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ONTARIO MATHEMATICS GAZETTE

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E D I T O R I A L

Here we are back again with the second issue of the Ontario Mathematics Gazette. Thanks to contributions from many sources, the first issue was very successful; we are grateful to the many people who gave so generously of their time and knowledge.

It would probably not be inappropriate if I were to make a few remarks about the commonly-heard phrase "Modern Mathematics". Recently, there has been a great deal of talk about a "revolution in mathematics", and a spate of new programmes has appeared in the United States; each of these programmes prides itself on being non-traditional, and claims that it will cure the ills of mathematical instruction. Unfortunately, not all of these programmes seem to be in agreement. Each seems to emphasize a different gimmick and one is left wondering whether absolute truth resides in "placeholders" or in "truth sets" or in some other marvellous new development.

In this issue of the Gazette, we publish, with permission, a reprint of a recent article from the American Mathematical Monthly; in this article, a number of distinguished mathematicians sound a note of warning. In particular, they point out that mathematics, despite any fashions of an ephemeral nature that may crop up, is still based on a firm foundation in Algebra, Geometry, Trigonometry, and Calculus. Personally, I regard their article as extremely valuable; by all means, let us use any individual ideas that are good in any of the "modernistic" suggestions; but let us also not throw out ideas simply because they were known in 1700. It is one of the great strengths of mathematics that facts which were "true" in 1700 are still true; this is not so in many branches of science. Indeed, I wonder if mathematics in the United States would not benefit more if some of the zealous reformers who are stressing the virtues of truth-tables and other "modern" ideas would expend part of their energies in seeing that the high school mathematics curricula in all states included the basic Algebra, Geometry, and Trigonometry which is taught in Ontario Grade 13. This is not the case now. I suspect that one reason for the emphasis on non-essentials such as truth-tables is that it is easier to teach such items to very weak students than it is to teach them some good solid algebra. But should we allow our curricula to be influenced too much by a desire to educate the uneducable?

There is another article which appeared recently which I would like to draw to the attention of all readers of the Ontario Mathematics Gazette. This is "The Narrow Mathematician" by Professor C.B. Allendoerfer in the American Mathematical Monthly for June-July, 1962 (Volume 69, Number 6). Professor Allendoerfer is certainly sympathetic to any tendencies which

lead to modernization and improvement of mathematical curricula. But he warns of the danger of putting too much emphasis on superficialities rather than on mathematical substance. Indeed, he states that "the formalistic approach to mathematics can be overdone", and warns of "several of its excesses".

The excesses mentioned by Professor Allendoerfer are, I believe, worth summarizing. First, he regrets that some teachers "have identified the reform movement with a fancy language which only they can speak"; he points out that "the villain of this piece is the language of sets" and that "too much too early or too little too late are equally to be deplored".

Secondly, Professor Allendoerfer questions some jargon that is creeping into mathematics. "Have we really improved the clarity of the idea of a function when we define it as a 'set of ordered pairs, etc.' when we might with equal rigor define it in terms of its domain, range, and rule?" Or have we gained anything by "the replacement of the universally understood phrase 'the solution of an equation' by its modern counterpart 'the truth set of an open sentence'"? Your editor agrees that the answer is "No".

Finally, Professor Allendoerfer criticizes over-emphasis upon axiomatic systems at the high school level; time should not be spent on "developing unmotivated foundations". "When one already understands almost everything about a subject, its axiomatic formulation is a thing of beauty and is essential for rigor. But to the novice axioms are frequently no help at all".

These few remarks can not begin to do justice to all the thought-provoking ideas in Professor Allendoerfer's article; let me commend it to your attention.

And now, best wishes until Issue Number Three reaches you. Since that will probably be shortly after the new year, let me wish you an early "Merry Christmas".

Ralph G. Stanton

Editor-in-Chief

POSTSCRIPT: Our Problems section has had to be very abbreviated this time; but watch for next issue!

ON THE MATHEMATICS CURRICULUM OF THE HIGH SCHOOL*

The following memorandum was composed by several of the undersigned and sent to 75 mathematicians in the United States and Canada. No attempt was made to amass a large number of signatures by canvassing the entire mathematical community. Rather, the objective was to obtain a modest number from men with mathematical competence, background, and experience and from various geographical locations. A few of the undersigned, whose support is indeed welcomed, volunteered their names when they learned about the memorandum from a colleague.

The mathematicians of this country now have a more favourable climate in which to develop and gain acceptance of improvements in mathematics education. Indeed a number of groups have recognized the opportunity and are working hard and with the best of intentions to utilize it.

It would, however, be a tragedy if the curriculum reform should be misdirected and the golden opportunity wasted. There are, unfortunately, factors and forces in the current scene which may lead us astray. Mathematicians, reacting to the dominance of education by professional educators who may have stressed pedagogy at the expense of content, may now stress content at the expense of pedagogy and be equally ineffective. Mathematicians may unconsciously assume that all young people should like what present day mathematicians like or that the only students worth cultivating are those who might become professional mathematicians. The need to learn much more mathematics today than in the past may cause us to seek shortcuts which, however, could do more harm than good.

In view of the possible pitfalls it may be helpful to formulate what appear to us to be fundamental principles and practical guidelines.

1. For whom. The mathematics curriculum of the high school should provide for the needs of all students: it should contribute to the cultural background of the general student and offer professional preparation to the future users of mathematics, that is, engineers and scientists, taking into account both the physical sciences which are the basis of our technological civilization, and the social sciences which may need progressively more mathematics in the future. While providing for the other students, the curriculum can also offer the most essential materials to the future mathematicians. Yet to offer such subjects to all students as could interest only the small minority of prospective mathematicians is wasteful and amounts to ignoring the needs of the scientific community and of society as a whole.

2. Knowing is doing. In mathematics, knowledge of any value is never possession of information, but "know-how". To know mathematics means to be able to do mathematics: to use mathematical language with some fluency, to do problems, to criticize arguments, to find proofs and, what may be the most important activity, to recognize a mathematical concept in, or to extract it from, a given concrete situation.

Therefore, to introduce new concepts without a sufficient background of concrete facts, to introduce unifying concepts where there is no experience to unify, or to harp on the introduced concepts without concrete applications which would challenge the students, is worse than useless: premature formalization may lead to sterility; premature introduction of abstractions meets resistance especially from critical minds who, before accepting an abstraction, wish to know why it is relevant and how it could be used.

3. Mathematics and science. In its cultural significance as well as in its practical use, mathematics is linked to the other sciences and the other sciences are linked to mathematics, which is their language and their essential instrument. Mathematics separated from the other sciences loses one of its most important sources of interest and motivation.

4. Inductive approach and formal proofs. Mathematical thinking is not just deductive reasoning; it does not consist merely in formal proofs. The mental processes which suggest what to prove and how to prove it are as much a part of mathematical thinking as the proof that eventually results from them. Extracting the appropriate concept from a concrete situation, generalizing from observed cases, inductive arguments, arguments by analogy, and intuitive grounds for an emerging conjecture are mathematical modes of thinking. Indeed, without some experience with such "informal" thought processes the student cannot understand the true role of formal, rigorous proof which was so well described by Hadamard: "The object of mathematical rigour is to sanction and legitimatize the conquests of intuition, and there never was any other object for it."

There are several levels of rigor. The student should learn to appreciate, to find and to criticize proofs on the level corresponding to his experience and background. If pushed prematurely to a too formal level he may get discouraged and disgusted. Moreover the feeling for rigor can be much better learned from examples wherein the proof settles genuine difficulties than from hair-splitting or endless harping on trivialities.

5. Genetic method. "It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely

assimilated when it is in the nascent state" wrote James Clerk Maxwell. There were some inspired teachers, such as Ernst Mach, who in order to explain an idea referred to its genesis and retraced the historical formation of the idea. This may suggest a general principle: The best way to guide the mental development of the individual is to let him retrace the mental development of the race - retrace its great lines, of course, and not the thousand errors of detail.

The genetic principle may safeguard us from a common confusion: If A is logically prior to B in a certain system, B may still justifiably precede A in teaching, especially if B has preceded A in history. On the whole, we may expect greater success by following suggestions from the genetic principle than from the purely formal approach to mathematics.

6. "Traditional" mathematics. The teaching of mathematics in the elementary and secondary schools lags far behind present day requirements and highly needs essential improvement: we emphatically subscribe to this almost universally accepted opinion. Yet the often heard assertion that the subject matter taught in the secondary schools is obsolete should be closely scrutinized and should not be taken simply at face value. Elementary algebra, plane and solid geometry, trigonometry, analytic geometry and the calculus are still fundamental, as they were fifty or a hundred years ago: future users of mathematics must learn all these subjects whether they are preparing to become mathematicians, physical scientists, social scientists or engineers, and all these subjects can offer cultural values to the general students. The traditional high school curriculum comprises all these subjects, except calculus, to some extent; to drop any one of them would be disastrous.

What is bad in the present high school curriculum is not so much the subject-matter presented as the isolation of mathematics from other domains of knowledge and inquiry, especially from the physical sciences, and the isolation of the various subjects offered from each other; even the techniques and theorems within the same subject appear as isolated, disconnected tricks to the student, who is left in the dark about the origin and the purpose of the manipulations and facts that he is supposed to learn by rote. And so, unfortunately, it often happens that the material offered appears as useless and boring, except perhaps, to the few prospective mathematicians who may persist despite the curriculum.

7. "Modern" mathematics. In view of the lack of connection between the various parts of the present curricula, the groups working on new curricula may be well advised in seeking to introduce unifying general concepts. We think, too that judicious use of sets and of the language and

concepts of abstract algebra may bring more coherence and unity into the high school curriculum. Yet, the spirit of modern mathematics cannot be taught by merely repeating its terminology. Consistently with our principles, we wish that the introduction of new terms and concepts should be preceded by sufficient concrete preparation and followed by genuine, challenging applications and not by thin and pointless material: one must motivate and apply a new concept if one wishes to convince an intelligent youngster that the concept warrants attention.

We cannot enter here into detailed analysis of the proposed new curricula, but we cannot leave unsaid that, in judging them on the basis of the guidelines stated above (Sections 1-5), we find points with which we cannot agree.

Of course, not all mathematicians have the same taste. Mathematics has many aspects. It can be regarded as an instrument to understand the world around us: mathematics presumably possessed this value for Archimedes and Newton. Mathematics can also be regarded as a game with arbitrary rules where the principal consideration is to stick to the rules of the game: some such view may be considered suitable for certain problems of foundations. There are several other aspects of mathematics, and a professional mathematician may favour any one. Yet when it comes to teaching, the choice is not a mere matter of taste. We may expect that an intelligent youngster would want to explore the world around him, but we cannot expect him to learn arbitrary rules: why just these and not others?

At any rate, we fervently wish much success to the workers on the new curricula. We wish especially that the new curricula should reflect more the connection between mathematics and science and carefully heed the distinction between matters logically prior and matters which should have priority in teaching. Only in this way can we hope that the basic values of mathematics, its real meaning, purpose, and usefulness will be made accessible to all students, including of course, the prospective mathematicians. The recently expressed "wide spread concern about a trend to excessive emphasis on abstraction in the teaching of mathematics to engineers" points in the same direction.**

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** First Summer Study Group in Theoretical and Applied Mechanics Curricula, Boulder, Colorado, June, 1961.

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GEOMETRIC VARIATIONS OF THE THEME OF SINE 2A

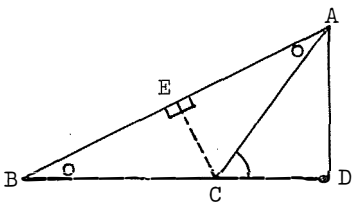
F. C. Asbury
Ontario Mathematics Commission

One of the attractions of Synthetic Geometry is its capacity for providing harmless entertainment in the working out of alternative proofs for various theorems. The proofs one establishes will probably not be new, but if they are original as far as one can recall, they provide a certain measure of satisfaction. In this unassuming spirit the following variations are offered. They may provide material for extracurricular suggestions to interested mathematical students in the senior forms.

Geometric Proofs for the Double Angle Formulae

Various geometric occurrences of an angle and its double suggest themselves as starting points.

I Take first the case of the exterior angle of an isosceles triangle.



$$\sin 2A = \sin ACD = \frac{AD}{AC} = \frac{AD}{AB} \cdot \frac{AB}{AC}$$

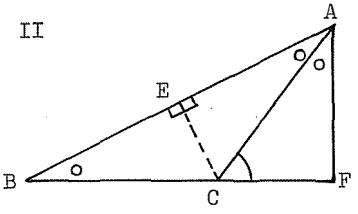
$$= \frac{AD}{AB} \cdot \frac{2AE}{AC}$$

$$= 2 \sin A \cos A.$$

In an analogous manner, it can be shown without much difficulty that

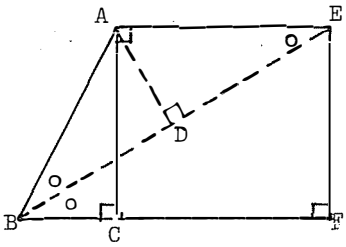
$$\cos 2A = 2 \cos^2 A - 1.$$

II

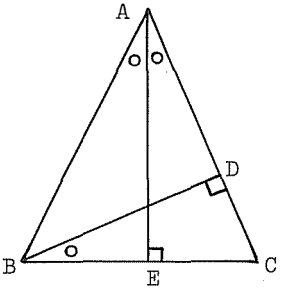


As a variation on this situation, apply the Sine Rule to the $\triangle ACF$, where $\angle ACF = 2A$, $\angle CAF = A$ and use the Angle Bisector Theorem.

III We may instead start with the (double) angle and bisect it. The proof, with the aid of the accompanying diagram, is not hard to work out.



IV Again, starting with the vertical angle of an isosceles triangle ABC,



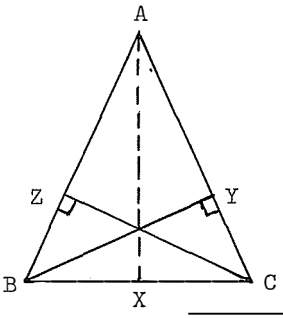
$$\sin 2x = \frac{BD}{AB} = \frac{BD}{BC} \cdot \frac{2BE}{AB}$$

$$= 2 \cos x \sin x$$

$$\cos 2x = \frac{AD}{AB} = \frac{AC - CD}{AB} = \text{etc.}$$

$$= 1 - 2 \sin^2 x$$

V A variation on IV. Where O is the orthocentre of isosceles $\triangle$ ABC,



$\angle$ BOC = $\angle$ ZOY = 180° - A = 2 $\angle$ C

Hence $\angle$ XOC = $\angle$ C

$$\sin 2C = \sin BOC = \frac{CY}{CO} = \frac{CY}{BC} \cdot \frac{BC}{CO}$$

$$= \frac{CY}{BC} \cdot \frac{2CX}{CO}$$

$$= 2 \cos C \cdot 2 \sin C$$

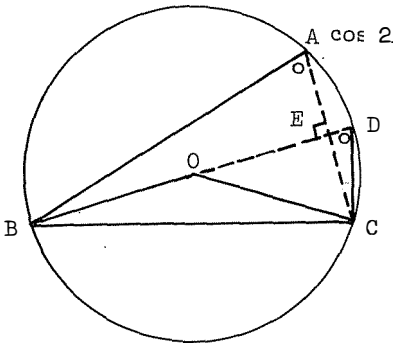
$$= 2 \sin C \cos C$$

VI Another interesting starting point is the relationship between angles at the centre and circumference of a circle, standing on the same arc.

$$\sin 2A = \sin BOC = \frac{CE}{CO} = \frac{2CE}{BD}$$

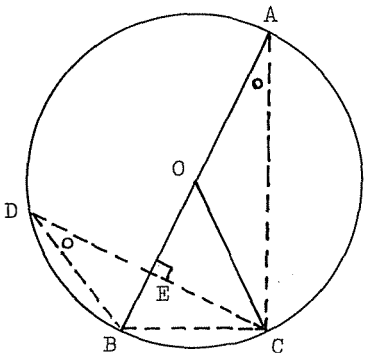
$$= \frac{2CE}{CD} \cdot \frac{CD}{BD}$$

$$= 2 \sin A \cos A$$



$$\cos 2A = \cos BOC = \frac{-OE}{OC} = \frac{BO - BE}{OC}$$
$$= 1 - \frac{BE}{OC} = 1 - \frac{2BE}{BD}$$
$$= 1 - 2 \cdot \frac{BE}{BC} \cdot \frac{BC}{BD}$$
$$= 1 - 2 \sin^2 A$$

If $2A$ is acute, as in the final diagram, the proofs for $\sin 2A$ and $\cos 2A$ follow readily. For $\tan 2A$ we have:



$$\tan 2A = \frac{CE}{OE} = \frac{CE}{AE} \cdot \frac{AE}{OE} = \frac{\frac{CE}{AE}}{\frac{OE}{AE}}$$
$$= \frac{2\frac{CE}{AE}}{2\frac{OE}{AE}} = \frac{2 \cdot \frac{CE}{AE}}{\frac{AE - BE}{AE}}$$
$$= \frac{2\frac{CE}{AE}}{1 - \frac{BE}{DE} \cdot \frac{DE}{AE}} = \frac{2 \tan A}{1 - \tan^2 A}$$

It is readily admitted that this last is somewhat "double-dummy" play!

Ontario Association of Teachers of Mathematics and Physics

Message from the President

Dear Fellow Members:

"The object of the Association shall be the promotion of the study and teaching of Mathematics and Physics." So reads the first draft of the Constitution of the Mathematics and Physics Association of Ontario made in 1891 as found on page six of our "Historical Highlights" published last spring.

The primary means used by our Association to fulfill this objective has been the Easter meeting and the newsletter. The

growth of the newsletter into the Ontario Mathematics Gazette has been a step forward, for through it the Ontario Mathematics Commission will help us to keep in touch with new developments in Mathematics curriculum and Mathematics teaching. A careful perusal by each one of us of the Ontario Mathematics Gazette and other publications, such as The Mathematics Teacher of the National Council of Teachers of Mathematics with which we are affiliated, seems to me to be the kind of study best suited to this time of changing curriculum.

As my current contribution to our attainment of our objective, I shall pass along to you the contents of a letter written to me last year by our Past Honorary President, Dr. Samuel Beatty. In this letter Dr. Beatty was writing about his speech of welcome to the delegates attending the 1961 Summer Meeting of the N. C. T. M. In it he says "I expressed a few sentiments on what is apparently involved in good teaching of Mathematics. First and foremost I put the quality which good teachers have of being boosters, not wet blankets, in their contacts with their students, while taking it for granted throughout that they would know the subject matter, no matter how it was thrown back at them. In short, I guess I was making a case for morale-building as the constant preoccupation of the good teacher. I have always felt that a good teacher is like a good parent, in feeling akin to every last pupil in his class and finding himself free to approve or criticize the individual student's efforts, and all this done in a way to encourage the student to seek the satisfaction that comes from doing his best. Failing this bond of sympathy and going in for a study of the subject in all its aspects, there can still be the personal problems of students resenting being dragged along they know not where. What I feared but did not really pin-point, is the tendency now in University teaching to be too learned and professional. Books are surely learned and professional enough, and the good teacher knows how to create a common sense feel for a subject that the strict and sequential exposition cannot give - unless, perhaps, to the very best students, who are never a problem anyway but a joy."

I hope that this year your problems will be confined to those found in the text books and that all of your students will be joys to behold.

Sincerely,

Father John C. Egsgard, C.S.B.,
President.

THE PEAUCELLIER CELL

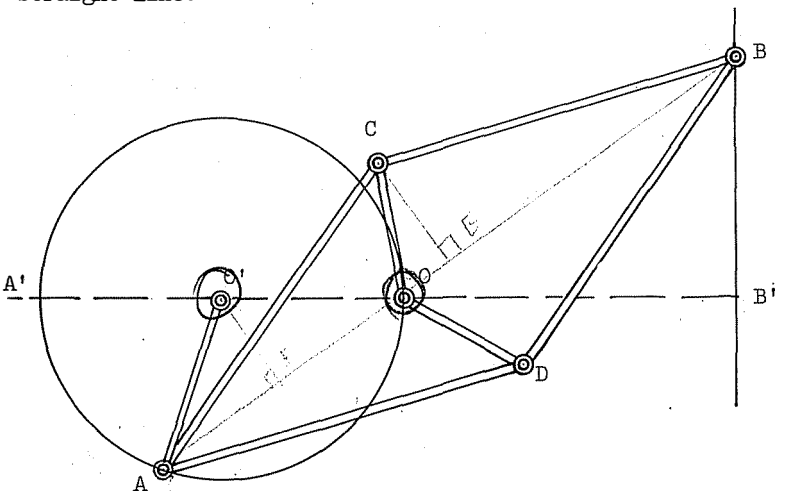
Edward Evans, Guelph

This instrument, shown in the accompanying diagram, is a device which transforms a circular motion of the end A into a linear motion of the end B.

The proof of this fact is readily obtained either by synthetic or analytic geometry, or by trigonometry, as shown in this article. Hence, constructing the instrument and finding the proof could form an interesting project for pupils of Grades 11 to 13.

The instrument was first invented in 1864 by a French army Lieutenant, Peaucellier. (See A Source Book of Mathematics by David Eugene Smith; see also Linkages in the Encyclopaedia Britannica.) He was seeking an improvement on the linkage in Watt's steam engine. The particular interest of this device for geometers is that it enables one to produce a theoretically straight line without using a guiding straight edge.

The cell consists of six movable rods, forming a rhombus ADBC with two rods of equal length, CO and DO, within the rhombus. A seventh rod, O'A, is attached at A. When in use, the points O and O' are held fixed, at a distance apart equal to the length of the rod O'A. Then, as the end A rotates in a circle about O', the end B will be forced to travel in a straight line.



The proofs which follow appear to me to provide the simplest method of attack in each case, although naturally they are not the only proofs that can be devised.

In these proofs, I have let each side of the rhombus be a,

OC = OD = b, and O'A = OO' = p. Also BB' is drawn perpendicular to OO'.

I. Proof by Synthetic Geometry

Join AB (by symmetry, AB passes through O), and draw CE and O'F perpendicular to AB.

Then BO = AO + 2 OE; hence AO . OB = AO^2 + 2 AO . OE. From triangle ACO, $a^2 = b^2 + AO^2$ + 2 AO . OE. Also, from the similar triangles OFO' and OBB', we obtain AO. OB = 2p . OB', whence ap . OB' = $a^2 - b^2$. Therefore OB' = $(a^2 - b^2)/2p$, which is a constant, regardless of the position of A on the circle.

II. Proof by Trigonometry

Calling the angle O'OA = θ, and the angle COB = α, and using the law of cosines on the triangles OAC and OBC, we obtain by subtraction and factoring BO - AO = 2b cos α. Also AO = 2p cos θ. Hence BO = 2b cos α + 2p cos θ.

Substituting 2p cos θ for AO in the cosine formula for triangle OAC gives us a value for cos α in terms of cos θ and the constants. When we substitute this into the expression for BO above, and note that OB' = OB cos θ, we arrive at the same value for OB' as in the geometric proof above.

Note: It is also possible, by resolving the displacements of A and of the mid-point M of AB into vertical and horizontal components, to prove that when O'A rotates through an angle 2θ about O', from the position O'A', then the horizontal displacement of B is zero.

III. Proof by Analytic Geometry

This proof is somewhat longer, but a suitable choice of method avoids troublesome quadratic equations.

Starting with the origin at O, the line A'OB' as the X-axis, and taking the line AB as y = mx, we solve for the intersections with the circle, and obtain the coordinates of A as $[-2p/(1+m^2), -2mp/(1+m^2)]$.

Now, CD is the common chord of the circle having centre A and radius a, and the circle having centre O and radius b. Its equation is found to be

$$4p(x + my + p) = (a^2 - b^2)(1 + m^2).$$

Since we have a rhombus, the mid-point of AB must lie on

this line CD. Letting (X,Y) be the coordinates of B, and noting also that B is on the line $y = mx$, we find the coordinates of the mid-point of AB. Substituting these into the equation of the line CD, and eliminating Y, we obtain $X = (a^2 - b^2)/2p$ as before.

Since X is constant, the locus of B is a straight line perpendicular to OO'.

Note 1: Since the lengths p, a, and b can be chosen arbitrarily several special cases can be obtained; in any case, the length of the line that B moves on is limited, and A cannot traverse the full circle.

Note 2: Since this article was written, Professor Petrie has called to my attention the fact that other synthetic proofs of Peaucellier's Cell appear in Plane Geometry for Advanced Students by C.V. Durell and in Euclid's Elements of Geometry by H.M. Taylor.

Ontario Association of Teachers of Mathematics and Physics

Ontario Educational Association

W. R. Cunnington
York Mills Collegiate Institute

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The Executive hopes that the members were well satisfied with the convention last Easter. We would be pleased to hear your reactions - if you found the meetings valuable, let us know. Likewise, if there was something about the convention that you would like to see improved, please be sure to pass your ideas along. As you see, our Councillors are chosen from scattered parts of the province. This is so they can keep the Executive aware of your wishes and thus make the Section speak for its members. A letter to the Secretary, or to any Councillor, who will pass it on, will certainly be brought to the attention of the Executive.

Plans are already under way for the 1963 meetings, and we hope to be able to tell you most of the details in our next issue.

Some people claim that the part of the convention most valuable to them is the opportunity to exchange ideas with old and new friends between meetings. This is easier to do now that the luncheon is in such a convenient location and with the introduction of the evening reception which proved so popular this year.

Others enjoy the stimulation which they receive from the various talks. These were supplemented this year by a group of seminars which were very well received.

For those who were not able to attend the meetings we are providing summaries of two of the addresses, namely,

I. Julius H. Hlavaty: Mathematics Education in Transition (reported by Howard Russell)
II. W.R. Sawyer: Rocketry (reported by J. McKnight).

In our next issue some of the others will be printed.

MATHEMATICS EDUCATION IN TRANSITION

In his opening remarks Professor Hlavaty made a brief reference to what has been termed the information explosion of the past fifty years. He emphasized developments in the areas of mathematics and science, pointing to such fields as automation, operations research, digital computers, quality control, linear programming, and space exploration as illustrations of the phenomenal advances which have taken place very recently. It would seem to be a consequence that education should be influenced profoundly.

The need for changes in education, especially in mathematics and science, was Professor Hlavaty's next point. He used two specific examples to illustrate the general trend of criticism of education which has emerged in the last ten years. The New York Times headline, "Teachers Chided on Mathematics; Study finds them deficient, curriculum outmoded, and pupils hostile or bored" introduced a front page article reviewing a report submitted to the Educational Testing Service in 1956. The other criticism came from the pen of Morris Kline, of New York University, who wrote, "... in all courses we are still emphasizing meaningless techniques ..." and "These topics (Horner's method, solution of oblique triangles, etc.) are dry, boring, useless, and no argument, not even mental discipline, will ever make such material palatable or pregnant with significance. This muscular mathematics develops nothing."

These criticisms, and many others, led the College Entrance Examination Board in 1956 to set up the Commission on Mathematics to make a comprehensive study of the problems of mathematical education in the high schools, and to make recommendations for a programme for pre-college mathematics.

Under the leadership of Professor Albert Tucker, Chairman of the Mathematics Department of Princeton University, the Commission worked vigorously, and its methods are worth noting. Other mathematicians, educators, and teachers were solicited for their suggestions and proposals. The Commission studied and analyzed the many projects and experiments that were already underway - the Illinois Programme, the Ball State Teachers College Plan, The Maryland Plan, the New York State Syllabus, and others. Outlines and drafts were submitted to country-wide discussions at professional meetings and in professional journals. All reactions, even the most negative, were utilized in re-framing the proposals.

What is the programme that the Commission recommends? First of all some general principles in which the Commission believes:

1. Strong preparation in both concepts and skills;
2. Study of the nature and role of deductive reasoning in algebra and geometry;
3. Appreciation of mathematical structure and patterns;

in algebra, say, the number system and its laws; in geometry, the logical structure of undefined terms, definitions, axioms, and theorems.

4. Judicious use of unifying ideas such as sets, and the language and concepts of abstract algebra.

What are the specific features of the recommendations of the Commission for the high schools? They may be summarized as follows.

1. In elementary algebra - the laws of the number system, use of unifying ideas such as sets, variables, relations, and functions, the study of inequalities as well as equations.

2. In the tenth year - incorporate with plane geometry an introduction to the concepts of coordinate geometry, and take up the essentials of solid geometry and space perception by a variety of intuitive techniques.

3. In the eleventh year - a combination of algebra and trigonometry, emphasizing coordinates, vectors, and complex numbers, and de-emphasizing solution of triangles.

4. In the twelfth year there is room for experimentation and for specialization which takes into account future plans of the students. The alternatives proposed by the Commission are:

a) Elementary Functions - the traditional course in Advanced Algebra refined by clearer concepts and vocabulary, and laying a foundation for college-level work in Calculus and Analytic Geometry.

[Editorial note: in the United States, the study of Analytic Geometry is frequently left until college; it would seem that Professor Tucker's Commission might well have suggested that all high school students study Analytic Geometry.]

b) Probability and Statistics - This was one area in which the Commission found experimental work under way in the schools. It therefore prepared a sample text-book for this course - the little gray book that became very popular in teacher-training institutes.

[Editorial note: we shall attempt to have an article on this topic in an early issue.]

c) An Introduction to the concepts and techniques of Modern Abstract Algebra.

d) The Advanced Placement Programme in Analytic Geometry and Calculus. This is a programme for a college-level course in the senior year of the high school for gifted students who have been able to fit into their first three years all the work suggested for the first four years, as well as some work on Elementary Functions.

The most profound, and potentially the most significant effect of the Commission has been on the writers of commercially produced text-books. Every new or revised text of the most recent period deliberately seeks to incorporate at least some of the recommendations of the Commission, and the authors explicitly declare so in their introductions.

Many new teachers, finding themselves in the middle of the extensive discussion on what is going on about the evolving programme in high school mathematics, sometimes ask: "What is the future programme in mathematics? Is it the Commission's programme? is it the SMSG programme? is it the Illinois programme?"

There is a simple answer to the question - it is none of these. In the first place, the various recommendations are not competing with each other for adoption - there is a great area of agreement among them. In the second place, the question often indicates a basic misunderstanding of the process of curriculum reconstruction. The curriculum of the future will evolve from the current rethinking that is going on. Teachers will be studying aspects of various programmes and they will be trying them out in their own teaching. Syllabus committees will be drawing up syllabi for their own schools or communities, incorporating in them what they like and feel they can do.

Professor Hlavaty concluded his presentation with a reference to the article "On The Mathematics Curriculum of The High School" in the March 1962 issue of the Mathematics Teacher. Although this article by sixty-five mathematics professors is critical of some phases of the "reform" movement in mathematics education, nevertheless the signers "fervently wish much success to the workers on the new curricula" and they think "the judicious use of sets and the language and concepts of abstract algebra may bring more coherence and unity into the high school curriculum."

Howard Russell

Address of Dr. W. R. Sawyer (April 25, 1962)

"Rocketry"

Dr. Sawyer emphasized at the outset the vastness of the topic and proposed to deal with Rocketry insofar as it applied to the exploration of the moon.

Why all the fuss about lunar exploration?

1. It is there!
2. It represents a frontier not yet "cracked"!
3. Important in the "cold" war and possibly in a "hot" war.
4. It represents a truly great quest.

The speaker made it clear that the "parameter" of prime importance was VELOCITY, and put forward the following figures:

Height of Trajectory in nautical miles	1500	6000	0300	Moon	Escape
Velocity in Km./sec.	4.5	7.0	7.6	11.1	11.2

0300 is interpreted as Orbit 300 n. mi. above surface of earth.

Dr. Sawyer then developed some of the basic mathematics of rocket theory.



Combustion Chamber

The basic equation may be written

$$M\dot{V} = -\dot{M}V_e + (P_e - P_a)A_e - Mg \sin\theta - V_D$$

where M represents mass, $\dot{M} = dM/dt$, P_e represents "escape pressure", P_a "atmospheric" pressure, V_D = "velocity drag" (which is negligible), $\dot{M}V_e$ = "momentum thrust", A_e = "escape area", $(P_e - P_a)A_e$ = "pressure thrust".
θ = angle of projection.

On a "test pad", thrust

$$F = -\dot{M}V_e + (P_e - P_a)A_e$$

$$= -Mc, \text{ where } c = \text{"effective exhaust velocity"}.$$

(Note: $\dot{M}$ is negative since M decreases.)

The basic equation may then be rewritten

$$M\dot{V} \text{ or } M\frac{dV}{dt} = -c\frac{dM}{dt} - Mg \sin\theta$$

$$M\,dV = -c\,dM - Mg \sin\theta\,dt$$

$$dV = -c\frac{dM}{M} - g \sin\theta\,dt$$

$$\int_{V_0}^{V_B} dV = -c\int_{M_0}^{M_B}\frac{dM}{M} - g \sin\theta \int_{t_0}^{t_B} dt$$

$$V_B - V_0 = -c\,(\log_e M_B - \log_e M_0) - g \sin\theta\,(t_B - t_0)$$

$$V_B = c \log_e \frac{M_0}{M_B} - g\, t_B \sin\theta$$

$$V_B = 2.3\, c \log_{10} \frac{M_0}{M_B} - g\, t_B \sin\theta$$

V_B = "all burnt velocity" ; $t_B \doteq 100$ sec.

Dr. Sawyer recalled that to produce a large value of V_B was the main objective, and pointed out that this could be brought about by a large "c" and/or a large "mass ratio" M_0/M_B. He first considered the mass ratio M_0/M_B, and quoted the relationship between M_0/M_B and the "payload ratio" λ, namely,

$$\frac{M_0}{M_B} = \frac{1}{\lambda + e(1-\lambda)}\ , \quad \text{where} \quad e = \frac{1}{6}\ .$$

Thus, a simple calculation showed

M.R.	2.0	3.0	4.0	5.0	5.7
λ	0.40	0.20	0.10	0.04	0.01

Obviously, increasing the M.R. had the undesirable effect of reducing sharply the payload ratio.

The speaker noted briefly at this point the advantage of a multistage rocket in increasing "λ", but asserted that this involved many complications, a 4-stage rocket being considered "optimum".

The question "how about c?" was then discussed as the logical alternative. Taking $M_0/M_B = 4/1$, then

$$V_B = 2.3\, c \log 4 = 2.3 \times .6 \times c.$$

The speaker here defined "specific thrust" T_S as the ratio of thrust (in pounds) to the rate at which fuel was burnt (in pounds per second), and gave some interesting comparative figures concerning "T_S". In 1232, the Chinese employed gunpowder for which $T_S = 100$. In 1926, Oxygen and Alcohol raised T_S to 260. (This was the fuel used in the famous V_2.) Oxygen and Kerosene give $T_S = 264$; liquid Oxygen and liquid Ammonia furnish a still larger T_S. The speaker at this point noted that it was simply a matter of applying "freshman"

chemistry to predict better fuel combinations.

Liquid Oxygen and Liquid Hydrogen: $T_S = 364$;

Liquid Fluorine and Liquid Hydrogen: $T_S = 373$;

Ozone + Fluorine and Hydrogen: $T_S = 383$;

Ozone + Hydrogen: $T_S = 393$.

(Ozone, however, is very unstable, and its use is accompanied by extreme danger!)

At this juncture, Dr. Sawyer made an interesting observation, namely that the Russian method of increasing the payload is to build bigger rockets, while the American method is to develop "more sophisticated" fuels with high "T_S".

Dr. Sawyer made brief reference to some of the "big boosters" of the present and the future, for example: Atlas, in project "Mercury", and in future project "Ranger" [to pick a spot on the moon for "Surveyor" (see below) to land]; Titan II, in future project "Gemini" (in which two men will occupy the capsule); Centaur, in project "Surveyor" (an unmanned rocket, designed to land on the moon, drill holes, signal information re material on the moon by radio - activating it by neutron bombardment, etc.). Centaur will also figure in project "Mariner", designed to approach Mars, and in project "Advent" which will be a space station fixed in the sky, that is, its period of revolution in orbit will equal the earth's period of rotation.
Saturn, to be used in project "Apollo", will put three men in space, while Advance Saturn will be employed in assembling a large space station.
The ultimate aim at present is to produce NOVA, which will carry a manned space vehicle directly to the moon in 1970.

Dr. Sawyer made brief reference to nuclear propulsion, wherein T_S could range from 400 to 1200. (Recall that for chemical propulsion, $T_S < 400$.) This could increase the payload of NOVA twofold or threefold. He described very briefly the Ion Engine, which could produce a small extra thrust (say 1 lb.) for a long period of time (60 days).

Dr. Sawyer concluded his very interesting and informative address by answering questions from the floor.

John McKnight

FUNCTIONS OF COMPOUND ANGLES

The method of the present note is undoubtedly already used by many teachers as an alternative to the diagram approach. With a little practice, and with facility in using the functions of 0°, 90°, etc., students can give a rapid mental development of any of the usual compound angle formulae.

The formulae

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B,$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

provide a very simple approach to the functions of $-x$, $90^\circ \pm x$, $180^\circ \pm x$, $270^\circ \pm x$, etc. It is assumed that the functions of 0°, 90°, 180°, 270° have already been determined.

Problem. Find the functions of $180^\circ + x$.

Solution.
$$\begin{aligned} \sin (180^\circ + x) &= \sin 180^\circ \cos x + \cos 180^\circ \sin x \\ &= 0 (\cos x) + (-1) \sin x \\ &= - \sin x \end{aligned}$$

$$\begin{aligned} \cos (180^\circ + x) &= \cos 180^\circ (\cos x) - \sin 180^\circ \sin x \\ &= (-1) \cos x - 0 (\sin x) \\ &= - \cos x. \end{aligned}$$

The tangent is then easily found:

$$\tan (180^\circ + x) = \frac{\sin (180^\circ + x)}{\cos (180^\circ + x)} = \frac{- \sin x}{- \cos x} = \tan x.$$

The three reciprocal functions are then:

$$\operatorname{cosec} (180^\circ + x) = \frac{1}{\sin (180^\circ + x)} = \frac{1}{- \sin x} = - \operatorname{cosec} x,$$

$$\sec (180^\circ + x) = - \sec x,$$

$$\sec (180^\circ + x) = + \cot x.$$

Problem. Express the functions of $x - 270^\circ$ in terms of the of x.

Solution.
$$\begin{aligned} \sin (x - 270^\circ) &= \sin x \cos 270^\circ - \cos x \sin 270^\circ \\ &= \sin x (0) - \cos x (-1) \end{aligned}$$

$$= \cos x.$$

$$\cos (x - 270^\circ) = \cos x \cos 270^\circ + \sin x \sin 270^\circ$$

$$= \cos x (0) + \sin x (-1)$$

$$= - \sin x.$$

Then $$\tan (x - 270^\circ) = \frac{\cos x}{-\sin x} = - \cot x,$$

$$\operatorname{cosec} (x - 270^\circ) = \frac{1}{\cos x} = \sec x,$$

$$\sec (x - 270^\circ) = \frac{1}{- \sin x} = - \operatorname{cosec} x,$$

$$\cot (x - 270^\circ) = \frac{1}{- \cot x} = - \tan x.$$

B O O K R E V I E W S

Modern Analytical Trigonometry.

by Julian D. Mancill. viii + 329 pp. Dodd, Mead and Company Inc., 1960. $4.50.

The author's purpose in writing the text is expressed in the Preface, "To provide a trigonometry which is teachable and at the same time mathematically trustworthy". He also goes on to state "That one cannot justify the teaching of trigonometry solely on the basis of its practical application. Its justification also lies in its value as a terminal subject and in its value as a preparatory subject."

The first two chapters are preliminary in character containing what are termed Important Concepts in Plane Geometry and Introductory Analytic Geometry. The author discusses the concepts of sets, subsets, one-to-one correspondence, the foundations of Euclidean plane geometry and generally applies "Modern" mathematics to the basic propositions in analytic geometry. The sub sequent chapters develop the theory and applications of trigonometry in a very interesting manner. The early chapters cover the Ontario course in Grade 13 Trigonometry in an up-to-date fashion and the later chapters include much that might be given to special students interested in writing the Problems Paper for a university scholarship.

One interesting feature at the end of many chapters is a "self-test" group of problems which the student can administer to himself. Letters of the Greek alphabet are used freely to represent angles and these letters are tabulated on a page at the beginning of the book. Answers are provided for the odd

numbered questions only, where applicable. The tables at the back of the test include I Trigonometric functions to four figures for angles to the nearest ten minutes

II Logarithms of numbers to five figures

III Logarithms of trigonometric functions except secant and cosecant to five figures. Special interpolation methods for angle to 3 degrees and from 87 degrees to 90 degrees are explained.

The writer would certainly recommend this text-book as a splendid source for supplementary material and for some alternative proofs.

T.H. Eberlee, Etobicoke Collegiate Inst., Islington.

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Understanding Numeration Systems.

by D.A. Johnson and W.H. Glenn, 56 pages, Webster Publishing Company, 1960. $1.00.

This soft-covered booklet is one of a series "Exploring Mathematics on Your Own", designed for high school students who want more information than is provided in their texts. It is complete in itself, and includes the following sections:

Numerals Past and Present
Numeration with Five Symbols
Numeration with the Dozen System
Numeration with the System of Twos
Some Number Base Ideas

The material is easy to read, and to aid the reader in understanding it, numerous exercises are included. In addition, sclutions to the exercises are provided, as well as a short bibliography for those who want to read even more widely.

L.G.H.

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Analytic Geometry.

by Wray G. Brady and Maynard J. Mansfield. Little Brown and Company, Toronto, 1961. ix + 244 pages. $5.00.

This book covers the usual content of a first course in analytic geometry: straight lines; conics; quadratic forms in the plane; sketching of algebraic, trigonometric, logarithmic and exponential curves; polar coordinates; loci; lines and planes in three dimensions; sketching of three-dimensional surfaces and curves.

As is customary, no use is made of calculus. There is scarcely any discussion of tangents and not much about asymptotes. A loose account is presented of horizontal and vertical asymptotes to an algebraic curve, without the inclusion of a definition of asymptote - it has been left to the reader to infer a definition from the discussion of the asymptotes of an hyperbola. The treatment of tangents is confined to a prescription for finding the tangents at the origin to an algebraic curve - this is given without defining the tangent, and without explaining how the method could be applied to determine tangents at any other point. There is nothing about points of inflection, cusps, poles and polars, or radical axes. Radian measure of an angle is introduced by referring to arc length without defining the latter or establishing its existence. Curvature is not mentioned.

The authors begin with an introductory chapter on set theory and the algebraic laws of the number system. The idea is laudable but the execution is bad. The concepts of 'mapping of a set A onto a set B' and of 'correspondence between a set A and a set B' are both defined as what is commonly understood to be a relation between the elements of A and those of B, i.e., a subset C of the Cartesian product $A \times B$. This is appropriate to a correspondence but not to a mapping, as a mapping is generally supposed to be one-valued. The definition of a finite set, as one that can be put in one-to-one correspondence with a terminating set of positive integers, is circular. The treatment of inequalities is based soley on the trichotomy law for the real number system. Hence, for example, theorem 7.4 (page 14), "If $a > 0$, then $-a < 0$", the proof of which is left to the student, cannot be proved at all.

These blunders are probably not of much importance, as algebra will have been treated in another course. The offending chapters could simply be left out. No real use is made of the set theory in themain part of the book, and the geometrical ideas which are connected with algebraic inequalities - betweenness, orientation, inside and outside, for example - are left to the geometrical intuition of the student rather than based upon an algebraic foundation. What is upsetting, however, is the authors' definition of a curve as a correspondence between real numbers which does not contain any non-trivial rectangle. In view of their definition of correspondence, the set of points with at least one irrational coordinate would be a curve, although its complement is countable. The set of points with at least one rational coordinate would also be a curve, and one-dimensional at that. By a suitable modification of the Peano construction of a square-filling curve, one can construct a curve which is a continuous image of the unit interval, satisfies the authors' definition and meets every straight line infinitely many times in the unit square. It is true that the authors say, in a footnote, that they "realize this as a somewhat restricted definition. It is used for pedagogical reasons." They mean, presumably, that it is too general. Actually, it is not used at all (how could it be?). It is nowhere proved that the curves

actually discussed in the book satisfy this definition (this would be trivial), nor is any general property of a curve derived from it (this would be impossible). What the pedagogical reasons might have been I shudder to think.

Symmetry is defined with respect to a point and with respect to a line. Surprisingly no mention is made of rotational symmetry even in the chapter on polar coordinates. The idea of a group of symmetries is absent. The omission is hard to understand since, for example, transformations of coordinates are used to reduce the equation of the general quadratic, and the invariance of the discriminant of the quadratic under rotation of coordinates is used in classifying the results.

The book is beautifully produced with excellent diagrams and plenty of white between the lines. There is a very large number of exercises, with answers, but scarcely any real problems. The book is not a bad introduction to the shapes, sizes, and equations of the simpler curves. As a work of mathematics it is uninspiring.

B. Abrahamson, University of Toronto.

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Senior Technical Mathematics.

edited by A.H. Heywood. MacMillan Company of Canada, 1961, 558 pages. $4.50.

Designed for a Grade Eleven and Twelve programme, this text has proved excellent. The material relates closely to the requirements of Curriculum 1 and S.27 and consists of short courses in the following fields:

Algebra, logarithms, and slide rule
Approximate Numbers, mensuration
Trigonometry
Statics
Geometry.

It is necessary to comment on these various fields by topics:

Algebra: It approaches the Grade Twelve General course in depth. The work on functions, pp. 92-3, is confusing and in need of revision. On page 99, the idea of tangent curves and double roots should be emphasized rather than avoided.

Chap. VIII - the topic of series would need more justification by application than the presence of this chapter warrants.

Approximate Numbers: Subsequent chapters do not consistently adhere to the limits of accuracy suggested.

Trigonometry: Considerably more work with actual compass bearings would be desirable. To be more practical, the work on

measuring of heights should include three-dimensional cases. The diagrams of this course present some excellent questions and could be extended. However, where curves appear, more indication as to type and tangency should be given.

Statics: The topic covers the departmental requirements. A reduction of the work with levers and pulleys, which is largely a repetition of Grade Eleven Physics material, might permit an introduction to the study of power and energy. This topic would be invaluable for students proceeding further, in particular toward Stationary Engineering examinations.

Geometry: The controversial theorems on congruency and parallelism should be taken as postulates. Some knowledge of similar figures should be included and related to the topic of ratio and proportion. Although this section appears at the end of the text, I believe that it should replace the Trigonometry of Grade Eleven, which at best meets with limited acceptance.

J.D. Smith, Selkirk Collegiate and Vocational Inst.,
Fort William.

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Number Principles and Patterns.

by A. Archer. Ginn and Company, 1961, 68 pages.

This booklet deals to a limited extent with the properties of numbers. Moving rapidly from the basic laws to casting out nines, then casting out elevens, much of the material and order of material is irrelevant. Certain topics with historical reference would be of value to students of a mathematics club.

J.D. Smith, Selkirk Collegiate and Vocational Inst.,
Fort William.

THE IDENTITY $x = a^{\log_a x}$

We define the logarithm of x to base a as the power to which we must raise a to produce x. Thus, if

$$a^p = x,$$

p is the logarithm of x to base a.

Substituting, we find

$$a^{\log_a x} = x ;$$

in words, if we take a and raise it to "the power to which we must raise a to produce x", then we indeed produce x.

This is perhaps an exercise in tongue-twisting similar to the statement "The street which I must walk down in order to walk down King Street is King Street"!

Problem. Evaluate $10^{\log_{10}\sin\theta}$.

Solution. The base 10 is raised to the power to which 10 must be raised to give $\sin\theta$; so the answer is $\sin\theta$.

Problem. Evaluate $6^{\log_6 13}$.

Solution. The base 6 is raised to the power to which 6 must be raised to give 13; so the answer is 13.

Problem. Evaluate $(.01)^{\log_{10}16}$.

Solution.
$$\begin{aligned}(.01)^{\log_{10}16} &= (10^{-2})^{\log_{10}16} \\ &= 10^{-2\log_{10}16} \\ &= 10^{\log_{10}(16^{-2})} \\ &= 16^{-2} = 1/256.\end{aligned}$$

Problem. Evaluate $(\sqrt[3]{10})^{\log_{10}343}$.

Solution.
$$\begin{aligned}(\sqrt[3]{10})^{\log_{10}343} &= (10^{1/3})^{\log_{10}343} \\ &= 10^{(1/3)\log_{10}343} \\ &= 10^{\log_{10}(343)^{1/3}} \\ &= 343^{1/3} \\ &= 7.\end{aligned}$$

Problem. Evaluate $100^{\log_{10}\cos\theta}$, given that $\cos 2\theta = \frac{1}{7}$.

Solution. $100^{\log_{10}\cos\theta} = (10^2)^{\log_{10}\cos\theta}$

$= 10^{2\log_{10}\cos\theta}$

$= 10^{\log_{10}(\cos^2\theta)}$

$= \cos^2\theta$

$= \frac{1}{2}(1 + \cos 2\theta) = \frac{4}{7}.$

The last problem suggests many related problems.

OUTLINE FOR GRADE 12

The following material is published according to the motion of the Ontario Mathematics Commission at its April meeting (p. 12 of minutes) when it was requested that "chapter and section headings be published in the Gazette together with suitable preamble and the intimation that more detailed outlines were available on request".

It was also requested (p. 11 of minutes) "that the preamble to the Grade 12 material make clear that these outlines are proposals for the preparation of experimental material and not proposals of courses to be immediately instituted in the schools."

In short, these are "proposals for experimentation, and anyone receiving them [must] be aware of this". "The set-up as finally arrived at [may] be entirely different from present proposals."

CHAPTER II - THE QUADRATIC FUNCTION AND QUADRATIC EQUATION

Section 2.1 - Review

Section 2.2 - The Quadratic Function

Section 2.3 - The Quadratic Equation

CHAPTER III - COMPLEX NUMBERS

Section 3.1 - Introduction

Section 3.2 - Extension of the Number System

Section 3.3 - Geometric Representation of Complex Numbers

Section 3.4 - Completion of discussion of character of the roots of $ax^2 + bx + c = 0$, when a, b, c are real, $a \neq 0$.

Section 3.5 - Sum and product of roots of $ax^2 + bx + c = 0$. Simple exercises based on $x_1 + x_2 = b/a$, $x_1 x_2 = c/a$.

CHAPTER IV - OTHER QUADRATIC RELATIONS AND THEIR GRAPHS

Section 4.1 - The Circle

Section 4.2 - Inductive Study of the Parabola

Section 4.3 - Inductive Study of the Ellipse

Section 4.4 - Inductive Study of the Hyperbola

Section 4.5 - A Pair of Straight Lines

CHAPTER V - COMPOUND RELATIONS

Section 5.1 - Introduction

Section 5.2 - Relations Defined by Two Linear Equations (Review)

Section 5.3 - Systems of Linear Equations in Three Variables

Section 5.4 - The General Solution of a System of Two Linear Equations

Section 5.5 - Relations Defined by Linear Equalities

Section 5.6 - Relations Defined by One Linear and One Quadratic Equation

Section 5.7 - Relations Defined by Two Quadratic Equations

Section 5.8 - Other Special Types

Section 5.9 - Other Relations Defined by Two Quadratic Equations

Section 5.10 - (Optional) Relations Defined by Quadratic Inequalities

CHAPTER VI - INDICES AND EXPONENTS

Section 6.1 - Introduction

Section 6.2 - Definition of a^x, $a \in R$, $x \in N$

Section 6.3 - Formal Proofs of the Laws of Indices

Section 6.4 - Extension of the Definition of a^x to all Integral Exponents

Section 6.5 - Roots

Section 6.6 - Extension of the Definition of a^x to all Rational Exponents

Section 6.7 - Summary of the Laws of Exponents for Rational Indices

Section 6.8 - Extension of the Definition of a^x to irrational exponents

Section 6.9 - Order Relations in Exponential forms (Optional)

Section 6.10 - Graphs of Exponential Functions (Optional)

CHAPTER VII - LOGARITHMS

Section 7.1 - Introduction

Section 7.2 - Definition of a logarithm

Section 7.3 - Systems of Logarithms

Section 7.4 - Characteristic of the Logarithm

Section 7.5 - Mantissa of the Logarithm and Use of Tables

Section 7.6 - Antilogarithms

Section 7.7 - Multiplication Using Common Logarithms

Section 7.8 - Computations Using Logarithms

Section 7.9 - (Optional) Logarithmic and Exponential Equations

Section 7.10 - Graph of $\log_{10} x$

Section 7.11 - Develop the logarithm laws for the general cases

Section 7.12 - Develop $\log_b M = (\log_a M) \times \frac{1}{\log_a b}$

CHAPTER VIII - TRIGONOMETRY

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Section 8.2 - Radian Measure

Section 8.3 - Trigonometric Functions for the General Angle

Section 8.4 - Angles Related to the Acute Angle x

Section 8.5 - Numerical Values of the Trigonometric Functions of angles of any Magnitude

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Section 8.7 - Graphs of the Trigonometric Functions

Section 8.8 - Identities

Section 8.9 - Solution of Triangles

CHAPTER IX - INTRODUCTION TO SEQUENCES, SERIES AND MATHEMATICAL INDUCTION

Section 9.1 - Definitions of Sequence and Series

Section 9.2 - Problems on Writing Terms of a Sequence (or series) Beyond Terms Already Known

Section 9.3 - Finding Particular Terms of Given Sequences and Series

Section 9.4 - Finding the nth Term of a Sequence (or series) by Relating to the Fundamental Sequence

Section 9.5 - Predicting the Sum of a Series by Observation of Patterns and Subsequent Educated Guess

Section 9.6 - Introduction of Mathematical Induction to Prove Conclusions Predicted in Section 9.5

O.M.C. Editorial

R.B. Gwilliam, Ryerson Institute of Technology

The Ontario Mathematics Commission was addressed by the retiring president, Prof. A.J. Coleman of Queen's University, at its second annual meeting held in Toronto last month. In his review of the work of the commission and its committees during the past two years he recalled that the original stimulus for the formation of a commission came from the Ontario Teachers' Federation which had been generous in its financial and moral support from the start. Dr. Coleman discussed the aims and objectives of the commission and noted the the production of experimental text material for the secondary schools had been successful and that with the co-operation of the Department of Education and several secondary schools in the province trials of the "modern mathematics" had been made. He hoped that the commission would encourage authors to write experimental materials for the elementary as well as for the secondary schools.

Prof. Coleman mentioned that the "Gazette" had met with a favourable reception and hoped that it would serve as a means of maintaining communication between teachers and students of mathematics in Ontario. There were two objectives of the commission, however, which Dr. Coleman felt should be given more emphasis in the coming year; enrichment programmes for superior students and "in service" training courses for teachers in the elementary and secondary schools.

The commission unanimously supported a motion which expressed the thanks of the commission and its members to the retiring president for his guidance and inspiration during his two terms of office.

The commission elected the following members to executive office for the year 1962-63.

Past President.....	Prof. A.J. Coleman
President.....	Prof. H.A. Elliott
Vice-President.....	J.N.C. Sharp
Secretary-Treasurer...	F.C. Asbury
Members.....	Dr. P.R. Beesak
	Rev. J.C. Egsgard
	J.C. Gardner
	J.F. Kinlin

Some More Difficult Locus Problems

E. Orlando, Timmins High and Vocational School

The following series of four problems (related in method of solution) may be of use to members of Mathematics Clubs. They are:

(1) Find the locus of the point of intersection of mutually perpendicular tangents to a parabola.

(2) Same as (1), for the hyperbola.

(3) If P_1P_2 is a variable chord of the circle $x^2 + y^2 = a^2$ which is always parallel to the y axis, find the locus of P, the intersection of P_1A_1 and P_2A_2; A_1 and A_2 are the points $(a,0)$ and $(-a,0)$ respectively (Problems, 1961).

(4) Find the equation of the locus of the point of intersection of tangents to the circle $x^2 + y^2 = a^2$ that cut off an intercept of length $2a$ units from the line $x - a = 0$.

These four problems really exemplify this type of locus problem. The first is rather easy, but it can be used to get across the idea that we are seeking a result which is independent of the parameter m. Hence it is impressed upon the student that, in this sort of problem, we try to eliminate the parameter by any legitimate method. Number 4 is particularly interesting.

Symmetric Functions of Roots of Equations
An Alternative Approach

A recent examination posed the problem: If p and q are the roots of the equation

(1) $$7x^2 - 5x - 1 = 0,$$

form the quadratic equation whose roots are p^{-3} and q^{-3}.

The standard solution is well-known, namely,

$$p + q = 5/7, \quad pq = -1/7.$$

Then $$p^{-3} + q^{-3} = (p^3 + q^3)/p^3q^3$$

$$= [(p+q)^3 - 3pq(p+q)]/p^3q^3$$

$$= -140$$

$$p^{-3}q^{-3} = -343$$

The required equation is

$$x^2 + 140x - 343 = 0$$

It is interesting to solve the problem by introducing the quantity $y = x^{-3}$; then we can formally write $x = y^{-1/3}$ and obtain

$$7y^{-2/3} - 5y^{-1/3} - 1 = 0,$$

$$7 = 5y^{1/3} + y^{2/3}.$$

Cubing to simplify, we obtain

$$343 = 125y + 75y^{4/3} + 15y^{5/3} + y^2$$

$$= 125y + 15y(5y^{1/3} + y^{2/3}) + y^2$$

$$= 125y + 15y(7) + y^2.$$

We thus see that y satisfies the relationship

$$(2) \qquad y^2 + 140y - 343 = 0.$$

Notice that (1) is an equation satisfied only by p and q; similarly, (2) is an equation satisfied only by p^{-3} and q^{-3}.

PROBLEMS

Problem 3 was poorly worded. It is herewith reworded.

"A tower of height h stands on a level plane. Points A and B lie in a straight line through the foot of the tower. AB subtends an angle α at the top of the tower and an angle 2α at a point exactly $h/4$ units up the tower. Express AB in terms of α and h."

Problem 5 (Proposed by Norman Gillespie):

A drunk falls down 4, 7, 12, 19, ... times an hour when he has consumed 20, 21, 22, 23, ... ounces of alcohol respectively. How much may he consume and still get home if it takes him 1/2 second to fall down and 12 seconds to get up? Assuming that his staggering rate is 2 mph and that he lives 1/2 mile from the local pub, find how many minutes it will take him to get home after consuming 30 ounces of alcohol.