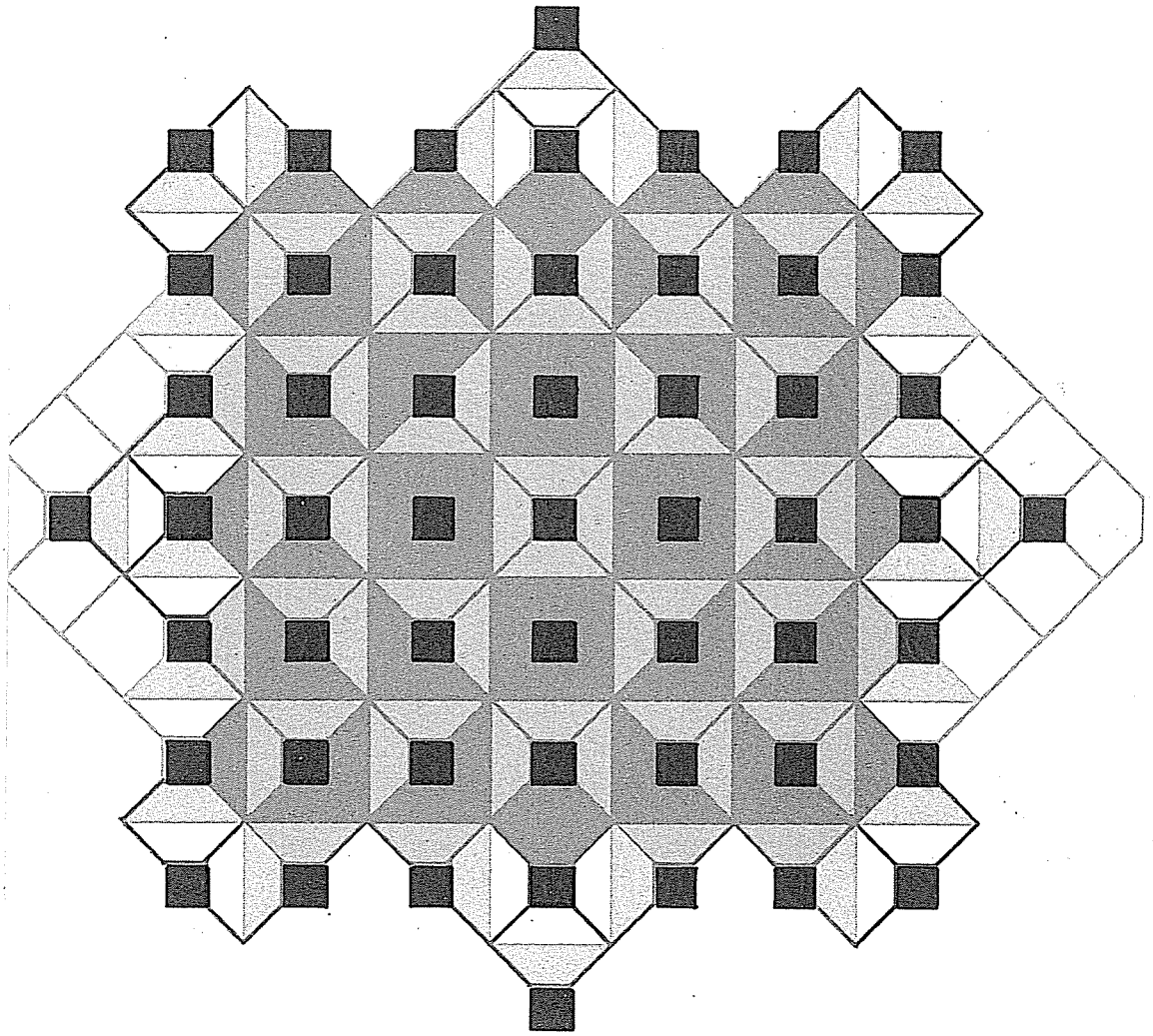
ontario mathematics gazette

VOLUME 19 **NUMBER 3** **MARCH, 1981**



THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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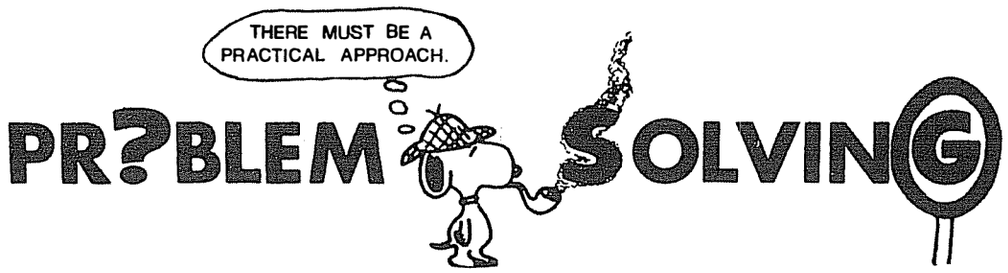
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MARCH 1981 ISSN 0030-3011



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Learning to solve problems is the principal reason for studying Mathematics.

The Ontario Mathematics Gazette is produced by
J.B. POLE PRINTING
Box 69, Forest, Ontario
(519) 873-5112

EDITORIAL

In February, 1962, the first issue of the Ontario Mathematics Gazette was published. Over the years it has attempted to increase liaison among all the teachers of mathematics. Under the successive editors

Dr. Ralph Stanton
Dr. J.E. Hogarth
Dr. D.H. Crawford
Dr. T.M. Davison
Dr. W. Eames and Dr. J.S. Griffith
Professor A. Harris

and their editorial boards, the journal has continually offered a wide range of stimulating and informative articles. As I join the ranks of former editors, I must state that it has been a good experience. Before I write this last editorial I cannot help but remember the many letters written, the exchange of views from writers not only here in Ontario, and other provinces, but also in the United States, England and even Australia. Most of all I recall the support given by the associate editors, contributing editors and the editorial board, who have been my willing and able co-workers. May the new editor have just as good a group.

One year ago, I promised further comments on the theme Problem Solving. Since that time the Ontario Intermediate Mathematics Guideline, the NCTM Agenda For Action and the National Council of Supervisors of Mathematics have all recommended that problem solving be the focus of school mathemtics. I would suggest that it may be the ultimate goal but it has also been the most elusive goal of education.

If our students had relatively high I.Q. scores, high reading comprehension scores, high scores in computation and high spatial aptitude scores, we probably would have little difficulty reaching our problem solving goals. Since few of us can select such a class, what then can we do to help? Marilyn Suydam gives specific suggestions in her article, "Untangling Clues from Research on Problem Solving"[1], in the 1980 NCTM yearbook. It seems to me her concluding statement hints at that all important aspect ATTITUDE. "If we present problems to children that challenge them to think and if we develop the idea that they must search for a solution, we shall have taken a big step toward promoting better problem solving at all levels." What we do in the classroom and how we do it influences whether or not students feel secure enough to venture. If we are curious and enjoy solving puzzles, one might hope that this attitude would rub off on the students we teach.

Most teachers agree that exposure to more problems would be a worthwhile step to improving problem solving. This then becomes a question of time. Are we using it effectively and well? A paper by Dr. Berliner at the San Diego annual meeting brought the following concerns to my attention. Using a school year of approximately 180 days, he found the **allocated time** for mathematics ranged from:

	hours per year	minutes per day
Grade 2	81 - 159	27 - 53
Grade 5	69 - 183	23 - 61

He further spoke of engaged time. **Engaged time** was measured by directly timing the amount of time spent studying mathematics. This percentage of allocated time was then multiplied by a reasonable estimate of the functional school year days; approximately 150 days. I am sure your school year also consists of test days, teacher or pupil absence, field trips, organizational days, etc. The engaged time ranged from:

	hours per year	minutes per day
Grade 2	48 - 103	19 - 41
Grade 5	43 - 123	17 - 49

If we add one more variable ie. the "difficulty" of the material that is attended to by the student, Dr. Berliner obtained what he termed **Academic Learning Time.** He found this figure ranged from:

	hours per week	minutes per day
Grade 2	33 - 58	13 - 23
Grade 5	18 - 53	7 - 21

This was his estimate of the effective learning time used by these teachers. If problem solving is considered one of the "difficult" areas in the mathematics curriculum there may indeed be insufficient time for students to learn problem solving strategies effectively.

I have no answers for this difficulty. Principals and supervisors have adequate reasons to do their part in minimizing disruptions. We teachers should examine time management in our own classroom. Speeding up transition time, avoiding distractions and interruptions, setting time-saving routines, and giving special attention to the difficulty level of material would be good starting points.

Perhaps we should see problem solving as a part of the larger problem facing teachers generally - the teaching of thinking. To exhort students to "Think Hard!" does little to help students who do not perform adequately. Teaching to develop thinking skills is not the same as providing instruction on facts or helping students learn to calculate - to get correct answers. It is insufficient to know that a student is operating at a formal operational level. We can recognize that the child uses logic based upon his experiences. But are we clear on the systematic instruction required to help the child through the years to the stage of formal thought? In broad strokes of the brush, I would suggest these three sequential phases: exploration, concept development, and application. The teacher's role in exploration surely is to allow the students to verbalize their experiences for this is the necessary ingredient for intellectual development. This allows logical structures to be examined. It sets the stage for the more structured lessons and applications where the student can test the learning. In this broader context, may I recommend the work of Popp, Robinson and Robinson[2] of O.I.S.E. who have described the basic thinking skills in the following terms: Basic Operations, Integrated Operations, and Complex Strategies.

Lastly then, a word about applications. They can be used either to establish a need for a new concept or to consolidate these topics in solving problems. One of the best illustrations of applications I have read recently was in the article "Applications for the Classroom - Any Grade"[3] by Peter Weygang found in the 1979 NCTM Yearbook. Do borrow a copy and read the article. It is most worthwhile.

Teaching problem solving is not easy. It takes commitment. It takes time. It takes faith. But there are great rewards.

Bob Smith

1. Suydam, Marilyn N., **Problem Solving in School Mathematics,** N.C.T.M., 1980 Yearbook.
2. Popp, L., Robinson, P., **The Basic Thinking Skills,** St. Catharines, Ontario; Niagara Centre, O.I.S.E., July 1974.
3. Weygan, P., **Applications in School Mathematics,** N.C.T.M., 1979 Yearbook.

LETTERS TO THE EDITOR

Ms. Lorna Morrow, Pres. O.A.M.E.
c/o North York Board of Education

The University Schools
Faculty of Education
University of Toronto

Dear Lorna:

I have delayed writing this letter to you simply because our school only last week held its first mathematics awards assembly. During that short yet important assembly we presented the awards that U.T.S. had received in the contests we participated in last year. I was happy to present the Father Faught shield on your behalf and also to mention the books that we have added to our mathematics library. I also read your letter to the students and made specific reference to Father Faught and his important influence in the mathematics of Ontario. Through you I wish to thank O.A.M.E. for both their interest and concern in what I feel is an important feature of student mathematics in our country today.

Best personal wishes,
G.A. Fleming

Dear Bob:

Faculty of Education
University of British Columbia

In the last issue of the OMG a brief statement was made that "anyone lucky enough to attend the Las Vegas Calculator Conference" should send a report or share the results. As you can see from the enclosed letter, I was lucky enough and wrote to the NCTM with some suggestions. Instead of a report, I am sending a copy of my letter to you for your information and in case anyone should ask. There were several people from Canada among the 80 participants, so perhaps you may have heard from some others.

Dear Prof. Sobel:

I should like to make some comments and suggestions following my attendance at the Leadership Conference on Calculators recently held in Las Vegas.

For many of the 80 participants this conference was enlightening and informative. For others it was somewhat less productive. The degree of usefulness depended largely on the previous experience of participants. Those who have had considerable experience with calculators found that this conference was a review of ideas with dissemination of materials that were not new. Before proceeding further I wish to make it clear that the persons who presented the program did an excellent job especially for persons relatively new to calculators. It is now clear to me that this conference was aimed at such persons, not at experienced people. I had hoped to engage more actively in discussion or work sessions where the knotty problems associated with calculators could be seriously discussed.

As it developed, on the first day, a number of the people at the conference recognized that such problems would not be the focus of this conference. As a result, informal, round table discussions were organized by some participants immediately after the programs on the first two days. These sessions provided some degree of satisfaction, but they were too short to accomplish much toward the solution of the big problems. I believe that such discussion and work sessions, intensively carried out over a longer period (length of this conference) might be of considerable benefit. If we look at the 18 recommendations in the Report of the Conference on Needed Research and Development on Hand-Held Calculators in School Mathematics, (1976) we can find a great deal that needs to be thoroughly examined. As I look at these recommendations I sense that much less has been done than needs to be done. The set of recommendations on curriculum materials appears to be particularly critical, yet from my observations has been barely touched. Materials which have been developed tend to be attached to present day curricula with little that truly exploits the calculator as a tool in a continuing, sequential, logically organized development of some skill area. For example, does there exist a calculator unit of the development of concepts and applications of percent? When will we begin teaching decimals earlier, not in some fashion attached to the present curriculum, but following a well organized sequence of instruction in which the calculator is used a major cog rather than an attachment?

To some the NCTM Agenda for Action for the 1980's appears to be a major step toward calculator respectability in schools. I would disagree. It is not enough for the NCTM to suggest what should happen. I think the NCTM needs to initiate more effective action by **producing** "curriculum materials that integrate and require the use of the calculator and computer in diverse and imaginative ways." (p. 9) Of course, saying is easier than doing. But it is precisely that sort of thing which a "Leadership Conference" might consider with the objective to produce a foundation or structure or coordinated set of ideas on which a task force or writing committee could build, to produce some actual calculation based curricular materials. It seems to me that utilization of calculators has proceeded at an extremely sluggish rate, partly because of inertia in our schools but also because we have not come face to face with the real world and the recognition that we have available a tool of far greater significance than we are willing to admit. I say this even though the research on calculators has not produced dramatic effects. The fact remains that very little research has produced negative effects with calculators, most of the results being either non-significant or in favour of the calculator. Only very recently have we seen a bold (yet almost restrained) suggestion that we should abandon the teaching of fractions with unlike denominators in elementary schools (Wheatley, 1980).

In summary, the Las Vegas type conference served a purpose, beneficial for many but one which might be better deployed locally, in different centers in North America. However, more valuble, might be a true Leadership Conference on calculators in which experienced persons who know what the problems are, could address the problems. The object would be to produce some suggestions and/or materials that would enable teachers who were willing but unsure, to more aggressively begin the utilization of calculators in schools. This might provide a catalyst or impetus for other teachers and school boards to open their eyes to the writing on the wall. If the NCTM produced a well coordinated unit on percent concepts and skills which made use of calculators, we might be amazed at the number of teachers who would request such materials and use them in their classes.

Sincerely,
Walter Szetela, Ed. D.

Editor's Note: Great idea, Walter. What you suggest can be done. O.M.C. did it in the 60's. Time, money and secretarial help were available. Suggestions in the form of notes for the Intermediate Guidelines have appeared but no materials or finished Support Documents such as those which have helped interpret other guidelines.

PROBLEM SOLVING ARTICLES FROM "THE ARITHMETIC TEACHER"

1980

Mar.	p. 18-22	Analogy & Problem Solving -A Tool for Helping Children to Develop a Better Concept of Capacity
Apr.	p. 6-9	But It Can't Be Done
Sept.	p. 8-14	Solving Verbal Problems: Results & Implications from National Assessment
Sept.	p. 34-38	Problem Solving Plus Role Playing Equal Interest Plus Basic Skills in Mathematics
Oct.	p. 16-20	Frame Geometry: An Example of Posing and Solving Problems
Oct.	p. 20-40	Making a Problem of Junior High School Mathematics
Oct.	p. 48-55	Problem Solving for Teachers
Nov.	p. 48-54	A New Look at an Old Weight Problem

1979

Sept.	p. 2	Problem of Solving Story Problems
Jan.	p. 5	Almanac Facts

1978

Dec.	p. 11	More Problems Please
May	p. 39	Using Tables to Solve Some Geometry Problems
May	p. 35	Simulating Problem Solving and Classroom Settings

1977

Dec.	p. 51	Problem Solving: Some Considerations
Oct.	p. 28	Language Experience Approach
Apr.	p. 333	Problem Solver
Apr.	p. 309	Problem Posters
Apr.	p. 286	Making Money Meaningful
Apr.	p. 272	Headlines
Feb.	p. 115	Measuring, Science & History

1976

May	p. 389	Notes on Word Problems

1975

Nov.	p. 591	Decoding Student Names
Nov.	p. 586	Notes on Area and Perimeter
May	p. 414	Strategies For Solving Verbal Problems

1974

Nov.	p.g 633	Creative Ability In Math
May	p. 377	Efficiency Problems
May	p. 409	Student Problem Solving Contest
Apr.	p. 315	Team Experience In Problem Solving (6, 7 and 8)
Mar.	p. 209	Problem-Solving Activities
Jan.	p. 33	Word Problems - A Mirror of Society

1973

Dec.	p. 663	Problem-Solving Activities
Feb.	p. 97	Please Give Us More Story Problems

1972

Feb.	p. 131	Problems Associated With The Reading of Arithmetic

1971

Apr.	p. 223	Improving Math Verbal Problems Solving Ability

1970

Nov.	p. 621	Problem Solving For Elementary Grades
Nov.	p. 575	Procedures For Teaching Reading In Math
Jan.	p. 7	Problem Solving - The Math Lab

1969

Jan.	p. 29	Problem Solving Using The Sphere
Jan.	p. 25	Problem Solving In Special Education
Jan.	p. 21	Problem Solving
Jan.	p. 17	Improving Problem - Solving Skills
Jan.	p. 11	Problem Solving - Cognitive
Jan.	p. 7	Problem Solving - Nonverbal Problems
Nov.	p. 577	Solving Story Problems And Liking it

MATHWAYS 6 PAGE 32

Why do you need to know the length of _____ 1 side of a _____ to find its perimeter?

Answer: Just square.

Ministry of Education Ontario

Authorized by the Minister of Education.
Hon. Bette Stephenson, M.D., Minister
Dr. H. K. Fisher, Deputy Minister

1980

MATHEMATICS

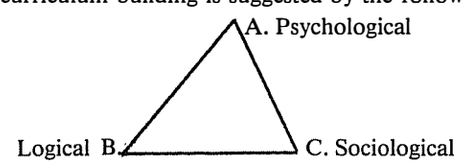
CURRICULUM GUIDELINE FOR THE INTERMEDIATE DIVISION

COMMENT ON THE INTERMEDIATE MATHEMATICS GUIDELINE

R.S. Smith, Faculty of Education, The University of Western Ontario

Many things must be considered when building a curriculum. The 1980 curriculum guideline for mathematics is the result of many years of work. Let me be one of the many to congratulate not only the authors but also all those who had any part in its construction.

A model of mathematics curriculum building is suggested by the following triangle:



If we look back at the "New Math" curriculum the I-12, A, B, C, and D documents, I think we would all agree that the emphasis was on the logical base. The precision of language, the emphasis upon structure and deductive logic of the 60's does not have a primary role for the '80's. Such statements as "Generally the underlying structures of mathematics need not be studied in abstract form in the Intermediate Division" and "Experiential approaches are fundamental to the implementation strategies of courses based on this guideline" suggest the emphasis has swung to the psychological base. I did not see any mention of Jerome Bruner's dictum that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development." Instead there are references to Piagetian Stages of Learning. Specific suggestions are "the use of mathematical models and the experiential approaches." This latter requires 1) a move from a formal classroom with much talk by the teachers; 2) more opportunities for pupils to act on physical materials; 3) much more social intercourse, as verbal language is an important influence in the development of concrete-operational and formal thought.

We are cautioned however that "the experiential approach must be complemented by the development of skills and concepts needed to solve problems that arise in the real world." The document exhorts us to place a major emphasis upon applications, e.g. "everyone uses mathematics in some form or other in everyday life, . . . learn the basic skills needed for handling real-life applications, . . this guideline places a clear-cut emphasis on applications." The centre of balance has shifted toward the sociological base. The forces of change are pulling us toward the points A and C and away from B.

I agree whole heartedly with this shift in emphasis. I feel the document gives a good balance between core topics and optional topics and sufficient time for implementation. No one can disagree with statements such as: "courses should take into account the different needs of students, ... not all students in a particular course will need the same detailed development of the core topics, . . . the program must provide students with opportunities to experience success if their interest and enthusiasm are to be maintained." These are easy to state but not so easy to accomplish. If students are to gain "an integrated view of mathematics in which concepts of arithmetic, measurement and geometry are interwoven with practical experiences and uses of mathematics," it will be as a result of the stated guidelines but, I believe, even more as a result of the work of dedicated teachers. I must therefore look ahead to the implementation of this forty-four page document. After due considera-

tion, I make the following four observations:

1. For the teacher of the Primary and Junior Division, the Ministry of Education produced a companion booklet to **The Formative Years.** It is **Education in the Primary and Junior Division.** It provides an extensive philosophical basis and rationale for the program of these divisions. No such document has yet been produced to help teachers in the Intermediate Division. The Ontario Mathematics Coordinators' Association did write **A Rationale for Elementary and Secondary School Mathematics** which provided a good starting point. It gave us a reasoned, logical framework in its overviews and even practical helps in its statements of some sample objectives for the various grade levels. The dialogue has begun. It is to be hoped it will continue and that at some future date the Ministry's proposed document, **"The Intermediate Years"** will be completed and will be as useful as **Education in the Primary and Junior Division.**

2. In the few meetings held by the Ministry officials to help in implementation, strong suggestions of a cyclic review (a monitoring of implementation of the guidelines) were made with 1984 as the year. Obviously this necessitates immediate consultation among the teachers of mathematics in both elementary and secondary schools. I do hope these Intermediate Mathematics Committees are successful. Much depends on their planning. I wish them well, particularly in the County School Boards.

3. Streaming is a method to deal with individual differences. Perhaps the five streams now set forth for the Grades 9 and 10 will do the job. Well prepared mathematics teachers are also a key factor in dealing with this problem. If we agree that three quarters of the teachers in Grades 7 and 8 are generalists we must be concerned about helping them. I would suggest that student teachers graduating with a teaching certificate endorsed for the Junior Intermediate Division have less preparation for teaching mathematics today than the little they received formerly. Perhaps well-honed support documents illustrating various teaching strategies will yet appear to help the hard pressed "generalist" teacher. It is a fact as well, that there is a shortage of math teachers in Ontario as well as in the United States and Great Britain. What should be done to help overcome this problem?

4. When the New Mathematics was introduced there were summer courses, in-service days and many meetings to try to help teachers. Although there are some courses available I have not seen any great interest in these courses. Perhaps this suggests that we are ready to implement the guidelines using resources now in our own areas. I hope this is so and that these resources are sufficient.

SOLVING MULTI-STEP MATHEMATICAL WORD PROBLEMS[1]

James M. Sherrill, The University of British Columbia

[1] The project was funded by the Educational Research Institute of British Columbia, the Social Sciences Grants Committee of the University of British Columbia, and the Youth Employment Programme of the federal government.

In the study the term "problem solving" was synonymous with "solving certain types of word problems found in mathematics textbooks at the specified grade level." The certain "types" are explained below.

One of the disturbing results of the 1977 B.C. Mathematics Assessment was the differential performance of students on one-step and multi-step mathematical word problems. A one-step mathematical word problem is a problem that can be solved by combining specified numbers using one operation. An example is given in Figure 1.

> There are 25 members in the volleyball club. If the cost for each uniform is $24, how much would it cost to buy new uniforms for all the club members?

Figure 1: One-Step Mathematical Word Problem

To solve the problem in Figure 1 the student is required to combine 25 and 24 using multiplication.

A multi-step mathematical word problem is, of course, a problem requiring more than one step for solution. There are, however, many types of multi-step problems of which only two were used in the present study. An example of a Type I multi-step mathematical word problem is presented in Figure 2.

> Yesterday, the whale in the aquarium ate a total of 98 fish in three meals. She ate 32 fish at the first meal and 25 fish at the second meal. How many fish did she eat for her third meal?

Figure 2: Type I Multi-Step Mathematical Word Problem

To solve the problem in Figure 2, the student is required to combine 32 and 25 using addition and, then to combine the result of the addition with 98 using subtraction. In Type I problems the specified numbers must be combined, but using more than one operation.

An example of a Type II multi-step mathematical word problem is presented in Figure 3.

> What is the area of a square which has a perimeter of 12 cm.?

Figure 3: Type II Multi-Step Mathematical Word Problem

To solve the problem in Figure 3, the student is required to generate the value for the side of the square using a number (4) that is not specified; then use that value to find the area. In a Type II problem one of the numbers needed to solve the problem must be generated using a number which is not specified.

On the grade 4 test of the 1977 B.C. Mathematics Assessment the average performance on the one-step problems was 80.7%; the average performance on the multi-step problems was 48.7% --- a differential performance of 32%! On the grade 8 test of the 1977 B.C. Mathematics Assessment the average performance on the one-step problems was 73.3%; the average performance on the multi-step problems was 44.8% --- a differential performance of 28.5! Such drastic differential performances prompted the authors of the B.C. Mathematics Assessment Reports (Robitaille and Sherrill, 1977) to make Recommendation 4-7 which suggested that students get many more experiences solving multi-step mathematical word problems.

While the B.C. Mathematics Assessment had shown the existence of a difficulty with solving multi-step mathematical word problems it was beyond the scope of the study to investigate causes for the difficulty or make specific suggestion to the teachers of B.C. as to how the difficulty could be alleviated. The main objective of the present study was to serve as a first step in the implementation of Recommendation 4-7.

Unlike many problem-solving studies, the present study did not use novel or unusual problems. All the problems used were ones that had been selected from materials suitable for the grade 7 students in the study. Over 600 word problems were collected that fit the description of one of the three types of problems used in the study. From this pool of problems the ones selected met three criteria:

1. From the original item three forms could be created --- one form for each type of word problem used in the study.
2. All three forms of the item could be worded in the same physical format.
3. All three forms of the item had equal levels of difficulty with respect to computation.

Two problem-solving tests and a computation test were developed. The initial problem-solving test consisted of four different physical settings yielding four one-step problems and four each of the two types of multi-step problems. The second problem-solving test contained three different physical settings. The 42-item computation test was created based on the computation required to solve the 12 word problems. One hundred and twenty items were field-tested and the final selection consisted of thirty-six problems (three forms of each of twelve word problems).

Using the first problem-solving test, the computation test, and data provided by the schools, two groups of students were formed. **Both** groups had to do well on the computation test, **both** groups had to make a perfect score on the one-step problems, and **both** groups had to be above average

readers.

From the students meeting the above criteria the final sample was constructed. The sample was not a random sample. From the specified students, nine were picked who had scored 7-8 out of 8 multi-step problems and nine were picked who had scored 0-3. The sample size was decided upon after a discussion with Nick Herscovics of Concordia University who has had considerable experience with studies in which data are collected through interviews.

The description of the two groups is given in Table 1.

Table 1
Criteria Scores for the Interview Groups

	Computation Test	Reading Level	One-Step Problems	Multi-Step Problems
Group A	35.0	8.7	4.0	3.0
Group B	35.0	8.7	4.0	7.1

As one can see from Table 1 the students were not average students. **Both** groups scored above 80% on the computation required for the problem-solving tests; **both** groups scored perfect on the one-step problems; **both** groups were above average readers. As the 1977 B.C. Mathematics Assessment had demonstrated, however, one group performed poorly on multi-step problems.

Each student was then interviewed in a problem-solving situation using the second problem-solving test. During the interview, protocol construction, and data analysis phases of the study, the staff involved in the study did not know to which group any student belonged.

The interviews used the "Think Aloud" procedure. This procedure has been tested in many similar studies. For a complete description of the procedures used in the interviews and the entire study one can write ERIBC, Suite 400, 515 West 10th Ave., Vancouver, B.C. and request a cope of **ERIBC REPORTS 80:18.** Since the report is 52 pages long, many of the details had to be omitted from this article.

Finally, from the interviews typed protocols were created by the person who did the interviews and the data in the protocols were analyzed.

Some of the Results

This section contains a paragraph or two concerning some of the different variables that were analyzed.

The success rate on the problems reflects how the samples were constructed and the items selected. The success rates by group are presented in Table 2.

Table 2
Success Rates as Percentages

	One-Step Problems	Type I Problems	Type II Problems
Group A	89	72	44
Group B	100	94	78

The success rate decreases for both groups for the three types of problems and the differential performance between the two groups increases.

The time the students spent on the problems yielded some predictable results and some mild surprises. As expected, the students spent more time on the more difficult items. Not as expected, every student persevered on all items. The most extreme case was a student who spent almost half an hour on one problem! The greatest amount of time spent by someone who correctly solved the same problem was less than five minutes. There was no time limit.

Each student was required to read the problem to begin his/her work. A record was kept of how

many times a student re-read a problem. Re-reading the problem meant reading the entire problem, not just skimming the problem for facts or key words. This is especially important for multi-step problems. Consider the problem in Figure 4.

> A contractor is paid $18,000 in 12 equal payments (one each month). How much will the contractor have been paid in 5 months?

Figure 4: Type I Multi-Step Problem

Of the students who missed the problem in Figure 4, a common "solution" was to divide 18,000 by 12 and stop. They skimmed the problem for the question which was worded by many of the students as, "How much was he paid?". They skimmed the problem for the numbers, divided the two numbers, and stopped.

The most commonly re-read problem yielded a median of only 1.17 re-reads.

The number of times the students attempted each problem was tallied. Since each student attempted each problem at least once, the number of re-attempts was analyzed. A "re-attempt" was defined to be the situation where a student started the problem from a beginning step. Of the 17 problems missed by Group A, only 7 were attempted more than once; of the 5 problems missed by Group B, **none** were attempted more than once.

Everyone in the interview phase of the study was "good" at arithmetic. The two groups averaged 83% on the test covering the arithmetic involved in solving the word problems. However, it is still surprising that of the 22 problems missed only 3 "solutions" involved computational errors. A much larger problem was not knowing the numbers and/or operations with which to apply their computational skills.

After obtaining an answer for a problem students should be encouraged to check their answers. Each group faced 54 problems (6 problems X 9 students). Of the 54 problems faced by Group A, 2 were checked; 9 solutions were checked by Group B. All 11 problems for which solution were checked were eventually solved correctly.

There were three types of checks of an answer. By far the most common type was using the inverse operation, a method that simply checks the arithmetic. The second type was to simply review the steps. The third type was to compare the answer to what was asked for in the problem and see if the answer seemed reasonable. Two students obtained an answer of 15,000 when dividing 18,000 by 12. One student, in checking his answer rejected it since it was unreasonable to expect that a contractor paid $18,000 in one year would get $15,000 each month. The other student rejected the answer since 15,000 is too large to be the answer in a division problem involving 18,000.

The decision was made in the design stage of the study to use the same physical setting for each group of the three types of problems. So, for example, there was a one-step, Type I, and Type II problem involving a contractor being paid $18,000 in one year. This decision was made to avoid the criticism that the physical setting of a problem yielded differential results, not the type of problems. The decision, however, yielded a surprising result in itself. Even though the student were forewarned and warned again when the situation occurred, many insisted that all three problems had to be solved the same way and should even have the same answer because all three problems involved a contractor being paid the same amount over the same time period.

Almost all of the current research in mathematical problem solving deals with the use of heuristics. Many definitions of "heuristics" are given in the literature. The elements that are common to the definitions are that a heuristic . . .

1. ... in something --- device, decision mechanism, path, procedure, etc.
2. ... helps is solving a problem.
3. ... offers no guarantee of success.

Of the eight most commonly observed heuristics in previous research studies only one was observed to any great extent and that was Uses Algorithms. Any set of problems will have characteristics that obviate the use of certain heuristics. It may not be that the students do not have all eight heuristics in their repertoire, it may be that the problems found in current elementary school mathematics textbooks do not require such heuristics for solution.

The pattern that is clear is that the problem-solving strategy most prevalent among those students interviewed is as follows:

1. Search for key words.
2. Select algorithms based on key words.
3. Apply the algorithms.

Suggestions to Teachers

Based on the results of the study, some reported here and many others found in the complete report, I would offer the following suggestions to teachers of elementary school mathematics concerning the teaching of problem solving.

1. Searching for key words (clue words or phrases) is still an important method for determining any computational algorithms that may be needed to solve the word problems found in the current elementary school mathematics textbooks, BUT there is a caution. When re-reading a problem to gain more information or to re-organize the information **skimming for facts may mislead a student.** After skimming for details the student should be encouraged to re-read the entire problem to put the facts (or key words) in the context of the entire problem.

2. We must communicate to students that it is not wrong to use trial and error. Much can be learned about some problems by taking a numeric example, trying it, and seeing if it tells us something about the structure or solution of the problem. Some students appear to have the image of problems that to solve one, one must simply pick an algorithm, apply it, and the result is the answer to the problem.

3. Students need to be encouraged (or continue to be encouraged) to check their work and their answers.

All the suggestions are, of course, interrelated. One process for checking one's work is to re-read the problem to be sure the solution process being used matches the problem. To check arithmetic applying the inverse operation is a useful and used technique. Checking one's answer for reasonableness against what is asked for in the problem is a quick and simple step.

4. Some students have the perception that a problem is something one solves on the first attempt or never. They must be encouraged to try a problem again if they are not successful on the first attempt.

5. Given that the emphasis on problem solving is beginning and will continue to increase in the 1980's in the elementary school mathematics curriculum, the students need more experience in solving word problems. The results of the present study indicate that students need experience in solving several problems using the same physical setting. They need to be rid of the syndrome observed in the interviews that two problems that have the same physical setting must be the same problem or at least solved using the same method.

6. The final suggestion and, in the opinion of this researcher and teacher, the most important one is that students need much, much more experience with multi-step problems.

Some students selected an algorithm, used the algorithm to combine two numbers, and stopped. Such a technique is highly successful with one-step problems, but not with multi-step problems.

If we want to improve our students' ability to solve mathematical word problems, we must give them many opportunities to solve problems that require more than applying one algorithm to two numbers and the result is the answer to the word problem.

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MATCHING STUDENTS AND READING MATERIALS IN MATHEMATICS USING THE CLOZE PROCEDURE



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Introduction

There appears to be a consensus among educators that a large proportion of the information that students are expected to acquire from about grade four in school is transmitted to them through written materials. In this connection, the role of the teacher becomes one of facilitating reading through selection of appropriate books, assigning reading passages, and evaluating the student's understanding after reading. The success of the information transfer depends to a large extent on whether the students possess adequate linguistic ability to comprehend the language of the reading materials or conversely, on whether the materials are written at a level of linguistic complexity appropriate to the language ability of the students.

Since so much time is allocated to reading assignments in the various subjects, one mandate of teachers is to ensure that appropriate reading materials are assigned so that the student's gain the maximum amount of information with a minimum of time and effort. The problem of identifying materials accessible to students is compounded as students progress through school and are exposed to a variety of different subjects. From personal communications with teachers, there is ample evidence to suggest that most textbooks are too difficult for the majority of students. This finding has led educators from all subjects areas to search for methods to determine a better match between students and reading materials. For example, Kirkwood and Wolfe (1980) [language arts and social studies], Thelen, (1974) [science], and Hater and Kane, (1975) [mathematics] have all reported that the cloze procedure was an effective method of determining the difficulty of textbooks in the respective subject areas which they studied. The purpose of this brief paper is to describe the cloze procedure and how it can be used by classroom teachers to improve the match between students' reading ability and the readability of their math textbooks.

The Cloze Procedure

Essentially any process in which students are requested to recover original text from text which has been altered or "mutilated", by deleting words is called cloze testing. Performance on cloze tests is indexed by how much of the original text the student can recover. (Taylor, 1953).

There are different cloze testing procedures, involving different deletion rates, use of systematic or random deletion, deletion of only certain kinds of words, etc. There are also procedures for scoring the tests, including the acceptance of only exact replacements or the acceptance of synonyms for deleted words. The method used most often is to delete every fifth word. In this traditional procedure every fifth word is deleted and only the exact replacements are accepted as correct responses. Bormuth (1977) has shown that different deletion rates produce highly correlated scores. While the mean score levels differ according to the kind of deletion or the kind of acceptable replacements, the ranking of readers and the ranking of the passages are relatively unaffected.

The traditional cloze procedure operates with mutilated text and requires the student to replace deleted words exactly. Cloze tests measure literal comprehension because the process of identifying the correct word simulates the process readers go through in reading the passage. Successful substitution for the deleted parts is possible because of the cues in the original language, but the students must use all the syntactic and semantic cues which are visible in the remaining graphic information. In other words, a reader must interact with the content and structure of the passage as far as it can be seen in order to consider possible substitutions. The cloze procedure measures overall concept load and the overall syntactic and semantic processing that must be employed in reading and understanding the passage (Taylor, 1953). As such, the standard open-ended cloze procedure provides a general measure of the student's understanding of a given passage. In fact, the work of Bormuth (1968) has found that a particular level of cloze response corresponds to a fixed amount of information gain, regardless of whether one deals with fluent readers of difficult passages or with poor readers of easy passages. This means that the cloze test response proportion is an absolute

indication of the usefulness of the passage for the student.

The criterion level which Bormuth suggested is 35% of the cloze response which is the minimum for which a passage will be useful to students in terms of basic understanding. Cloze scores lower than this criterion will effectively yield no information gain or furthermore, there may be an unwillingness on the part of the student to spend the time necessary to extract information. Bormuth finds that 40% or 50% is probably ideal and that 50% and above means that extracting information will be easier.

Constructing a Cloze Passage for Classroom Use

The following steps were utilized by Kirkwood and Wolfe (1980) in constructing cloze passages for a recent literacy survey completed in Ontario. The steps used for the construction and administration of the cloze tests were developed initially by Bormuth (1975).

(1) Select a minimum three passages at random from the textbook. One passage from each of the beginning, middle, and end of the book should be selected. Try and get at least 250 words of connected discourse for each passage. In mathematics, the introduction to chapters or pages containing word problems usually contain at least 250 words and correspondingly would be suitable.

(2) Starting on the tenth word of the first sentence, mark every fifth word for deletion. Use a felt marker to circle or underline the words to be deleted.

(3) Retype the passage replacing the underlined or circled words with a blank line of 15 typewriter spaces. (See example at the end of this article).

(4) Duplicate the passages for your class. Preferably, photocopy the passages, however, the mimeo stencil is also adequate.

(5) Give the three different passages to each of your students. The time required to complete three cloze passages containing 50 items is approximately 45 minutes. (See directions for administration with the cloze example).

(6) To correct the passages, make an overhead containing the correct responses. Have students exchange papers and mark the correct responses. Remember that only the **exact** replacement is acceptable. However, spelling does not count if the word resembles the correct response and is semantically correct.

(7) Count the number of correct responses and calculate the percentage of items scored correctly for each passage.

(8) Determine the percentage of students who scored above and below the 35% criterion. This will give two important pieces of information (a) students who will experience difficulty with textbook (assuming that they did not reach criterion on any of the passages) and (b) the variability between passages (whether there exists any differences in readability between the passages in the same book).

Conclusions

The cloze procedure appears to be a straightforward, efficient method of determining the suitability of reading materials across grades and subjects. The major advantages of the cloze procedure are that it is inexpensive to construct, easy to administer and score, and it is selected from materials that students actually use in classroom situations.

Instructions for Administration of Cloze Tests

Since this is a new kind of test that you will be writing, it will be helpful if you would read the directions very carefully. You will notice that every fifth word has been left out and your task is to guess what word was left out and then to write that word in the blank space. Here are some hints to help you with this test:

(1) Only one word can be written in each blank.
(2) Do not be afraid to guess.
(3) Try to fill in every blank.
(4) Skip over the hard blanks and then come back to them when you have finished.

(5) Most of the blanks can be answered with ordinary words but a few will be:

numbers like . . . . . 2345 or 68.00 or 1981
contractions like . . . . . can't or weren't
abbreviations like . . . . . Mrs. or Ont.
parts of hyphenated words like self- in the word self-made

Here is a sample of one of the tests. Fill each blank with the word you think has been left out. The answers are written at the bottom of the page so you can check your answers.

Editor's Note: Further examples will be used as fillers in this issue.

Mathways 6 Page 303

We use a random sampling of data to represent _____ larger body of data. _____ which month of the _____ were the greatest number _____ students born?
If there _____ 500 students in your _____, it would take a _____ time to ask each _____ your question. Instead you _____ only a sample of _____ students and predict the _____ for the whole group _____ the results of the _____.
For a sample of _____ students, you would ask _____ tenth student. For a _____ of 100 students you _____ ask every fifth student. _____ sample should not be _____ small.

Answers:
(a, In, year, of, were, school, long, student, ask, the, answers, from, sample, 50, every, sample, would, Your, too)

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A PROBLEM AND SOME IMPLICATIONS

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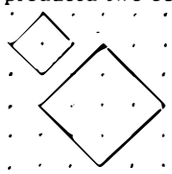
It was by no means the first time I had given out some geoboards and asked, "How many different squares can you make?"

On this occasion the geoboards were 6 x 6, instead of the 5 x 5 ones I was accustomed to, and the teachers receiving both them and the question were attending the first session of a week's summer course at the University of Western Ontario. We were mainly concerned with junior grades, but though this affected subsequent discussion it did not in any way influence the first two actions, which always in my experience seem to be the same, whatever the interests and mathematical backgrounds of the teachers.

They began by creating squares which had their sides parallel to the sides of the geoboard.

Then they asked what I meant by 'different'.

I never know quite how to deal best with either of these. But I knew that the blockage about the non-oblique squares would eventually disappear, because it always does; and, sure enough, after some time Doreen produced two oblique ones,



and Rose, noticing this, copied these and produced some others.

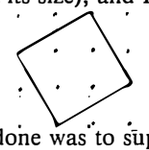


The question about the interpretation of 'different' is more difficult to cope with. I can ask, weakly, "What do **you** mean by different?" but eventually I have to discuss the idea that squares can be different both in size and position, and the answer you obtained to the original question would depend on whether you included the second or not. Keith and Vince worked on different portions for the non-oblique squares.

For the girls the initial random exploration had produced a complexity of elastic bands that made counting difficult. We removed them and began again, more systematically, counting the squares as they made them.

The initial exploration is important. It is not always possible to see an organization straight away. One needs to play around with the problem first, get a feel for it, become aware of the possibilities. Only then can one impose any system.

This process continues until a satisfactory system is found, and this now arose almost by accident. Doreen was describing one of her squares to me so that I could copy it onto my board, and she struggled with "3 along, 1 up" and similar descriptions of successive sides. We discussed the idea that if we fixed one of the sides then we fixed the square (or at least its size), and I formalized the usual notation for "2 along, 1 up", so that this square



became identified with (2,1).

I emphasized that this was Doreen's idea, and that all I had done was to supply a fairly standard notation.

Now we organized the number pairs for the other squares we had found. Numbers are easier to put in order than shapes, even in the comparative complexity of pairs. This enabled us to find a further two squares to add to the nine that everyone had previously agreed upon.

Admittedly, that previous agreement on an answer of nine had not been altogether convincing, but what does one do about it? Most times in the classroom if students come to a wrong conclusion the teacher tells them they are wrong. In this way students came to rely on the teacher's authority rather than their own; they never learn to recognize errors; they never have to justify their conclusions.

In any case, counting is one of the most difficult of mathematical activities. It is difficult for young children, counting perhaps some cubes on a table. They usually begin in the middle, and count in a sort of spiral, leaving some out and counting some twice. If they move them to one side, and then transfer them one by one to the other side, it is easier to establish a one-to-one correspondence between cubes and successive numerals. Or if the cubes are arranged in a straight line, a suitable order is established that makes the correspondence easier. But these are counting skills that teachers are not often aware of. In the case of our squares, the problem was to find a suitable order, and this first method mapped each square onto an order pair of numbers, and it was easier to order those.

(0,1) (1,1) (2,2)

(0,2) (1,2) (2,3)

(0,3) (1,3)

(0,4) (1,4)

(0,5)

It now became easier to deal with the ordered pairs than to cope with the squares, especially when we noticed that the maximum total for the two numbers was 5, a fact which we explained geometrically. Now it became possible, **without** elastic bands, to identify those squares possible on smaller geoboards, and to add those squares possible on larger boards.

This interpolation and extrapolation produced a sequence of results,

2 x 2	1	6 x 6	11
3 x 3	3	7 x 7	15
4 x 4	5	8 x 8	19
5 x 5	8	etc.	

which in turn embodied both a hypothesis and an algorithm for continuing the sequence, so we felt that we could, with little difficulty, decide how many squares were possible on a 100 x 100 geoboard. The power of this situation was suddenly immense.

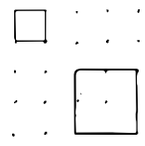
Had the teachers been intermediate teachers we could have separated the 'even' and 'odd' boards and produced two algebraic formulas for the respective sub-sequences. These would not, of course, provide any new hypothesis; they would merely have provided algorithms for calculating the number of squares on any size board, algorithms that were more efficient than the recurrence relation in the original sequence.

Instead, I asked if there were other ways of ordering the squares. The length of the side was suggested. But there were complications about choosing units, especially if the natural geoboard unit of 2 cm between adjacent pins was used, and one could not measure accurately enough. (Pythagoras, which does not feature in junior grades, appeared to be unknown at this stage.)

Were there other ways of ordering squares?
There did not appear to be.

I do not like to impose my own ideas on any class, but on this occasion I thought it might be profitable to allow myself the luxury of a gigantic hint!

I made these two squares,



and asked how much bigger the larger square was. It was a loosely formed question, but Keith immediately answered "Four", and the idea of area dawned.

We now needed methods for finding areas of the oblique squares, which took some time, but eventually we were able to attach an area to each corresponding ordered pair.

(0,1) →	1	(1,1) →	2	(2,2) →	8
(0,2) →	4	(1,2) →	5	(2,3) →	13
(0,3) →	9	(1,3) →	10		
(0,4) →	16	(1,4) →	17		
(0,5) →	25				

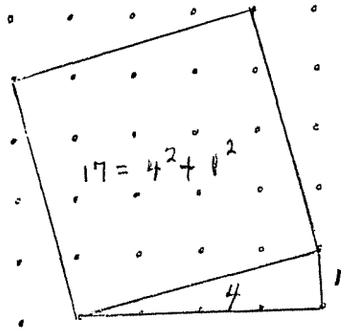
There were number sequences everywhere, and we extrapolated with more feelings of power.

The left-hand column was easily replaced by the squares, 1^2, 2^2, 3^2, 4^2, 5^2, and subsequent discus-

sion revealed the other columns as sums of appropriate squares, with the general result,

$$(m, n) \rightarrow m^2 + n^2$$

It was now easier to draw a typical square and remind everyone of Pythagoras' theorem.



We put the areas in order:

1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, . . .

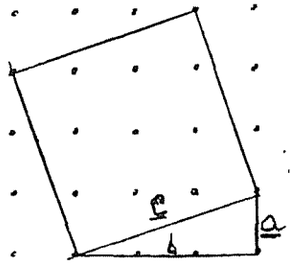
Why no 3? That was easy to explain, and so were some of the other missing numbers. And it looked, for instance, as if we would be missing areasof 3, 7, 11, 15, 19, . . . Various results in number theory were possible, including Fermat's Theorem, but we had gone far enough.

A worry was expressed about spending all this time (it had taken us two and a half hours) on something that "was not in the curriculum". There were two complementary answers that I could provide.

Firstly, half of one's mathematical education is concerned not with mathematical content - concepts, facts, skills, techniques and algorithms - but with developing abilities involved in solving problems. These include: understanding and identifying the problem; deciding on its limitations; collecting information, organizing it and perhaps as a result collecting more information; forming hypotheses, testing them, rejecting them if necessary and forming new ones; interpolating and extrapolating; choosing and using appropriate notation; generalizing; knowing when the problem has been solved and interpreting the answer; creating further problems. None of these items, incidentally, seems to appear on curriculums.

Secondly, although a teacher may not present the problem of the squares in order specifically to cover any topic that does appear on the curriculum, a large number of curriculum topics are involved. Some of these will perhaps have already been covered, and are reviewed, not artificially in some sort of routine test, but naturally in a context in which they are being used as part of the problem-solving process. Here we have used simple number bonds, number patterns and sequences, properties of squares, area.

Other topics are anticipated rather than learnt. Although coordinates for points in the plane are not used as such, the notation and the idea are developed in a simple and natural way, and when students meet coordinates more formally they will benefit from this previous experience. Pythagoras' theorem is not made explicit, but given some algebraic experience it is but a short step from the special case given above to a general one,



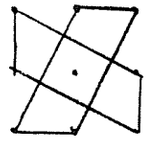
in which the square whose area is C^2 is seen to equal to the whole square, $(a + b)^2$, less four right triangles which can be arranged to form two rectangles each of area ab; thus

$$C^2 = (a + b)^2 - 2\,ab$$

$$= a^2 + b^2$$

with a little standard manipulation. The number theory involving sums of squares and primes may not be met for some time, if at all by most students, but here is a situation in which it occurs naturally, and the resourceful teachers may choose at least to discuss some of the interesting hypotheses.

We never did go back to the men's interpretation of the problem and count the different positions as well. Nor did anyone think, as others have done, of producing squares this way.



But that is another story. (1)

(1) Casey S. Clemens, B. Metzk: "The Area of a Square," **Mathematics Teaching** No. 8's. Dec/78.



THE CREATIVE PROBLEM SOLVING APPROACH IN MATHEMATICS

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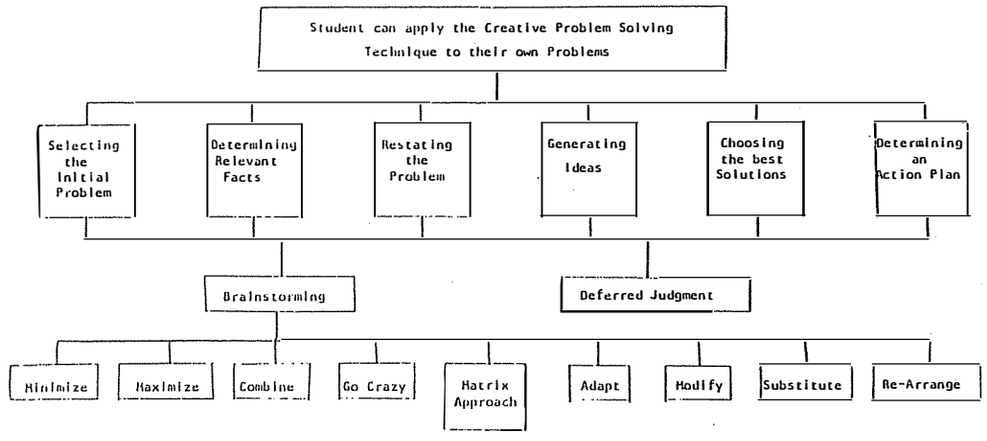
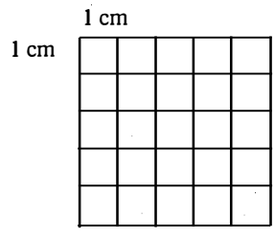


FIGURE 1: Using Task Analysis to Structure the Technique of Creative Problem Solving

Figure 1 shows the basic CPS model. Once the original problem has been defined, there are five steps between problem and action which include fact finding, problem finding, idea finding, solution finding and acceptance finding. Underlying each step is brainstorming and deferred judgment. The latter involves training the children to accept any and all ideas without judging them prematurely. This involves attitude retraining which can be difficult. Brainstorming is used at each stage to generate as many ideas as possible and the children are urged to 'use their brains' and give any idea even if it seems crazy. After a stage is brainstormed, the group selects the best answer and then develops it at the next stage until an action plan is formulated.

ORIGINAL PROBLEM: How many squares are there in the following figure?



Fact Finding	Problem Finding	Idea Finding	Solution Finding	Acceptance Finding (proving the answer)
A square has 4 equal sides and 4 right angles. There can be squares inside of squares.	1. How might I count the squares in the figure? 2. How might I count squares of every size? 3. How might I discover a trick to the question? 4. How might I discover the relationship between the number of squares on each side and the total number of squares?	1. Be systematic and count 1x1, 2x2, 3x3, etc. 2. Keep a chart of the numbers. 3. Put a pin in each square to keep track. 4. Count across and up. 5. Cut out a square of each size and move it around. 6. Imagine you cut a square of each kind and are moving it around. 7. Count each size square by moving cardboard squares around.	Follow through with idea 7 to find the answer. There are: 25 1 cm squares 16 2 cm squares 9 3 cm squares 4 4 cm squares 1 5 cm square 55 squares total	Show how it conforms to a pattern. $5^2, 4^2, 3^2, 2^2, 1^2$. The number of squares in any square equals the sum of the squared numbers from 1 to x where x is the number of squares Total $= 1^2 + 2^2 + 3^2 + 4^2 + 5^2$ $= 1 + 4 + 9 + 16 + 25$ $= 55$

Figure 2: Use of Problem Solving Techniques with an Academic Problem

Figure 2 demonstrates how the process is used and developed. At the fact finding stage, children list the facts pertinent to solving the problem. The next stage asks the children to look at the original problem and then re-state it to find different ways of looking at it. Once several problem statements have come forth, the group selects the one that states the problem as specifically and usefully as possible. In figure 2, the first problem statement 'How might I count the squares in the figure?' was chosen as the best one to work on and so several ideas were generated to solve that problem. After all the ideas were listed, they were discussed, ranked and the seventh (Count each size square by moving cardboard squares around) was chosen. This idea is an example of how two ideas can be combined to form a new and often better one. At the solution finding stage, the cardboard squares of each size are used as an aid to counting out each size square. The final stage of acceptance finding is in this case a second solution that can act as a proof for the answer found in the previous stage. The value in the fifth stage lies not only in the proof but in its ability to demonstrate to students, who prefer the security of lower level solutions such as concrete aids and counting, that there is a higher level solution possible.

Discussion

This method is initially more time consuming than the traditional approach to problem solving. However, there are several advantages to the creative problem solving approach. One that has already been mentioned is the possibility of showing children how problems can be solved at at least two levels of thinking.

ORIGINAL PROBLEM: Petunia was visiting a farm and noticed that the farmer raised only rabbits and chickens. She counted and among the two kinds of animals, she saw 70 heads and 220 feet. How many rabbits and how may chickens did the farmer have?

APPROPRIATE STUDENT(S): Grade 4 - 8, individual or small group

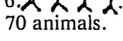
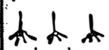
Fact Finding	Problem Finding	Idea Finding	Solution Finding	Acceptance Finding
70 heads; therefore there are 70 animals. There are 220 feet or 110 pairs of feet. Chickens have 2 feet or 1 pair each. Rabbits have 4 feet or 2 pair each.	*The problem or ideas chosen by the group for further consideration.	1. Half chickens and half rabbits. 2. Divide 70 heads into 220 feet. 3. All chickens 110, all rabbits 55. Ration 2:1, somewhere in between. 4. Write an equation with x equalling number of chickens and y equalling number of rabbits.* 5. Try different numbers until one set works. 6. Draw stick chickens and add as many extra legs as you have.* 7. Pair stick figures until you run out of animals. 8. Ask the farmer how many he had. 9. Ask the rabbits.	4. Equation Therefore there are 30 chickens and 40 rabbits. 6.  . . . 70 animals. There are 220-140 = 80 legs left. Add two extra legs to as many chickens as possible.  . . 40 animals Therefore there are 40 four legged animals (rabbits) and 30 two legged animals (chickens)	Put numbers back into equation to check your answer. $x = 70 - y$ $2x + 4y = 220$ $2(70-y) + 4y = 220$ $2y = 80$ $y = 40$ $x = 30$ Scan figures to make sure all legs are accounted for.

FIGURE 3: Using CPS to Solve a Mathematical Problem.
This is an Adaptation of a Problem Developed by Noller, Heintz and Bleuler (1978).

Figure 3 gives a second example of how both concrete and abstract solutions can be used to find answers in unique ways. The technique itself supports the concept that there are many ways of looking at and solving a problem. Many children become fixated on simple (frequently rote) methods of solution finding and lose (or don't develop) the ability to approach problem solving and mathematics generally in a more flexible manner. The CPS technique exposes them to a method that encourages them to think in more flexible and hopefully more creative ways. The technique also involves a great deal of talking about the problem which can support a greater understanding of basic mathematical concepts and procedures. Finally, the method has potential with exceptional children in several ways. The gifted children can use it to support and train their creative thinking at higher conceptual levels. Children with average intelligence, who have learning difficulties and have had failure experiences in mathematics, can be reintroduced to some aspects by this novel approach. In this regard it has been found that a method that stresses 'using your brain' can be highly motivating for learning with disabled children (Crealock, 1980).

The method is not suggested as a replacement for traditional approaches but rather as an alternate approach that can be a useful part of the mathematics program.

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INTERVIEW

Taped Talks with Bob Smith



Dr. Carole Greenes, Dr. George Immerzeel

Carole Greenes is Associate Professor of Mathematics Education at the Boston University School of Education. She is no stranger to Ontario audiences who have enjoyed her talks at our conventions. George Immerzeel is Professor of Mathematics at the University of Northern Iowa and Director of the Iowa Problem Solving Project. It was my pleasure to interview both, seated at the edge of the putting green in the garden of the Prince Hotel, during the Toronto Convention.

R.S. In Ontario, the Recommendations for Intermediate Division Mathematics known as the Pogue Report, stated strongly that problem solving must be at the heart of any mathematics program. In fact, at the N.C.T.M. Convention held in Seattle, the first recommendation in **An Agenda for Action** was that "problem solving" be the focus of school mathematics in the 1980's."

C.F. That is right.

R.S. Why is problem solving so much in people's minds? It isn't new.

C.G. No, as a matter of fact, problem solving has been emphasized since the turn of the century. In 1901 John Dewey presented his five-step model for solving problems. That model was then refined in the 20's and the 30's, and presented to mathematics educators in the mid-forties by George Polya in his book, **How to Solve It.** So the model has been around for quite a while. And story problems which are thought by many to be synonymous with problem solving have been with us as long as mathematics has been taught. But I think the real recent interest was motivated by the results of the National Assessment of Educational Progress' second assessment of student abilities in mathematics. This was carried out in the United States in 1977-78 with students, ages nine, thirteen and seventeen years old. The results of the assessment indicated that students' computational abilities had not changed significantly from the first assessment six years earlier. However, there was a significant decline in their ability to solve problems that involved the application of a variety of mathematical skills and concepts.

R.S. George, as chairman of the Committee for the Mathematics Curriculum of The 80's, one of the strongest points you made was that problem solving must become the focus of school mathematics for the 1980's. Perhaps you would like to add just a word or two to what Carole has said.

G.I. The reason that the emphasis is on problem solving is an outgrowth not only of the N.A.E.P. data that Carole has described, but may also be attributed to the N.C.T.M.'s P.R.I.S.M. Project. The primary purpose of P.R.I.S.M. was to establish priorities in school mathematics by analyzing the responses to surveys which were sent to various populations. Problem solving was identified as a priority issue by all groups surveyed.

C.G. The real pressure is coming from society. Advances in science and technology and the low cost availability of the calculator and computer are removing the necessity for performing lengthy computations. Our curriculum must be reorganized so that students will be taught skills that allow them to solve a wide variety of types of problems. The curriculum must not concentrate solely on computation.

G.I. Well, Carole, I believe that the basic skills movement was considerably misunderstood by many people. It isn't really useful to be able to multiply unless you know **when** to use multiplication. I think we educators are recognizing the importance of applications.

R.S. What we are saying here, it seems to me, is that a problem faced in the textbook of the past is not the problem solving that society is saying we have to help children do. We must help them to develop various strategies. Does your TOPS (Techniques of Problem Solving) help do this?

C.G. There are two reasons that motivated us to develop the TOPS program. First, the National Science Foundation studies showed that children are not interested in studying mathemaics and feel that it is very limited in scope. Teachers, too, are disinterested in teaching mathematics. Thus we felt a need to make our curriculum more interesting; to get people involved in mathematics, using mathematics, appreciating mathematics, and learning mathematics. The second reason for our program dealt with the paucity of problem solving material in textbooks. George found that in most grade 3-8 textbooks, about 7% of the books is devoted to the solution of story problems; 93% is devoted to skill or computation development. Further, in the 7% dealing with problem solving, the majority of lessons are simply exercises in prose. The title of the page might be "Using Addition" and the student simply has to extract the numeric data and add. There is no need to identify relevant data or the appropriate computational procedure. The decision making aspect of problem solving is not present! So we wanted to find a way to develop alternatives to the traditional textbook approach.

G.I. Students need to be exposed to many different types of problems. Because of individual differences in abilities and skill levels, teachers need a bank of problems not only of different types, but of varying difficulty. Card decks allow for problem variety and flexibility. Several students can use one card deck simultaneously, selecting those problems of interest and of appropriate difficulty. At each grade level (3-9 and secondary), the teacher has 200 problems. On a given day the teacher may select a set of 25 problems that were designed to match or complement the curriculum. Within those twenty-five problems, there may be problems appropriate for the gifted as well as problems for the slow learner. In each deck the cards are separated into three difficulty levels.

R.S. Is this going to meet the need when pupils come to the National Educational Assessment again in 1985?

C.G. We certainly hope these materials will provide students with experiences that are necessary for becoming better problem solvers, experiences that are not present in sufficient quantity in textbooks. We have identified five different skills or strategies that are important for children to experience. The first is to identify relevant data. Frequently data necessary to solve a problem is not presented directly in the problem statement but must be found in a drawing, in a graph, or in a table. Or perhaps, the data must be obtained from an outside resource such as a map or a reference book. The other strategies include guess and check, data organization, deductive and inductive logic, and of course, choosing the correct operation or operations.

G.I. These 1,600 cards are only the beginning of the problem solving materials that we are developing. Currently, we are designing instructional books. We designed the cards first in response to teacher requests for more problems.

C.G. And, it has been demonstrated that the more problems students solve, and the greater variety of problems they experience, the better problem solvers they become.

G.I. Let met tell you too a little about a study we did last year that involved problem cards with 200 classrooms in the state of Iowa. We found that in Grades 5 and 6 statistically significant positive changes came about after four weeks of activity with problem cards and with materials to support them. I think that cards used on a much broader base, for a much longer period of time, and at different grade levels, will show significant change in results as measured by the National Assessment.

R.S. Are there significant changes in the teaching of problem solving?

G.I. Well, certainly in the past we have tried to develop one or two strategies for problem solving. In fact, for a while, the equation was the principle method for solving problems. Teachers are now beginning to recognize that there are at least a half dozen viable problem solving methods. We as educators have to take responsibility for developing those methods. It is fine to think that students might draw a diagram or a picture to solve a problem. But unless somebody does build that approach into a program and encourages the student to use it, then it will not become an effective technique. That is also true for other strategies such as guess and test, or the use of logic. The fact that we are interested in increasing the set of problem solving tasks or skills that we teach students to use to solve problems is a very significant change.

C.G. We are also seeing changes in content in terms of a greater number of applications of mathematics in the biological sciences, in the physical sciences, in medicine, art and music.

R.S. Problem solving certainly goes beyond the discipline of mathematics. The idea of systematic inquiry, of decision making, comes up in a number of subject disciplines. Let's talk for a while about the characteristics of a good problem solver. It seems to me that in addition to knowing different skills and strategies, a good problem solver must be inquisitive and persistent.

G.I. Yes, in fact, Carole has identified the ability to stick to a problem long enough to get the problem solved as a characteristic of the gifted student in mathematics.

C.G. Studying the gifted we found that the qualities or characteristics of the gifted parallel the characteristics or qualities of a good problem solver. And, it is true that gifted children exhibit persistence of task. In fact, to identify gifted students, we look not only for high levels of creativity and intelligence but also for persistence in task solution. Persistence of task is a very interesting phenomenon. When working with students at the other of the spectrum, the disabled learners, we found that many of these children could not seem to deal with any kind of symbol manipulation - whether it be language or mathematical symbols - because of what appeared to be lack of persistence or short attention spans. However, when these very same children were involved in free play, they exhibited lengthy attention spans and great perseverance. If students are involved and enjoy what they are doing, if they can relate to the situation, they will "stick-to-it" for longer periods of time.

R.S. Surely we are now talking about the affective domain and how one does generate a good atmosphere in the classroom. Traditionally, problem solving has not been a particularly happy or successful scene.

G.I. Materials that allow for choice and allow for involvement make a lot of difference in attitudes.

R.S. Then, the key ingredient is the classroom as a workshop, a classroom where children do things other than just pencil-paper basic computation.

G.I. Absolutely.

C.G. We have used TOPS cards in many classrooms. Student select the problems **they** would **like** to solve. Since the answers to many of the problems are not immediately available by the direct application of some computational algorithm, the students have to decide what to do, then try out their plan. If it doesn't work, they have to try an alternative approach. Problem solving requires "time for thinking." The teacher has to wait to allow the students to do much of the exploration on their own. That "waiting" time is really foreign to many teachers who for many years have used a lecture approaeh and intervened frequently in order to elicit student responses to questions.

R.S. It makes sense that we rededicate ourselves to the view that problem solving be at the centre of our mathematics program because decision making is critical in all life. Thank you George and Carole.

MATHWAYS 6 PAGE 38

Main Street School has 18 rooms with an average_____ 28 students per room. _____ many students attend the _____?

Answer: Of How School.



ALMOST ROOTS AND ALMOST FACTORS

C.W. Zeller and M.L. Kestell, Morning Star Secondary School, Mississauga



The grade 12 program in Ontario schools lends itself well to the development of an integrated approach to the teaching of number theory, algebra, and geometry.

In future, when programming has a much stronger hold on our lives, number theory may, for a while, take over the curriculum as set theory did ten years ago. Even though this may not be a recommended course of action, it is difficult to see a way to avoid it. The electronic revolution is upon us. We must stop sticking our heads in the sand and hoping that it will go away. The hand-held calculator and the microprocessor have already had a greater impact on our lives than did the typewriter and the printing press. It is imperative for us to learn to use the microprocessor and the calculator as tools just as humans once had to learn to use and later appreciate paper. The two cases are part of the same evolutionary process.

At Morning Star we are attempting to get in step pedagogically with this technological revolution. Our grade 12 program represents our best attempt. It is a program which has "grown up" with the students. People currently enrolled in advanced mathematics at this level have been using scientific calculators with one memory pod, daily, for two years. It has been the desire of the members of the department to teach students to use all of the capacities of their own machines effectively. In order to do this, number theory reasoning sometimes finds its way into solutions. Students learn, intuitively, the methods of numerical analysis. According to the Van Nostrand Reinhold Concise Encyclopedia of Mathematics, "In numerical mathematics all statements must be arrived at with the aid of rational numbers and as a rule, for example, when a digital computer is used, with only finitely many. In numerical analysis the formulation of a procedure for the solution of a mathematical problem therefore requires setting up a model, . . ." Watch a student, who has mastered her machine, work through some problems and you will see procedures being formulated! In a time when "mathematics is becoming the Latin of the day" surely we cannot harm students by encouraging its further study.

Our grade 12 course could be called a "Pre-Calculus Mathematics" program. It concentrates on the examination of:

a) quadratic functions - their transformations, roots, and the nature of their roots,
b) trigonometric functions - their properties, transformations, and identities,
c) exponential functions - their uniqueness with respect to operations and their transformations, and
d) combinations of these types of functions and further a general study of polynomials.

The course covers the traditional analytic geometry topics connected with such studies by examining, in detail, the interaction of the algebra and the geometry. The uniqueness comes in the integration of those with numerical methods.

To get to the heart of the issue, let us consider $p(x) = x^3 - 6x^2 - 2x + 1$. Find the roots of $p(x)$.

In nested form $p(x) = x(x^2 - 6x - 2) + 1$

$= x(x(x - 6) - 2) + 1$

$= x - 6]\ x - 2]\ x + 1$

where] means, multiply the entire quantity to the left of the sign by the single next value.

Now, observe $p(0) = 1$ and $p(1) = -6$. Thus the root ,x, lies between 0 and 1, or $x \in (0,1)$.

So, $x_0 = 0.5$ (guess) $p(x_0) = -1.375$. so, $x \in (0.0.5)$

$x_1 = 0.25$ $p(x_1) = 0.1406$ $x \in (0.25, 0.5)$

$x_2 = 0.375$	$p(x_2) = -0.54$	$x \in (0.25, 0.375)$
$x_3 = 0.31$	$p(x_3) = -0.18$	$x \in (0.25, 0.31)$
$x_4 = 0.28$	$p(x_4) = -0.014$	$x \in (0.25, 0.28)$
$x_5 = 0.266$	$p(x_5) = 0.064$	$x \in (0.266, 0.28)$
$x_6 = 0.273$	$p(x_6) = 0.028$	$x \in (0.273, 0.28)$
$x_7 = 0.2765$	$p(x_7) = 0.009$	$x \in (0.2765, 0.28)$
$x_8 = 0.285$	$p(x_8) = -0.034$	

$\therefore$ $x = 0.28$ (to 2 decimal places) as $|p(x_7)| < |p(x_8)|$ suggesting the second decimal place will be an 8.

The finger format for the solution goes as follows:

Step 1. 0.05 [STO]

Step 2. [RCL] [-] 6 [X] [RCL] [-] 2 [X] [RCL] [+] 1 [=]

Check for the appropriate interval, then continue.

Step 3. [RCL] [+] p [-] 2 [=] [STO] [p, the endpoint of the new interval]

Repeat steps 2 and 3 until appropriate accuracy is achieved.

Next we use x - 0.28 as an "almost" factor, and by long division, find the remaining "almost" factor.

$$
\begin{array}{r}
x^2 - 5.72x - 3.6016 \\
x - 0.28 \;\overline{)\; x^3 - 6x^2 - 2x + 1} \\
\underline{x^3 - 0.28x^2} \\
-5.72x^2 - 2x \\
\underline{-5.72x^2 + 1.6016x} \\
-3.6016x + 1 \\
\underline{-3.6016x + 1.008448} \\
-0.008448
\end{array}
$$

So, $p(x) = (x - 0.28)(x^2 - 5.72x - 3.6016) - 0.008448$. To approximate remaining roots, set $x^2 - 5.72x - 3.6016 = 0$ and use the quadratic formula.

$$x = \frac{5.72 \pm \sqrt{(-5.72)^2 - 4(1)(-3.6016)}}{2}$$

$$= \frac{5.72 \pm \sqrt{47.1248}}{2}$$

$x = 6.29$ or $x = -0.57$

So, $p(x) \simeq 0$ for $x = -0.57$, or $x = -0.28$, or $x = 6.29$.

At an early stage, we reduce the notation of long division by introducing synthetic division. A reduction process that can be used to explain is shown:

Refer to the division,

$x - 0.28 \;\overline{)\; x^3 - 6x^2 - 2x + 1}$

and notice the repetition involved in the recording. Reduce the repetition by writing only coefficients.

$$
\begin{array}{r}
1 - 5.72 - 3.6016 \\
1 - 0.28 \;\overline{)\; 1 - 6 - 2 + 1} \\
\underline{1 - 0.28} \\
-5.72 - 2 \\
\underline{-5.72 + 1.6016} \\
-3.6016 + 1 \\
\underline{-3.6016 + 1.008} \\
-0.008
\end{array}
$$

And reduce again by omitting repetition to show,

$$-0.28 \quad \begin{array}{|cccc|} \hline 1 & -6 & -2 & +1 \\ & -0.28 & +1.6016 & +1.008 \\ \hline \end{array}$$

$$1 \quad -5.72 \quad -3.6016 \quad -0.008$$

A further sophistication can be added by taking the roots as discovered, leaving the sign the same, and adding throughout. The polished form shows:

$$\begin{array}{|cccc|} \hline 1 & -6 & -2 & +1 \\ & +0.28 & -1.6016 & -1.008 \\ \hline \end{array} \qquad (0.28)$$

$$1 \quad -5.72 \quad -3.6016 \quad -0.008$$

The root is shown to the right and the coefficients of the remaining factor, along the bottom.

After numerous computations of this "accuracy-open" algorithm, the generalization could be presented and explored.

FUTURE MEETINGS

Local	**International**
Executive - April 3	N.C.T.M. 59th Annual, 1981
Annual Convention - May 8-10	April 22-25, St. Louis
Canterbury Inn - Sarnia	N.C.T.M. 6th Annual, 1982
Leadership Conference - OAME	April 14-17, Toronto
Ramada Inn - October 1981	R.C.D.P.M.
Theme "Mathematics Across the Curriculum"	April 18-19, Hershey, Penn.

TEACHING PROBLEM SOLVING

Donald R. Woods, Cameron M. Crowe, Terrence W. Hoffman and Joseph D. Wright, McMaster University

One of the first challenges in talking about problem solving is to ensure that we are all talking about the same thing. Although many complicated definitions have been proposed we define it simply as the activity whereby a **best** value is determined for an unknown, subject to a specific set of conditions. This definition is general enough to include the following extremes in problem statements: "I have just locked my keys in my car; now what do I do?" and "Find the general solution of the following differential equation y'' + 4y = sec2x". This "problem solving" or activity that we do to answer such problem statements is, like mathematics, a subject discipline. It contains a set of skills that can be taught, yet we teachers and cognitive pyschologists are just now beginning to see the structure in that discipline. But what is the structure of that discipline? What skills does a good problem solver need? What skills should we teach? Based on our research of the past six years we now see expert problem solvers as possessing the following general skills or attributes:

1. prerequisite knowledge. (You cannot solve problems without some background information or skill at collecting the necessary information.)

2. an organized approach or strategy that is generally applicable to all types of problems.

3. an ability to create, analyze, simplify and generalize.

4. an awareness of heuristics or hints that work for their own preferred style of solving problems.

Such a broad list; you might be tempted to ask "What else could any person have?" Yet, if, in our analysis of how to improve problem solving, we omit consideration of one of these, we may find that

our efforts are in vain. For example, in a pilot project we tried to improve freshman engineering students' skills in solving problems by emphasizing the application of an organized approach or strategy. This failed to result in improvement in the students' ability until we backtracked and helped them to do a better job of organizing the background knowledge. So what can we do?

To teach problem solving requires more than assigning many problems and working many examples in class. It takes a careful selection of some long range objectives, of the problems to be used as the vehicle and of the classroom setting. Whether we are going to focus on improving a few or many skills in the host of skills needed; whether we take 5 minutes or 30 minutes a day - this is not a vital issue. What is important is that we make a conscious effort to improve problem solving skills, that we have a plan, that we build up the skills gradually in order and that we are patient with ourselves and with the students.

Where do we start? Six years ago we asked ourselves that question. We spent the intervening years observing a group of students during their four years in the Chemical Engineering program at McMaster Universty. (1)(2) From this we have developed a structured set of learning objectives (3) that we believe are as as applicable to high schools as to college or universities, as applicable to Mathematics as to Chemical Engineering. The starting objectives are:

1. to increase students' awareness of the process they use to solve problems. (4)
2. to memorize a strategy for solving problems.
3. to be able to apply the first step - define

Details of the other learning objectives are given elsewhere. (3)

To increase students' awareness of the process probably the pair method is the most useful. In this method, described by Whimbey (5) and Whimbey and Lochhead (6), one student solves aloud a 2 to 5 minute problem while the other listens and encourages the problem solver to verbalize everything, and to work carefully. The problems chosen for the medium are often mathematical ones. Bauman (7), in particular, has developed a full set of materials around very simple mathematical concepts.

Students need to be given a common strategy for solving any problems. Such a strategy provides a consistent methodology for breaking the problem solving process into subsections, and gives a common terminology that allows students and teachers to talk about the process. Having a strategy slows the problem solving process down and so helps the problem solver be more accurate and methodical, and helps problem solvers to overcome the initial panic of "how can I ever solve that problem". The strategy also helps a problem solver to locate where he or she is in the process. A common strategy is essential for group problem solving. Which of the hundred or so strategies described in the literature should we use? We have found that it does not make that much difference just so long as you work with one consistently. Ideally all teachers in the same school should agree upon one. This recommendation is not that outlandish because most of the strategies we have evaluated are essentially the same and eventually individuals will evolve their own strategy. At the beginning though, it is best to start with a common one, if for no other reason than, to have a common language.

Polya's four step strategy (8), or minor variations on it, is a good place to start: Step 1. Define; Step 2. Plan; Step 3. Do It; Step 4. Look Back. (For engineering or physics problems we use a five step strategy of Define, Explore, Plan, Do It and Look Back (9)).

Our approach in selecting a starting strategy was to ask expert problem solvers to solve a variety of problems using a detailed but general strategy of Polya that we supplied. We asked these experts to identify what they did in addition to the activities described by Polya, and to identify what they did not do. The experts solved everyday problems, freshman physics problems, engineering troubleshooting problems and practical real world engineering problems. The results were:

1. All four steps in Polya's strategy were used.

2. The steps were not always used in sequence in that often when Planning, the experts would go back to the Define step.

3. Experts did another clearly identified step part way between the define step and the plan step. During this activity they:

a. tried to solve simplified versions of the problem,
b. tried to discover what the problem **really** was about,
c. tried to discover what additional information was needed and what information was not needed,
d. tried to see how the problem responded when we "shoved" it one way or another,
e. tried to discover if the problem as stated was the real problem,
f. tried to discover what was **really** important in the problem.

We call this activity the "Think About It" or "Explore" step. This separate step does four important things.

a. identifies Step 1, Define, as an activity demanding only analytical thinking. The objective is to **understand the problem as it is given to us.** We are not to identify the **real** problem. We are not to try to see what all is involved. Understanding the given problem is a simply activity that is easy to learn; it also calms down the problem solver and helps to overcome the initial panic.

b. legalizes in a problem solver's mind the mental tasks of "playing around with the problem" before really settling in to solve it. It focusses attention on this activity as being important.

c. separates the types of skills needed. Exploring the problem combines creative, analytical, generalizing and simplifying activities. The activities are complex and differ greatly from individual to individual, whereas defining the problem requires analytical skill only and all problem solvers follow the same procedures.

d. separates known tactics from exploring ideas. Developing the Plan can be done by following three or four well-known tactics. Which ones are used depends more on the type of problem than on the personal preference of the problem solver.

The results of this analysis are given in Table 1. The general steps are always valid. The details (given as a, b, c ...) may not be universally applicable and are given to provide a flavour of the activity being done. The basic five steps are:

Define
Explore
Plan
Do It
Look Back

Here are some McMaster students' comments about this choice of general strategy:

> "I found it very confusing at the beginning of the year when each of the professors gave advice about problem solving. But they all fit into Dr. Woods' outline of problem solving". [sic: the above 5 step strategy]

> "All the professors when they present lectures have their own method of problem solving. You see it in Chemistry; the professor will give you a type of outline every time he approaches a problem. He'll say 'write down the electron equations and the mass balance equation'. He goes through a system. Each professor takes some problem solving method and applies it to his own subject so that when we attend that class we see each professor on his own. It's very nice learning a method [sic: the above 5 step strategy] so that you can see how the professor is doing and what he is doing in general. That way we have something to refer to so that we known where he is".

TABLE 1 - The McMaster Five-Step Strategy

1. DEFINE: Understand the stated problem.
   a. Understand the words, drawings.
   b. Identify the unknown or state objective.
   c. Draw a diagram or visual representation.
   d. Isolate the system and identify inputs and outputs - as stated in the problem.
   e. Identify the knowns: input, outputs, identified laws, assumptions.

f. Identify the stated constraints and list the inferred constraints.
g. Identify the stated criteria and list the inferred criteria.

2. EXPLORE
   a. Recall past related problems or experience.
   b. Recall pertinent theory and fundamentals.
   c. Identify tentative pertinent relationships between inputs, outputs and unknowns.
   d. Hypothesize, visualize, idealize, generalize and simplify.
   e. Gain overall appreciation of the problem by solving an extremely simple version of it, and determine "about" what the answer is.
   f. Translate problem into another form.
   g. Discover what the real problem is, i.e. what is the real unknown. Consider both short time and long time implications.
   h. Identify meaningful criteria.
   i. Identify real constraints.
   j. Choose a basis or reference set of conditions.
   k. Collect missing information.
   l. Guess the answer or result.
   m. Select useful heuristics and hints.

3. PLAN
   a. Generate alternative ways that the objective can be achieved. (Consider both technical (content, what?) and mathematical (process, how?) aspects).
   b. Identify problem type and select among working backwards, subproblem, heuristic or contradiction tactics.
   c. Map out the procedure (algorithm) to be used.
   d. Assemble resources needed.

4. DO IT
   a. Follow the procedure developed under the plan; use the resources available.
   b. Calculate criterion functions.
   c. Compare alternatives and select the best.

5. LOOK BACK
   a. Check that the solution is blunder free.
   b. Check reasonableness of results.
   c. Check that criteria and constraints are satisfied.
   d. Check procedure.
   e. Study related problems (every day, desert island and technical/engineering).
   f. Identify experience factors to be memorized.
   g. Identify what has been learned about problem solving from the problem just solved.
   h. Communicate results.

Whether you start with Polya's four-step or the McMaster five-step strategy is not that vital. What is important is that you start with one; cast all your examples into that format, plaster the walls with the strategy, display it, talk about it, require that the students use it and show that they have used it; ask them to recall the strategy and its components.

Resources Available

Some may be interested in ideas about how to overcome other challenges to teaching problem solving in the context of mathematics. In this section, we list resources we think you might find useful. Some of this material has not been published yet, and you have to contact the appropriate author listed below directly.

Topic	Resource
Heuristics on how to solve mathematical problems.	1. Schoenfeld's papers (10). Although Polya (8) introduced the idea of heuristics, Schoenfeld continues to pioneer the actual in-class teaching of heuristics especially as they apply to mathematics. His papers should be required reading.

2. Jim Kaput (11) has written an interesting article emphasizing the difficulty students have conceptualizing and relating symbols and actions.

3. Rutherford Aris (12) lists 12 heuristics and illustrates how these apply to college or university level mathematics.

Translating concepts into visual, symbolic, word or graphical forms.

The HELP(P) physics program (13) in the UK has developed several units to improve a student's ability to translate ideas from one form to another.

The very active group of Clement, Lochhead and co-workers (14) at the University of Massachusetts have zeroed in on one particlar difficulty of translating words into equations. Much of their work concentrates on discovering difficulties; not all the details of how to overcome the difficulties have been worked out. George Monk (15) has developed materials in this area.

Strategies

Strategies specific to solving different types of mathematical problems are given by Jenson and Jeffries (16).

Explore/Plan

Himmelblau and Bischoff (17) gives a useful classification of mathematical methods that are available.

Planning

Polya plots are an intriguing way (9) for students to see the structure in sets of equations. For more details on their use see Brown (18).

General

Rubinstein's (19) book gives a wonderful overview of logic, computers, probability, modeling, probabalistic models, decision-making models and other applications of mathematics in the context of solving real life problems. William's (20) book explores game theory. Adam's (21) book is fun for creativity. Servais and Varga's book (22) is full of ideas.
For general insight into the problem solving process, the articles by Larkin (23) and Reif (24) should be consulted.

Finally, there are two newsletters devoted solely to sharing ideas on how to teach problem solving. These are "Problem Solving" from the Franklin Institute Press, P.O. Box 2266, Philadelphia, PA 19103, contact Ms. Julia Hough, and "PS News" from the Dept. of Chemical Engineering, McMaster University, Hamilton, Ontario, L8S 4L7, contact Mr. Don Woods.

Summary

Problem solving can be taught. We defined problem solving, suggested that we start improving students' skill by increasing their awareness of the process and then by using a strategy consistently. Ideas were given on how to do this. Many references were given for materials and articles for improving other aspects to problem solving.

Acknowledgement

Our work at McMaster is financially supported by an initial grant from the Ontario Universities Program for Instructional Development and by grants from McMaster University.

Editor's Note: References are available from either the authors or myself.

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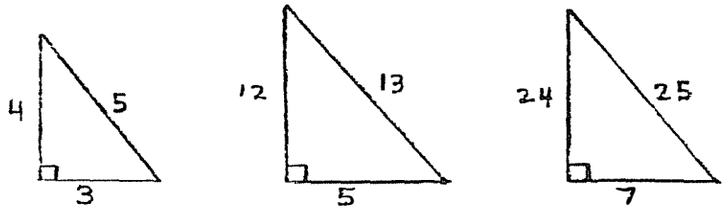
The average traffic flow on a highway is 341 _____ a day. How many _____ will drive on the _____ in a 30 d _____?

Answers: vehicles, vehicles, highway, month.

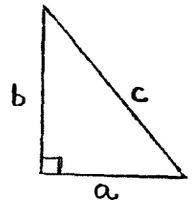
AN INVESTIGATION OF RIGHT-ANGLED TRIANGES

Paul Crones, Port Colborne Secondary School

Many mathematics students are fairly familiar with the right-angled triangles having sides;



If we then drew as the general case of the above

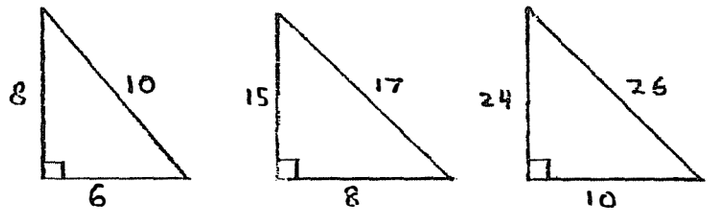


where a, b, c $\in$ N
$a < b$

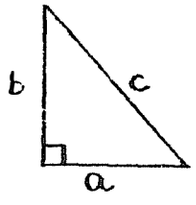
and asked them to list as generalizations any patterns they saw, some responses might be

(i) $c - b = 1$
(ii) a increases by 2 each time
(iii) a is always odd, b is always even
(iv) $a^2 = b + c$
(v) a, b, c have no factors in common (we will now identify such an a, b, c as a primitive Pythagorean triple)

So as not to leave their students with the idea that these generalizations are always true, some teachers might now challenge them with the following triangles:



and again ask for generalizations based on



a, b, c, $\in$ N
$a < b$

We would now certainly obtain

(i) $c - b = 2$
(ii) a increases by 2 each time
(iii) a, b are both even or a is even and b is odd. We note further that when a, b are both even we are

dealing with a multiple of a previous primitive Pythagorean triple. For example, 6, 8, 10 is simply twice 3, 4, 5. When a is even and b is odd, however, we have a "new" primitive Pythagorean triple not included in our original list

(iv) $a2 \neq b + c$ **but** $a^2 = 2(b + c)$

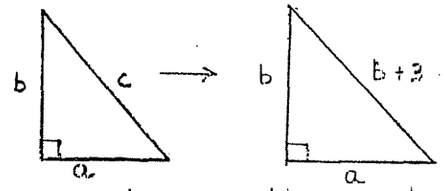
(v) some a, b, c, have factors in common and are multiples of previous cases having no factors in common. Others have no factors in common and are PPT (primitive pythagorean triples). We might summarize our results thus far in the following table:

c - b = 1		c - b = 2
3, 4, 5	x 2 →	6, 8, 10
		8, 15, 17
5, 12, 13	x 2 →	10, 24, 26
		12, 35, 37
7, 24, 25	x 2 →	14, 48, 50
		16, 63, 65
⋮		⋮

Every triple above is a PPT.

Every **second** triple above is a PPT

What predictions can we make for the case where c - b = 3? Will every **third** one be a PPT?



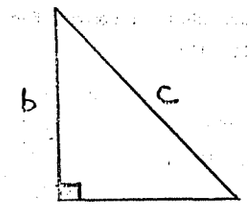
Hence $a^2 + b^2 = (b + 3)^2$
$= b^2 + 6b + 9$
$a^2 = 6b + 9$
$a = \sqrt{6b + 9}$

If b	6b	6b + 9	$a = \sqrt{6b + 9}$	b + 3	Triple	PPT
= 1	6	15	—	—	—	—
2	12	21	—	—	—	—
3	18	27	—	—	—	—
⋮	⋮	⋮	⋮ —	⋮ —	⋮ —	⋮ —
12	72	81	9	15	9, 12, 15	No → 3(3,4,5)
⋮ —	⋮ —	⋮ —	⋮ —	⋮ —	⋮ —	⋮ —
36	216	225	15	39	15, 36, 39	No → 3(5, 12, 13)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
72	432	441	21	75	21, 72, 75	No → 3(7, 24, 25)

Are there **any** PPT's having c - b = 3?
If not, then the question must arise "For what value of c - b will the next "new" PPT occur? A development similar to the foregoing will find none for c - b = 4, 5, 6, and 7. But for c - b = 8 we find

a, b, c		
20, 21, 29		
24, 32, 40	= 8(3, 4, 5)	c - b = 1 case
28, 45, 53		
32, 60, 68	= 4(8, 15, 17)	c - b = 2 case
36, 77, 85		
40, 96, 104	= 8(5, 12, 13)	c - b = 1 case
44, 117, 125		
48, 140, 148	= (12, 35, 37)	c - b = 2 case

Is it possible to discover for what values of c - b we will obtain further PPT?



$$a^2 + b^2 = c^2$$
$$b^2 = c^2 - a^2$$
$$bb = (c - a)(c + a)$$
$$\frac{b}{c + a} = \frac{c - a}{b}$$

Let $\frac{b}{c + a} = \frac{u}{v}$ and $\frac{c - a}{b} = \frac{u}{v}$

$bv = uc + ua$ $\qquad vc - av = ub$

$ua - bv = -uc$ (1) $\qquad av + ub = vc$ (2)

Solving for a: (1) u $\qquad u^2a - ubv = -u^2c$ (3)

(2) v $\qquad v^2a + ubv = v^2c$ (4)

(3) + (4) $\qquad v^2a + u^2a = v^2c - u^2c$

$$a(v^2 + u^2) = c(v^2 - u^2)$$

$$\boxed{a = \frac{c(v^2 - u^2)}{v^2 + u^2}}$$

Solving for b: (1) v $\qquad uva - bv^2 = -uvc$ (5)

(2) u $\qquad uva + u^2b = uvc$ (6)

(6) - (5) $\qquad u^2b + bv^2 = 2uvc$

$$b(v^2 + u^2) = 2uvc$$

$$\boxed{b = \frac{2uvc}{v^2 + u^2}}$$

If we set $\frac{c}{v^2 + u^2} = k$ then $\qquad a = k(v^2 - u^2)$

$b = k(2uv)$

$c = k(v^2 + u^2)$

Hence Pythagorean triples a,b,c are all proportional to $v^2 - u^2$, $2uv$, $v^2 + u^2$, with the primitives being given by $v^2 - u^2$, $2uv$, $v^2 + u^2$, and the multiples of the primitives by $k(v^2 - u^2)$, $k(2uv)$, $k(v^2 + u^2)$.

Now for any PPT, a, b can have no common factor, for if $a = km$ and $b = kn$ then since $a^2 + b^2 = c^2$ we have $\qquad c^2 = k^2m^2 + k^2n^2$

$$= k^2(m^2 + n^2)$$

$c = k\sqrt{(m^2 + n^2)}$, so that a, b, c would have a common factor k and could not constitute a PPT.

Similarly, if u and v have a common factor P, say $u = p\alpha$ and $v = p\beta$, then since for a PPT

$a = v^2 - u^2 = p^2\beta^2 - p^2\alpha^2 = p^2(\beta^2 - \alpha^2)$

$b = 2uv = 2p\alpha p\beta = 2p^2\alpha\beta$

$c = v^2 + u^2 = p^2\beta^2 + p^2\alpha^2 = p^2(\beta^2 + \alpha^2)$

and a, b, c would have a common factor p^2, and would not be a PPT.

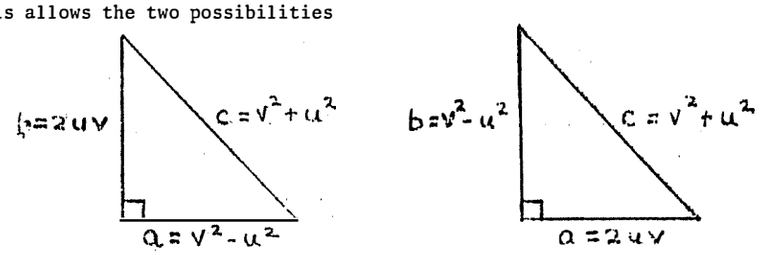
So for any PPT, u and v have no common factor. It follows immediately then that u and v cannot both be even, for they would then share a common factor of 2. But could they both be odd, say $u = 2e + 1$, $v = 2f + 1$? Then

$a = v^2 - u^2$
$= 4f^2 + 4f + 1 - 4e^2 - 4e - 1$
$= 2(2f^2 + 2f - 2e^2 - 2e)$

$b = 2uv$
$= 2(2e + 1)(2f + 1$
$= 2(4ef + 2e + 2f + 1)$

$c = v^2 + u^2$
$= 4f^2 + 4f + 1 + 4e^2 + 4e + 1$
$= 4f^2 + 4f + 4e^2 + 4e + 2$
$= 2(2f^2 + 2f + 2e^2 + 2e + 1)$

and a,b,c would have a common factor of 2 and would not be a PPT. Thus we have that all PPT are of the form $v^2 - u^2$, $2uv$, $v^2 + u^2$
where u,v have no common factor and one of u,v is even and the other odd.
But this allows the two possibilities



giving

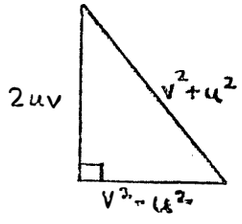
$c - b = v^2 + u^2 - 2uv$
$= (v - u)^2$

or

$c - b = v^2 + u^2 - (v^2 - u^2)$
$= 2u^2$

Hence there are only two classes of c - b, namely perfect squares and twice perfect squares. Further, if (i) v is odd, say $2g + 1$, and u is even, say $2h$, then
$v - u = 2g + 1 - 2h$ and $(v - u)^2 = 4g^2 + 1 + 4h^2 + 4g - 4h - 8gh$
$= 2(2g^2 + 2h^2 + 2g - 2h - 4gh) + 1$, which is odd, and if (ii) v is even, say $2k$, and u is odd, say $2\ell + 1$, then $v - u = 2k - 2\ell - 1$ and $(v - u)^2 = 4k^2 + 4\ell^2 + 1 - 8k\ell + 4\ell - 4k = 2(2k^2 + 2\ell^2 - 4k\ell + 2\ell - 2k) + 1$, which is odd, so that $(v - u)^2$ is always an odd perfect square.

Finally then, those values of c - b giving rise to PPT are
$c - b = 2u^2 \rightarrow 2, 8, 18, 32, 50, 72, \ldots$ and $c - b = (v - u)^2 \rightarrow 1, 9, 25, 49, 81, 121, \ldots$
where the specific PPT are



c - b = 1

u = 1, v = 2
3, 4, 5
u = 2, v = 3
5, 12, 13
u = 3, v = 4
7, 24, 25
u = 4, v = 5
9, 40, 41
u = 5, v = 6
11, 60, 61

c - b = 9

u = 4, v = 7
33, 56, 65
u = 5, v = 8
39, 80, 89

u = 7, v = 10
51, 140, 149
u = 8, v = 11
57, 176, 185
u = 10, v = 13
69, 260, 269

c - b = 25

u = 4, v = 9
65, 72, 97
u = 6, v = 11
85, 132, 157
u = 7, v = 12
95, 168, 193
u = 8, v = 13
105, 208, 233
u = 9, v = 14
115, 252, 277

c - b = 49

u = 5, v = 12
119, 120, 169

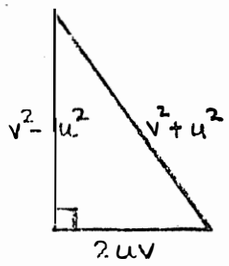
c - b = 81

u = 7, v = 16
207, 224, 305

u = 6, v = 13
133, 156, 205
u = 8, v = 15
161, 240, 289
u = 9, v = 16
175, 288, 337
u = 10, v = 17
189, 340, 389

u = 8, v = 17
225, 272, 353
u = 10, v = 19
261, 380, 461
u = 11, v = 20
279, 440, 521
u = 13, v = 22
315, 572, 653

c - b = 2



u = 1, v = 4
8, 15, 17
u = 1, v = 6
12, 35, 37
u = 1, v = 8
16, 63, 65
u = 1, v = 10
20, 99, 101
u = 1, v = 12
24, 143, 145

c - b = 8

u = 2, v = 5
20, 21, 29
u = 2, v = 7
28, 45, 53
u = 2, v = 9
36, 77, 85
u = 2, v = 11
44, 117, 125
u = 2, v = 13
52, 165, 173

c - b = 18

u = 3, v = 8
48, 55, 73
u = 3, v = 10
60, 91, 109
u = 3, v = 14
84, 187, 205
u = 3, v = 16
96, 247, 265
u = 3, v = 20
120, 391,409

c - b = 32

u = 4, v = 11
88, 105, 137
u = 4, v = 13
104, 153, 185
u = 4, v = 15
120, 209, 241
u = 4, v = 17
136, 273, 305
u = 4, v = 19
152, 345, 377

c - b = 50

u = 5, v = 14
140, 171, 221
u = 5, v = 16
160, 231, 281
u = 5, v = 18
180, 299, 349
u = 5, v = 22
220, 459, 509
u = 5, v = 24
240, 551, 601

Remember that u, v have no common factor and that one is even and the other odd.

This kind of problem can give students a very concrete example of what mathematical investigation is all about, and lets them move through the stages of examining specific cases for patterns, conjecturing hypotheses, and then attempting to prove their assertions in mathematical form. At the same time, the teacher has a gold mine of topics available in this one problem, only **some** of which are the Pythagorean theorem, inequalities, common factoring, substitution, common multiples, tables of values, square roots, manipulation of formulae, logic, solving simultaneous equations in two unknowns, proportion, and number theory. It provided me with countless hours of enjoyment, for which I am greatly in debt to the expertise and advise of my brother, Peter Cronin, of Leaside High School, and two of my colleagues, Horst Gyetko and Dave McPherson, of Port Colborne High School.

PERCENTAGE ALGEBRA



Norman M. Rice, Queen's University, Kingston

If a merchant increases the price of an article by 20%, and then increases it a further 10%, the combined increase is not 30%. In this note we look at the algebra involved in such combinations of percentages. It is closely related to addition, but is not quite the same. As we will see, it provides a simple and concrete but nontrivial example of an algebraic group operation.

First answer for yourself the following questions. No pencils or paper please; just off the top of your head. I asked these questions in the coffee lounge of the Queen's Mathematics and Statistics Department a few weeks ago and got a startling variety of off-the-cuff answers.

1. If a merchant increases the price of an article by 20% and then by a further 10% what is the combined increase? more than 30% less than 30%?

2. If a second merchant increases the price of a similar article by 10% and then by a further 20% is this combined increase the same as the first merchant's? more than the first merchant's? less than the first merchant's?

3. If the first merchant increases the original price of the article by 20% and then decreases the new price by 20%, is the final price the same as the original price? more than the original price? less than the original price? By what percentage should the merchant decrease the new price to bring it back down to the original price? 20%? more than 20%? less than 20%?

Well, arithmetic shows that if the original price of the article was P, then the price after the first merchant's repeated increases is $P(1 + .20)(1 + .10) = P(1.32)$, representing a 32% combined increase. The second merchant's final price after increases of 10% and 20% is $P(1 + .10)(1 + .20) = P(1.32)$, also representing a 32% combined increase. (Many people seem surprised at first that the two combined increases are equal.)

Next, a 20% reduction following a 20% increase yields a final price of $P(1 + .20)(1 - .20)$ which is $P(.96)$, 4% less than the original price. To get back to the original price after a 20% increase it turns out as we'll see that you need a 16⅔% reduction.

More generally the three questions above ask:

1. If an increase of a% is followed by an increase of b%, what is the combined increase? **We will denote this combined increase by a⊕b%.**

2. Is $a \oplus b = b \oplus a$? i.e. Is an a% increase followed by a b% increase the same as a b% increase followed by an a% increase?

3. For any given a, does $a \oplus (-a) = 0$? i.e. is an a% increase followed by an a% decrease the same as no change at all? i.e. a 0% increase? If not, for what change b% will $a \oplus b = 0$?

Once the questions are asked in this general form it is easy to answer them:

1. For an article with an original price of P, the final price after increases of a% and b% will be

$$P\left(1+\frac{a}{100}\right)\left(1+\frac{b}{100}\right) = P\left(1+\frac{a}{100}+\frac{b}{100}+\frac{ab}{10000}\right)$$

This is supposed to be the same as the original price increased by a⊕b% which is

$$P\left(1+\frac{a \oplus b}{100}\right) .$$

Thus $$\frac{a \oplus b}{100} = \frac{a}{100} + \frac{b}{100} + \frac{ab}{10000}$$

and so we get our first formula: $$a \oplus b = a+b+\frac{ab}{100} \quad \cdots\cdots\cdots\cdots \quad (1)$$

So for example, $$20 \oplus 10 = 20+10+\frac{200}{100} = 32 ,$$

and $$20 \oplus (-20) = 20-20-\frac{400}{100} = -4$$ as we saw earlier.

2. The answer to the second question (about commutivity) is "Yes", since it is clear from the general formula (1) that $a \oplus b = b \oplus a$.

3. For the third question, we see from formula (1) that $a \oplus (-a) = -a^2/100$, which will never be zero unless $a = 0$. However, we can use formula (1) to find a number b for which $a \oplus b = 0$; indeed from $0 = a \oplus b = a + b + ab/100 = a + b(1 + a/100)$ we conclude that $b = -a/(1 + a/100) = -100a/(100 + a)$. If we **denote this number** b **by** ⊖a we have

$a \oplus (\ominus a) = 0$

$$\ominus a = \frac{-100a}{100+a}$$

for example $\ominus 20 = \frac{-2000}{120} = -16\frac{2}{3}$ as claimed earlier; and it is easy to verify that $20 \oplus (-16\frac{2}{3}) = 0$.

Considering the whole situation a bit more abstractly, we now have a **binary operation** $\oplus$, **a neutral or identity element** 0(since $a \oplus o = a$), and for every element a an **inverse** $\ominus a$ such that $a \oplus (\ominus a) = 0$. The one remaining requirement that the system form a **group** is that the operation $\oplus$ be **associative,** i.e. that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$. This can be verified in a straightforward if slightly tedious manner using equation (1), or it can be seen immediately be noting that both $a \oplus (b \oplus c)$ and $(a \oplus b) \oplus c$ represent a combined increase of a% followed by b% followed by c%.

We now have a group, and in fact it is a **commutative group** since $a \oplus b = b \oplus a$. But there is one last matter to be resolved: just what are the **elements of the group?** (Logically this is the question that should have been resolved first if we were defining a group; but we are not defining a group, we are discovering one.) Now, since you can increase a price by any percentage (20%, 100%, 250%, etc.) and can decrease it by any percentage up to 100%, it would appear that the elements of the group should be all numbers greater than or equal to -100. But there is a slight difficulty, namely -100 does not have an inverse in the group because if a price is lowered by 100% (i.e. to 0) there is no percentage increase that will bring it back up to the original price. (One also notices that formula(2) for $\ominus a$ breaks down for a = -100.) Consequently we must restrict the elements of the group to be all **numbers greater than -100.**

To summarize, we now see that percentage arithmetic forms a commutative group G whose elements are the real numbers $a > -100$, and whose operation $\oplus$ is defined by

$$a \oplus b = a + b + \frac{ab}{100} \quad ;$$

The neutral (identity) element of the group is 0, and the inverse $\ominus a$ of any element a can be calculated by

$$\ominus a = \frac{-100a}{100+a}. \quad \ldots\ldots\ldots\ldots \quad (2)$$

* * * * * * * * * * * * * *

Finally, here are some exercices involving equations (1) and (2).

1. Verify that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

2. Verify that $\ominus(\ominus a) = a$.

3. Defining "subtraction" $a \ominus b$ as $a \ominus b = a \oplus (\ominus b)$, show that
$$a \ominus b = \frac{100(a-b)}{100+b}.$$
4. Show that $a \ominus b\%$ is the percentage change that must follow a change of b% to yield a net change of a%. i.e. verify that $b \oplus (a \ominus b) = a$.

5. Verify that (i) $\ominus(a \oplus b) = (\ominus a) \oplus (\ominus b)$
(ii) $\ominus(a \ominus b) = (\ominus a) \oplus b = b \ominus a$

and interpret these equations in terms of percentage changes.

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PROBLEM SOLVING USING TRANSFORMATION

by D. Moalem, N.S.W. Australia

Editor's Note: Many of us will remember Dan from his tour about the province with George Scroggie and talks at the Hamilton Conference. Dan suggests other solutions are possible and would be quite happy to hear from you.

Dear Bob:

I have seen the completed Intermediate Curriculum Guidelines and must congratulate your curriculum editorial panel for an excellent effort. It has many constructive features most important of which are (1) flexibility sot that it can be adapted to a variety of ability levels

(ii) helpful hints for teachers

It is a most forward looking programme and should be a pace-setter in your part of the world.

I have received several enquiries about the geometry section of the guidelines particularly on testing this subject for the age group 14 - 16 years. As you have asked me for further contributions to your gazette on transformations, perhaps my reply should concentrate on this facet. Let me first emphasize that transformations should not replace Euclidean Geometry but rather supplement it together with analytic, vecton and matrix geometry. The emphasis in the eighties is on problem solving and the wider the range of skills and strategies available to students, the better will be their capacity to solve problems.

Below are a selection of questions covering a range of types useful for testing. They allow students to mix the approaches when solving. The solutions given have a transformations slant but other solutions are possible.

1.

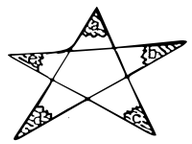


Fig. 1(a)

The figure shows a pentagram. What is the sum of the angles a,b,c,d and e?

Solution. One of the simplest solutions is obtained by translating each of the angles to a point O as shown in fig. 1(b). Using the vertically opposite angles formed we get $2(a+b+c+d+e) = 360°$.

$\therefore\ a+b+c+d+e = 180°$.

Combinations of translations, rotations and reflections are most useful in computing the angles in plane figures.

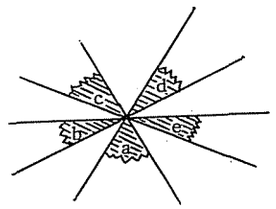


Fig. 1(b)

2.

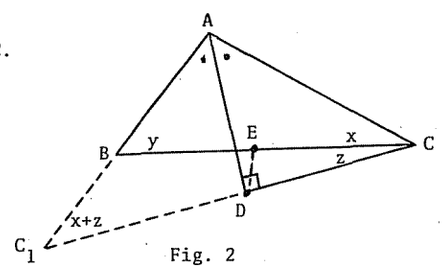


Fig. 2

In fig. 2, ΔABC has AD bisecting $B\hat{A}C$. $CD \perp AD$ and E is the midpoint of BC.

Show that i) $DE = \frac{1}{2}(AC - AB)$

ii) $A\hat{C}D = \frac{1}{2}(A\hat{B}C + A\hat{C}B)$

iii) $B\hat{C}D = \frac{1}{2}(A\hat{B}C - A\hat{C}B)$

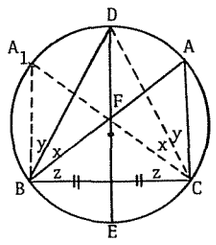
Solution: An angle bisector suggests a reflection in this bisector of ΔADC to give ΔAC_1D.

(i) D,E are midpoints of C_1C and BC. $\therefore\ DE = \frac{1}{2}BC_1 = \frac{1}{2}(AC_1 - AB) = \frac{1}{2}(AC - AB)$
(midpoint theorem + invariance of length under reflection $AC = AC_1$).

(ii) $\frac{1}{2}(A\hat{B}C + A\hat{C}B) = \frac{1}{2}(y + x) = \frac{1}{2}(x + z + z + x) = x + z = A\hat{C}D$. [Note: ΔACC_1 is ososceles and $A\hat{B}C = B\hat{C}_1C + B\hat{C}C_1$).]

(iii) $\frac{1}{2}(ABC - A\hat{C}B) = \frac{1}{2}(y - x) = \frac{1}{2}(x + z + z - x) = z = B\hat{C}D$. [Note: the mixture used of transformation, Euclidean and algebraic techniques.]

3.



In fig. 3, ΔABC is inscribed in the circle. DE bisects BC at right angles. Show that

i) $D\hat{B}A = \frac{1}{2}(A\hat{C}B - A\hat{B}C)$

ii) The bisector of $B\hat{A}C$ passes through E

iii) DC bisects $A\hat{C}F$

iv) EC bisects $A\hat{C}F$ externally.

Solution: Reflect $D\hat{B}A$ in diameter DE to give $D\hat{C}A_1$.

(i) $\frac{1}{2}(A\hat{C}B - A\hat{B}C) = \frac{1}{2}(x + y + z - z) = \frac{1}{2}(x + y) = \frac{1}{2}(2x) = x$. [$D\hat{B}A = D\hat{C}A$ subtended to the circumference by the same arc AD. $\therefore$ $x = y$.]

(ii) arc BE = arc CE (images of one another when reflected in line DE).
$\therefore$ $B\hat{A}E = C\hat{A}E$ (subtended to the circumference by equal arcs)
$\therefore$ AE bisects $B\hat{A}C$.

(iii) DC bisects $A\hat{C}F$ since $x = y$.

(iv) $D\hat{C}E = 90°$ (angle in a semi-circle). $\therefore$ EC is the external bisector of $A\hat{C}F$. The internal and external bisectors are perpendicular to each other.

4.

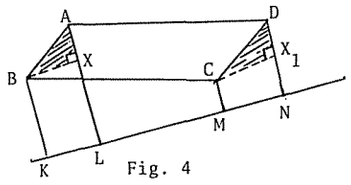
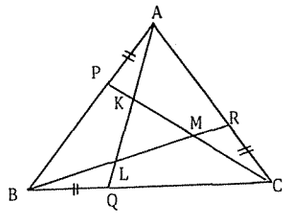


Fig. 4

In fig. 4, ABCD is a parallelogram. KLMN is any line, BK, AL, CM and DN are perpendiculars to this line. Show that $BK + DN = AL + CM$.

Solution: Translate right angled triangle ABX by $\vec{BC}$ to give ΔDCX_1. $\therefore$ $AX = DX_1$ (length preserved under translation).
Now BXLK and CX_1NM are rectangles.
$\therefore$ $BK = XL$, $CM = X_1N$ (opposite sides of a rectangle)
$\therefore$ $BK + DN = XL + XN_1 + DX_1 = XL + CM + AX$
$= (XL + AX) + CM = AL + CM$.

5.



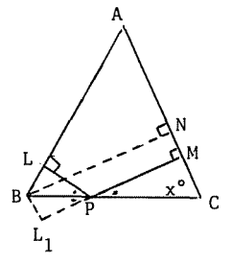
Given: ΔABC is equilateral. P,Q,R are points on the sides such that $AP = BQ = CR$. AQ, BR, CP form ΔKLM.
Show that ΔPQR and ΔKLM are equilateral.
Solution: Under successive 120° rotations about the circumcentre of ΔABC.
$A \to B \to C \to A$. $\therefore$ $AB \to BC \to CA \to AB$.
Any subset of AB maps onto a congruent subset of BC.
$\therefore$ $kAB \to kBC \to kCA$ (k = positive rational number)
$\therefore$ $AP \to BQ \to CR$.

But we know that $A \to B \to C \to A$. $\therefore$ $P \to Q \to R \to P$. $\therefore$ ΔPQR has rotational symmetry of order 3. $\therefore$ ΔPQR is equilateral.

(i) $\therefore$ AQ → BR → CP → AQ under this rotation.
$\therefore$ AQ ∩ BR → BR ∩ CP → CP ∩ AQ → AQ ∩ BR. $\therefore$ L → M → K → L. $\therefore$ ΔKLM is equilateral (rotation symmetry of order 3).

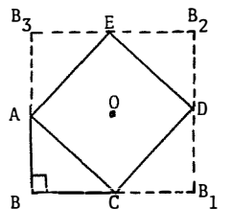
Alternately (ii) Because AQ → BR → CP → AQ under the 120° rotation, then the angles between the pairs of lines AQ and BR, BR and CP, CP and AR are each 120°. (Using the fact that the angle between a line segment and its image under rotation is equal to the angle of rotation.) i.e., $A\hat{L}B = B\hat{M}C = C\hat{K}A = 120°$. $\therefore$ $K\hat{L}M = L\hat{M}K = M\hat{K}L = 60°$.
$\therefore$ ΔKLM is equilateral.

6.



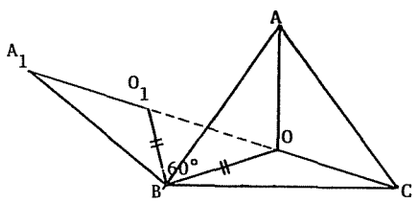
Given: ΔABC is isosceles with AB = AC. P is any point on BC, perpendiculars are drawn from P to AB and AC meeting them in L and M.
Show that for all positions of P on BC, PL + PM is a constant.
Solution: Reflect ΔBLP in BC to give ΔBL_1P. Since $B\hat{P}L = B\hat{P}L_1 = M\hat{P}C = (90 - x)°$, then M, P and L_1 are collinear and BNML is a rectangle.
$\therefore$ $PM + PL_1 = BN = PM + PL$ for all positions of P on BC.

7.



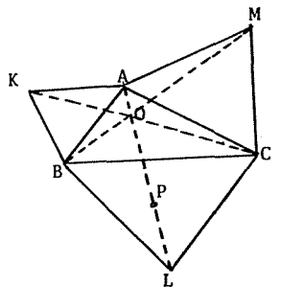
Given: ΔABC is right angled at B. ACDE is a square drawn on side AC. O is the centre of the square.
Show that OB bisects $A\hat{B}C$.
Solution: Rotate about O, the triangle ABC through successive angle of 90° to form square $BB_1B_2B_3$.
$\therefore$ O is the intersection of diagonals BB_2 and B_1B_3 of the square. $\therefore$ OB bisects $A\hat{B}C$ (the diagonals of a square bisect the angles of the square).

8.



Find a point O within a scalene triangle ABC such that OA + OB + OC is least.
Solution: Rotate ΔBOA about B through 60° to give ΔBO_1A_1. Now BO = BO_1 and $O\hat{B}O_1 = 60°$. $\therefore$ ΔOBO_1 is equilateral, i.e., OO' = OB. Now OA + OB + OC = O_1A_1 + OO_1 + OC. This is best when A_1, O_1, O and C are collinear. i.e., $C\hat{O}B = A_1\hat{O}_1B = A\hat{O}B = A\hat{O}C = 120°$. (This point is called the Fermat point.)

9.



Show that by drawing equilateral triangles KAB, LBC, MCA on the sides of ΔABC, then the intersection of KC, LA and MB gives the Fermat point.
Solution: Rotate KC about A through 60° anticlockwise. $\therefore$ K → B and C → M under this rotation.
$\therefore$ KC → BM. The angle between a line segment and its image under rotation is equal to the angle of rotation. $\therefore$ $K\hat{O}B = 60°$, i.e., $K\hat{O}M = 120°$. Similarly by rotating AL about B and BM about C through 60°, we get $K\hat{O}L = L\hat{O}M = 120°$.

To show that O is the point of concurrency of KC, MB and AL, take O as the point of intersection of BM and KC. Now KAOB and AOCM are cyclic quadrilaterals. AO produced to P forms $P\hat{O}B = P\hat{O}C = 60°$ both adjacent to 120° angles. Also LO bisects $B\hat{A}C$ because $L\hat{O}B = L\hat{C}B = 60°$ and $L\hat{O}C = L\hat{B}C = 60°$ (angles in the same segment of a circle).

$\therefore$ LO and POA are the same straight line, hence O is the point of concurrency of AL, BM, CK, and is the Fermat point.

MEMBERSHIP RENEWAL

By the time you have received this issue of the Gazette your OAME membership will be within approximately a month of expiration. We trust you will be attending the Annual Meeting at the Canterbury Inn in Sarnia and when you send your registration fee for the conference you will include $15.00 for your membership. You will receive your receipt, card and seal with your registration package.

Or perhaps you will wish to visit us at the registration desk and purchase your renewal directly from us.

If you are not able to attend the Annual Meeting OAME would appreciate you forwarding your renewal fee prior to May 1st to save the Association sending a reminder. The information required is listed below.

Name_____

Same address as before _____ or new address as listed below

_____

Chafter Affiliation (if applicable) _____

School _____ Elem _____ Sec. _____ Univ. or CATT _____ Other _____

Please enclose cheque or money order for $15.00 payable to OAME. Mail to OAME, 247 Bright St., Sarnia, Ont., N7T 4E9. Renewal notices will be sent out on May 1st to all members who have not renewed.

STRATEGIES FOR SOLVING A TEXT BOOK PROBLEM

Dr. H.L. Ridge, Faculty of Education, University of Toronto

Editor's Note: Brownell, William A. - in 1942 wrote in a classic paper "Practical Suggestions for Developing Ability in Problem Solving" that to be most fruitful, practice in problem solving should not consist of repeated experiences in solving the same problems with the same techniques, **but** should consist of the solution of different problems by the same techniques and the application of different techniques to the same problems. Larry's article illustrates different approaches.

Problem: How far is a man's home from downtown if he walks downtown at 4 mph and returns at once by subway at 12 mph and is away from home for only 2 hours.

.....

The strategies to be considered can be summarized as

Arithmetic or numerical

Graphical: coordinate geometry, functional analysis

Diagrammatic: vector, accurate construction, trigonometry

Average rate of speed.

I ARITHMETIC OR NUMERICAL STRATEGIES

1. Ratio and proportion

a) Walking time is 3 times longer.

$\therefore$ Walking takes up $\frac{3}{4}$ total time.

$\frac{3}{4} \times 2 = 1\frac{1}{2}$

$\therefore$ Walking distance = $1\frac{1}{2}$ hr × 4 mph

= 6 miles.

b) Break 2 hr into 16 parts (suggested by 4 + 12). Each part is $\frac{2}{16}$ of an hr.

Walking requires $\frac{3}{4}$ of those 16 parts.

$\therefore$ Walking time is $\frac{3}{4} \times 16 \times \frac{2}{16}$ hr

= $1\frac{1}{2}$ hours.

2. Systematic trial

Trial	Time (hr.) W	Time (hr.) S	Distance (mi.) W	Distance (mi.) S (must be the same)
1 →	2	0	8	0
2 →	1	1	4	12

Try between trials 1 and 2

3 → $1\frac{1}{2}$ $\frac{1}{2}$ 6 6 ✓

3. False position

The trip takes 2 hr and the distances are equal.

Time (hr) W	Time (hr) S	Distance (mi) W	Distance (mi) S
$\frac{1}{3}$	$\frac{5}{3}$	$1\frac{1}{3}$	20
$\frac{2}{3}$	$\frac{4}{3}$	$2\frac{2}{3}$	16
1	1	4	12
$\frac{4}{3}$	$\frac{2}{3}$	$5\frac{1}{3}$	8
$\frac{5}{3}$	$\frac{1}{3}$	$6\frac{2}{3}$	4

For the distances to be equal the walking time must be between $\frac{4}{3}$ and $\frac{5}{3}$ hr and the distance must be between $5\frac{1}{3}$ and $6\frac{2}{3}$ mi.

In fact the average of $5\frac{1}{3}$ and $6\frac{2}{3}$ is 6 while the average of 4 and 8 is also 6. This suggests that the walking time is the average of $\frac{4}{3}$ hr and $\frac{5}{3}$ mi; namely $\frac{3}{2}$ hr.

4. By pattern

DISTANCE (1 way) (mi)	TIME (hr) W	S	TOTAL
1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{3}$
2	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$
3	$\frac{3}{4}$	$\frac{1}{4}$	1

The pattern generates the ordered pairs (D,T), $(1,\frac{1}{3})$, $(2,\frac{2}{3})$, (3,1), $(4,\frac{4}{3})$, $(5,\frac{5}{3})$ and (6,2), the last of which solves the problem. The pattern can be expressed as D = 3T.

Note: This relationship between one way distance and total time* has not been evident in any of the preceding treatments.

Such a pattern is most convenient for extrapolation to longer time periods. A total of 3 hr. implies that 9 mi. are covered each way.

*This relationship will be dealt with later in more detail under "Average rate".

II ALGEBRAIC STRATEGIES

1. Using total time of 2 hours

TWO VARIABLES	ONE VARIABLE
a) Let x mi. be the walking distance and y mi. the subway distance	Let x mi. be the distance one-way.
$\frac{x}{4} + \frac{y}{12} = 2$, $y = x$	$\frac{x}{4} + \frac{x}{12} = 2$*
$x = 6$	$x = 6$.
$y = 6$.	

In this case the root(s) of the equation(s) give the distance(s) directly.

*Note 1: The role of units analysis in understanding the meaning of this equation

$$\frac{x \text{ mi}}{4 \text{ mi/hr}} + \frac{x \text{ mi}}{12 \text{ mi/hr}} = \frac{x \text{ mi} \times \text{hr}}{4 \text{ mi}} + \frac{x \text{ mi} \times \text{hr}}{12 \text{ mi}}$$

$$= \frac{x}{4} \text{ hr} + \frac{x}{12} \text{ hr}$$

$$= (\frac{x}{4} + \frac{x}{12}) \text{ hr}$$

Such an analysis reinforces the $\frac{\text{DISTANCE}}{\text{RATE}}$ = TIME relationship.

Note 2: $\frac{x}{4} + \frac{x}{12} = 2$ is more generally a rates model.

Eg: Widgets are priced at 4 for a dollar.
Smidgets are priced at 12 for a dollar.
An equal number of widgets and smidgets are purchased for a total of \$2. How many of each item are purchased?

$$\frac{\text{x widgets}}{\text{4 widgets/dollar}} + \frac{\text{x smidgets}}{\text{12 smidgets/dollar}}$$

$$= \$(\frac{x}{4} + \frac{x}{12})$$

b) Let t hr. represent the subway time and v hr. the walking time.

$$\left.\begin{aligned} t + v &= 2 \\ v &= 3t \end{aligned}\right\} \Longrightarrow$$

$$t = \frac{1}{2},\ v = 1\frac{1}{2}$$

Let t hr. represent the subway time. Then 3t hr. is the walking time.

$$t + 3t = 2$$
$$t = \frac{1}{2}$$

In this case the root(s) of the equation(s) have to be multiplied by the rate in order to give the distance(s).

2. Using equal distances

TWO VARIABLES	ONE VARIABLE
a) Let t hr. be the walking time and d mi. be the walking distance	Let t hr. be the walking time. Then (2 - t) hr. is the subway time.
$\left.\begin{aligned} d &= 4t \\ d &= 12(2-t) \end{aligned}\right\} \Longrightarrow$	$4t = 12(2-t)$
$t = 1\frac{1}{2},\ d = 6$	$t = 1\frac{1}{2}$

In the two variable version, one of the roots gives the distance. In the one variable case, the distance must be calculated.

b) Let x mi. represent the walking distance and y mi. the subway distance.

$$\left.\begin{aligned} x + y &= 12^* \\ x &= y \end{aligned}\right\} \Longrightarrow \quad x + x = 12,\ x = 6$$

$x = 6,\ y = 6$

Let x mi. represent the distance one way.

*The 12 arises from 2 × average speed (6 mph). The calculation of average speed is dealt with at the end of this presentation. In either case, the distance is given directly.

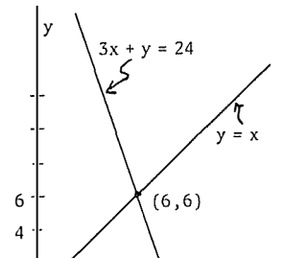
III GRAPHICAL STRATEGIES

1. Using coordinate geometry

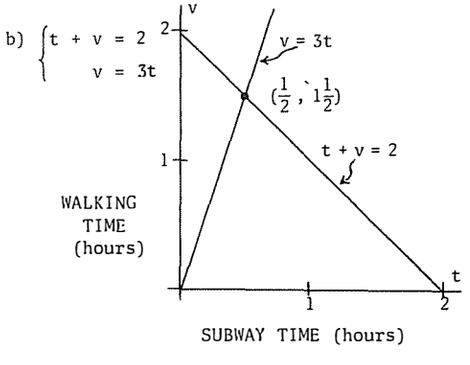
Each of the 2 variable approaches under ALGEBRAIC STRATEGIES give rise to a graphical interpretation.

a) $\left.\begin{aligned} \frac{x}{4} + \frac{y}{12} &= 2 \\ y &= x \end{aligned}\right\}$

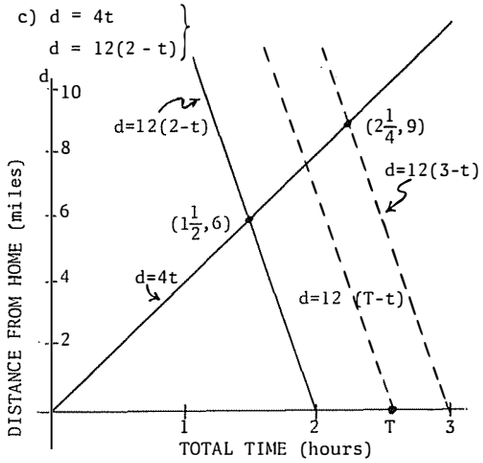
$\Leftrightarrow \left.\begin{aligned} 3x + y &= 24 \\ y &= x \end{aligned}\right\}$



b) $\left.\begin{aligned} t + v &= 2 \\ v &= 3t \end{aligned}\right\}$



c) d = 4t
d = 12(2 - t)



NOTES:

(i) The line d = 12(2 - t) can be determined on the basis of transforming the line d = 12t which lies on the origin with slope 12.

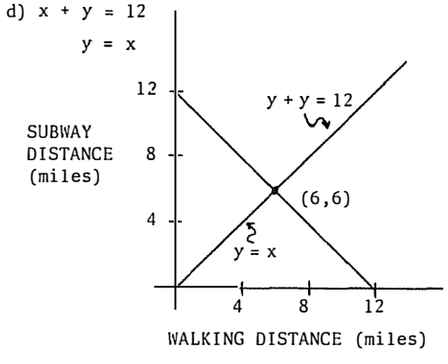
d = 12 (t - 2) is the TRANSLATION image of d = 12t on the point (2,0).

d = -12 (t - 2) or d = 12 (2 - t) is the REFLECTION image of d = 12 (t - 2) in the line t = 2.

(ii) This system readily permits determination of distance from home for any total time period T hours.

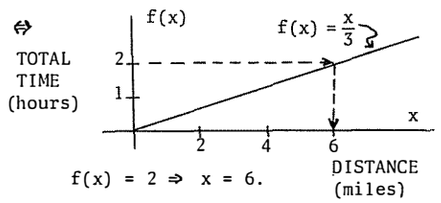
d = 12(T - t) is the family of lines parallel to d = 12(2 - t) on point (T,0). The line for T = 3, d = 12(3 - t), is shown dotted in. It yields a distance of 9 miles.

d) x + y = 12
y = x



2. Using functional analysis

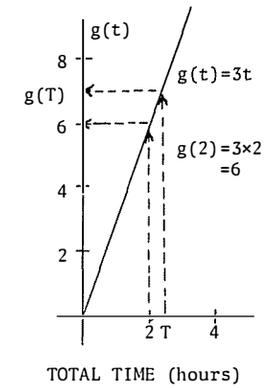
a) Consider time f as a function of distance x. $f(x) = \frac{x}{4} + \frac{x}{12} \Leftrightarrow f(x) = \frac{x}{3}$.



$f(x) = 2 \Rightarrow x = 6$.

b) Consider distance g as a function of total time t.

According to the pattern investigated in item I, (4), g(t) = 3t.



Note 1: f and g are inverse functions.

Note 2: Function g accomplishes with a single line what the system

d = 4t
d = 12(2 - t)
} III (1), (c)

permitted in the way of distance determination from any total time T hours as shown above).

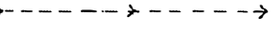
Note 3: Derivation of this function is considered under "Average rate of speed".

IV DIAGRAMMATIC STRATEGIES

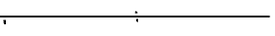
1. Using vectors

a) Use a vector to represent the distance covered at each rate

WALKING (4 mph)

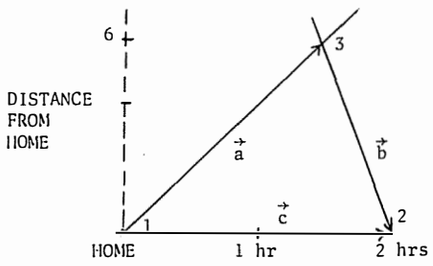


SUBWAY (12 mph)



Clearly, 3 of ——→ are needed to cover the same distance as the subway. Hence 4 units of time t hours are involved over the whole trip. . . each unit of time is $\frac{1}{2}$ hr.

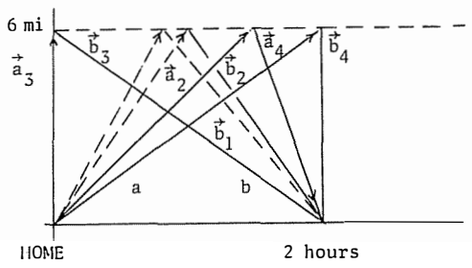
b) Use velocity vectors in 2-space. Let $\vec{a}$ represent walking velocity, $\vec{b}$ the return subway velocity, and $\vec{c}$ the zero rate (stay at home for 2 hours).



Order of construction:

1. initial point and direction of $\vec{a}$
2. terminal point and direction of $\vec{b}$
3. initial point of $\vec{b}$ and terminal point of $\vec{a}$ (at the point of intersection).

Once the distance of 6 mi is determined, vector sums can be used to show the various combinations of "walking" and "subway" rates which result in the same total time of 2 hours. The technique shown below generates a whole family of similar problems. Granted, suitable contexts would have to be created as the comparative rates varied.



Note 1: $\vec{a}_2$ and $\vec{b}_2$ are of special significance which will be discussed under "Average rate of speed".

Note 2: $\vec{a}_3$, $\vec{b}_3$ and $\vec{a}_4$, $\vec{b}_4$ each represent limiting cases which involve "instantaneous speed".

2. Using accurate construction

(Thanks to Bob Saler)

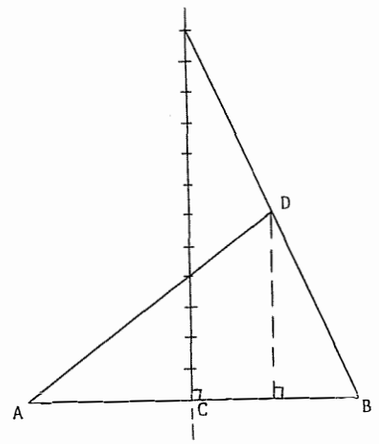


Figure 3

Draw a horizontal line segment AB (Fig. 3) of any length. It represents the total time. Bisect AB at C. Then AC represents the units of any size on this perpendicular. Draw a line through A and the fourth arc on the perpendicular. This line represents the walking speed of the man. Draw a line through B and the 12th arc on the perpendicular. This line represents the speed of a subway. Where the two lines meet is the point (D) where the man stops walking and starts riding a subway, without delay, of course. The distance from point D to the line segment AB is the distance from his home to downtown. This distance could be read off on the scale of the perpendicular line to AB at C. Q.E.F.

The above method can be adapted to any pair of rates and any total time. For time greater than 2 hours it would be convenient to erect a second 12 segment line at a point 1 hour before the total time.

3. Using Trigonometry

See ADDENDUM at the end of the article.

V AVERAGE RATE OF SPEED

1. Fallacy of averaging rates of speed

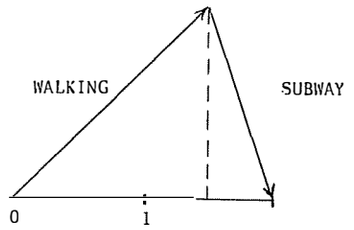
Many students are lured into the trap of stating that the overall average rate of speed is the average of the stated speeds. In this case $\frac{1}{2}(4 + 12)\text{mph} = 8$ mph.

Students should be encouraged to check the validity of such a conclusion.

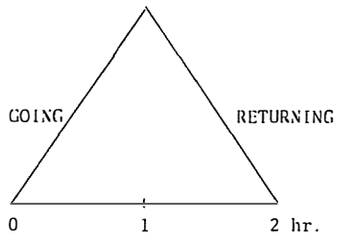
The man would travel a total of 8

mph × 2 hr. or 16 mi. Distance from home would then be 8 mi. At a rate of 4 mph it would take him the full 2 hr just to walk downtown. Clearly, something is 'askew'.

To 'see' that the arithmetic mean of the speeds is incorrect in this case, consider the graph in III (1) (c) or IV (1) (b)



It is because the times of walking and riding are not equal that the arithmetic mean does not work. In the situation shown below, times going and coming would be equal. In addition, the speeds are the same. On one hand, the overall speed can be calculated as the average of the two (equal) speeds, but such a calculation becomes redundant when the speeds are the same.



2. Calculation of average rate of speed

Average speed over an interval of time means $\frac{\text{TOTAL DISTANCE}}{\text{TOTAL TIME}}$. Since we don't know the total distance in this, some may think that the average overall speed cannot be determined. Such is not the case.

Knowing the speeds for both parts of the trip and that the distances are equal is sufficient.

Let the distance each way be d mi. Then walking time is $\frac{d}{4}$ hr and subway time is $\frac{d}{12}$ hr. Total distance is 2d mi. Total time is $(\frac{d}{4} + \frac{d}{12})$ hr. Average rate of speed is $\frac{2d}{\frac{d}{4} + \frac{d}{12}}$ mph $= \frac{2d}{\frac{d}{3}}$ mph $= 6$ mph. Hence, total distance (2d mi) is 6 times the time expended (t hr). That is

$$2d = 6t$$

or $$d = 3t$$

as arose in the "pattern" approach in part I and in the functional approach in part III.

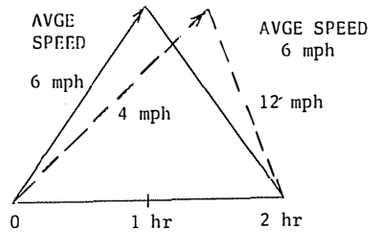
Note 1: In this case the average rate was numerically equal to the distance. The reason is that

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}$$

$$= \frac{2d}{2} \text{ mph} = d \text{ mph}.$$

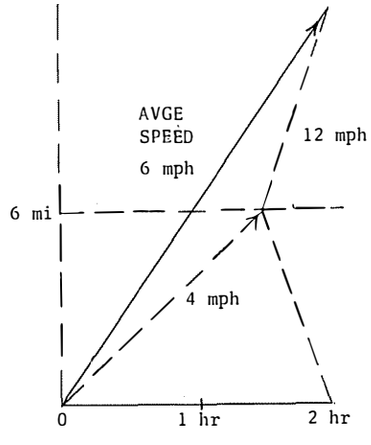
In general, average speed $= \frac{2d}{t}$ mph.

Note 2: In item IV (1) (b), vectors $\vec{a}_2$ and $\vec{b}_2$ were said to have special significance.



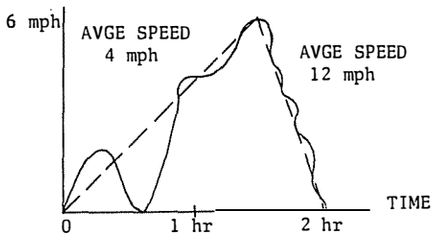
In this case, the speeds are the same and hence represent the average speed 6 mph.

The graphical relation of average speed to individual speeds can be seen better by considering the reflection of the return trip, as if the second 6 mi were farther away from home.



Note 3: In all of the work to this point, the 4 mph and 12 mph have been assumed to be constant rates of speed. From a practical point of view these speeds should be considered average walking and average subway speeds.

The actual distance time graph might be quite irregular as shown below but the average speed is shown as constant throughout.



Interpretation of the actual distance time graphs makes for interesting conjectures. It appears that, for example, that our walker was somewhat forgetful.

Having students draw a variety of distance time graphs just within the constraints of "downtown and back" into 2 hours from any fixed distance can produce an amazing variety of situations--but in <u>all</u> of these cases, the average speed remains the same because of the fixed distance and time intervals.

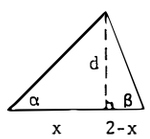
Note 4: The average speed for walking @ k mph and returning @ m mph over a distance d is $\frac{2d}{\frac{d}{k} + \frac{d}{m}}$ mph = $\frac{2km}{k + m}$ mph. The total distance over t hr. is then given by

$$2d = \frac{2km}{k + m} \cdot t \ .$$

Hence, the one way distance is given by

$$d = \frac{km}{k + m} \cdot t$$

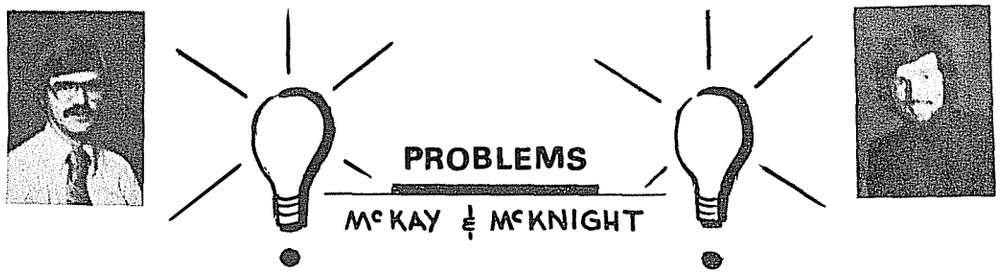
<u>ADDENDUM:</u> IV (3) Using trigonometry (variation on vector solution)



$$\left.\begin{aligned} d &= x \tan \alpha \\ d &= (2 - x) \tan \beta \end{aligned}\right\}$$

$$\left.\begin{aligned} d &= 4x \\ d &= 12(2 - x) \end{aligned}\right\}$$

as in previous solutions.



1. UP TO GRADE 10

19.3.1 A committee of three people, Jim, Sue and Al meets and agrees that Jim report back every 8 days, Sue every 12 days and Al every 15 days. Find the least number of days before Al again meets with Jim and Sue. (All three together).

19.3.2 Find all real values of x which satisfy

$$\frac{⅓ + 1/x}{⅓ - 1/x} \geq -1$$

19.3.3 If a certain number were reduced by x and the remainder were then multiplied by x, the result would be the same as reducing the original number by y and multiplying the resulting remainder by y. Express the original number in simplest form in terms of x and y.

19.3.4 An equilateral triange has sides equal to 2 units. A point x is taken a units along an altitude (from the vertex). Determine the sum of the perpendicular drawn from x to the 3 sides.

II. UP TO GRADE 13

19.3.5 Determine which is larger $9\sqrt{9!}$ or $10\sqrt{10!}$

19.3.6 One side of a triange is 10 cm. longer than another and the angle between them is 60 degrees. Two circles are drawn with these sides as diameters.. One of the points of intersection of the two circles is the common vertex. How far from the third side is the other point of intersection?

19.3.7 One root of $x^2 + kx + 5i = 1$ is $1 - i$, where i is the imaginery unit. Find the value of k.

19.3.8 A car accelerates uniformly from rest to 8 k km/h in k/5 minutes. It continues at that speed for k minutes, then decelerates uniformly and takes another k/5 minutes to come to rest, having travelled exactly k - 1 kilometres altogether. The trip took an exact number of minutes. How many?

III. ARE YOU READY FOR A CHALLENGE?

19.3.9 The longer base of trapezoid ABCD is AB. Rays AD and BC meet at E forming ΔABE. A line, parallel to AB, meets AD at F and BC at G. A perpendicular from G intersects AE at P, and a perpendicular from F intersects BE at Q. If BC = 6, PG = 9 and FQ = 12, find AD.

PLUS + + +
AN INFORMATION COLUMN



Ed Barbeau

81.1 **Provincial Mathematics Programs in Canada (1979-79)**

A draft document, entitled **Provincial Mathematics Programs** in Canada as of 1978-79, has been prepared by LaJune Naud, Consultant in Mathematics for the Nova Scotia Department of Education, for the Curriculum Committee of the Council of Ministers of Education of Canada. The data was obtained in part from the official provincial guidelines and in part through direct communication with each department/ministry. In Grades 1-6, it is noted generally that there is less emphasis on abstraction, more exploration of space and shape, more recourse to basic scales and a tendency to postpone work on fractions. In Grades 7-9, more emphasis is detected on problem solving, application, interdisciplinary aspects and basic numeric skills. Treatment of algebraic manipulation, factoring, graphing and statistics varies from province to province. In the upper school, trigonometry is spread over several years. Transformations are increasingly taught and there is a move towards the emphases of Grades 7-9. Lists of topics common to all curricula are given. Appendices: Summaries of answers to questionaires by provincial officials, Lists of guides and texts, Summary of programs by provinces.

81.2 **Provincial contests and exams**

A recent issue of **Delta-K,** the journal of the Alberta Teachers' Association Council edited by Ed Carriger (R.R. No. 1, Site 2, Box 4, Bluffton, Alta. T0C 0M0) carries a report on the C.M.S. 1980 Alberta High School Prize examination. This contest, won by Robert Morewood of Medicine Hat, consists of two papers: (1) 20 multiple-choice questions, (2) 6 problems **(Delta-K** publishes solutions). One of the problems asked the contestants to show that a solid figure all of whose cross-sections are circles must be a --- (the reader will undoubtedly identify the missing word).

In the same issue is a report on a diagnostic tests of 25 multiple-choice questions given to students in introductory calculus courses at the University of Alberta. In the most poorly answered question, the respondent had to identify a parabola through (-1,0), (0,-1), (1,0) (diagrammed) as one of: y = (x - 1)2, y = x2 - 1, y2 = 1 - x2, y = l - x2, none of the foregoing. On the other hand, 89% of 2071 students identified the real values of k for which kx2 + kx + 1 had not real roots.

The question paper for the 1977 Concours mathemtique de l'A.M.Q appears in **La Gazette des Sciences Math9matiques du Qu9bec,** redacteur en Chef Marc Bourdeau, Ecole Polytechnique, C.P.

6079, Succ. A. Montreal H3C 3A7. There are seven problems each labelled with a title. Sample: "La boule de billard" Considerons une table de billard de 360 cm. par 180 cm. dont les six poches A, B, C, D, E et F sont disposees comme dans la figure. (A, B, C, D, E, F in clockwise order; B midway between A and C, E midway between D and F.) Une balle est frappe au point 0, centre du carr9 ABEF, et rebondit pour la premiere fois au point, P, a 48 cm. du trou B (et 132 cm. du trou A). Dans quelle poche ira-t-elle tomber (on negligera la friction, le rayon de la boule et de l'entree des poches).

81.3 **Canadian Mathematics Education Study Group/Groupe canadien d'etude en didactique des mathematiques**

1980 Meeting: June 5th to 9th at the University of Alberta. Open to mathematicians and mathematics educators. Various groups will study mathematics education research, aspects of teacher training, mathematics and language, the history and pedagogy of mathematics, etc., Guest speaker: Dr. Jeremy Kilpatrick, University of Georgia. Information from J. Hillel, Department of Mathematics, Concordia University, 7141 Sherbrooks Street West, Montreal, Quebec H4B 1R6.

SOLUTION TO PROBLEM SET 19.3

19.3.1 For Jim $8 = 2^3$, for Sue $12 = 2^2 \cdot 3$, for Al $15 = 3 \cdot 5$. Least common multiple for 8, 12, 15 $= 2^3 \cdot 3 \cdot 5 = 120$. They meet in 120 days.

19.3.2 $\dfrac{\frac{x+3}{3x}}{\frac{x-3}{3x}} + 1 \geq -1 + 1$. Thus $\dfrac{2x}{x-3} \geq 0$

CASE 1. $2x \geq 0$ and $x - 3 > 0$ CASE 2. $2x \leq 0$ and $x - 3 < 0$

$x > 3$ or $x < 0$.

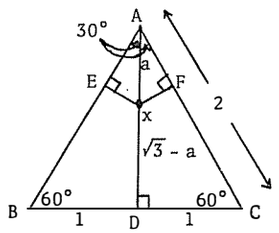
19.3.3 Let N = the number. Then $x(N - x) = y(N - y)$

$xN - x^2 = yN - y^2$

$N(x - y) = x^2 - y^2$

$\therefore N = x + y$.

19.3.4



In ΔABD, BD = 1 and $AD = \sqrt{3}$. Thus $xD = \sqrt{3} - a$.

Now $\Delta AEX \sim \Delta ADB$. $\therefore \frac{a}{2} = \frac{Ex}{1}$. Thus

$Ex = \frac{a}{2}$ and also $Fx = \frac{a}{2}$.

$Ex + Fx + Dx = 2(\frac{a}{2}) + \sqrt{3} - a = \sqrt{3}$.

19.3.5 $\sqrt[9]{9!} = (9!)^{1/9} = (9!)^{10/90} = \sqrt[90]{(9!)^{10}} = \sqrt[90]{9!\,(9!)^9}$

$\sqrt[10]{10!} = (10!)^{1/10} = (10!)^{9/90} = \sqrt[90]{(10!)^9} = \sqrt[90]{(10 \cdot 9!)^9} = \sqrt[90]{10^9 (9!)^9}$

Now since $9! < 10^9$ then $\sqrt[10]{10!} > \sqrt[9]{9!}$.

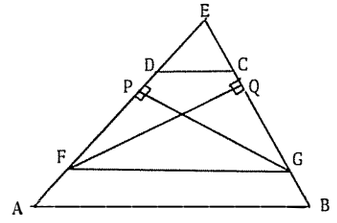
19.3.6 At zero distance. Regardless of the angles or lengths of the sides, if a line is drawn connecting the two points of intersection, it will be perpendicular to the third side. (Recall that an angle inscribed in a semi-circle is a right angle). Hence the point of intersection will lie on the third side.

19.3.7 Since the produce of the route is 5i - 1, the second root is $\frac{5i - 1}{1 - i} = -3 + 2i$. Then, since the sum of the roots of the equation is -k, we have that $k = -(3 + 2i + 1 - i) = -(-2 + i) = 2 - i$.

19.3.8 The car can be assumed to travel 4k km/h throughout the acceleration and decelera-

tion period, thus travelling $\frac{k^2}{75}$ kilometres at each end of the trip. In between it goes 8k km/h for k minutes, thus going $\frac{2k^2}{15}$ kilometres in the middle. Altogether it travels $\frac{2k^2}{75} + \frac{2k^2}{15}$ kilometres. If this is equal to k - 1, $4k^2 - 25k + 25 = 0$ from which k = 5 or 5/4. Since the number of minutes is an integer, k must be 5 and the trip took 7 minutes.

19.3.9



Area of $\Delta FEG = \frac{1}{2}PG \cdot EF = \frac{1}{2}FQ \cdot EG$

$\therefore \frac{PG}{FQ} = \frac{EG}{EF}$.

Now $\frac{EG}{EF} = \frac{CB}{AD}$ (parallel lines divide proportionally)

$\therefore \frac{PG}{FQ} = \frac{CB}{AD}$

i.e., $\frac{9}{12} = \frac{6}{AD}$ $\therefore$ AD = 8

ANNOUNCEMENT

The Publications Committee is searching for editors.

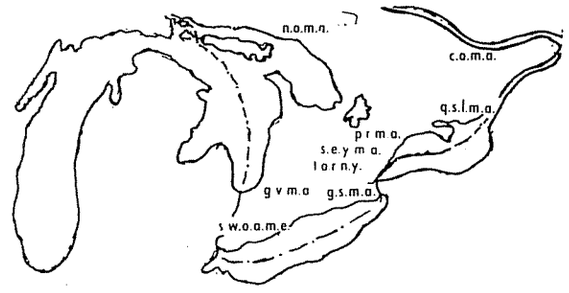
1. Editor needed for the Ontario Mathematics Gazette. The journal is published 3 times a year - Sept. 1, Nov. 1, Mar. 1. The term is for 2 years with a possible extension.

2. Editor needed for Abacus. The newsletter is published 4 times a year - Sept., Nov., Feb., Apr. The term of office is for 2 years with a possible extension.

If interested, please send application with all relevent information to:

Chairperson, Publications Committee
Mrs. S. McPhail

CHAPTER ITEMS



At the January meeting, **Curricula and Assessment** was the theme. Jim MacLean spoke on The Formative Years, George Scroggie - The Guidelines with a panel of persons answering questions regarding The Assessment Instrument Pool Concept. As a result of this meeting, a C.O.M.A. Examining Board was set up to provide two annual examinations at the Gr. 12 and Gr. 13 level in English and French for students who make application for examination.

Golden Section

In co-operation with the Lincoln County Board of Education, a Childhood Education Conference will be presented on April 11th at Beamsville Secondary School. There will be a choice of 14 workshops and Dr. John Bates will be the Key note speaker on **Problem Solving: The Key To Mathematics.**



Grand Valley

The Association looks forward to its next meeting on April 25 when **Monday Morning Mathematics** will be a theme.

It is rumoured that the people who hosted the last O.A.T.M. conference titled **Waterloo in 72** are interested in hosting a Grand Valley in 83 convention. This will mark ten years of Math conventions for me so I look forward to G.V. in 83.

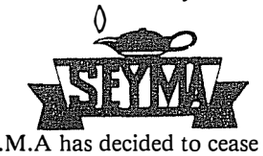
NOMA

Preparations are now underway for the annual conference to be held Sept. 29-30 in North Bay. Watch for further details in the September issue.

Pine Ridge

It is believed that Pine River Chapter has become the Mathematics Teachers of Durham.

A very successful dinner meeting was held at Ernestown Secondary School on February 10. Mrs. S. Hornbeck spoke on Mathematics for the Gifted, Mr. D. Crawford spoke about Measuring and Evaluating in Intermediate Division, Mr. D. Reynolds presented ideas about computer use, Mr. R. McPhee's talk dealt with Statistics and Interpreting Data and Mr. R. Corbeil with Mr. F. Leduc suggested practical ideas on Transformational Geometry.



It is with regret that I hear that S.E.Y.M.A has decided to cease operation. The decision was taken at a poorly attended executive meeting on Jan. 13. Reasons for the decision given were the following: declining membership, work pressure on executive members, and competition with Board of Education sponsored activities.





ACROSS THE EDITOR'S DESK

By Bob Smith

1. For the 60th Annual Meeting, the N.C.T.M. Convention to be held in Toronto, 14-17 April 1982, a last call for your participation. Three major themes will be included covering "Motivation", "Problem Solving, and "The Role of Technology". If interested, please send five copies, typed and double spaced of the title, a brief program description, proposer's name and school and a preferred mailing address. The committee is also looking for "Research Abstracts" and "Short Subjects". Proposals for the latter should be in triplicate, typed and double spaced and contain 1) title 2) proposer's name and school 3) N.C.T.M. membership number 4) preferred mailing address and phone number 5) interest level 6) a brief overview of the idea (not more than 200 words). Reviewing teams will notify participants on or before July 1, 1981.

2. ICOTS ... INTERNATIONAL CONFERENCE on TEACHING STATISTICS

The **First International Conference on Teaching Statistics** will be held in Sheffield, England, from 8-13 August, 1982. For a copy of the first announcement write to:

The Conference Secretary, ICOTS,
The University,
Dept. of Probability & Statistics,
Sheffield S3 74H, England.

3. BCAMT π — 9TH SUMMER WORKSHOP

Place: Cariboo College **Dates:** Aug. 31, 1981 - Sept. 2, 1981

4. The 8th National Conference on Diagnostic & Prescriptive Mathematics will be held April 26-28, 1981 at the Hershey Hotel, Hershey, Penn. The 1982 convention will be held at Niagara Falls hosted by SUNY Buffalo.

5. The thirteenth Canadian Mathematical Olympiad will be held on Wednesday, May 6, 1981 at schools across Canada. Copies of nomination forms can be obtained from:

The Olympiad Committee, DR. G.J. Butler,
Dept. of Mathematics, Univ. of Alberta,
Edmonton, Alberta, T6G 2G1

Entries must be received not later than April 2, 1981.

6. Last but not least let me remind you of our own conference. The Eighth Annual conference of O.A.M.E./A.O.E.M. will be held at the Canterbury Inn, Sarnia, May 8-10. David and Sharon McPhail, Conference Co-chair persons and Tomm Griffiths, Program Chairman, assure me that there will be a first rate program. There are good jogging trails near both the St. Clair River and Lake Huron, so do join us.