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ONTARIO MATHEMATICS GAZETTE

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EDITORIAL

By the time you receive this issue of the Gazette, it will be "Happy New Year"; also, we shall be working on the next issue of the Gazette.

This month, I want to say a few words about "multiplicity of approaches". This is a somewhat vague topic, but it is a timely one. Perhaps I should illustrate what I mean be an example.

We all know that, in developing the number system, it is possible to start with the positive integers, then introduce the positive rationals, and finally introduce zero and the negative rationals. On the other hand, it is possible to start with the positive integers, first introduce zero and the negative integers, and finally bring in the system of positive and negative rationals.

Now both of these alternative approaches are sound. Either may, for practical reasons, be preferred in certain circumstances or with certain groups, or even just for the sake of variety. Personally, I hope that we will, as time goes on, be able to select from among texts embodying both approaches. There is no intrinsic superiority in either.

Indeed, a knowledge of more than one approach often casts more light on a subject; the more angles a teacher can view a subject from, the better insight he will have. The great Gauss strongly urged the virtues of providing several alternative proofs of important results, and himself gave several proofs of his law of quadratic reciprocity. Knowing several approaches to a result helps one to avoid the pitfall encountered by the teacher who, familiar only with the angle - bisector approach to the Theorem that "the angles opposite the equal sides of an isosceles triangle are equal", assured an enterprising student who had herself discovered the Euclidean "reflection" proof that this latter proof was wrong!

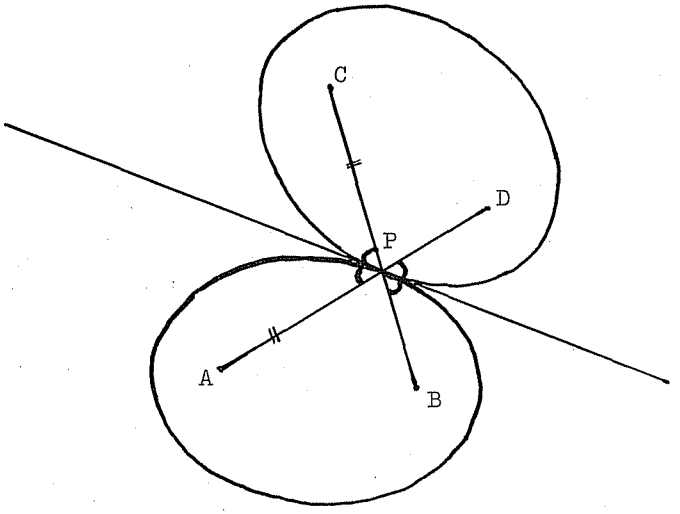
Last issue, I quoted Professor Allendoerfer who defended the "domain, range, and rule" approach to functions as compared with an exclusive use of the "number pair" approach. This month, we are fortunate to have an article by Professor Staal who points out that there are excellent grounds for not defining functions in terms of "sets of ordered pairs", but for accepting both functions and sets as primitive undefined elements at the secondary level. This discussion should be thought-provoking, and will help us to realize that there is no dogmatically correct orthodox approach to many mathematical problems. The ensuing mathematics, the meat of the subject, is the same, no matter what our pedagogic approach; but the roads to this mathematical salvation are many.

I had better end up by wishing you a prosperous and successful term before I get too deeply involved in philosophizing! I shall look forward to seeing many of you at the Easter meeting.

Ralph G. Stanton

Editor-in-Chief

PROBLEMS SOLUTIONS



Problem I, proposed by Mrs. Leppard: If an ellipse rolls without slipping on another ellipse of the same size so that the two ellipses touch at corresponding points, prove that the locus of a focus of the moving ellipse is a circle with centre at a focus of the fixed ellipse.

Solution by Norman Miller: The variable ellipse with foci C and D rolls on the equal fixed ellipse with foci A and B and major axis 2a. The point of contact P is a corresponding point on the two curves. The two ellipses are symmetrical with respect to their common tangent at P. From this property and the fact that focal chords to a point make equal angles with the tangent to that point, it follows that the four angles marked in the figure are equal. Hence BPC and APD are straight lines. Then BC = DP + PC = 2a and also AD = 2a. The conclusion follows.

Also solved by: M.C. England, Charles Plourde.

THE THREE FAMOUS PROBLEMS OF GEOMETRY IN THE HISTORY OF MATHEMATICS

(Lecture given at the Easter Convention of the Ontario Association of Teachers of Mathematics and Physics, Tuesday, April 24th, 1962)

C.J. Scriba, University of Toronto

I

A discussion of the history of the three famous problems in geometry will lead us through more than 2000 years of mathematical history, and will require us to pay some attention to algebraic considerations as well. That a single subject, in fact, a rather specific set of three problems, has been able to interest mathematicians for such a long time, and has stimulated some excellent work by outstanding mathematicians, seems to justify the choice of it as subject for a general lecture such as this. My choice is far from being original: some of you undoubtedly know of the celebrated lectures which the leading German educator in mathematics around the turn of the century, Felix Klein, gave some 65 years ago, and which are available in English translation in a little paperback bearing the title "Famous Problems of Elementary Geometry". Klein, who was a first-rate mathematician himself, aimed at giving a complete treatment, with rigorous proofs, in as elementary a form as possible. L.E. Dickson's article, in the "Monographs on Modern Mathematics", is based on Klein's exposition. Other writers at the beginning of the century, for instance, D.E. Smith and E.W. Hobson, devoted an article to the Quadrature of the Circle, and most books on the history of mathematics or general introductions to mathematics spare at least some lines, if not a number of pages, to comment on the classical problems.

My goal to-day can not be a very ambitious one. After a brief introduction concerning the nature of the problem, I want to indicate how the shift from geometry to algebra came about and in what sense a study of the algebraic equivalent of the geometrical problems finally led to a definite answer. It was not before the end of the 19th century that this answer was established, a fact which explains why so many writers devoted time and energy to a discussion of the problems some 60 years ago. In the last part of this lecture, I intend to exhibit some of the geometric constructions which have been devised during the ages in attempts to solve the three famous problems.

As so often, the history of our problems takes us back to Euclid's Elements of Geometry, compiled about 325 B.C. You are familiar with the axiomatic set-up of Euclid's work; all that concerns us here is the first three of his postu-

lates which require that:

1) a straight line can be drawn from any given point to any other point;
2) a given line segment can be extended in either direction;
3) a circle can be drawn with any given centre and with any given radius.

It does not require much intelligence to see that the only tools needed to carry out the constructions here postulated are an unmarked straight edge and a compass. As a consequence, a compass may then also be used to transfer the length of a given segment from one line to another.

However, a straight edge with rulings is not admitted; its use would make constructions possible which can not be made by means of an unmarked straight edge and a compass. It is important to note this difference; in fact, the whole nature of the problems depends on the tools which are permitted. Perhaps the best way to view the situation is to take it as a game whose basic rules are those just postulated: the only actions allowed in the game are the elementary geometrical constructions which use only the unmarked ruler and a compass in the sense postulated by Euclid. This, at least, was the view taken by the Greeks and later by other geometers. (To outwit the "smart" students, we should perhaps add that in this game pencil and paper are to be supplied by the examiner!)

After so much introductory talk, it is time to get down to the problems themselves. It is required to give ruler and compass constructions which accomplish:

1) the quadrature of the circle;
2) the trisection of an arbitrary angle;
3) the duplication of the cube.

In other words, Problem One calls for the construction of a square equal in area to a given circle; Problem Two requires us to find a method of trisecting any arbitrary given angle; Problem Three requires us to construct the side of a cube whose volume shall be twice the volume of a given cube.

The third problem is also known as the "Delian Problem". One version of the story is that the citizens of Athens appealed to the oracle at Delos asking what they should do to stop the plague that raged in their city. The oracle replied that they should double the size of the altar of Apollo. This altar, unluckily for the Athenians, had the shape of a cube. Whether or not Apollo's altar even was replaced by a bigger one, I do not know, but the plague must eventually have come to an end since Athens continued to flourish.

In all these problems, the ancients sought in vain for a solution with Euclidean tools, and the celebrity of these problems is mostly due to the fact that nobody succeeded in solving them; so they remained a challenge for each new generation.

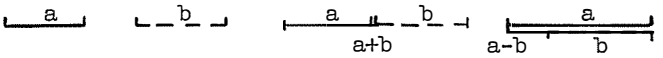
It is true that a great number of constructions have been proposed in ancient and modern times, but they all have one of two short-comings. Either they adhere to the restriction of using only a straight edge and a compass (in which case they are only approximations); or they give exact solutions (in which case they use some additional appliance besides those just mentioned).

It may be well at this point to dwell for a moment upon the difference between practical and theoretical constructions. Constructions which satisfy any practical need can always be given, and their accuracy may be estimated. We are interested in the theoretical possibility or impossibility of giving an exact ruler and compass construction for the solution of a certain problem. For instance, it is rather obvious that a hyperbola can not be drawn accurately by means of a straight edge and a compass alone. And yet any child can draw a hyperbola by drawing a number of tangent lines in a simple systematic way without any additional tools. It is the theoretical aspect only to which the following remarks apply.

II

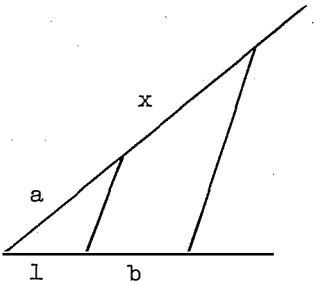
To approach an answer as to whether or not theoretically exact constructions for our problems can be given, let us analyze (forgetting the actual historical development for a while) what kind of constructions can be made by the straight edge and compass alone; let us then compare the results with what is required in the three classical problems. We can best appreciate the range of ruler and compass constructions by translating the results into algebraic terminology.

First of all, a little experimentation will yield constructions for each of the four basic operations, that is, addition, subtraction, multiplication, and division. Given two segments of lengths a and b, with $a \geq b$, $a + b$ and $a - b$ are immediately found.

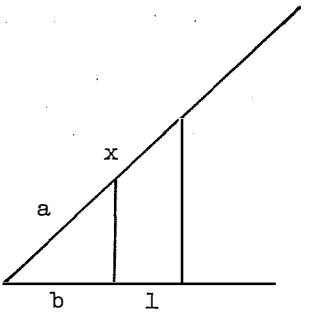


To construct the product ab and the quotient a/b, we need, in addition to a and b, a given segment of unit length 1. Mark off these segments on the legs of an arbitrary angle, as shown, and draw a pair of parallel lines. The fourth segment thus determined is of length ab in one case, and of length a/b in the other.

The numbers derivable from unity by these four operations are either integers or fractions; they belong to the "field of rational numbers". Applying any of the four basic operations to them will not produce any other kind of numbers; the result will always be in the rational number field.

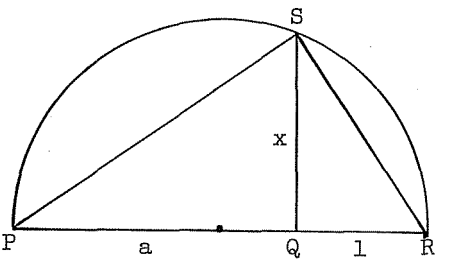


$\frac{a}{1} = \frac{x}{b}, \quad x = ab$



$\frac{a}{b} = \frac{x}{1}, \quad x = \frac{a}{b}$

Furthermore, we can also extract square roots geometrically as follows. To find a segment of length $\sqrt{a}$, draw a semicircle on a line segment PQR of length $a + 1$ ($PQ = a$, $QR = 1$), and erect a perpendicular at Q which meets the semicircle at S. The segment $QS = x$ will have the length $\sqrt{a}$.



$\frac{a}{x} = \frac{x}{1}, \quad x^2 = a, \quad x = \sqrt{a}$

Having shown that these five operations, together with their repetitions and combinations, can be carried out by ruler and compass constructions, let me now state that thereby all possibilities are exhausted. Under the given restrictions, we can only draw straight lines and circles. But the intersection of two lines, of a line with a circle, or of two circles, will lead to at most a quadratic equation in one unknown (once the second unknown has been eliminated); such a quadratic equation does not produce any irrational

numbers except square roots. Of course, one may repeatedly extract square roots in a more complicated construction, and thus be led to numbers of the type

$$\sqrt{2+\sqrt{3}-\sqrt{5}},\quad \sqrt{\sqrt{\sqrt{7}}}=\sqrt[8]{7},\quad \frac{\sqrt{\sqrt{2+\sqrt{3}}-17}}{14-\sqrt[4]{5}},\ \text{etc.}$$

But a third, fifth, or sixth root will never be produced in these constructions. This result can be summarized in

Theorem I: An analytic expression can be constructed by Euclidean tools if and only if it can be derived from known quantities by a finite number of rational operations and extractions of square roots.

On the other hand, corresponding to each such expression, there is a "simplest" equation which has this expression as a root. Corresponding to the rational number b/a, we have the equation $ax - b = 0$ satisfied by $x = b/a$. Corresponding to simple quadratic irrational number $\sqrt{a}$, we have the equation $x^2 - a = 0$, and no linear equation with integral coefficients will have $\sqrt{a}$ as its root. The expression $\sqrt{3+2\sqrt{2}}$ is a solution of $x^4 - 6x^2 + 1 = 0$, but of no equation of lower degree with integral coefficients. Let us call an equation, none of whose roots satisfy another equation of lower degree, "irreducible".

Example. If $x = \sqrt{3\sqrt{2}+2\sqrt{3}}$ is given, we find

$x^2 = 3\sqrt{2} + 2\sqrt{3}$, $x^4 = 30 + 12\sqrt{6}$, $(x^4-30)^2 = 864$ or

$x^8 - 60x^4 + 36 = 0$.

You may in the same way check that the equation

$$256x^8 - 1024x^7 + 1792x^6 - 1792x^5 - 22880x^4 + 47552x^3$$
$$- 35888x^2 + 11984x + 21001 = 0$$

is satisfied by

$$x = \frac{1}{2}(1 + \sqrt{5}\,\sqrt{3\sqrt{2} - 2\sqrt{3}}).$$

These examples must suffice for now to convince you of the following fact: an analytic expression composed of rational operations and square roots gives rise to an irreducible equation whose degree is a power of 2, say 2^h. For further details, you may consult the books by Klein, Courant and Robbins, and Dresden. If, therefore, an irreducible equation is given with a degree different from 2^h ($h = 1, 2, \ldots$), its solution must involve roots other than square roots. Thus we have

Theorem II: If an irreducible equation is not of degree 2^h, where h is a positive integer, it can not be solved by square roots.

We shall base the following discussion of the three famous problems on these two theorems.

III

As the first problem, consider the trisection of an arbitrary given angle. Let the angle α which we want to trisect by Euclidean tools be specified by its cosine. We start from the trigonometrical formula

$$\cos \alpha = 4 \cos^3 (\alpha/3) - 3 \cos (\alpha/3).$$

Here $\cos (\alpha/3)$ is to be determined from the given value of $\cos \alpha$. Now to show the impossibility of the trisection of a general angle, it is sufficient to demonstrate that one particular angle can not be trisected, since the general method would have to cover every single example. As a particular angle, I select $\alpha = 60^\circ$; then $\cos \alpha = 1/2$. Using $z = \cos (\alpha/3)$ for the unknown, the equation becomes

$$8z^3 - 6z = 1.$$

To simplify further, let $y = 2z$; we obtain

$$y^3 - 3y - 1 = 0.$$

If the angle of 60° can be trisected in the prescribed way, the solution y of this equation must be a real number obtainable by the four rational operations and the extraction of square roots. In other words, the cubic equation in y must be reducible.

To show that this is not the case, let us proceed in two steps. First, I will demonstrate that y can not be a rational number, that is, an integer or a fraction. Suppose on the contrary, that $y = p/q$, where p and q have no common factor. By substitution we arrive at

$$p^3 - 3pq^2 - q^3 = 0;$$

hence $p(p^2 - 3q^2) = q^3$ or $q^2(3p + q) = p^3$.

The first equation tells that p is a factor of q^3, hence of q; since p and q were assumed to have no common factor, $p = \pm 1$. From the second equation, we see that q must be a factor of p; therefore $q = \pm 1$. So if there is a rational solution y, it must be ± 1. It is clear that neither value satisfies the given equation.

Next let us show that a quadratic irrationality can not satisfy the equation

$$y^3 - 3y - 1 = 0.$$

Suppose $y = a + b\sqrt{c}$, where a and b are rational, $b \neq 0$, c is an integer differeing from 0 and 1 and having no square factor. Substituting, we find

$$(a^3 + 3ab^2c - 3a - 1) + (3a^2b + b^3c - 3b)\sqrt{c} = 0;$$

this is only possible when the two expression in the brackets vanish. The second one gives $b^2c = 3 - 3a^2$; when this is

substituted into the first expression one obtains $-8a^3 + 6a - 1 = 0$. With $x = -2a$, this gives

$$x^3 - 3x - 1 = 0,$$

and this equation ought to be satisfied by a rational value of x. We have earlier shown that this is not so. Hence a can not be rational; since this is contrary to our assumption, we see that $y = a + b\sqrt{c}$ is not a possible solution; so the equation in y is irreducible. Since its degree is 3, which is not a power of 2, it can not be solved by square roots (Theorem II). It follows by Theorem I that the solution is not constructible by Euclidean tools. The trisection of an arbitrary given angle by straight edge and compass alone is thus impossible.

In spite of this fact the problem continues to attract a great deal of attention. Arnold Dresden, in his book "An Invitation to Mathematics", has this to say: "It is probably no exaggeration to say that every year the mathematics departments of colleges and universities throughout this country receive numerous "Solutions"of this problem. They come from bakers and butchers and candlestick makers. Not infrequently a high school student succeeds in making the front page of a local newspaper with the announcement of his discovery. It is a sad commentary on the teaching of mathematics in the schools that interested and ambitious pupils are allowed to waste their energies on so futile an undertaking."

In the Delian problem, we are given a cube of side a, say, and have to find one of side x; thus $x^3 = 2a^3$ or $x^3 - 2a^3 = 0$. Set $x = ay$ to give

$$y^3 - 2 = 0.$$

To save time, I will simply state that this cubic equation is irreducible, (this may be verified by trying out expressions for y which are of exactly the same form as those used for the equation in y for the angle trisection problem). Since its degree is not a power of 2, its solution $\sqrt[3]{2}$ is not constructible, and hence $a\sqrt[3]{2}$ can never be found exactly by means of a ruler and a compass. Thus the final answer to the Delian problem is also negative.

Before we turn to the third problem, let me mention an observation made by Hippocrates of Chios more than 100 years before Euclid. He noticed that the duplication of the cube is equivalent to the determination of two mean proportionals between two given numbers. If a and b are two given positive numbers, then x and y are the two mean proportionals between a and b if

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}.$$

From this we obtain $ay = x^2$, $bx = y^2$, hence $x^4 = a^2y^2 = a^2bx$, $x^3 = a^2b$. If $b = 2a$, $x^3 = 2a^3$. Thus the Delian problem is equivalent to the determination of two mean proportionals between a and $2a$, and we shall later see how this fact can be used to give a construction by means of conic sections.

Our last problem is considerably more difficult, and I can not hope even to give you an outline of the proof of its impossibility. If, as usual, the ratio of the area A of a circle of radius r and a square of side r is denoted by π, so that $A = \pi r^2$, it is required to find a square of side x which has equal area:

$$x^2 = \pi r^2, \quad x = r\sqrt{\pi}.$$

Thus the constructibility depends upon the nature of the number $\sqrt{\pi}$. If it is rational or can be found by using square roots, it will be constructible; otherwise, it will not.

Towards the end of the 18th century, Lambert and Legendre showed that π is not a rational number, but it took another century until the problem was finally settled when Lindemann, in 1882, by work based on investigations of Hermite, exhibited the transcendental nature of π. By this, we mean that π can not be the solution of any algebraic equation with rational coefficients, no matter how high its degree may be. The equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

will not be satisfied by π for any choice of coefficients and degree, as long as the coefficients are integers $(n > 0)$. A simplified version of the proof may be found in the monographs by Klein and D.E. Smith, but the fact that π does not satisfy any algebraic equation calls for different and more advanced methods than those applicable to the other two problems.

The quadrature of the circle distinguishes itself from the other two problems for one other reason. The number π appears in many different places in mathematics not having any obvious connection with the area of the circle. This became particularly obvious in the 17th and 18th centuries when analysis took such great steps forward. New roads were opened up by the method of infinite series. For instance, Leibniz found in 1673 that the infinite series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

has the sum $\pi/4$. Other more rapidly convergent series enabled eager calculators to compute more and more decimal places of π. In 1699, 70 places were known; in 1706, 100; in 1719, 112; in 1789, 126; in 1844, 200; in 1853, 440; in

1855, 500; in 1873, 700 (of which only 528 were correct); in 1949, an electronic computer, in about 70 hours, gave π to 2035 places. [After the lecture, I was told that recently more than a million decimals of π have been computed.]

In passing, let me mention the guild of the circle squarers; they belong in the same category as the angle trisectors. In his "Budget of Paradoxes", published in 1872, A. De Morgan states: "Before proceeding to open the Budget, I say something on my personal knowledge of the class of discoverers who square the circle, upset Newton, etc. I suspect I know more of the English class than any man in Britain. I never kept any reckoning; but I know that one year with another - and less of late years than in earlier time - I have talked to more than five in each year, giving more than a hundred and fifty specimens. Of this I am sure, that it is my own fault if they have not been a thousand. Nobody knows how they swarm, except those to whom they naturally resort. They are in all ranks and occupations, of all ages and characters. They are very earnest people, and their purpose is bona fide the dissemination of their paradoxes. A great many - the mass, indeed - are illiterate, and a great many waste their means, and are in or approaching penury. ... I will not, from henceforward, talk to any squarer of the circle, trisector of the angle, duplicator of the cube, constructor of perpetual motion, subverter of gravitation, stagnator of the earth, builder of the universe, etc."

The climax in the history of the circle squarers was perhaps reached in 1897, when House Bill No. 246 of the Indiana State Legislature was passed by the House, reading in part: "Be it enacted by the General Assembly of the State of Indiana: It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side." Fortunately, the bill was shelved by the Senate in spite of the energetic backing of the State Superintendent of Public Instruction.

IV

In this section, some examples of ancient and modern constructions were discussed.

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GRADE XIII PROBLEMS, 1962

The answers to the even-numbered problems will appear in the February issue of the Gazette.

1. Given that the two arithmetic progressions whose sums to n terms are $29n - 2n^2$ and $(17n + 3n^2)/4$ have a common rth term, find this term.

The first terms of the arithmetic progressions are 27 and 5 respectively. Let the common differences be denoted by d_1, d_2.

The nth term of the first progression is given by

$$t_n = 27 + (n-1)\, d_1 .$$

The sum of the first n terms, S_n, is thus given by

$$S_n = \tfrac{1}{2} n\, [27 + 27 + (n-1)\, d_1] = 29n - 2n^2 .$$

Equating powers of n and n^2 yields

$$(1) \qquad \tfrac{1}{2} d_1 = -2.$$

In the same way, using the second progression,

$$(2) \qquad \tfrac{1}{2} d_2 = 3/4.$$

The first two progressions thus have

$$a = 27, \; d_1 = -4$$

and

$$a = 5, \; d_2 = 3/2$$

respectively. Equating the r'th terms, we obtain

$$27 + (r-1)(-4) = 5 + (r-1)(3/2).$$

Then $22 = (r-1)(11/2)$; $r = 5$.

Thus the common term is the fifth, and its value is 11.

3. Show that if x is real

$$\frac{x^2 - 4x + 1}{x^2 - 6x + 3}$$

can assume any real value.

Let r be any real number. It is necessary to show that a real number x exists such that

$$(1) \qquad r = \frac{x^2 - 4x + 1}{x^2 - 6x + 3}$$

or

$$(2) \qquad x^2(r - 1) + 2x(2 - 3r) + 3r - 1 = 0$$

The equation (2), considered as a quadratic in x, has discriminant 4Δ, where

$$\Delta = (2 - 3r)^2 - (r - 1)(3r - 1)$$

$$= 6(r - 2/3)^2 + 1/3,$$

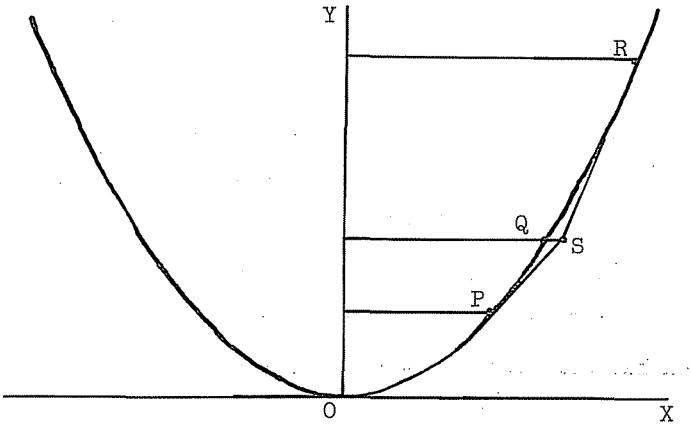
which is positive for all values of r. It follows that there exist real values of x which satisfy (2), and hence (1). In fact if

$$x_1 = (3r - 2 + \sqrt{\Delta})/2(r - 1) \quad \text{or}$$

$$x_2 = (3r - 2 - \sqrt{\Delta})/2(r - 1)$$

is substituted for x, the given expression assumes the value r.

5. Let P, Q, R be three points on a parabola whose distances from the axis are in geometric progression. Prove that the tangents to the parabola at P and R intersect on the line through Q perpendicular to the axis.



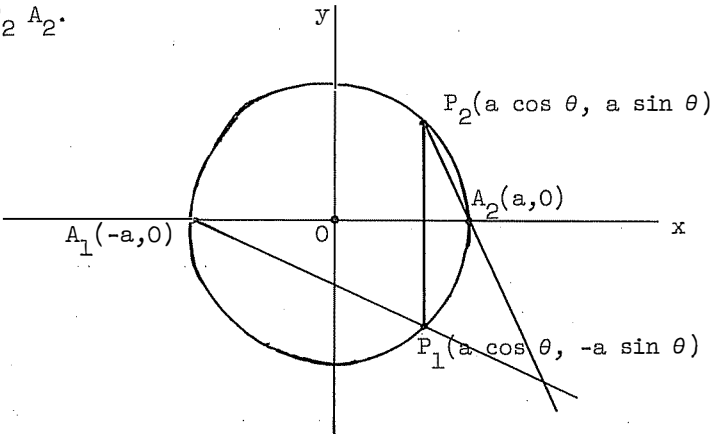
Any parabola can be put into the form $y = p x^2$ for a proper choice of p. Let the points P, Q, R have coordinates (a, pa^2), (b, pb^2), (c, pc^2) respectively. The distances a,b,c of the points P, Q, R from the axis will be in geometric progression provided $ac = b^2$. It is sufficient then to prove that the tangents at P and R intersect on the line $y = pb^2 = pac$.

But the equations of the tangents at P and R are given by

$$y + pa^2 = 2apx$$
$$y + pc^2 = 2cpx$$

which intersect at the point $S[\frac{1}{2}(a+c), pac]$ which lies on the line $y = pac$.

7. <u>The points A_1, A_2 are the ends of the diameter of slope zero of $x^2 + y^2 = a^2$. A variable line parallel to the y-axis meets $x^2 + y^2 = a^2$ at P_1, P_2. Find and identify the locus of the point of intersection of $P_1 A_1$ and $P_2 A_2$.</u>



The coordinates of A_1, A_2 are $(-a,0)$ and $(a,0)$. Let the coordinates of P_1, P_2 be $(a \cos \theta, - a \sin \theta)$ and $(a \cos \theta, a \sin \theta)$ respectively. The equations of $A_1 P_1$ and $A_2 P_2$ are then given by

$$(1 + \cos \theta)y \pm \sin \theta(x + a) = 0$$
$$(1 - \cos \theta)y - \sin \theta(x - a) = 0$$

which has solution $(a \sec \theta, - a \tan \theta)$.

The locus of Q, the point of intersection of $P_1 A_1$ and $P_2 A_2$, is obtained by moving $P_1 P_2$ parallel to the y-axis or equivalently by varying θ, and has parametric equations

$$x = a \sec \theta \qquad y = - a \tan \theta$$

The standard equation obtained by eliminating θ is $x^2 - y^2 = a^2$, and is easy recognized as a rectangular hyperbola.

Notice that only the portion of the hyperbola lying in the second and fourth quadrants is obtained if the points P_1, P_2 are labelled as in the diagram. The remainder of the hyperbola is obtained by interchanging the points P_1 and P_2.

9. Show that

$$4(\sin^2 A + \sin^2 B + \sin^2 C)(\cos^2 A + \cos^2 B + \cos^2 C) - (\sin 2A + \sin 2B + \sin 2C)^2$$

is never negative and, without the use of tables, find its value when $A = 80^\circ$, $B = 50^\circ$, and $C = 20^\circ$.

Denote the expression by E. Then, using the double angle formula $\sin 2\theta = 2 \sin \theta \cos \theta$, we multiply out and rearrange to give

$$\begin{aligned} E &= 4(\sin^2 A + \sin^2 B + \sin^2 C)(\cos^2 A + \cos^2 B + \cos^2 C) \\ &\quad - 4(\sin A \cos A + \sin B \cos B + \sin C \cos C)^2 \\ &= 4(\sin^2 A \cos^2 B + \sin^2 B \cos^2 A - 2 \sin A \cos A \sin B \cos B) \\ &\quad + 4(\sin^2 B \cos^2 C + \sin^2 C \cos^2 B - 2 \sin B \cos B \sin C \cos C) \\ &\quad + 4(\sin^2 C \cos^2 A + \sin^2 A \cos^2 C - 2 \sin C \cos C \sin A \cos A) \\ &= 4[\sin^2(A-B) + \sin^2(B-C) + \sin^2(C-A)] > 0. \end{aligned}$$

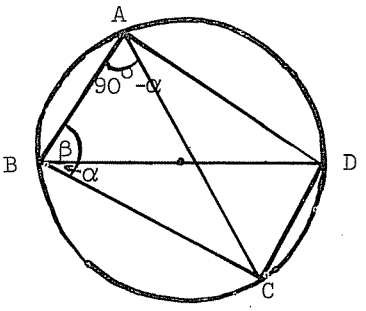
In the special case given, $E = 4(2 \sin^2 30^\circ + \sin^2 60^\circ) = 5$.

11. When a quadrilateral ABCD is inscribed in a circle, the following relation (Ptolemy's Theorem) holds among rectangles:

$$AB.CD + AD.BC = AC.BD$$

Consider the special case in which the diagonal BD is a diameter of the circle and denote the angles DBC and ABD by α and β respectively. Deduce from Ptolemy's Theorem the formula

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$



Since triangles BCD and BAD are right-angled at C and A respectively, it follows that

(1) $\frac{CD}{BD} = \sin\alpha$, $\frac{BC}{BD} = \cos\alpha$, $\frac{AD}{BD} = \sin\beta$, $\frac{AB}{BD} = \cos\beta$.

In $\triangle ABC$, $< ABC = \alpha + \beta$ and $< A = 90^\circ - \alpha$.

By the sine rule,

(2) $$\frac{\sin(\alpha+\beta)}{AC} = \frac{\sin(90^\circ - \alpha)}{BC} = \frac{\cos\alpha}{BC},$$

so that, using (1),

(3) $$\sin(\alpha+\beta) = \frac{AC}{BC} \cdot \frac{BC}{BD} = \frac{AC}{BD}.$$

Now dividing the relation among the rectangles in Ptolemy's Theorem by BD.BD we have after rearranging the terms

(4) $$\frac{AC}{BD} = \frac{CD}{BD} \cdot \frac{AB}{BD} + \frac{BC}{BD} \cdot \frac{AD}{BD}$$

But (4) is the law

$$\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

when the substitutions (1) and (3) are made.

Ontario Association of Teachers of Mathematics and Physics

Ontario Educational Association

November 22, 1962.

Dear Members:

The Executive and Councillors of the Ontario Association of Teachers of Mathematics and Physics are busy planning YOUR Easter Convention, and are hoping that 1963 will be the best meeting yet.

Dr. Max. Beberman, a dynamic teacher from the University of Illinois will be the main speaker and his topics "Positive and Negative Numbers and How To Get Them from the Real World" and "Functions and Geometry" should evoke your interest and bring you to Toronto. Dr. Beberman recently addressed the Mathematics Council of the Toronto Board of Education and his talks were stimulating and practical.

Dr. Norman Miller, who since retirement from the Department of Mathematics of Queen's University has been working with mathematically bright students in the High Schools of Kingston, will be on hand to tell us "How to Enrich the Programme for the Better than Average Student".

Bruce MacLean, Professor of Methods, Ontario College of Education, will speak on "Programmed Learning and its Implications for the Teachers of Mathematics".

James MacLachlan, who is head of the Science Department, W.A. Porter Collegiate Institute, Scarborough, and a co-author of a new text-book for Grade XI Physics, will speak on "Mathematical Implications of the Grade XI Physics Course".

You'll notice that the last two speakers are attempting to implicate you, so be on hand at the Ontario College of Education. Please note that this year there is a change in the time of our meeting. The first session will begin at 12.45 p.m. on Tuesday, April the sixteenth, and will last until 4.45 p.m. On Tuesday evening a reception will be held from 8 to 10 in the New Education Centre of the Toronto Board of Education. Husbands and wives are invited and here you may meet the Convention Speakers, chat with friends, and even do a bit of problem solving in the corner!

Wednesday's programme will begin at 9 a.m. and will finish with the Annual Luncheon in Bloor Street United Church. Seminars are being planned for either Tuesday or Wednesday and they will be arranged according to topic rather than by grades.

The stage is being set for you. Do not fail!

Sincerely,

Jean Leppard, Publicity Representative.

P.S. - A complete programme and more biographical details concerning the speakers will appear in the next issue of the Gazette.

Letters to the Editor

Dear Mr. Editor:

While glancing through the Mathematics books on sale at the University of Toronto Press, I recently hit upon a fascinating book by H.D. Forder, entitled "The Foundations of Euclidean Geometry". I would think that many of your readers would be interested in it, even though it was published in 1927 for the first time. It now exists as a paperback in the Dover Series.

Forder's book is an extremely detailed examination of the axioms of Euclidean geometry. Very few people would now have the courage to plod through it in complete detail, however, any one who is worried about the axiomatics of geometry will almost certainly find great help in this book. In addition, there are a number of fascinating sidelines which will supply material for Problems classes.

Of particular interest to us in Ontario is the following quotation from page 91:

> "As a method of proof, we must of course utterly rule out the method of superposition, the fallacies inherent in which have been frequently shewn ... it is a scandal that in some public examinations questions are still set, which demand formal proofs involving this vicious method; a method of which Euclid himself was clearly ashamed, as he only used it twice, and was prepared to make any detour to avoid it."

Yours sincerely,

A.J. Coleman.

Sets and Functions in Elementary Mathematics

R.A. Staal
University of New Brunswick

A conspicuous feature of many recent attempts to revise school mathematics, especially in North America, has been an emphasis on the concept of a set as the basic element of mathematics. Indeed, a more or less standard new approach has been developed known as "Modern Mathematics Based on Sets" (or some similar description). In this article we will

discuss some of the issues which arise in connection with this trend in introductory mathematics, our conclusion being that at best it is being handled rather badly with unnecessary and unfortunate by-products, and that at worst it is fundamentally mistaken and should be abandoned altogether.

For those who might feel an uneasy premonition at this point, we hasten to add that we shall not be attacking the edifice of set theory, nor the claim that mathematics can in some sense be founded on the notion of a set. Nor will we enter into dispute on intuitionistic grounds, or take any stand but the "classical" attitude toward ordinary mathematics (not that this would be without value).

Another point to be cleared up at the outset is that, except for the present paragraph, no notice will be taken of the really bad examples of the textbook racketeering that is fast becoming a major industry, stimulated no doubt by the possibility of vast sales among the school population. These deserve no more than a satirical treatment, yet, numerically, they dominate the textbook field, and their influence should not be underestimated. They have been soundly criticized on the grounds of replacement of content by jargon and pointless special symbolism, lack of real mathematical motivation, transparent disguise of trivialities by "modern mathematical" terminology, spurious generality and the false implication that what is presented as "modern" usage is what one actually finds predominant in recent literature. Nothing further need be added here, except to express the hope that these criticisms will receive far more attention than they have so far. The mere fact that a trend is basically unsound, or is even a travesty on educational principles, is no guarantee that it will not eventually, for a time at least, attain a dominating position in our schools, and the trend to which we refer is well on its way to such fulfillment.

Turning aside from the more questionable literature, let us now consider the common features of the more reputable attempts to produce a modern introduction to mathematics which is based on sets.

The genesis of these attempts seems to be somewhat as follows. It is noted that set-theoretical language has been a conspicuous feature of recent (for the past 50 years) developments in the foundations of mathematics. Problems concerning paradoxes, the continuum hypothesis, the axiom of choice, and other fundamental matters have often been dealt with in terms of an axiomatic set theory. In such a theory the concept of a set and of membership in a set is basic, in the sense of being undefined, other than by the axioms: the concepts of cardinal number and function are derived - a function, for example, is defined as a particular kind of set of ordered pairs, and ordered pairs are themselves defined in terms of unordered sets. It is also noted that set theory (transfinite numbers, analytic and projective sets, etc.) has become a major branch of mathematics, non-trivial portions of

which have applications in other regions of mathematics.

The conclusion is drawn that elementary mathematics can be unified, brought up to date, and otherwise improved by explicitly basing it upon the concept of a set, and deriving other concepts, when needed, from this. In particular, the concept of a function, with its obvious abstractness when presented as a correspondence, or mapping, is to be introduced as a set of ordered pairs, this being considered to be a more uniform and "concrete" approach.

It is encouraging to think that, in the organization of elementary mathematical instruction, some notice is being taken of what has recently been going on in mathematics itself. While acknowledging the excellence of the motivation, however, we wish to call attention to various pitfalls in the particular programme referred to above - pitfalls which we feel have on the whole not been successfully avoided, and which by their number and location suggest that perhaps the wrong road is being taken altogether.

Possibly the most obvious kind of error is to end up with nothing more than a transplanting of terminology and notation into a context in which they are without purpose and serve merely as an exotic distraction from the real business at hand. Excellent examples are the "set-maker operation" and the terminology of "truth-sets". One sees exercises such as the following:

"Find $\{x \mid x^2 + x - 6 = 0\}$"

"What is the truth-set of $x > 2$?"

Another is the terminology of "open sentences". This is particularly offensive, for the noun "sentence" is dealt with at some length in grammar instruction and is there given the usual dictionary definition as being a number of words expressing a complete thought, and therefore (if declarative) being either true or false, and yet the student is expected to think of an "open sentence" not as a sentence (such as he is used to) which is open (in some sense or other) but as something else. The difficulty is usually compounded by a very difficult and unconvincing salestalk about sentences with pronouns, such as "He was 14 years old" when "He" is described as being a non-mathematical example of a variable, in spite of the fact that such sentences are usually used in a context where "He" is an actual name, and not a variable at all, e.g., "John was in High School. He was 14 years old. (Compare with "If a boy is in High School, then he is 14 years old.") If one wants to get across the idea of an open sentence, it would be much better to use a more descriptive phrase such as "sentence form". Here the noun "form" suggests something from everyday experience (cement form, dress form, etc.) and "sentence" used as an adjective suggests that when the form is filled in some way it produces a sentence. To a mathematician such matters may be quite irrelevant, but is unforgivably

simple-minded to think that this is the case with children in school.

This error is closely connected to the trap of ultra - long - range motivation, in which the author, being a specialist in some field, thinks (without thinking) of each pupil as a potential specialist in this same field (or at least as a potential mathematician) and regards every step along the road as an opportunity for preparation for the great day ahead. (See the review by R.L. Goodstein, in the Mathematical Gazette, Feb. 1962, pp. 69-72, of the OEEC publications "New Thinking in School Mathematics" and "Synopses for Modern Secondary School Mathematics"). When a greater variety of specialists get in on the game, this will become more obvious. Some of our children will be trained from the start as finitists, others will think of functions as tabulations of ordered pairs, while others, in preparation for certain types of computation, will regard functions as finite expressions involving variables. The result of such postponed motivation is too often a distaste for what is actually being done at the time, and the enthusiasm of the writer or teacher is no consolation. Nor is it any help for the writer to admit the triviality of "exercises" such as we have quoted above, and to exhort the student to think of the (unstated) advantages he will receive later on, or to ask for a simple faith that what is apparently trivial will later on be seen as incorporating a profound analysis of mathematical principles which can not be appreciated at this early stage, but which can at least be prepared for. The student, if he has any sense, will turn his thoughts elsewhere.

A teacher is also to some extent responsible for pitfalls into which the student himself may stray. One of these is the mistaken impression that the use of sets in mathematics is something new and characterizes modern mathematics. This is quite false, and results from the confusion of use with formal use, and possibly also from the fact that set theory (like many other branches of mathematics) has grown in size and in the scope of its applications.

A more sophisticated error lies in assuming that what are taken as primitive undefined elements in an axiomatic theory are intended to correspond to the simplest intuitive concepts, and are therefore ideally suited as starting-points in elementary instruction. This is a point which merits special attention, and it is here where we feel that "Modern Mathematics Based on Sets" has fallen most seriously into error.

First, we question the assumption that the reduction of the number of basic notions, which is such an important part of a formal development, is obviously a good thing in elementary mathematics. No one has yet (we haven't, however, read all the latest publications) proposed the use of joint denial as a "unifying concept" at this level, and it seems at least plausible that the reasons for this might apply just as

well in the case of functions and sets. It has long been customary to talk, informally at least, about both functions (operations, relations) and sets (classes, collections) in elementary mathematics. We suggest that this is so because both can arise idependently out of the student's experience, as far as he is aware, and that a premature attempt to show dependence only creates an artificial situation which really adds nothing to the student's appreciation of what he is doing. (It is the task of the teacher to be aware of such dependences, and to use this awareness in smoothing the way, but such smoothing should be done without undue forcing.)

Secondly, if we turn to the standard works in set theory and foundations themselves, it is by no means clearly established that the concept of a set is fundamental in the sense of being conceptually primitive. For one thing, any purely axiomatic treatment is, by its very nature, neutral on this point. (See Fraenkel-Bar Hillel, "Foundations of Set Theory", North Holland, 1958, p. 164). For another, most authors refrain from getting involved at all in the region where intuition and formal systems meet. And finally, when the approach is from the logical-philosophical side, (which, if any, should be closest to teaching and its problems) there is a general tendency to introduce the concept of a function first, and then to get sets and classes out of propositional functions considered in extension. (See Church, "Introduction to Mathematical Logic", Princeton, and Quine "Mathematical Logic", Harvard). Nearly everyone seems to be familiar with the "class of all classes ..." definition of a cardinal number, but it does not seem to be so well known that Russell at the same time emphasized that classes, like descriptions, were eliminable. None of this establishes the priority of either the function or the set (nor does it show that neither can have priority), but where the angels themselves fear to tread, perhaps we should not be too hasty in making blueprints for reform.

Finally, the step from the prominence of set theory to "Modern Mathematics Based on Sets" results in part from succumbing to the enticing thought that a set is just as concrete as its elements. Attention is frequently called to the abstractness involved in the concept of a function, considered as a mapping or correspondence, and to the inadequacy of the way out through forms in a specified language. It is then remarked that "we can find a way out of this difficulty by defining a function simply as a particular kind of set of ordered pairs" (the underlining is ours). Now, if it were merely being said that it is better to reduce all our difficulties to one, then there would be less objection, but it is usually not at all clear that this is intended. Rather, the concept of a set is introduced by means of "all the chairs in this room", "the people in your family", etc., with no words of caution whatsoever, and then when the function is reduced to a class of ordered pairs, the impression is given that a difficulty has been eliminated, rather than merely re-located,

and this is not so. (See Suppes, "Introduction to Logic", 1957, van Nostrand, pp. 229-230; also Weyl, "Philosophy of Mathematics and Natural Sciences", 1949, Princeton, p. 12.)

Thus, the question of what concept or concepts should be taken as basic is by no means answered by referring to recent literature in the study of foundations, although relevant factors are of course brought out. It is not at all clear whether the function or the set should be taken as primitive concept, or possibly even both might be used in the early stages. This is something which will have to be decided at the conceptual level where the student first meets these ideas.

It is the writer's opinion that when the issue is faced at this level, without the unwarranted assumption of sanction from above, the advantages are in fact heavily on the side of presenting the function as a primitive idea, expressed in some such terms as "correspondence", "operation", or "mapping". This does not necessarily mean that sets must then be introduced in elementary mathematics (unless one regards a curve as a set of points) and there is therefore no harm in leaving analysis of the concept until later (e.g., consider "The solutions of $x^2 + x - 6 = 0$ are $-3, +2$". There is no need here to investigate in what sense, if any, -3 and $+2$ form a whole, and the advantages of writing this as "$\{-3, +2\} = \{x \mid x^2 + x - 6 = 0\}$" are not at all obvious.) On the other hand, if functions (which <u>are</u> treated formally as mathematical objects) are to be introduced as sets, then much more has to be said.

This is, of course, not a mathematical thesis, to be rigorously defended, but reasons for supporting it can be given.

First, one is primarily interested (a non-mathematical concept which we should not ignore!) in functions rather than in sets at this stage, and a direct route to the notion is therefore preferable.

Secondly, the great majority of functions met with arise out of a given operation (e.g., squaring) or form (e.g., a polynomial form) rather than out of an exhibited class of ordered pairs (even this, some would claim, can only be done via a function). In the case of functions defined over an infinite range, this is forced on us.

Thirdly, even when the range is finite, the approach via sets is of dubious value, especially if it encourages the student to confuse a set with a physical conglobulation. For example, in order to demonstrate that a random (in some sense or other) pairing, not necessarily corresponding to any simple form or description such as "$2x - 5$" or "square of" can be used to set up a function, a tabulation such as

x	1	2	3	4	5
$f(x)$	2	1	3	3	4

is presented, and a discussion follows such as "The above table determines ordered pairs which when formed into a set, constitute a function, f. For $x = 1, 2, 3, 4$ and 5, $f(x)$ is the second number of the ordered pair in f whose first element is x." In our opinion it would be equally rigorous and more to the point to say "The above table determines a function, or correspondence, which associates with each number x in the upper line the number $f(x)$ appearing directly beneath it." It is in the topic of functions and their graphs that we inevitably come closest to the approach which we are criticizing, for we certainly want to associate with every function f the set of all points (x,y) in a plane for which $y = f(x)$. We also want to be able to say that every set of points which satisfies the usual single-valuedness condition yields a function. But there is an appreciable step between this and defining a function as a set of ordered pairs, and there is certainly no need to become at all involved in the other practices of Modern Mathematics Based on Sets.

In summary, the programme known as Modern Mathematics Based on Sets is here criticized on the grounds that:

(a) contrary to the impression that is often given, it is not universally sanctioned by recent studies in set theory and foundations.

(b) it results in a purposeless and confusing transplanting of terminology and notion.

(c) the immediate motivation of the student is replaced by the ultra-long-range motivation of the teacher, without adequate compensation.

(d) it carries a misleading impression of newness.

(e) it forces a "unified" approach prematurely.

(f) it encourages the debatable attitude toward sets that they are as "concrete" as their elements, and provide a way out of the problem of an arbitrary correspondence.

We also suggest that it is not through mere ignorance that many mathematicians continue in their elementary courses to introduce the function as a primitive concept and to carry out the innocent operations on sets required at this level in an unpretentious and informal manner.

PROBLEMS WITH PHYSICAL QUANTITIES

James MacLachlan, W.A. Porter Collegiate

Many teachers are aware of different approaches to problems in algebra and physics classes. Mathematics teachers may be inclined to think that physics teachers treat problems in an inelegant and unrigorous way. They may be right, but I wish to illustrate a necessary distinction that requires a

difference in approach.

The concern of mathematics teachers is directed toward numbers and relations between them. Physics teachers, on the other hand, are concerned with the relations between physical quantities. Physical quantities (described by number and unit) are not simply numbers, and can not be treated as if they were. In elementary mechanics, we define u as the velocity of an object at a particular instant; v, the velocity after time t has elapsed; and a, the acceleration of the object. The relation between the quantities (from the definition of acceleration) is $v = u + at$. How should one present a solution to the problem: What is the attained velocity of an object with an acceleration of 7 feet per second per second ($7\ \text{ft/sec}^2$) 2 seconds after the instant when its velocity was 5 years per second (5 yd/sec)? Consider straight line motion.

(a) In an algebra class this problem has to be turned into a relation between numbers: Let the number of feet per second in the attained velocity be v, equal to the number of feet per second in the initial velocity u increased by the product of the number of feet per second per second in the acceleration a and the number of seconds in the time t. Since the number of yards per second in the initial velocity is 5, then $u = 5(3)$, since the number of feet in 1 yard is 3. Then, $a = 7$, $t = 2$, and

$$\begin{aligned} v &= u + at \\ &= 15 + 7(2) \\ &= 15 + 14 \\ &= 29. \end{aligned}$$

The attained velocity is 29 feet per second. But is it not also true that the attained velocity is 5 yd/sec + 14 ft/sec? In other words, is it not possible that the relation is true for units other than those specified in the preamble (even mixed units)?

(b) Here is the method found in a widely used physics text: Given, $u = 5$ yd/sec $= 15$ ft/sec, $a = 7\ \text{ft/sec}^2$, $t = 2$ sec, then,

$$\begin{aligned} v &= u + at \\ &= 15 + 7(2) \\ &= 15 + 14 \\ &= 29. \end{aligned}$$

The attained velocity is 29 ft/sec. This solution bears some similarity to (a), but is $u = 15$, or is $u = 15$ ft/sec? Another text goes so far as to write 15 + 14 = 29 ft/sec. The equality is simply not true, but the same error (though thinly veiled) occurs in the above solution.

(c) Why not simply state the symbols in the relations

of physics represent physical quantities? Then every substitution requires both a number and a unit. Given, $u = 5$ yd/sec, $a = 7$ ft/sec^2, $t = 2$ sec; then,

$$\begin{aligned} v &= u + at \\ &= 5 \frac{\text{yd}}{\text{sec}} + \left(7 \frac{\text{ft}}{\text{sec}^2}\right)(2 \text{ sec}) \\ &= 5 \frac{\text{yd}}{\text{sec}} \left(\frac{3 \text{ ft}}{1 \text{ yd}}\right) + 14 \frac{\text{ft(sec)}}{\text{sec}^2} \\ &= 15 \text{ ft/sec} + 14 \text{ ft/sec} \\ &= 29 \text{ ft/sec}. \end{aligned}$$

Here, units are being manipulated in a manner analogous to the manipulation of numbers, and the unit ratio "3 ft/1 yd" is used to produce consistent units.

When students include units in the relations of physics, they can readily see whether indicated additions can, in fact, be performed. And units can be used to check relations, in a form of dimensional analysis. For example, in another relation where s represents the displacement of an object,

$$s = ut + \frac{1}{2} at^2,$$

each term has the dimension of length. In MKS units,

$$s[\text{metre}] = y[\frac{\text{metre}}{\text{sec}}][\text{sec}] + z[\frac{\text{metre}}{\text{sec}^2}][\text{sec}]^2.$$

The same approach can be used to show that the famous quantity mc^2 has the dimensions of energy. $E = mc^2$ is physically true for any units of mass and velocity (of light). For example, the rest energy of 1 ounce of matter is 3.46×10^{10} ounce mile^2/sec^2. However unfamiliar it may be, the unit does have the dimensions of energy.

I do not expect that this use of physical quantities will find any currency in algebra classes. However, realizing that students bring their algebraic notions into physics classes, I would like algebra teachers to leave an opening for physics teachers to say: "You have been taught how to solve problems in algebra using a certain format; since we are dealing here with physical quantities, I recommend that you use this format." Is it possible for a teacher to avoid making a general condemnation of all approaches other than his own, and to assist students to see value in another approach?

THE TORONTO SEMINARS

John del Grande, Newmarket

The Toronto Seminars in mathematics attract more than 200 teachers to each of the three meetings a year. These Seminars are probably among the most popular local mathematics meetings in Canada.

In order to encourage other areas to start meetings of this kind, the executive of the OATMP requested that a short history of the seminars be written.

In the spring of 1958, Dr. A.J. Coleman, then at the University of Toronto, invited several teachers of mathematics to an informal get-together. During the course of the evening, one of the guests was reminiscing about the "good old days" when some Toronto teachers met occasionally on Saturday morning under the leadership of that great teacher, Dr. J.T. Jenkins. At that moment, the Toronto Seminars were created.

Dr. Coleman was appointed chairman, and H.E. Totton suggested we use Forest Hill Collegiate for the meetings. Mr. Kinlin volunteered to supply a list of schools. Others volunteered to spread the word and act in any capacity required.

The initial meeting was planned for about 40 teachers. The university professors were asked to give moral support to the project. The aims of the Seminars were outlined as follows:

(1) to promote discussions between the University and High School teachers on problems of common interest;
(2) to study current proposals in mathematics being made in other countries;
(3) to arrange study groups at which teachers could "talk shop".

The meetings were planned as follows:

day - usually Thursday
time - 4:00 p.m.
place - Forest Hill Collegiate
Number of meetings - 4 a year (now 3)
4:00 - 4:30 - coffee
4:30 - 6:00 - lecture and discussion usually by a professor
6:00 - 7:00 - supper
7:00 - 8:00 - 5 seminar groups for grades 9 - 13

For each of the seminar groups a chairman was appointed for the year. It was his responsibility to acquire speakers for topics at the grade level of his group. In most case, practising teachers have been used as speakers.

The teachers of each school in Toronto and area were contacted through the heads of the mathematics departments.

Replies were requested stating the number that expected to attend and the number requesting supper. To the amazement of the executive, about 150 teachers accepted the invitation. The first meeting was a great success.

Many teachers were interested in more than one seminar group and found it difficult to decide which to attend. To solve this, Father J.C. Egsgard and four of his staff volunteered to take detailed notes at each seminar and make them available at the next meeting. These minutes enabled each teacher to keep in touch with all the seminar groups.

Interested people from outside centres were invited and attended regularly. Teachers travelled from places as Orillia, Barrie, Oshawa, Hamilton, Peterborough, and Kitchener.

The present seminars are held at the Education Centre in Toronto. Invitations are still being sent, upon request, to anyone, anywhere. The seminars have become a place to meet old friends, make new ones, and see what is going on in the outside world.

The executive of the OATMP strongly urges any group that wishes to organize a series of seminars to contact them for advice and assistance. With proper planning and dedicated people, seminars can be immensely successful.

Some Remarks on Asymptotes

G. Berman and R.G. Stanton

1. Introduction. For over twenty years, the analytic geometry of the straight line, circle, parabola, ellipse, and hyperbola has been taught in our secondary schools. Work with the asymptotes of the hyperbola has been a standard part of this programme. It is therefore rather surprising to find an inadequate treatment of asymptotes, based on a fallacious definition, in both of the textbooks commonly used in Ontario. Perhaps this confusion has arisen because mathematicians use the word asymptotic in several ways. In any case, it is time that the proper definition of asymptotes as "tangents at infinity" appeared in Ontario textbooks; there is little point in drilling our students on material they will have to unlearn later if they are to consider asymptotes of curves more complicated than the conic sections.

2. Asymptotic Functions. Throughout this article, x will denote a real variable. We say that two functions $f(x)$ and $g(x)$ are asymptotic to one another if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

We also write

$$f(x) \sim g(x),$$

and say "f(x) is asymptotic to g(x)".

It must be stressed that $f(x) \sim g(x)$ does not mean that f(x) and g(x) are approximately equal for large x. For instance, if

$$f(x) = x^2 + 3x, \qquad g(x) = x^2,$$

we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(1 + \frac{3}{x}\right) = 1;$$

hence $f(x) \sim g(x)$. But, in this case,

$$f(x) - g(x) = 3x,$$

which becomes arbitrarily large as $x \to \infty$. This situation is illustrated in Figure 1.

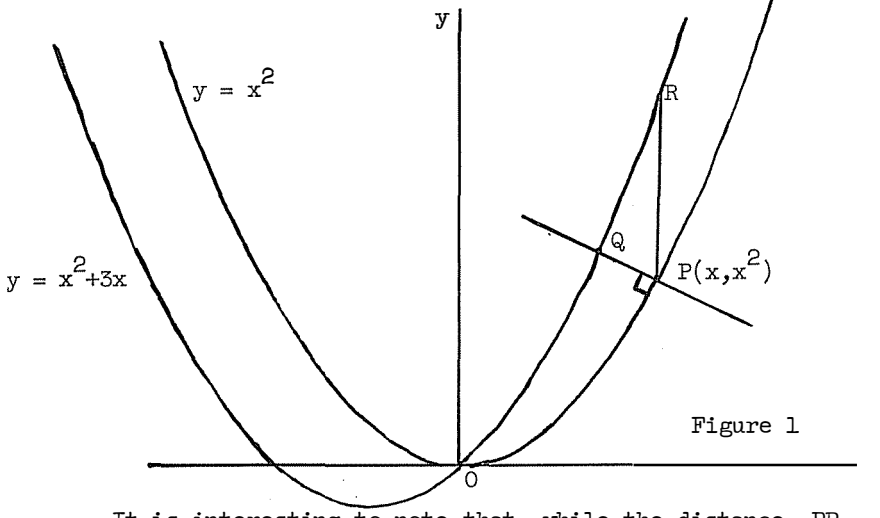


Figure 1

It is interesting to note that, while the distance PR = 3x becomes infinite as $x \to \infty$, the distance PQ measured along the normal at P to the curve $y = g(x)$ approaches 3/2.

While it is not true to say that f(x) - g(x) approaches zero as $x \to \infty$, it is true that the percentage difference between f(x) and g(x) approaches zero. For

$$\lim_{x\to\infty} \frac{f(x) - g(x)}{g(x)} = \lim_{x\to\infty} \left[\frac{f(x)}{g(x)} - 1\right] = 0.$$

Probably the most famous of all asymptotic relationships is the Stirling Formula for Large Factorials; it reads

$$x! \sim \sqrt{2\pi}\ x^{x+\frac{1}{2}}\ e^{-x}$$

or

$$\lim_{x\to\infty} \frac{\sqrt{2\pi}\ x^{x+\frac{1}{2}}\ e^{-x}}{x!} = 1.$$

The absolute difference between $x!$ and $\sqrt{2\pi}\ x^{x+\frac{1}{2}}\ e^{-x}$ increases slowly as $x \to \infty$, but the percentage difference approaches zero.

Since the statement

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

is equivalent to the statement

$$\lim_{x\to\infty} [\log f(x) - \log g(x)] = 0,$$

we see that we may use the sign $\doteq$ of approximate equality and say that if $f(x) \sim g(x)$, then $\log f(x) \doteq \log g(x)$. So if two functions are asymptotic, their difference need not approach zero, but the difference of their logarithms will approach zero.

3. Pseudo-Asymptotes. Suppose that two functions $f(x)$ and $g(x)$ are such that

$$\lim_{x\to\infty} [f(x) - g(x)] = 0.$$

In this case, as $x \to \infty$, the curves $y = f(x)$ and $y = g(x)$ approach one another arbitrarily closely in position. We may then say we are dealing with pseudo-asymptotic curves, for example,

$$f(x) = x^2 + x^{-1}, \quad g(x) = x^2$$

in Figure 2.

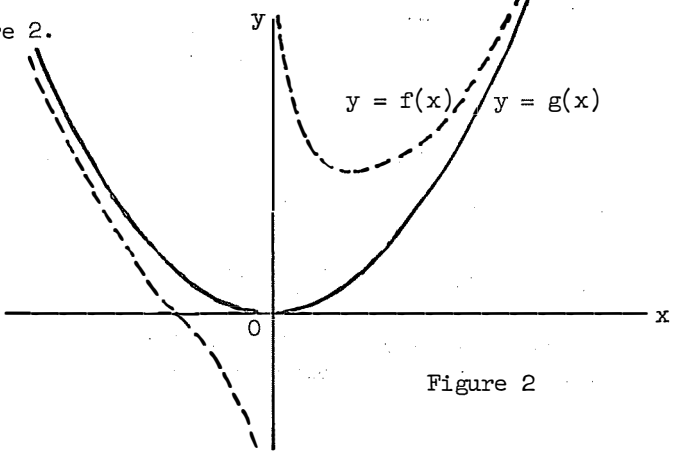


Figure 2

In case one of the pseudo-asymptotic curves is a straight line, we may say it is a pseudo-asymptote of the other curve. We shall delay illustrations until Section 3, but must bear in mind that, for a line to be a pseudo-asymptote to a curve, we only require that the distance between the line and the curve approach zero as $x \to \infty$; the curve approximates to the pseudo-asymptote in position.

4. True Asymptotes. Two curves are said to be geometrical asymptotes of one another if

$$f(x) - g(x) \to 0, \quad f'(x) - g'(x) \to 0$$

for $x \to \infty$. In short, the curves must approach one another both in position and in direction. Obviously, this is a stronger condition than the condition for pseudo-asymptoticity; if two curves are asymptotic, they are automatically pseudo-asymptotic.

We have talked about curves in general, but the most important case is when the one curve is a straight line. In this case, we specialize to the

DEFINITION: An asymptote to a curve is the limiting position (if any) assumed by a tangent at a point P on a curve as the point of contact P recedes arbitrarily far from the origin along the curve.

We might then say that an asymptote is a "tangent at infinity" or a "tangent with point of contact at infinity", provided we use these phrases to mean the longer locution "limiting position of a tangent as the point of contact recedes arbitrarily far from the origin".

This has been the standard meaning of an asymptote for many years. Let us now investigate a few interesting examples.

Example 1: $f(x) = \dfrac{\sin x}{x}$, $g(x) = 0$.

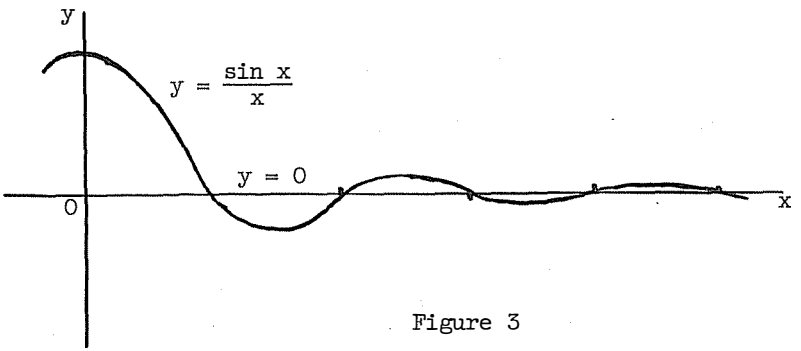


Figure 3

It is obvious that

$$\lim_{x \to \infty} [f(x) - g(x)] = 0,$$

since $f(x)$ becomes arbitrarily small as $x \to \infty$. So the 2 curves shown, one of which is the x-axis, approach one another in position as $x \to \infty$.

If we compute

$$f'(x) = \frac{x \cos x - \sin x}{x^2} = \frac{\cos x}{x} - \frac{\sin x}{x^2}, \quad g'(x) = 0,$$

we see that

$$\lim_{x \to \infty} [f'(x) - g'(x)] = 0$$

also.

In short, the two curves approach one another, both in position and in direction. So the x-axis is a true geometrical asymptote for the curve $y = f(x)$ in this example.

Example 2: $f(x) = \frac{\sin(x^2)}{x}$, $g(x) = 0$

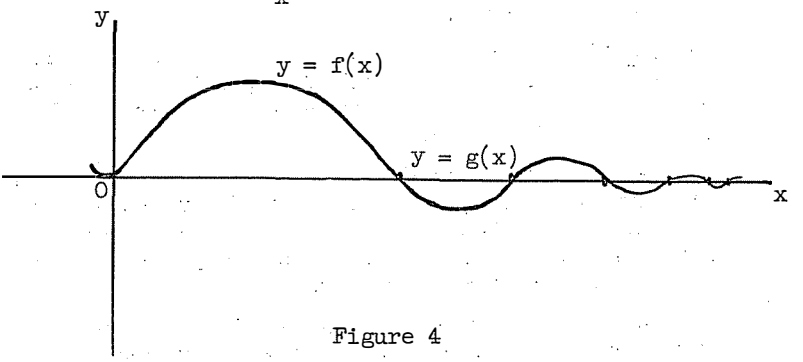


Figure 4

It is again clear that, as $x \to \infty$,

$$f(x) - g(x) \to 0.$$

So the curve approaches the x axis in position. But

$$f'(x) = \frac{2x^2 \cos(x^2) - \sin(x^2)}{x^2} = 2\cos(x^2) - \frac{\sin(x^2)}{x^2}.$$

It follows that, as $x \to \infty$, $f'(x)$ does not approach zero, but oscillates wildly back and forth between -2 and +2. In short, while the curve $y = f(x)$ approaches the x axis in position, it does not flatten out and it does not approach the x axis in direction. So the x axis is only a pseudo-asymptote of the curve. It is a line which the curve approaches arbitarily closely, but it is not the limiting position of a tangent line to the curve.

Example 3: As a corollary to the preceding examples, it is trivial that the curves

$$y = x^2 + \frac{\sin x}{x} \quad \text{and} \quad y = x^2$$

are geometrically asymptotic, whereas

$$y = x^2 + \frac{\sin(x^2)}{x} \quad \text{and} \quad y = x^2$$

are only pseudo-asymptotic.

5. Asymptotes of the Hyperbola. The equation of the asymptotes of the hyperbola are readily developed from the definition given in Section 4. We take the hyperbola in standard form as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(see Figure 5).

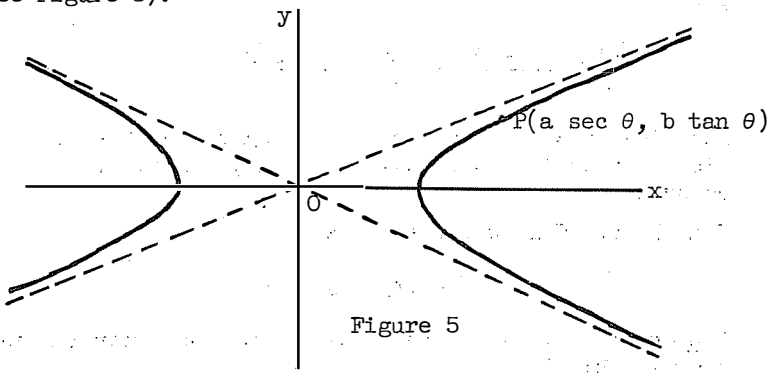


Figure 5

Let a general point P on the hyperbola be given in parametric form as $(a \sec \theta, b \tan \theta)$. Then the tangent at P is

$$\frac{x(a \sec \theta)}{a^2} - \frac{y(b \tan \theta)}{b^2} = 1,$$

and the equation simplifies to

$$bx - ay \sin \theta = ab \cos \theta,$$

or to

$$y = \frac{bx}{a \sin \theta} - \frac{b \cos \theta}{\sin \theta}.$$

This latter equation shows that, for a fixed abscissa x, the ordinate y of a point on the tangent line depends not only on the abscissa x, but on the parameter θ.

Now let P recede along the first-quadrantal branch of the hyperbola; this can be achieved by letting $\theta \longrightarrow \frac{1}{2}\pi$.

Since x is fixed (though arbitrary), we immediately obtain

$$y = \frac{bx}{a}$$

as the required asymptote.

Similarly, we can obtain the asymptote

$$y = -\frac{bx}{a}$$

by allowing P to recede to infinity along the fourth-quadrantal branch of the curve $(\theta \longrightarrow -\frac{1}{2}\pi)$.

6. Asymptotes to More General Curves. We content ourselves with two examples which illustrate general methods.

Example 4: Consider the curve

$$y = \frac{2x + 5}{x - 3}.$$

The line $x - 3 = 0$ is a vertical asymptote. Also,

$$\lim_{x\to\infty} \frac{2x + 5}{x - 3} = \lim_{x\to\infty} \frac{2 + 5/x}{1 - 3/x} = 2.$$

The line $y = 2$ is a horizontal asymptote (the slope conditions are easily checked). Actually, Figure 6, the curve is an hyperbola.

Example 5: Consider the curve

$$x^3 - 3xy + y^3 = 0.$$

Suppose that we solve this curve with the general line $y = mx + b$; then

$$x^3 - 3x(mx + b) + (mx + b)^3 = 0,$$

that is,

$$x^3(1+m^3) + x^2(-3m + 3m^2b) + x(-3b + 3mb^2) + b^3 = 0.$$

In general, there are 3 roots for this equation; if it is to be an asymptote, two of these roots must coincide at infinity. Thus the coefficients of x^3 and x^2 must approach zero as the general line approaches an asymptote. From

$$1 + m^3 = -3m + 3m^2b = 0,$$

we obtain $m = -1$, $b = -1$. Hence the line

$$y = -x$$

is the only asymptote.

This situation is illustrated in Figure 7; the curve, known as the Folium of Descartes, actually has a triple contact with its tangent at infinity.

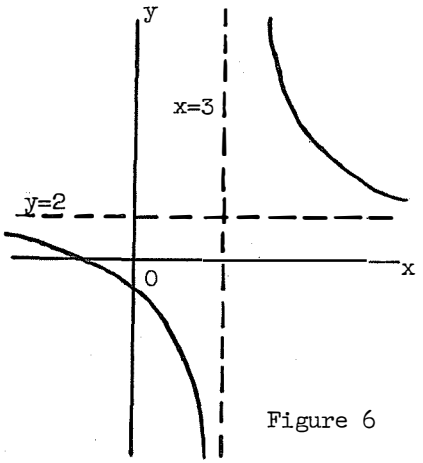


Figure 6

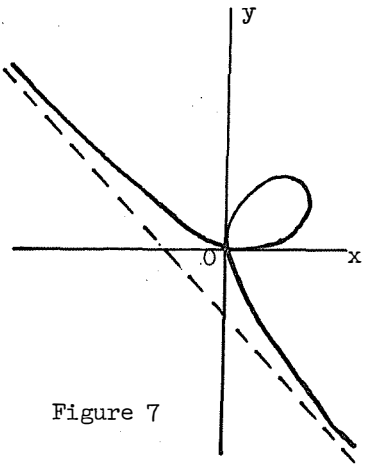


Figure 7

The method of this example is quite general. A tangent is characterized by the fact that, when its equation is solved with the equation of the curve, two identical roots are obtained; for an asymptote, these two roots must be identical and infinite (Try this method on the hyperbola).

VIEWS FROM ABROAD

V. Linis, University of Ottawa

Switzerland. Dr. L. Locher-Ernst is a prominent figure in Swiss mathematical education. He is Principal of the Technical High School in Winterthur, editor of an internationally known periodical devoted to elementary mathematics [Elemente der Mathematik, Basel] and author of several books and many articles. His publication On Thoughtlessness in the Treatment of Mathematics [Von der Gedankenlosigkeit in der Behandlung der Mathematik, Elem. Math. 16 (1961), 97-102, 121-123; 17 (1962), 1-3] is an interesting and provocative piece of writing although the content is largely "negative", i.e., it is a sharp critique of certain trends in American mathematical education. A more recent article [Neue Gestaltungen in der Behandlung der Mathematik, Elem. Math. 17 (1962), 25-27] has a positive approach and contains some worth-while suggestions concerning the mathematical curriculum in secondary schools. Loosely translated the main points are:

Arithmetic and Algebra.

1. Greater emphasis on recursive computation.
2. Development of the natural number concept.
3. Arithmetical operations as the central theme in

the construction of number systems.

4. Introduction to Boolean algebra and its applications in set theory, number theory, and formal logic. These topics should serve as a prelude to formalism.
5. Simple principles of electronic computers which can be explained on the basis of point 4.

Geometry.

1. Experimental acquaintance with geometric forms, including many curves.
2. Development of plane (later, space) projective geometry, stressing the duality principle, and using drawings extensively to verify and discover new geometric situations. Specialization to affine and metric geometries.
3. The process of proving is first replace by "reasonable evidence", and is strengthened later but without any pretense of strong scientific form.

In addition, Locher-Ernst points out the linear continum as an example for development of modern mathematical thinking. As applications he mentions statistics and mathematical geography (sic!).

For the Canadian reader, the novelty lies in the suggestions for geometry. On this continent, the only alternatives thus far have been "Euclid's" geometry versus euclidean (Hilbert's) geometry. Except for a rather brief glance on a similar proposal at Lakefield by myself, no serious discussion has taken place of the merits or demerits of the projective approach to geometry. There is already some groundwork done in this direction in Swiss schools, mainly under the influence of mathematician and philosopher Gonseth who is also co-author of several text-books (with Moser and Marti). It would be worthwhile to explore this new avenue in the teaching of geometry.

B O O K R E V I E W S

Algebra 1.

by Charles F. Brumfiel, Robert E. Eicholz, and Merrill E. Shanks, (Reading, Mass., Addison-Wesley Publishing Company Inc., 1961) Cloth, xi + 371 pp.

The material of this text was used during the past three years in normal grade nine classes in average high schools in connection with the Ball State Teachers' College. The material, in mimeographed form, was also used in other classes by teachers with no special training in the postulational approach. This approach and this material has been reported to stimulate both teacher and pupil and reportedly was well received.

The emphasis in this book is on structure. This begins with a study of the counting numbers including zero for which a few properties, known from arithmetic, are postulated. The number system is gradually extended through integers, fractions, rationals, and an introduction to reals with repeated contact with the same set of postulates. The introduction to reals is by way of a chapter concerned with $\sqrt{2}$ and manipulations using it and a chapter concerned with decimal representations up to and including the completeness theorem. Thus students are gradually led to appreciate the rules of algebra and also to grasp the nature of a deductive proof. Deductive proofs are rarely included in the text; definitions and theorems are stated in abstract form following exercise material leading to the conclusion.

The abstract method is used entirely, but students are led to the abstract conclusion by doing a large number of example problems both numerical and algebraic. The stress throughout is on the discovery method. Thus the book contains a large number of exercises each with numerous problems. These problems review earlier material and also lead gradually to the next idea to be introduced. This new idea is then named and discussed abstractly in a brief explanatory section.

The ratio of exercise material to explanatory abstract material is approximately 2:1 by line space. Thus this text differs from most others the reviewer has met dealing with the postulational approach. There would be apparently a large amount of time spent on student work and discussion and a minimum of presentation of new material by the teacher.

The text differs from others also in that, following the two introductory chapters dealing with number concepts for the counting numbers and the use of algebraic symbols, there is a complete chapter on logic. This latter chapter seems rather difficult for grade 9, but with careful analysis of this material later difficulties could undoubtedly usually be overcome much more easily than in traditional texts.

The book is written to be read by the student, and better-than-average students could work ahead on their own with little difficulty.

Throughout, emphasis is placed on exact use of words, precise language, appreciation of mathematical structure, and drill on basic skills.

Each chapter ends with a summary of material contained therein and a review exercise with a large number of problems reviewing that chapter as well as preceding chapters. Besides there are two appendices; the first, a review of postulates of each of the number systems; the second, a thirty-page set of review problems set apart chapter by chapter.

It is the author's view that the book contains more than enough material for the average student for a year's work. Thus it seems rather odd that the final two chapters dealing with quadratic equations and similar triangles and trigonometry are included when these topics are not on the course of study for grade nine. Besides this, other topics are noted as difficult and may be omitted without disturbing the continuity.

This book is not just another textbook dealing with the postulational approach to mathematics. Instead it brings mathematics to life in a way so that students can gain a maximum of understanding and interest with a minimum of teacher time spent on actually presenting new material. The reviewer thinks that the text is the most stimulating, most interesting, and most informative one that he has met so far that deals with this material.

J.F. Chambers, Jarvis Collegiate Inst., Toronto.

Teacher's Manual and Answers for Algebra 1.

by Charles E. Brumfiel, Robert E. Eicholz, Merrill E. Shanks. (Reading, Mass., Addison-Wesley Publishing Company Inc., 1961) Paper, v + 136 pp.

This teacher's guide to Algebra 1 gives greater detail on the selection of topics and the emphasis to be given to the topics in Algebra 1 (see above). It outlines section by section the ideas to be stressed, the difficulties to be overcome, and the methods by which these difficulties may be overcome.

With each chapter a number of sample test questions are given and their answers are included in the following section. These give a comprehensive review of the material of the chapter concerned.

Included in pages 69 through 136 are the answers to the problems from the text book itself. These are not included in the text. In many cases complete solutions are given, but of necessity only one of several possible solutions is indicated. This guide would be invaluable to a teacher who has had no previous knowledge of the abstract approach and would indeed be very helpful to any teacher using the text.

J.F. Chambers, Jarvis Collegiate Inst., Toronto.

And then there was the Grade 13 student, brought up in a modernistic school, who hadn't studied the development of the formula for the sum of the squares of the first n natural numbers. When he found it on the departmental exam, his solution read "Look, look, look. Damn, damn, damn".

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