continuing reassuring research evidence on this point. Educators need to consider carefully ways of assuring parents that calculators can be used in developing a wide range of mathematical ideas which will promote mathematical achievement.

References

Burt, Bruce, C. (editor). **Calculators: Readings from the Arithmetic Teacher and the Mathematics Teacher.** Reston, Virginia: National Council of Teachers of Mathematics, 1979.

Suydam, Marilyn, N. **Calculators: A Categorized Compilation of References.** Columbus, Ohio: ERIC Clearninghouse for Science, Mathematics, and Environmental Education, June 1979. ERIC: ED 171 152.

Readers interested in receiving complete reference bulletins and materials like the review I just summarized can write to:

Calculator Information Centre
1200 Chambers Road
Columbus, Ohio 43212 U.S.A.

and just ask to be put on the mailing list.

R.D.C.

CALCULATORS AND PROBLEM SOLVING WITH HIGH ABILITY STUDENTS IN ELEMENTARY GRADES

Walter Szetela, The University of British Columbia



ABSTRACT:

The investigator and the regular teacher taught 34 students enrolled in an enrichment program for bright students in three classes for four weeks, four days each week. The main focus of instruction was problem solving with the aid of mini-calculators. One class had 8 students from grades 3 and 4, the second class had 16 students from grade 5 and 6, and the third class had 10 grade 7 students. The regular teacher taught the first two days each week with materials selected by the investigator who taught the third and fourth days each week. Students were given daily practice using the calculator for computations but mainly for solving problems.

At the end of the fourth week students were given tests of problem solving. They worked problems without calculators on one test, but they used calculators on the parallel test. Problems were arranged so that no bias in problem difficulty would occur. The results showed that students solved more problems when they used the calculators than when they used pencil and paper only. However, the percentage of correct solutions for the grade 5-6 group and grade 7 group did not differ appreciably from the percent of problems solved correctly with only paper and pencil. Thus while students could attempt many more problems when they used calculators, they also made many more

mistakes. With paer and pencil a total of 95 problems were correctly solved while 132 problems were correctly solved when calculators were used. Of the problems attempted 67.9 percent were correctly solved when only pencil and paper were used. When calculators were used the percent of correct solutions was 73.3. The investigator suggests that a possible reason for the maintenance of a high error rate even when calculators were used, is the pressure of the factor of time is as much as many more problems were given to the two groups that failed to lower its error rate significantly. Further study focused on this factor is suggested.

This paper is a report of an informal study, the third of a series of studies on calculators in mathematics teaching by the investigator. In the two previous studies, the first with grade 9 and 10 students in 1977, the second with grade 7 students in 1978, results indicated that students who use calculators while learning mathematics do at least as well and sometimes better than students who do not use calculators. The totality of research to date provides encouraging evidence that calculators may well be a viable tool in mathematics learning. The present study is an exploratory venture with students of high ability, enrolled in an enrichment program, in grades three to seven. The main focus of the study was on problem solving with calculators. Such studies are needed to assist school boards and curriculum committees in setting up guidelines and policies for incidental use of calculators in the immediate future. For the long term these studies will provide information which will help in the planning of curriculum changes and programs that will improve mathematics learning by exploiting the uses of calculators as teaching tools.

Subjects and Instructional Activities

Subjects for this study were 34 students in grades three to seven from Hawthorne Elementary School in the school district of Delta, British Columbia. All of the students were enrolled in an enrichment program for high ability students taught by Mrs. Judy Coleman. Students were released from their regular classes for one period four days each week in three groups. There were 8 students from grades three and four in one group, 16 students from grades five and six in another group, and 10 students from grade seven in the third group. Each of these groups met with the instructor for one period between nine o'clock and noon from Monday through Thursday for four weeks. The instruction was given on Mondays and Tuesdays by Mrs. Coleman and on Wednesdays and Thursdays by the investigator.

Initially students were provided with four function calculators and they engaged in exercises with elementary computations to become familiar with calculator operations. Subsequently, students were given more realistic problems from such sources as newspaper advertisements, commercially available calculator problem cards, calculator activity books, and problems prepared by the investigator. The instruction was quite informal, a situation made possible because of the small classes, high ability of the students, and high motivation. Although calculators were used most of the time, students were occasionally directed to solve problems with paper and pencil and to check their answers with calculators. Instruction was also given on rounding off numbers with students in grades five to seven. The nature of the problems in grades three and four did not necessitate such instruction.

Although many of the calculator activities, especially calculator games were entertaining, the emphasis was placed on problem solving with calculators, especially in the last two weeks. After some informal activities with calculators, students would be given a sheet of five problems to solve. They were directed to solve some of the problems by calculator and others with paper and pencil only. Even when students solved problems with calculators, they were directed to record the data and the equations used to solve the problems. Papers were checked daily and returned to the students.

Tests

On the fourth day of the fourth week the three

A man goes to a store and says, "Give me as much as I have and I will spend ten dollars with you." It is done and the man repeats the operation at a second and third store, and has no money remaining. What did he start with?

groups were given a test on problem solving with half the problems to be worked using pencil and paper only and half the problems to be worked with the aid of calculators. The students started with the problems to be done with pencil and paper only. Immediately following that set, the students began working a parallel set of problems using calculators. The students in grades three and four were given five problems for each of the two testing situations. In the other two groups, the students were given ten problems for each of the two testing situations.

In order to control for possible bias on the difficulty level of the problems, the problems were arranged on the test sheets so that half the students in each group would solve one set by pencil and paper only, and half the students in each group would solve the same set using calculators. For the other set of problems the subgroups of students were reversed, but again for that set of problems, one group used paper and pencil only while the other group used calculators. The tests were scored for number of correct methods of solution and for correct answers. The tests for each group are atttached as an Appendix to this report.

Results

In all three groups the students solved more problems correctly when they used calculators than when they used only pencil and paper. Similarly, in all three groups the correct method of solution was used more frequently by students using calculators than with pencil and paper only. These results are summarized in Table 1.

Summary Of Results Of Problem Solving Tests With Paper And Pencil Only And With Calculators

Grade Group	N	Number of Problems with Correct Method		Number of Problems with Correct Answer	
		Using Pencil and Paper	Using Calculator	Using Pencil and Paper	Using Calculator
3-4	8	31	40	29	38
5-6	16	52	70	42	61
-7	10	29	38	24	33
Total	34	112	148	95	132

Table 1

At first observation of Table 1 the results appear to clearly favor the use of calculators over use of paper and pencil only in problem solving. However, the students who used calculators also solved more problem incorrectly than they did when using only paper and pencil. Although the error rate for the grade 3-4 group decreased significantly when calculators were used, the error rate for the other two groups remained about the same. The percentage scores showing the rate of problems solved correctly to problems attempted are shown in Table 2.

Summary Of Percentage Scores On Problem Solving Tests With Paper And Pencil Only And With Calculators

Grade	N	Total Number of Test Problems	**Pencil and Paper Group**			**Calculator Group**		
			Problems Tried	Problem Correct	Percent Correct	Problems Tried	Problems Correct	Percent Correct
3-4	8	40	37	29	78.4	40	38	95.0
5-6	16	160	64	42	65.6	85	61	71.8
7	10	100	39	24	61.5	55	33	60.0
Total	34	300	140	95	67.9	180	132	73.3

Table 2

Discussion

Some critics of calculators have facetiously remarked that with calculators students can now obtain wrong answers faster. In this study it is a fact that except for the grade 3-4 group, the error rate in solving problems did not differ appreciably between problems worked by calculator and those worked with paper and pencil only. It is certainly important to be able to solve problems in greater numbers, but it is even more important to solve problems correctly. This concern therefore focuses attention on possible reasons for failure of the error

rate to decrease significantly with the grade 5-6 and grade 7 groups.

A possible explanation for this result may be found by an inspection of Table 2. One observes that the percent of problems attempted by students in the grade 5-6 and grade 7 groups was small compared to the percent of problems attempted by the grade 3-4 group. With pencil and paper only, about 40% of the problems were attempted by these two groups, and with calculators about 54% of the problems were attempted by these small groups. Thus, the number of problems on the tests were well in excess of the number that could reasonably be done in the testing period. This served to place a severe pressure of time upon the students who may have tried to work far more rapidly than they might in a test that could be finished within the period. Such pressure will affect students whether they have electronic aids or not. In the case of the grade 3-4 students there was apparently no such pressure of time. Each grade 3-4 student was given only 5 problems instead of 10 as were given to the other two groups. Even when using pencil and paper only, all except three of the forty problems were attempted by the grade 3-4 group, and all forty were finished by the calculator group. With calculators this group solved all problems correctly except two while when using pencil and paper eleven of the forty problems were solved incorrectly. Whether or not the pressure of time is the key factor in the failure of students to reduce their error rate when using calculators is a matter for future study.

Limitation of this Study

This study was undertaken with a special group of students, assigned to a program of enrichment classes because of their superior ability. The number of students in the study, 34, was small, and one group in particular had only 8 students. Instruction was conducted quite informally and split between the regular teacher and the investigator. Any generalizations from this study are at best risky. The study should be treated as one in which a small number of students with untypical abilities provides another small piece of information that may lay the basis for more extensive and generalizable studies that may ultimately produce information to assist school boards and teachers in designing their instructional programs in mathematics.

References

Sharp, J.N.C. **The Calculator Workbox.** Don Mills, Ontario: Addison-Wesley, 1977.

Szetela, W. Hand-Held Calculators in the Learning of Trigonometric Ratios. **Journal for Research in Mathematics Education** 10: 111-118 March 1979.

_____Calculators and the Learning of Ratios in Grade Seven. **Journal for Research in Mathematics Education** (in press).

Vervoort, G & Mason, D. **Calculator Activities for the Classroom.** Toronto: Copp-Clark Publishing Co., 1977.

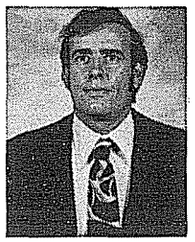
FUTURE MEETINGS

Local
Executive - Nov. 14
Council - Feb. 14
Executive - Apr. 3
Annual Convention - Apr. 8-10
Canterbury Inn - Sarnia

International
N.C.T.M. 59th Annual, 1981
April 22-25, St. Louis
N.C.T.M. 60th Annual, 1982
Apr. 14-17, Toronto

Would anyone who was lucky enough to attend the Leadership Conference in Las Vegas, of all places, share the results with the rest of us? It was called **Calculators in Mathematics Instruction** and ran Dec. 17-20.

USING CALCULATORS WITH STUDENTS WITH MATHEMATICS LEARNING DIFFICULTIES



by Larry L. Houser, Carroll County Board of Education, Westminster, Maryland

Consider The Following:

A commonly cited fallacy regarding the sequencing of mathematics instruction is that all students, including students with mathematics learning difficulties, must memorize certain number facts and accurately perform basic computational skills before proceeding to other mathematics. National Council of Teachers of Mathematics President Shirley Hill (1979) stated that "Much good and challenging mathematical problem solving can proceed without awaiting complete skill mastery. The danger in the 'prerequisite' fallacy is to be seen in the practice of keeping students solely in remedial work on skills until minimal-competency tests are passed." She added that problem solving and reasoning skills are more likely to be useful to learning disadvantaged students in their future lives than are basic computational skills.

The research of Denckla (1979) and others dealing with brain hemisphericity and mathematics learning disabilities at the Children's Hospital Medical Center in Boston, Massachusetts lends further support to the notion that students who may never learn step-by-step arithmetical procedures often exhibit surprising problem solving ability.

Wheatly (1979) indicated that the left hemisphere of the brain controls language while the right hemisphere facilitates wholistic type solutions to problems and spatial ability. A strong right hemisphere and a weak left hemisphere is a possible explanation for unexpected problem solving ability by some students who are poor computers. The brain hemisphere theory seems to lend further support to the notion that the mastery of all computational skills is not prerequisite to problem solving.

The NACOME Report (1975) discussed the role of calculators for use with students with mathematical learning difficulties. The report stated that "arithmetic proficiency has commonly been assumed as an unavoidable prerequisite to conceptual study and application of mathematical ideas. This practice has condemned many low achieving students to a succession of general mathematics courses that begin with and seldom progress beyond drill in arithmetic skills. Providing these students with calculators has the potential to open a rich new supply of important mathematical ideas for these students . . ."

It is the purpose of this article to describe how the hand-held calculator can be used to "open doors" for students with mathematics learning problems. The following three areas will be discussed:

1. The calculator as a motivator.
2. The calculator in problem solving.
3. The calculator and large number exercises and number patterns.

THE CALCULATOR AS A MOTIVATOR

Success and novelty can be powerful motivators and the calculator can be used to provide both. With a small amount of calculator instruction the students can master all of the skills of arithmetic that may previously have caused a great deal of difficulty. Allowing students to check home assignments with a calculator provides the opportunity for success as does allowing one student to have a calculator as the class "judge" during some lesson.

Calculator games can add variety to the math classroom. The numerals 8, 0, 5, 7, 6, 1, 3 when inverted look like the alphabet letters B, O, S, L, G, I. E. Exercises that can interest students can be generated with this letter, numeral relationship.

Example 1

The object of the game is to get 65 on the calculator display. The first player enters a non-zero,

single digit number. The second player adds a single digit number to the display with players taking alternate turns. The winner is the player who gets the display to show 65 without "busting".

Example 2

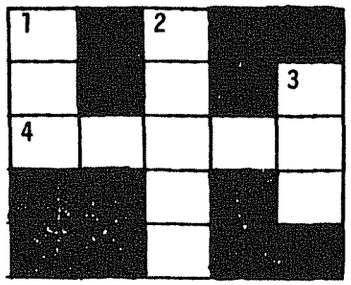
Enter 0.7734 on your display and tip your calculator upside down for a greeting. Answer: Hello

Example 3

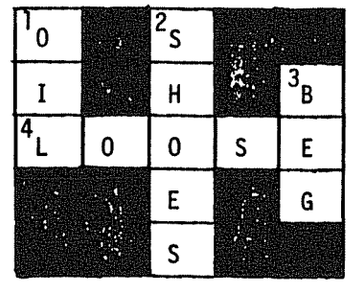
Work the following problems. Enter your answer on your calculator, tip upside down, and enter the word in the crossword puzzle.

Across
4. 35000 + 7*

Down
1. 742 - 32
2. 10609 x 5
3. 1276 ÷ 2*



Answers



*Would the starred items have been done more quickly using the calculator or mentally.

THE CALCULATOR AND PROBLEM SOLVING

A faulty assumption is that calculators replace thinking on the part of students. As with most educational tools the benefits of the calculator depend upon how it is used. The calculator, if used properly, can stimulate and extend problem solving and higher level cognitive processes.

Example 4

A. Make the numeral "5" appear on the display by only using the "2" button and the =, -, x, ÷ buttons.
One Answer: $\frac{2 + 2 + 2 + 2 + 2}{2} =$

B. The above solution required 13 "punches" of the calculator. Is there a solution requiring fewer "punches"?
Another Answer: $\frac{2}{2} + 2 + 2$

Further questions such as "How many solutions are there?", "Is this the simplest solution?" can encourage further problem solving activities by students.

Example 5

Make the numberal 1234 appear on the display without punching a "1', "2", "3" or "4".

One Answer: $\frac{5555}{5} + 55 + 55 + 5 + 8$

Can you solve the problem in fewer punches? Can you solve the problem using only the "5" button? Can you solve the problem by using only the addition operation?

LARGE NUMBER EXERCISES AND NUMBER PATTERSN

The world of large numbers and number patterns can be investigated by students with poor computational skills if the calculator is available as a tool. These problems enable students to make conjectures and to test them.

Example 6

The light from the sun reaches the earth in approximately 8 seconds. How long would it take: A) an automobile traveling 100 km/h to travel from the sun to the earth (149500000 km) and B) a jet airplane traveling at (900) km/h?

Answer to A - 149500000 ÷ 100 ÷ 24 ÷ 365 yrs. = 171 years
Answer to B - 149500000 ÷ 900 ÷ 24 ÷ 365 yrs. = 19 years

Example 7

The human heart is a strong pump. If the heart beats one time per second how often does it beat per year?

Answer: 60 x 60 x 24 x 365 = 31536000

Example 8

What is your age in years when you are a billion seconds old?

Answer: 100000000 ÷ 31536000 = 31.7 years

Example 9

The perimeter of Ontario is approximately 5800 km, the population is approximately 8503300. If the residents of Ontario were to stand around the border to protect the state from wheat thieves, how much of the border would each have to guard?

Answer: 5800 x 1000 = 0.68 m. per person
8503 x 300
m. per person

Example 10

There are approximately 12905450 registered motor vehicles in Canada. The distance around the earth at the equator is about 40000 km. If the average vehicle is 4.8 m. in length how many times could the vehicles extend around the earth bumper to bumper.

Answer 12905450 x 4.8 = 1.5
4000 x 1000

Example 11

(1 x 9) + 2 = 11
(12 x 9) + 3 = 111
(123 x 9) + 4 = 1111
(1234 x 9) + 5 = 11111

Example 12

9 x 9	= 81
99 x 99	= 9801
999 x 999	= 998001
9999 x 9999	= 99980001

Example 13

101 x 222	= 22422
101 x 2222	= 224422
101 x 22222	= 2244422
101 x 222222	= 22444422

The pattern in example 14 breaks down afte two cases. This should encourage students to be more cautious in their conjectures.

Example 14

(20 + 25) = 2025
(30 + 25) = 3025
(40 + 25) ≠ 4025

Conclusion

It often indicated that the attempts by learning disabled students to master school learning tasks can be a traumatic, frustrating experience, but that, fortunately, life will be much better for them. Hopefully, the hand-held calculator can "make life better" for the student with learning difficulties while the student is in school as well as in later life.

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HAND-HELD CALCULATORS GRADE 7-8

Paul Lessard, Mathematics Consultant, Hamilton-Wentworth Roman Catholic Separate School Board

- What are the crucial benefits? -

Motivating	Reinforcement
Clarifying	Mainstream
Generating	Enrichment

As members of the educational community it is one of our most basic responsibilities to be academically informed beyond surface myths and to be able to describe rationally the pros and cons of all materials that will ultimately affect the educational process. If we are to maintain this leadership role in the learning process then we must professionally evaluate the educational value of the calculator for today's student and we must translate our findings into useful information for the guidance of our partners in education.

Society has at times suggested that today's children are illiterate and very poor at basic

mathematics. Those who cherish these myths are ready to tell us that the calculator will sever the thinking process and leave us with a host of non-readers, who are mathematically illiterate and incapable of thought. Therefore parents will be asking many calculator related questions such as:

Can she bring her calculator to school?
What kind of calculator should I buy for him?
Should we let him use our calculator at home?
How often do they use calculators in school?
What does the Ministry of Education say about calculators?
Are calculators good?
Are calculators bad?
Is there an incorrect way to use calculators?
Is there a correct way to use calculators?
"My son is so poor in mathematics, I wonder if it will help if I buy him a calculator?"
What calculator program have you tested?
What group of students would you recommend calculators for?
Will mathematics programs change drastically due to calculators?

These questions and many more are evolving in the taxpayers' mind as consumers are continually bombarded with the calculator reality. This is most vividly illustrated when publishers are contemplating inserting a "throw-away calculator" in the back cover of their $12.00 textbook. Some may suggest that school boards give their Grade 8 students a calculator to keep as part of an **Introductory Calculator Course.** These and many more societal nuances have the mysterious ability to influence people in a subliminal way; hence I am recommending conscious awareness, critical evaluation and the formal educational sharing of the possible judicious use of this societal instrument.

Reinforcement

The proper use of the calculator in the classroom initiates an analysis that emphasises variety in solution, and this focus on variety is reinforced by the notion of efficiency. Calculators become motivators, clarifiers and generators at all levels of Intermediate Mathematics.

Using the calculator with students who hate mathematics and/or who are very weak in the mathematics processes require an emphasis on how to prepare a message for a calculator. This emphasis necessitates allocating generous amounts of time for decision making. The motivation to decide should replace the usual avoidance technique that some of the weaker students have learned to depend on. With the calculator as motivator many of the essentially basic skills can be revisited with those students who have benefitied minimally from the sum total of their previous mathematics instruction. The following list includes some suggestions for consideration re: calculator curriculum at the reinforcement level:

* Reads Numbers Properly
* Estimates
* Orders Numbers
* Distinguishes the Best Discount (50%, 27%, etc.)
* Rounds Off

Within the calculator context students who usually avoid their assignments may be motivated to participate. This participation may result in the clarification of some fairly powerful concepts vis-a-vis the examination of very simple problems within the calculator mode.

Focus No. 1

Elaine bought 4 cans of pop at 40c each.
How much did she spend?

Write a message for your calculator that will organize a solution for this question.

How do you enter 40c into your calculator?

CALCULATOR MESSAGES
(a) 0.40 + 0.40 + 0.40 + 0.40 =
(b) 0.40 x 4 =
(c) 4 x 0.40 =
(d) .40 + .40 + .40 + .40 =
(e) .40 x 4 =
(f) .4 x 4 =
(g) .4 + .4 + .4 + .4 =

This simple question can result in:

* clarifying repeated addition as multiplication
* clarifying the commutative property of multiplication

* suggesting efficiency by leaving off certain zeros
* deciding which zeros can be left off
* suggesting the importance of considering more than one solution

Focus No. 2

A series of simple questions can generate some essentially basic opportunities.

(a) 40 x 32

(b) 6521 x 48

(c) 516 x 28

(d) 52625 x 19

(e) 3465 x 56

SPECIFIC SKILL REINFORCEMENT

* Order the questions by estimating their answers.
* Write the calculator message for each question.
* Compute the answers.
* Read the answers properly.
* Order the answers from largest to smallest.
* Identify the metric space in 52 625.

This traditionally basic exercise in computation becomes a rich dialogue that promotes the development of "meaning", essentially basic for daily life.

Focus No. 3

Percent as an area of focus in reinforcement mathematics highlights the importance of a frame of reference. Many of the weaker students have not had the chance to interact with situations involving percent. The calculator generates a greater possibility for their involvement with some aspects of percent and the teacher can tailor this involvement to reinforce some essentially basic skills for living.

PERCENT

50% off

How much would you save on a $29.90 purchase?

Write the calculator message;
.5 x 29.95 =

Typical student responses will help clarify some initial understanding of percent.

"50% off means ½ off"

What does 12% off the list price of your favourite car mean as a savings to you?

These three areas of focus describe some useful opportunities for utilizing the calculator in reinforcement mathematics. This utilization may eliminate the tendency to drill all those skills that haven't been mastered in eight years and may encourage an emphasis on deciding what is the most crucial agenda in the precious little time available. If these crucial decisions are appropriate, we may succeed in assisting more students to develop competencies for life by capitalizing on the hand-held calculator in the Intermediate years.

Mainstream

Using calculators with students in the mainstream of mathematics education in Gr. 7 and 8 signals two central issues for consideration.

I The proper efficient and thoughtful use of the hand-held calculator.

II How calculators strengthen the present curriculum.

The hand-held calculator presents a rich opportunity to strengthen a school's credibility with the taxpayer by focusing in on a two or three week unit on the efficient use of the calculator. Here is a chance for Grade 7 and 8 students to impress their parents by clarifying the proper use of a calculator and being able to demonstrate the various differences amongst calculators. This type of experience should do much to strengthen a student's confidence in technology.

I have found that embarking on this path with a common calculator model makes it easier to react halfway through the course to the individual characteristics of a variety of models. Students should have two basic goals, i.e.

(a) proficiency with a common model
(b) proficiency with their own calculator

CALCULATOR LESSONS

* Numbers and operations
* Different ways to clear the calculator
* Handling large numbers
* Calculator overload
* Memory
* Fractions

The calculator can strengthen the basic curriculum in two significant ways:

1) relating to specific objectives
2) requiring a process that demands estimation and emphasizes decision making.

SPECIFIC TOPICS	PROCESSES
PERCENT	Uses different ways of calculating percent. Selecting the most efficient way.
EXPONENTS	Recognizes meaning of 2^5. 2 x = = = = 2 is a factor five times.
INTEGERS	The importance of having an idea about the correct answer.
FRACTIONS	Relationship to decimals.
PATTERNS - Number Awareness	Deciding the next number in a series through analysis.

In summary, a well planned unit in calculator studies represent a responsible service to the community and strengthens the existing curriculum significantly. I am not suggesting that the calculator has to be an integral part of every mathematics lesson in Grade 7 and 8 but that it can be used wisely from at least two significant points of view in mainstream mathematics education.

Enrichment

Enrichment mathematics in Grade 7 and 8 heightens the credibility of the unique usefulness of a calculator generated curriculum. In this instance the calculator opens the door to a generous collection of curriculum possibilities for those confident Intermediate students who are always looking for more challenging math activities that pay dividends. Using the calculator with these enriched students results in an emphasis on the following categories.

ENRICHMENT CALCULATOR CATEGORIES
Pattern Analysis
Decision Making
Trigger Questions
Complicated Numbers
Number Chains
Applications

The student with calculator in hand has unlimited opportunities to investigate, collect data, and describe sufficient solutions.

SAMPLE ENRICHMENT OPPORTUNITIES
What percentage of the newspaper is illustrations?
Is gold jewelry a good investment? (Compare the price of gold to 14-karat gold.)
How many words are in the Saturday edition of your newspaper?
We can't wait for the weekend! What percentage of the days of the year do the weekends represent?
What are the most frequently used letters of the alphabet?

SAMPLE ENRICHMENT OPPORTUNITIES
Create number chain novelties like the following: 1) Record the number of the month you were born 2) x 1 0 3) + 1 0 4) x 1 0 5) + your age 6) - 1 0 0
Money Market - Futures
Best Buy
Mass and Forces
Credit Card Math
Energy Costs
Compound Interest
Calories and Your Calculator

In addition to these more specific categories of enrichment oriented calculator curriculum the history and future of calculators provide an interesting springboard for research. How has the calculator progressed and where might it be going?

The benefits of calculator generated learning activities in 7 and 8 mathematics classes are reflected most vividly in that enormous range of explorations that become a reality vis-a-vis the hand-held calculator and the resultant personal discoveries that build mathematical foundations. Let's deliver some calculator related mathematics to today's student for tomorrow.

A merchant whose goods were marked so as to made 60% gave his customers 40% discount. What percent profit did he make?

CALCULATOR ERRORS

Robert Mura - Laval University, Quebec City
Charles Laywine and Eric Muller - Brock University, St. Catharines

Parts or all of this article will appear in French in "Bulletin de liaison du groupe des responsables en mathematique au secondaire du Quebec (GRMS)".

INTRODUCTION

Calculators are being used more and more to motivate and explore mathematical concepts. Calculators are tools designed to perform numerical computations and like any other tool they have their limitations. Both people and machines have limitations. In most instances human limitations in performing correct numerical computations are unpredictable while for calculators the limitations are generally deterministic. In some cases the calculator has been programmed to indicate that a limit has been reached (for instance, the word ERROR or the letter E may be displayed, or the display may start blinking, . . .); in other cases a number is produced without any indication that it is incorrect.

First of all we wish to underline that, under normal school use, the numerical implications of calculator limitations will appear to be small. The numerical implications of four figure logarithms and the slide rule are relatively larger and yet these were used successfully for decades. What's all the fuss about then? We expect you will gain in two ways by working through this article.

(a) You will improve your competence in the use of calculators and you will gain confidence by being able to analyze cases where a displayed answer is in fact incorrect.

(b) You will explore once more some of the properties of the number system from a calculator point of view.

We have divided this article into four parts. The first "A Calculator Arithmetic" is for anyone who can do arithmetic. The short second part on "Calculator Approximations" looks at some simple concepts of errors and gives examples of cases where even a small error can make a big difference. The third "Calculator Error in more Advanced Mathematical Problems" begins in each of its two sections with an assumed knowledge of intermediate level mathematics and develops to problems at the advanced secondary school level. We conclude with a short section on "The Accuracy of a Result Depends (Also) on the Accuracy of the Data".

It is now time to take out your calculator. You'll need it to work through the examples with us. The little machine you now hold in your hand appears to take care of all the computations you need. It certainly does them faster and appears to do them more reliably. (It never daydreams in the middle of long division or loses digits in the addition of a long list or numbers!) Does it have any limitations? Is its arithmetic the same good old arithmetic taught at school?

Well, the digits and the operations signs on the keyboard look familiar, and so do the numbers appearing on the display; the results look right . . .

Let's take a second look at those results. Try to divide 1 by 3 on your calculator : the result is 0.333 333 3. The correct result, of course, is ⅓ or 0.3. The difference between the correct result and the result displayed by the calculator (in this case 0.000 000 03 or $\frac{1}{3} \times 10^{-7}$) is called the *error* committed by the calculator.

It is important at this point to stress the difference between an error in the above sense and a mistake. The result displayed by the calculator was not due to your pressing the wrong key, nor to some mechanical or electronic failure. As far as the calculator is concerned, 0.333 333 3 is the correct answer to the division $1 \div 3$. The fact is that the rational number ⅓ does not exist in the numerical universe of the calculator.

Here is another example. Have you asked yourself what is the largest number that can be entered in your calculator? It shouldn't be hard to find: probably something like 99 999 999. Now ask yourself: does your own numerical universe allow for a "largest number"?

1. A CALCULATOR ARITHMETIC

(a) What kind of numbers does a calculator work with and display?

The two simple observations above prompt the following question: what number system does a calculator work with? How does this system fit into the traditional picture of natural numbers, integers, rational numbers and real

numbers?

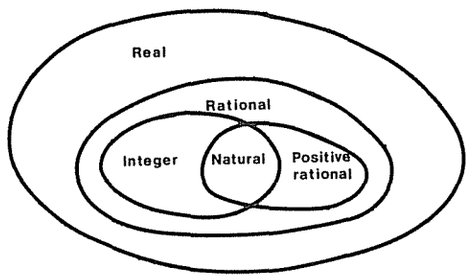


Figure 1 - Traditional number systems

We shall assume that your calculator is a simple 4 function 8-digit display calculator. If you have a more sophisticated "scientific" calculator don't feel left out - all examples followed by a (*) are rewritten for your calculator in the Appendix, at the end of this article.

Once the question is raised, it does not take long to establish that the calculator works with only a *finite* set of numbers (a small subset of the rational numbers). Moreover, these "calculator numbers" are not homogeneously distributed: they are crowded together around zero, while they become more and more sparse as you move further away from zero toward either the largest (positive) or the smallest (negative) number in the system.

To see this, try to work out the following example on your calculator.

*Example 1**

Pair of numbers	Find the difference between the two numbers.	Can you find a third number between the two numbers in the pair?
22.222 222 and 22.222 223	0.000 001	there isn't any
2.222 222 and 2.222 223	0.000 001	e.g. 2.222 222 5

This example demonstrates that even though the difference between the numbers in each of the two pairs is the same, in the first case there is no number between them, while in the second case there are several.

After realizing that the calculator number system is not the same as any of the usual ones, one may wonder how a real number, for instance, is translated into a calculator number. There are basically two ways: either the number is *truncated* or it is *rounded off.*

Try to divide 2 by 3 on your calculator. The correct result is 0.6. If your calculator display 0.666 666 6, the result has been truncated: the calculator has carried out the division until it obtained the first 8 digits of the quotient and then stopped. The other possibility is that your calculator displays 0.666 666 7: in this case the result has been rounded off. In order to be able to do this, the calculator has carried out thd division until it obtained 9 or more digits of the quotient and the extra digist have been used to round off the result displayed. (+)

Here is a trick to find out how many *extra digits* your calculator carries. Enter "88 888 88 + 0.123 456 7 - 88 888 888" in your calculator. Notice that the sum of 88 888 888 + 0.123 456 7, which is displayed after the "-" key has been pressed, is 88 888 888. If the final reslt is 0, you calculator has no extra digits; if the result is 0.1, your calculator has 1 extra digit; if the result is 0.12, your calculator has 2 extra digits; etc. Can you see why this is so?

(b) **What kind of rules apply in calculator arithmetic?**

After having established what kind of numbers a calculator works with, it may be worth investigating the structure of this new number system. Do the familiar laws, such as associativity and commutativity, still hold?

Some calculators exhibit a mixed kind of behaviour: although they do carry some extra digits, they don't always use them to round off the results displayed.

Examples 2 to 8 below illustrate how most of the properties which characterize a field structure may fail to hold in a calculator number system. Such examples may need to be adapted to the specific type of calculator one wants to examine. We assume your calculator has no extra digits beyond those 8 displayed.

In examples 2 to 4 we explore the properties of "+"

*Example 2**

60 000 000 + 70 000 000
The calculator number system is not closed under addition.

*Example 3**

10 000 000 + 0.4 = 10 000 000
The cancellation law does not hold for addition.

*Example 4**

10 000 000 + (0.4 + 0.4 + 0.4) = 10 000 0001.
(((10 000 000 + 0.4) + 0.4) + 0.4) = 10 000 000.
Addition is not associative.

Examples 3, 4 and examples 8, 10, 11 to follow illustrate two typical sources of errors: additions of terms that have a very different order of magnitude and subtractions of terms that are very large and almost equal. As far as possible, one should try to avoid such situations altogether (e.g. by rearranging the order of a series of computations). If this is not possible, then one should be particularly wary of the results obtained.

In examples 5 to 8 we explore the properties of "x".

*Example 5**

What is the multiplicate inverse of
20 000 000?
Not all non-zero numbers have a multiplicative inverse (or the inverse may be displayed as zero).

*Example 6**

What is the square of 10 000?

This shows that the calculator number system is not closed under multiplication. Similarly taking the square root of 0.000 000 1 yields zero: thus the square function is not the reciprocal of the square root function.

*Example 7**

(99 999 999 x 0.000 4) x 0.000 4 = 15.999 999.
99 999 999 x (0.000 4 x 0.000 4) = 9.999 999 9.

This example shows that multiplication on the calculator is not associative.

*Example 8**

(19 999 999 x 1.000 000 1) - (19 999 999 999 x 1) = 1
19 999 999 x (1.000 000 1 - 1) = 1.999 999 9.
Multiplication is not distributive.

Collecting curiosities such as the above may be quite amusing. The exercise can also be useful from a pedagogical point of view, for, by looking at the peculiarities of the calculator arithmetic, attention is drawn to properties of the usual number system.

2. CALCULATOR APPROXIMATIONS

Upon first realizing that the calculator arithmetic is not the usual one, one may start doubting the value of calculators. There really is no reason to panic; as was mentioned in the introduction, the differences between the two arithmetics (i.e. the errors committed by the calculator) are *usually quite small*. In fact the results displayed by a calculator are usually much more accurate than what we would accept as a satisfactory approximation, were we to carry out the computations by hand.

The main point to keep in mind is that *the results produced by a calculator are approximate ones* (even though they look so neat on the display!)

Approximate computation can be and has been studied independently of calculators: readers who have taken a course in numerical analysis will probably be familiar with all the topics touched in this article. Calculators, however, vastly increase the importance of this subject for a much larger public.

We stated already that errors are usually quite small. This implies that occasionally situations arise in which they are not quite so small. Before looking at specific examples, we need to introduce the notion of *relative error*: this is the ratio between the (absolute) error and the correct value. (In cases where the correct value is zero we will take the relative error to be the ratio between the error and the approximate value.)

The idea behind the notion of relative error is that obtaining, for instance, 1 001 instead of 1 000, is not as bad as obtaining 2 instead of 1. Although in both cases the absolute error is 1, in the first case the relative error is 0.001 or 0.1%, while in the second case it is equal to 1 or 100%. Thus in example 3 (Appendix 1) the error is 4×10^{90}, which is certainly very large, however the relative error is of the order of 10^{-8}, which is quite small.

Evan as we study approximations we should be aware that there are many mathematical problems in which an approximate answer just won't do, no matter how good the approximation is. This applies, for instance, to problems asking for a clear-cut yes-or-no answer, like the one illustrated below.

*Example 9**

This example works with many scientific calculators as well. Use Example 9 in the Appendix only if you have a 10-digit display calculator.)

Does 539 divide 66 541 707?

Dividing 66 541 707 by 539 on the calculator,

we obtain 123 454. This seems to indicate that 539 is indeed a divisor of 66 541 707. However by multiplying 123 454 by 539, we see that this is not the case. In fact the correct quotient, rounded off at four decimal places, is 123 434.001 9.

Although the quotient computed by the calculator is a very good approximation of the exact quotient, it still leads to the wrong conclusion.

3. CALCULATOR ERRORS IN MORE ADVANCED MATHEMATICAL PROBLEMS.

In the first two sections we have discussed errors which one might describe as intrinsic to the calculator because they were a consequence of the number system used by the calculator.

The calculator is a very potent tool and greatly expands the domain of problems one can tackle, from those attempted with only paper and pencil and perhaps tables and slide rule. But in doing so, the calculator exposes us to a new class of computational problems which depend on both the accuracy of original data and the stability of the computational process or algorithm used. In sections 3 and 4, we have given examples of such cases, which have long been of interest to numerical analysts.

We begin by looking at:

(a) Roots of Polynomial Equations

*Example 10**

Does the equation
$1\,000.2\,x^2 - 2\,000.2\,x + 1\,000 = 0$
have one or two roots?

If we compute the discriminant of the equation on the calculator, we find it to be zero. This ought to indicate that the equation has a double root. However this conclusion is false: the correct value of the discriminant is a small, but positive one, hence the equation has two distinct roots.

Example 11

A striking example of the sensitivity of some polynomials to what might appear to be a negligible round-off error is given by Wilkinson [1]. Consider the equation.

$(x + 1)(x + 2)(x + 3) \ldots (x + 20) = x^{20} + 210x^{19} + \ldots + 20! = 0$ with roots -1, -2, . . . , -20.

Our intuition would expect the modification of the coefficient of x^{19} from 210 to 210.000000119 to have a negligible effect on the roots. Since this alteration represents a change of unit in the tenth significant figure (or one unit in the seventh decimal place) of only one of 21 coefficients it is certainly less than any error we might expect if the coefficients were to be found experimentally or by prior calculation. Amazingly, the roots of the polynomial with the one modified coefficient are, to 5 decimals.

-1.00000	-10.09527 + 0.643501 i
-2.00000	-11.79363 + 1.652331 i
-3.00000	-13.99236 + 2.518831 i
-4.00000	-16.73074 + 2.812621 i
-5.00000	-19.50244 + 1.940331 i
-6.00000	
-6.99970	
-8.00727	
-8.91725	
-20.84691	

Another area where numerical analysts tread carefully is in the

(b) Solutions of Linear Equations

[1] Wilkinson: "The Evaluation of Zeros of Ill-Conditioned Polynomials", *Num. Math.*, 1, 150-180 (1959).

*Example 12**

Consider the following system of two linear equations in two unknowns.

$0.000\,000\,1\,x + 10\,y = 10$
$0.1\,x + 0.1\,y = 0.2$

Multiplying the first equation by 10^6 and subtracting it from the second one, we obtain

$0.000\,000\,1\,x + 10\,y = 10$
$-10\,000\,000\,y = -10\,000\,000$

This leads to the solution $y = 1$ and $x = 0$. The correct values, rounded off at seven decimal places, are $x = 1.000\,000\,1$ and $y = 0.999\,999\,9$. Hence there is an absolute error of the order of 1 and a relative error of 100% in the value of x.

The above example shows what can happen when the limits of the calculator are surpassed. There are also problems which occur even when the calculator limits are not exceeded.

Geometrically the solution on 2 linear equations in 2 unknowns corresponds to determining the point of intersection of 2 lines. If the lines are close to perpendicular, as in Figure 2, any small uncertainty in the position of the lines, due to experimental or round-off error in the determination of the coefficients, will have a correspondingly small effect on the solution.

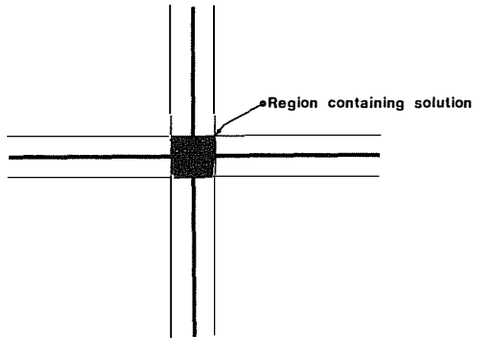


Figure 2 - Graph of Well-Conditioned Equations

On the other hand, if the lines are close to being parallel, Figure 3 shows how a large error may arise in the solution. In this case the equations are called "ill-conditioned".

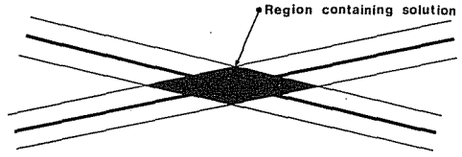


Figure 3 - Graph of Ill-Conditioned Equations

Example 13

Consider the following 3 linear equations in 3 unknowns

$$10.03x + 7.91y + 7.03z = 24.97$$
$$8x + 6.3y + 5.59z = 17.66$$
$$7.1x + 5.6y + 4.97z = 17.66$$

Inspection verifies that the exact solution is $x = 1$ $y = 1$ and $z = 1$.

Solving, on a calculator with 8-digit display, produces the following results:

```
x +  0.7886341 y + 0.7008973 z =  2.4895314

    -0.009072  y - 0.0171784 z = -0.026251

     0.00069   y - 0.006370  z = -0.015672

x +  0.788634  y + 0.7008973 z =  2.4895314

               y + 1.8935626 z =  2.8936287

                 - 0.0076936 z = -0.0176934

x =  2.029914

y = -1.4611024

z =  2.2997556
```

Here we see that using the accuracy of an 8-digit display calculator still gives relative errors greater than 100% and does not provide an answer correct even to 1 figure.

We conclude by examining the fact that

4. THE ACCURACY OF A RESULT DEPENDS (ALSO) ON THE ACCURACY OF THE DATA.

Example 14

Suppose we want to compute the square root of 2.3. We can do so by entering 2.3 in a calculator and then pressing the square root key. The display reads 1.516 575. How accurate is this result?

The answer depends on the interpretation we give to 2.3. If by 2.3 we mean the number 2.300 000 . . ., then the result is accurate to six decimal places. On the other hand, if 2.3 is itself the result of a computation or of a measurement, it should be interpreted as representing some unspecified number between 2.25 and 2.35. In this case, all we can say about the square root of 2.3 is that it lies between 1.5 and 1.532 and 1.532 971, and thus only the first two digits of the result produced by the calculator are significant.

Example 15

Consider the calculation x - y where x = 10.5 and y = 10.0. Suppose the possible error in both x and y is .1.

$$x - y = 10.5 - 10.0 = .5$$

But the correct result could be anywhere between 10.4 - 10.1 = .3 and 10.6 - 9.9 = .7



The result could best be described as 10.5(+0.1) - 10.0(+0.1 = 0.5(+0.2)

Hence the most trivial of calculations involving 2 numbers with errors of about 1% has produced a result with an error of 40%.

Example 16

Approximation of the slope of a function is liable to reflect and magnify the errors discussed in Example 15.

Suppose we wish to find the slope between 2 points, whose coordinates are known to a tolerance of 0.1.

if P_1 = (10.0, 74.7) and P_2 = (10.5, 76.3) then, as seen in Example 15,

$\Delta x = 0.5(+0.2)$; and $\Delta y = 76.3(+0.1) - 74.7(+0.1) = 1.6(+0.2)$

The slope then lies somewhere between

$$\frac{1.6 - 0.2}{0.5 + 0.2} = 2.0$$

and $$\frac{1.6 + 0.2}{0.5 - 0.2} = 6.0$$

If we use 4 as our approximation of the slope, the possible error of +2 is in percentage terms 50%. Hence the errors of 1% in the x coordinates and 0.13% in the y values, have after 2 subtractions and 1 division, given an error of 50% in the result. If we calculate the slope directly as

$\frac{\Delta y}{\Delta x} = \frac{1.6}{0.5} = 3.2$ the maximum error could be

$$\frac{6.0 - 3.2}{3.2} \times 100 = 87.5\%.$$

Conclusion

In this article we have shown the computational errors using calculators arise from a number of factors beside those of human error. By example we have considered cases where

(a) the limits of the calculator are surpassed,

(b) the algorithm is unstable, and

(c) the accuracy of the input data limits the accuracy of the final answer. In general one should be aware that a result cannot be more accurate than the data that produced it: if the data are approximate, the result is likely to contain meaningless digits.

Appendix

Corresponding Exercises for scientific calculators.

Example 1

Pair of numbers	Find the difference between the two numbers.	Can you find a third number between the two numbers in the pair?
2.222 222 2 10^9 and 2.222 222 3 10^9	100	there isn't any
200 and 300	100	e.g. 250

Examples 2 to 8 assume the scientific calculator has a 2-digit exponent and carries 5 extra digits beyond the 8-digit mantissa.

Example 2

$6 \times 10^{99} + 7 \times 10^{99} =$

The calculator number system is not closed under addition.

Example 3

$1 \times 10^{98} + 4 \times 10^{90} = 1 \times 10^{98}$

The cancellation law does not hold for addi tion.

Example 4

$1 \times 10^{98} + (4 \times 10^{90} + 4 \times 10^{90} + 4 \times 10^{90}) = 1.000\ 000\ 1 \times 10^{98}.$

$(((1 \times 10^{98} + 4 \times 10^{90}) \div 4 \times 10^{90}) + 4 \times 10^{90}) = 1 \times 10^{98}.$

Addition is not associative.

Example 5

What is the multiplicative inverse of 2×10^{99}?

Not all non-zero numbers have a muntiplicative inverse (or the inverse may be displayed as zero).

Example 6

What is the square of 3×10^{-50}?

The calculator number system is not closed under multiplication.

Example 7

$10^{50} \times (10^{50} \times 10^{-30}) = 10^{70}.$

$(10^{50} \times 10^{50}) \times 10^{-30}$ does not exist.

Multiplication is not associative.

Example 8

$(1.9 \times 10^{12} + 1) \times (1 + 10^{-12}) - (1.9 \times 10^{12} + 1) \times 1 = 1.$

$(1.9 \times 10^{12} + 1) \times (1 + 10^{-12} - 1) = 1,9.$

Multiplication is not distributive.

Example 9

Does 539 divide 6 654 318 287?

Dividing 6 654 318 287 by 539 on the calculator, yields 12 345 674. This seems to indicate that 539 is indeed a divisor of 6 654 318 287. However by multiplying 12 345 674 by 539, we see that this is not the case. In fact the correct quotient, rounded off at four decimal places, in 12 345 674.001 9.

Example 10

Does the equation
$10\,000\,002\,x^2 - 20\,000\,002 \times 10\,000\,000 = 0$
have one or two roots?

If we compute the discriminant of the equation on the calculator, we find it to be zero. This ought to indicate that the equation has a unique solution. However this conclusion is false: the correct value of the discriminant is a small, but positive one, hence the equation has two distinct roots.

Example 14

Consider the following system of two linear equations in two unknowns.

$$10^{-13}\,x + y = 1$$

$$x + y = 2$$

Multiplying the first equation by 10^{13} and subtracting in from the second one, we obtain

$$10^{-13}\,x + y = 1$$

$$-10^{13}\,y = -10^{13}$$

This leads to the solution $y = 1$ and $x = 0$. The correct values, rounded off at thirteen decimal places, are $x = 1.000\,000\,000\,000\,1$ and $y = 0.999\,999\,999\,999\,9$. Hence there is an absolute error of the order of 1 and a relative error of 100% in the value of x.

THE MINI-CALCULATOR IN SCHOOL MATHEMATICS

by Don Craig, Three Rivers High School, Trois-Rivieres, Que.

Preamble

In 1965 Texas Instruments Incorporated was working on an experimental four-function pocket calculator based on a single integrated circuit. A patent on the device dates back from 1967. By 1971 the pocket calculator became available to the general public at a price under $200. By early 1976 some 100 million units had been marketed and the low end of the price range was well below $20. This dramatic increase will have a revolutionary effect on the learning of mathematics. It is our job as teachers to control the use of mini-calculators in such a way that the effects of their use will be most advantageous to the students.

In September 1974 the Board of Directors for the National Council of Teachers of Mathematics adopted the following position statement:

"With the decrease in cost of the mini-calculator, it accessibility to students at all levels is increasing rapidly. Mathematics teachers should recognize the potential contribution of this calculator as a valuable instructional aid. In the classroom, the mini-calculator should be used in imaginative ways to reinforce learning, and to motivate the learner as he becomes proficient in mathematics."

In 1975, the National Advisory Committee on Mathematics Education appointed by the Conference Board of the Mathematical Sciences and aided by a grant from the National Science Foundation, published their report which examines the status of mathematics teaching in grades K - 12, and predicts future developments. In their report they recommend that "NO LATER THAN THE END OF THE EIGHTH GRADE ALL STUDENTS SHOULD HAVE ACCESS TO A CALCULATOR FOR ALL MATHEMATICS WORK."

In October 1974 the National Association of Secondary School Principals encouraged administrators to view the mini-calculator, not as a technological curiosity, but as an essential element in the newest mathematics, and urged them to make funds available for buying mini-calculators for classroom use (NASSP Curriculum Report, October 1974).

This article will concentrate on the uses of the mini-calculator in school mathematics.

USES OF THE MINI-CALCULATOR IN SCHOOL MATHEMATICS

Before using the mini-calculator in school mathematics, it must be absolutely clear that the fundamental objectives for which we introduce the mini-calculator into the classroom are:

1) TO HELP STUDENTS LEARN MATHEMATICS.

2) TO HELP THE STUDENTS TO PRACTICE MATHEMATICS.

With this in mind there is an almost limitless number of situations in which the mini-calculator can be a significant aid. Here is an outline of some of the main uses which can be made of these little machines.

A. It is clear that the MOTIVATIONAL FACTOR associated with calculators is important. Their popularity outside of school indicates their potential in school. They are, most obviously, TIME SAVERS which will allow us to teach more topics more effectively. They SIMPLIFY awkward calculations which children have shied away from in the past. BIG NUMBERS which have always fascinated children are more accessible. They DEVLOP CONFIDENCE in mathematics and, as a side effect, in machines, for students of all abilities.

B. They provide better access to many PROBLEM SOLVING TECHNIQUES than most students have had in the past. Children can GENERATE THEIR OWN DATA and ANALYSE it in more ways. They can prepare TABLES for present and future use. TRIAL AND ERROR techniques can be used to much greater advantage. Mini-calculators can be very helpful in dealing with problems related to GRAPHING. A study by George Imerzeel has shown students can solve up to three times as many problems in a given time period, when using mini-calculators.

C. In a world where STATISTICS are becoming more and more important it is clear that mini-calculators open this branch of mathematics up to the students. Hitherto inaccessible problems relating to PERMUTATIONS AND COMBINATIONS can now be handled with relative ease. This naturally facilitates dealing with any problems related to PROBABILITY.

D. In the past we have avoided REAL LIFE PROBLEMS due to the awkward numbers which usually resulted. This is no longer necessary. CONSUMER PROBLEMS, MARKETING PROBLEMS and FINANCIAL PROBLEMS can be approached with actual figures. Problems about SPORTS and other GAMES can be better handled with mini-calculators.

E. The introduction of the mini-calculator into the classroom will result in an emphasis on VERIFICATION TECHNIQUES. ESTIMATING, CHECKING and DEBUGGING SKILLS will become more prominent than ever before.

F. A great deal of insight into many of the theroems of NUMBER THEORY may be derived with the use of a mini-calculator. Questions dealing with DIVISIBILITY, FACTORING and PRIMES are often easily solved.

G. The pocket calculator can be used profitably in the study of the properties of the REAL NUMBERS and their important subsets. The FIELD PROPERTIES of Real numbers can be explored. Several properties of RATIONAL NUMBERS can be investigated. There are advantages to using mini-calculators to study POWERS AND ROOTS. There are several problems relating to the set of INTEGERS which could also be analysed and solved with the use of a small calculator. Some insight into the concept of a LIMIT could be gained with the aid of a mini-calculator.

H. Early in their mathematical careers most students become familiar with the so-called "FUNCTION MACHINE". A mini-calculator is exactly that and as such it can be used to aid and emphasize several important learning concepts and techniques. EVALUATION OF FUNCTIONS can teach PLANNING BEFORE EXECUTION. Such plans should probably take the form of some type of FLOW CHART. The importance of an ORDER OF OPERATIONS can be emphasized with the use of a pocket calculator. The important concept of INVERSE OPERATIONS can easily be studied with a small calculator.

I. Calculators can be used to increase understanding of many useful ALGORITHMS. Not only can we use these little machines to study the STANDARD ALGORITHMS but also, there is much good mathematics involved in devising CALCULATOR ALGORITHMS and "EMERGENCY ALGORITHMS", that is algorithms which can be used to perform certain sequences of operations if one or more keys on a calculator are inoperative.

J. The mini-calculator can be used to provide a considerable amount of DRILL on several aspects of mathematics. This is usually done through one of several GAMES which have been specifically designed for, or adapted to, the pocket calculator.

K. Finally, the small calculator can be used extensively in many of the OTHER SUBJECT AREAS besides Mathematics. There are several applications in SCIENCE, SOCIAL SCIENCES and PHYSICAL EDUCATION, to name but a few.

"In conclusion, it should be clear that

calculators can function as useful teaching aids at virtually every level of the instructional process. Refusal to allow their presence in school would mean that the school would be the only place left where mathematics was still being done uniquely by hand. On the other hand, if we allow calculators in the classroom and if we encourage proper use of them, then we can improve our students' understanding and appreciation of mathematics significantly."

SOME THINGS YOU CAN DO WITH YOUR CALCULATOR (and some you can't)

James Templeton, Department of Industrial Engineering, University of Toronto

This article discusses the real number system and some of the computational difficulties that arise in using real numbers. Most of the discussion deals with a particular real number, the very important and well-known real number π (= 22/7 approximately).

1. The Real Number System-Integer, Rational, Algebraic and Transcendental Numbers.

If we think of the integers ... 2, -1, 0, 1, 2, ... as represented by equally spaced points on an infinite straight line, then all the points on the line (including the integer points) represent real numbers. Besides the integers, there are other classes of real numbers which are important in mathematics. One such class of numbers is the class of rational numbers (quotients obtained by dividing an integer by a non-zero integer). Real numbers can also be defined in terms of limits of sequences of rational numbers. Another important class of real numbers consists of the real algebraic numbers, which are real roots of non-zero polynomials all of whose coefficients are integers.

Examples: 2 is an integer, and is also rational and algebraic, since it is a root of $x - 2 = 0$; 2/3 is rational and also algebraic, since it is a root of the equation $3x = 2$; $\sqrt{2}$ is algebraic, since it is a root of $x^2 = 2$; $x^5 - x - 5 = 0$ has a real root near +1.4519, and that root is also a real algebraic number.

The class of algebraic numbers is a very large one, and it is natural to ask whether this class is large enough to contain all the real numbers. The answer is that not all real numbers are algebraic. A real number that is not algebraic is called a transcendental number. The class of transcendental numbers is, surprisingly, much larger than the class of real

algebraic numbers. Although "almost all" real numbers are transcendental, it was quite difficult for the pioneers in this branch of mathematics to find examples of transcendental numbers. For further discussion of the real number system, including algebraic and transcendental numbers, see for example Chapter II of Courant and Robbins (1947) or Chapters 16 and 18 of Stein (1976).

The first number to be proved transcendental was constructed by Liouville in 1844, and is defined as the sum of a very rapidly converging series. Then in 1873 Hermite published a proof that e(= 2.71828...), the base of natural logarithms, is transcendental. Finally in 1882 Lindemann - but what Lindemann did in 1882 belongs in the next section.

2. Anyone for π? or, Have I Refuted Lindemann with my Pocket Calculator?

My pocket calculator claims to be "scientific",and has a number of buttons marked sin, cos, e^x, x^2, y^x, $\sqrt[x]{y}$, etc. to prove it. One button is marked simply "π", and if I push it the calculator displays 3.141592654. If I use the y^x button to calculate π^4, and then calculate $242\pi^4 + 113\pi - 23928$, the result is -0.000 000 11, which is a small negative number. If on the other hand I find π^4 by pushing the π button and then pushing the x^2 button twice, I find $242(\pi^2)^2 + 113\pi - 23928 = +0.000\ 000\ 08$, a small positive number.

(Try the calculation on your calculator and see what you get -- your answer may not agree with either of my answers.) The calculator, even though it carries 13 places internally, appears to have made a roundoff error, since it gives both a positive and a negative answer to the same problem. Neither the positive nor the negative answer to our problem can be relied on as accurate. Since we can't be sure whether the value of $242\pi^4 + 113\pi - 23928$ is positive or negative, could this value be exactly zero?

Let us try the problem again on a more expensive and newer card-programmable calculator. If we read in the "polynomial evaluation" program and the coefficients of the polynomial f(.) defined by $f(x) = 242x^4 + 113x - 23928$, we can evaluate f(x) for any given real x which can be entered in the calculator. For example f(3) = -3987 and f(4) = +38476. If we ask the calculator to evaluate f(π), we find the answer is given as f(π) = 0. It is fairly easy to show that the equation f(x) = 0 has exactly one positive root, one negative root, and one pair of conjugate complex roots. This suggests that the number π could perhaps be defined

as "the positive root of the equation $242x^4 + 113x - 23928 = 0$". Could this definition be correct? If it is, then we have refuted Lindemann, who claimed in 1882 to have proved that π is a transcendental number, or in other words that no polynomial equation with integer coefficients can have π as a root. If Lindemann had had a modern pocket calculator in 1882, would he have withdrawn the research paper that made him famous? Would he have worked out and published our little fourth-degree equation instead?

The answer to the above questions is NO. Lindemann was right, and his proof applies to all polynomial equations with integer coefficients, including ours. If our equation appears to have π as a root, then we must conclude that $f(\pi) = 242\pi^4 + 113\pi - 23928$ is only approximately, and not exactly, equal to zero. The calculated result $f(\pi) = 0$ must be due to the limited accuracy of the calculators used. The two calculators use 12-or 13- digit numbers internally, and display10-digit results. While 10-place accuracy is more than we need for most purposes, it is not at all the same as perfect accuracy.

(Note on mathematical proofs: How do I know that Lindemann was right? I have not myself read his paper, and any mathematician (including Lindemann in his time) can make mistakes. Some research papers contain mistakes which may be overlooked by the referees and editors who critically examine papers before accepting them for publication. Nevertheless, in the case of a famous paper like Lindemann's, one can have a high degree of confidence that the paper has been read carefully by many capable mathematicians since publication so that erros which might have occurred would (almost certainly) have been detected. Furthermore, there are now other proofs of the same result.)

3. Computing f(x) for $x = \pi$.

Our calculated value of $f(\pi)$ is subject to errors from two sources:

1) we calculate with an approximate value of π given by the calculator, which is accurate to at most 12D, or 13 digits in all including the 3 which precedes the decimal point; and

2) further errors due to rounding off calculations to 12 or 13 places will occur in the evaluation of the polynomial. It may be useful to remark that roundoff error can be demonstrated very simply in the calculator. Examples: 1/3 is represented in the calculator by .33.....3, using as many 3's as the calculator can carry in its register, but we know that 1/3 is a non-terminating decimal fraction; my calculator gives $\sin 45^\circ - \cos 45^\circ = 1 \times 10^{-12}$, but the

true value is $\sin 45^\circ - \cos 45^\circ = 1/\sqrt{2} - 1/\sqrt{2} = 0$.

If our equation does not give the exact value of π, how close does it come? To answer this question, we need a more accurate calculation of $f(\pi)$, with a smaller error due to rounding off in the calculator. One way of doing this is to use the approximations $\pi \approx 3.14159$ and $\pi^4 \approx 97.40909$ as follows:

We want to calculate $f(\pi) = 242\pi^4 + 113\pi - 23928$, accurately.

First find

$$242(97.40909) + 113(3.14159) - 23928 = -5.5 \times 10^{-4}$$

This calculation should be free from roundoff error, since the numbers calculated have at most 10 digits. Note that the number calculated is <u>not</u> exactly equal to f(3.14159).

Then find

$$242(\pi^4 - 97.40909) + 113(\pi - 3.14159) = +5.5008609 \times 10^{-4}$$

This calculation is subject to roundoff error, but since we have 10 D values of π and π^4, we should retain 5-place accuracy after subtracting 5 D approximations to π^4 and π.

Now add the two previous results and get

$$242\pi^4 + 113\pi - 23928 = +8.609 \times 10^{-8}$$

where at least the first digit on the right should be correct.

Now that we have a value for $f(\pi)$, how can we find how close π is to the positive root $r_1 \approx \pi$ for which $f(r_1) = 0$ exactly? To do this let $r_1 = \pi + \delta$, where we know that δ is small, and then find δ by setting $f(\pi + \delta) = 0$ and solving for δ. Since δ is very small, δ^2 and higher powers of δ will be small enough to be neglected at least in our first attempt. We write

$$\begin{aligned} f(\pi + \delta) &= 242(\pi + \delta)^4 + 113(\pi + \delta) - 23928 \\ &= 242\,(\pi^4 + 4\pi^3\delta + 6\pi^2\delta^2 + 4\pi\delta^3 + \delta^4) + 113(\pi + \delta) - 23928 \end{aligned}$$

where we expand $(\pi + \delta)^4$ by the binomial theorem or by "brute force" (straightforward multiplication)

$$= (242\pi^4 + 113\pi - 23928) + (968\pi^3 + 113)\delta + \text{terms in } \delta^2 \text{ and higher powers of } \delta$$

$$\approx f(\pi) + (968\pi^3 + 113)\delta$$

The coefficient of δ is $(968\pi^3 + 113)$, which is simply the derivative of $f(x)$ with respect to x, evaluated at $x = \pi$. Now set $f(\pi + \delta) = 0$ and solve the

(approximate) linear equation for δ. The result is $\delta = -f(\pi)/(968\pi^3 + 113) = -2.86 \times 10^{-12}$. (A more accurate calculation, using 18 D tabulated values of π and π^4, gives $\delta = -2.80 \times 10^{-12}$.)

Our conclusion is then that the positive root r_1 of the equation $f(x) = 0$ is not equal to π, but approximates π with an error which is less than 3×10^{-12} in absolute value.

"Good" and "Bad" Approximations to π.

Many approximations to π are known. A very familiar one is 22/7; some less familiar ones are $\sqrt{10}$, 355/113 and 88 arc cot 28. All these and others may be found in Chapter XII of Rouse Ball and Coxeter (1974), along with the error-free (except for roundoff) formula $\pi = 4$ arc tan 1. It seems reasonable to call an approximate value "good" if it is both simple and accurate. We may be willing to accept some loss of simplicity to gain more accuracy, but it is not clear how much. One way of giving a precise definition of a "good" approximation is to use a particular approximation, or class of approximations, as a standard of comparison. Let us therefore classify approximations by the number of decimal digits required to write them, and say that a "good" approximation is one which is more accurate than the "trivial" approximation obtained by rounding off π to the same number of places. Thus, 22/7 = 3.142857 ... (which requires 3 digits 2,2,7) would be compared to the "trivial" 3-digit approximation 3.14. Since $|\pi - 3.14| = 0.00159 \ldots$ and $|\pi - 22/7| = 0.00126 \ldots$, we say that 22/7 is a "good" approximation to π since .00126 < .00159.

Approximations which are neither "good" nor "trivial" will be called "bad". "Good" approximations to π include all those given above and also $\sqrt{2} + \sqrt{3}$, $3 + \sqrt{2}/10$, $(2143/22)^{\frac{1}{4}}$, and others. "Bad"approximations include 25/8 (used by surveyors in ancient Rome), 223/71 (given by Archimedes as a lower bound for π), and many others. One might object that by counting only the digits used in an approximation and ignoring square roots, circular functions, etc. we have adopted a somewhat biased criterion, but let us keep it for the moment at least.

Now consider again the approximation derived from the quartic equation $242x^4 + 113x - 23928 = 0$. This equation requires one digit for the exponent in x^4 and 11 digits for the coefficients, or 12 digits altogether if we do not count the zero on the right side.

The positive root of the equation differs from π by $|\delta| \approx 3 \times 10^{-12}$. Now, from Table 1.1 of Abramowitz and Stegun (1965), π = 3.14159 26535 89793 correct to 15 D (16 digits altogether). Hence, we can find the error in the trivial approximation using n digits with $n \leq 15$. For n = 12, the error is 2.1×10^{-13}. Thus the positive root of our equation is a "bad" approximation.

Note: The approximation based on our quartic equation although "bad" by our criterion, is nevertheless accurate, perhaps surprisingly so. The equation was found by combining two of the other approximations given above, using a weighted sum to obtain approximate cancellation of errors of opposite sign and nearly equal magnitude. The reader is invited to repeat the construction of this equation, or to improve on it by constructing a better one.

I acknowledge helpful comments by my colleague Scott Rogers and by an anonymous referee.

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Don't forget the March 1981 issue will be devoted to this topic. Watch for it, and submit your articles on problem solving soon.

MICROCOMPUTERS AND HYPRACTIVE CHILDREN

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Introduction

Dramatic recent advances in computer technology have made powerful microcomputers potentially affordable for widespread use in schools and clinics. The overall aim of our research project is to determine how best to capitalize on the potential benefits of these computers for hyperactive and other attention deficient children. This report is a brief summary of some pilot work and an initial experimental study conducted at the Child Devlopment Clinic, Hospital for Sick Children. This research compared children's performance on arithmetic problems administered by computer with problems given in a standard paper and pencil format.

The children in the project were referred to the hospital Clinic for problems of hyperactivity. They participated in the computer study during their free time between their scheduled appointments in which they were being assessed for drug effects on their hyperactivity. In all, there were 4 females and 31 males ranging in age from 6 to 14 years. Since the evaluation of drug effects was not part of the computer study, all children were tested either while not on medication or immediately after they had taken their medication.

Seventeen children served as pilot subjects in the initial phase of designing the basic computer programs and determining which aspects of the programs would be beneficial and which aspects had to be modified. Observations and interviews with these children resulted in significant changes in the programs, including a modification of the display to make it more readable, altering timing parameters related to the delivery of feedback, and adding special graphics and prompt messages. The detailed characteristics of the program are outlined below.

In the experimental study proper, a total of 18 children did addition problems both on a PET microcomputer and with paper and pencil. The two tasks were run on separate days. In each case, the children were asked to "do as many problems as you want and stop when you think you have done enough." The difficulty level of the problems was individually adjusted to be in the middle range of difficulty for any particular child. The study compared the rate of problem solving, accuracy and number of problems attempted in the computer format and the prepared pencil format.

The Program

The computer program used in this study had a number of characteristics that should be considered in the design of any program for impaired children:

(1) The level of problem difficulty could be individually tailored to a particular child. In the case of simple arithmetic, this meant we (or anyone who might subsequently use the program) had the option of specifying whether or not carrying was required, the number of addends, and the number of digits in each addend. The computer would then randomly generate any number of questions within the specified constraints.

(2) The display format on the computer was designed to be as easily readable as possible for a youngster (e.g., extra spacing between characters; vertical problem format; prompt arrows).

(3) In order to avoid confusion for children just learning arithmetic, the answer format on the computer was designed to be as similar to the pencil and paper format as possible. For example, the answers were entered into the computer from right to left rather than the usual computer format of left to right; carrying could be marked above the appropriate column of numbers with a special key; another special key allowed the child to "erase" mistakes.

(4) Problem solving was completely self-paced: when the child was satisfied with an answer, he or she could press a button to get the answer recorded and obtain feedback; when ready to proceed to the next problem, the child could press another key and the computer would proceed.

(5) Some modest motivational features were incorporated such as graphic displays (i.e., a series of robot-like characters) that appeared following correct answers. For incorrect answers, the message "Wrong, the correct answer is ..." was given.

(6) There were several built-in messages that were specifically related to the child's problem

of hyperactivity - special messages appeared whenever a child answered incorrectly too quickly, took an inordinately long time to respond on a particular problem or simply made too many inappropriate button presses (i.e., "STOP IT!").

In addition to controlling all of the problem generation, the display and the feedback material, the computer automatically recorded the child's response, the time taken to make the response, the time taken to request another problem, the number of erasures, if any, and the occurence of any special messages.

The problems for the pencil and paper test were generated by the same computer algorithm so that the difficulty levels and format of the problems could be kept as comparable as possible. During the paper and pencil tests, an observer, located behind a one-way mirror, recorded the time the child spent on each problem.

Results and Conclusions

The results were very clear-cut. There were no differences between paper and pencil and computer work in the proportion correct, the average time to do problems, or the average time between problems. The differences appeared in the number of problems the children voluntarily chose to do in the two mediums. On the average, the same children working on the same level of problems did almost twice as many problems on the computer as they did with paper and pencil. The table below gives the averages across the 18 children.

	# Problems done	% Correct	Average time to do problems	Average time between problems
Computer	31.4	.74	37.8 seconds	6.8 seconds
Paper & Pencil	17.6	.76	36.5 seconds	6.6 seconds

Apparently, hyperactive children are willing to spend significantly more time working problems on the computer, without any significant loss of accuracy or speed. Moreover, the interview data strongly confirms this preference for the computer. Among the comments given by the children were that the problems seemed to be easier on the computer, it was nice not to have to write down the answers, and they liked the rapid feedback. Many were eager to return and wanted to know when they would have a chance to work on the computer again. None of them requested more paper and pencil drill.

Although the results to date are at best suggestive and must be treated with caution until they can be replicated and extended, we are optimistic about the potential use of the microcomputer, both for clinical assessment and as a classroom tool. It is useful as a clinical assessment tool because it can present drill and practice type tasks (arithmetic drill; verbal learning tasks; etc.) automatically, and can very precisely and efficiently record information about accuracy and response time. Apparently, the medium itself does not affect the accuracy with which a child will do a certain set of arithmetic problems, nor does it distort the time parameters of the child's response. It does provide a motivating test situation in which the child is willing to work almost twice as long.

As a classroom tool for working with hyperactive children, the results would also seem to be quite promising. Again, it provides a way of doing drill and practice which the child finds motivating, can do with minimal adult supervision, and can be tailored to a child's particular performance level and rate of progress. The fact that he or she will voluntarily spend almost twice as long at the task at a sitting, should also greatly simplfiy classroom management and scheduling problems and in the long run, perhaps increase the rate at which the child acquires the skills.

Future Research

The results we have obtained to date are suggestive rather than conclusive. They are sufficiently positive, however, to warrant some follow-up studies to clarify some of the implications of using computers with hyperactive children in either a clinical or a classroom context. The most obvious question is the durability of the computer's appeal for these hyperactive children. Is this apparent willingness to work much longer with a computer than with paper and pencil simply a novelty effect or will it persist? The answer to this question is important for both clinical and educational uses.

Ed. Note: This past summer, Dr. Peter Lindsay et al. did a survey of microcomputer use in Ontario schools. Examples from this survey follow.

***Content Being Taught** (more than one possible for a specific application)	
Perceptual training	5.1%
Computation Skills	28.7%
Mathematical Reasoning	21.0%
Reading	6.4%
Computer Programming	51.0%
Other (e.g., "Informatics"; foreign language; chemistry)	12.7%

***Most Valuable Aspects of Computerized Instruction**	(Percentage time ranked First)
Instant and Consistent Feedback	37.3
Step by Step Instruction	3.0
Motivating the Student	32.8
Individualized Programming	23.9
Other	3.0

***Type of Program**	(Percentage of time ranked First)
Drill and Practice	31.8%
Simulation	13.6%
Games	5.9%
Introduction of New Material (i.e. computer programming)	38.3%
Other	10.4%

***Type of Training of Primary Instructor**	
Undergraduate	54.6%
Graduate	11.7%
Community College	0.6%
Inservice	7.8%
Self Taught	24.0%
Other	1.3%

COMPUTER ASSISTED REMEDIATION AND EVALUATION (The CARE-OAIP Intermediate Mathematics Project)

by Janis Gershman and Evannah Sakamoto

O.I.S.E.

The most extensive computer assisted instruction development effort ever undertaken for Ontario secondary schools has recently reached a conclusion. The Computer Assisted Remediation and Evaluation project (CARE), funded by the Ontario Ministry of Education, extended over the two-year period May 1978 to April 1980. CARE had two major objectives: to create computer assisted instruction (CAI) sequences for selected topics in Intermediate Mathematics (Grades 7-10); and as a parallel product of the CAI development process, to produce and validate test items suitable for inclusion in the Ontario Assessment Instrument Pool (OAIP).

Course Development

CARE was organized as a co-operative development project, modelled after earlier CAI development efforts undertaken by the Ontario Institute for Studies in Education (OISE). Subcontracts were written by the OISE to five representatives boards of education across Ontario:

Carleton
Hastings County
Lakehead
London
North York

These subcontracts provided for participation in both the course development and the field trials of the materials created. The CAI course authoring was the responsiblity of the OISE research officers and teacher representatives from the five boards. These groups of teachers created CAI sequences in Intermediate Mathematics covering approximately 30% of the new Ontario Intermediate Mathematics Guidelines. A Subject Advisory Committee comprised of curriculum experts from Ontario provided some guidance in the course development.

The material developed under the CARE contract was combined with topics previously developed by a pilot North York-OISE-Toronto project to form the following courses:

- Arithmetic (Whole Numbers, Fractions and Percent, Application) (23 topics)
- Integers and Rationals (9 topics)
- Algebra (12 topics)
- Probability (7 topics)

- Measurement (7 topics)
- MATHPRO (Mathematiques Programmees par Ordinateur) (28 topics)

Test items from these courses were field-tested, analyzed to varying degrees, and contributed for use to the Ontario Assessment Instrument Pool.

All of the courses were adapted for computer use by the OISE Instructional Progamming Group using the CAN-8 computer language and are being run on a Honeywell Series 60 Level 6 minicomputer located at the OISE.

The CAI courses are designed to individualize the instructional process by rapidly diagnosing skill deficiencies and providing remedial lessons and activities if required. Computer dependent techniques such as random problem generation, a variable-length testing procedure, instant answer analysis and feedback, and performance-dependent routing through a hierarchical structure of objectives are used. Games are a popular feature unique to the intermediate courses. They are available for most of the topics, and utilize the math skills taught in each particular topic. In addition, detailed student progress reports are generated and made available on a weekly basis to the teachers.

Project Participants

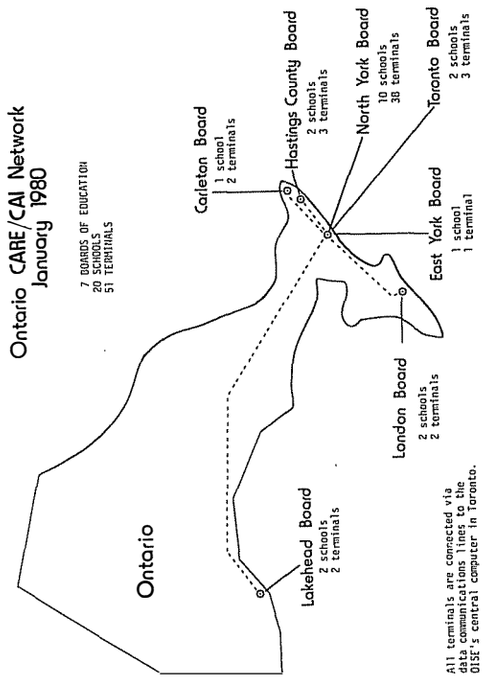
Pilot tests of the CAI material were carried out of seventeen schools in the participating boards (Carleton, Hastings, Lakehead, London, and North York) and at three more schools in the East York and Toronto Boards. A 51-terminal network was established, connecting these twenty schools to the OISE computer (see Map). Over this network, CARE was accessed by almost 10,000 students across Ontario during the two years of field trips.

The schools using CARE included a public school, a half-dozen junior high schools, and secondary schools of various types, including a French language school. Both advanced and remedial students, and a few handicapped students were included in the group that used CARE. Installations ranged in size from one to twenty terminals and were a mixture of standard paper-copy and CRT or TV-type terminals.

Usage

Usage of the terminals varied considerably from school to school. In the experimental period between September 1979 and January 1980, each terminal was used an average of 10 hours per week. This amount varied from 3 to 18 hours depending on the school. Individual students spent an average of 4 hours at the terminal in the 4 month period. This figure also varied dramatically from half an hour to 50 hours. Both teachers and students indicated in the attitude surveys that the ideal usage was approximately 1½ hours per week per student. Teachers however, were reluctant to release students from more than half a class period per week in order to use the CARE programs and students were required to find terminal time outside of class hours. The immense popularity of the terminals made this a difficult task for some students.

Some schools developed unique solutions to the problem of terminal usage. One that is noteworthy is the parent volunteer system at Don Mills Junior High School. In this program, parents whose services were recruited via the school newsletter, came in on a voluntary basis during evenings and Saturdays to supervise the students at the six terminals. Because of this parent participation, the terminals at Don Mills were available at times when they otherwise would have been inaccessible.



Evaluation: Achievement Gains

To measure achievement gains due to the use of CARE, paper and pencil tests were administered to students in Grades 7 to 10 at the participating schools. Of two thousand

students tested, 1082 went on to use the CARE Arithmetic material and the remaining students acted as an informal comparison group. The average test score in September 1979 was 43% and at the eight schools where both CARE and non-CARE students were tested, there was no difference between the scores of the CARE group and the comparison group.

However, after the January 1980 test administration, it was determined that CARE students not only improved significantly from pre-test to post-test, but also improved significantly more than non-CARE students at the same school (t=4.9, p .001). Moreover, when CARE students completing a substantial portion of the Arithmetic material (19 topics out of 32) are compared with the other students, an even greater difference in favour of the CARE group is evident. The table below illustrates these differences in gain scores.

Achievement Gains in Other Intermediate Courses

Course	No. of Students Completing Course by June 1980	Average Pre-test Score	Average Post-test Score*
Probability	324	44%	82%
Measurement	164	37%	86%
Algebra	125	69%	83%

*computer-administered post-test

Achievement Gains - Intermediate Arithmetic

Percentage Gain from Pre-test to Post-test

Students* (n)

Comparison Group (611) 12%

Total CARE Group (361) 18%

CARE group completing 19 or more topics 21%

*These students were at the 8 schools where both CARE and non-CARE students were tested.

Although the major evaluation effort was directed towards the use of Arithmetic material, a measure of achievement gain was determined for students completing the other CARE courses. These results are listed accordingly.

Evaluation: Attitude

The attitudes of both students and teachers towards the use of CARE were measured and found to be very enthusiastic.

Of the 831 students who completed the attitude survey, 97% of the students described their CARE experience as "fun" and 94% rated it "interesting".

91% of the students felt that they improved in Mathematics either a lot or a little and 72% of the students said that they liked Math better than before using CARE.

The students expressed little difficulty with either the use of the terminal, the language used, or the course material in general.

Student responses to the open-ended questions on the survey were very favourable. The most common response to "What did you dislike about learning Math this way?" was "Nothing". Some of the most frequent responses to "What did you like?" were:

- You can learn at your own rate.
- It's fun.
- It's a change from the classroom.
- It's easy to understand.
- No teacher to yell at you.
- No writing - just thinking.

Many students also felt that they had learned about computers as a result of using the CARE material.

Teacher response to CARE was equally favourable. Over 90% of the 42 teachers perceived at least moderate gain in students' attitudes towards computers, motivation to learn, attitude towards Math, and achievement in Math.

The teachers indicated that having CARE as a teaching aid increased their workload but also increased their teaching effectiveness. More than half the teachers felt that they spent more individual time with students using CARE than they did before. Teachers were divided in their opinions as to which type of student benefits most from CARE. Their responses were equally distributed among "higher ability", "lower ability", and "all students equally".

Most of the teachers' concerns about CARE

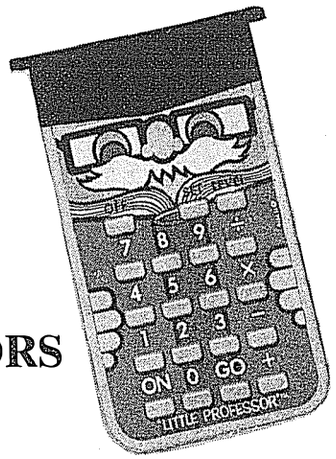
related to technical problems, scheduling terminal usage, and integrating CARE with the regular curriculum. Some of the "best features" of CARE as listed by the teachers were:

- it's a good individualized program
- it motivates disinterested students
- it progresses at a student's own rate
- it is effective with English as a Second Language students.

Conclusion

The CARE project has resulted in the parallel creation of a substantial body of computer assisted instruction and an almost infinite number of associated test items. This material in Intermediate Mathematics was developed and field-tested by more than 70 teachers at 20 schools in Ontario, and from September 1978 to January 1980 was accessed by almost 10,000 students. Evaluation of the effectiveness of the CAI indicated that CARE had a positive effect on both student achievement and student attitude. In addition, there was an overwhelming favourable response to CARE by those who used it. All of the teachers and 96% of the students surveyed wanted to continue their use of CARE.





FUN CALCULATORS

BOOK REVIEWS

FIFTEEN THOUSAND HOURS. by Michael Rutter et. al. (Harvard University Press: 1979) Reviewed by John Willensky of Algoma District.

This book is more than an isolated study of twelve English high schools, it is part of an ongoing debate on school achievement. Up to this point, as Rutter outlines in his useful survey of the recent literature, the major studies have not been too encouraging. Coleman's study (1966) found that differences in schools had little effect on achievement; though this subsequently proved to be limited to effects on verbal ability, it did give pause to educators. Jencks (1972) had similar conclusions: the quality of schools did little to effect *cognitive inequality,* and luck was still a major factor in future success. Yet Rutter and five colleagues believe that they have found it otherwise.

Their title is meant to make it obvious. If one spends, as children do, 1,500 hours being schooled, then how it is done must surely make a difference. A difference in behaviour, attendance and achievement that they hold can be traced to certain school practices. A differences in behaviour, for instance, that suggests the same student would be five times as likely to get into trouble at the worst school than at the best. And the best consists of a staff that (a) teaches a full period after arriving on time, (b) is regular in assigning and marking work, (c) has praise and time for the students, (d) is overseen by a concerned administration, and (e) works in a school of shared norms.

It may seem as obvious as an old school inspector's report, yet consider what it does not affirm:

regular corporal punishment for instance, or more importantly, the fact that the pupil/teacher ratio and preparation time do not figure in the factors that positively influence schools. Rutter suggests that this is one of the helpful benefits of his study, and one can well imagine it being used to *prove,* once more, that smaller classes and preparation time are not justified. But it need not do that. For Rutter has found that nothing makes a good school like a well-prepared staff with time for individual students, that nothing will support this approach to teaching better than ensuring reasonably-sized classes and time to prepare. The debate on school achievement will go on; the lesson is that the facts do not speak for themselves, they beg interpretation.

INTERVIEW

Taped Talks with Bob Smith

. . . Dr. James M. Sherrill

Dr. James Sherrill is a Professor (Mathematics Education) at the University of British Columbia. Along with Dr. David Robitaille he has been involved in testing programs conducted by the Learning Assessment Branch of the B.C. Ministry of Education. It was my pleasure to interview Jim at the Hyatt Regency Hotel, Vancouver during the 1980 R.C.D.P.M. Convention. As Jim talked, I was able to enjoy the view along Burrard St. with its flowering trees across the beautiful blue Burrard Inlet to the snow topped Grouse Mountain. It is hard to beat spring in Vancouver. But . . . to the interview -

R.S.: What is the British Columbia Mathematics Assessment (B.C.M.A.)?

J.S.: The B.C.M.A. was funded under contract by the B.C. Ministry of Education. The purpose of the 1977 B.C.M.A. was to measure minimal competency of children at Grade 4, Grade 8 and Grade 12 levels. The curriculum for mathematics was stated in 1972. The B.C.M.A. gave an overall picture of how the students in B.C. were achieving goals of the curriculum.

R.S. When was this testing done?

J.S. In March of 1977, over 100,000 students were tested. This has given us base-line data. The next testing will take place in March of 1981, and it will be interesting as we will be looking for any changes in the pattern.

R.S. What are the topics assessed?

J.S. They were restricted to the essential concepts and skills of mathematics at the three levels selected for testing. The Grade 4 test was designed as an-end-of-primary education test. Similarly the Grade 8 was designed as an end-of-elementary education test and the Grade 12 test was designed as an end-of-secondary test.

The actual Item Specification Model is an adaption from Jim Wilson's article in the Handbook on Formative and Summative Evaluation of Student Learning.

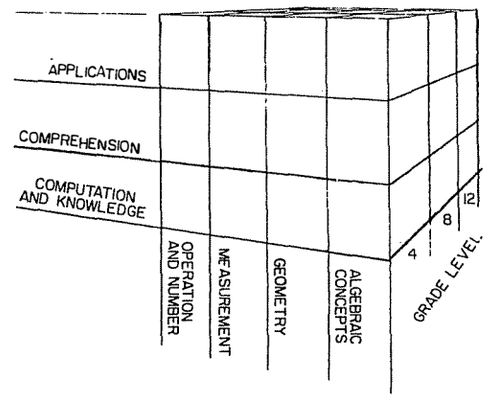


Figure 1 - Item Specification Model

J.S. It portrays the three Grade Levels, the four content areas and most important the three levels of cognitive behavior. In this hierarchy, the first level, Compulation, tests such areas as knowledge of facts, of terminology, and ability to use standard algorithms. The second level, Comprehension, included understanding of concepts, rules and ability to read and interpret problems. Applications required the ability to analyze data, recognize patterns and solve problems.

R.S. A most useful model, I understand all of the items were curriculum based and that each item developed went through a three-stage

editorial process. What did you do after the machine-scored results became available?

J.S. Three 15 member Interpretation Panels, one for each grade tested, were appointed by the Ministry of Education. Each panel included seven teachers of mathematics, two supervisors of instruction, two teacher educators, two school trustees, and two members of the public at large. Before examining the results they stated what a satisfactory performance rate should be for each item. Following a procedure designed to assist them in arriving at consensus the panelists eventually assigned for each item a rating scale:

Strength	S
Very satisfactory	VS
Satisfactory	SA
Marginally satisfactory	MS
Weakness	W

R.S. An interesting procedure. I am particularly pleased to see the general public and trustees represented on these panels. What general picture emerged from their ratings?

J.S. A good one. At the Grade 4 level, the performance on 77% of the items were judged satisfactory or better with only 6% judged as a weakness. At the Grade 8 level, the performance on 65% of the times were judged satisfactory or better with only 10% judged as a weakness. At the Grade 12 level, the performance on 69% of the items were judged satisfactory or better with only 10% judged as a weakness.

The strengths at the Grade 4 level were knowledge of the basic facts and problem solving. Both are surprising if one reads the professional journals. Problem solving is considered the most difficult topic in the curriculum to teach, yet the Grade 4 performance was satisfactory or better on all but one item. Both the Grade 8 and 12 levels performed well in the knowledge domain on their respective tests. Geometry and Problem Solving were the weak areas for Grades 8 and 12.

R.S. Geometry is a particular interest of mine. How did you test it since so many of the objectives do not always lend themselves to paper-and-pencil testing?

J.S. One of the restrictions of large scale testing is for an item to be in this assessment it had to be a paper-and-pencil item. We also found from our instructional practices questionaire that teachers placed geometry at the bottom of their priority list.

R.S. At the N.C.T.M. meeting in Seattle, President Shirley Hill in announcing the recommendations for School Mathematics of the 1980's placed problem solving at the head of the NCTM list of priorities. The N.A.E.P. found this to be an area of weakness. So your results also show a need. What did you do about it?

J.S. I gathered maximum funding from the Educational Research Institute of British Columbia, U.B.C.'s Social Sciences Grant Committee, and Youth Employment Program of the Federal Government - a total of $11,000. I then surveyed the literature for lots and lots of problems. What had really startled me from the results at Gr. 4 and Gr. 8 was the tremendous differential in performance from what are called one-step problems and the multi-step problems.

R.S. So you now have a research project titled "Solving Multi-step Problems".

J.S. Yes. For the study I found two groups of children with similar scores on a computational math test - 36 out of 42 questions correct on average. This test contained the kinds of computations needed in the word problem tests. I then looked at their reading score. Both groups were one grade level above average in reading. I gave them a one-step problem solving test and both groups solved all the problems. I then gave the multi-step problems test. One group scored 3 out of 8, the other scored 7 out of 8. Thus we have two groups of children who are above average in computation, above average in reading and a perfect score in solving one-step problems. The one disturbing factor is their difference in solving multi-step problems.

R.S. How do you intend to find the factors which caused the difference?

J.S. We have hired and trained interviewers. The interviews are being done blind in that the interviewer does not know whether they are working with a good problem solver or a poor problem solver. These data are sitting on my desk at the university awaiting analysis. My report to the funding agencies is due in the middle of July.

R.S. Would you be good enough to write an article describing the study in further detail, your findings and hopefully outlining some suggestions for teachers? This would be a welcome article for our Math. '81 issue titled "Problem Solving Part II".

J.S. Glad to do so for you, Bob.



TEACHING THE TRICKS

by Sharon Ross, California State University

Helping children memorize the basic facts of arithmetic is a challenge that faces every teacher of elementary mathematics. While some children seem to memorize the combinations independently through repeated exposure to assigned worksheets and activities, many others, perhaps as many as half of our school population, continue to use inefficient counting strategies long after those strategies seem appropriate.

An instructional program to help all children to memorize includes several components:

1. The use of concrete materials to develop prerequisite understanding of the four operations.

2. an understanding of what is meant by "memorizing" a fact such as 6 + 6 = 12;

3. **instruction** in the various thinking strategies which cluster related facts and

4. appropriate drill to reinforce the learning of clusters of facts.

While the prerequisite understanding of the meaning, or action, which defines the four operations is not within the scope of this article, the remaining components will all be addressed.

Many children, when presented with the task of memorizing basic facts, understand the task simply as a **speed** task. They will try to, for example, count on their fingers **very** quickly. They do not understand that a true **change** in thinking behaviour is required, not a speeding up of old behaviours. It is helpful to spend some time discussing with children what it means to memorize. Bob Wirtz has suggested using the language of "rolling off the tongue." When a fact is memorized it comes "rolling off the tongue" without any hesitation. Examples of things children have memorized such as songs, nursery rhymes, television commercials and perhaps "1 + 1 is 2" can help children to clarify the task.

Children (and their teachers) also need to realize that the **unit** of memory is the **whole** fact, for example "three times three is nine." Children need repeated practice in reciting the whole fact, not just the "answer". Many opportunities for such recitations can be added to the daily classroom, allowing children to repeat a particular fact at least twice, since the first time only those children who already know the fact will in fact be practicing it.

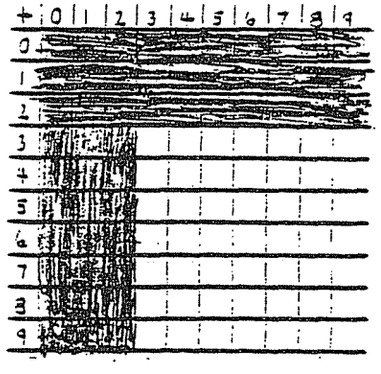
When confronted with the task of memorizing one hundred addition facts and one hundred multiplication facts, not to mention subtraction and division, children are overwhelmed. The size of the task can be substantially reduced by providing instruction in **thinking strategies,** those "tricks" which successful learners use to group facts into related clusters. Such an instructional program utilizes teacher-directed lessons about thinking strategies. Facts are examined in related clusters and the "tricks" for memorizing them are discussed with a wide variety of language.

The instructional program described is sequenced to begin with those clusters of facts which children find it easiest to remember and progresses systematically through clusters which become more difficult. The sequence is based on both research and the experience of classroom teachers. Subtraction and division are included in the total program through an emphasis on full fact families. Oral language is a dominant emphasis throughout the lessons; when children can talk their way from what they know to what they don't know, they have accomplished THINKING!

THINKING STRATEGIES FOR ADDITION

Tricks for Zeros, Ones and Twos

Fifty-one of the 100 addition facts include at least one addend which is a zero, a one, or a two. These addition facts are the easiest to learn; when this group is learned, the student is more than half way there!



The One's Trick

When adding the number "one", the answer is the next number.

The Zero Trick

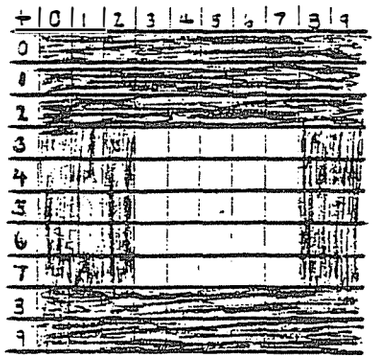
When adding zero, the answer is the other number.

The Two's Trick

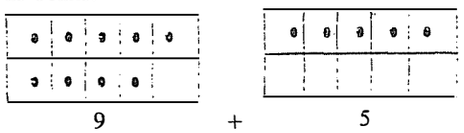
When adding two, look at the other number and count on two more.

The Nine's Trick

When adding nine, add ten plus one less than the other number. For example, 9 + 7 THINK: "One less than 7 is **6.** Ten plus 6 is **16.**"



The nine's strategy for addition can be discovered by every child. Use a manipulative which clearly shows ten, such as a bean frame. A bean frame is a 2 x 5 matrix cut from chicken wire, strawberry baskets, egg cartons, etc., which can be filled with countable objects such as beans.

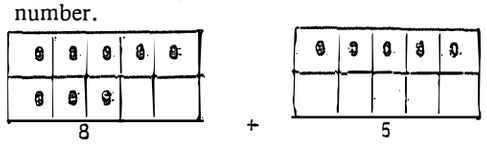


9 + 5

The language which accompanies such a presentation of 9 + 5 would be something like, "Take one bean from the five, put it with the nine and you have fourteen."

The Eight's Trick

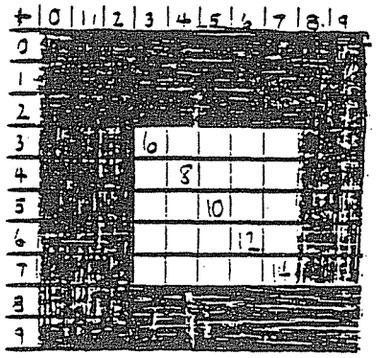
An extension of the nine's trick is often used by children to do the eights. When adding eight, add ten plus two less than the other number.



8 + 5

"Put two beans from the five with the eight to make ten. The sum is ten plus three or thirteen." This strategy involves counting back to get **two less** than the five.

Doubles



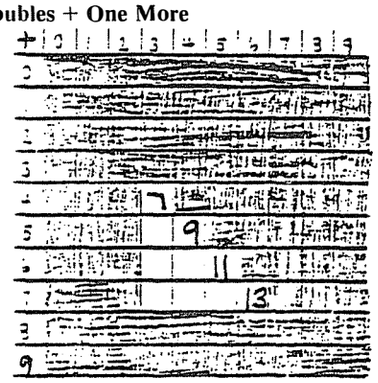
The doubles are easiest of all the rote learned facts because only two numbers, rather than three, need be recalled. Once the doubles are memorized, other facts can be related to the doubles.

For example: 6 + 6 = 12

The Addition Twins

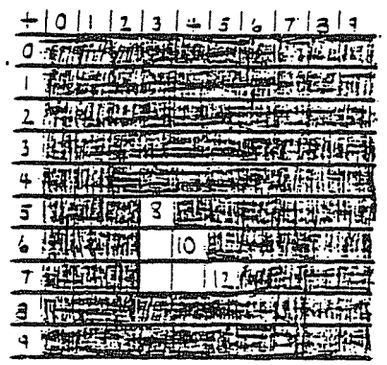
The order of the addends is unimportant to the sum. (3 + 2 = 2 + 3) The commutative property is not useful to many young learners without much discussion and instruction. Present students with many examples of twins, allow them time to find sums, then ask students how they solved each problem. Discussion should include adding on from the larger number. For example, 8 + 2 = 2 + 8. Solve by counting "eight, nine, ten," not by starting with two and adding eight more.

Doubles + One More

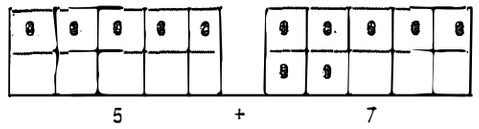


"If 6 + 6 = 12, then 6 + 7 is just one more, or thirteen." Students discover this principle when allowed to explore the relationship manipulatively.

Take From the Rich/Give to the Poor



Those facts where the addends differ by two, for example, 5 + 7, can be related to the doubles with another manipulative lesson. If one object is moved from the seven (the "rich") to the five (the "poor"), the result is the familiar double, 6 + 6 = 12.



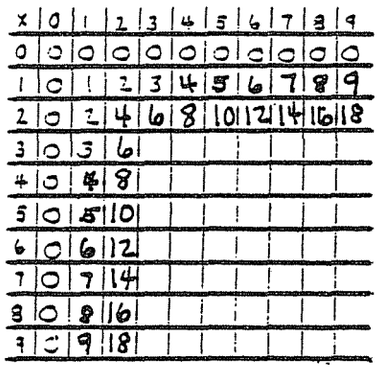
The Toughies

When all of the above-mentioned strategies are used, **only three facts remain!**

6 + 3 = 9
7 + 3 = 10
7 + 4 = 11

THINKING STRATEGIES FOR MULTIPLICATION

Tricks for Zeros, Ones and Twos



Fifty-one of the 100 multiplication facts include at least one factor which is a zero, a one, or a two. These multiplication facts are the easiest to learn; when this group is learned, the student is more than half-way there!

Twos

When you multiply by two it's just like the doubles from addition.

For example, 2 x 6 is the same as 6 + 6.

Ones

When you multiply by one, the answer is the other number.

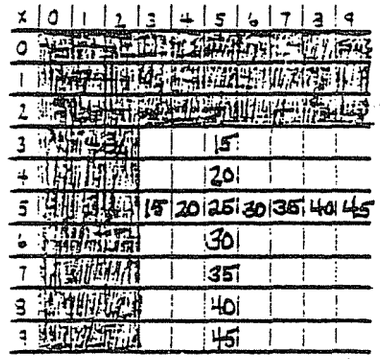
For example, 3 x 1 is 3.

Zeros

When you multiply by zero, the answer is always zero.

For example, 3 x 0 is 0.

The Five's Trick



The fives are easy because counting by fives is easy. This is probably due to the rhythmic, lyrical quality of the chant. Children who can already tell time on a conventional clock can use this skill to learn their five's facts. For example, when the big hand points to "4", it is **twenty** after the hour.

The Nine's Trick



The nine's trick is "tricky" to put into words, but "cinchy" when you get the hang of it. Study the patterns you find in the table below.

1 x 9 = 9
2 x 9 = 18
3 x 9 = 27
4 x 9 = 36
5 x 9 = 45
6 x 9 = 54
7 x 9 = 63
8 x 9 = 72
9 x 9 = 81

Looking down the answers, in the tens place the numbers increase by one, in the ones place they decrease by one.

The tens digit in the answer is one less than the number being multiplied. For example, in 7 x 9 = 63; the "6" in sixty is one less than the "7" in 7 x 9.

The two digits in the answer add up to nine.

Now these patterns are interesting in themselves, but with some additional development they provide a "Nine's Trick."

When multiplying by nine, first find one less than the other number (for the tens place); then add what you need to make nine.

For example, 9 x 7 THINK: "One less than 7 is **6. What do I add to get 9? 3.** Answer is 63."

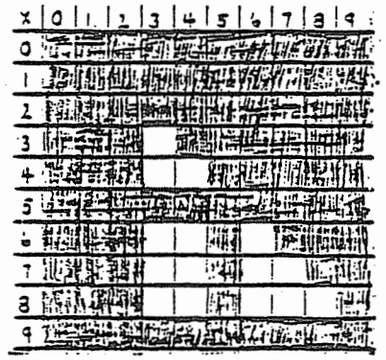
The nine's trick can also be taught as a finger-multiplying trick. Number your fingers from one to ten. Fold down the finger you are multiplying by and the tens digit will be on the left and the ones digit on the right.



For example: 3 x 9 = Fold down the third finger. Two on the left, seven on the right!



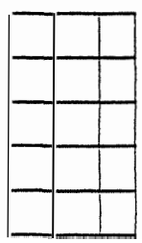
The Multiplication Twins



The order of the factors is unimportant. 3 x 5 is the same as 5 x 3. In developing this lesson, the use of rectangular arrays is useful. For example, in an arrangement such as --



and



-- are more readily seen as equivalent than a display of three sets of five objects compared with five sets of three objects. Children enjoy making up their own names for this property. Examples I have run across are the "upside down trick," the "switching trick," and the "twin's trick."

Use of all the above-mentioned tricks reduces the number of facts to be memorized from the original one hundred to fifteen.

Square Numbers

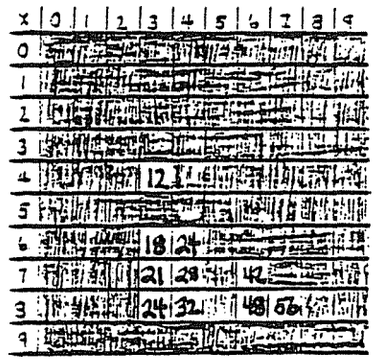


Building rectangular arrays for facts such as 3 x 3, 4 x 4, 5 x 5, etc. will yield squares. This is the origin of the term "squared" in the expression.

$3^2 = 9$ or $4^2 = 16$.

These facts which yield square numbers are the easiest to learn by rote memory because there are only two numbers in the combination. This is a far easier task than learning to associate three numbers.

Ten Toughies



At this point there are exactly ten facts remaining.

Doubling Trick



The doubling trick allows you to go from a fact you know to one you don't know. Seven of the ten toughies can be thought of as doubles.

4 x 3 is the double of 2 x 3
6 x 3 is the double of 3 x 3
8 x 3 is the double of 4 x 3
6 x 4 is the double of 3 x 4
4 x 7 is the double of 2 x 7
6 x 7 is the double of 3 x 7
6 x 8 is the double of 3 x 8

None of these require "carrying" in the doubling process.

Conclusion

By now your sleeve has few tricks remaining, although student demand may cause you to reach up that sleeve for one or two more. You might try 7 x 8 = 56 - look at that! 5, 6, 7, 8! Pretty tricky!

Students will continue to share their tricks with you and with each other. Surely you'll discuss the adding-on strategy for getting from what you know to what you don't know.

> For example: 6 x 8 = ? "I know 5 x 8 is forty so six eights must be eight more or forty-eight."

By presenting the facts to be learned in related cluster and allowing full discussion in the **children's own language, all** of the children in your class will learn what the "tricky" students have always been able to figure out - that there are some great short-cuts to getting those multiplication facts to roll off your tongue!

Further Reading

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STOCKING A MATHEMATICS CENTER

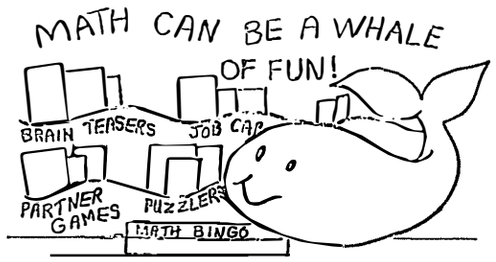


by Ann Lukasevich, Faculty of Education, University of British Columbia

> Mathematics is not merely counting, measuring or manipulating formulae but in its essence, is a way of thinking -- of deductive and inductive reasoning. It is an area of dynamic changes and exciting new discoveries. (Gallagher, 1965)

An imaginative, well stocked, well planned mathematics center can do much to increase mathematical skills and promote more positive attitudes towards mathematics learning. It can become one of the most treasured spots in the elementary classroom, producing larger numbers of math loving students. It can be a place where all students go to increase their mastery in basic computation -- a key goal of the elementary math program. This goal can best be achieved by providing the center with a large number of alternatives for repetition -- task cards, math games, number puzzles, manipulative puzzles, work sheets and listening drill tapes. For it is through the provision of such alternatives that slower students will get the additional drill practice they so desperately need without becoming frustrated by numerous repetitions of the same basic number facts. For those students requiring enrichment, more challenging materials such as puzzles and brain teasers constitute one of the favorite forms of enrichment in mathematics. They can be easily added to any mathematics center. One of the great benefits derived from this kind of enrichment is the motivation such puzzles arouse in students. The mathematics center can also be a place in which opportunity is abundant for the student to explore thought-provocative material which leads to the discovery of certain mathematical principles and concepts. In addition, supplementary materials involving extended applications of mathematics and extended topics in mathematics may be added. In this way, the math center will become "a place where imagination and creative ideas are born and grow, where thoughts and feelings can become certainties, where many young minds decide that mathematics is certainly more than practicing "sums" that teacher assigns (Charbonneau 1972)." Such a center can serve as a positive addition to a more structured mathematics program, or become the focal point of a less structured mathematics program. It can work effectively at all grade levels in the elementary school, and can be established in traditional as well as open classrooms. The choice is up to the teacher. If they are used as a supplement to the curriculum, they should be used in a positive, relevant manner.

An excellent location for this center is a corner of the classroom away from the flow of student traffic. It should be near bulletin boards or display areas if possible. An attractive sign or caption such as MATH CAN BE A



WHALE OF FUN! could be the central focus of the center. (Fig. 1) It should be equipped with a large work table or work area, some chairs, shelves, an abundant supply of manipulative materials, paper, and a variety of writing tools. The success of the center will depend to a great degree on the abundance and variety of materials supplied. The teacher might include teacher made and commercial made games, books of math, puzzles, task cards, and cassettes with math drills at various speeds. Their format and content will depend on the students who will be using the center. The teacher must be acutely aware of and sensitive to the needs of the individual students with whom she will be working -- the gifted, the average, and the slow learner. She must allow these students to work independently or in small groups at activities tailored to their individual interests, needs and abilities. This will be the teacher's greatest challenge. It will require a lot of planning, hard work and commitment on the teacher's part if the center is to become a flexible, creative laboratory where children can experiment, practice skills and learn at their own pace and level. The center can offer introductory resources for youngsters who are just beginning to learn about a special topic; are experiencing some difficulty in mastering elements of the topic; advanced resources for the gifted or very interested students of mathematics. While some of these resources will remain in the center all year, most of them will remain in the center for only a short period of time. This means that the teacher must have a large number of varied resources if this mathematics venture is to become successful, exciting, rewarding and challenging.

A key component of the mathematics center should be an area devoted to appropriate fun games. These games may be commercially purchased or made by the teacher, parent aides, or older students. If the games do not serve a purpose in attaining mathematical objectives, they should not be considered part of the mathematics program. Some commercial games which either provide practice in number facts or help develop concepts that are important to the math program of the child are Battleship, Auction, Nim, Equations, Tuf, Math Rummy, Ratio Rounds, Hi-Q, Quizmo and Twixt. Additional games can also be made by the teacher to make drill more palatable, to reward effort, and to change classroom pace. They can also be made to develop higher level thinking. The following games are examples of ones that can easily be made by classroom teachers.

1. **Dominoes.** Dominoes are great for drilling any operation. Make a set with the basic facts on one end and the answer on the other.

2. **Bingo.** This popular game may be played in a variety of forms. Often the caller calls a number question and the players must find the answer on their cards. It may also be played so that the caller calls the answer, and the player covers the number question. This is a more difficult task.

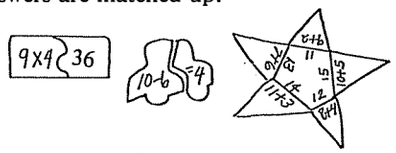
3. **Concentration.** It may be used on almost any grade level. It consists of 20 cards, half questions and half answers. The players take turns trying to match facts with their corresponding answers. The player with the most cards at the end of the game wins.

4. **Make a Dollar.** Put 1c, 5c, 10c, 25c, 50c, $1.00 on a cube. Children take turns tossing and adding to get a dollar. The first one to get exactly one dollar wins. They can also play to $2, $3 or $5. This game could also be used for addition of fractions to make a whole.

Another option which works well in the mathematics center is the manipulative puzzle. This can have two or more parts and is similar to a jigsaw puzzle in format. Matching boards are often included in this category. Both are excellent ways to develop specific computational skills. The following are activities which would fit into this category.

1. **Match-up Boards.** Pizza sheets are divided into sections. Problems or number facts are put in each wedge section. Clothespins with answers are provided to be clipped to the wedges that they match.

2. **Manipulative Puzzles.** These can be designed in a variety of shapes. Questions and answers are matched up.



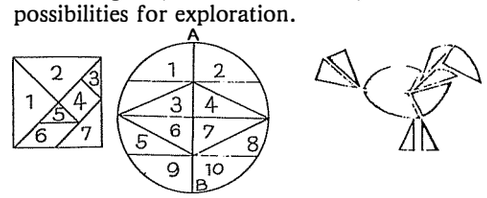
Another popular option which may be added to the center is what I called geometric tidbits. Materials of this type may be either purchased or made. They give children a chance to discover characteristics of specific shapes through direct examination and manipulations of models, geoboards, tangrams, and other concrete objects. The following activities would be examples of this type of activity.

1. **Geometric Jigsaw Puzzles.** Using construction paper, cut a variety of puzzles using the geometric shapes that have been studied. The larger shape could then be cut in smaller shapes of the same type. For example, one large triangle could be cut into six smaller triangles and placed in an envelope. The children would solve the puzzles by using the small triangles to make one large triangle.

2. **Make the Design.** Parquetry Blocks offer a unique way for young children to learn about geometric shapes. Teachers can construct task cards showing unusual shapes which utilize three or four of the parquetry shapes to make a simple design. The children are asked to repeat the designs on the cards using the real Parquetry Blocks.

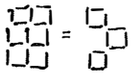
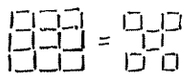
3. **Geoboards.** They can be a challenge at almost any grade level. Younger students may be asked to make as many different 3-sided figures as they can on a geoboard and then transfer the figures to geodot paper. With older children, unusual shapes may be constructed and the boundaries and areas determined by partitioning off units and square units and counting. Children can also create their own shapes and challenge their peers.

4. **Tangrams.** They offer children the chance to engage in puzzles and creative endeavor. The common tangram illustrated below consists of seven parts. These are to be arranged so as to form imaginative and fanciful figures. It has been somewhat overexploited. The circular tangram, illustrated below, is rich in possibilities for exploration.



5. **Stamp Problems.** These can become quite fascinating as the number rises. Children can first try to discover how many different ways they can buy three attached stamps (there are six possibilities). After this first problem has been solved, they can go on to four attached stamps and five attached stamps. Their solutions can be drawn on graph paper.

6. **Matchstick Puzzles.** These can be solved using toothpicks. The children are instructed on task cards to use 17 matchsticks to form six squares. They are then instructed to remove five of the sticks so as to leave only half as many squares. As a second challenge, they could be asked to use twenty-four sticks to make nine squares. They are then asked to remove four sticks so that there are five-ninths as many squares.

1.  2. 

Another challenging option which should be in all mathematics centers is the brain teaser. These should vary in difficulty if individual class needs are to be met. The following brain teasers would be examples of this type of activity.

1. Two-thirds of my brother's age is five-twelfths of mine, and I am nine years older than he is. How old are we? (He's fifteen; I'm twenty-four).

2. I have in hand 10 current Canadian coins. Yet, I lack change for a nickel, dime, quarter, half dollar and dollar. What coins have I? (Four pennies, four dimes, a quarter and a half dollar).

3. Can you split 100 into four parts so that when you add 4 to one part, you get the same answer as when you multiply 4 by another part, or subtract 4 from another part, or divide the last part by 4?) (Four parts include 64, 20, 12, 4).

4. In my carpenter shop I make only three-legged stools and four-legged tables. One day I looked at my day's output and counted 31 legs. How many stools and how many tables did I complete that day? (Three possible answers: 1 stook and 7 tables, 5 stools and 4 tables, 9 stools and 1 table).

5. A frog is at the bottom of a 32 foot well. Each day he jumps up 5 feet but during the night he slides down two feet. How many days will it take him to jump out of the well? Remember the frog must jump five feet each day. (10th day).

6. A group of 30 students were divided into two groups so that the first group had 8 more than the second group. How many in each group? (11 and 19).

7. Each day Tom gets to work on time his father gives him $10 but he takes away $15 each day Tom is late. After twenty days Tom has neither gained or lost any money. How many days did he get to work on time? (12 days).

8. A farmer's chickens and sheep have 24 heads and 76 feet. How many chickens does he have, and how many sheep? (14 sheep and 10 chickens).

9. A horse travels half a trip at 12 miles per hour and half at 4 miles per hour. What is its average speed? (6 miles per hour).

10. A man paid out $63 in six bills, none of them $1 bills. How did he do it? (He used a $50 bill, one $5 bill and four $2 bills).

In addition to these brain teasers, the center should also contain number puzzles such as number crosswords, magic squares and triangles, star-burst puzzles and cross number puzzles. These should vary in difficulty if individual student needs are to be met.

Drill tapes can also be developed for use in the mathematics center or in the listening center. These should represent different levels of mastery such as easy addition, hard addition, easy multiplication, hard multiplication. For each set of problems, several tapes can be prepared, each with time intervals between problems varying from ten, six, to three seconds. The children begin with the tape that has the longest intervals and progress to the one with the shortest intervals. A three second response is a very efficient immediate recall. Six seconds is adequate for most children.

The above suggestions for stocking a mathematics center are only a beginning. As you work with the children and as they work in the center, "things" will come to mind. Remember to begin slowly, with a small amount of equipment used for short periods of a day, and then work up to larger periods of time. **Do not feel that you must start with a packed center of materials!**

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"BEHAVIORAL INDICATION OF MATH ANXIETY"

an excerpt from

WHAT CAN WE DO ABOUT MATH ANXIETY?

Dr. Robert J. Sovchik, Associate Professor of Education
Dr. Gary W. Kane, Associate Professor of Education
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Many students of mathematics suffer from a mysterious malady called mathematical anxiety. "Freezing up" on a mathematics test or feeling extremely nervous when doing mathematics are two symptoms associated with mathematical anxiety. As Lewis Aiken (1970) has observed "anxiety and attitude are related but one generally thinks of an attitude as being a bit less visceral and exciting than anxiety". In general, most writers view anxiety as an unpleasant experience often accompanied by feelings of uneasiness and bodily tension. Students experiencing mathematical anxiety may experience performance problems in mathematics and they may avoid the aversive condition of studying mathematics.

Characteristics of Mathematical Anxiety

Various sources in psychology such as Phillips, Martin and Meyer (1972) were examined in order to draw up a list of traits related to anxious individuals. These traits are not considered to be separate and unique categories, but they can offer some help to an observant classroom teacher. Observation of several of these traits may indicate that a student is experiencing mathematical anxiety. Psychological traits, along with specific symptons relating to mathematical anxiety, are presented below:

1. Cautiousness and perseveration

Cautiousness and perseveration can be valuable when doing mathematical work. Obviously some mathematical problems require careful work, but when extreme caution is observed, this may indicate a sympton of mathematical anxiety. Perseveration can be observed when the same response is used to solve many different problems. For example, suppose a child was studying multiplication and obtaining products for 24. If, rather close

$$\begin{array}{r} x2 \\ \hline 48 \end{array}$$

in time, we observe the child solving 35; we

$$\begin{array}{r} +\,4 \\ \hline 79 \end{array}$$

may conjecture that the child is perseverating by using the same multiplicative procedure for an addition example.

2. Dependency, direction seeking, conforming tendencies

A child who continuously asks for help when doing mathematics may be indicating another sympton of mathematical anxiety. The "is this right" syndrome may be indicating a lack of confidence with mathematics.

3. Reduced responsiveness to the environment

Not paying attention may have a purpose. By avoiding the mathematical topic the child can then explain failure in a more rational way. Paying attention and then failing requires realistic assessment by the student. By not paying attention, we can avoid a realistic assessment of the consequences of our behavior. Culturally, we also avoid saying that taboo phrase, "I don't know".

4. Deterioration in complex, intellectual and problem solving processes

Sometimes children will perform calculations well but they will have difficulty with mathematical problem solving. It may be that the anxiety and perceived loss of self esteem operate to reduce problem solving effectiveness.

5. Extreme fear of failure

Children who are unwilling to take a moderate risk or make a guess when solving mathematical problems may be very concerned with failure. Avoidance of this failure may be observed in some of the very quiet children.

6. Rejection by family, especially the mother

Newspaper estimates state that for every two marriage licenses granted, one divorce paper is filed. Mathematics teachers should attempt to understand that this situation may affect the school performance of a child. Even in the case of the "healthy" family, a child's perceived rejection may affect his anxiety level and his mathematical performance.

7. Inadequate congruence between expectations and realistic talents

Realistic goals are important. A child needs to develop self diagnostic tools to appraise, refine and further develop his mathematical ability.

8. Hostility

Hostility may be a result of anxiety. Perhaps some children who are loud or abusive during mathematics class are really very frightened about their mathematical abilities.

9. Physiological symptoms, such as hair pulling, eye ticcs, dilated pupils, headaches, stomach aches, loss of sleep, increased respiration and nightmares

Physiological symptoms observed during mathematics work may reflect a high degree of anxiety about mathematics learning.

10. Compulsive behaviors

Some compulsive behaviors can be helpful to students. For example, extreme neatness in a scientist may be a desired trait. On the other hand, a mathematics student was once observed to compulsively play with his glasses during a mathematics test. After asking the youngster why his performance on the test was low, the youngster replied "my glasses don't fit". Compulsively thinking about his glasses was a non-task related behavior and it tended to decrease the student's mathematical performance.

11. Avoidance behaviors

Avoidance behaviors may be observed when assignments are not turned in, courses are dropped or simply not taken, etc.

12. Low self-esteem

Low self-esteem manifests itself when remarks such as "I am just not any good at mathematics". Some students will consistently depreciate their mathematical performance.

Ed. Note: The full paper, which includes teacher strategies to overcome pupil anxiety, is available from the author for those interested.

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INSIGHTS INTO STUDENTS' ERRONEOUS SOLUTION TECHNIQUES FOR ADDING FRACTIONS



by Dr. Mary Ann DeVincenzo, Fordham University, New York City, New York

In a recent research study[1] ninth grade elementary algebra students were observed to apply certain faulty algorithms on written diagnostic tests in arithmetic and algebra. Solutions of 1600 students from 14 schools in New York City helped to identify instances in which a faulty algorithm was applied with a certain degree of consistency in arithmetic and algebra addition of fractions problems. Ninety-three students were randomly selected to be interviewed from those students who demonstrated a consistent use of a faulty algorithm on the written diagnostic tests. The purposes of the interview sessions were: (a) to help establish the reliability of proper recognition of errors from written solutions and (b) to

[1] The research study "An Investigation of the Relation Between Elementary Algebra Students' Errors in Arithmetic and Algebra in Selected Types of Problems" was completed for a doctoral dissertation in Mathematics Education at New York University by Dr. Mary Ann DeVincenzo.

better understand the processes which accompany erroneous solution techniques. In a typical Observation-Interview session students were presented with problems, each one written on a separate sheet of paper and asked to "think aloud" as the problem was solved. The researcher intervened only in instances where additional clarification was needed. Information that was obtained from these interview sessions about specific errors in adding fractions will be presented in this article.

Error EA1: Adding numerators and adding denominators
($\frac{1}{2} + \frac{1}{3} = 2/5$).

There were 29 students who applied EA1 during the Observation - Interview sessions. The position of the "plus sign" and the horizontal format of the problem were the most common reasons supplied (in 27 instances) for selecting this solution technique, as illustrated in the following excerpts:

Because they're across from each other and the plus sign is in the middle (Student 719)

I guess cause the figures are lined up straight across. (Student 179)

That's the way it looked right to me. Cause that's how I was taught to do it, to just add the numbers. Just put a dividing line because it's only logical to go across. (Student 1262)

It was also noted during the Observation-Interview sessions that solution technique EA1 was based upon algorithms used for problem types other than addition of fractions. One student explained that the numerators and denominators were added across because "when adding whole numbers you just add across." At this point the student supplied an example of a whole number addition computation: $2 + 3 = 5$. In another instance a student's justification for using solution technique EA1 is reported: "you go across in multiplication so I thought you'd go across in addition."

It may appear from the excerpts presented here, that the horizontal format of the addition of fractions problems prompted students to initially consider EA1 as a solution technique. However, many students were observed during the interviews to attempt to apply the standard algorithm in a vertical format, although unsuccessfully, before adding across. An example of this occurrence is illustrated in Figure 1 as the solution is explained:

Do the following addition:

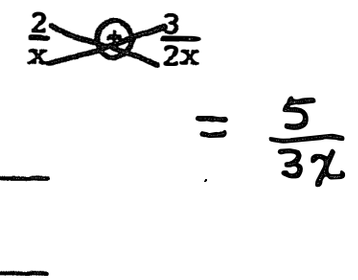


Figure 1

Student: I'm trying to find a common denominator, find the common denominator and get to the final answer. [At this point the student draws two lines to the left of the problem and attempts to solve the problem vertically. (See Figure 1)] I added two and three and got a five and I said 2x plus x is 3x.

Interviewer: Why did you think of doing this problem another way?

Student: Well, if you do it that way [adding across] you didn't like cross-multiply or cross-out. [The student now draws diagonal lines in the problem. (See Figure 1)].

Interviewer: Why did you decide to add two and three and 2x and x?

Student: The positive [the student circles the "+" in the problem (See Figure 1)] and the x and the 2x. This is simple. It's a simpler way without having to do all that work.

This student's reference to "cross-multiply" as a step in the reconstruction of the standard algorithm for adding fractions may be surprising. However, many students were observed to produce equivalent fractions from "cross-multiplying". In this case, the steps of the standard algorithm could not be completed without a common denominator.

In many instances during the Observation-Interview sessions it appeared that EA1 was supplied as a replacement for the standard algorithm, which could not successfully be retrieved. This tendency of students to consider the standard algorithm before eventually applying solution technique EA could **not** have been discerned from an analysis of only written computations.

The observed relationship between an unsuccessful completion of the standard algorithm and utilization of faulty algorithm EA1 leades to the following question: If the

original statement of the addition of fractions problems was **not** presented in a horizontal format, would students tend to continue to resort to using EA1?

Error EA2: Finding a common denominator and adding the original numerators (½ + ⅓ = 2/6).

There were six instances in which EA2 was utilized as a solution technique during the Observtion-Interview sessions. It is not coincidental that the computations used in EA2 are some of the same computations found in the standard algorithm. The following excerpts reveal that students were applying EA2 as an abbreviated version of the partially forgotten standard algorithm.

Because I guess it just came naturally to multiply the 5 and the 3 [denominators] . . . I forgot how I did the top . . . I did the numerator different from the denominators (Student 179).

(Ed. Note: Student 179 applied error EA2 to the problem 1/5 + ⅔, which explains the '5' and '3' he refers to. The author noted that it was not unusual for a student to try more than one technique on the same or similar problems, thus student 179 trying EA1 on one problem, and EA2 on a similar problem.)

I looked at the fraction, right and you're supposed to be multiplying fractions by the denominator, that's why (Student 1507).

These excerpts demonstrate that the computations in EA2 were generally applied in a rote, mechanical fashion. Segments of the standard algorithm as well as the complete standard algorithm for adding fractions were applied in an automatic manner. The following excerpts demonstrate a lack of understanding by many students who correctly added two fractions:

The method is to divide and then multiply (Student 783).

It's just something I follow (Student 948).

This is how we learned to do it (Student 359).

I don't understand it, but it works (Student 730).

The discussion presented here indicates that forgetting parts of the standard algorithm may have resulted in the creation of solution technique EA2.

Error EA6: Multiplying each fraction by a common denominator (1/5 + ⅔ = (15)1/5 + 15⅔ = 3 + 10 = 13).

Solution technique EA6 was described in 4 instances during the interview sessions to be based upon a technique used to solve fractional equations. A similarity between the fractions in addition of fractions and fractional equations problems was recognized and conveyed by the following exercepts:

Sometimes we have a fraction and we try to do it as a whole number . . . when it's easier for the equation (Student 1093).

First I was thinking of adding these but I wouldn't get a complete answer . . . and I was thinking of another problem that I did in class . . . The answer I want to get is x equals something (Student 807).

Analysis of data for the diagnostic tests revealed that there were no instances of EA6 reported in one particular school. Upon interviewing the teachers of the students who participated in the research study at this school it was discovered that the standard algorithm of multiplying both sides of a fractional equation by a common denominator was **not** presented as a solution technique. Consequently, "crossing-out" with a common denominator was not familiar to these students and was not inappropriately adapted for addition of fractions problems.

The adaptation of standard algorithms as erroneous solution techniques suggests that certain standard algorithms may be conducive to be used as faulty algorithms in other types of problems and therefore selections and presentations of standard algorithms should be reevaluated.

The excerpts presented here can suggest ways that mathematics teachers can better understand their students' solution techniques. Information obtained from listening to students' explanations of their computations can help to suggest proper instruction that will improve the teaching and learning of mathematics since it has been shown that: (a) written computations do not always reveal the specific problem which a student is experiencing, (b) the same faulty algorithm may be accompanied by different reasoning, (c) "correct" computations do not necessarily represent correct mathematical reasoning and (d) several solution techniques are sometimes considered before computations are recorded and one solution method selected.

References

The complete study is available from University Microfilms (number 80 - 17,494).

SIMPLE TESTS FOR DIVISIBILITY

by Doug Woods, Toronto, Ontario

Does 7 divide 12,345 evenly? A simple way to answer this question is to apply the following test:

1) take the last digit (5) of the number to be tested and double it;

2) subtract this result (10) from the other digits of the number (from 1,234);

3) if you can see that the resulting number (1224) is divisible by 7 then stop because the original number will also be divisible by 7. If it is not obvious that 1,224 is or is not divisable my 7, then go back to step 1) and test 1,224.

122 - (2 x 4) = 114
Testing 114 gives
11 - (2 x 4) = 3

Since 3 is not divisible by 7, it follows that 114, 1,224 and the original number 12,345 are not divisible by 7.

Does 13 divide 1,559 evenly? The rule for testing divisibility by 13 is

1) multiply the last digit by 4;

2) add this result to the other digits of the number;

3) is the result divisible by 13?

159 + (4 x 9) = 195

Since it is not obvious that 195 is divisible by 13, test 195 in the same way.

19 + (4 x 5) = 39

Since 39 is divisible by 13, it follows that 195 and the original number 1,599 are also divisible by 13.

All of the tests described in this paper follow the general pattern of the tests above.

1) The last digit of the number to be tested is multiplied by some factor;

2) the result is either added to or subtracted from the remaining digits of the number being tested;

3) the newly generated number is inspected for divisibility if it is not obvious whether it is divisible or not, then the new number itself is tested. This process continues until divisibility is easily determined.

A convenient way to write down the numbers involved in performing the tests is as follows.

Is 867 divisible by 7? (Recall that the test for 7 is 'double the last digit and subtract from the rest of the number.')

```
 861
-2    (subtract 2 x 1 from rest of number)
 84
-8    (subtract 2 x 4 from rest of number)
 0
```

Since 0 is divisible by 7, both 84 and 861 are also divisible.

Is 34,567 divisible by 13? (Recall that the test for 13 is 'multiply the last digit by 4 and add to the rest of the number.')

```
  34 567
+   28    (add 4 x 7 to rest of number)
  34 84
+  1 6    (add 4 x 4 to rest of number)
  36 4
+ 16      (add 4 x 4 to rest of number)
  52
+ 8       (add 4 x 2 to rest of number)
  13
```

Since 13 is divisible by 13, the numbers 52, 364, 3,484 and 34,567 are also divisible.

Once the pattern of this type of simple test is understood, it remains to give a rule for generating the paricular test for a given divisor.

The rule for testing for divisibility by N is:

1) Form the lowest multiple of N (call it MN) which ends in 1 or 9 (this restricts N to numbers ending in the digits 1, 3, 7, 9);

2) if MN ends in 1, the rule is 'multiply by (MN-1)/10 and subtract';

3) if MN ends in 9, the rule is 'multiply by (MN+1)/10 and add'.

For example: 7 x 3 = 21 ends in 1
(21 - 1)/10 = 2

Therefore the rule for 7 is 'multiply by 2 and

subtract'.

$$3 \times 3 = 9 \text{ ends in } 9$$
$$(9 + 1)/10 = 1$$

Therefore the rule for 3 is 'multiply by 1 and add'. (This is the same as finding the digital root.)

PROOF

S is the divisor ending in 1, 3, 7 or 9 which we are using and N is the number we want to test for divisibility by S.

Form the product TS which is a multiple of S (generally the smallest is best) ending in 1 or 9. (This proviso is necessary in order that the number R, which will be formed below, be an integer.)

The number N can be written in general form as

$$N = a_k 10^k + a_{k-1} 10^{k-1} + \ldots + a_1 10 + a_0$$

The number formed from N by removing its last digit and adding or subtracting some multiple of that digit will be called R. R can be written in general form as

$$R = a_k 10^{k-1} + a_{k-1} 10^{k-2} + \ldots + a_1 \pm \frac{(TS \pm 1)a_0}{10}$$

(Here, if TS ends in 1, then R has $- \frac{(TS - 1)a_0}{10}$ and if TS ends in 9, then R has $+ \frac{(TS + 1)a_0}{10}$.)

As pointed out above, R will be an integer iff TS ends in 1 or 9.

$$10R = a_k 10^k + a_{k-1} 10^{k-1} + \ldots + a_1 10 \pm (TS \pm 1)a_0$$
$$= a_k 10^k + a_{k-1} 10^{k-1} + \ldots + a_1 10 + a_0 \pm TSa_0$$

By simple substitution

$$10R = N \pm TSa_0$$
$$N = 10R \mp TSa_0$$

Now 1. S divides N iff S divides 10R
2. S divides 10R iff S divides R
3. S divides N iff S divides R

which completes the proof.

The general method will not work for divisors ending in 0, 2, 4, 5, 6 or 8 because of the factor 10 appearing in line 2 above. If S = 12, for example, the statement "S divides 10R if S divides R" may be false since 12 and 10 have a common factor.

The general divisibility test described above works, therefore, only for divisors ending in 1, 3, 7 or 9, as mentioned before. Despite its limited range though, students may find several of the tests useful in their arithmetic work. More advanced students may find the proof interesting.

RESEARCH COLUMN

William Higginson, Queen's University

Young Children's Acquisition of Language and Mathematics: Some Recent Findings from Geneva.

In August of this year some 3000 mathematics educators from all over the world gathered at the University of California at Berkeley for an International Congress on Mathematical Education. This meeting was the fourth in a series which began in Lyon in 1968 and continued with Congresses in Exeter in 1972 and Karlsruhe in 1976. Plans are underway for ICME 5 which will be held in Adelaide, Australia in 1984.

Participants in the Berkeley Congress spent most of the six working days of the meeting in specialized small group sessions. On four occasions, however, the whole Congress was addressed by a distinguished educator. On the afternoon of Sunday, August 10th, Professor Hans Freudenthal, the eminent Dutch mathematician spoke on "Major Problems of Mathematics Education". On Monday, Dr. Mimi Sinclair, Professor of Psycholinguistics at the University of Geneva, gave a lecture entitled "Young Children's Acquisition of Language and Understanding of Mathematics". On Wednesday Dr. Seymour Papert of MIT spoke on "The Computer as Carrier of Mathematical Culture" and on Friday Professor Hua Loo-Keng of Academica Sinica, Peking talked about "Some experiences in Popularizing Mathematical Methods". Each of these lectures had many positive features. Freudenthal's reflections, based on some fifty years of active involvement in mathematical research and

mathematics eduction, were quite provocative as were Papert's speculations about the long range impact of computers on mathematics instruction. It seemed, however, that it was the lecture of Professor Sinclair which caught the mood of the Congress most accurately and which showed clearly how first-class research, even in a different discipline, can have important implications for the teaching of mathematics. It is the purpose of this brief review to report on some of the main features of Dr. Sinclair's lecture.

Because of the important role she has played in the Genevan school it is not surprising that her position is basically Piagetian. Since her research interests are in the area of the early development of language she refers mainly to research carried out with children under the age of ten. Her remarks about mathematics are in almost all cases remarks about arithmetic or even pre-arithmetic. The extent to which her finding could be extrapolated to older learners and to other areas of mathematics is just one of a number of questions begged by her observations. The following are excerpts from her paper[1] .

. . . young children can only learn arithmetic if they can attach meaning to numerals and equations. Arithmetic, like reading and writing, has to do with the extraction and construction of meanings.

. . . This first idea of letters carrying a meaning has nothing to do with letters as representing sounds. Curiously enough, children think at first that the meaning of the squiggles has to do with some quantitative property of what is symbolized: when there are three dogs in a picture, they think that there should be three squiggles in the writing underneath it, each squiggle representing one dog; they also may think that **elephant** should be written with more letters than **butterfly,** and that the name of a child three years old should have three letters. Thus, in the search for the meaning of written texts, the first link children think of is in the domain of number and measurement, **not** in that of sound.

. . . Consequently, there are two still very wide-spread tacit assumptions that I think should be questioned, and that can be formulated as follows:

1) Before the age at which children generally are taught arithmetic at school, they have no capacity for dealing with numerical operations.

2) Being able to use the notational system for arithmetic operations is no different from understanding them.

It may surprise you that I want to question the first assumption as it is often thought to be supported by, or even to stem from, Piaget's number conservation task. The argument goes: children who do not know that the cardinality of a collection of discrete objects in independent of spatial arrangement cannot do arithmetic. Such children have no "number concept" -- Piaget dixit. Though numerosity-conservation is an important acquisition, it is certainly not the conceptual watershed it is often made out to be. I will first quote some evidence showing that certain important ingredients of number concepts are present before conservation, and, second, that long after conservation is acquired, some apparently equally important ingredients are still not mastered. These were the experimental results that led Piaget and his colleagues to speak about the "slow arithmetization" of number - a process that takes many years.

. . . A study of Sastre and Moreno on children's spontaneous ways of symbolizing numberical quantities brought to light several different levels of representation. The children were interviewed on various tasks, in pairs. In one of these tasks the experimenter placed a number of pieces of candy in front of the child on the table, and asked him to use paper and pencil to show how much candy there was "so that your friend will be able to use your paper to put out just as many". The following levels were found:

1) any drawing, having nothing to do with the number of objects; a house, a tree, a flower, a person, etc.

2) the drawing of one object which has number characteristics similar to that of the number of elements presented: for example, a hand for 5 elements presented, an octopus for 8, a flower with 6 petals for 6, etc.

3) the same number of drawings as elements: 5 people or 5 trees, for example, for 5 elements;

4) a drawing of the objects themselves, in their correct number: five pieces of candy;

5) numerals 1, 2, 3, 4, 5, 6, etc. as counting numbers, i.e. **1, 2, 3, 4, 5, 6** written down for 6 elements;

6) a single numeral, 5, 6, etc. written to represent the number of elements as a cardinal number.

. . . To sum up, just as for the teaching of reading and writing, the largely tacit and

widespread assumptions on which the teaching of written arithmetic is based appear to be incorrect. Available research suggest the following conclusions:

1) Children's acquisition of number concepts and numerical operations (as apart from the notational system) is a lengthy process, stretching over many years, of which many details are still unknown.

2) Just as for written language, young children have theories about written numerals and equations. These theories may not fit the symbolical nature of the systems elaborated by humanity over many centuries. In both domains children seem to re-invent or to reconstruct the systems their societies have adopted.

If, as I have argued, children actively reconstruct the notational systems that symbolize spoken language and numerical operations, we should compare the two notational systems as to their differences and similarities, with a view to helping children avoid unnecessary confusions and so as to profit from progress in one system when building up the other.

Concluding Remarks

If we accept that very young children, already at the age of three or so, have ideas about the letters and numbers they cannot help but see around them, and that they try to make sense of their observations, we have to ask ourselves how we can help them in this difficult task. It seems to me that in the first place we ourselves should become more aware of the confusing nature of the various kinds of information the child encounters in his everyday life. Secondly, much more research should be devoted to the theories the child elaborates about the notational systems. Thirdly, especially for arithmetic, it should be realized that the conventional notational system of written numerals and equations has to be conceptualized by the child as well as the nature of the quantifying operations it symbolizes. Unless the child can bring together his ideas on numerical operations and his ideas on the notation, he cannot make sense of the latter, and will be reduced to either learning to perform tricks correctly, or to becoming despondent and hostile.

1. Professor Sinclair's paper and the invited reaction paper, "Threeks, Rainbrellas and Stunks" by W. Higginson will appear in the Conference Proceedings which are expected in 1981. Dr. Sinclair's paper may, in the meantime, appear in the new international journal, **For the Learning of Mathematics** which is edited by Professor David Wheeler of Montreal's Concrodia University.

INFINITE GEOMETRIC SERIES MADE CHILDISH

J. Rubinstein, Northview Heights S.S., North York

I truly enjoyed several of the problem-solving articles in the March 1980 issue of the Gazette. In particular, Ann Lukasevich's emphasis on oral problems and Dave Alexander's discussion of mental imagery reminded me of a rewarding experience I had a few years ago, and I describe it below.

It is commonly accepted that each new concept cannot be understood by children until they reach an appropriate stage of development; a considerable body of research (e.g. by Piaget) lends support to this belief. However, it is not clear to me whether these learning restrictions reside in the concepts themselves, or rather in the symbolisms normally associated with them. In the case that follows, rather abstract concepts were easily handled by a child, much earlier than he was "supposed" to.

The child, my son Eddie, 6 years old at the time, has two younger brothers and is of average intelligence. He was familiar with addition, but only barely with multiplication. He had also been exposed to the concepts "one half", "one quarter", "one eighth", etc.

At dinner time, I, who have never taught at the Elementary Level, was trying to introduce the concept of "one third", and the following dialogue developed:

Me: "If you split an apple among 2 persons, how much will each get?"

Eddie: "One half"

Me: "And if you split it among 4 persons?"

Eddie: "One quarter"

Me: "And among 3 persons?"

Eddie: "One quarter"

Me: "Isn't there one quarter left over?"

Eddie: "Yes, but you can split **it** into 4 parts and give one to each person."

Me: "O.K., but doesn't that again leave off one quarter of one quarter?"

Eddie: "Yes, but you can split it again into 4 parts."

Me: "For how long would you have to go o n ? ' '

Eddie: "Forever"

Me: "Would you then have used up the whole apple?"

Eddie: "Yes."

Me: "And would each person then have the same amount of apple?"

Eddie: "Yes."

The conversation then drifted into the atomic nature of matter. I have taken a few liberties with the dialogue, but in essence it was as printed here. I do not remember whether I told him that each person ended up with one third of the apple, but I certainly found a novel way of showing that:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{3} \quad . \tag{1}$$

This result would normally be studied in a grade 13 course after using considerable time and effort for developing the concepts of sequence, geometric sequence, series, infinite series, limit, etc. Eddie, ignorant even of the sybol "½" for the concept "one half", was, however, able to develop and comprehend several highly sophisticated concepts, with a modicum of symbolism-free guidance.

My Calculus class was really charmed when I described the above dialogue.

Although I did not pursue it at the time, it is clear that the above method can be readily generalized for any infinite convergent geometric series with a rational positive common ratio. For instance, dividing an object into fifths for distribution among 4 people gives

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{1}{4} , \tag{2}$$

or dividing it into nths for distribution among (n - 1) people gives

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots = \frac{1}{n-1} \quad ; \tag{3}$$

similarly, dividing the object into fifths for distribution among 3 people leaves 2 pieces unused, which we divide again into fifths, and so on, obtaining

$$\frac{1}{5} + \frac{2}{5}\times\frac{1}{5} + \frac{2}{25}\times 2 \times \frac{1}{5} + \cdots = \frac{1}{3} \text{ , i.e.} \tag{4}$$

$$\frac{1}{5} + \frac{1}{5}\times\left(\frac{2}{5}\right) + \frac{1}{5}\times\left(\frac{2}{5}\right)^2 + \cdots = \frac{1}{3} \quad ; \tag{5}$$

or, dividing it into nths for distribution among n - k (k + n) people gives

$$\frac{1}{n} + \frac{k}{n}\times\frac{1}{n} + \frac{k}{n^2}\times k \times \frac{1}{n} + \cdots = \frac{1}{n-k} \text{ , i.e.} \tag{6}$$

$$\frac{1}{n} + \frac{1}{n}\left(\frac{k}{n}\right) + \frac{1}{n}\left(\frac{k}{n}\right)^2 + \cdots = \frac{1}{n-k} \quad . \tag{7}$$

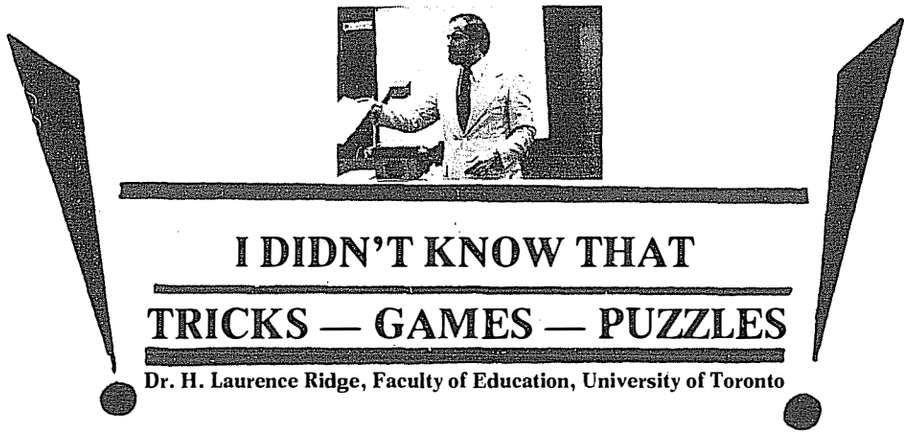
If the terms of (7) are multipled by an arbitrary number c, setting $c/n = a$ and $k/n = r$ gives the familiar result

$$a + ar + ar^2 + \cdots = \frac{a}{1-r} \quad . \tag{8}$$

Notice that (4) and (6) are written in a manner that reflects the reasoning used, while (5) and (7) clearly exhibit the geometric pattern.

The procedure developed in the above dialogue and formalae (1)-(7) is basically an extension of Zeno's Paradox, with the "paradox" aspect ignored rather than emphasized. The method certainly lacks mathematical rigour, and the result (8) has only been established for $r \in \mathbb{Q}$, $0 \le r < 1$, instead of the more general result, i.e. for $r \in \mathbb{R}, -1 < r < 1$, or $r \in \mathbb{C}$, $|r| < 1$. It could, however, serve as an easy introduction to the topic even in early high school grades, and, in grade 13, as motivation for developing the tools needed for a more comprehensive treatment. In purely numerical form, (3) could be used in the elementary grades - e.g. as oral problem solving, to motivate work with fractions, to illustrate succesive approximations, to apply a calculator, etc.

Thus, two long held beliefs of mine have been reinforced by the above experience:
that teaching a subject produces, in the teacher, a deeper understanding of the subject;
that almost any topic, with adequate symbolism, can be made comprehensible to almost anybody.



I DIDN'T KNOW THAT

TRICKS — GAMES — PUZZLES

Dr. H. Laurence Ridge, Faculty of Education, University of Toronto

This issue's column maintains the problem solving motif from September. The problem is to find a winning strategy for the board game "SATURATION". Here are the rules.

Two players alternate with an "X" or an "O" in any unmarked square (Fig. 1) so long as two squares with a common side don't contain the same mark.

The object of the game is to force your opponent into a position so that he/she has no legal move.

The winner is the last player to move.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Fig. 1

I was introduced to the game by Steve Krulik of Temple University at the 1977 NCTM Annual Meeting in Cincinnati. It also appeared in the November 1976 issue of School Science and Mathematics in the article "Mathematics Class + Strategy Games = Problem Solving" by Krulik and Wilderman. Here are some sample moves.

A_K and B_K represent the first and second players' Kth move, respectively. A_1 on space 6 means that A cannot play on spaces 2, 5, 7, 10 on any subsequent move (Fig. 2).

1	2	3	4
5	6 A_1	7	8
9	10	11	12
13	14	15	16

Fig. 2

All of these spaces are still open to B however. With B_1 in space 7, B cannot play on spaces 3, 8, 11 on any subsequent move (Fig. 3).

If play continues as shown in Fig. 4, A, is the last move since B cannot play on spaces 10, 13, or 15. Hence A is the winner.

1	2	3	4
5	6 A_1	7 B_1	8
9	10	11	12
13	14	15	16

Fig. 3

Does A have the advantage or did B just play a bad game? Which player has an advantage if any? What makes for a good or bad move? What rule or pattern of moves will tell a player exactly where to play on each move? (Such a rule is a "strategy". If A win is guaranteed by following the rule, it becomes a "winning strategy".) Before reading on, try the game a few times to get some feeling for it.

A_4^1	B_6^2	A_5^3	B_4^4
B_5^5	A_1^6	B_1^7	A_6^8
A_3^9	10	A_7^{11}	B_2^{12}
13	B_3^{14}	15	A_2^{16}

Fig. 4

Now you're ready to tackle the problem of investigating winning strategies. When you have given it your best shot, read on to see how mathematics teachers in the MScT Problem Solving Course (1979-80) responded. What follows is a synopsis of their solutions accompanied by editorial comment.

THE "SATURATION" GAME

I. STRATEGIES FOR A 4 BY 4 GAME BOARD - 2nd player

1. Line Symmetry

Think of a vertical line separating the board into 2 congruent parts (Fig. 1).

Each of B's moves is the reflection image of A's preceding move in the vertical line.

(Alternatively, a horizontal reflecting line can be used - Fig. 2.)

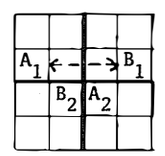


Fig. 1

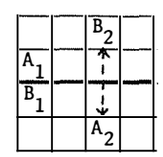


Fig. 2

Rationale: By the properties of reflection, for each space filled by A there is a unique space for B.

The only time that B could not move arises when A has not been able to make a move, in which case B wins.

A_1	B_5	A_5	B_1
B_6	A_2	B_2	A_6
A_3	B_7	A_7	B_3
B_8	A_4	B_4	A_8

Fig. 3

Note: A complete game (full board) as shown in Fig. 3 is possible. In this case A cannot play by virtue of the full board as opposed to being faced by open but illegal spaces. In either event A cannot play and hence loses.

---

2. Point Symmetry

Think of a point at the centre of the board.

Each of B's moves is the reflection image of A's preceding move in the centre point. (This is a combination of the horizontal and vertical symmetry in strategy 1). See Fig. 4.

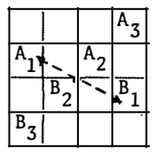


Fig. 4

Rationale: Same as that for line symmetry.

Note: A full board is impossible, as illustrated in Fig. 5 The maximum number of moves is 6 for each player (or 12 in all). Each player can fill 2 spaces in every row but one (located centrally). A fills 2 spaces in rows 1, 2, 4 while B fills 2 spaces in rows 1, 3, 4.

A_1	B_5	A_3	B_6
	A_2		A_4
B_4		B_2	
A_6	B_3	A_5	B_1

Fig. 5

Variations

(i) Number the spaces as shown below in Fig. 6.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Fig. 6

B moves so that the sum of the numbers in the spaces occupied by B and by A on its preceding move is 17.

(ii) Think of coordinate system placed on the grid as in Fig. 7.

(-2,2)	(-1,2)	(1,2)	(2,2)
(-2,1)	(-1,1)	(1,1)	(2,1)
(-2,-1)	(-1,-1)	(-1,-1)	(2,-1)
(-2,-2)	(-1,-2)	(1,-2)	(2,-2)

Fig. 7

B moves to space (-a,-b) when A has played space (a,b), where $a,b \in \{-2,-1,1,2\}$.

2. Quadrant Strategy

Think of the board broken into four quadrants as shown in Fig. 8. By considering that A starts in a particular quadrant there is no loss of generality since the board can be rotated. There are three starting positions to consider: positions 1, 2, 4 (Fig. 9) (spaces 2 and 3 are equivalent).

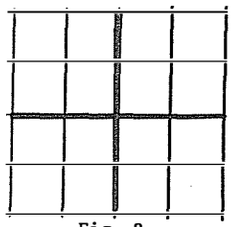


Fig. 8

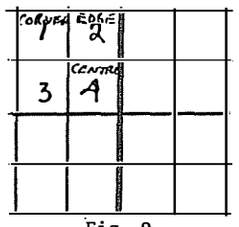


Fig. 9

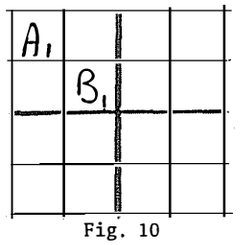


Fig. 10

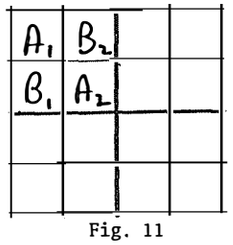


Fig. 11

Fig. 10 shows a "corner start". B can 'win' the quadrant via pattern 1 (Fig. 10) or via pattern 2 (Fig. 11).

Clearly the moves in the other quadrants have to be thought out carefully to overcome "problem patterns" related to A's opening.

Fig. 12 shows the "problem pattern" for the opening in Fig. 10. B should not occupy either shaded square in the diagonally opposite quadrant.

Fig. 13 shows that A can win if B does occupy one of these spaces. (In this case on A's 5th move.)

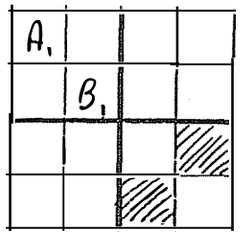


Fig. 12

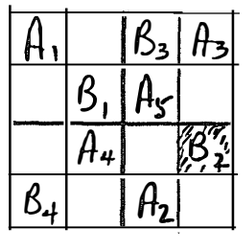


Fig. 13

The "problem patterns" for B when A makes a "centre start" are shown in Fig. 14.

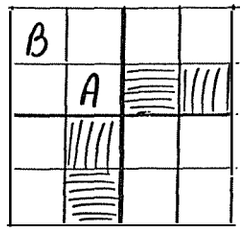


Fig. 14

B cannot play in both squares shaded ||||| or in both squares shaded ≡.

The "problem patterns" for B when A makes an "edge start" are shown in Fig. 15.

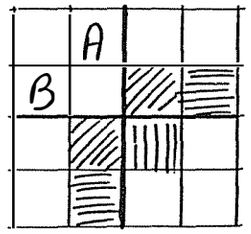


Fig. 15

B cannot play in ||||| nor in both ////// squares nor in both ≡ squares.

In general, B plays in the same quadrant as A has just played in. B uses pattern 1: A\|/B , or pattern 2: A\|B/ or A/\|B, depending on the particular "problem patterns".

Author's Comments

While the quadrant strategy sounds complex it is relatively easy to use as one knows how to start and where the potential "problem situation" can arise. Since the board is not too large, one can play to avoid the "problem situations" without having to memorize the moves".

---

4. Configuration Strategy (Brian Lapcevic)

As noted in the quadrant strategy, A has one of three openings: corner 

, edge 

, or centre 

. For each of these openings B can move one of the configurations in Figs. 16 and 17.

From either of these configurations (or their reflection or rotation), B wins.

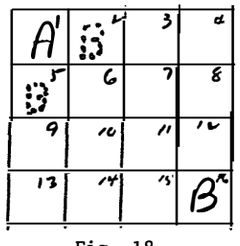


Fig. 18

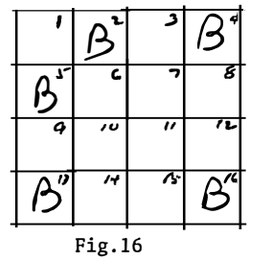


Fig.16

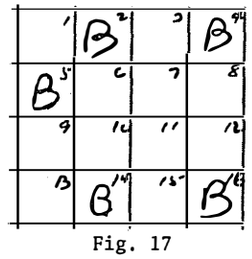


Fig. 17

As an illustration, say A makes a corner start in square 1 (Fig. 18), B takes 16 (2 and 5 being assured). If A takes 13, B takes 4 (14 being assured). If A doesn't take 13, B does, then takes either 4 or 8 (since A cannot take both).

That B must win from either configuration must be established by cases.

That one or the other configuration can arise from an edge or corner start can be demonstrated.

II. CONSIDERATION OF OTHER PLAYING BOARDS (Barney Matisko)

1. n by n board for any even n. Strategies 1 and 2 hold. Strategies 3 and 4 would have to be developed for particular n.

Note: In strategy 2 (point symmetry), the maximum number of moves is n(n - 1). This was illustrated in Fig. 5 for the 4 by 4 case. Fig. 19 shows the 6 by 6 case.

A	B	A	B	A	B
B	A	B	A	B	A
A		A		A	
	B		B		B
B	A	B	A	B	A
A	B	A	B	A	B

Fig. 19

In general, each of the n rows but one central row can contain $\frac{n}{2}$ A's. Also each of the n rows but the other central row contain $\frac{n}{2}$ B's.

Hence the total number of moves is
$(n - 1)\frac{n}{2} + (n - 1)\frac{n}{2} = n(n - 1)$.

---

2. m by n board for any even m, n. Strategies 1 and 2 hold.

---

3. n by n, n odd. For n = 3 for each of the three openings: centre (Fig. 20), corner (Fig. 21), edge (Fig. 22) B has a winning strategy.

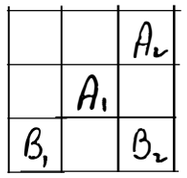


Fig. 20

Don't let A have opposite corners.

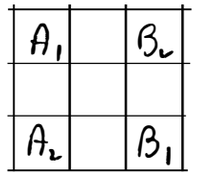


Fig. 21

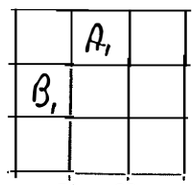
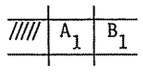


Fig. 22

Move as per reflection in a diagonal.

For n = 5: (i) A taking centre then using centre symmetry strategy fails for A when B moves adjacent to centre.



=> A_2 can't be in shaded space.

(ii) A's best bet is to keep to the odd squares where squares are numbered 1 2 3 4 5 6 7 8 9 10.

(iii) B still appears to have advantage.

Author Question: For greater odd n, say 7 × 7, would the prolonged game offset any equalization of odds between A and B?

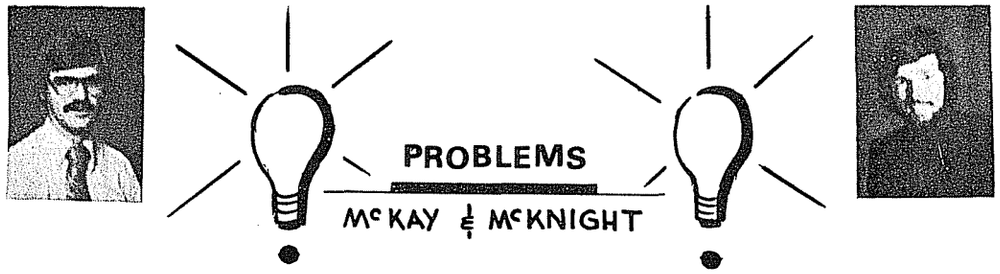
4. m by n, m, n of opposite parity. - Still open for investigation!!

Almost as interesting as the winning strategies are the attempts that don't always win. Then the fun starts in trying to make up sample games as counterexamples. Try it with your classes and mathematics clubs! Happy playing!

Recreationally yours,

H.L.R.

If a herring and a half cost a cent and a half, what would a dozen and a half cost?



I Up to Grade 10

19.2.1 Each of a group of 50 girls is blonde or brunette and is blue or brown eyed. If 14 are blue-eyed blondes, 31 are brunettes and 18 are brown eyed, find the number of brown-eyed brunettes.

19.2.2 A function f, defined on the real numbers, satisfies

$$f(x + y) = f(x) + f(y)$$

If $f(1) = 1$, find the value of $f(\frac{1}{2})$.

19.2.3 The area of an isosceles right triangle is 8. Finds its perimeter.

19.2.4 Al, Bill and Carl each bought manure in different sized bags. Al's bag was 20% less expensive than Bill's bag, but contained 20% more manure than Carl's bag. Carl's bag contained 50% less manure than Bill's bag, but cost 33 1/3% less than Al's bag. Which person bought his manure at the cheapest cost per unit weight?

II Up to Grade 13

19.2.5 Find all real solutions of $\cos x + \cos 5x + \cos 7x = 3$

19.2.6 In $\triangle ABC$, the bisector of angle B intersects AC at D and intersects the circumcircle of $\triangle ABC$ at E. If AB = 35, BC = 22, BD = 22, find DE.

19.2.7 Show that the product of the digits of any number containing more than 2 digits is less than the number itself.

19.2.8 By the time the radius of a certain pearl has increased 1 mm, the area will have increased as much (in mm^2) as the volume (in mm^3).
If the pearl is an exact sphere, what is its radius now?

III Are you ready for a challenge ?

19.2.9 Find rational numbers x, y such that

$$\sqrt{(2\sqrt{3} - \sqrt{3})} = x^{\frac{1}{4}} - y^{\frac{1}{4}}$$

ONE PROBLEM FORM
CRUX MATHEMATICORUM

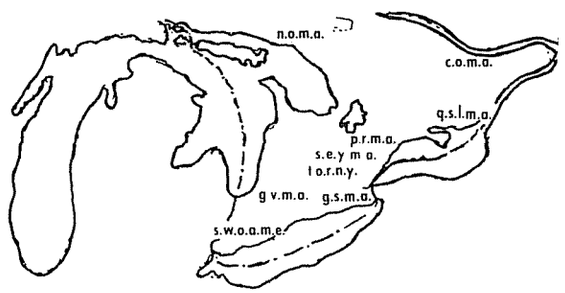
102 (1976: 5) *Proposé par Leo Sauve, Collège Algonquin.*

Si, dans un ΔABC, on a $a = 4$, $b = 5$, et $c = 6$, montrer que $c = 2A$.

If in ΔABC $a = 4$, $b = 5$, $c = 6$, show that $c = 2A$.

The solution may be found in issue #4 of volume 2 (April, 1976). Write to CRUX MATHEMATICORUM, 200 Lees Avenue, Ottawa K1S 0C5 for back issues and bound volumes. Current costs are $1 and $10 respectively.

CHAPTER ITEMS



A successful dinner meeting was held Thursday, November 13 at Algonquin College, Ottawa. Don Craig shared his experiences in the intermediate classroom by posing a problem which had many fascinating discoveries in it for students. Peter Ulrich illustrated how microcomputers could be used in the primary and junior divisions and Fred Marshall brought a report on the Algonquin Pilot Test run for the Regional Examining Board.

NOMA

The Fall Conference was held at Laurentian University the weekend of Oct. 3 in the midst of a whirling snowstorm. Problem Solving was the theme with Frank Ebos as the guest speaker. There was ample opportunity to examine current materials from the publishers as well as testing out the computers. Workshops and speakers for each division were at work Saturday morning.

GOLDEN SECTION ★

The Golden Section held as evening workshops at McMaster University on Friday, November 14. A highlight of the program was the panel discussion on Math '80 with panelists Dave Didur (Hamilton Board of Education), Tom Davidson (McMaster University) and Jack Worster (Stelco).

CURRENT ACTIVITIES

- This year's fall general meeting is scheduled for Thursday, November 6, 1980. Our topic is "Consumer Education in the Mathematics Class". A group of practising professionals will speak to the questions "What everyone should know about _____?" with individual sessions on Life Insurance, Mortgages, Home and Car Insurance, Savings and Loans and Stocks and Bonds.

- Our latest publication, "Recreational Math Problems for High School Students, Part II" will be available later this year.

- Work has been initiated on three possible new publications.
  - sample exams for Math 151 and Math 251
  - a report on a junior math league in Waterloo County
  - computer programmes for use in the math class



Advice from retired Math teacher Dorothy Bevan to Dave Davidson.



Workshopping with Neil Williamson.

G.V.M.A. PUBLICATIONS

Over the past seven years the G.V.M.A. has produced a number of publications. Below is a list of the publications available and their costs:

		G.V.M.A. Member Price
Handbook of Trigonometry (1975)	$2.00	$1.50
A Preparedness Test of Grade Nine Students (1973)	2.00	1.50
Sample Grade 13 Examinations	1.00	.50
Grade 11 (Level 5) Sample Examinations	1.00	.50
Reprints of previous Gauss Contest papers	.15 each	.15
Cumulative Testing	1.50	1.00
Recreational Math Problems for High School Students	2.00	1.50

Solutions to the Handbook of Trigonometry	2.50	2.25
A Handbook of Geometry	3.00	2.50
"Lies, Damn Lies and Statistics" (1979)	5.00	4.00
The Game "SEARCH" by G.V.M.A. - 1 game (2 players) This game (a mathematics variation of battleship and mastermind) is excellent for developing analytical thinking required for certain kinds of problem solving. It also has the potential as an aid for introducing some mathematical topics such as symmetry, transformations, etc.	3.00	3.00

To receive these publications send your order and payment (cheques and money orders payable to the **"Grand Valley Mathematics Association")** to:

Mr. R.G. Scoins
Faculty of Mathematics
University of Waterloo

The co-chairman of last year's conference was very pleased to hear from the treasurer that approximately $5000 was made and are happy to be able presently to make this available to OAME. Any helpful suggestions from the groups who have suffered conference fatigue, but have survived or been revived would be welcome.

The fall conference was held on October 9 at York University. "Microcomputer in our society" was the title for the session. A winter seminar is planned for Jan. 14 on the topic, "Teaching the gifted has advantages for teaching children at all ages" with Henry Courthey and Brendan Kelly as guest speakers.

Announcing: THE JOHN G. McCARTNEY SCHOLARSHIP

1. The amount of this scholarship shall be $400.00.

2. This scholarship shall be awarded to the applicant from Ontario who demonstrates the highest proficiency in three Grade XIII mathematics courses and who has demonstrated outstanding mathematical insight.

*3. This scholarship shall be administered by the Secretary Treasurer of OAME/AOEM.

4. (a) Applications must be made by the school, on behalf of the student currently enrolled in that school who is deemed most outstanding in mathematics, and who is going on to post-secondary study in a program that is primarily mathematics related.

(b) Only one application will be accepted from each Ontario school.

(c) All applications must be signed by the principal of the school or his designate.

5. The deadline for applications shall be June 1, 1980.

6. (a) All applications will be reviewed by the John G. McCartney Scholarship Committee.

(b) The winner shall be contacted by July 1, 1980 and the winner's name shall be published in the next issue of the Ontario Mathematics Gazette.

This scholarship is given under the auspices of the OAME and funded by Science Research Associates (Canada) Limited.

* Application forms available from OAME/AOEM, 247 Bright St., Sarnia, Ont. N7T 4E9.

[Ed. Note: The history of the John G. McCartney (1939-1977) Scholarship can be found on page 50 of the September, 1979 OMG.]

NOMINATION NOTICE TO MEMBERS
ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

At the Annual Meeting of OAME in Sarnia in May, 1981 we shall elect the following:

1 President Elect
3 Executive Members (Directors) and approximately
10 Councillors

If you would like to be considered by the nominating committee to fill one of these positions, or wish to nominate someone else for consideration, please complete and mail the form below.

Note the following:
1. Information about the nominee should accompany the nomination.
2. A summary of the qualifications of the nominee will be published with the March issue of the Ontario Mathematics Gazette.

Deadline for return: 1981 02 01

To: Mr. Alex Norrie

I nominate_____(name)

_____(professional address)

_____

For the position of: President Elect
Director
Councillor

Name of Nominator: _____

Professional Address: _____

_____

NOTE: This form does not have to be removed from the journal, as long as the above information is sent to Alex by the deadline. Thank you!

SOLUTIONS TO PROBLEMS SET 19.2

Up to 10

19.2.1

	Blue	Brown
Blonde	14	18
Brunette	31	x
	T	

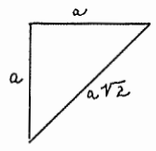
total

$14 + 31 + 18 - x = 50$

$x = 13$

19.2.2 Since $f(\frac{1}{2} + \frac{1}{2}) = f(1) = 1 = f(\frac{1}{2}) + f(\frac{1}{2}) = 2f(\frac{1}{2}) \therefore f(\frac{1}{2}) = \frac{1}{2}$

19.2.3



$1/2\ a^2 = 8$

$a^2 = 16$

$a = 4$

$\therefore P = 8 + 4\sqrt{2}$

$P = 2a + a\sqrt{2}$

19.2.4 A = amount of manure

C = unit cost of manure (total)

	A	C	Unit Cost
AL	6/5 x	4/5 y	2/3 y/x
BILL	2 x	y	1/2 y/x
CARL	x	8/15y	8/15y/x

$\therefore$ Bill bought his manure at the cheapest cost per unit weight.

Up to 13

19.2.5 Since the maximum value of each term is 1, each term must have this value -

$\cos x = 1 \Rightarrow x = 2k\pi$

19.2.6 In the diagram since $\angle DCB = \angle AEB$

$\angle ABE = \angle DBC$

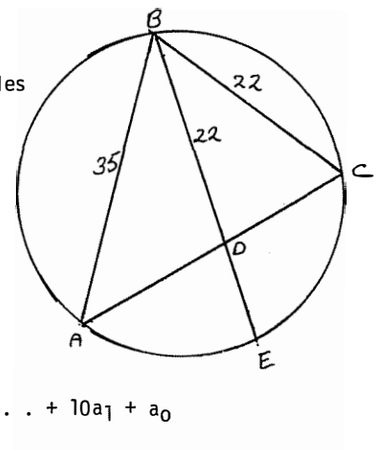
$\therefore \triangle ABE \sim \triangle DBC$

Since $\triangle DBC$ is isosceles, $\triangle ABE$ is isosceles

$\therefore AB = 35$

$\therefore EB = 35 \Rightarrow 22 + DE = 35$

$DE = 13$



19.2.7 Let the number be $a_n 10^n + a_{n-1} 10^{n-1} + \ldots + 10a_1 + a_0$

where $0 \leq a_i \leq 9$ and $n \geq 1$

the product $= a_n \cdot a_{n-1} \cdot a_{n-2} \ldots a_1 \cdot a_0$

and $a_n \cdot a_{n-1} \cdot a_{n-2} \cdots a_1 \cdot a_0 < a_n \cdot \underbrace{10 \cdot 10 \cdot 10 \cdots 10}_{n \text{ times}}$

or $< a_n \cdot 10^n$

which is the first term in the number

$\therefore$ the product is less than the number

19.2.8 We have $4\pi (r-1)^2 - 4\pi r^2 = \frac{4\pi}{3}(r-1)^3 - \frac{4\pi}{3} r^3$

from which $3r^2 - 3r - 2 = 0$

$r = 1/2 + 1/6\sqrt{33} \simeq 1.457$ mm

CHALLENGE

19.2.9
$$2\sqrt{3} - 3 = \sqrt{3}(2 - \sqrt{3})$$
$$= \sqrt{\frac{3}{2}}(4 - 2\sqrt{3})$$
$$= \sqrt{\frac{3}{2}}(\sqrt{3} - 1)^2$$
$$= \sqrt{\frac{3}{4}}(\sqrt{3} - 1)^2$$

Hence $\sqrt{(2\sqrt{3} - 3}} = \left(\frac{3}{4}\right)^{\frac{1}{4}} (\sqrt{3} - 1)$

$$= \left(\frac{27}{4}\right)^{\frac{1}{4}} - \left(\frac{3}{4}\right)^{\frac{1}{4}}$$

$\therefore$ $x = \frac{27}{4}$, $y = \frac{3}{4}$

MATHEMATICAL APPLICATIONS

by Peter Weygong

Ed. Note: Mathematical applications will hopefully become a regular feature of the OMG. Authors are invited to send in any examples they have.

Mathematical applications can be broken down into three steps: -

1) The Problem
2) The Mathematical Model
3) The Application

The Problem

The problem may arise in industry, commerce, sociology, or in a game setting such as the Knights Tour in chess. Typically the problems are poorly expressed and the mathematician must do some verbal probing in oder to make things clear.

The Mathematical Model

The second task is to regenerate the problem in a mathematical form. It may be necessary to take measurements, to collect data, to look for spatial relationships and so on. This information must all be squeezed together and represented by a functional relationship, a network, a statistical function, equation of best fit or other suitable model.

The first model is usually an over simplification of the real problem, and is based upon

simplifying assumptions such as "frictionless pulleys", or "massless strings". The basic model may be manipulated at will to give more and more accurate representations of the problem.

The Application

This third step is, to most people, the real application. This is where we use the model to make predictions about the real world. For example, mortgage payments are calculated using the appropriate formula, or model. We seldom sit down in a room full of dollar bills, and work out the payments with real money. Similarly an architect "builds" a house firstly in his mind, then on paper as a series of blueprints, and finally in bricks and mortar. The house may be altered quite easily at the mathematical model stage (blueprint) - it is quite another matter once the house has been built.

Example 1

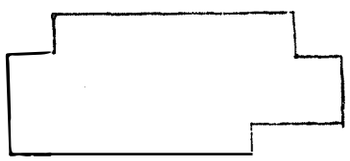
Problem: - To find the perimeter of a room when the edges of the room are inaccessible.

Some of my own real world mathematical applications have their origin in some genuine personal problem. In this instance I had moved to Kenora from Sudbury and wanted to order materials to complete the recreation room in my new house.

In particular I wanted to find the perimeter of the room. This could not be measured directly because my furniture and other bric-a-brac was piled waist deep all over!

Model

Most rooms are rectilinear with alcoves and so on. The diagram is a geometric model of the plan view of a real room.



The perimeter of the figure is a **model** of the perimeter of the room.

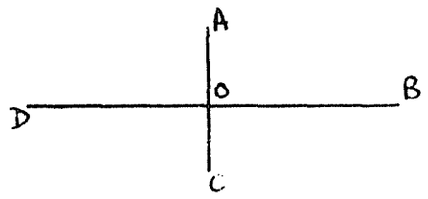
Manipulation

The model is now manipulated. This is the great advantage of mathematical models; they may be manipulated, dissected, reconstituted and so on with great ease. Much more so than the real life object. In practice the manipulation is a rather unstructured task. One just keeps squeezing and twisting the model in the hope that some useful property will emerge -like the genie from the lamp.

The first manipulations is to translate certain portions of the figure parallel to themselves. In this way we form a rectangle which has the same perimeter as the model. We can say that the rectangle is a model of the perimeter of the figure.



Now we do a horizontal and vertical translation as shown.

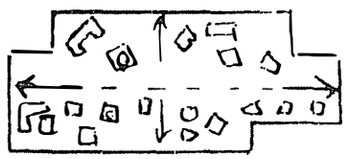


Solution

It is apparent that the path AOBOCODOA is the required perimeter.

Application of result

It appears that we may find the perimeter of a room by measuring the widest part and the longest part. This is much easier than crawling around the skirting board jotting down measurements on the back of a computer card!



In my particular case I was able to make enough space to take the two measurements, even though most of the walls of the room were inacessible. I am pleased to report that by so doing I was able to take advantage of a limited time sale of wall panels (light oak) - and saved a considerable sum by so doing. The wastage was less than half a sheet!

Example 2

Problem: - Five men want to hire a boat to go fishing for Yellow Pickerel on the Lake of the Woods. How big should the boat be?

Model

The U.S.A. coastguards have done some research on the stability of small craft, and have produced the following algebraic model.

P = the maximum number of passengers in good weather.
L = length of boat in metres.
W = width of boat in metres. $P = L \times W$

This is still a rather complicated model to use. One can imagine the five fishermen going around a dock measuring lengths and widths until the product is 5 - and wasting good fishing time!

A little more research produces another empirical relationship, which is true for most runabouts.

$$W = 0.4 \times L$$

Now we manipulate our two algebraic models and produce, by substitution,

$$P = 0.4 \times L^2$$

$$L = \sqrt{\frac{P}{0.4}}$$

$$L = \sqrt{2.5P}$$

We now have a useful rule of thumb.

"In calm weather the length of the boat, in metres, should not be less than the square root of two and a half times the number of passengers."

Application

Using the value of P = 5 we have

$$L = \sqrt{2.5 \times 5}$$
$$= \sqrt{12.5}$$
$$= 3.5 \text{ metres}$$

Example 3

Problem: - What is the maximum weight that can be lifted by a human? (This has an application to industries in which crates and boxes are manhandled)

Model

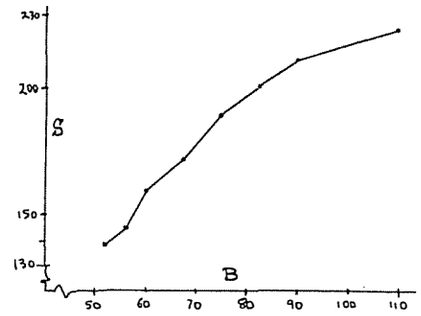
This is a difficult problem. Obviously the maximum weight depends, in some way, on the size and strength of the individual weightlifters. We know that big people are, in general, stronger than small people and our first simplistic model would be.

$S = kB$ S = lifing strength = weight lifted

B = bodyweight of lifter

This model is checked against relevant data, such as the world records for weightlifting.

Class	Bodyweight	Jerk
Flyweight	52 kg	137.7 k
Bantam	56	145.2
Feather	60	160.2
Light	67.5	172.8
Middle	75	190.3
Light-heavy	82.5	202.9
Middle-heavy	90	212.8
Heavy	110	225.4



From the graph we see that the relationship is not linear, and thus our simplistic model will not do. We now build a more sophisticated model using ideas brom biomechanics; as follows.

1) Assume that the lifting strength (S) is proportional to the cross sectional area (A) of the muscle tissue.

$$S = k_1 A$$

This is the same relationship as for the strength of ropes and wires.

2) Assume that the cross sectional area of the muscles is proportional to the square of the linear dimensions of the body (H).

$$A = k_2 H^2$$

3) Assume that the linear dimensions are proportional to the cube root of the body mass.

$$H = k_3 B^{\frac{1}{3}}$$

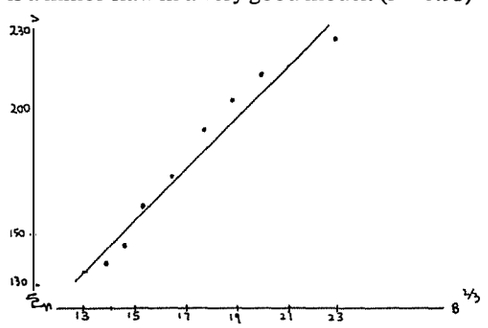
Now these three models are manipulated by repeated substitution.

$$S = k_1 A$$
$$= k_1 k_2 B^2$$
$$= k_1 k_2 k_3 (B^{\frac{1}{3}})^2$$
$$= kB^{\frac{2}{3}}$$

This model lifts the data much better; in fact when we do a least squares fit of the data we obtain the relationship: -

$$S = 10.27 \times B^{⅔} + 0.92$$

We can see at a glance that this model has an amusing little error, namely that a massless weightlifter could lift 0.92 Kg!! However, this is a minor flaw in a very good model. (r = 0.95)



We now have a model which we can use as follows. The maximum lift we can expect from a man having a body mass of 80 kg is 191.6 kg.

Example 4: -

Problem: - How much power is generated by a windmill.

Model: -

This is a somewhat difficult problem. The power generated depends upon the mechanical efficiency of the machine, the area swept out by the vanes, and the windspeed. However, I will only attack the power - wind speed relationship.

Data for the Kidco Downwind Windcharger suggests that:

$$P = 0.3 \times V^3$$

P = power in watts
V = wind velocity in km/h

I don't have the supporting data for this algebraic model, however, it should not be too hard for a grade 13 physics student to establish the $P \propto V^3$ using fairly simple equipment.

It is apparent from the model that relatively minor changes in wind speed produce large changes in power. In fact doubling the wind speed increases the power by a factor of eight:

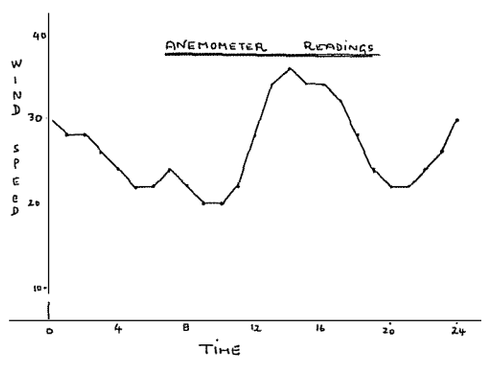
V = 20 Km/h P = 0.3 x 20³ = 240 watts
V = 40 Km/h P = 0.3 x 40³ = 1920 watts

The model predicts that the average wind speed is not a good indication of the suitability of a given location for a wind generator. What we must calculate is

$$V_a = \sqrt[3]{\frac{\sum V^3}{N}}$$

Application:

Calculate the effective average wind speed for the 24 hour period shown on the graph.



In this instance the average wind speed is about 26 km/h; the effective wind speed is a trifle higher at about 27 km/h.

The significance of this investigation is rather practical, and it is that bad weather, such as winter storms, are often accompanied by high winds. Thus the very time that we need additional energy for heating our homes, then that is the very time that the wind generator is most productive.

Example 5

Problem: -

To devise a simple method for mixing different qualities of milk in order to obtain a supply with a specific butter-fat content.

The problem is restated in more precise terms:

X litres of milk with P% butter-fat
+ Y litres of milk with Q% butter-fat
Make (X + Y) litres of milk with Z% butter-fat

The value of Z is determined by the milk marketing board. Obviously if $P > Q$ then $P > Z > Q$

Model: -

Equating the mass of butter-fat before and after mixing gives: -

$$XP + YQ = (X + Y)\,Z$$

Manipultion produces:

$$XP + YQ = XZ + YZ$$
$$\text{and } X\,(P - Z) = Y(Z - Q)$$

which is equivalent to saying that the butter-fat lost by the richer milk is equal to the butter-fat gained by the poorer milk.

Since we are only interested in the **ratio** of the components, we have

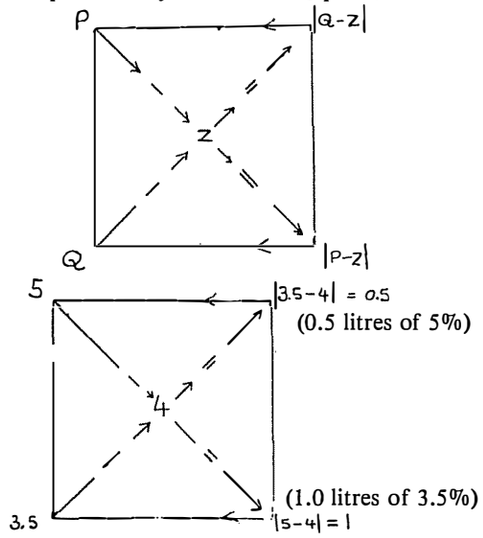
Application

Suppose that we have two supplies of milk with 5%, and 3.5% butter-fat, respectively. How should these supplies be mixed to produce a product with 4% butter fat?

$$\frac{X}{Y} = \frac{(4 - 3.5)}{(5 - 4)} = \frac{0.5}{1}$$

Thus 0.5 litres of 5% milk plus 1 litre of 3.5% milk will produce 1.5 litres of 4% milk.

In the field this relationship is expressed in a simple form by the Pearson Square.



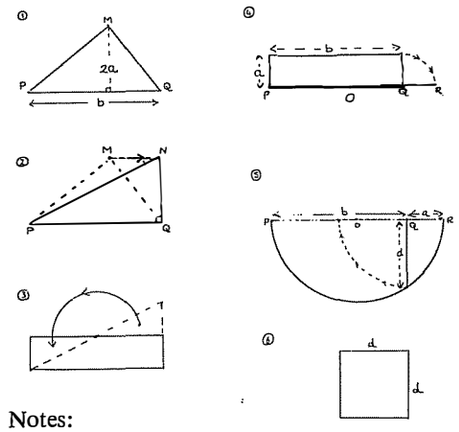
The Pearson Square model is a very useful procedure for workers in the milk industry.

Example 6

Problem: - Find a square which has the same area as a given triangle.

Model: -

This is an exercise in model manipulation using **the** properties of transformation geometry. **The** complete sequence of steps is given below.



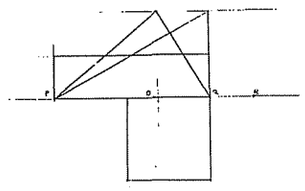
Notes:

Step #2 is a shear in the X direction with PQ on the X axis.

Step #4 - the point O is found by reflecting P onto R.

Step #5 uses the intersecting perpendicular chord theorem, thus $ab = d^2$ and ab is the area of the triangle, so d must be the length of the side of the square.

The whole procedure can be done using only the properties of reflection, and some construction techniques with a Mira.



Application: -

As far as I know there is no application for this model. However it does show that we can construct a line which is equal in length to a radical.

Traditionally we do this for numbers which can be expressed as the **sum** of squares.

e.g. $\sqrt{5} \approx \sqrt{2^2 + 1^2}$

and then we construct a right angle triangle

with sides 2 and 1. The diagonal has the required length.

The present model will work for any number which can be factored.

e.g. $\sqrt{15} = \sqrt{5 \times 3}$

and thus we start with a triangle which has a base length of 5 and a height of 6.

Example 7

Problem: - To find the best strategy in a two person game in which each person has two alternative plays.

This may be clarified by an example.

Player 'R' has two cards; a '2' and a '4'.
Player 'C' has two cards; a '3' and a '5'.

Each player turns over a card simultaneously.
If the sum is '7' then R wins $2.
If the sum is '5' then C wins $1.
If the sum is '9' then C wins $3.

This information is represented in a 'Payoff Matrix'.

R \ C	+	3	5
	2	-$1	$2
	4	$2	-$3

Note that the payoff is given for the **row** player (R).

Model

Consider a game with the following payoff matrix:

R \ C	?	Y1	Y2
	X1	a	b
	X2	c	d

Thus if R plays X1 and C plays Y1 then R wins 'a' units.

The players will not play the same card all the time, in fact there is a probability for each of the two alternatives.

Thus we assume that:
R plays the alternative X1 with a probability p
R X2 (1-p)
C plays the alternative Y1 with a probability q
C Y2 (1-q)

Since the choices are made simultaneously they are obviously independent. Thus the probability of any given pair occuring is given by the product of the separate probabilities.

Thus: X1 and Y1 have a probability of occurrence of p.q.

The results are conveniently represented by probability matrix.

Prob.	Y1	Y2
X1	p.q	p.(1-q)
X2	(1-p).q	(1-p).(1.q)

Over a long period of time player R should win p.q.a. as a result of the pair X1, Y1 and p.(1-q),b from the pair X1, Y2 and so on. The total winnings for R will approach the value:

$W = p.q.a. + p.(1-q).b + (1-p).q.c + (1-p).(1-q).d$

So far we have put the problem in a good mathematical form, however, we are no closer to the real problem, which is to determine the optimum values for 'p' and 'q'.

We wonder what kind of manipulation of the model will lead us down a fruitful path. I discovered, after many dead-ends, that partial differentiation was the way to go.

$$\frac{\partial W}{\partial p} = q.a + b - q.b - q.c - (1-q).d$$

$$= q\{(a+d) - (b+c)\} + (b-d)$$

For stationary values $\frac{\partial W}{\partial p} = 0$

Therefore $q = \frac{(d-b)}{(a+d) - (b+c)}$

Similarly $(1-q) = \frac{(a-c)}{(a+d) - (b+c)}$

$$p = \frac{(d-c)}{(a+d) - (b+c)}$$

$$(1-p) = \frac{(a-b)}{(a+d) - (b+c)}$$

Application:

In our problem example we find that:

$$q = \frac{-5}{-4 - 4} = \frac{5}{8} = 0.625$$

$$(1-q) = \frac{3}{8} = 0.375$$

$$p = \frac{-5}{-4 - 4} = \frac{5}{8} = 0.625$$

$$(1-p) = \frac{3}{8} = 0.375$$

In practice there are some simple rules which allow us to discover the correct strategy directly from the payoff matrix.





ACROSS THE EDITOR'S DESK

by Bob Smith

1. The long awaited **Mathematics Curriculum Guideline for the Intermediate Division** has appeared. Your Mathematics Council had an orientation to the document presented by Education Officer Dino Doltori at their meeting of October 18. I am reserving my comments on it until the March issue. I would be happy to hear your comments and observations. Do write me a letter.

2. At the same meeting Education Officer Jim MacLean spoke about the Ontario Assessment Instrument Pool. The hard copy examples of these instruments should be in your possession now. They were written specifically for selected topics from the Intermediate Mathematics Guideline. Do see how or if O.A.I.P. can be a help in your situation.

3. For the N.C.T.M. convention in Seattle, **An Agenda For Action** was unveiled. It outlines future directions for educational programs and represents an agenda for a decade of action. A must for anyone concerned about the directions mathematics program should be taking. Obtainable from N.C.T.M. for $1.00.

4. Closer to home and more immediate is our involvement in the 60th Annual Meeting to be held in Toronto 14-17, April 1982. Three major themes will be included covering "Motivation", "Problem Solving" and the "Role Technology". If interested in participating, send submission to J.A. Rudnich not later than 13, March 1981.

The program committee is also looking for "Short Subjects". Proposals should be submitted in triplicate, typed and double spaced and should contain 1) title, 2) proposer's name and school, 3) NCTM membership number, 4) preferred mailing address and phone number, 5) interest level, 6) brief overview of idea (not more than 200 words).

Both should be sent to:
Jesse A. Rudnick,
Temple University

Chairman of Local Arrangements
Prof. D. Alexander
University of Toronto

5. The International Association for the Evaluation of Educational Achievement is conducting a second International Study of Mathematics. O.I.S.E. is the Ontario Coordinating Centre and funding is being provided by Ministry of Education. Analysis will concern itself with the interrelationship between the Intended Curriculum (Curriculum Analysis), the Implemented Curriculum (Classroom Process) and Senior levels. A new newsletter from O.I.S.E. will contain further details. Watch for **Ontario Assessment.**

6. That you may keep informed on microcomputer developments, the following organizations may be helpful:

a) Educational Computing Organization of Ontario,
sec. R.W. Sash, George Brown College

This organization publishes a newsletter ECOO quarterly. Membership $10.00.

b) Ontario Society for Microcomputers in Education O.S.I.M.E.
sec. Hal Mueller, Sheridan College,
Trafalgar Road, Oakville L6H 2L1

This organization is particularly concerned with microcomputers.

c) The College of Education of the Ohio State University has begun a project to develop and disseminate exemplary curriculum materials in which high technology is used to teach basic mathematical skills including problem solving, estimation, computer literacy, etc. Funded by the U.S. Department of Education, the project will collect and evaluate existing educational software for minicomputers (e.g. APPLE, TRS-80, PET, etc.) and select high quality units for inclusion. Individuals or groups who have developed programs related to mathematics at the upper elementary school level are invited to submit them for possible inclusion for national dissemination. To have materials considered, send a cassette tape or floppy disk together with a printout, machine documentation and any related information to Dr. Suzanne K. Damarin, TABS Project, Arps Hall 202-A, 1945 N. High Street, Columbus, Ohio 43210. For further information write or call Dr. Damarin at (614) 422-1257.