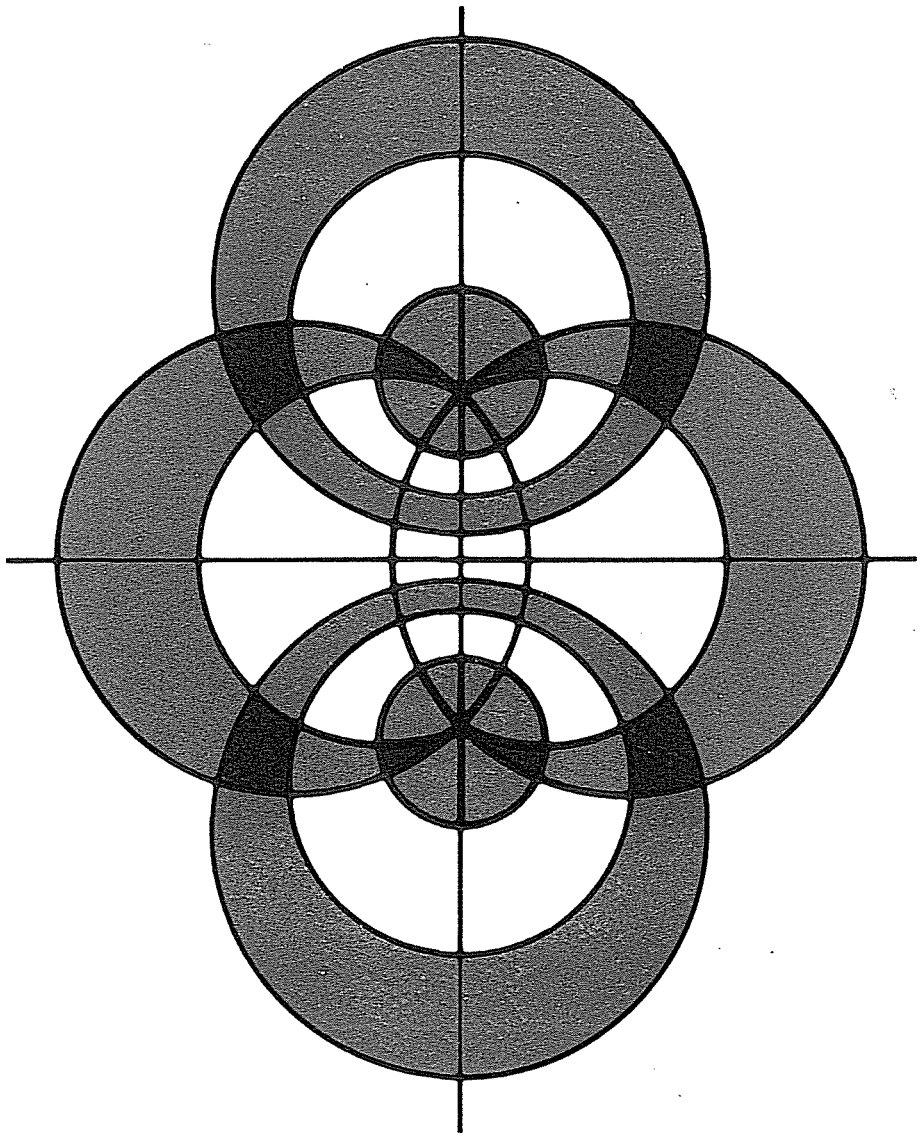
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EDITORIAL

It is well known among educators that the classroom teacher plays a pivotal role in the education of his students. Science and mathematics teachers are instrumental in motivating students in relation to their advancement levels in the courses they teach and in their decisions about whether to take more advanced courses in science and mathematics. Given this central importance of mathematics teachers, there are factors of concern in both Britain and the United States, which are particularly topical to Ontario mathematics educators in the light of the SERP recommendations.

In the report by the National Science Foundation and the Department of Education **Science and Engineering Education for the 1980's and Beyond** it was noted that in the United States there is a significant erosion of unfilled positions, particularly at the Secondary School level. Because of demographic projections, the perceptions that there are few openings for new teachers, the number of people who seek an educational career has markedly decreased. Many mathematics teaching positions in Secondary Schools are vacant and projections indicate further substantial erosion. The report notes, that when positions cannot be filled through new hiring of qualified instructors they are often filled by teachers with lower specialist qualifications, or by a transfer of tenured teachers from other subject areas. Another concern is the erosion of support systems for teachers. Supervision at the Secondary School level has been reduced because of financial retrenchment or has been shifted to administrative tasks from instructional support. Very few persons are available outside the classroom to provide quality control and assist teachers with pedagogical problems.

Although the report reflects concern of the educational system in the United States, there is no doubt that some of their findings apply also to mathematics - science education in Ontario. All projections show not only for Ontario, but all of North America, a deficiency of computer science and computer engineering graduates for many years to come. Ontario can also not afford that a substantial percentage of its mathematics teaching positions are filled by non-specialists.

Many educational reports indicate that the international trade position of highly developed countries is linked in no small part to the considerable scientific and mathematical literacy of its population. For example the Finniston Report, an inquiry into the engineering profession in Great Britain, laid the decline in the ability of Britain to compete in international trade on the lack of vigor in the engineering profession and recommended among other things that " ... all pupils should be strongly encouraged to maintain the study of mathematics and physics at least until after '0' level" (the competence examination given at the end of secondary schooling). It noted further, that the level of mathematics-science education of the average Japanese worker is markedly higher than his U.K. counterpart; that in Germany a student in modern languages in Secondary School learns enough science and mathematics to be able to attend engineering school and can compete effectively with science majors; that the Soviet Union's science and mathematics curricula surpasses that of any other country in scope and content.

There is no doubt that Ontario has a fine primary and secondary educational system. Its standards are high and its instructors are dedicated professionals. As we are revising and reviewing mathematics education in Ontario, let us begin early to instill in our students, a positive attitude in their liking for mathematics and attain a high level of science and mathematics literacy of all our graduates. Perhaps we can learn from others as we develop a mathematics-science curriculum. There is much talent and ability in our young people. We must prepare them for the future.

MATHEMATICS EDUCATION

FACTORS AFFECTING THE PARTICIPATION OF YOUNG WOMEN IN MATHEMATICS: AN OVERVIEW OF RESEARCH FINDINGS

by

Marijane Werner, O.P., Ph.D.,

Mathematical understanding and skill are essential to every individual's meaningful and productive life in today's world. At present, females are underrepresented in mathematics courses and in careers for which mathematics is a prerequisite. In delineating directions that mathematics programs should be taking in the 1980's, recommendation 6 of **An Agenda for Action** (NCTM, 1980, p. 17) states that "more mathematics study must be required of all students and a flexible curriculum with a greater range of options should be designed to accommodate the diverse needs of

the student population." Perhaps the key word in this recommendation is the word "require". It would be naive to suggest that continuing to provide mathematics courses as electives would effect equality of opportunity for all.

The National Institute of Education (1977) has identified the goal of improving mathematics literacy among females as an important educational objective. It has developed a program in the area of women and mathematics as a strategy to increase the educational and occupational choices of women. Improved mathematical education is crucial to the movement of women into new and economically more attractive employment opportunities. Yet current research (Armstrong, 1980; Donady and Tobias, 1977; Fennema and Sherman, 1976, 1977; NIE, 1977; Sells, 1976) shows that fewer females than males choose to study mathematics beyond the minimal school requirements. Regrettably among those young people who are qualified intellectually, many more girls than boys are making this decision. Armstrong (1980, p.1) states that "from the eighth grade on, female participation in mathematics declines steadily until, by high school graduation, male students outnumber women in mathematics classes." Thus, pre-college women continue to preclude themselves from entering a number of male-dominated and high-paying occupations which require some mathematical competence.

Attempts to explain the lack of participation of women in mathematics range, along a continuum, from Rousseau's theory that women's nervous systems were too delicate, their heads too small and their reasoning capacities too limited to the more sophisticated genetic studies of our day that characterize brain dominance as favoring the male in the areas of spatial visualization, abstract logic and inductive/deductive thought. (Sherman and Fennema, 1977) Supposedly women shy away from these "unfeminine" modes of thought which are viewed as required disciplines for success in mathematics. Other studies have focussed on the interaction of many factors which influence the socialization process of females in our society from infancy through adolescence.

Research (Aiken, 1976; Armstrong, 1980; Banks, 1970; Faumrind, 1972; Crosswhite, 1975; Fennema and Sherman, 1977) indicates that attitudes play an important role in electing school requirements and in acquiring mathematical knowlege. Therefore, we can conclude that it is important to study attitudes toward mathematics in order to improve the learning of mathematics.

One important factor that appears to be affecting teenage girls' attitudes toward mathematics is the societal stereotyping of mathematics as a male domain. During adolescence a most important developmental task is that of establishing an appropriate sexual identity. If females perceive mathematics as a male activity, they will not be encouraged to put forth much effort to learn mathematics (Fennema and Sherman, 1977; Stein and Smithells, 1969; Werner, 1980). In the socialization process within our culture, both sexes are oriented toward two different kinds of achievement, boys in achieving on tasks that deal with inanimate objects or impersonal ideas and girls on tasks that involve interaction with people. Another related hypothesis is that when the two sexes are working on a task, boys tend to be intrinsically interested in the task itself, whereas girls work primarily for the praise and approval of others (or to avoid their disapproval). Given these existing socialization patterns which stereotype the male as the aggressor, girls tend to be carefully trained in non-aggression throughout childhood; otherwise, they will be doomed to failure and disappointment as adults in their encounters with men. Baumrind (1972) concludes that such circumstantial evidence leads one to look elsewhere than in genetic endowment for the absence of women in the competitive professions in America today. He strongly suggests that societies have the option of minimizing, rather than maximizing, sex differences through their socialization practice.

Lynn Fox (1977) authored a lengthy review of literature on the effects of sex role socialization on mathematics participation and achievement. She summarized the conclusions which she reached as a result of that literature search as follows:

1. Girls receive less encouragement than boys from parents, teachers, guidance counselors and peers to pursue advanced mathematics courses in high school.

2. Girls are less likely than boys to perceive the usefulness or value of high school mathematics courses to their future goals, perhaps partly because teenage girls are not encouraged to aspire to and plan for professional mathematics/mathematics-related careers.

3. Girls have less self-confidence as learners of mathematics than boys.

4. Mathematics classes are commonly thought of as male domains and girls in adolescence may avoid them for this reason.

The message that girls can't, or shouldn't, or don't need to excel in mathematical endeavors is still well-entrenched, not only in American folklore, but also in the reality of the home and the classroom (Burton, 1978).

Several reviews of studies which investigated the relationship of pupils' attitudes toward mathematics and pupil achievement in mathematics produced contradictory conclusions (Aiken, 1976; Callahan and Glennon, 1975; Suydam and Weaver, 1975). "There is no consistent body of research evidence to support the popular belief that there is a significant positive relationship between pupil attitudes towards mathematics and pupil achievement in mathematics. ... We have little research basis for believing that these two things are related" (Suydam and Weaver, 1975, pp. 1-3). Callahan and Glennon (1975) agree with Suydam and Weaver when they admit the constraints they experienced "to present compelling research evidence ... that positive attitudes play an important role in contributing to mathematics achievement" (p. 80). However, Aiken (1976) states: "When attitude scores are used as predictors of achievement in mathematics, a low but significant positive correlation is usually found at the elementary, secondary, college undergraduate and post-graduate levels" (p. 295). Fennema (1977) believes that serious flaws in the afore-mentioned studies result from the impreciseness of the definitions of "attitude" and "mathematics" and from the wide range of ages of the subjects used in these studies. Additional studies, using national data, need to be conducted regarding the interrelationship of attitudes toward and achievement in mathematics in order to resolve this lack of agreement.

The interrelationship of anxiety and mathematics learning has been explored by a variety of methodologies and with instruments purported to measure debilitative or facilitative anxiety in general and/or specific to mathematics. Crosswhite (1972) reported that debilitating anxieties increased for females across grade levels four through ten. In a study by Callahan and Glennon (1975), high anxiety was found to be associated with lower achievement in mathematics and they concluded that anxiety and achievement in mathematics are inversely related.

In the Fennema-Sherman study (1976), the investigators attempted to measure both confidence and anxiety. While it might be possible to talk about the two traits independently, Fennema and Sherman believed that the two traits, in relation to mathematics, appeared to be very similar. They found that a high rating on the confidence scale correlated ($r = 0.89$) with a low rating on the anxiety scale. In the Fennema-Sherman study, at each grade level from six to twelve, boys were significantly more confident in their abilities to deal with mathematics than were girls. Fennema claims that more information is badly needed concerning females' feelings of confidence, what contributed to its development and possibilities of intervening to change low self-esteem. Mathematics learning is cumulative and if a low level of confidence is operative in the sixth grade, the learning that takes place during junior and senior high school years, as well as one's willingness to elect to study mathematics, will have been affected. If females could be prevented from developing feelings of anxiety or lack of confidence, more positive attitudes toward studying mathematics might result. Presently, however, the impact of this confidence-anxiety dimension as it relates to mathematics learning on factors that influence course selection beyond minimal requirement by teenage females is uncertain.

In a report of a two-year study (Armstrong, 1980) funded by the National Institute of Education for the purpose of identifying the most important factors related to the problem of women's participation in mathematics and determining the relative importance of those factors to guide future research and intervention efforts, three groups of variables were found to have the greatest effect on participation: (1) positive attitudes toward mathematics, (2) perceived need for mathematics for future career and educational plans and (3) influence of significant others, including parents, teachers and counselors." The results of this study imply that intervention strategies aimed at instilling and sustaining the positive influences of these factors are good prospects for maximizing participation in mathematics (Armstrong, 1980, p. ix). The strategies which are suggested at the end of the report could have significant influence in assuring that mathematics attracts both females and males. The results of this study are especially significant because the findings are based on a survey of a national sample of 13 year-olds and high school seniors. Such a large-scale survey can detect the small but stable factors overlooked by previous studies using limited samples which frequently reflected inconclusive and sometimes contradictory, findings due to lack of statistical power. In summarizing the conclusions of the study, Armstrong (1980) states:

Participation in mathematics is mostly an issue of awareness. If parents, teachers and counselors understand and transmit to students the necessity of taking mathematics to keep their options open and if they have the same high expectations for women as they have for men, then a basis for equal opportunity in scientific and technical fields will exist -- for men and women (p. 31).

To this end, funds have been awarded by the National Science Foundation, the National Institute of Education and the Women's Educational Equity Act Program to ensure that young women obtain the high school mathematics required to prepare for and to enter mathematics-related occupations. One such effective intervention program, Multiplying Options and Subtracting Bias (Fennema, Becker, Wolleat and Pedro, 1980), was developed and evaluated with funding from the U.S. Office of Education's Educational Equity Program. The M.O.S.B. program was designed and developed to change the beliefs of female high school students, their male peers, mathematics teachers and counselors about women and mathematics. Because of the difficulty of obtaining data, this study omits parents as a target group. Because most of the desired change was in the affective domain, a motion medium was selected. Four 30-minute videotapes were developed. Each tape focused on a specific target group. Included in each videotape were vignettes designed to change attitudes, to provide information about sex-related differences in mathematics and about career and educational relevance of mathematics and to suggest activities for involving each target group in a 2-hour workshop. The nine participating high schools located in urban, suburban and rural communities in Minnesota and Wisconsin represented a broad spectrum of socioeconomic conditions, were of differing sizes, were willing to take part in the lengthy data collections and had a larger percentage of males than females enrolled in advanced mathematics classes.

In schools where the M.O.S.B. intervention program was implemented, the amount of mathematics that female high school students planned to study was significantly increased. Although male peers changed their plans for the study of mathematics after viewing the videotape, the increase in the amount of mathematics they planned to take in high school was not as great as that of females' plans. Changes in the experimental male peers' knowledge about sex-related differences in mathematics were significant. An increase in their stereotyping of mathematics as a male domain may be explained by the increased awareness of this problem that was brought out by the videotape. The experimental math teachers and counselors, as a group, changed little after viewing the respective videotapes.

Fennema, Becker, Wolleat and Pedro (1981) conclude that exposure to the M.O.S.B. series can substantially influence students' attitudes about mathematics and their willingness to take more mathematics courses in high school. The results of this study suggest than an educational intervention designed to remedy inequities in the study of high school mathematics can be effective. Areas for intervention could include programs which help young women realize the appropriateness of pursuing a career in a mathematical or technical field. The Math/Science Network in the San Francisco Bay area has pioneered intervention programs to widen options and improve the career awareness of students. Articulate women in non-traditional fields attend conferences for high school girls and discuss aspects of their careers and the training required. Classroom activities encourage students to consider as many career options as possible, to learn about careers related to their interests and to determine and take the necessary mathematics and science courses that these careers will require (Armstrong, 1980).

Recent studies (Armstrong, 1980; Fennema, Becker, Wolleat and Pedro, 1981; Fennema and Sherman, 1977; Werner, 1980) emphasize the role of parents, teachers and counselors as significant change in agents in terms of career awareness and educational aspirations. Students' educational aspirations are frequently a reflection of their parents' aspirations for them. Parental encouragement and counselor encouragement seem to affect the amount of mathematics that is taken by high school students.

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The whole art of teaching is only the art of awakening the natural curiosity of young minds for the purpose of satisfying it afterwards.

Anatole France

To teach is to learn twice.

Joseph Joubert

A teacher affects eternity; he can never tell where his influence stops.

Henry Adams

The decent docent doesn't doze:
He teaches standing on his toes.
His student doesn't doze -- and does
And that's what teaching is and was.

David McCord; What Cheer

If we wish to foresee the future of mathematics our proper course is to study the history and present condition of the science.

Henri Poincare

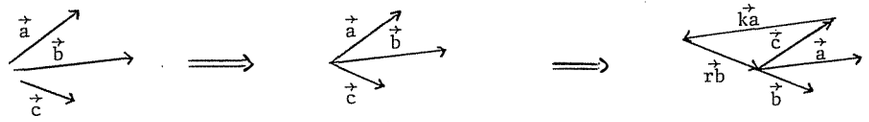
A STRATEGY FOR TEACHING THE TOPIC OF LINEAR INDEPENDENCE IN SECONDARY SCHOOLS

by
C. Pinnegar

One topic that consistently causes difficulty in grade 13 Algebra is that of linear independence. This is not to say that students cannot grind out an answer if the question concerns some aspect of manipulating numerical coefficients. Generally, they can do this well. Unfortunately, until five years ago I don't think my students understood the geometrical or real world significance of the statement $\vec{ma} + \vec{nb} + \vec{kc} = \vec{0}$ for different values of the scalars and vectors. At that time I placed the following innocuous question on a test. "Comment fully on the following partial statement, 'three linearly independent vectors in R^2 ...'" The answers I received convinced me that the students had memorized a methodology but had no idea of what it meant.

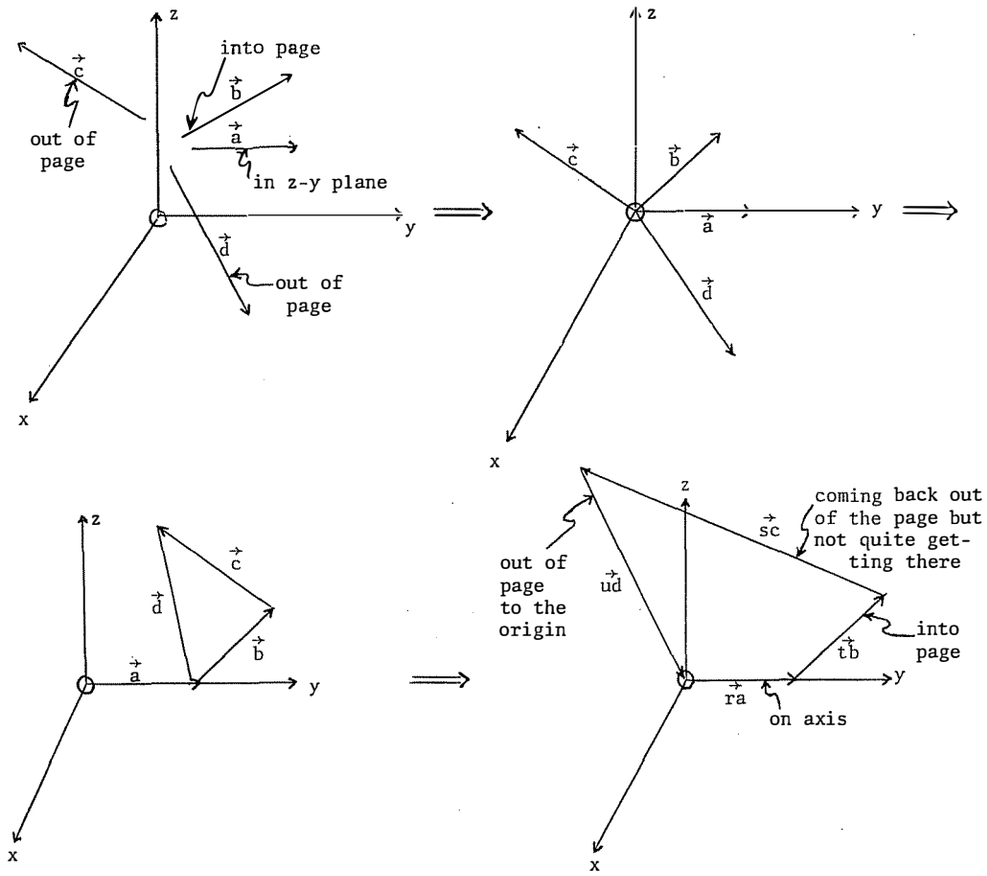
To verify my opinion I assigned them the following question. Are four vectors in R^3 linearly dependent or linearly independent? Very few students could think their way through the problem. At this juncture, I decided to change my methodology. First I would attack the concept geometrically and only then would I introduce the definitional aspect.

Begin this series of lessons by drawing two non-parallel vectors $\vec{a}$ and $\vec{b}$ on the board. Review the concept of translating vectors and do so with $\vec{a}$ and $\vec{b}$ so that they have a common point. Using the parallelogram law (or triangle law) draw in a vector $\vec{a} + \vec{b}$ and call it $\vec{c}$, thus $\vec{c} = \vec{a} + \vec{b}$. Re-examine what you did. You started with two vectors and produced a third. Tell the class that this is an example of a "linear combination". Now you have two further questions to answer. Can <u>any</u> three vectors in the plane of the blackboard (R^2) be written as a linear combination and if so are the numerical coefficients of the right side always one? After all, the three vectors we used were certainly generated by only two. This is a good place to let your class brainstorm and with a little prodding they can attach the second part successfully. Take any three vectors $\vec{m}$, $\vec{n}$ and $\vec{p}$. Does $\vec{m} = \vec{n} + \vec{p}$? If it does what are the implications? Well, since $\vec{m}$ is any vector, one conclusion is that all vectors equal $\vec{n} + \vec{p}$ and thus there is only one vector in R^2, i.e., $\vec{n} + \vec{p}$. Thus it appears that $\vec{m} = \vec{n} + \vec{p}$ is not a general statement and we should look for a geometric interpretation which will allow us to answer both the above questions. I suggest the following well-known approach.



From this diagram it is evident that $-\vec{c} = \vec{ka} + \vec{rb}$, m, r $\in$ R; m, r $\neq$ 0 and that k and a are not necessarily one. In fact one is negative and the other positive.

Does the approach follow in R^3. Draw four vectors in R^3 and see if you can make this type of linear combination. I would suggest you use an overhead projector with prepared transparencies for this. Different colours will help.



Notice that I have been careful here to have a vector $\vec{ud}$ although I could easily produce $\vec{d}$. Thus $-\vec{ud} = \vec{ra} + \vec{tb} + \vec{sc}$ which is analogous to one previous combination in R^2, i.e., $-\vec{c} = \vec{ka} + \vec{rb}$. The students can easily generalize that some $\vec{fc} = \vec{ga} + \vec{hb}$ in R^2 from one example in R^3 and thus the vector on the left does not have to have numerical coefficient one either. You might have your students graph two or three similar sets of vectors in R^2 and R^3 so that they can recognize the manipulations involved. Make sure they apply them on a Cartesian grid.

Have your students examine their sketches closely. Ask them to consider their linear combinations as a trip. Where did they go? Immediately they will all see that they ended up where they started. What, then, is the magnitude of the trip vector? Clearly, zero. It is evident then that these linear combinations can be described as "making the zero vector." Thus $-\vec{ud} = \vec{ra} + \vec{tb} + \vec{sc}$ can be written $\vec{0} = \vec{ra} + \vec{tb} + \vec{sc} + \vec{ud}$. Recall that the final $\vec{0}$ depended on the fact that all or some of the scalars r,t,s and n be non-zero. Name such a combination of vectors "linearly dependent".

At this stage you pray that some bright student will remark that there is another way to produce a $\vec{0}$. If not, ask the class if there is another way, and discuss the possibility of all the scalars being zero. This combination of vectors gave the $\vec{0}$ independent of the vectors themselves, so we will name it "linearly independent".

With these combinations in mind you can now go <u>back</u> to the cases of two parallel vectors and two non-parallel vectors in R^2 and discuss the geometric interpretations of

linear independence and linear dependence respectively. This is the reverse of the usual dynamics of these lessons.

From this point, the definitional aspect of the lessons can be introduced and discussed. To see if your students have learned the basics ask them the two questions posed earlier or ask them to describe briefly the diagram of #9 p. 188 Algebra - Del Grande et al. (2nd ed.). This question was #2 in the exercises of the first edition and always posed difficulty. Now the student should be aware that; if you start with three linearly independent vectors you must be in R^3; if you start in R^2 and draw three vectors they must be linearly dependent; if you draw four vectors in R^3 they must be linearly dependent; if you are in R^3 and you want three vectors to be linearly dependent, then you must draw them in the same plane in R^3.

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ERROR ANALYSIS IN MATHEMATICS EDUCATION OR I SPY WITH MY LITTLE EYE SOMETHING THAT MAY BE WRONG; AND WHAT AM I GOING TO DO ABOUT IT?

by
Max Rittner

Piaget was struck by the observation that the character of errors made by children held as much as the nature of their correct answers.

- L.S. Shulman, 1970

Discovering the method that a student may use to solve a problem is probably one of the more challenging aspects of teaching. No matter whether the strategy is completely or partially correct, the underlying principle is that some cognitive reasoning has occured and it is our duty as educators 'o facilitate these processes.

If teachers became more knowledgeable of incorrect solution strategies, then the possible planning of instruction to remediate these difficulties would be greatly enhanced. Ginsberg (1976), Light and Reynolds (1972) clearly point out the importance of being aware of this kind of information in diagnosing student learning difficulties.

Diagnosing a students' strengths and weaknesses will only be of benefit to the student if the teacher has the appropriate tools to provide remediation. Remediation (Sovchik, 1978) may be complicated by variables in the school and home environment. A remedial teacher may need to have several options and may need to consider many factors to arrive at the best form of "preventive medicine".

In essence, the learning of mathematical concepts is of the same nature as the acquisition of foreign language. Solving problems (Radatz, 1979) requires the student to use their analytical and developmental skills, i.e. obtaining, processing, retaining and reproducing the information given in the English language; thereby translating and adapting it into the usage of the mathematical language. A misunderstanding of the semantics of mathematics would seriously affect the remediation process.

This paper sets out to present several issues surrounding the topic of error analysis and remediation techniques. Let us begin with a brief historical approach of the international philosophies that directed their attention to our discussion. From there we will proceed to address the major issues involved:

a) do's and don't's of diagnosis
b) difficulties that were consistent throughout all operations
c) error analysis of each individual operation with whole numbers and some.remediation strategies:

i) addition
ii) subtraction
iii) multiplication
iv) division

In 1925, Buswell and Judd cited more than thirty studies that were directly concerned with the diagnosis of arithmetic errors. At the same time, Weimer (1925) and Seemann (1929) in Germany were conducting studies of their own. In approximately 1931, Cooke selected students from the Peabody Demonstration School, for a study to diagnose the difficulties in arithmetic. Each student was assessed by the Buswell-John Diagnostic Chart. This chart can still be considered a comprehensive outline for a diagnostician. Later Grossnickle (1935) distinguished between constant errors and errors due to chance.

The analysis of students' errors, due to different philosophies in Europe and the United States, had different starting points and interests. "The methods and hypotheses of American research on errors have long been oriented toward behaviourism, whereas in Europe aspects of Gestalt theory and ideas of pedagogical reformers have been influential; in the Soviet Union the conditions of the changed educational system and the curriculum were important factors. These differences may be one reason why research findings on errors in mathematics learning were seldom exchanged across national boundaries". (Radatz, 1979, p. 163)

Between 1973 and 1974, Cox expanded the two types of errors that Grossnickle outlined, to three basic types of computational errors. In each type, systematic, random or careless; Cox had set a certain criterion sample of five questions. The greater the sample, the greater the likelihood to distinguish between error types. Grossnickle and Reckzeh (1973) confirmed this by stating that a reliable diagnosis required a minimum of three samples.

Although most of the research has been concerned with the diagnosis of student errors, some (Callahan, 1977; Gregory, 1979) have also suggested that our diagnostic tools have some serious design and content defects that may precipitate further student errors.

Scheer (1980) provides teachers with twelve guidelines that they may use to maintain a high level of quality in their diagnostic techniques.

(1) Do be accepting of the child's failures as well as of the child's right answers. Body language and non-verbal communications play an important part here. Human nature has it that we smile when a correct answer is given; but we frown or sigh or uncross our legs or raise an eyebrow or simply do nothing when an incorrect answer is given. Children frequently pick up these cues and may be negatively affected by behaviours that suggest disapproval. If we want to know what children are thinking - what inadequate rules or strategies they may be using - we need to help them feel good about themselves and be willing to share their thinking with us.

(2) Don't cue the child when he or she gives an incorrect answer, even when you know that the child knows the right answer. Remember, you are collecting data about the child's way of doing things; you are not teaching in the usual sense.

(3) Do be thorough; gain as much information as possible. Remediation based on a hasty, inadequate diagnosis will not be effective. You may waste time in teaching the child something he or she already knows or something for which he or she does not have the necessary prerequisite skills.

(4) Don't interupt a student when he or she makes an error. The temptation to correct students' errors is frequently difficult to resist. We must remember, however, that our immediate aim is to find out what the errors are and why the child makes them. Teaching comes later.

(5) Do keep a record of your sessions (with a cassette tape or video tape recorder, if available). We cannot rely always on our memories and a complete record of the diagnostic sessions is valuable reference as you analyze problem areas and plan remediation activities.

(6) Don't hurry the student. The purpose of diagnostic testing is to discover the strengths and weaknesses that a child has in mathematics, not to see how rapidly he or she can do something.

(7) Avoid answering direct questions, such as, "Was that right?" or, "How am I doing?" A standard non-ego-damaging response, which usually satisfies the child is, "You're doing fine". Negative remarks at an interim point could discourage or inhibit the child in his or her responses to the remaining items in the diagnosis.

(8) Don't lead the child through a series of questions that will ultimately lead to the response for which you are looking. "Why did you say that" or "Why did you do that?" asked in an easy conversational tone of voice will elicit the thought patterns you're after. Questioning techniques are of utmost importance during diagnosis.

(9) Ask the child to "think out loud". We may observe that a child consistently makes an error in written work, but we need to know why he or she is doing so. Asking the child to think out loud removes a lot of the guesswork from our job.

(10) Repeat the child's answer when the child is responding orally in the diagnostic process. Repeating the answer that a child has given serves a twofold purpose. First, many children tend to speak softly, mumble or cover their mouths while responing and we don't always hear correctly what the child has said. Repetition of the child's answer will give him or her opportunity to tell us whether or not we heard correctly. Second, it gives us something factual to say, something that is neither negative nor positive and therefore, will not influence the outcome of the diagnosis. (One thing to beware of here is voice intonation - the tone of voice that says, "Are you sure", and non-verbal cues, like the raised eyebrow.

(11) Encourage children to cross out and start over, rathern than to erase, when they are working through written exercises. The written record will facilitate your diagnosis by enabling you to analyze the child's thought process.

(12) Make a note of behaviour patterns. Observation plays an important role in diagnosis. Squinting, nail biting, fidgeting, sitting on hands (may indicate counting on fingers), daydreaming and gazing out the window may all imply things about the child that will provide useful information for your diagnosis.

(Sheer, 1980, p.p. 18-9)

Although the title of the essay may seem somewhat comical in nature; it has relevance to the teachers' most valuable skill: the power of observation. Not only should we be observing behaviours of anxiety, inattentiveness or boredom as Scheer suggests; we should also be aware of the unique strategies students use to cope with problem-solving tasks. Children may:

(1) use finger or dots,
(2) look for lines which can be on the floor, a ruler, paper or clock,
(3) use teeth, eyes, nose and toes.

Pincus (1975) outlines five major areas in which students generally have difficulty:

(1) Poor understanding of the symantics of numbers and place value.
(2) Inadequate mastery of basic facts.
(3) Poor alignment of digits in column arrangement.
(4) Poor penmanship, so that the child misreads his own work.
(5) Failure to estimate or check.

In addition, some students have a poor understanding of equal signs in equations.

Let us now consider the systematic errors that are specific to certain operations. Following each type of error, some suggestions for remediation will be included.

Errors Specific to Addition (Pincus, 1975)

1. **Difficulties with exchange.** This may include both digits as units (a), no carrying of units (b), or the wrong figure is exchange (c).

eg.

a)	b)	c)
74	74	75
+87	+87	+87
1511	261	171

In all operations, Pincus suggests that the teacher ask the student to verbalize the procedure that they chose to solve the problem. This may provide the teacher with a preliminary tool, before instituting remediation with techniques such as Dienes' block concrete materials, place value charts, etc.

The teacher should review in written and oral form the concept of place value.

2. **Inconsistency in adding.** Subtraction may be implemented by the student when a smaller digit is located in the top position.

e.g.

a)	b)
53	237
+ 71	+ 152
24	329

The student should review the commutative property with the teacher.

3. **Inability to add to higher decade numbers.**

eg. 3 + 4, 13 + 4, 23 + 4, . . .

The use of number lines and the emphasis of patterns may suggest a solution.

4. **Difficulty with zero.**

eg.

a)	b)
38	407
+ 40	+ 321
70	708

The student possibly has confused multiplication with addition. However, if this was not the case, review the concept of zero with the student in all digit locations using abstract, symbolic or concrete form, whichever is most appropriate.

Errors Specific to Subtraction
(Pincus, 1975; Engelhardt, 1977)

Students may have more difficulty in general with subtraction since little emphasis is placed on the concept of the inversive operation of addition. If students are expected to orally express sequence of numbers in an increasing fashion, then one should expect that students be aware of sequencing numbers in a decreasing order. Possibly this would add to the familiarity of subtraction to students.

1. **Disregard of symbol.**

56
- 31
87

Oral examination may quickly differentiate careless error from conceptual error. Estimation is essential in teaching all operations. Estimation provides the student with a reasonable solution.

2. **Inappropriate inversion.**

43
- 19
36

The student has difficulty subtracting a greater number from a lesser. The student possibly does not understand the concept of carrying. Remediation may include expanded notation, estimation and concrete materials (if required).

3. **Neglect of the one extra digit in the minuend.**

eg.

a)	b)
374	374
- 57	- 87
17	87

Request the student to estimate, identify and determine the meaning of each digit; and possibly the concept of zero, could be reviewed as well.

4. **Decreasing a number as though carrying was required.**

eg. a)
$$\begin{array}{r} {\scriptstyle 7}\\ 8\not{9}\\ -\,43\\ \hline 36 \end{array}$$

b)
$$\begin{array}{r} {\scriptstyle 6}\\ 7\not{3}\\ -\,41\\ \hline 22 \end{array}$$

Remediation may proceed as follows: an oral explanation of procedure by student; estimation; expanded notation and/or use of concrete materials.

5. **Increasing a number without carrying.**

eg. a)
$$\begin{array}{r} 42\\ -\,16\\ \hline 36 \end{array}$$

b)
$$\begin{array}{r} 637\\ -\,118\\ \hline 529 \end{array}$$

This error is similar to error number 4. Since the student has not shown any rough work, this may indicate a careless error. Remediation would consist of an oral explanation, estimation and possibly expanded notation.

6. **Difficulty with multiple exchanges.** (especially when zero is involved)

eg. a)
$$\begin{array}{r} {\scriptstyle 3}\\ 4\not{0}7\\ -\,139\\ \hline 278 \end{array}$$

b)
$$\begin{array}{r} {\scriptstyle 5}\\ 6\not{0}0\\ -\,241\\ \hline 369 \end{array}$$

The student is mechanically aware that borrowing is required; but from where? Expanded notation may be required and the concept of zero as a digit may need some explanation.

Errors Specific to Multiplication
(Pincus, 1975; Engelhardt, 1977; Kilian, 1980)

1. **Inappropriate inversion.**

eg.
$$\begin{array}{r} 37\\ \times\ 4\\ \hline 202 \end{array}$$

As in subtraction, the student has carried the inappropriate digit to the following column. Decomposition of multiplication steps may be helpful; expansion of digits, labeling columns as ones, tens, hundreds; may further remediate the students' difficulty.

2. **Zero Difficulties.**

eg. a)
$$\begin{array}{r} 507\\ \times\ 4\\ \hline 3508 \end{array}$$

b)
$$\begin{array}{r} 307\\ \times\ 2\\ \hline 604 \end{array}$$

Often when students have been taught the properties of zero in addition and multiplication, there will be a tendency to interchange the rules within each operation. A more descriptive explanation may be required to distinguish the differences between the two operations. Solving problems that may be relevant to the students' home/school environment may be helpful. Once an appropriate explanation has been made, drill work should follow to reinforce the concept.

3. **Multiples only by unit digit with a multiple digit multiplier - disregards other digits**

eg. a)
$$\begin{array}{r} 37\\ \times\ 2\\ \hline 74 \end{array}$$

b)
$$\begin{array}{r} 372\\ \times\ 34\\ \hline 1488 \end{array}$$

Estimation may be the quickest solution. A more in-depth examination of each factor may be necessary to distinguish the value of each digit.

4. **Failure to indent partial products.**

eg. 273
x 32
546
819
1365

Remediation may include estimation, review of place value (as an example), and/or decomposition of factors.

5. **Failure to use simplest method with multiples of 10.**

eg. 300 x 473 → 473 x 300

Teachers should review the commutative property with the student.

6. **Difficulty in multiplication in horizontal form.**

5 x 327 =

Graph paper and having the student rewrite the numbers in vertical form may remediate the difficulty.

Errors Specific to Division
(Pincus, 1975)

1. **Incorrect placement of figures in quotient.**

$$7\overline{)372} \quad 53R1$$

Before any remediation is initiated, the teacher should investigate whether the student has done the example on a scrap piece of paper and indiscriminantly placed the digit on the division symbol. If not, the teacher may require the student to use graph paper to align the quotient in the appropriate columns.

2. **Zero difficulties.**

a) $3\overline{)210}$ $\quad 7$ b) $5\overline{)3105}$ $\quad 63$

Once again estimation may be the key to our solution. However, if this method is inaffective, then a possible return to a longer form of division demonstrating the concept of zero may be necessary.

3. **Incomplete division.**

$$5\overline{)75} \quad 13R10$$

Remediation would include an oral explanation by the student and possibly the revision of the concept "highest common factor".

The previous discussion concerning general and specific errors to given operations, are only a small sample of the variety of possibilities that children may develop. Cox (1975) suggests that teachers look for patterns in the material that students hand in. This attitude would compliment our expectations of students to search for patterns in operations with numbers. Thompson (1978) also believes that the use of any concrete material be parallel to the mental process required to solve a problem in a more abstract manner.

Although we may consider our teaching role as successful if remediation has been attained; we

must consider the long term comprehension of concepts. What might be remediation this week, or for that matter this year, may be a reoccuring problem for one of our colleagues and students next term. Consistently we must rehearse concepts and procedures whether at a fixed interval of time, or at a varient interval of time with our students. The strategies we use to remediate our students' difficulties should be at their level of need and comprehension.

If we ask; "what will be the retention level that will determine an understanding of concepts in our classroom"; and respond "a standardized achievement test"; then as Callahan (1977) stated "we should stop and smell the roses" and spend some time in using our common sense in interpreting test results. A pencil-paper test may be one of the most valid indication that a teacher can have. Considering the teacher is aware of his/her students' abilities, then an accurate description of achievement may be outlined by an informal test.

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PRACTICAL TESTING FOR MATHEMATICS

by

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In 1977, the Department of Education and Science, U.K., recognized the need for practical testing and proceeded to include it in their assessment work called Assessment of Performance Unit (APU). Considerable experience had been gained as a result of the Tests of Attainment in Mathematics in Schools (TAMS) commissioned in 1972 and developed by the National Foundation for Educational Research (NFER). A.P.U. practical tests included the following topics: methods of computation, fraction of continuous and discrete objects, number patterns, problem solving strategies and attributes as well as measures and geometry. The APU group offered the following reasons to justify the practical mode of testing. I agree with the reasons but would invert the order of importance.

Pencil and paper tests containing multiple choice items, fill-the-blank items, computational items and/or problem solving items are not the whole answer in monitoring mathematical programs. I propose that there must be practical testing as well. Practical tests should become an integral part of any program of assessment of mathematics. Items available from a bank, such as that offered by the Ontario Assessment Instrument Pool, are not meeting the needs as expressed in local Board of Education, or even Ministry of Education, curriculum objectives.

"...1. Because pupils are at various stages in the development of their thinking from concrete to abstract, 'practical testing' could possibly allow comment on how far pupils have travelled

along the route towards abstraction and also contribute to our understanding of the processes involved, at least in a mathematical context.

2. Some topics, for example, the use of calculating aids and measuring instruments, or work with three dimensional models, are particularly suited to the practical mode of testing and some such items might have to be excluded if testing were restricted to 'pencil and paper'.

3. Assessment should reflect what is happening in schools on a day-to-day basis, including the use of apparatus, practical activities, topic work and other components of good practice.

4. Some pupils are placed at a distinct disadvantage by only doing written tests. Slow and poor readers, poorly motivated pupils and slow learners are often discouraged, even to the extent of making no response whatsoever. We need to give pupils an opportunity to show themselves in their best possible light; it is more likely that a truer profile will be obtained by assessing them in a variety of modes. It is interesting to note that even in formal written examinations, the 'Applicable Mathematics' papers at GCE A/O Level use the idea of cues (see glossary) with corresponding penalties and their use is advocated in part of the N and F Commissioned Groups Report.

5. Mathematics is, in part, a 'service' subject which crosses subject boundaries, while at the same time it is a communication skill. Developing pupils' ability to communicate with their teachers and their peers, through speech and in writing is an essential facet of teachers' work in schools. In science, for example, the work is frequently of a practical nature and the pupil should be able to report his findings orally and in writing and also be able to illustrate his ideas further.

6. Practical mathematics corresponds to the way in which mathematics arises in adult life. ..." [1]

Since 1977, I have been using practical tests as a part of a development program in Measurement and Geometry in the city of London (see article "Achievement in Measurement and Geometry in the Junior Division", **Ontario Mathematics Gazette**, Vol. 17, No. 3, March 1979). This spring, a criterion referenced test based on the London Board of Education Mathematics Core Guidelines, Gr. 7, containing some 106 items was used by four testers. The items were grouped under four sections as follows:

1 Measurement - linear, perimeter, area, volume, angles
2 2-D shapes
3 3-D shapes
4 Symmetry, Congruence and Transformations

Each of these sections was set up so that on the left side of the page the materials were listed in brackets. Under this, appeared the question or statement the tester would use. Directly opposite each question was a statement of the objective being measured by that item. Below is an example. Anecdotal comments come from tape recordings and testers' observations. Where questions were the same as those used in Gr. 6 testing of June, 1979, the results are also shown.

Test Instrument (Grade Seven)		Interview Technique Criterion Referenced		
Linear	Measurement		Per Cent Correct Gr. 6	7
(a line drawn 6.7 cm a ruler, mm smallest unit markings start at zero)				
T. With what unit do you measure a line?		1. States a linear unit	85	95

It seems that pupils not only have a grasp of a standard unit of linear length but now are thinking in S.I. units. The slogan **THINK METRIC** has happened! In the National Assessment of Educational Progress (NAEP) in the United States, 80% of 13 year olds chose centimetres from a multiple choice item as the correct unit to measure the length of your thumb. In a study of children's comprehension of content in reading, using the Cloze Procedure (Kirkwood and Laws, in press) it

was found that children tended to conceptualize linear measurement in terms of metric units. Children, when faced with a blank in a sentence, did write metre in place of yard. From the APU survey it was found that "about 80 percent correctly selected units of mass (kg) and units of temperature from lists containing capacity, length and areas measures." [2]

Knowledge of and thus recall of the other units of measure was not as good. To think metric is one thing; to think in two and three dimensions is another. The following table indicates this:

95% of pupils in Gr. 7 correctly gave a linear unit, e.g. 1 cm
72% of pupils in Gr. 7 correctly gave an area unit, e.g. 1 cm^2
72% of pupils in Gr. 7 correctly gave an angle unit, e.g. 1^0
54% of pupils in Gr. 7 correctly gave a volume unit, e.g. 1 cm^3

Some of the answers written for a unit of measure were:

Area - 1 sq. cm , 1 $^{sq.}$ cm , 1^2 cm , 1 cm^4, 1 cm.
Volume - 1 cu. cm , 1^c cm , 1 qu. cm , 1% cm.
Angle - 1 cm , 1%, ^{01}.

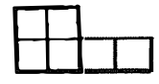
Estimation is a practical objective, so students were asked to estimate both a straight line and a curved line. As it happened APU did the same even though the lengths of the lines were not the same as the ones used here. 80% of the Grade 7 pupils were able to draw a 1 cm. line. The APU survey showed that "80% chose the most realistic alternative for the likely height of a table in a multiple choice item."[3]

	Percent Correct
3. Estimates length of straight line (actual length 6.7 cm - allowed 5-8 cm) (In England 81% of pupils estimated the length of a 4.5 cm. correctly, allowed - 3-6 cm)	69
9. Estimates length of a curved line (actual length 11 cm - allowed 9-13 cm) (In England 80% estimated the length of a curved line 12 cm correctly (allowed 8-15 cm).	80

After estimating each length, pupils were then asked to measure the lines. Measuring a straight line involves aligning the ruler, placing the zero and then reading the scale. Several children placed the '1' on the ruler against the end of the line, giving an answer of 7.7 cm. Several others started measuring at the very end of the ruler. When following the curve with string, most pupils were quite careful. Some pupils, however, tried to measure the curve with a ruler, summing short line segments. Some marked the string where the line ended while others just held the place with their fingers before setting the string against the ruler. One pupil measured a 9 cm. length of string, placed it along the curved line and then estimated the remaining 2 cm. for an acceptable answer.

4. Measures length of the straight line correctly. (In England 86% measured a 4.5 cm straight line correctly).	33	74
10. Measures length of the curved line correctly. (In England 78% measured a 12 cm line correctly - 0.5 cm).		80

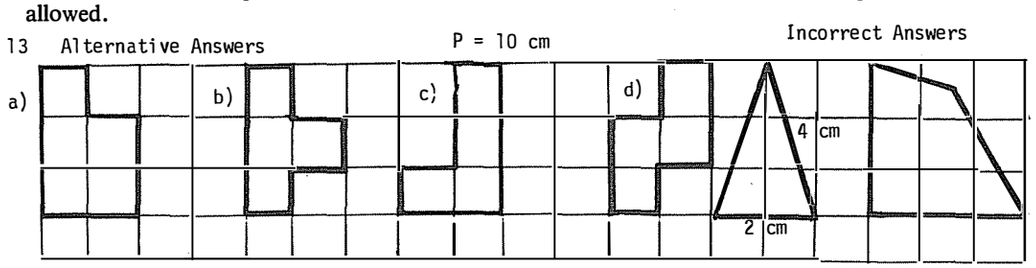
In testing the idea of perimeter, this figure was drawn on a cm^2 grid. 74% of the students counted the units to give us a perimeter of 12 cm. Some gave us an answer of six centimetres which indicated a confusion between perimeter and area. It seemed wise to probe more deeply their understanding. A synthesis question is challenging, allows for creative behaviour and often gives insight into pupils thinking about concepts. Thus, the following items:



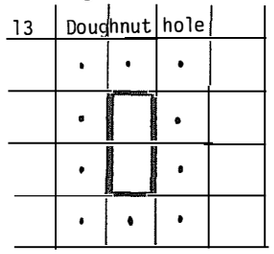
(a ruler, cm^2 paper)	Percent Correct	
Grade	6	7
13. Draw a figure whose perimeter is 10 cm.	52	66

The easiest and most common figures drawn here were a 1 x 4 and a 2 x 3 rectangle. Yet an irregular figure with slanted lines was attempted as well as a triangle, disregarding the cm unit grid.

Four other correct responses are shown below. Transformations of these were also given and were allowed.

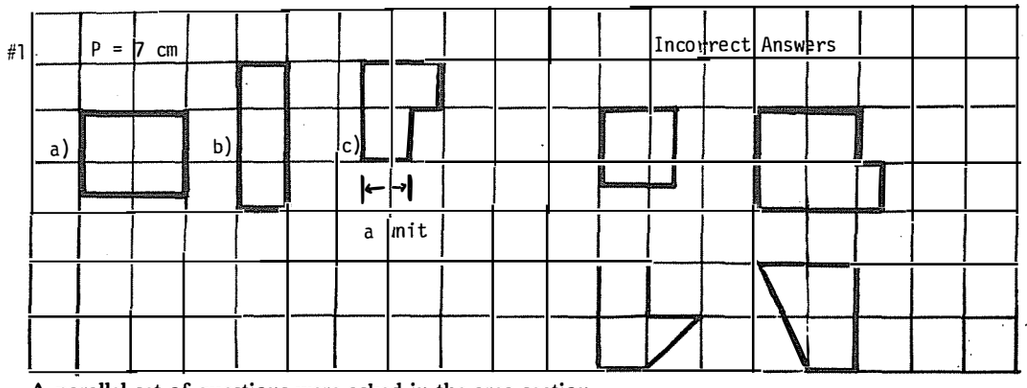


One pupil drew a doughnut hole and dotted the squares about the hole as he counted the squares declaring that was the perimeter. In England, 23% of the pupils gave an answer to perimeter questions suggesting they could not distinguish between the notions of 'area and perimeter'.



14. Draw a figure whose perimeter is 7 cm. 10 15

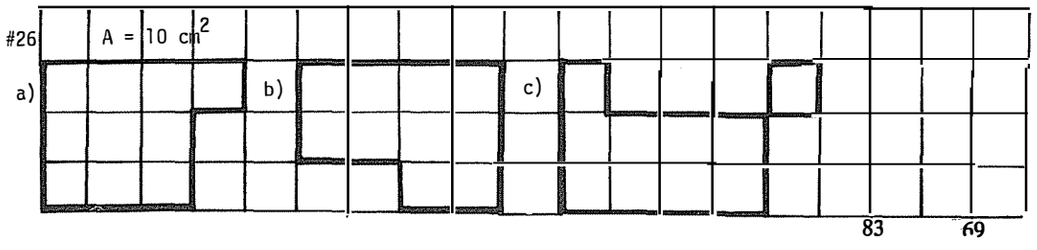
This question presented the further challenge, i.e. to visualize a part unit. The few pupils who answered it correctly, provided us with some very interesting ideas and these might well be examined further. Did they go about making the figure in an adult manner? Did they divide the seven by two, thus arriving at the sum of the two sides of a figure, in this case, three and a half centimetres? Our observations overwhelmingly suggest that a trial-and-error method was used. The incorrect examples, shown below, illustrate the child's discovery of a half unit but inability to arrive at a correct combination. One lad used as his unit, half centimetre (14 c.) and then proceeded to solve the question in this manner. Quite a different approach!



A parallel set of questions were asked in the area section.

26. Draw a figure whose area is 10 cm^2.

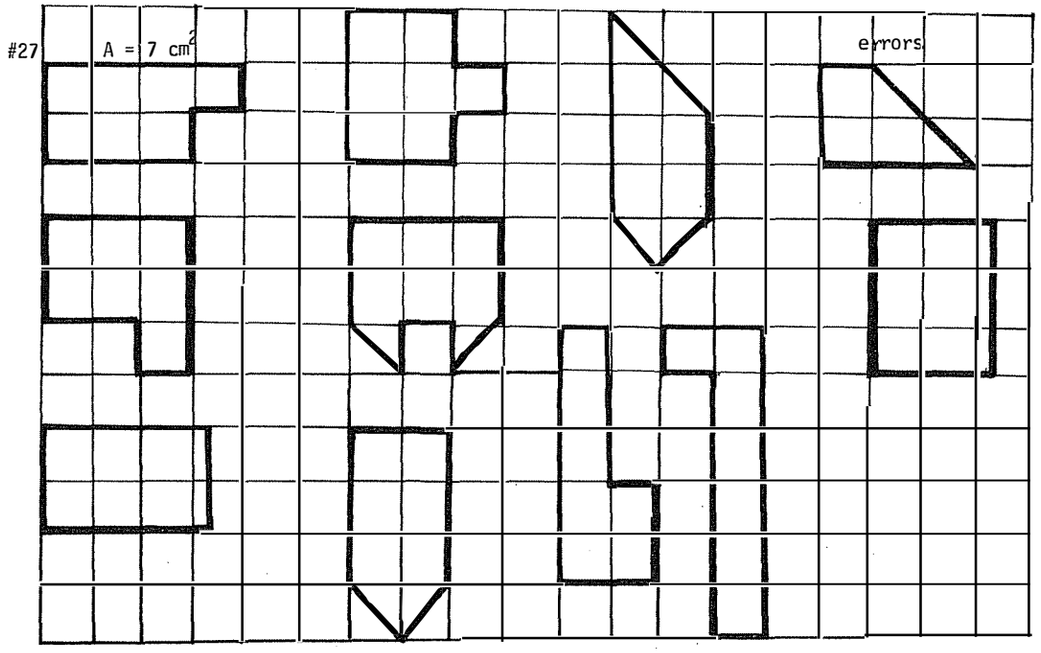
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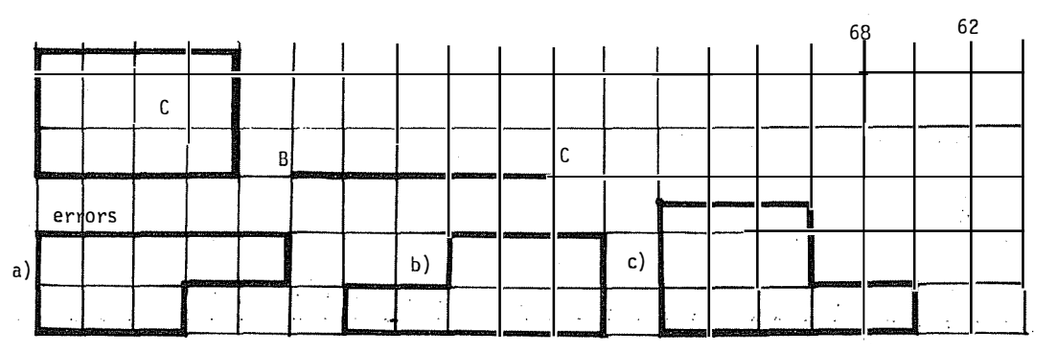
83 69

27. Draw a figure whose area is 7 cm ^{2}.

Surprisingly, no one chose to draw the obvious rectangle - a 1 x 7. This item did, however, give full reign to originality as the results below indicate. One student used half linear unit and drew a rectangle 2 x 3.5. Many more realized a 0.5 cm ^{2} might be useful.



30. Draw a rectangle with the same area as (c) but using BC as one side.



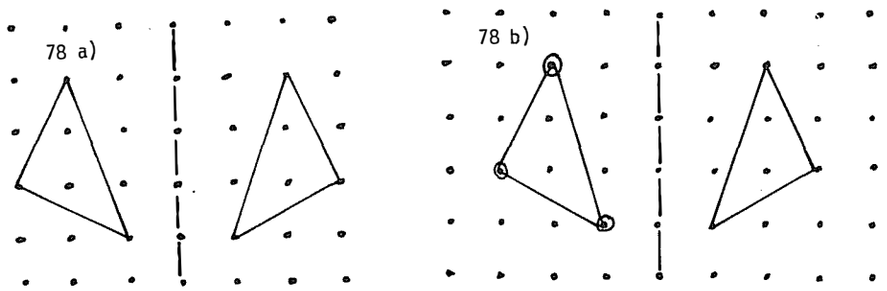
In this case, the area had to be found and a relationship between an area of 10 cm ^{2} and the length of one side of a new figure which was given as 5 cm. The obvious error was to extend the top line of the original figure giving a figure of area 12.5 cm ^{2}. Other errors were more individual.

These practical tests showed clearly that children do adopt different approaches to solving concrete tasks. An interviewer can well afford the time to carry on a prolonged conversation with a child. Although it is desirable for testers to stick to a script for reliability purposes, testers may feel justified in rephrasing words, in following a different line that looks more productive or even in abandoning the question altogether. Talking with children with problems in mathematics might well clear up misconceptions. Often children clear up their difficulties simply when voicing their thoughts. A practical way of obtaining the time for interviews is to extend team teaching concept. It might well include one teacher assessing and talking to individual children whilst the other teacher copes with the whole group.

TRANSFORMATIONS

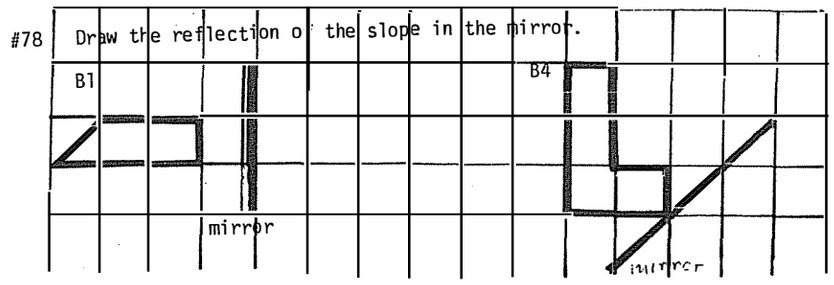
The interviews and questions posed on slides, turns and flips, although done with pencil and paper for record purposes, were very practical in nature. After observing some pupils it became obvious that they were seeing the MIRA for the first time. Some of these were reluctant to handle it. One pupil however was disdainful of the instrument preferring to draw images as he conceptualized them. Another pupil enjoyed the work so much he wanted to buy the MIRA from us so he might play with it at home. Others experimented with it, turning first the paper then the MIRA in his experiments. A few even thought aloud while they worked thus providing us with a commentary of their thoughts. A few sat, thought, then picked up the MIRA only after they had decided what to do with it.

78. Draw the image. Draws correct image given the object and a vertical line of symmetry. Object and line of symmetry given on dotty paper. Trial item done before as experiment. Pupil allowed to use the MIRA if desired.	Grade	Percent Correct 6	7
		98	95

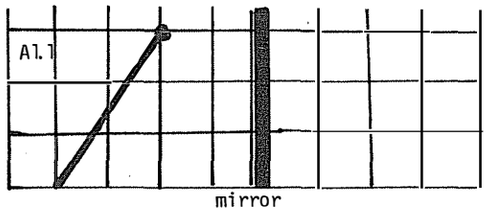


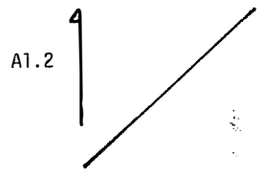
Most pupils tried to draw the image free hand as shown in 78a above, after looking through the MIRA. We accepted this even though it gave a rough image, not an exact one. A few students looked through the MIRA, marked the vertices and joined these with a ruler thus making a more accurate image (78b). One pupil found the vertices in the image by mapping, i.e. "is one space to the right of the line of symmetry, therefore A^1 is one to the left of the line of symmetry, etc." He did not call the points A and A^1 but they were the vertices he pointed out to me.

In England, 79% drew the reflection in a vertically placed mirror line correctly (B1). But if the mirror line was placed diagonally ((B4) the facility value dropped sharply to 14%.



In the CSMS study, pupils were asked to draw reflections both with and without the benefit of grid lines. In item A1.1, 84% drew the reflection correctly on the grid. In item A1.2 where no grid was given, only 23% succeeded.





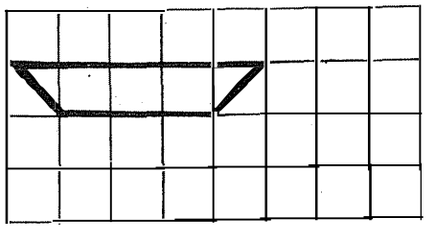
(A good discussion of children's errors on reflection items is to be found on pgs. 141-145 in the book by Hart, K., et al., **Children's Understanding of Mathematics: 11-16).**

Practical testing allows the interviewer an opportunity to have mathematical conversations with children. Developing the pupil's ability to communicate with their peers and teachers is essential. Even common (to us) measurement language such as perimeter and volume may have different meanings in the pupil's mind. One child, when asked what volume means, replied that it was the name on the T.V. knob. It is clear in my mind that opportunities similar to the next item will help in the formation, or perhaps more so in the consolidation, of transformation concepts.

Pupils were given an isosceles trapezoid of unit length which I called a boat and asked to give instructions which would move the boat to different positions on the grid. They were asked to take the role of teacher while I took the role of student. This was probably a first venture of this type for them. During this activity the interviewer had the opportunity to prompt the child so that he might show his knowledge of the following objectives.

OBJECT

To move the original shape (the boat) to an image position on the grid.



OBJECTIVES

95. Describes a slide: in terms of direction.	56	79
96. in terms of distance.	32	56
97. Describes a turn: in terms of direction	56	74
98. in terms of amount.	36	72
99. Describes a flip.	48	62

In describing a slide in terms of distance, the common error was to state a slide of 3 spaces but accept a slide of 8 spaces as correct. For turns, pupils had difficulty in telling us what direction to turn, wanting to say such things as "go to the right" or "two spaces left." Most pupils merely pointed to an intersection of lines rather than marking or naming a turn centre. The same type of difficulty appeared for flip line. The pupils' language was imprecise, perhaps even a first attempt. As these efforts are a beginning point, teachers play an important role. Providing further experiences is essential as is the listening. Perhaps it will be necessary to clarify language, to introduce technical terms as they are needed, perhaps to question so misconceptions can be cleared up, but most of all to listed and assess the learning that is taking place. These teaching/learning situations allow the pupil to perfect his language by using it. Frank Smith, in his book **Reading Without Nonsense,** states that we learn to read by reading. I would offer a parallel truism for mathematics teachers, "we learn to use the language of mathematics as we talk it." The time for practical testing and prolonged conversations with children has come.

FOOTNOTES

1. Department of Education and Science, London, England. **Assessment of Performance Unit, Monitoring Mathematics,** p. 9.

2. Ibid., p. 32.

3. Foxman, D., et al., Mathematical Development, **Primary Survey Report, No. 1,** Department of Education and Science, Welsh Office, p. 33.

4. CSMS - Concepts in Secondary Mathematics and Science based at Chelsea College, University of London, was a research programme funded from 1974-1979. Items were written by the CSMS team, tried on interview with children, then replaced or revised. Interviews were used to find methods used and errors made by children. These, in turn, were used to interpret the results of the written tests.

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MATHEMATICAL EXPOSITIONS

MATHEMATICS AND THE SWIMMING COACH

by
L. Keith Roy

A swimming coach wishes to choose 4 of her 6 swimmers to compete in the 4 x 100 m medley relay race. Each competitor will swim a distance of 100 m , the first using the backstroke, the second using the breaststroke, the third using the butterfly and the fourth freestyle. In an attempt to determine which swimmer whould be assigned which stroke, all six were timed using each of the 4 strokes and the results in seconds were recorded.

		Stroke			
		(1)	(2)	(3)	(4)
Swimmer	Agnes	82	89	75	70
	Betty	77	91	72	70
	Cathy	78	88	70	68
	Donna	81	90	72	70
	Ethel	77	93	73	71
	Flo	79	90	73	69

Table 1. Times in seconds for (1)-backstroke, (2)-breaststroke, (3)-butterfly, (4)-freestyle, for each of the 6 swimmers.

The coach wishes to use the information in table 1 to assign individuals to strokes in such a way as to minimize the relay team's total time.

In a small problem such as this it might be possible to deduce the solution using the method of trial and error. For larger, more complicated problems a procedure for selecting the proper assignments of individuals to tasks is required. If we could convert table 1 into a table which contained positive numbers and some zeros (but no negative numbers) and if we could make all assignments of swimmer to strike only where zero times appeared in the new table, then the relay team's total time for the new table would certainly be the smallest possible, namely zero. We can manufacture some zeros for our new table by adding or subtracting any constant to/from all entries in any row or column of table 1. This has the effect of changing the values of the entries in such rows or columns but maintains the difference between the entries.

Our procedure will also require that there be the same number of rows as columns in our table, so that all swimmers can be assigned to something. The result is table 2, formed by adding 2 extra columns to table 1.

		(1)	(2)	(3)	(4)	(5)	(6)
Swimmer	Agnes	82	89	75	70	0	0
	Betty	77*	91	72	70	0	0
	Cathy	78	88*	70*	68*	0	0
	Donna	81	90	72	70	0	0
	Ethel	77*	93	73	71	0	0
	Flo	79	90	73	69	0	0

Table 2. Times in seconds for (1)-backstroke, (2)-breaststroke, (3)-butterfly, (4)-freestyle, (5) & (6)-cheerleaders.
* indicates the smallest entry in each column.

In the above table, the swimmers that do not make the relay team, the cheerleaders, will not contribute to the team's total medley relay time and hence zero times appear in columns (5) and (6).

To produce more zeros, subtract the smallest entry in a column from all entries in that column. Do this for each column. The result is:

	(1)	(2)	(3)	(4)	(5)	(6)
Agnes	5	1	5	2	0	0
Betty	0	3	2	2	0	0
Cathy	1	0	0	0	0	0
Donna	4	2	2	2	0	0
Ethel	0	5	3	3	0	0
Flo	2	2	3	1	0	0

Table 3. obtained from table 2 by subtracting the smallest entry in a column from all entries in that column.

Note that it is still not possible to make all assignments where zeros appear. Only two of Agnes, Donna and Flo can be assigned as cheerleaders, Betty and Ethel can't both be assigned to the back stroke and Cathy can be assigned to only one of butterfly, breaststroke and freestyle.

Also note that it is useless (in this case) to do the same trick with rows that we did with columns. Since the smallest entry in each row is a zero, subtracting it from each entry in its row produces no additional zeros. To manufacture still more zeros without losing many of our existing zeros, we draw lines through some rows and columns in the table so that all zeros are covered by at least one line. The number of lines required to do this task equals the number of assignments that can be made (without conflicts) in positions where zeros appear. Since, in table 3 we could, for instance, assign Agnes and Donna as cheerleaders, Betty to the backstroke and Cathy to the butterfly, we know that 4 lines are required to cover the zeros. There may be more than one way to cover all zeros with lines and how we choose to do so could influence how quickly we arrive at a solution. The only possibility for our example is:

	(1)	(2)	(3)	(4)	(5)	(6)
Agnes	5	1*	5	2	0	0
Betty	0	3	2	2	0	0
Cathy	1	0	0	0	0	0
Donna	4	2	2	2	0	0
Ethel	0	5	3	3	0	0
Flo	2	2	3	1*	0	0

Table 4. Obtained from table 3 by covering all zeros with lines. * indicates the smallest uncovered entry.

We select the smallest uncovered entry, 1, subtract it from every entry in the table and then add it to each entry where lines intersect. The effect of these two actions is to subtract our smallest uncovered entry to every entry where lines intersect. The result is:

	(1)	(2)	(3)	(4)	(5)	(6)
Agnes	5	0	4	1	0	0
Betty	0	2	1	1	0	0
Cathy	2	0	0	0	1	1
Donna	4	1	1	1	0	0
Ethel	1	4	2	2	0	0
Flo	2	1	2	0	0	0

Table 5. Obtained from table 4 by subtracting the smallest uncovered entry in table 3 from all uncovered entries and adding it to all entries where lines intersect.

It is now possible to make all assignments where zero entries appear in table 5. One of Betty and Ethel swims the backstroke while the other is assigned as a cheerleader, Agnes swims the breaststroke, Cathy the butterfly, Flo the freestyle and Donna is assigned as a cheerleader. Going back to table 2 we see that the total time for the team will be $77 + 89 + 70 + 69 = 305$ seconds. It it still had not been possible to make all assignments where zeros appear we would simply repeat our method of covering all zeros with lines and manufacture more zeros.

Question: How should our problem be solved if Donna (the coach's daughter) must be a member of the medley relay team? There are 8 possible team combinations with the smallest total time. Can you find them all? (Hint: in table 2 make the entries for Donna in columns (5) and (6) so large that Donna could never be assigned as a cheerleader).

ANSWERS TO QUESTION

Backstroke	Betty	Ethel	Betty	Ethel	Betty	Ethel	Betty	Ethel
Breaststroke	Agnes	Agnes	Donna	Donna	Agnes	Agnes	Cathy	Cathy
Butterfly	Cathy	Cathy	Cathy	Cathy	Donna	Donna	Donna	Donna
Freestyle	Donna	Donna	Flo	Flo	Cathy	Cathy	Flo	Flo

All of the above teams have a total time of 306 seconds.

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PYTHAGORAS REVISITED

by

Hans P. Heinig

The best known of all theorems in mathematics is the theorem of Pythagoras. It states that

(1) $a^2 + b^2 = c^2$,

where a and b are the lengths of the two sides of a rectangle and c the length of the diagonal. A more traditional formulation of the theorem is: <u>In right triangles the square of the side subtending the right angle is equal to the sum of the squares of the sides containing the right angle.</u>

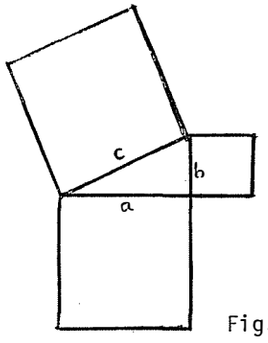


Fig. 1

Although the name Pythagoras (c. 585-497 BC) is attached to this theorem, it was known to the Babylonians during Hammurabi's reign (c. 1750 BC) ([6, p. 36]). In fact they solved the much harder

problem of finding all triples (a,b,c), a,b,c non zero integers for which $a^2 + b^2 = c^2$ holds. A proof of Pythagoras' theorem is given in Euclid's Elements Book 1, Proposition 47([4]) and is followed by this converse: <u>If a triangle with side lengths a,b,c satisfies $a^2 + b^2 = c^2$, then the triangle is a right triangle.</u>

Many proofs of Pythagoras' theorem are known, the simple one sketched below is due to Bhaskara (c. 1300 A.D.) ([9, p. 71-81]) although the special case $a = b$ appears already in Plato's Meno ([7, 82-86]).

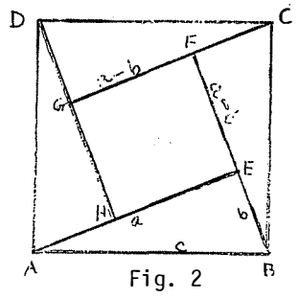


Fig. 2

Construct a square of side length c and let $a \geq b$ (see Figure 2). Place ΔABE as in Figure 2 and extend BE to F so that $|BF| = a$ ($|BF|$ = length from B to F). Extend CF to G, so that $|CG| = a$ and DG to H so that $|DH| = a$. Then the square EFGH has side length $a - b$, because the areas of indicated similar triangles are the same. Therefore,

$$c^2 = 4 \cdot (\text{area of } \Delta ABE) + (\text{area of square } EFGH)$$
$$= 2ab + (a - b)^2 = a^2 + b^2,$$

which proves the theorem.

The theorem generalizes easily to three or higher dimensions. Thus, if a,b,c are the sidelengths of a rectangular box, then the length of the diagonal is $d = \sqrt{a^2 + b^2 + c^2}$; and if $a_1, a_2, \ldots, a_n$ are the dimension of a rectangular hyperbox, then the 'diagonal' length is given by $\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$. There is another elementary generalization of Pythagoras' theorem to higher dimensions which is not in Euclid's Elements, and indeed seem to have been largely overlooked in the literature. It is the object of this article to discuss these generalizations.

We call a tetrahedron <u>a right tetrahedron</u> if three of its four faces are right triangles. In three dimensional Euclidean

space - denoted by R^3 - a right tetrahedron is a solid bounded by the planes $x = 0$, $y = 0$, $z = 0$ and the plane determined by the points $P_1 = (a,0,0)$, $P_2 = (0,b,0)$ and $P_3 = (0,0,c)$, $abc \neq 0$, (See Figure 3). We also denote by O the origin $(0,0,0)$.

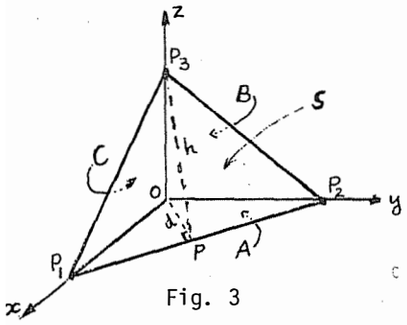


Fig. 3

Let A,B,C denote the right triangles of the tetrahedron determined by OP_1P_2, OP_2P_3 and OP_3P_1, respectively, having areas $|A|$, $|B|$, $|C|$. Let S denote the subtended triangle determined by $P_1P_2P_3$, then the geometric generalization of Pythagoras' theorem is the following:

Theorem 1. In a right tetrahedron the sum of the squares of the areas of the right triangles is equal to the square of the area of the subtended triangle. That is,

(2) $|S|^2 = |A|^2 + |B|^2 + |C|^2$.

Proof. It is obvious from Figure 3 that $|A| = ab/2$, $|B| = bc/2$, $|C| = ac/2$ and $|S| = (h\sqrt{a^2 + b^2})/2$, where h is the height of S. Let d be the length of the line OP which is perpendicular to P_1P_2. Then $|A| = (d\sqrt{a^2 + b^2})/2 = ab/2$ which implies that $d = ab/\sqrt{a^2 + b^2}$. Now by the theorem of Pythagoras $h = \sqrt{d^2 + c^2} = \sqrt{(ab)^2/(a^2 + b^2) + c^2} = \sqrt{a^2b^2 + a^2c^2 + b^2c^2} \,/\, \sqrt{a^2 + b^2}$, so that $|S|^2 = (a^2b^2 + a^2c^2 + b^2c^2)/4 = |A|^2 + |B|^2 + |C|^2$.

Notice that distance in Pythagoras' theorem has been generalized to area in this theorem.

In order to extend this theorem to even higher dimensions we give another proof of Theorem 1 which is less visual but motivates the higher dimensional situation.

We may consider P_1, P_2, P_3 to be vectors, then $P_1 - P_2 = (a,-b,0)$

$P_1 - P_3 = (a,0,-c)$ are vectors in the triangle S. We wish to obtain the equation of the plane containing triangle S. Let $\vec{n} = (\alpha_1,\alpha_2,\alpha_3)$ be the normal vector emanating from this plane, then this vector is orthogonal (perpendicular) to any vector in the plane and hence their dot-product is zero. That is,

$$0 = \vec{n} \cdot (P_1 - P_2) = (\alpha_1,\alpha_2,\alpha_3) \cdot (a,-b,0) = \alpha_1 a - \alpha_2 b$$

$$0 = \vec{n} \cdot (P_1 - P_3) = (\alpha_1,\alpha_2,\alpha_3) \cdot (a,0,-c) = \alpha_1 a - \alpha_3 c.$$

Let $\alpha_1 = 1$, then $\alpha_2 = a/b$ and $\alpha_3 = a/c$ so that $\vec{n} = (1,a/b,a/c)$.

Now if $X = (x,y,z)$ is any point in the plane of S then $X - P_1 = (x - a, y, z)$ is a vector in this plane and therefore

$$0 = \vec{n} \cdot (X - P_1) = (1, a/b, a/c) \cdot (x - a, y, z) = x-a + ay/b + az/c,$$

that is $z = c - cx/a - cy/b$ is the equation of the plane S. (For details of vector analysis see e.g. [5]). Using calculus one now sees that the area of S is given by the double integral

$$|S| = \iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dxdy,$$

where A is the xy-region defined by triangle A. The partial derivatives are

$$\frac{\partial z}{\partial x} = -c/a, \qquad \frac{\partial z}{\partial y} = -c/b$$

and since $|A| = ab/2$ it follows that

$$|S| = \sqrt{1 + (c/a)^2 + (c/b)^2} \iint_A dxdy = |A| \sqrt{a^2b^2 + b^2c^2 + a^2c^2}\ /(ab)$$

$$= \sqrt{a^2b^2 + a^2c^2 + b^2c^2}\ /2.$$

This yields equation (2) on squaring. Thus we have proved Theorem 1 by a different method.

This argument is now generalized to construct a right hyper-tetrahedron and then to obtain a formula similar to equation (2).

Let $P_1 = (a,0,0,0)$, $P_2 = (0,b,0,0)$, $P_3 = (0,0,c,0)$, $P_4 = (0,0,0,d)$ and $0 = (0,0,0,0)$. In analogy with the three

dimensional case, let A be the solid $P_1P_2P_3O$, B the solid $P_1P_2P_4O$, C the solid $P_1P_3P_4O$, and D the solid $P_2P_3P_4O$. The 'subtended' hyperplane S is defined by $P_1P_2P_3P_4$. This defines a right hypertetrahedron. Using calculus and the previous argument we see that the volume of the solid A is given by

$$|A| = \iiint_A dxdydz = \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dzdydx$$

$$= \int_0^a \int_0^{b(1-x/a)} c(1 - x/a - y/b)dydx = \frac{abc}{6}.$$

Similarly, one obtains $|B| = \frac{abd}{6}$, $|C| = \frac{acd}{6}$, $|D| = \frac{bcd}{6}$, so that the four dimensional generalization of Theorem 1 is

Theorem 2.

(3) $|S|^2 = |A|^2 + |B|^2 + |C|^2 + |D|^2$

Proof. We proceed as in the proof of Theorem 1. First we find the 'normal' to the hyperplane S, then the equation of the hyperplane and finally its 'volume' which yields equation (3).

Let $\vec{n} = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ be the normal emanating from S and $P_1 - P_2 = (a, -b, 0, 0)$, $P_1 - P_3 = (a, 0, -c, 0)$, $P_1 - P_4 = (a, 0, 0, -d)$ vectors in the hyperplane S. Since their dot-product with $\vec{n}$ is zero we obtain

$$0 = \vec{n} \cdot (P_1-P_2) = a\alpha_1-b\alpha_2 = \vec{n} \cdot (P_1-P_3) = a\alpha_1-c\alpha_3 = \vec{n}\cdot(P_1-P_4) = a\alpha_1-d\alpha_4.$$

Let $\alpha_1 = 1$, then $\alpha_2 = a/b$, $\alpha_3 = a/c$ and $\alpha_4 = a/d$ so that $\vec{n} = (1, a/b, a/c, a/d)$. Now if $X = (x,y,z,w)$ is a point in S, then $X - P_1 = (x-a, y, z,w)$ is a vector in S and since $\vec{n}$ is perpendicular to this vector we get

$$0 = \vec{n} \cdot (X - P_1) = (1, a/b, a/c, a/d) \cdot (x-a, y, z, w)$$

$$= x-a + ay/b + az/c + aw/d.$$ Hence the equation of the hyperplane S is $w = d(1 - x/a - y/b - z/c)$ and the 'volume' of the hyperplane is

$$|S| = \iiint_A \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2}\, dxdydz$$

$$= \sqrt{1 + (d/a)^2 + (d/b)^2 + (d/c)^2} \iiint_A dxdydz$$

$$= \frac{abc}{6} \sqrt{1 + (d/a)^2 + (d/b)^2 + (d/c)^2} .$$

Therefore

$$|S|^2 = \frac{1}{36} [(abc)^2 + (bcd)^2 + (acd)^2 + (abd)^2]$$

$$= |A|^2 + |B|^2 + |C|^2 + |D|^2$$

which proves Theorem 2.

It is not difficult to see now how to proceed in the n-dimensional case, we therefore omit the more technical argument.

There is a familiar generalization of the theorem of Pythagoras which is called the cosine law of triangles. It states that

(4) $c^2 = a^2 + b^2 - 2ab \cos C$

where a,b,c are the sidelengths of a triangle (in the plane) and C the angle subtended by side c. It is clear that (4) implies the converse of Pythagoras' theorem, for if a triangle with sidelengths a,b,c satisfies $a^2 + b^2 = c^2$ then by (4) $\cos C = 0$ so that $C = \pi/2$ and hence the triangle is a right triangle. Of course this is not how Euclid proved it. It is natural to ask: What is the generalization of the law of consines (4) along the lines of Theorem 1? Our last theorem will answer this question for a typical but not the most general case. We consider a tetrahedron in which two (and not three) of the adjacent triangles to the subtended triangle are not right triangles. This particular case is easier to visualize and easier to prove.

Theorem 3. In the tetrahedron of Figure 4 the equality

(5) $|S|^2 = |A|^2 + |B|^2 + |C|^2 - 2|A||C|\cos\beta\sin\alpha - 2|B||C|\cos\alpha\sin\beta$

holds where A,B,C,S are the indicated triangular regions and α, β

the indicated angles.

Proof. It is obvious that $|A| = ac/2$, $|B| = ec/2$

$|C| = \sqrt{a^2 + b^2} \cdot \sqrt{d^2 + e^2}\,/2$; $\sin\alpha = a/\sqrt{a^2 + b^2}$, $\cos\alpha = b/\sqrt{a^2 + b^2}$,

$\sin\beta = e/\sqrt{d^2 + e^2}$, $\cos\beta = d/\sqrt{d^2 + e^2}$. Let A' be the perpendicular projection of S on the xy-plane so that $|A'| = a(c-d)/2$. As in the proof of Theorem 1 we find the equation of the plane S from which we obtain $|S|$ on integrating and hence the result.

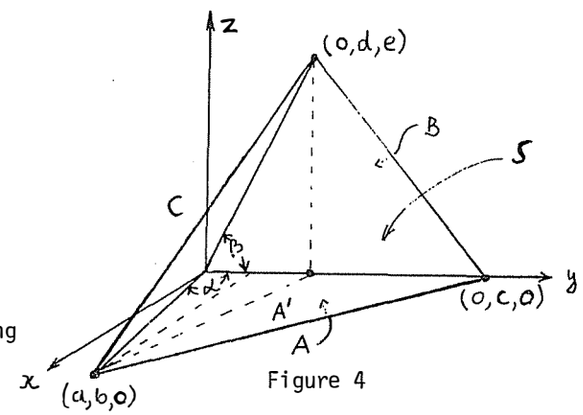


Figure 4

Let $\vec{n} = (\alpha_1,\alpha_2,\alpha_3)$ be the normal of S, $P = (a, b-c, 0)$ and $Q = (0, d-c, e)$ two vectors in S. Then

$$0 = \vec{n} \cdot P = \alpha_1 a + \alpha_2(b - c) = \vec{n} \cdot Q = \alpha_2(d - c) + \alpha_3 e$$

and if $\alpha_1 = 1$, then $\alpha_2 = a/(c-b)$ and $\alpha_3 = a(c-d)/[e(c-b)]$, so that $\vec{n} = (1, a/(c-b), a(c-d)/[e(c-b)])$. Let (x,y,z) be a point in S, then $(x, y-c, z)$ is a vector in S so that

$$0 = \vec{n} \cdot (x, y-c, z) = x + a(y - c)/(c - b) + a(c - d)z/[e(c - b)]$$

or

$$z = e(c - b)x/[a(d - c)] + e(y - c)/(d - c)$$

is the equation of the plane containing S. Calculus now shows that

$$\frac{\partial z}{\partial x} = e(c-b)/[a(d-c)], \quad \frac{\partial z}{\partial y} = e/(d-c)$$

and

$$|S| = \iint_{A'} \sqrt{1 + \frac{\partial z}{\partial x}^2 + \frac{\partial z}{\partial y}^2}\, dxdy$$

$$= \sqrt{1 + [e(c-b)/a(d-c)]^2 + [e/(d-c)]^2}\ |A'|$$

$$= \frac{1}{|d-c|a}\sqrt{a^2(d-c)^2 + e^2(c-b)^2 + a^2e^2}\,.\,\frac{(c-d)a}{2}$$

$$= \sqrt{a^2(d-c)^2 + e^2(c-b)^2 + a^2e^2}\,/2.$$

An easy calculation now shows that

$$|S|^2 - |A|^2 - |B|^2 - |C|^2 = \tfrac{1}{4}(-2cda^2 - 2cbe^2)$$

$$= -|A|ad - |B|be = -|A|a\cos\beta \cdot \sqrt{d^2+e^2} - |B|b\sin\beta\sqrt{d^2+e^2}$$

$$= -2|A||C|\sin\alpha\cos\beta - 2|B||C|\cos\alpha\sin\beta$$

so that the result follows.

The converse, namely, that if a tetrahedron of the type given in Figure 4 satisfies

$$|C|^2 = |A|^2 + |B|^2 + |C|^2$$

then it is a right tetrahedron, follows now from Theorem 3.

There are of course other questions which seem to be open. Let me conclude with only one. In the non-Euclidean Geometry - called the Bolyai-Labachevsky geometry - the Pythagorean formula has the form

$$2(e^{c/k} + e^{-c/k}) = (e^{a/k} + e^{-a/k}) + (e^{b/k} + e^{-b/k})$$

where a, b and c are the sides of a 'triangle', k a certain fixed constant and e the base of the natural logarithm. The question is the following: Is there an analogue of Theorem 1 in this non-Euclidean geometry and if so what is it?

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OF TIME, SPACE AND DIGITAL CLOCKS *)

by
Peter Donnelly

Move over, Einstein! I have just discovered a new dimension of the time-space relationship. While it is not earth-shaking and most unlikely to result in my receipt of next year's Nobel Prize for Physics, it certainly shook me. I have just realized that *time is space. Ergo,* digital clocks make me late.

Let me explain, Like most *Playboy* generation North American males I surround myself with the toys that are the trappings of modern technology. I have a miniature calculator in my office that also tells me the time, day and date, functions as a stop watch and lap timer, plays a series of tones as I balance my cheque book and wakes me from my afternoon nap to the electronic notes of "Fur Elise" and "Tarantella Napoletana."

I also have a digital clock radio by me bedside and it was while I was lying awake one night, mesmerized by the numbers flipping over, the minutes of my life skipping away, that I realized that this digital duo were turning me into a tardy person.

It occurred to me that the numbers have no meaning. What is 12:47? What is 4:33? They are numbers that have no relationship whatsoever to time in my conception. Because time is space and until the appearance of digital clocks and watches, it always has been.

For early man the passage of time was represented by the changing seasons, the migration of animals and birds, the growth of crops and the movement of the sun across the sky. The latter phenomenon became the basis for the first formal means of measuring time, the gnomon. The gnomon, which was later refined into the sundial, was a stick planted in the ground - the length and direction of the shadow determined the time of day.

Other early means of measuring time included the clepsydrae or water clocks, hour glasses and the burning of knotted ropes and incense tapers in the far East. In every case the passage of time was represented by movement through space - a moving shadow, the passage of water from one vessel to another, the changing level of oil or height of a candle, or the movement of a flame along a rope or incense taper.

European monks (Benedictine) were apparently the first people to live by the clock. They divided the day into seven devotional periods, canonical hours, each marked by the ringing of the monastery bells. But their means of determining time were troublesome. Water clocks were prone to freezing, clouds and darkness rendered the sundial useless and hourglasses, lamp and candle clocks needed constant tending. The mechanical clock was an invention whose time had come and that time came in the thirteenth century.

Again, the passage of time was represented by movement through space - movement of the hands of a clock through the 360 degrees of a circle. It is no accident that both hours and degrees are divided into minutes and seconds because time and space are analogous.

Nowhere is this analogy more apparent than in Hopi culture where no distinction is made between time and space - long ago and far away are the same. And child psychologists have determined that the concepts of time and space appear to develop together in our early years, a parallelism that we

*) This article appeared in CAUT Bulletin ACPU, September 1981.

retain in our adult lives. Our concern with travel, our love of punctuality and our particular style of sports all emphasize this parallel.

Distance becomes translated into time in so much of our conversation about travel:
"How far is it to the Chateau Laurier?"
"Are you walking or driving?"
"Walking?"
"Oh, about twenty minutes."

The question concerns distance, the response is given in time and we accept it as a valid answer. We invariably think in terms of travel time rather than distance whether we are driving from Toronto to Tampa Bay or flying from Vancouver to London. And I have never met a commuter who told me how far it was from home to work - travel time is the basis for comparison.

Time and space are key elements in all of our most popular sports. Football is governed by the chain and the clock and even the North American Soccer League has felt constrained to introduce a semi-official time clock. But the ubiquitous jogger best expresses the parallelism of time and space by employing the concepts interchangeably ("I was out for 45 minutes"/"I ran five miles"), or by combining the concepts in constant references to pace.

Punctuality must rank very highly in the hierarchy of North American virtues, perhaps even next to cleanliness. Promptness, keeping to a schedule, being *on time* - all ways of indicating that a person is in a certain space at a certain time. And therein lies my problem.

When I look at my old-fashioned, windup watch I see time as space. If my watch tells me that it is ten minutes to eleven and I have a class to teach in another building at eleven o'clock, the length of the arc on my watch from ten to twelve becomes a model of the distance from my office to the classroom. The time it takes for the minute hand to move from ten to twelve is the time that it will take me to walk to my classroom. I keep pace with the minute hand and I know that I have to leave now in order to be on time.

When I look at my digital calculator watch I see a number - 10:50. What is 10:50? It does not trigger any relationship between time and space in my mind and . . . I'm late for class again!

As I gradually began to realize what was happening I became more and more enraged about all digital clocks and watches. They were destroying my sense of time. But I could not understand why sports clocks were not having the same effect. Although digital clocks are widely used they have the effect of enhancing our sense of time and our excitement near the end of a close game.

The solution, like all the best solutions, was only too obvious. Sports clocks work backwards! They count down to zero at a constant rate, thereby giving us a very clear idea of how much time is left.

Now, if only I could get my calculator clock to work backwards. Maybe if I turn the batteries around . . ..

Professor Donnelly,
School of Physical Education & Athletics,
McMaster University.

When we cannot use the compass of mathematics or the torch of experience . . . it is certain that we cannot take a single step forward.

Voltaire

Without the concepts, methods and results found and developed by previous generations right down to Greek antiquity one cannot understand either the aims or the achievements of mathematics in the last fifty years.

Herman Weyl

This, therefore, is mathematics: she reminds you of the invisible form of the soul; she gives life to her own discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas; she abolishes oblivion and ignorance which are ours by birth.

Proclus

DEPARTMENTS

PROBLEMS? DID YOU SAY

by
Pat Yip

20.2.1. Straight-edge and Compass (submitted by G. Field)

Given two intersecting straight lines and a point P, construct a circle passing through P and having the straight lines as tangents.

20.2.2. Numbers and More

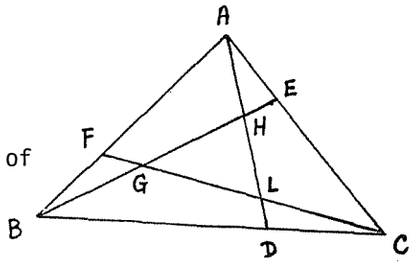
A six digit number is formed by repeating a three digit number; for example 256, 256 or 678, 678. Find a common divisor for all such numbers.

20.2.3. Triangles Galore

In the given diagram,

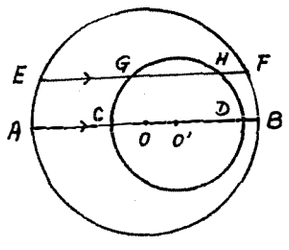
$CD = \frac{1}{3}CB$, $AE = \frac{1}{3}AC$, $BF = \frac{1}{3}BA$.

Find a relation between the area of ΔHLG and the area of ΔABC.



Solutions to problems in Vol. 20 #1.

20.1.1. (Solution by proposer; also solved by H. Derynck, Westgate Collegiate and Vocational Institute, Thunder Bay, Ont.)



To show: $(AB)^2 + (GH)^2 = (EF)^2 + (CD)^2$

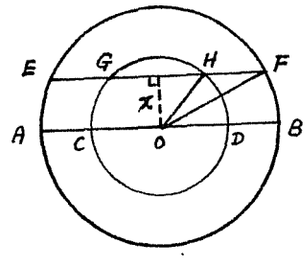
Proof: Let $AB = 2R$, $CD = 2r$, $EF = 2S$ and $GH = 2s$.

Then we are to show that $(2R)^2 + (2s)^2 = (2r)^2 + (2S)^2$

OR $R^2 - S^2 = r^2 - s^2$.

(Note that the length of the chord GH is unchanged if O' is moved to O.)

Let O and O' by coincident.



From diagram $x^2 = R^2 - (\frac{1}{2}EF)^2 = R^2-S^2$

$x^2 = r^2 - (\frac{1}{2}GH)^2 = r^2-s^2$

$R^2-S^2 = r^2- s^2$ as required.

(Derynck's solution is by co-ordinate geometry).

20.1.2. (Solution by the proposer). To show that 16 and 18 are the only "square" number and "rectangular" number respectively.

a) Let n be a natural number, and the side of a square.
Then the square number is $4n = n^2$, $n = 4$, $n^2 = 16$.

b) Let m and n be natural numbers such that $n > m$, and be sides of a rectangle. The rectangle number is then $mn = 2(m + n)$.

$$2 = \frac{n}{(1+\frac{n}{m})} \quad \text{from whence} \quad n > 2.$$

Now $$\frac{n}{\frac{n}{m}+\frac{n}{m}} < \frac{n}{1+\frac{n}{m}} = 2 \quad \text{since} \quad n > m,$$

$$\frac{m}{2} < 2 \quad \text{or} \quad m < 4.$$

We need only examine $m = 1, 2, 3$ and $n > 2$, which satisfy $\frac{n}{1+\frac{n}{m}} = 2$.

(i) $m=1$, $\frac{n}{1+n} < 2$ for $n > 2$

(ii) $m=2$, $\frac{n}{1+\frac{n}{3}} = \frac{2n}{2+n} < 2$ for $n>2$

(iii) $m=3$, $\frac{n}{1+\frac{n}{3}} = \frac{3n}{3+n} = 3- \frac{9}{3+n}$

$= 2$ iff $n = 6$.

$m=3$, $n=6$ is the only solution, giving $mn = 18$.

20.1.3. To find a relation between the height of a regular tetrahedron and the perpendicular distances from a point inside, to the four faces.

Let H be the height, A be the area of each face and h_1, h_2, h_3, h_4 be the perpendicular distances from O inside.

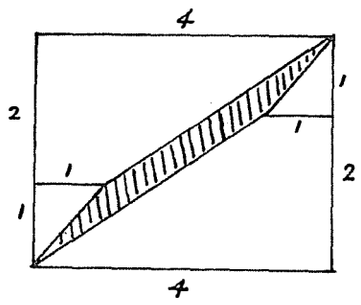
Volume of the tetrahedron = $\frac{1}{3}$ AH.

Sum of the volumes of the smaller tetrahedra, with O as the common apex = $\frac{1}{3}Ah_1 + \frac{1}{3}Ah_2 + \frac{1}{3}Ah_3 + \frac{1}{3}Ah_4$.

$H = h_1 + h_2 + h_3 + h_4$.

20.1.4. To find the hole in using 11 square units to cover up 12 units.

A properly drawn diagram makes the discrepency obvious:



20.1.5. To solve for $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}$

Obviously: $x = \sqrt{1 + x}$, $x^2 = 1 + x$

or $x^2 - x - 1 = 0$ $x = \frac{1}{2} \pm \sqrt{\frac{5}{2}}$ is the exact solution. $x = \frac{1}{2}(1 + \sqrt{5})$ if only the positive value is used, as required.

The ideal condition
Would be, I admit, that men should be right by instinct;
But since we are likely to go astray,
The reasonable thing is to learn from those who can teach.

Sophocles, Antigone, Ode II

INTERVIEW

with
Bob Smith

Dr. Jim Heddens is Professor of Mathematics Education at Kent State University where his graduate program trains teachers to qualify for the Ohio State Mathematics Clinician certificate. Ohio is the only state at the present time which has certification for remedial mathematics although Florida, Arizona, Kansas and Maryland are considering standards for such certification. It was my pleasure to visit Jim at the Mathematics Clinic in the University School in April to discuss diagnostic and remedial mathematics.

R.S. - Jim, somewhere in your career you became interested in diagnostic and remedial arithmetic. What made you take up this cause?

J.H. - I was a mathematics major in Iowa, then did my undergraduate work with Henry Van Engen at the University of Northern Iowa. Over the years I taught at all levels and I kept noticing children who were having difficulty with mathematics. I kept wondering why they were having so much difficulty. I was drawn to those that I felt really needed help. From that it was a natural move into teacher education where I have pioneered courses in remedial mathematics teaching. The reading people had put together a remedial reading program so it seemed to me that a similar program was needed in mathematics.

R.S. - What are some of the basic elements in your program?

J.H. - First, we feel we need to reinforce the mathematics content which students bring with them to the University. They must feel confident in mathematics of the elementary and secondary school as pupils of any grade may need their assistance. So, in consultation with the Department of Mathematics at Kent State University, Arithmetic, Algebra and Geometry Courses were prepared. These courses aim at the basic understanding of Mathematics - not just another course on calculating, not a prerequisite to a university mathematics course but courses valuable for those who will need to teach the discipline.

R.S. - I have often said that none of us know enough Mathematics, so I am glad to hear your courses are written with teachers in mind. What next?

J.H. - Second, for me, mathematical structure is the bottom line. For the past 15 years we have been trying to sequence a checklist of mathematical concepts. We tried to take direction from theorists such as: **Piaget, Gagne, Bruner** and **Skinner** and many others. Even with the help of their theories, I am not sure that we really know how children learn. So over the years, we have kept case studies of the children who attend our clinic. These have become very detailed in terms of the ways children function. At the same time, I have become convinced that every year every child ought to have something to do involving number and place value. There should be some learning with respect to the operations, with geometry and that these topics should be hit at every grade. So that there will be a different level of sophistication, we developed a summary of mathematical concepts organized under appropriate sub-headings named Concept Clusters. Items were sequenced in logical order but since children do not learn mathematics in a linear fashion, concepts should be introduced, developed, and continually reinforced through a spiral curriculum.

The following is an example taken from our book.[1]

E. CONCEPT CLUSTER-PLACE VALUE

1. Distinguishing the difference between ten ones and one ten, using counters and rubber bands.

2. Writing the appropriate numeral in expanded notation and standard notation, given the model of a one or two-place number.

3. Using counters and rubber bands to model the number represented by a numeral (1-99) in standard notation.

4. Rewriting the numeral in expanded notation relating the word-name for place value with the numeral (34, 3 tens, 4 ones, and 30 + 4).

5. Extending 1, 2, 3, above through hundreds place.

6. Renaming numerals in several different ways (273 can be renamed as 273 ones; 27 tens, 3 ones, 2 hundreds, 7 tens, 3 ones).

7. Renaming numerals with zeros as place holders in several different ways. (300 can be renamed as 300 ones; 30 tens, 0 ones; 3 hundreds, 0 tens, 0 ones).

8. Extending the concept in #1, 2, 3 through thousands place.

9. Extending the concepts in #6, 7 through thousands place.

10. Approximating numbers to the nearest:
    a. ten
    b. hundred
    c. thousands
    d. extend

11. Generalizing the relationship between any two adjacent place values:
    a. whole numbers (right to left)
    b. whole numbers (left to right)
    c. rational numbers expressed as decimals (left to right)
    d. rational numbers expressed as decimals (right to left)

12. Generalizing the relationship between any two non-adjacent place values:
    a. whole numbers (right to left)
    b. whole numbers (left to right)
    c. rational numbers expressed as decimals (left to right)
    d. rational numbers expressed as decimals (right to left)

13. Expressing expanded notation in exponential form:
    a. positive integral exponents
    b. zero as an exponent
    c. negative integral exponents

Introduction	In Process	Mastery

R.S. - I understand that successive teachers would update a student's chart periodically. What next?

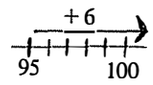
J.H. - We believe children learn from concrete examples, through a semi-concrete and semi-abstract stage to the abstract level. I realize both Piaget and Bruner suggest only three levels. Various names have been given to the middle level: **mental imagery, iconic, concrete-operational,** yet in my writing, I have kept to the ideas of semi-concrete and semi-abstract. Usually the former is a picture, as a picture of an automobile stands for the real automobile. When I use something that stands for that picture, such as a counter, it seems to me you have another step in the mental process. Much emphasis is placed on these stages in our program and, in particular, the developing of mental images.

R.S. - How do you know when a child has a mental image in his mind?

J.H. - He talks about it, especially if we encourage him and allow for discussion. From students' talk we can look at their thought process. We need to get into children's minds and examine how they think mathematically. Many function mechanically only. Piaget's theory of accommodation and assimilation suggests ways to help such children. When we present an idea to a child and it fits into his logical structure, he can assimilate it . . . it makes sense to him. When we present an idea

which does not fit into his logical structure, he must go back and reorganize or accomodate in his thought process in order to get that new idea to fit. Confront a child making a pattern error with a situation where the answer and the pattern error are at odds and he must make an accommodation, e.g.

$$\begin{array}{r} 6 \\ +\,95 \\ \hline 911 \end{array} \qquad \begin{array}{r} 47 \\ +\,18 \\ \hline 515 \end{array} \qquad \begin{array}{r} 26 \\ +\,97 \\ \hline 1113 \end{array}$$



R.S. - Very interesting. But you still have yet to say anything about the clinical components.

J.H. - Right. We have a clinical practicum in diagnosis and one in correction. In the former we assure that students will have the skill to administer standardized diagnostic instruments, to interpret the data, to devise informal diagnostic tasks and to prescribe effective corrective teaching strategies. One idea I have used effectively in our clinic is to have two graduate students work with one child. One college student is the clinician or diagnostician, the other is the observer. The observer keeps a detailed record of the child's behaviour. How does the child respond? Does the child look at you when answering or look down? Such behaviour patterns may help to suggest meanings. The two college students are to analyse the child's difficulties in mathematics. It is very difficult for us as teachers to hear what children are saying. We ask a question. We have a predetermined answer in our mind and too often, if the child does not give that answer, we forget what the child has said. I don't believe children automatically give wrong answers. In his mind the child believes the answer to be correct. We need to do more work with wrong answers. Allow the child time to talk, to explain his thinking. It helps him clarify what he has done, and reveals his thoughts to you.

R.S. - Yes, but it doesn't help the teacher who is attempting to get on with the lesson in her classroom.

J.H. - What is the value to advance the lesson if the pupil is not understanding it? It seems to me that we need to spend more time with basic understandings and less time on calculation. We need far more verbalization by the children to find out what they are thinking. Then we must deal with the way they are thinking to try to get the mathematics across. Many teachers think that they can follow a textbook, teach sequential lessons. I think the children will tell us what they need tomorrow if we are observant. The kinds of difficulties displayed today really will dictate what kind of help I must give tomorrow.

R.S. - Jim, you were the prime mover and organizer of the Research Council for Diagnostic and Prescriptive Mathematics.

J.H. - Yes, it all started here at Kent State University. Our first conference took place here and I was the first president. It is my hope that council will not only be a forum for discussing findings in the field but that it might organize the research and development work to prevent overlap. It seems to me that different people can be doing different segments and thus move research ahead much more quickly.

R.S. - As a leader in the field, what are your plans for the future?

J.H. - I feel we need more sophisticated diagnostic tools so I'll be working on new tools for diagnosing children's mathematics difficulties as well as writing about this area.

R.S. - Could you give us just a preview of the tools which you see coming?

J.H. - Diagnostic instruments need to be much more detailed than what we have at the present time, and they need to be organized differently. When we are diagnosing children, we have voluminous material to diagnose. You need a systematic technique. Let me give a quick example. Let's consider a concept cluster of mathematics content, say addition, and I could sequence it logically from beginning to end. Then I might use a diagnostic technique in which I might take every 10th entry and develop a diagnostic item to which a child is to react. The clinician observes the child's behaviour not considering if answers are right or wrong. When I find that he has a series of correct answers and an incorrect answer, I can begin to zero in on the items that are between the selected items. I think we also have to examine the prerequisite knowledges. If the child misses an example in multiplication, did the child miss it because he didn't know what multiplication was? Did he miss it because he doesn't know the basis facts? Did he miss it because he used improper place values? Did he miss it because he couldn't regroup properly? Did he miss it because he couldn't add correctly when he had found the partial products? You see, there may be many reasons why children miss mathematics examples.

R.S. - Thank you Jim, I'll look forward to reading your new book.

1. Underhill, R.G., Uprichard, A.E., Heddens, J.W., **Diagnosing Mathematical Difficulties.** Columbus, Ohio, Charles E. Merrill Pub. Co. 1979, p. 277.

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THE PROPOSED COMPUTER SCIENCE CURRICULUM

by
Charles Pinnegar

Three weeks ago, while attending a Ministry of Education summer course, I was given a copy of the Summer, 1981 issue of "Computers in Education". On page 1, much to my surprise, I learned that a small committee chaired by a business teacher from Peel County had produced "Stage One" of the new computer studies/guideline. Now, I pride myself in keeping in close contact with my discipline, but I had no idea this guideline was in progress. Upon discussing this with several colleagues, I found none better informed that I.

The final draft of this guide -- "Stage Two" -- is due for completion in late May or early June, 1982. It occurred to me that before that date many computer science teachers, through the pages of the *Gazette,* could state their opinions on the crucial issues facing the Stage Two committee. The purpose of this article, then, is to provide a focus for this discussion. Hopefully, the Ministry guideline committee will consider our INPUT.

1. *What should be the principal objectives of this guide?*
According to the article in *Computers in Education,* "computer awareness in Grade 9 or 10", an awareness of the "social issues related to the computer and about the responsibilities of the individual in the computer age" and a definition of "the role of both business mathematics and technology in teaching about the computer", are three objectives. What is computer awareness? Can you teach it without programming skills? At this stage in history, we do not know clearly what the social issues are. We have no historical perspective since we are a **part** of the computer revolution. It is my opinion that educators need more concrete, realistic objectives to work toward. I suggest that our primary objective is that every student -- advanced, general and occupational -- at some time during his secondary school years be exposed to a "hands-on" experience with a computer. This should be through a structured course of study and not just as an "extra".

2. *What types of courses will these objectives lead to?*
I believe that one computer studies course which is to cover all students, is a ludicrous idea. Different groups will be expected to reach different levels of achievement, but these levels will be related to them and not to other groups. All three levels should be expected to write basic programmes. However, those students who demonstrate a flair for programming should have an advanced course to strive toward. The preliminary course for general and advanced students should include all the usual keywords and as far as the use of double-subscripted arrays. The advanced course should stress file-handling and give the student a better idea of what happens inside the computer. The course for occupational students could be incorporated into their present mathematics courses. Stress would be placed on solving problems with the computer which the student has met in his math or science courses. This computer component would not be a full course but would be part of the existing course.

3. *What skills should we expect?*
Each student should be able to write and understand elementary programming in the BASIC language. Moreover, he should be able to apply these skills to practical, everyday problems. In addition, he should know when and if the computer should be used as a problem-solving aid.

4. *At what grade levels should computer science be taught?*
I cannot see the advantage of teaching "computer awareness" in grade 9 or 10. If you teach a preliminary course in grade 11 for advanced and general students, then there is a vast reservoir of mathematics and science problems waiting for computer application. Then, a grade 12 advanced course can challenge the skilled programmer.

5. *What is the relation between computer science and data processing?*
This is one area to which the committee must address itself, since any lack of definition will result in these two disciplines being treated as synonymous. I suggest that data processing applications follow a preliminary computer science course if the data course stresses the use of the computer. The topic of file handling may be placed here and not in an advanced computer science course.

6. *How prescriptive should the guideline be?*
I favour a guideline that lays out exactly what is to be taught as core and what may be taught as optional.

7. *Does the "hands-on" experience in the principal objective exclude other disciplines from participating?*
Initially, this is probably true since the necessary expertise in other disciplines may not yet be available. However, this should not be allowed to continue. One of the recommendations this committee should make to the Ministry is the formation of another committee in each subject area to study the ramifications of the computer revolution in their disciplines.

Of course, these are only the issues I see as important. Perhaps you totally disagree. Fine, at least write a letter to the Editor of the *Gazette* and state your opinion. Remember, there is not much time until the "Stage Two" curriculum is produced. How will it affect you?

Head of Mathematics,
Dunnville Secondary School

CANADA ENTERS THE INTERNATIONAL MATHEMATICAL OLYMPIAD

by
Ed Barbeau

After years of waiting, the Canadian Mathematical Society decided this year to send a team of high school students to the International Mathematical Olympiad. This year's competition was in the United States.

Professor Geoff Buter, of the University of Alberta, looked after all the arrangements, including the choice of the team. Eight students were selected to go to the U.S.A. for the I.M.O.:

David Ash, from Fort William C.I. in Thunder Bay, now living in Peterborough;
Arthur Barager, from Old Scona Academic High in Edmonton;
David Bernier, from College des Jesuits in Quebec City;
John Bowman, from Old Scona Academic High in Edmonton;
John Chew, from University of Toronto Schools;
Georges Gonthier, from Montreal, but attending Lycee Louis-le-Grand in Paris, France;
Cary Timar, from Sarnia, but attending University of Toronto Schools;
Julian West, from St. George's School in Vancouver.

Professor Butler, assisted by Professor Ed Barbeau, conducted a training session at the University of Saskatchewan and accompanied the team to the United States.

Twenty-seven countries took part in the I.M.O. A full delegation consisted of a team of eight students plus two adults; however some countries sent fewer students. Each country had in advance submitted up to five questions and this list had been reduced by the host country to a set of nineteen possibilities, from which six were to be selected for the competition by a jury containing one adult from each country.

On July 12, the students travelled to Georgetown, Washington D.C., the site of the contest. The following day, after a brief opening ceremony, they were given the first paper - three problems to be done in 4½ hours. There was a second paper on Tuesday. Marking was done by each student's own adult delegates and verified by panels. Of 185 students, no fewer than 26 achieved a perfect score of 42. About half the students received a first, second or third prize, a first prize going to those with 41 or 42 points. Each prize - winner received a medal, along with either a digital watch, HP calculator

or a backpack emblazoned with the I.M.O. emblem. The awards ceremony took place on July 19 at the National Academy of Sciences in Washington.

Students were scored individually. Unofficially, the total scores for the top eight countries were:

United States	314	Bulgaria	287
Federal Republic of Germany	312	Poland	259
United Kingdom	301	Canada	249
Austria	290	Yugoslavia	246

The U.S.S.R., shorthanded with only six team members, had to be satisfied with 230 points and ninth place; its average score per student however, was between that of West Germany and Britain. The Hungarian team was a small powerhouse with four students who scored 41, 42, 41, 40. The highest average was achieved by Luxembourg with only one student, who wrote a perfect paper. Seven gold medals were given to contries with at least two perfect scores: Austria, Canada, France, West Germany, U.K., U.S.A., U.S.S.R.

We were delighted with the performance of the Canadians. First prizes went to David Ash and George Gonthier, who wrote perfect papers. Second prizes went to Cary Timar (37) and John Chew (35) and a third prize went to Julian West (29). The most prominent high school in the Olympiad was Lycee Louis-le-Grand in Paris with seven student participants, - five on the French team, one from Luxembourg and G. Gonthier from Montreal - accumulating a total score of 217.

Our country has now taken the plunge into international competition and showed that it can hold its own very well. The question is: will we stay in? I hope we can. Hungary has tentatively offered to host next year's event, but sending a Canadian team over to Europe requires a great deal of money.

Anyone interested in helping to identify and train future competitors, make a donation or help in fund raising should get in touch with E.J. Barbeau, Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 1A1 (416-978-8601).

XXII International Mathematical Olympiad

Washington, D.C., U.S.A.

July 13, 1981

Time: Four hours and thirty minutes.

Each problem will receive equal weight.

1. P is a point inside a given triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least.

2. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Also consider the least number in each of these subsets. $F(n, r)$ denotes the arithmetic mean

of these least numbers; prove that

$$F(n, r) = \frac{n+1}{r+1}.$$

3. Determine the maximum value of $m^2 + n^2$, where m and n are integers satisfying $m, n \in \{1, 2, \ldots, 1981\}$ and $(n^2 - mn - m^2)^2 = 1$.

XXII International Mathematical Olympiad

Washington, D.C., U.S.A.

July 14, 1981

Time: Four hours and thirty minutes.

Each problem will receive equal weight.

4. (a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n-1$ numbers?

   (b) For which values of $n > 2$ is there exactly one set having the stated property?

5. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle, and the point O are collinear.

6. The function $f(x, y)$ satisfies

   (1) $f(0, y) = y+1$,

   (2) $f(x+1, 0) = f(x, 1)$,

   (3) $f(x+1, y+1) = f(x, f(x+1, y))$,

   for all non-negative integers x, y. Determine $f(4, 1981)$.

Department of Mathematics
University of Toronto
Toronto, Ontario, M5S 1A1

THE JOHN G. McCARTNEY SCHOLARSHIP

This Scholarship valued at $400.00 has been established to honour John G. McCartney (1939-1977) in recognition of the many innovative contributions he made to learning in the field of mathematics.

John was born in Glasgow, Scotland and came to Canada with his parents as a young boy. He attended secondary school and teachers college in Peterborough and studied at Queens University in Kingston. In 1967, after having taught both in Kingston and Stoney Creek, he joined Science Research Associates as editor of Educational Programmes and advanced quickly to become Vice-President of Development. John McCartney was a highly productive and innovative person who instigated numerous educational programmes which at the time of his untimely death in 1977 were used in forty countries around the world.

The McCartney Scholarship is awarded to an Ontario student who demonstrates the highest proficiency in three Grade 13 mathematics courses and shows profound mathematical insight. Application forms for the scholarship are sent to all Heads of Mathematics of Ontario Secondary Schools. However, only one application per school may be made on behalf of a student who is enrolled in that school and who intends to proceed towards post-secondary study in a mathematically oriented field. The Scholarship is administered by the Secretary-Treasurer of the O.A.M.E./A.O.E.M., Morley MacGregor. The judges this year were Paul Pogue of Barrie and Tom Griffith of London. The deadline for applications is June 1.

The 1981 winner of the McCartney Scholarship is **Gregory Plaxton,** of London, Ontario. Greg attended London French - and Bishop Townshend Elementary Schools and subsequently Central Secondary School in London. Principal Roy Yates of Central Secondary calls him the "most exceptional student I've come across in 27 years" a sentiment that is echoed by the head of Central's math department, J.R. Abel. With 100% in Calculus and 99% in both Algebra and Functions and Relations, his grade 13 marks were indeed impressive. Morley MacGregor and his wife Mona got to know Greg very well last summer and found this outstanding young scholar a very interesting and highly intelligent man.

Greg is now enrolled in an Engineering course at the University of Toronto. Congratulations to him for a fine achievement and best wishes for the future.

Hans P. Heinig

BOOK REVIEWS

TEACHING SECONDARY SCHOOL MATHEMATICS TECHNIQUES AND ENRICHMENT UNITS

Alfred S. Posamentier and Jay Stepelman. 1981, 408 pages, $19.80 paperback. ISNB 0-675-08033-9. Charles E. Merrill Publishing Division, Bell and Howell Ltd., 230 Barmac Drive, Weston, Ontario, M9L 2X5.

Both of the authors of this book are involved in education in New York City; Dr. Posamentier is Professor of Mathematics at The City College, City University of New York and Mr. Stepelman is Supervisor of Mathematics at George Washington High School. In writing this book they have drawn upon the experiences of colleagues and experts in the field of education.

This book was primarily written as a textbook for teacher training but is also designed as a reference in a departmental or personal mathematics library. It is in the context of the later use that this review is written.

As the full title suggests this book is written in two parts and they are quite independent of one another. The 'techniques' portion of the book entitled **Teaching Mathematics** includes 16 chapters. The authors deal with the standard areas such as planning lessons, questioning, motivation, tests and classroom management as well as special focuses like problem solving, calculators and computers, teacher evaluation and dealing with mathematics anxiety. The 'enrichment units' portion of the book entitled **Enrichment Units for the Secondary School Classroom** includes 118 enrichment units

for students in the Intermediate and Senior Division.

The very first chapter of the first part of the book is called Planning a Lesson and an experienced teacher might not expect too much from it. However, through the use of examples of both good and bad lesson plans and a variety of suggestion, the authors succeed in both supporting the good practices of teachers and challenging them to further refine their methodology. Unfortunately, not all chapters on methodology are equally stimulating. Other chapters which might be particularly valuable to teachers were one concerning evaluation and one dealing with mathematics anxiety.

Some of the weakness of this book, such as the use of Imperial Units and the obvious American frame of reference can be easily adjusted for by the user, however, a more serious weakness is the index. The index includes a complete listing of topics from the first part of the book but has two shortcomings. Firstly, the book contains several bibliographies, lists of resources and project suggestions which are difficult to locate through the index. Secondly, the entire second part of the book (the enrichment units) is excluded from the index. While this weakness does not detract from the content it certainly will make it more difficult to use as a handy reference.

In their introduction to the second part of the books, the authors have pointed out that we often consider enrichment as the special topics for gifted students but we should attempt to provide enrichment for all our students. They have then proceeded to provide 118 units and have attempted to classify them from grades 7-12 and as suitable for remedial, average or gifted classes. Some of these classifications seem to be a little optimistic, however, they do admit teachers will have to take the basic idea and modify it somewhat for some classes.

In general, the units are well written, providing sufficient background for the teacher, suggested teaching strategies and follow-up questions.

Despite the weakness mentioned earlier, this book is a worthwhile reference for individual teachers and a source of in-service for mathematics departments.

Trevor Thomson,
Mathematics Consultant,
Hamilton Board of Education,
Hamilton, Ontario

THE FIRST SYSTEM OF WEIGHTED DIFFERENTIAL & INTEGRAL CALCULUS

J. Grossman, M. Grossman and R. Katz; Archimedes Foundation, Rockport Mass. 1980 (pp. 55) $3.00.

In this short soft cover book the authors discuss a weighted calculus which is more general than the traditional calculus. However, with this generalization much of the simplicity of calculus is sacrificed and the relationship between integral - defined as limit of partial sums - and derivative is lost.

Let w be an arbitrary selected integrable function of a real variable, then the w-derivative of a function f at a point a is defined by

$$(\underline{\underline{D}}f)(a) = \lim_{x \to a} \frac{f(x) - f(a)}{W(x)}, \qquad W(x) = \int_a^x w(t)\,dt,$$

provided this limit exists. This derivative is more general than the usual derivative in the sense that $\underline{\underline{D}}f = Df$ if $w \equiv 1$. The derivative $\underline{\underline{D}}$ is a linear operator, however the authors do not give the product or quotient rule, nor do they give the formula corresponding to the chain rule. Further the derivatives $\underline{\underline{D}}$ and D are **not** comparable, because if

$f(x) = x^{3/2}$, then $(Df)(0) = 0$, however, if $w(x) = x$ then

$(\underline{\underline{D}}f)(0) = 2\lim_{x \to 0} \frac{1}{\sqrt{x}}$ which does not exist. Conversely, $g(x) = |x|$,

as is well known, fails to have a derivative at zero. However, if one takes $w(x) = 1$ if $x \geq 0$ and -1 if $x < 0$, then w is integrable and $(\underline{\underline{D}}g)(0) = 1$. This example seems to contradict a statement on p. 19 that

"if either (derivative) exists, then so does the other".

The Fundamental Theorem of Calculus relates the concept of area defined by a function to the antiderivative of that function. The integral is of course here defined as limit of partial sums. If one would define integral by

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is the antiderivative of f, then it would not be clear why the left side of the equation should represent an area. Certainly Archimedes would be most unhappy with this definition, moreover the Fundamental Theorem of Calculus would be a triviality. But this is precisely the definition of weighted integral. Thus

$$\int_r^s f = h(s) - h(r)$$

where h is the w-antiderivative of f. Naturally the Fundamental theorem of w-calculus holds. The w-integral is related loosely to the Riemann-Stieltjes integral - which of course is defined as limit of sums. It is regrettable that the authors did not choose to give conditions on f and w for which the integral exists nor did they include any new applications. It is in this direction where the book might have been informative.

Sometimes the text is confusing, thus in the preface it is assumed the reader need only have a working knowledge of rudiments of calculus while on p. 29 the reader is assumed to know Chapter 5 and 6 of *Non-Newtonian Calculus*. If this is not enough, there is no reference of the text *Non-Newtonian Calculus*.

Perhaps there is a need for a book on weighted calculus from an elementary viewpoint, I am not convinced however that this need is satisfied with this booklet.

H.P. Heinig,
Department of Mathematical Science,
McMaster University,
Hamilton, Ontario

COUNCIL COMMENT

by
Bob Robinson

Ed. Note:
In order to keep you informed with regard to the workings of the OAME Council, we will be including this feature in the issue following Council meetings. The Editorial Panel hopes that you find this section of interest. Your comments are welcomed and should be addressed to:

The Editor
Ontario Mathematics Gazette
c/o Mathematics Department
McMaster University
Hamilton, Ontario
L8S 4K1

COUNCIL MEETING

The meeting was held Saturday, 1981 10 17 at the Ramada Inn in Mississauga.

CHAPTERS

COMA (Carleton-Ottawa Mathematics Association):

The president of COMA is Ann Jones who can be reached at the Ministry of Education, Eastern Ontario Region in Ottawa. A membership drive is currently under way. The COMA Council includes representatives from private schools as well as boards, colleges and universities.

COMA has volunteered to undertake the "Mathematics Interface Project" sponsored and funded

by the Ministry of Education and the Ministry of Colleges and Universities. The project, the only one of its kind in Ontario, will study the mathematics curriculum at the level of transition from secondary school to the college and to the university level. The chairman of the committee is Dr. Joe Byrnes and a report of the conclusions and proceedings is due to the Ministry for June, 1983.

A Grade 13 pilot examination was held in the spring and a questionnaire was also circulated with results reported. Berry Nichole is chairman of the Examination Board.

COMA has made a submission to SERP which Fred Maskell (Algonquin College) co-ordinated.

Fred Maskell continues as the editor of Crux Mathematicorum. Those of you interested in subscribing to this interesting journal can contact him at Algonquin College. Of the subscribers 25% are from overseas, 35% - U.S.A. and 44% - Canada.

A finance committee has been established and it now has charitable status. Bruce Armstrong (Carleton Board) is chairman.

Golden Section (Hamilton-Niagara Region):

The Hamilton section will hold a spring meeting mainly on the topic of computers.

GVMA (Grand Valley Mathematics Association):

The fall General Meeting is scheduled for Wednesday, 1981 11 04 in the evening. The theme is "Gold Mines and Nuggets". The pre-dinner part of the program offers a choice of five one-hour presentations which range from enrichment to survival topics. The after dinner part of the program includes a major presentation and 16 fifteen-minute "Nuggets" grouped into four themes. For further information, contact Tom Rajnovich, (519) 885-4620.

GVMA's newest publications are entitled **Recreational Math Problems For High School, Part II** and **Grade Nine Sample Examinations.** For the price of these publications and a complete list of all their publications, write to: Mr. Ron Scoines, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1.

The Gauss contest continues to grow with close to 30,000 copies sold last year.

Donna Cousineau, (519) 669-5414 is the new chairperson of GVMA. A position of elementary co-ordinator has now been created on GVMA Council.

NOMA (Northern Ontario Mathematics Association):

The 12th Annual NOMA Fall Conference was held in North Bay and hosted by Bob Tuck. Over 50% of the attendees were from elementary staffs . . . Congratulations!! All reports indicate a most successful meeting.

Pine Ridge (Oshawa-Peterborough Region):

The Pine Ridge chapter is being revitalized. Paul LaFrance indicates that they hope to hold two workshops . . . one in April and one in June. Paul may be contacted at: (416) 725-6856 in Oshawa.

QSLMA (Quinte - St. Lawrence Mathematics Association):

By the time you read this, the Tuesday, 1981 10 27 meeting at Ernestown S.S. in Oddessa will be history. Here are some of the topics presented: "Hands-On Materials for Primary Maths", "Enrichment Activities in Geometry", "Computers As Simulators of Genetic Phenomena", "Maths At MacDonald's", "A Finite Geometry". The President for 19811/82 is Peter Battey-Pratt, Ernestown Secondary School.

SAME (Scarborough Association for Mathematics Education):

This chapter replaces SEYMA (Scarborough-East York Mathematics Assoc.). Jeanette Jones represented SAME at the Council Meeting.

TORNY (Toronto-North York):

Gordon Barry is the president of TORNY and he can be contacted at: (416) 292-7337. A very suc-

cessful half day conference was held on Saturday, 1981 10 03. The theme or topic was **Left Brain -Right Brain - What Are The Implications Of This Classification For Learning?** TORNY lined up speakers who made interesting and provacative presentations.

TORNY is planning another meeting for January. For more information, contact Gord Barry.

CONFERENCE GUIDELINES

OAME now has a set of Conference Guidelines. A copy of these guidelines may be obtained from the President or Secretary-Treasurer of OAME.

OAIP - ONTARIO ASSESSMENT INSTRUMENT POOL

There have been some financial constraints pertaining to the funding of OAIP. Items are being organized to be sent to schools. A committee is being formed to work on testing, evaluating and perhaps revising test items. Peter Ulrich (Ottawa) and Ron Sauer (Waterloo) have been appointed as OAME representatives on this Junior Mathematics Working Committee co-ordinated by Jim McLean, Ministry of Education.

SERP - SECONDARY EDUCATION REVIEW PROJECT

President George Knill received notification that his letter (see September, 1981 Gazette) had been received. More information may be available my mid November.

SENIOR CYCLICAL REVIEW

The Ministry has received the report and no answer is available at this time.

SIZE OF COUNCIL

The OAME council accepted in principle that the council be reduced in members from 40 to 30. A plan will be presented to the Executive at a later date.

1981 ANNUAL-SARNIA CONFERENCE

The final report was presented and council on behalf of all those who attended complimented Sharon and David McPhail and Mona and Morley MacGregor for their outstanding leadership which made this conference such a success.

1982 ANNUAL-TORONTO

Plans are on schedule for the NCTM Annual in Toronto (first time in Canada) from 1982 04 14-17 which will be hosted by OAME. If you wish to assist, contact the chairman of local arrangements:

Dave Alexander
Faculty of Education
University of Toronto
371 Bloor Street West
Toronto, Ontario
M5S 2R7

A poster, designed by a Barrie student, depicting this event is being distributed for advertising purposes. For more information, contact Paul Poque (705) 728-0134.

Ralph Connelly (NCTM Representative) has received a report of the costs of hotels and registration.

It was reported that much interest is being shown by our American colleagues to attend this exciting first time event in Canada. See you there!

NCTM

Ralph Connelly suggested that any orders for NCTM materials be sent through him as OAME gets a 20% rebate on NCTM publications sold. Cheques or money orders should be made payable to NCTM in U.S. funds or equivalent. Contact Ralph Connelly, NCTM Representative, 11 Berkwood

Place, Fonthill, Ontario, L0S 1E0.

The Nominations Committee of the NCTM has announced the eight candidates for next year's four positions on the Board of Directors. They are: Forest Coltharp, John Dossey, Patricia Hess, Henry Kepner, Bob Robinson, Wallace Rogelstad, Thomas Rowan and James Sherrill. Biographical data and an official ballot will be sent to each NCTM member in January.

The November issue of the **Mathematics Teacher** is devoted to micro-computers in Mathematics education. This would be a good time to join the NCTM and enjoy a reduced registration fee for the "Annual" in Toronto next April and receive an up-to-date listing of NCTM publications.

1983 ANNUAL

The 1983 Annual Conference of OAME will be held in Waterloo under the auspices of GVMA.

PUBLICITY AND MEMBERSHIP

A draft of a new membership form is being developed which will be ready for the February Council Meeting.

PUBLICATIONS

Claire Zeller (Peel) has been appointed editor of the **Abacus** for the next two years. Any articles or ideas for either the **Abacus** or **Gazette** should be sent directly to the editors. The Gazette editorial panel is looking for articles at the Elementary and High School levels especially in the Mathematics exposition area as described on page 3 of the September issue.

ABACUS Editor:

Mr. Claire Zeller
Math Head
Morning Star Secondary School

COMPUTERS

The OAME committee on computers feels their main aim is to investigate the use of the computer as a tool for teaching mathematics in the classroom. There is a vast amount of material being released at the present time. OAME is asking for representation on ECCO.

NEXT COUNCIL MEETING - SATURDAY, 1982 02 13

CALENDAR OF EVENTS

1981 11 04 **GVMA GENERAL MEETING:**
Forest Heights C.I., Kitchener; 4:30 - 8:30; short and long sessions; Computer Science (Tim Crawford), Recursions (Ian Goulden), Statistics (Doug Henderson); Math Discovery and Invention (Paul Schellenberg), Math A Second Language (Claire Zeller), Math Modelling (J.N. Kapur); Dinner included; GVMA members $9.00, non-members $10.00; Contact: Mr. T. Rejnovich, Bluevale C.I., 80 Bluevale Street North, Waterloo, Ontario, (519) 885-4620.

1981 11 05-07 **NCTM - HARTFORD MEETING:**
Six strands: problem solving, computers/microcomputers, calculators, mathematics for the gifted, basic competencies and urban education; Contact: NCTM, Reston, Virginia.

1981 11 12-14 **NCTM - MINNEAPOLIS/ST. PAUL MEETING:**
"Ten Commandments For Handling Surface Behaviour", "Gambling: Applied Probability" (by a professional gambler-turned-educator); "Mind Reading, Fire and Water", "Mathematics Learning From A Neuropsychological Perspective". Contact: NCTM, Reston, Virginia.

1981 12 02-06 **AMERICAN STATISTICAL ASSOCIATION - NCTM; LEADERSHIP CONFERENCE, STATISTICS AND PROBABILITY IN THE CLASSROOM;**

Williamsburg, Virginia. Contact: Dr. Edna Gross, Division of Teacher Education, 3086 Oliver Hall, Virginia Commonwealth University, Richmond, Virginia. 23284.

1981 12 10-12 **WINTER MEETING OF THE CANADIAN MATHEMATICAL SOCIETY:** is held at the University of Victoria, B.C.

1982 04 18-2 **RESEARCH COUNCIL FOR DIAGNOSTIC PRESCRIPTIVE MATHEMATICS (RCDPM) MEETING:**
State University of New York at Buffalo. The Program Committee solicits papers that address all aspects of mathematics instruction for exceptional learners, diagnostic assessment procedures and remedial instruction activities. Proposals of 1 to 2 page length are to include statement of purpose, educational significance, description of subjects research methodology, results and conclusions - should be sent by December 1981 to: RCDPM Program Committee, c/o L.G. Callahan, 505 Baldy Hall, State University of New York, at Buffalo, Amherst, New York, 14260.

1982 04 14-17 **NCTM 60TH ANNUAL MEETING:**
will be held in Toronto. For information: Dave Alexander, Faculty of Education, University of Toronto, 371 Bloor St. W., Toronto, Ontario M5S 2R7 or 1906 Association Drive, Reston, Virginia 220919

1982 04 22-23 **THIRTEENTH ANNUAL PITTSBURGH CONFERENCE ON MODELING AND SIMULATION:**
will be held at the School of Engineering, University of Pittsburgh. The Conference will place special emphasis on microprocessors and their applications. Papers are solicited on social, economic and globel modeling and simulation. Information: William G. Vogt, Modeling and Simulation Conference, 348 Benedum Engineering Hall, University of Pittsburgh, Pittsburgh PA. 15261.

1982 05 12 **QSLMA MEETING:**
Gananoque; 4:00 - 9:00. Contact: Peter Battey-Pratt (613) 386-3054 or David Handley (613) 962-9581.

1982 08 08-13 **FIRST INTERNATIONAL CONFERENCE ON TEACHING STATISTICS:**
Sheffield, England. Contact: Conference Secretary, ICOTS, The University, Department of Probability and Statistics, Sheffield S374H, England.

Thanks to Bob Robinson and Don Kapoor for providing much of this information. Send information pertaining to upcoming events to Bob Robinson, Hamilton Board of Education and/or Don Kapoor, Faculty of Education, University of Regina, or the editor.

MISCELLANEA

For the Learning of Mathematics - An International Journal of Mathematics Education
An international journal of Mathematical Education has been recently started by the FLM Publishing Association under the auspices of the Canadian Mathematics Education Study Group. This publication aims to promote inquiry and research at all levels of mathematics and mathematics education. For further information, write to:
Dr. David Wheeler, Editor, Dept. of Mathematics, Concordia University

About the Leapfrogs Group
Leapfrogs Group is a network of users and producers of books/materials for use in middle and secondary schools. Membership is open to all who wish to join and there is no membership fee. When you register you will be sent a list of the books and materials available in the current "Wave" of publication. You are then invited to maintain your membership by placing an order from at least one of the two bi-annual lists. For further information, write to:
Leapfrog Group, Cold Harbour,
Newton Street, Cyres, Exeter, Devon, U.K.

Gender and Mathematics - Canadian Report

The ERIC Center is planning to publish an International Review of Gender and Mathematics. If you know of any Canadian publications, researches, action programs, . . . on this subject, please write to:

Dr. Roberta Mura, Faculty of Education, Laval University,

Thanks to Don Kapoor for providing this information.

ACKNOWLEDGEMENT

I would like to thank Deane Maynard for her excellent typing and retyping of several articles in this issue.

H.P.H.

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