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ONTARIO MATHEMATICS GAZETTE

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Additional copies of the Gazette may be obtained from
Mr. F. C. Asbury

or from

Dr. R. G. Stanton,
University of Waterloo, Waterloo, Ontario.

E D I T O R I A L

It is with some trepidation that I look forward to the task of prefacing each issue of the Ontario Mathematics Gazette with an editorial. However, the first editorial is really an easy one to write; I shall simply tell you what the Ontario Mathematics Gazette is, and indicate what it aims to accomplish.

This first issue of the Ontario Mathematics Gazette is made possible through the financial assistance of the Ontario Mathematics Commission (through its Advisory Committee on Research and Publications) and the Ontario Association of Teachers of Mathematics and Physics. Indeed, the Gazette includes among its functions those previously performed by the OATMP Newsletter; it can be considered as a successor, in amplified form, of the Newsletter.

Among the aims of the Gazette will be that of increasing the liaison between secondary school teachers and students on the one hand and university professors on the other hand. It is hoped that portions of the Gazette will attract student readers; perhaps this aim could best be achieved if mathematics teachers would induce their boards to buy bulk lots of future Gazettes, since most boards probably have funds set aside for purchase of educational literature and materials.

The actual production of the Gazette will be made easier if teachers will be generous with suggestions for future features or articles. What would you like to see in the Gazette? We welcome articles, problems, correspondence; do not be shy about offering help. If we can't use your first submission, perhaps we can use your third! Any articles to be considered for publication should be sent to the Editor-in-Chief, who will pass them on to a suitable referee for judgment; correspondence on other matters should be addressed to the appropriate editor. In particular, any "newsy" material should be sent to the Newsletter Editor.

One question we would like to see answered is the following: how should the Gazette be financed? We hope to put out three or four issues per year once we get rolling. Would you favour a subscription of, say, one dollar per annum? Do you think school boards could be encouraged to subscribe for their teachers? Here is one area where we would welcome correspondence.

And now, before concluding, let me make the usual disclaimer. Any views expressed in articles are the views of the particular authors, and do not necessarily reflect the views of the Ontario Mathematics Commission, the Ontario Association of Teachers of Mathematics and Physics, or the Editorial Staff of the Ontario Mathematics Gazette.

This first issue has had little precedent to rely on. We are trying to develop a typically Canadian magazine which will differ from corresponding American publications in two respects: the percentage of articles on mathematics, rather than on "education", will be higher; the articles on purely American themes will be lacking. Naturally, we won't achieve perfection in one effort! But we count on you to help us; how do you like our first effort, and what suggestions have you for the future?

With best regards to present and future friends.

Ralph G. Stanton

Editor-in-Chief

SURD EQUATIONS AND EXTRANEOUS ROOTS

L. D. Auckland
St. Thomas Collegiate Institute

It is well known that the process of squaring in solving equations may introduce extraneous roots; for example, the equation

$$\sqrt{x + 21} + x = 9$$

becomes, on squaring,

$$(x + 21) = (9 - x)^2,$$

$$0 = x^2 - 19x + 60 = (x - 4)(x - 15).$$

The root 4 satisfies the given equation, since $\sqrt{25} + 4 = 9$; the root 15 does not satisfy the original equation, since $\sqrt{36} + 15 \neq 9$ (we recall that the surd sign refers to the positive root only).

It is the purpose of this note to show how the process of verification, which in more complicated examples can become very tedious, can be avoided by a consideration of the domain of the variables concerned. We shall select, as our illustration, Problem 3(a) from the 1955 Grade XIII Algebra paper, namely,

$$x^2 + 2\sqrt{x^2 - 3x - 1} = 3x + 16.$$

Let the surd be y; then

$$2y = (-x^2 + 3x + 1) + 15 = -y^2 + 15.$$

$$y^2 + 2y - 15 = (y + 5)(y - 3) = 0.$$

Putting $y = 3$, $x^2 - 3x - 1 = 9$; hence $x = 5$ or $x = -2$.

Putting $y = -5$, $x^2 - 3x - 1 = 25$; hence $x = \frac{1}{2}(3 \pm \sqrt{113})$.

Then comes verification; substitution in the original equation shows (easily) that 5 and -2 are roots; substitution in the original equation shows (not so easily) that $\frac{1}{2}(3 + \sqrt{113})$ and $\frac{1}{2}(3 - \sqrt{113})$ are not roots.

All this final substitution can be avoided if we change the format of the solution slightly.

Given: $x^2 + 2\sqrt{x^2 - 3x - 1} = 3x + 16$

Solution: Set $\sqrt{x^2 - 3x - 1} = y$, where y is a non-negative real number [the quadratic surd sign always refers to a non-negative number].

Then $2y = -y^2 + 15$,

$$y^2 + 2y - 15 = 0.$$

Then $y = 3$ (we automatically reject the value $y = -5$, since y can not be negative). Consequently,

$$x^2 - 3x - 1 = 9,$$

whence $x = 5$ or $x = -2$. We do not need to verify these values to know that they work; they must work, since they come from a positive value of the surd y.

The reader will notice that this problem might have had 4 solutions (if both values of y had been non-negative) or no solutions (if both values of y had been negative). Student exercise in this method can be provided by the following examples; the certainty as to which roots are valid roots, without the need for verification, saves much work.

(1) $10\sqrt{x^2 - 3x + 6} = x^2 - 3x + 30$ (4 roots)

(2) $5\sqrt{x^2 - 2x - 11} + x^2 - 2x = 5$ (no roots)

(3) $4\sqrt{x^2 - 7x + 1} + x^2 - 7x + 6 = 0$ (no roots)

(4) $2\sqrt{x^2 - 13x + 89} + 13x = x^2 + 54$ (2 roots)

Last Minute Notice

The speaker at the O.E.A. luncheon will be Dr. Harvey Gellman, president of a firm specializing in the application of electronic computers to business, engineering, and scientific activities.

Ontario Association of Teachers of Mathematics and Physics

Ontario Educational Association

Wilfred Cunnington,
York Mills Collegiate Institute

The Easter Convention will again be held in the Ontario College of Education building at the corner of Bloor Street and Spadina Avenue. Lectures will be in room 374; book displays and seminars will be in nearby rooms.

We have arranged for three main speakers: Dr. J. H. Hlavaty of De Witt Clinton High School and Columbia University; Col. W. R. Sawyer of The Royal Military College of Canada; and Dr. C. J. Scriba of the University of Toronto.

Julius H. Hlavaty

Dr. Hlavaty, in addition to being Chairman of the Department of Mathematics at De Witt Clinton High School and an Instructor at Teachers College, Columbia University, is a Director of the Mathematics Commission Programme of the College Entrance Examination Board. Many will recognize him as a National Teacher on the Continental Classroom which has been in operation since 1960.

His years of experience in High School classrooms, in teacher training, on advisory committees, and as a contributor to various mathematical publications, will no doubt help to make the coming convention one of the best ever.

Colonel W. R. Sawyer

Dr. Sawyer, the present Director of Studies at the Royal Military College, graduated from that institution with honours in 1924. Post graduate work at Queen's and McGill was followed by an active career in the fields of research, education, and the army. As General Staff Officer on the H.Q. of the First Canadian Army, he was responsible for all aspects of gas, smoke, and flame warfare and meteorology. He has been Scientific Adviser to the Atomic Energy Control Board, and is now a member of the Defence Research Advisory Committee on Chemical Warfare and Atomic Energy. His talk on Rocketry should be most interesting.

Dr. C. J. Scriba received his education in Germany at the universities of Marburg and Giessen, receiving his Ph.D. in 1957. A Fulbright Travel Scholarship brought him to this continent where he spent one year at the University of Kentucky, one year at the University of Massachusetts, and three years as a member of the Department of Mathematics at the University of Toronto. His research interests lie in the history of mathematics, particularly around the 17th century, when calculus and

analytic geometry were getting started.

Dr. Scriba has recently been awarded a scholarship under which he will spend two years at Oxford engaged in studies in the history of mathematics.

PROGRAMME

Tuesday, April 24, 1962

8:45 a.m.	Registration and examination of Book Displays
9:15 a.m.	Minutes; Financial Statement; President's Remarks
9:30 a.m.	Speaker from Department of Education: "Implications of the 3-Stream Programme for Mathematics"
10:15 a.m.	Dr. J. H. Hlavaty
11:15 a.m.	Father Egsgard: Report of the N.C.T.M. Summer Meeting
11:30 a.m.	Dr. Scriba: History of Mathematics
12:30 a.m.	Adjournment for lunch
1:00 p.m.	Luncheon
2:30 p.m.	College and Secondary Department Meeting
8:00-9:30 p.m.	Reception

Wednesday, April 25

9:00 a.m.	Examination of Book Displays; Report of Nominating Committee
9:15 a.m.	Dr. W. R. Sawyer: Rocketry
10:00 a.m.	Dr. J. H. Hlavaty
11:00 a.m.	Introduction of new executive; new business
11:10 a.m.	Seminars

- Grade 13: Analysis of Fractional Rational Functions (Tim Gibson-North Toronto Collegiate)
- Grade 12: Ratio and Proportion (Dave Alexander-University of Toronto Schools)
- Grade 10: Inequalities and Absolute Values (Fred Miller-Thistletown Collegiate)

The seminars will include a period for questions from the floor and general discussion.

Reception

All members and their wives are invited to attend the reception on Tuesday evening at the Faculty Club. This will give us all an opportunity to meet the speakers as well as old friends and classmates. The Faculty Club is located on Willcocks Street, near Spadina.

Luncheon

Arrangements have just been completed to have the luncheon in Bloor Street United Church, just across the road from O.C.E. on Bloor Street. The convenience of this location will, no doubt,

be welcomed by all members.

SYLVESTER'S PROBLEM OF COLLINEAR TRIADS

Ross Honsberger,
Northview Heights Collegiate Institute

In connection with the geometry of the lattice, the English mathematician James Sylvester (1814-1897) sought a finite set of distinct points in the plane, not all in a straight line, which possess the property that the straight line joining any two of them contains at least a third point of the set. This property is possessed by several pairs of points in the set A,B,...,H,I, shown in Figure 1 (G,H,I, are collinear by Pappus' theorem).

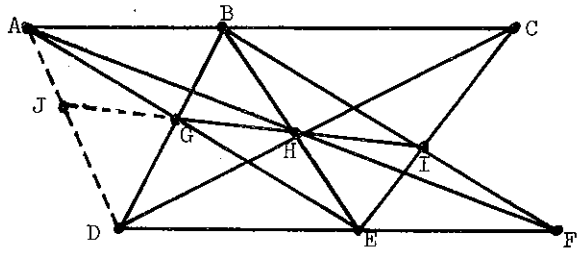
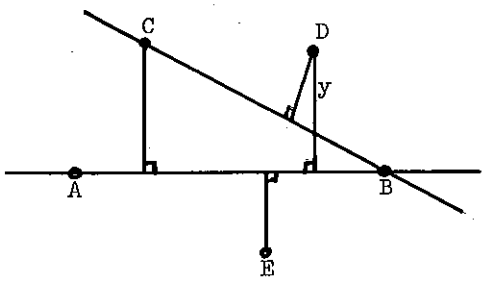


Figure 1-The Pappus Configuration

However, there are several lines such as AD or DI which contain only two points of the set. Of course, it is possible that an unsuccessful attempt might become successful by adding a few strategically placed points. In the above attempt, the point J, the intersection of AD and GH, might be included to correct the deficiency in the line AD. But, with J added, four new deficient pairs are formed - JB, JC, JE, JF. Sylvester was unable to solve the problem. Accordingly, in 1893, he proposed to the mathematical world the task of settling whether or not such a set of points was possible. The question was neglected till 1933, when T. Gallai (alias Grünwald) showed that for any finite set of distinct points in the plane, not all collinear, there exists a straight line containing exactly two of them. Gallai's proof for this is very complicated. Soon after his success, however, other simpler proofs were forthcoming. The following solution is due to L. M. Kelly.

Consider any finite set of distinct points in the plane, not all of which are collinear. Since they are not collinear, a line which joins a pair of them can not contain all the points. Thus, for each line joining a pair of points, there is at least one point which does not lie on the line. Consequently, a list of line-point combinations may be compiled in which

each line joining two of the points is associated with each of the points not lying on it (cf. Figure 2).



Line-point Combination	Distance
AB-C	x
AB-D	y
⋮	⋮
BC-A	z
⋮	⋮
DE-A	s
⋮	⋮

Figure 2.

Since the number of points in the set is finite, the number of line-point combinations, though perhaps very large, is finite also. Therefore, theoretically, a complete list of the combinations may be made and inspected.

Associated with each of these combinations is the perpendicular distance from the point of the combination to its line. These distances may be added to the list of combinations. The list of distances will have the same length as the list of combinations.

Since the list contains only a finite number of distances, there must be a distance for which there is none smaller. If the distances are all equal, then any one of them meets this requirement. Let a line-point combination for which the distance is minimal be AB-C, with distance p (cf. Figure 3).

The line AB contains only two points of the set (namely A and B), or it contains more than two. Let us suppose that there is at least a third point of the set, D, which lies on AB. Then, if K be the foot of the perpendicular from C to AB, the point D will put at least two points (of A,B,D) on one side of K, or it will divide A and B by coinciding with K (Figure 4). In the former case, the points might be distributed along the line in the order D, A, K, B, (two on one side of K). Thus the line-point combination CD-A would appear in the list of combinations, with its distance q, which clearly is smaller than the minimum distance p. Such an impossibility is encountered in each case for which at least two of A, B, D lie on one side of K. With D at K, and A and B on opposite sides of K, a similar contradiction is provided by either of the combinations CA-D or CB-D (Figure 4). Therefore, any combination yielding the minimum distance contains exactly two of the points.

Of course, there may be more than one line containing exactly

two of the points. Kelly and Moser proved that the number of such lines is at least three-sevenths the number of points in the set. Three-sevenths is the largest number which can be used in this theorem; for the vertices of a scalene triangle together with the midpoints of its sides and its centroid provide an example of a set of points for which the number of "two-point" lines is exactly three-sevenths the number of points (cf. Figure 5).

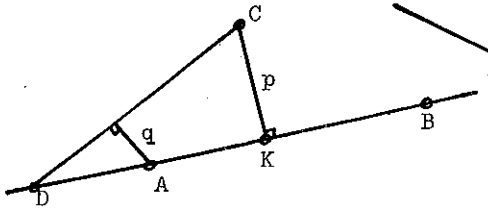


Figure 3.

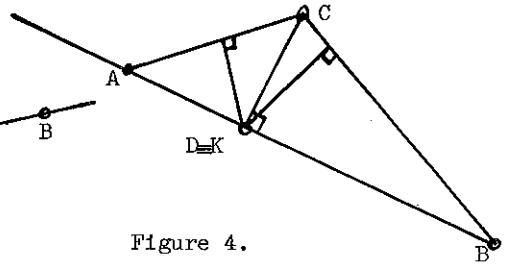


Figure 4.

In certain cases, all lines are "two-point" lines, as in Figure 6.

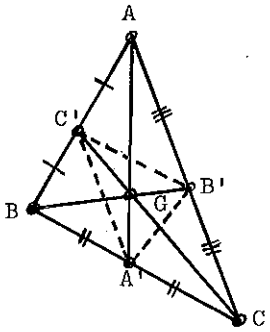


Figure 5.

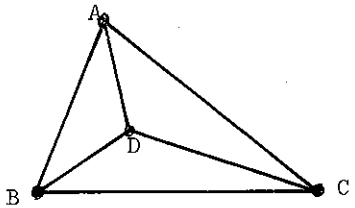


Figure 6.

Bibliographic Material

Howard Russell
East York Collegiate Institute

The majority of research studies in Mathematics Education which are undertaken or reported in North America are reviewed at the end of each decade in The Encyclopedia of Educational Research, and in the intervening years they are summarized periodically in the Review of Educational Research. The books and articles listed below constitute only a small sample of what may be found in these authoritative sources. The list will be continued in future issues to include some important European studies;

as well, it will include some newly completed American studies.

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AN INVITATION

The Executive and Councillors

of

The Mathematics and Physics Section

of

The Ontario Educational Association

request the pleasure of the company of

Members, their wives and husbands

at

A RECEPTION

on Tuesday, April the twenty-fourth

from eight o'clock to ten o'clock

THE FACULTY CLUB

University of Toronto

41 Willcocks Street

Parking: Willcocks Street, east of club.

PROBLEMS DEPARTMENT

This department will be devoted to the solution and discussion of problems of general interest which can be solved using secondary school level mathematics or elementary calculus. Readers are invited to submit either problems or solutions, or both. As many will be published in each issue as space permits. As a general rule, selected solutions will be published in the issue following the publication of the problem, together with the names of all readers who submitted correct solutions before the date of publication. Problems and solutions should be submitted to Dr. Gerald Berman, Department of Mathematics, The University of Waterloo, Waterloo, Ontario.

Grade 13 Annual Examinations, 1961

PROBLEMS

A regular feature of the Problems Department will be the solution of the Grade 13 PROBLEMS paper. The solutions below are not necessarily those given by the Department of Education. More than one solution is given where the editor found that different solutions seemed interesting or instructive. In the case of future exams, readers are invited to submit solutions and comments after the examination has taken place, and a selection of these will be published.

1. (a) What condition must be satisfied by the constants a,b,c for the quadratic function

$$f(x,y) = x^2 - y^2 + 2ax + 2by + c$$

to be the product of two linear factors?

(b) Indicate the nature of the graph of $f(x,y) = 0$ in the general case, and also when the condition referred to in part (a) is satisfied.

SOLUTION

(a) Completing the square on x and y yields

$$f(x,y) = (x+a)^2 - (y-b)^2 - (a^2-b^2-c).$$

The right hand side is the product of linear factors if and only if there are real numbers A and B such that

$$f(x,y) = [(x+a) + (y-b) + A][(x+a) - (y-b) + B].$$

The expressions for $f(x,y)$ are identical if and only if

$$A + B = 0, \quad A - B = 0, \quad AB = -(a^2-b^2-c) = -d, \text{ say;}$$

this implies $A = B = d = 0$. Thus $f(x,y)$ is the product of two linear factors if and only if

$$a^2 - b^2 - c = 0.$$

(b) Translate the origin to $(-a,b)$ by setting $x + a = X$, $y - b = Y$; then $f(x,y) = 0$ may be written in the form

$$X^2 - Y^2 = d.$$

If $d \neq 0$, the curve is a rectangular hyperbola with centre at $(-a,b)$; if $d > 0$, the major axis is parallel to the X axis, and if $d < 0$ the major axis is parallel to the Y axis. If $d = 0$, the curve consists of the pair of lines $X - Y = 0$, $X + Y = 0$, that is,

$$x - y = -a - b, \quad x + y = -a + b.$$

2. To what extent are x, y, z, which satisfy the equations

$$x + y + z = 2,$$

$$xy + yz + zx = 1,$$

determined by the condition that they be real.

SOLUTION 1

From the first equation, $z = 2 - x - y$; substitute in the second equation to give

$$x^2 + (y-2)x + (y-1)^2 = 0.$$

Since x is real, the discriminant is non-negative; hence

$$(y-2)^2 \geq 4(y-1)^2;$$

simplifying,

$$y(3y-4) \leq 0.$$

Thus y must lie between 0 and $4/3$. By symmetry, x and z must also lie in this range; thus

$$0 \leq x \leq 4/3, \quad 0 \leq y \leq 4/3, \quad 0 \leq z \leq 4/3.$$

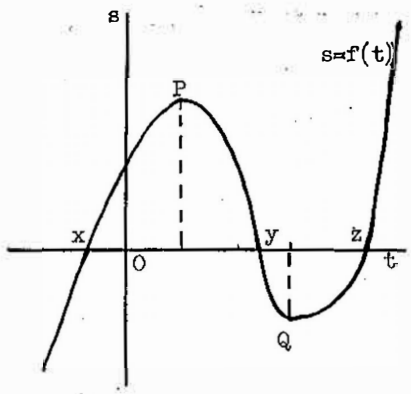
A typical solution might be $x = 4/3$, $y = z = 1/3$.

SOLUTION 2

More information is obtained by proceeding as follows. Let

$$f(t) = t^3 - 2t^2 + t - xyz = (t-x)(t-y)(t-z);$$

thus x, y, and z may be considered as the roots of the cubic equation $f(t) = 0$. Consider the curve $s = f(t)$, for x, y, z fixed. It is a cubic which cuts the t axis $(s=0)$ at $t = x$, $t = y$, $t = z$; s is positive for t a large positive number, and s is negative for t a large negative number. Furthermore



$f'(t) = 3t^2 - 4t + 1 = (3t-1)(t-1)$ vanishes at $t = 1$ and $t = 1/3$. Thus the point $P(\frac{1}{3}, \frac{4}{27} - xyz)$ must be a relative maximum and $Q(1, -xyz)$ a relative minimum. The roots will all be real if and only if P is not below the t axis and Q is not above the t axis (see the Figure). Thus

$$4/27 - xyz \geq 0, \quad -xyz < 0,$$

that is

$$0 \leq xyz \leq 4/27.$$

SOLUTION 3

It is not difficult to find all x, y, z which satisfy the equations. All real numbers which satisfy the linear equation may be written in the symmetric form

$$x = 2/3 + \ell t, \quad y = 2/3 + mt, \quad z = 2/3 + nt,$$

where ℓ, m, and n are any real numbers such that $\ell + m + n = 0$. Substituting these expressions into the second equation yields

$$t^2 = -1/3(\ell m + mn + n\ell) = 2/3(\ell^2 + m^2 + n^2),$$

noting that $0 = (\ell + m + n)^2 = \ell^2 + m^2 + n^2 + 2(\ell m + mn + n\ell)$. Thus, all the solutions of the equations are given by

$$x = 2/3 + \ell\sqrt{2/3(\ell^2+m^2+n^2)}, \quad y = 2/3 + m\sqrt{2/3(\ell^2+m^2+n^2)},$$

$$z = 2/3 + n\sqrt{2/3(\ell^2+m^2+n^2)},$$

where ℓ, m, and n are any real numbers satisfying $\ell+m+n = 0$.

The last solution is not as obscure as it may seem at first. Since

$$(x+y+z)^2 = x^2+y^2+z^2 + 2(xy+yz+zx),$$

the numbers x, y, z also satisfy the equation

$$x^2 + y^2 + z^2 = 2.$$

From analytic geometry, we know that this equation represents a sphere with centre at the origin and radius $\sqrt{2}$. Thus the solutions of the equations are the points of intersection of this sphere and the plane $x + y + z = 2$. This turns out to be a circle of radius $\frac{2}{\sqrt{3}}$ with centre at $(2/3, 2/3, 2/3)$.

3. If s_n denotes the sum of the first n natural numbers, find the sum of the infinite series

$$\frac{s_1}{1} + \frac{s_2}{2} + \frac{s_3}{4} + \frac{s_4}{8} + \cdots .$$

SOLUTION 1

It is known that $s_n = n(n+1)/2$; so the series may be rewritten in the form

(1) $s = 1.2(\frac{1}{2}) + 2.3(\frac{1}{2})^2 + \ldots + n(n+1)(\frac{1}{2})^n + \ldots$.

The coefficients are similar to binomial coefficients; in fact

$$\frac{1}{(1-x)^3} = \frac{1.2}{2} + \frac{2.3}{2}x + \ldots + \frac{n(n+1)}{2}x^{n-1} + \ldots .$$

If we set $x = 1/2$, we obtain $s = 1/(1 - \frac{1}{2})^3 = 8$.

SOLUTION 2

Denote the sum by s as before. Multiplication by $\frac{1}{2}$ yields

(2) $\frac{1}{2}s = 1.2(\frac{1}{2})^2 + (2.3)(\frac{1}{2})^3 + \ldots + (n-1)\,n(\frac{1}{2})^n + \ldots$.

Subtracting (2) term by term from the series (1) for s, we get

(3) $\frac{1}{2}s = s - \frac{1}{2}s = 2(\frac{1}{2}) + 4(\frac{1}{2})^2 + \ldots 2n(\frac{1}{2})^n + \ldots$.

Multiply expression (3) by $\frac{1}{2}$, and subtract the resulting expression from (3).

(4) $\frac{1}{2}s - \frac{1}{4}s = 1 + \frac{1}{2} + \ldots + (\frac{1}{2})^n + \ldots = 1/(1 - \frac{1}{2}) = 2$.

Using (4), we have $s = 8$, as before.

4. Prove that

$$\frac{m}{(m+1)(2m+1)} < \sum_{r=m+1}^{2m} \frac{1}{r^2} < \frac{1}{2m} ,$$

for all integers m, where

$$\sum_{r=m+1}^{2m} \frac{1}{r^2} = \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \ldots + \frac{1}{(2m)^2} .$$

SOLUTION 1

Since $n-1 < n < n+1$, it follows that

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} .$$

For convenience, set $s_m = \sum_{r=m+1}^{2m} \frac{1}{r^2}$. Then

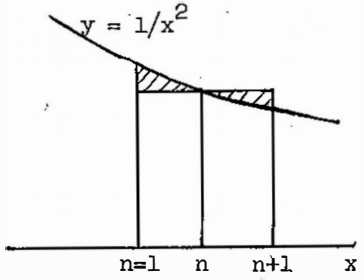
$$s_m < \sum_{r=m+1}^{2m} (\frac{1}{r-1} - \frac{1}{r}) = \frac{1}{m} - \frac{1}{2m} = \frac{1}{2m} ,$$

and

$$s_m > \sum_{m+1}^{2m} (\frac{1}{r} - \frac{1}{r+1}) = \frac{1}{m+1} - \frac{1}{2m+1} = \frac{m}{(m+1)(2m+1)} .$$

SOLUTION 2

Calculus may be used to solve this problem as follows. Consider the curve $y = 1/x^2$. The value of y decreases as x increases, so that the area between the x axis, the curve



$y = 1/x^2$, and the lines $x = n$, $x = n-1$ is greater than $1/n^2$; similarly the area between the x axis, the curve $y = 1/x^2$, and the lines $x = n$, $x = n+1$, is less than $1/n^2$. This is obvious from the diagram (each of the rectangles shown has area $1/n^2$).

Thus

$$\int_n^{n+1} \frac{dx}{x^2} < \frac{1}{n^2} < \int_{n-1}^{n} \frac{dx}{x^2} .$$

So that

$$s_m > \sum_{m+1}^{2m} \int_n^{n+1} \frac{dx}{x^2} = \int_{m+1}^{2m+1} \frac{dx}{x^2} = \frac{m}{(m+1)(2m+1)}$$

and

$$s_m < \sum_{m+1}^{2m} \int_{n-1}^{n} \frac{dx}{x^2} = \int_m^{2m} \frac{dx}{x^2} = \frac{1}{2m} .$$

5. (a) In a triangle with vertices (a,0), (-a,0), (0,b), prove that

(i) the medians concur in a point G (the centroid),

(ii) the right-bisectors of the sides concur in a point H (the circumcentre),

(iii) the altitudes concur in a point K (the orthocentre).

(b) Find the ratio HG:GK.

(c) Draw a figure to show the positions of these three points when a = b.

SOLUTION

Let A, A', and B denote the points with coordinates (a,0), (-a,0), and (0,b) respectively.

(i) Let M, M' denote the midpoints of AB and A'B respectively.
Then the coordinates of M and M' are $(\frac{1}{2}a, \frac{1}{2}b)$ and

$(-\frac{1}{2}a,\frac{1}{2}b)$ respectively, and the equations of the medians A'M and AM' are

$$3ay = b(x+a), \quad -3ay = b(x-a).$$

These lines intersect in the point G with coordinates $(0,b/3)$. The third median $x = 0$ also passes through G.

(ii) The slope of AB is $-b/a$. Thus the right bisector of AB passes through $M(\frac{1}{2}a,\frac{1}{2}b)$, and has slope a/b. Similarly, the right bisector of A'B passes through $(-\frac{1}{2}a,\frac{1}{2}b)$ and has slope $-a/b$. The corresponding equations are

$$b(2y-b) = a(2x-a),$$

$$b(2y-b) = -a(2x+a).$$

The intersection point H is $[0,(b^2-a^2)/2b]$. The third right bisector, $x = 0$, passes through H.

(iii) The altitude of the triangle through A' is clearly parallel to the right bisector through M, and thus has slope a/b. Similarly, the altitude through A has slope $-a/b$. The equations are

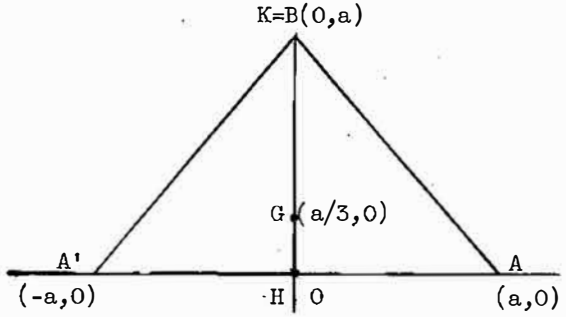
$$by = a(x+a), \quad by = -a(x-a),$$

and the point of intersection K is $(0,a^2/b)$. The third altitude, $x = 0$, passes through K.

(b) The points G, H, and K lie on the y axis. Thus $HG = b/3 - (b^2-a^2)/2b = (3a^2-b^2)/6b$, and $GK = (a^2/b - b/3) = (3a^2-b^2)/3b$; so $HG/GK = \frac{1}{2}$.

(c) In the special case $a = b$, the coordinates of the points G, H, K become $(0,a/3)$, $(0,0)$, $(0,a)$, as shown in the diagram.

[In this problem, it is sufficient to obtain equations relating to A. The equations relating to A' are then obtained by replacing a by -a.]



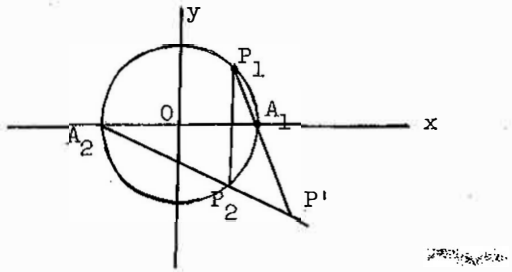
6. If the circle $x^2 + y^2 = r^2$ meets the x axis in A_1 and any chord parallel to the y axis in P_1 and P_2, prove that the locus of the point of intersection of A_1P_1 and A_2P_2 is the rectangular hyperbola $x^2 - y^2 = r^2$.

SOLUTION

The coordinates of A_1 and A_2 are $(r,0)$ and $(-r,0)$. Let $P_1(x_1,y_1)$ be any point on the circle, thus $x_1^2 + y_1^2 = r^2$. The coordinates of P_2 are $(x_1,-y_1)$. Let $P'(x',y')$ be the point of intersection of A_1P_1 and A_2P_2. Since P' lies on A_1P_1 and A_2P_2, we have

$$\text{slope } P_1A_1P' = \frac{y'}{x'+r} = \frac{y_1}{x_1+r};$$

$$\text{slope } A_2P_2P' = \frac{y'}{x'-r} = \frac{y_1}{x_1-r}.$$



Multiply the equations together, we find that

$$\frac{y'^2}{x'^2-r^2} = \frac{-y_1^2}{x_1^2-r^2} = \frac{-(r^2-x_1^2)}{x_1^2-r^2} = 1.$$

Thus (x',y') lies on the hyperbola $x^2 - y^2 = r^2$.

7. A chord PQ of a parabola cuts the x axis in R; tangents at P,Q intersect in T. Prove that RT is bisected by the tangent at the vertex.

SOLUTION 1

By choice of coordinate axes, the equation of the parabola can be put into the form $y^2 = 4ax$, and the points P and Q represented in terms of a parameter in the form $P = (au^2,2au)$,

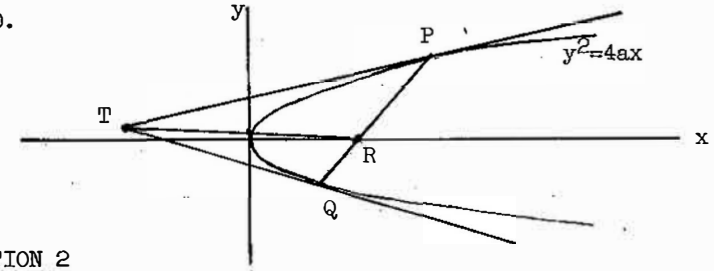
$Q = (av^2, 2av)$. The equations of the tangents at P and Q are then

$$x - uy + au^2 = 0, \quad x - vy + av^2 = 0$$

respectively, and the intersection point T has coordinates $[auv, a(u+v)]$. The equation of PQ is

$$2x - (u + v)y + 2auv = 0;$$

hence R, the point of intersection with $y = 0$, has coordinates $(-auv, 0)$. Thus the midpoint of RT has coordinates $[0, \frac{a}{2}(u+v)]$, and lies on the tangent at the vertex, namely, $x = 0$.



SOLUTION 2

A much simpler solution may be found by noting that PQ is the polar of T. Thus, if the coordinates of T are (x_0, y_0), the equation of PQ is

$$yy_0 = 2p(x + x_0).$$

Setting $y = 0$, we find that R has coordinates $(-x_0, 0)$; hence S, the midpoint of RT, has coordinates $(0, \frac{1}{2}y_0)$, that is, S is on the tangent at the vertex.

8. Tangents are drawn to the circle $x^2 + y^2 = 1$ from any point (h,k) on the fixed line $x = h$, where $h > 1$, and from the point $(-h,-k)$. Prove that the two vertices of the parallelogram formed by these tangents lie on the ellipse

$$x^2 + y^2(1 - 1/h^2) = 1$$

for all values of k.

SOLUTION

Let $Q(x',y')$ be one of the intersection points. The line through Q and (h,k), which has equation

$$(y'-k)x - (x'-h)y + kx' - hy' = 0,$$

must be tangent to the circle $x^2 + y^2 = 1$. Thus the perpendicular distance from the origin to this line must be 1; squaring, we obtain the condition

$$(1) \qquad (x'-h)^2 + (y'-k)^2 = (kx'-hy')^2.$$

Similarly, the condition that the line joining Q to $(-h,-k)$ be tangent to the circle yields (replace h and k by -h and -k)

$$(2) \qquad (x'+h)^2 + (y'+k)^2 = (kx'-hy')^2.$$

We need to get a relationship between x' and y' by eliminating k between (1) and (2). Adding and subtracting (1) and (2), we obtain

$$(3) \qquad x'^2 + y'^2 + h^2 + k^2 = (kx' - hy')^2,$$

$$(4) \qquad 0 = hx' + ky'.$$

Squaring (4) and adding to (3) yields on simplification

$$x'^2 + y'^2 - 1 = \frac{x'^2 + y'^2}{h^2 + k^2};$$

Using (4) again, the right hand side equals y'^2/h^2.

9. The inscribed circle of the triangle ABC touches the sides BC, CA, AB in the points P, Q, R respectively. Show that

$$\cos \tfrac{1}{2}A : \cos \tfrac{1}{2}B : \cos \tfrac{1}{2}C : = QR : RP : PQ.$$

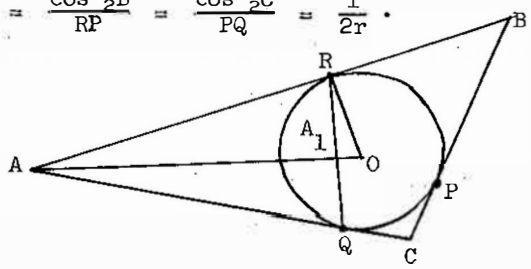
SOLUTION

Let O be the centre of the inscribed circle, and let OA cut QR in A_1. It is immediate that OA bisects the angle A, and the line segment QR. Further, since OAR is a right angled triangle, $< AOR = 90^\circ - \frac{1}{2}A$. Thus

$$QR = 2A_1R = 2r \sin (90^\circ - \tfrac{1}{2}A) = 2r \cos \tfrac{1}{2}A.$$

Similarly $RP = 2r \cos \frac{1}{2}B$ and $PQ = 2r \cos \frac{1}{2}C$, and we have

$$\frac{\cos \frac{1}{2}A}{QR} = \frac{\cos \frac{1}{2}B}{RP} = \frac{\cos \frac{1}{2}C}{PQ} = \frac{1}{2r}.$$



10. Given that one side of a triangle is twice as long as another side, show that the angle opposite the first-named side is more than twice the angle opposite the other side.

SOLUTION

Let the triangle ABC, with interior angles α, β, γ, respectively and sides $a = BC$, $b = CA$, $c = AB$ be such that $a = 2b$. We must show that $\alpha > 2\beta$, or equivalently, $\sin(\alpha - 2\beta) > 0$. The sine law shows that $\sin\alpha = 2\sin\beta$; hence

$$\begin{aligned}\sin(\alpha - 2\beta) &= \sin\alpha\cos 2\beta - \cos\alpha\sin 2\beta \\ &= \sin\alpha[1 - 2\sin^2\beta] - 2\cos\alpha\sin\beta\cos\beta \\ &= \sin\alpha - 2\sin\beta[\sin\beta\sin\alpha + \cos\alpha\cos\beta] \\ &= \sin\alpha - 2\sin\beta\cos(\alpha-\beta) \\ &= \sin\alpha - \sin\alpha\cos(\alpha-\beta) \\ &= \sin\alpha[1 - \cos(\alpha-\beta)].\end{aligned}$$

Since $\alpha - \beta > 0$, $\cos(\alpha-\beta) \neq 1$; hence $\sin(\alpha-2\beta) > 0$. This proves the result.

11. Two equal rectangles, both inscribed in the circle $x^2 + y^2 = 1$, with their axes of symmetry along the x axis and y axis, cross each other, forming a square ABCD which is common to both rectangles.

(a) If θ is the acute angle between the diagonal of one rectangle and its major axis of symmetry, find, in terms of θ, the total area of the four rectilinear figures exterior to ABCD.

(b) Find the value of $\tan\theta$ when this total area is a maximum.

SOLUTION 1

(a) Since the diagonals of the rectangles have length 2, it follows that the two sides have lengths $2\cos\theta$ and $2\sin\theta$, with $2\cos\theta > 2\sin\theta$. Thus the area of each rectangle is $4\sin\theta\cos\theta$, and the area of the square ABCD is $4\sin^2\theta$. The required area R, which is twice the difference between the area of one of the rectangles and the square is given by

$$R = 2(4\sin\theta\cos\theta - 4\sin^2\theta).$$

(b) Noting that $2\sin\theta\cos\theta = \sin 2\theta$, $-2\sin^2\theta = \cos 2\theta - 1$, and $\cos 45^\circ = \sin 45^\circ = 1/\sqrt{2}$, we can rewrite R in the form

$$R = 4(\sin 2\theta + \cos 2\theta - 1) = 4[\sqrt{2}\sin(2\theta + 45^\circ) - 1].$$

It follows that R is a maximum when $\sin(2\theta + 45^\circ) = 1$.

Thus $\theta = 22\frac{1}{2}^\circ$; $1 = \tan 45^\circ = 2\tan 22.5^\circ/(1-\tan^2 22.5^\circ)$.

Hence $\tan 22.5^\circ$ is the positive root of the equation $x^2 - 2x - 1 = 0$, that is, $\tan 22.5^\circ = -1 + \sqrt{5}$.

SOLUTION 2

Calculus can be used in (b) to find the maximum value of R. Thus

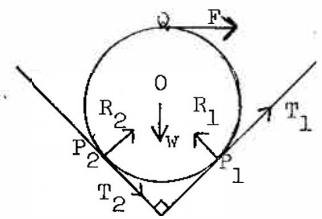
$$\begin{aligned}\frac{dR}{d\theta} &= 8(\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta) \\ &= 8(\cos 2\theta - \sin 2\theta).\end{aligned}$$

We see that $\frac{dR}{d\theta} = 0$ when $\cos 2\theta = \sin 2\theta$; hence $\tan 2\theta = 1$, $2\theta = 45^\circ$, as before.

12. A cylinder of weight w pounds rests with its axis horizontal in a trough made of two planes which are perpendicular to each other and equally inclined to the vertical. A horizontal force of F pounds, applied at the top of the cylinder and perpendicular to its axis, is just on the point of turning it. Assuming that the cylinder in turning remains in contact with both faces of the trough, and that the coefficient of friction between the cylinder and each of the two planes is μ, show that

$$F = \frac{w\mu\sqrt{2}}{\mu^2(1+\sqrt{2}) + 1}.$$

SOLUTION



Let the tangential forces at the points of contact be denoted by T_1 and T_2, and let R_1 and R_2 be the corresponding normal reactions as shown. The only other forces acting are the weight w, acting at the centre of the cylinder, and F at Q. Equating horizontal and vertical forces, yields

(1) $$F - R_1/\sqrt{2} - R_2/\sqrt{2} + T_1/\sqrt{2} + T_2/\sqrt{2} = 0$$

and

(2) $$w - R_1/\sqrt{2} - R_2/\sqrt{2} - T_1/\sqrt{2} + T_2/\sqrt{2} = 0.$$

Taking moments about O, yields

(3) $$Fa = T_1a + T_2a,$$

since R_1, R_2, and w act through O. Friction at P_1 and

P_2 yields

(4) $\quad T_1 = \mu R_1, \quad T_2 = \mu R_2.$

The required relation is obtained by eliminating R_1, R_2, T_1, and T_2 from these equations. From (1) and (3), we get

(5) $\quad (\sqrt{2} + 1)F = R_1 - R_2.$

(3) and (4) imply

(6) $\quad R_1 + R_2 = \frac{1}{\mu}(T_1 + T_2) = \frac{1}{\mu} F, \; R_1 - R_2 = \frac{1}{\mu}(T_1 - T_2).$

Substituting in (2) yields

$$\sqrt{2}w = \frac{1}{\mu}F + \mu(R_1 - R_2);$$

then, using (5), we obtain

$$\sqrt{2}w = \frac{1}{\mu} F + \mu(\sqrt{2} + 1)F.$$

This equation is easily solved for F in terms of w and μ.

BOOK REVIEWS

Sets, Sentences, and Operations.

by Donovan A. Johnson and William H. Glenn.

In the series "Exploring Mathematics on Your Own". Webster Publishing Company, St. Louis, 1960. 63pp. $1.05.

This booklet was written with the intention of introducing the terminology and symbolism of the set method and the use of Venn Diagrams to mathematical enthusiasts who have not previously been familiar with it. The material is quite within the grasp of most secondary school students and would serve as enrichment material for any classes not already studying the so-called new mathematics. For classes already studying this approach, the booklet might offer novel ideas and interesting problems. However, the terminology and symbolism is sometimes different from that presently in use in recent Ontario publications. Examples of such differences are "universe of a variable" rather than "domain of a variable", and frequently "{ }" for the empty set rather than "$\emptyset$". Each symbol and term is well explained; so no ambiguity should occur.

The booklet is set up in three sections: sets and set terminology; sets and solving problems; sets in geometry, algebra, and logic. A concise but entirely adequate description of each topic is followed by three to five exercises based on the material previously presented. There are frequent examples

of complete solutions to problems in the presentation part of a topic; for all exercise material, the answers are given at the back of the book.

Three-colour presentation gives the book an interesting appearance, and the red colour emphasizes the important ideas to be gained from the frequent diagrams throughout.

The booklet is well written and should provide a good understanding of set terminology, operations with sets, and use of sets and elementary mathematics in some practical problems. Lack of an index causes a slight handicap as far as referring to particular topics, but the small size of the book minimizes the inconvenience.

A short bibliography is given to lead interested readers to more thorough and more advanced books using this approach to mathematics.

J.F.Chambers, Jarvis Collegiate Institute, Toronto.

Some Views on Mathematical Education

Professor GEORGE POLYA was recently invited to address the Panel Discussion of the Canadian Mathematical Congress. He replied that he was extremely interested but to his deep regret had to sail for Europe on August 21. However, he kindly sent a message, reproduced below:

To the Panel Discussion of the Canadian
Mathematical Congress
Friday, September 8, 1961

I wish to congratulate the Canadian Mathematical Congress for initiating the panel discussion to which these lines are addressed.

In fact, there is virtually unanimous agreement that the high schools should teach more mathematics and should teach it better. Yet it is much less clear how this goal should be attained.

The various groups now working on the revision of the curriculum deserve the thanks of the public. Yet two points should be clear: First, the proposed programs are experimental; some of their proposals may be eventually adopted, several will be rejected. To regard any of their proposals as final at this time, before further discussion and experimentation, to accept any one as a dogma, would be a grievous error. Second, no change in the curriculum, however desirable it may be in itself,

can cure all the ills: the real bottleneck is, in my opinion, the insufficient training of the teachers. Hence my best wishes to your panel which has the important task to discuss remedies for the sorest point of the high school mathematics instruction.

I wish to submit two points to your consideration to which I attached the greatest weight in the last five or six years in which my teaching activity was almost exclusively devoted to high school teachers. Each high school teacher of mathematics should have, I think, opportunity in his training for two things:

(1) Some practive in solving problems which need only knowledge on the high school level, but some degree of independence and originality. In fact, how should the teacher recognize or direct the creativity of his students if he himself had never an opportunity to do something approaching creative work.

(2) To acquire knowledge of physical science, at least some serious understanding of elementary mechanics and optics. In fact, separated from science, mathematics loses one of its most important sources of interest and inspiration, not to mention usefulness.

I wish to mention that, in my instruction, I tried to take care of these two points which I consider as crucial by two courses. They are:

(i) Seminar in Problem Solving;

(ii) Mathematical Methods in Science.

My very best wishes for the success of your panel.

George Polya.

Some Remarks from our O.M.C. Editor

R.B.Gwilliam, Ryerson Institute of Technology

Prior to the formation of the Ontario Mathematics Commission in the summer of 1960, a conference had been held at Lakefield, under the sponsorship of the Ontario Teachers' Federation, to discuss various aspects of the curriculum and teaching of mathematics in the Ontario school system. The promoters of the Lakefield conference were motivated by various considerations which included the work of several groups in the United States and a feeling that major changes in the mathematics programmes might be desirable.

The Ontario Mathematics Commission inherited the "Lakefield

Proposals" which stated that, whilst the basic mathematical content of the curriculum was sound, some new topics and different approaches to existing topics should be tried in some schools on an experimental basis during the coming years. The production of the experimental texts at present in use in selected Grade IX and Grade X classes was a direct result of the "Lakefield Proposals". However, serious doubts were expressed by many members of the Commission about the desirability of using public funds - grants for the support of the Commission from the Department of Education - to encourage the production of textbooks by potential authors. There are some members who would like to see the Commission limit itself to producing monographs on mathematical topics and teaching methods, and to encouraging authors who are not members of the Commission to write books for approval by the Department of Education.

During its second year of life the O.M.C. altered its constitution to relegate the writing committee's work to a position subordinate to the work of the Advisory Committees on Secondary and Elementary Education. By this change it was hoped to encourage commission members to devote time to assessing the current curricula in foreign countries and to making predictions about the future of mathematical education in Ontario. The recent news of the 'Robarts Plan' makes this work most desirable.

Not unnaturally, the Commission and its works have been subjected to criticism; some of the more important ones are summarized below.

(a) The O.M.C. is not, in fact, commissioned to do anything. It has no 'terms of reference' and its work has not been officially coördinated with the curriculum branch of the Department of Education.

(b) The members of the Commission do not adequately represent all the groups and professional bodies concerned with mathematical education in the schools. Some persons have suggested that the universities should not be represented exclusively by members of Departments of Mathematics, but that Schools of Science and Engineering have interests in the work of the Commission.

(c) The O.M.C., to date, has been only slightly concerned with the mathematical training of the majority of school pupils who receive less than ten years of formal education.

(d) Consideration should be given to developing programmes of mathematics which would encourage rapid progress to creative levels of mathematics by exceptional pupils at high schools. It may be that some segregated schooling should be tried to reach this end.

Comments about 'Modern Mathematics' are not lacking; since the major productions of the Commission to date have been favourably disposed towards 'axiomatics' and 'set theory', it may be in order to quote some recent statements in connection with other considerations.

Professor George Polya in a recent letter to the Canadian Mathematical Congress stated that, in his view, the bottleneck in mathematical education is the insufficient training of teachers; Professor Polya's statement is reproduced in its entirety elsewhere in the Gazette.

Professor Andrew Gleason of Harvard, in Vol. 68 (No.9) of the American Mathematical Monthly, comments: "Among the so-called modern approaches to college mathematics are those that put altogether too much emphasis on formal logic and set theory. It does not take very much formal manipulation of nonsense statements or cup and cap signs to convince a bumptious freshman that mathematics is trivial and impractical. He is likely to drop the subject, read a semipopular book, and become a firm believer in the curves-on-pretzels canard. Let us not start any subject unless we can take it far enough to justify the effort. Set theory, for example, hardly qualifies on its own merits; so let us teach it only as a tool and shy away from rhapsodies on the uncountability of the real numbers."

It would thus seem that some aspects of the so-called modern approach have value, but one should take great pains to avoid extremes.

ABSOLUTE VALUES

J. Norman C. Sharp
Burnhamthorpe Collegiate Institute

1. Introduction

There are many instances in mathematics in which scalar quantities are used. These quantities have magnitude and no direction. In calculation of some of these, the computation yields a negative result, or it may yield a positive result, although our interpretation would be independent of sign. Examples are area, length, volume, displacement, etc. In some cases, we have placed a plus or minus sign in front of the numeral with instructions to use the sign which yields a positive result. A neater way of expressing this is to use the absolute value $|x|$, which is read as "the absolute value of x".

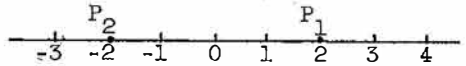
There are two definitions for $|x|$; either definition may be used with the same result. These are:

(1) $|x|$ is the greater of x and $-x$;

(2) $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$.

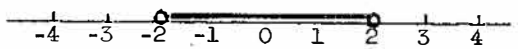
For example, $|3| = 3$ and $|-3| = 3$.

When considering distance in a one-dimensional graph, we can find two points such that $|x| = 2$. These are the two points P_1 and P_2



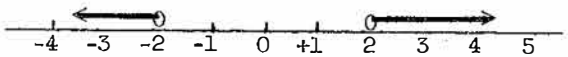
whose distance from 0 is 2. With centre at 0 and radius 2, the number line is cut in two places P_1 and P_2, that is, at 2 and -2. Thus if $|x| = 2$, then $x = 2$ or -2, and the solution is $\{2,-2\}$.

If $|x| < 2$, then with centre at 0 and any radius less than 2, points are located on the number line. When we consider all the radii less than 2, the set of points obtained consists of all points lying between -2 and 2.



This set is $\{x \mid -2 < x < 2\}$, and is an abbreviation of $\{x \mid x > -2$ and $x < 2\}$. If $|x| \leq 2$, there are two extra points included in the set; they are 2 and -2. Hence the set of points satisfying the condition is $\{x \mid -2 \leq x \leq 2\}$, and is an abbreviation of $\{x \mid x \geq -2$ and $x \leq 2\}$.

The next condition is $|x| > 2$. The set of points satisfying this condition would require centre at 0 and radius greater than 2. These are represented on the following graph.



This set includes all points not between nor including 2 and -2, and is written

$$\{x \mid x > 2 \quad \text{or} \quad x < -2\}$$

The three cases considered here are :

$$\{x \mid |x| = 2\} = \{x \mid x = 2 \quad \text{or} \quad x = -2\};$$
$$\{x \mid |x| < 2\} = \{x \mid -2 < x < 2\};$$
$$\{x \mid |x| > 2\} = \{x \mid x < -2 \quad \text{or} \quad x > 2\}.$$

2. Solving Equations and Inequalities Involving Absolute Values

Example 1. Solve for x: $|x - 2| = 5$.

We find $x - 2 = 5$ or $x - 2 = -5$;
$x = 7$ or $x = -3$.

Example 2. Solve for x: $|x - 2| < 5$.

$-5 < x-2 < 5$,

Hence $-3 < x < 7$.

Example 3. Solve for x: $|x - 2| > 5$.

$x - 2 > 5$ or $x - 2 < -5$;

Hence $x > 7$ or $x < -3$.

Other equations involving absolute values may be solved as the following examples illustrate.

Example 4. $|x - 3| = |3x + 7|$

Hence $x - 3 = 3x + 7$ or $x - 3 = -(3x + 7)$,

$2x = -10$ or $4x = -4$,

$x = -5$ or $x = -1$.

Substitution will show that -1 and -5 both satisfy the original equation.

An alternative solution to Example 4 would be:

$|x - 3|^2 = |3x + 7|^2$,

$(x - 3)^2 = (3x + 7)^2$,

$x^2 + 6x + 5 = 0$.

Hence $x = -5$ or $x = -1$.

Example 5. To solve $|x - 3| < |3x + 7|$, we must have a positive number $|3x + 7|$ greater than a positive number $|x - 3|$. This is equivalent to considering when $(3x + 7)^2$ is greater than $(x - 3)^2$, since each of these squares is a positive number. The solution is found by solving the inequality

$(3x + 7)^2 > (x - 3)^2$, that is, $(3x + 7)^2 - (x-3)^2 > 0$.

Hence $9x^2 + 42x + 49 - x^2 + 6x - 9 > 0$,

$8x^2 + 48x + 40 > 0$,

$x^2 + 6x + 5 > 0$,

$(x + 1)(x + 5) > 0$.

Hence x must lie outside of the interval between -1 and -5, that is, $x < -5$ or $x > -1$.

3. Absolute Values and Square Roots

A square root of a given number is a number which, when squared, equals the given number. Thus 5 is a square root of 25, since $5^2 = 25$; also, -5 is a square root of 25, since $(-5)^2 = 25$. By convention, we use the surd sign to denote the positive square root only; thus $\sqrt{25} = 5$, and $-\sqrt{25} = -5$. Since we

use the positive square root, we must be consistent and write $\sqrt{x^2} = |x|$, realizing that x may be positive or negative.

Example 6. $\sqrt{7^2} = 7$, $\sqrt{(-4)^2} = 4$.

Ex.7. Solve $\sqrt{x^2 - 2x + 1} = 5$; write $|x - 1| = 5$;

then $x - 1 = 5$ or $x - 1 = -5$,

$x = 6$ or -4.

The solution set is $\{6,-4\}$.

Example 8. To solve $\sqrt{x^2 - 4x + 4} + \sqrt{4x^2 - 12x + 9} = 10$, write $|x - 2| + |2x - 3| = 10$.

This equation is equivalent to four equations:

(1) $(x-2) + (2x-3) = 10$

$3x-5 = 10$

$3x = 15$

$x = 5$

(2) $(x-2)-(2x-3) = 10$

$-x+1 = 10$

$x = -9$

(3) $-(x-2)-(2x-3) = 10$

$-3x+5 = 10$

$-3x = 5$

$x = -5/3$

(4) $-(x-2)+(2x-3) = 10$

$x-1 = 10$

$x = 11$

There are four solutions to this equation, namely, 5, -9, -5/3, 11; these may be verified by substitution

4. Graphs Involving Absolute Values in Two Dimensions

Some interesting graphs can be obtained in two dimensions.

Example 9. Draw the graph of $y = |x|$.

This means y must always be positive or zero; with this restriction the equation is equivalent to the two equations $y = x$ or $y = -x$. Since y must be positive or zero, the graph is $y = x$ for $x \geq 0$ and $y = -x$ for $x \leq 0$ (Figure 1).

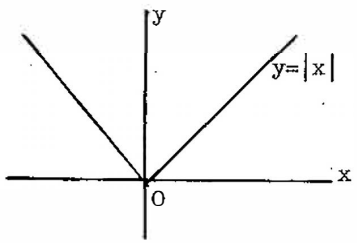


Figure 1

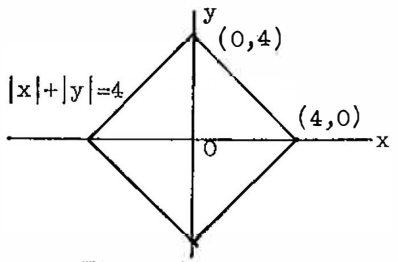


Figure 2

Example 10. Draw the graph of $|x| + |y| = 4$.

This is equivalent to the four equations $x + y = 4$, $x - y = 4$, $-x + y = 4$, $-x - y = 4$. However, $|x|$ and $|y|$ are never negative; hence the maximum value for each is 4. This means that $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$. With this restriction, the graphs of the four lines are cut off at $(4,0)$, $(-4,0)$, $(0,4)$, and $(0,-4)$. See Figure 2.

In many graphs there are points where abrupt changes take place. These can be called separation points. A few examples will illustrate this.

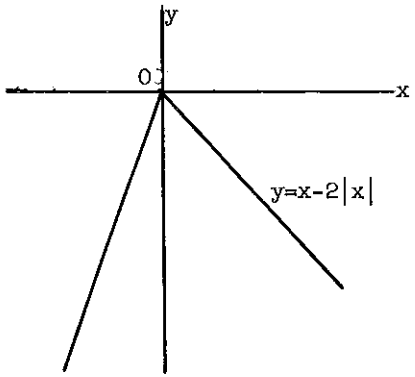


Figure 3.

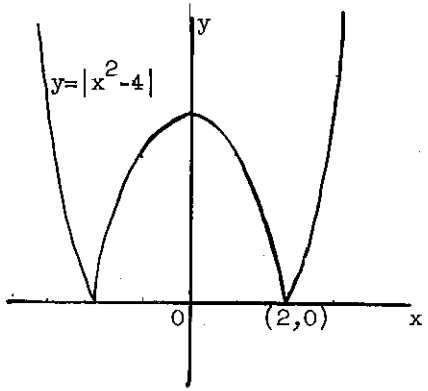


Figure 4.

Example 11. Draw the graph of $y = x - 2|x|$.

The separation point on this graph is $x = 0$. If $x \geq 0$, then the equation is equivalent to $y = -x$. If $x \leq 0$, then the equation is equivalent to $y = 3x$. See Figure 3.

Example 12. Draw the graph of $y = |x^2 - 4|$.

The separation points are at $x = 2$ and $x = -2$. If $x \geq 2$ or $x \leq -2$, then the equation is equivalent to $y = x^2 - 4$. If $-2 \leq x \leq 2$, then the equation is equivalent to $y = 4 - x^2$. See Figure 4.

Example 13. Draw the graph of $y = |x| + |x - 1|$.

The separation points are at 0 and 1; thus we consider the intervals $1 \leq x$, $0 \leq x \leq 1$, $x \leq 0$. The corresponding equations are

$y = 2x - 1$ if $1 \leq x$,

$y = 1$ if $0 \leq x \leq 1$,

$y = 1 - 2x$ if $x \leq 0$.

See Figure 5.

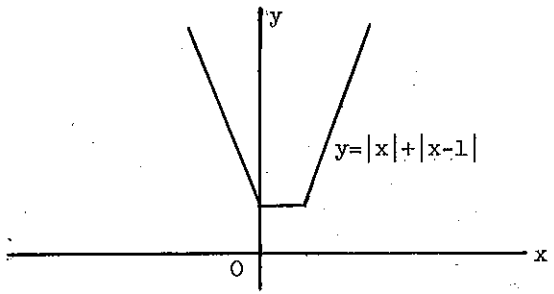
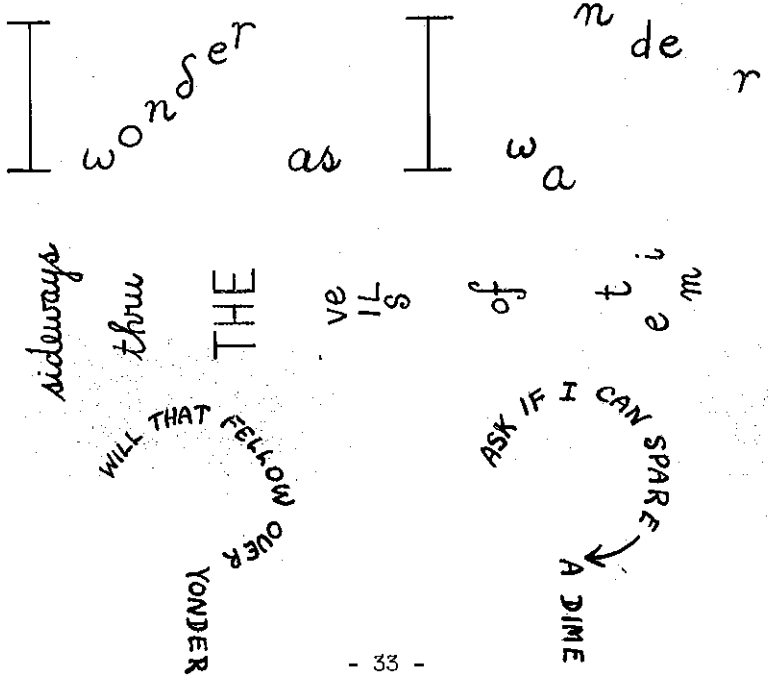


Figure 5.

The use of absolute value is becoming more important in secondary school mathematics, and practice in its use is necessary for an understanding of its applications.

P O E T R Y S E C T I O N

At the present time there seems to be an increasing interest in abstract art. We therefore feel that we are privileged to print this timely gem of abstract poetry by Professor K. D. Fryer. The poem is accompanied by six pages of interpretive criticism, as well as twenty three footnotes; because of exigencies of space, we are forced to omit the criticism and footnotes.



NOTICE

Readers of the Ontario Mathematics Gazette may be interested in becoming members of the Canadian Mathematical Congress. Membership is only $3.00 per annum, and entitles the member to receive the Canadian Mathematical Bulletin. Details about joining the Canadian Mathematical Congress can be obtained by writing to

Professor W. L. G. Williams,
Canadian Mathematical Congress

PROBLEMS SUBMITTED

Problem 1 (Mrs. Jean Leppard, Vaughan Road Collegiate Institute)

I should like to see a neat solution of a problem from the 1956 Problems paper: "If an ellipse rolls without slipping on another ellipse of the same size so that the two ellipses touch at corresponding points, prove that the locus of the focus of the moving ellipse, is a circle with centre at one of the foci of the fixed ellipse". I always feel a bit discursive when I write down a solution to this problem.

Problem 2 (Norman Gillespie, North Toronto Collegiate Institute)

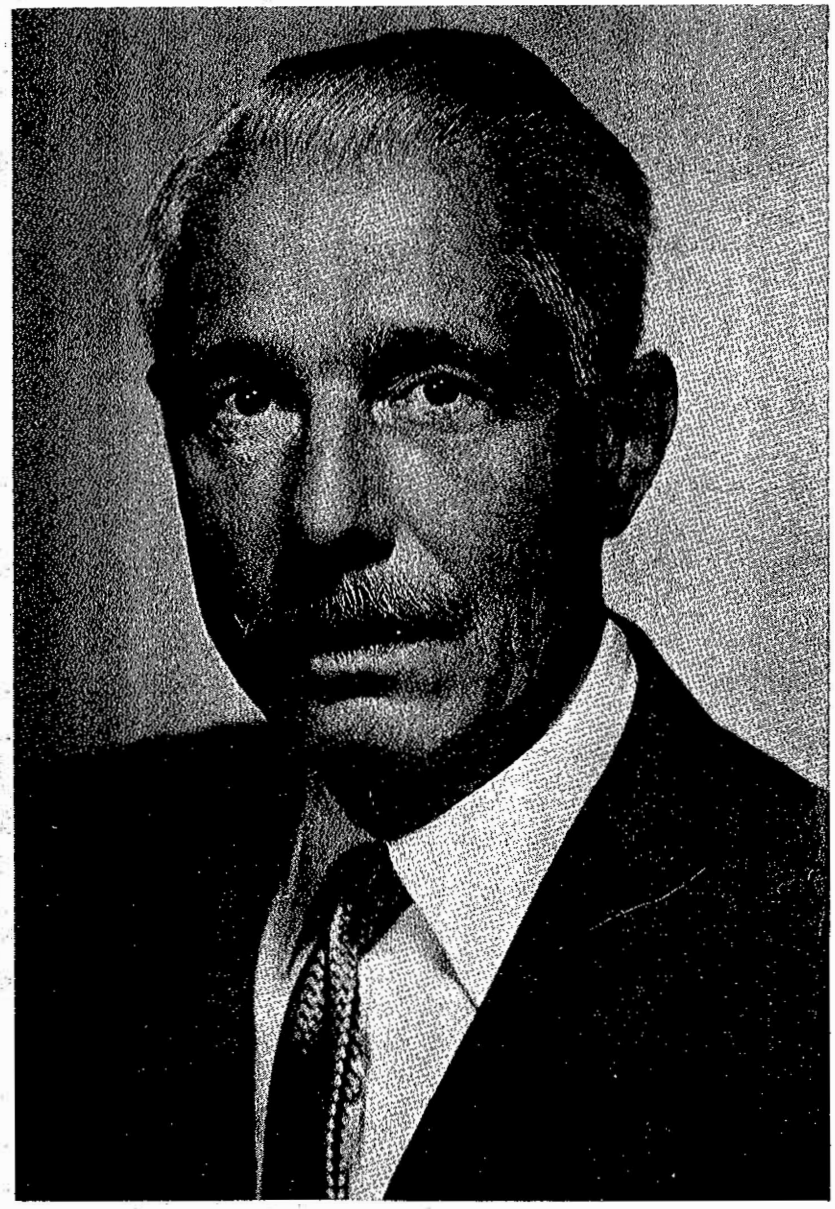
"A ray of light is sent through a small hole into a hollow sphere where it is reflected in an equatorial plane. The reflected rays are equal chords of the equatorial circle, each subtending an angle θ at the centre of the circle. What is the condition satisfied by θ if the ray eventually emerges from the hole through which it entered?

Problem 3 (R.A.German, student, University of Waterloo)

A tower of height h stands on a level plane. At a point A, the top of the tower subtends an angle α; at a point B, which lies in a straight line with A and the foot of the tower, an angle 2α is subtended at a point exactly $h/4$ units up the tower. Express AB in terms of α and h.

Problem 4 (R.A.German, student, University of Waterloo)

Find the relation which must hold if, in an ellipse, the normal at A (one end of the latus rectum) bisects the angle between the latus rectum and the line joining A to the centre of the ellipse.



DR. JULIUS H. HLAVATY
Director of the Commission on the Mathematics Programme of The College Entrance Examination Board, who will address the Easter Meeting of the Ontario Educational Association.
Photo by Fabian Bachrach.

BALANCING CHEMICAL EQUATIONS

Students who take both Algebra and Chemistry can be interested in the use of systems of algebraic equations to balance chemical equations; this avoids the oxidation-reduction technique used in first-year university.

Example 1. Ferric chloride and sulphuric acid react to give ferric sulphate and hydrogen chloride; balance the equation.

Using p, q, r, s as unknown integers, we write

$$p\ FeCl_3 + q\ H_2SO_4 \longrightarrow r\ Fe_2(SO_4)_3 + s\ HCl.$$

Comparing the amounts of Fe present on the left and right of the equation, we get $p = 2r$. Altogether, we get the equations

(1) $p = 2r$ (from Fe) (2) $3p = s$ (from Cl)

(3) $2q = s$ (from H) (4) $q = 3r$ (from S)

(5) $4q = 12r$ (from O)

These are easily reduced to

$s = 2q = 6r$ (from 3 and 4)

$s = 3p = 6r$ (from 1 and 2).

Since the solution must be integral, and is determined only up to proportionality, we may take $r = 1$; then $s = 6$, $p = 2$, $q = 3$.

This example has been rather simple, but the method shows to great advantage on more complex problems. Try it on

Example 2. $p\ H_2SO_4 + q\ HI \longrightarrow r\ H_2S + s\ I_2 + t\ H_2O$

Example 3. $p\ Cu + q\ HNO_3 \longrightarrow r\ Cu(NO_3)_2 + s\ NO + t\ H_2O$

Example 4. $p\ HCl + q\ KMnO_4 \longrightarrow r\ Cl_2 + s\ MnCl_2 + t\ KCl + u\ H_2O$

Example 5. $p\ KMnO_4 + q\ C_2H_2O_4 \longrightarrow r\ MnSO_4 + s\ CO_2 + t\ K_2SO_4 + u\ H_2O$

Example 6. $p\ NaCl + q\ KMnO_4 + r\ H_2SO_4 \longrightarrow s\ Cl_2 + t\ MnSO_4 + u\ Na_2SO_4 + v\ K_2SO_4 + w\ H_2O$

What did the ocean say to the beach?

Nothing; it only waved.