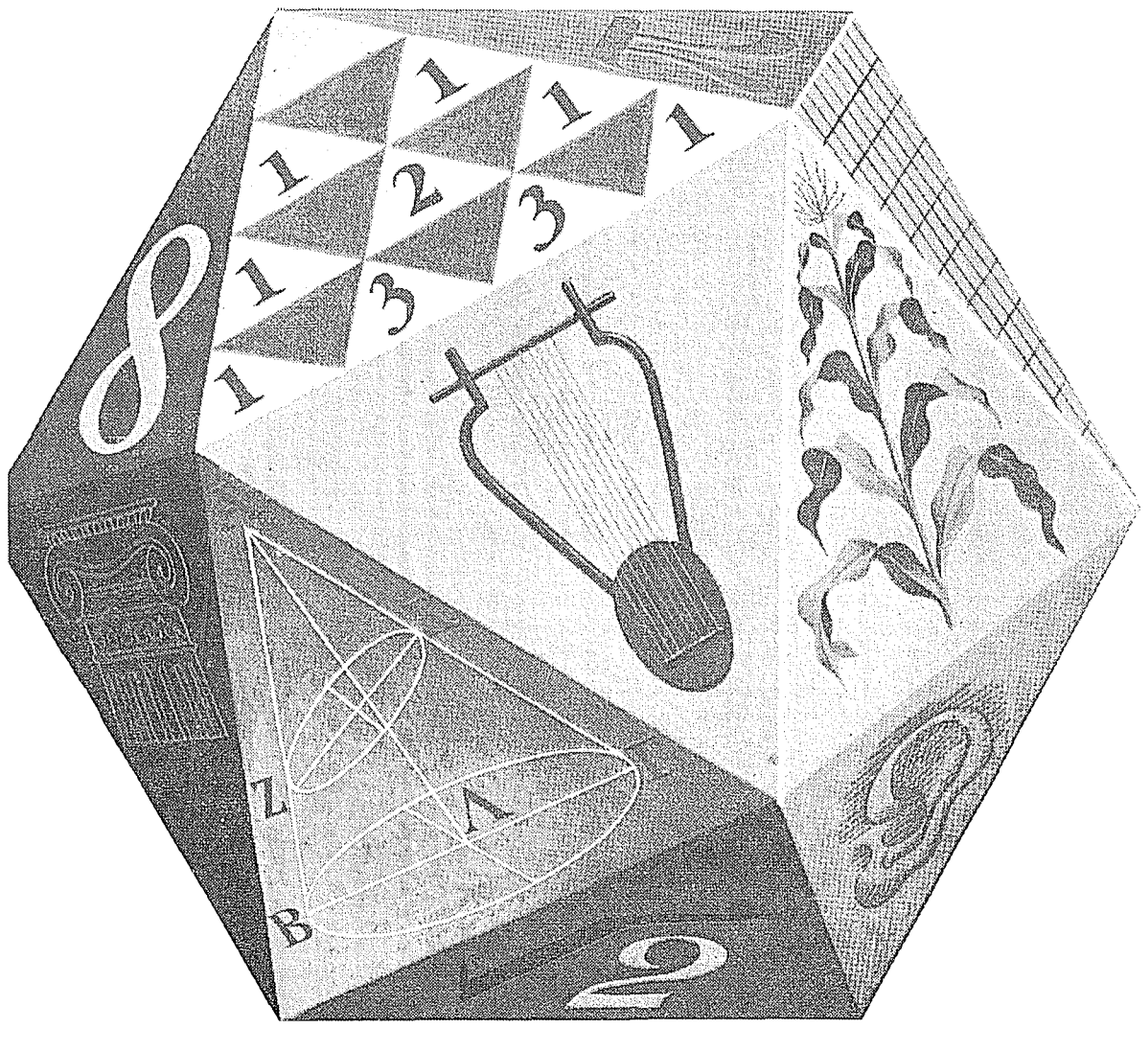
# ontario mathematics gazette

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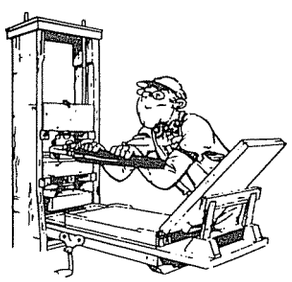
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# EDITORIAL

Well, three years at the helm of the **Gazette** have gone by. Mostly, it seems that time has gone by way too quickly. At times, though (like when you're frantically racing to meet the next publication deadline) it seems like you've been at it forever.

As this goes to press, the future of the editorship of the **Gazette** is still a bit of a question mark. There seem to be many willing **helpers,** but no one eager to be called **the** Editor. Hopefully this will be resolved in the near future.

You'll see a note in Bob Robinson's Council Comment about a possible new look on the horizon for the **Gazette.** This will hopefully have become a reality by the next **Gazette.**

As we reflect back on the last three years, and think of the objectives we had for the **Gazette** when we began as editors, there have been some notable successes and some unfortunate failures. We hoped to achieve more balance in the articles between elementary, intermediate and senior levels. We feel this has been quite successful, and this month's issue should attest to that. We hoped to build up enough articles so that we could consider putting out "special topics" issues when called for. This has proven painfully less than successful. The cupboard is close to bare. The luxury of an issue "in hand" was not to be. Perhaps, with the support of OAME's Council and membership, the next editor will come closer to this goal.

One must confess to feeling a bit of anguish when hearing a discussion of the need for change in the **Gazette** opened with remarks like "Anything's got to be better than what we have now." Feedback from the readers (assuming there **are** readers) has been almost non-existent, but that's not unusual for any publication.

As we close our tenure as editors, we hope that **you,** the OAME membership, will assume an **active** role in the organization. This might be by running for an office, by joining in the activities of the OAME affiliates, attending Conferences, and possibly by contributing to the Organization's publications. What we said at the beginning of our term bears repeating at the end. The **Gazette** is **your** publication and it will become whatever **you** decide it will become.

Ralph Connelly
Sam Pravica
Editors

# MATHEMATICS IN PRIMARY SCHOOLS -- A REMARKABLE DOCUMENT PART I

*by*
*William A. Ewbank*

Twenty years have elapsed since the publication in the United Kingdom of the Schools Council Bulletin No. 1, ***Mathematics in Primary Schools****, a book probably more quoted in the professional literature than any other single work; and if the term "classic" can be applied to the literature of mathematics education, surely a classic if ever there was one. The author, Miss Edith Biggs, HMI, Her Majesty's Staff Inspector for Mathematics in England and Wales, brought a vast wealth of experience to mathematics education, and an outstanding ability to inspire hundreds of in-service teachers with enthusiasm, and a determination to revolutionize the often dull and dreary mathematics class that the establishment and their own upbringing had thrust upon them. The book was placed into the hands of all elementary teachers in the U.K. by the government, and later a hardback for the N. American market was published, ***Freedom to Learn*** (Biggs and MacLean, 1969). This article will sample some of the challenging thoughts and notions of the original work, and perhaps light a spark or two where the dead hand of orthodoxy has engulfed, or reengulfed, the education scene.

A casual reading of ***Mathematics in Primary Schools*** will convince any reader that the work is no dull theoretical treatise. Children, their thoughts and endeavours, enlighten nearly every page. It is evident that Miss Biggs did a great deal more than "inspecting" and lecturing to adults, and the climate that made this possible was the whole movement towards the "new math" of the sixties, and in particular the remarkably successful marriage of Piagetian theories with practical activities in the classroom through the aegis of the Nuffield Mathematics Project.

Every book on teaching mathematics begins with a philosophy of education, and with the aims of teaching mathematics. Biggs is clear on this, "sound and lasting learning can be achieved only through active participation," (p. xv), and she is also clear on what this "active participation" could and should involve. And her aims:

> ...The first is to make children think for themselves. The second is to give children a knowledge and appreciation of mathematics as a creative subject; of its order and pattern (in number as well as in geometrical form); of its vital presence in everyday life and in the environment, not only in man-made things but in natural forms as well. Children who learn to appreciate mathematics in this way, as a subject in which they make their own exciting discoveries and which holds fun and fascination for them, have full confidence in their own abilities. The third aim is to give children a facility with number and quantity relationships or, more briefly, to give them skill in computation.
>
> (p. 10)

One cannot help feeling that the third aim, a worthy one by general consensus, has in N. America (in spite of the invention of the electronic calculator) been allowed to keep the paramount position it has held ever since consumer skills and the training of clerks became the need of society in the last century.

In achieving these aims, we are left in no doubt about the importance of two key components in mathematical learning, "experience" and "discussion,"

> ...the part which oral discussion and written recording can play ... should be assessed ... This oral discussion, between a teacher and a group of children, or between children, serves many purposes. The suggestion has already been made that the learning of concepts is accelerated by discussion. Children frequently learn by their attempts to put into words what they are doing, and what they have discovered ... a child can work through an assignment card and answer every question correctly and yet still fail to grasp important aspects of the concept. Such a failure is usually revealed only in discussion between teacher and child.
>
> (p. 12)

In considering this matter of workcards and discussion, we need to remember that the wind of change that brought in the open classroom in Britain twenty years ago, a movement which was able to thrive there for a combination of historical and organizational reasons, practically rendered the

* *"Primary school" -- the British term embraces Infant (ages 5-7) and Junior (ages 7-11) schools.*

textbook obsolete. And those who have never seen a successful open classroom in action, and who simply pursue the third aim stated above, would not see how there would be much in elementary mathematics to discuss, or if there were, how discipline could be maintained if students were allowed to discuss it. In her book, Biggs shows so many revealing instances of worthwhile activities, and how they are discussed, that one is compelled to the judgment -- perhaps it could work for me.

In making the teaching of mathematics really workable, however, there is some arresting advice that Biggs has:

> In the early stages discussion should take the place of written recording ... The signs + - x ÷ should be introduced fairly gradually ... and not until the children have had a very wide and varied experience of situation requiring these operations.
>
> (p. 13)

The term "situations" here clearly does not refer to drill exercises in a textbook or workbook, or on handout sheets. This brings us to the question of what the British call "apparatus," and what in N. America is referred to as "teaching devices" or "manipulatives." The book does not urge teachers to use apparatus. Everyone had it. Rather the question was, What to use, with whom, and when? Biggs says, "In the early stages of discovering number relationships, some form of structural apparatus is essential." (p. 113). Her example (which sounds comical in retrospect) is to stack twelve textbooks, two lots of six, three lots of four, etc. Nowadays we would do it with twelve calculators! "A plentiful supply of one inch squared paper, cardboard square inches and cubic inches in a number of colours have also proved most useful at different stages." (p. 113). Her own personal preference was for "apparatus made by the children themselves." (p. 113).

Miss Biggs was insistent that early mathematical experiences be "real life," and ones from which the concepts will be abstracted by the children when ready to do so:

> It is particularly important that children's first experiences with numbers, counting and the number sequence, should be with real things in the environment, including the weights and measures of everyday life, adapted for use by young children. This should precede discovery of number relationships from structural apparatus.
>
> (p. 17)

This, she believed, was true, not only for whole numbers, but for fractions:

> Only after wide experience are children ready to understand the significance of the symbols ½, ¼ and ¾, although they may have used the words, 'one half,' 'one quarter,' and 'three quarters' for a long time.
>
> Experience of varied aspects of the operations of addition, subtraction, multiplication and division as these arise in the real situations of the classroom.
>
> We have called these processes 'operations' because, in the first instance, they are (or should be) ***actions performed*** on real materials in natural situations which arise in the environment. (her italics)
>
> (p. 22)

In Biggs' view, "In most of their experiences the fractions they meet should be the simple ones of everyday life and should therefore normally be restricted to those in common use: ½, ¼, 1/8, 1/3, 1/6, 1/12." (p. 45).

The "new math" revolution brought in the idea of "other bases," and in 1964 Dr. Z.P. Dienes was introducing work with his multibase arithmetic blocks to 7-year olds. This was both enjoyable and educational, because children were able to get the feel (literally and figuratively) for place value without seeing it always in terms of "hundreds, tens, ones." Once the meaning of our orthodox numerals is grasped, however, Biggs warns about premature drill in problems like

$$\begin{array}{r} \text{T U} \\ 3\ 7 \\ +\ 2\ 8 \\ \hline \\ \hline \end{array}$$

and makes the startling suggestion that written work of this kind should not be assigned until

children have passed what she calls "a crucial test of readiness ... the ability to add two 2-figure numbers ***mentally*** by an efficient method." (my italics) (p. 27). I found that third graders can learn this skill, and learn it quickly, with the aid of a "hundred square," or 1-100 chart (Ewbank, 1981). Biggs repeats her claim, and clarifies it in ***Freedom to Learn:***

> When are children ready to profit from written practice using tens and units? There is a simple test of readiness for this. Ask the child to add in his head two numbers such as 47 and 35. Ask him to describe how he did this. Suppose he said: "40 and 30, that's 70, and 7, 77." If, in order to add 5 he now counts "78, 79, 80, 81, 82," he is not yet ready for written practice. He requires further use of the number line and strips to establish the pattern 7 + 5, 17 + 5, etc. more firmly.
>
> (Biggs and MacLean, p. 116)

Whether or not one is prepared to take this advice (mental before written) the question will then arise, How much written drill should be given? Biggs also feels strongly about this:

> It cannot be over-emphasised that if children are given written computation of this kind to do before they are ready for it, this is not only a waste of time but can be harmful because such children are forced to use inefficient 'counting on' or 'counting back' methods which, unless discovered, may persist for the rest of their lives. Many adults and school children (primary and secondary), among them some very intelligent ones, never lose the habit of counting in ones when asked to perform a piece of computation. This is often unsuspected by the teacher and can only be revealed when the child is asked to work an example such as 37 + 25 aloud. Very often, too, some of the basic number trios are not known so that the child has a double difficulty to deal with which can only be resolved by inefficient counting methods.
>
> (p. 27)

The matter of multiplication is also tied to the readiness test referred to:

> The readiness test ... is an important landmark in other ways also. For example, before children reach this stage, it is a sheer waste of time to try to get them to learn 'multiplication tables'. This does not mean that they will not understand the operation of multiplication. This understanding will have been based on a wide experience of multiplication (and division) at work in real situations. Nor does it mean that children will not know any of the multiplication facts ... it does mean that it is a waste of time to get children to learn multiplication facts systematically before they have achieved the facility with addition and subtraction implied by the readiness test. For example, if a child needs to know six 7s and already knows that five 7s are 35, provided that he understands what the operation of multiplication means, he will know that he must add another 7 to 35. But if he does not know that 35 + 7 = 42 without counting on in ones, he will be forced to count on to find the answer and may establish a habit of counting on. If teachers wait until children are ready, the learning of the multiplication facts will not be a burden.
>
> (p. 28)

A further aspect of readiness for learning tables is then given:

> Certainly, before being required to know any 'table', children ought to be able to count up in the number under consideration, starting from any number. For example, before learning the multiplication table of 4, it is useful to be able to count on in 4s from, say, 1; thus 1, 5, 9, 13, etc.; from 2; 2, 6, 10, 14 and from 3; 3, 7, 11, 15, etc. Counting on from 0 is the special case of the multiplication table. This necessitates some regular 'oral' practice, (not as a class for not all children will be ready at the same time), but in small groups or even individually when there is a special difficulty.
>
> (p. 28)

Biggs is also insistent upon the importance of commutativity for multiplication facts, "The teacher needs to make sure, at this stage, that the child has discovered the commutative law of multiplication and realises and appreciates its value." (p. 29) Notice "has discovered" rather than "has been taught." This emphasis crops up throughout the book, and many actual instances are cited of how children have discovered mathematics, given the "freedom to learn." One of the most striking is that of Peter (age ten), who discovered that the slope of the curve $Y = x^2$ is 2x, and became "intensely interested" in mathematical patterns -- a flying start not only in analytic geometry, but in the differential calculus! (p. 80)

Can children discover algorithms? Yes, but probably not the most efficient ones, that is, the traditional ones of our textbooks. An algorithm is a system or way of working something out on paper. The example of Peter is, of course, and extreme case, but Biggs believes children can compute without knowing standard procedures. In fact she believes they ought to be allowed to work things out their own way first:

> Teachers are urged to provide ... opportunities for the children they teach so that all may have the advantage of first attempting a solution for themselves. It cannot be over-emphasised that, provided the children's first attempts at written computation are made as records of their own experience, when an efficient method is eventually reached, far less practice will be required to attain and maintain efficiency in this process.
>
> (p. 36)

In the problem solving atmosphere of the open classroom, this is good advice, even if it were given before the days of the hand calculator. Biggs gives an instance of how some 7-year old and 8-year old girls, experimenting with 1-inch cubes, were asked by the teacher to find 17 times 16. Eventually they came up with 272. "With very little direction these girls had discovered the method of long multiplication for themselves. Here was nothing they did not understand and nothing they might forget, because they had invented a method for themselves." (p. 35) This revealing story, one of many dozens throughout the book, reminds us of how children can do complicated mathematics ***through the medium of*** manipulatives. And how skillful the teacher needs to be to ask the right question of the right group at the right time. Telling a whole class of third graders to work out 17 x 16 would go over like a lead balloon!

How can a mathematics class fail to be enjoyable when cubes, Cuisenaire rods (referred to by Biggs on several occasions), and all sorts of hands-on equipment are available? As she states:

> Children brought up to make their own discoveries in mathematics have complete confidence in their own abilities and show an evident enjoyment and appreciation of mathematics, even when it begins to become, as it must, more abstract. They, like Archimedes, can use their imagination in problem solving when materials are not available or when they no longer need these.
>
> (p. 49)

And it will certainly be more enjoyable for teachers (though not easier to teach well) for

> Those teachers who have already organised the work so that the children learn by their own efforts all bear testimony to the decrease in the amount of computational practice children require in order to attain and maintain computational efficiency.
>
> (p. 49)

***Mathematics in Primary Schools*** is a strong breath of fresh air today, as it was in 1965. In fact it is a blast at all the dull and unimaginative lessons, that, one fears, have for so long resisted the wind of change. In the second part of this article, we will focus on Edith Biggs' stimulating advice about geometry, and also quote some of her thoughts on games, and on evaluation.

(William Ewbank is a professor at Taylor University in Indiana)

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# MATHEMATICS IN PRIMARY SCHOOLS -- A REMARKABLE DOCUMENT PART 2

*by*
*William A. Ewbank*

In our previous article we had a glimpse of the philosophy of and methods advocated by Miss Edith Biggs in her post of Her Majesty's Staff Inspector for Mathematics, as seen in the British government publication **Mathematics in Primary Schools** (HMSO, 1965). We will now consider her views in three areas -- games, geometry, and evaluation.

Miss Biggs does not consider "games" as a separate category or phase of learning, but there is evidence that games have a part in her ideal mathematics class. Actual references to games are few, but it must be borne in mind that, in her view, mathematical discovery, with appropriate discussion and recording of such discoveries, should be interesting and exciting ***per se,*** "Every child should experience the joy of discovery" (Alfred North Whitehead, quoted by Biggs, P. 2). She has some advice about games in her discussion of multiplication:

> Practice is required to fix multiplication facts ... and a variety of methods is in use. It is at this stage that number games have a place. (It is important that number games should not be introduced at a stage when they would force children to use inefficient counting methods) ... At this stage, too, when some children require practice and some kind of incentive to make the effort to memorise the multiplication facts, games such as 'Snap' or 'Lotto' can sometimes supply the incentive ... Such games, however, have a limited value and must not be introduced too early.
>
> (p. 31)

By the time Edith Biggs teamed up with MacLean to publish in Canada their ***Freedom to Learn*** (1969), the authors were prepared to make some more categorical statements about games:

> Games are fun ... tremendously useful devices for developing skill in mathematics. Practice in computational skills is just as effective and much more palatable when disguised in a game context. Students devise and sharpen strategies of various kinds when confronted with problems that require advance planning involving the anticipated moves of their opponents....
>
> Mathematical games should be introduced when your pupils are ready for them. For example, if children are given a game like ***Snakes and Ladders*** to play before they know their simple addition facts, they will count in ones and achieve no useful purpose. When encouraged to calculate the number to which they should move without counting in ones, children are having worthwhile practice in addition without being aware of this. In a similar way, a game like ***Lotto*** ... may provide the necessary stimulus, if provided at a time when two children need an incentive to make an effort to memorize these facts.
>
> (***Freedom to Learn,*** p. 50)

This work also contains mention of ***Battleship, Tic Tac Toe, Tower of Hanoi, Mancala, Dr. Nim, Think a Dot,*** and ***Tuf.*** Moreover the bibliography in this book refers to eight titles under the heading RECREATION AND PUZZLES.

Both in photographs (black-and-white, and colour), and in the text, Biggs has much of interest to convey on the subject of shapes and space. "Geometry" is perhaps too technical a term to apply to the early experiences, such as that shown in one photograph of three "infants" enjoying filling plastic bottles with sand, and letting it run out, or others pasting newspaper on to cartons to make a skyscraper. "Such experience, and that provided by doll-making and dressing, is of value mathematically since the child not only looks closely at the shapes themselves but also intuitively strives to maintain the correct relationships in the shapes concerned so that the finished result is 'in proportion'." (p. 51)

In her ideas and examples of children's spatial activities, one is struck by the frequent use of everyday or junk material. Children are seen using cartons and boxes, paper-tearing, strips like ***Meccano (Erector),*** balsa wood, floor tiles, and tin cans. "Peter, aged 11, set out to discover why bees and wasps use hexagonal cells." (p. 55) -- perhaps the same Peter (a year younger) who reinvented calculus? Area, Biggs hints, could begin with leaves, and perimeter with string (p. 59). Modelling clay gives useful experience leading to the conservation of volume, but "Discussion between teacher

and children is essential if this experience is also to be used as 'conservation experience'. (p. 60) Symmetry is a key concept, and paper-cutting, mirrors, and shapes in the environment are mentioned: "It is most important that teachers should make the most of the activities of children which give them varied experience of symmetry ..." (p. 63) Experience with similarity is found in the construction of models, and in maps (p. 64).

Graphs also play an important part in Biggs' mathematics class, "Graphical representation makes an immediate appeal to children of all abilities and helps them to see relationships which they might not otherwise be able to appreciate." (p. 69) But, "It is of the utmost importance that, once a graph has been made, it should then be used for further discoveries." (p. 70)

As with number, in geometry Biggs urges that children make their own discoveries, and often they will go to surprising lengths, if allowed, like the class of ten-year-olds who, in order to find the relationship between the number of sides in a regular polygon, and its angle, carried it on to 450 000 000 sides! (p. 67). These investigations can often lead to graphs, "It is most important that children should undertake their own investigations for graphical representation ... individual personal investigations should be encouraged." (p. 70) Photographs in the book indicate that the results of these activities can usefully be posted on the classroom walls to interest and encourage the whole class.

With all this fun and activity going on (and hopefully fun for the teacher also), it will be difficult for the latter to have to consider the matter of evaluation ("assessment," a more usual term in the U.K.). First there is a warning:

> It is by no means unusual to find that at least one half of a class, at any stage in the primary or secondary field, are combining numbers (by addition, subtraction, multiplication or division) by counting on or back in ones. Yet such children may achieve full marks on a mechanical test if time is not taken into account. A written test on mechanical computation may be an invalid indication of mathematical attainment.
>
> (p. 94)

How then can children be evaluated? Of course there is no simple answer, but "there are certain definite mathematical concepts, the acquisition of which will be a landmark in the child's progress." (p. 95) One point to remember is that "An important aspect ... in assessment is not only the child's ability to think for himself (and hence to discover relationships) but also his ability to communicate or to record his discovery in symbolic form: in words, in numerals ... in tabular form, by a diagram or by a graph." (p. 95). But in any case, "It cannot be over-emphasised that, in all assessment, the teacher's personal record of a child's progress is of paramount importance." (p. 95). Biggs feels that there should be continuity in this record-keeping, even to the passing on of records between teachers and schools. And this leads to the further suggestion that teachers should make exchange visits between the different levels of schools.

Are Miss Biggs' many suggestions too radical and idealistic? The teachers who would have to change their ways of teaching mathematics are not mathematics specialists, nor a ***corps d'elite*** of professionals studying for the highest level of qualification. Biggs herself has expressed (in her latest book) disappointment at long-term results, "There was an enthusiastic response initially, and much interesting work was carried out with children during the first six month or so. But gradually the impetus was lost, probably as teachers came to the end of their mathematical resources." (Biggs, 1983) This is indeed discouraging. Nevertheless the spark of enthusiasm sometimes lies dormant for years, and when rekindled, can be infectious. If just one person in our great profession has made a success of teaching mathematics, and has ideas to share, there is hope for us all. And what is more important -- for the children.

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# ACTIVITIES FOR TEACHING LINE SYMMETRY

*by*
*Karla Tessaro*

Primary and junior school children need many direct experiences with geometry. Line symmetry is an important topic in geometry which children can learn rather easily when the method is adapted to their development. Through informal hands-on manipulations, children readily gain an intuitive feeling for symmetry. Children have to experience that which they are to later understand and apply.

This article attempts to compile a number of activities children can undertake to discover the meaning of line symmetry. These activities have several advantages. To begin with, symmetry is a highly motivating topic. Its constant presence in the environment, makes it of natural interest to children. Children enjoy doing the activities which in turn makes them feel comfortable with mathematics in general. Many of the paper folding and mirror activities also give children the practice they need to develop confidence with making and testing predictions, an important problem-solving skill. As well, the hands-on experience of symmetry activities provides a useful vehicle for developing spatial perceptions (perceptual constancy) and reasoning abilities not provided in other areas of the school curriculum.

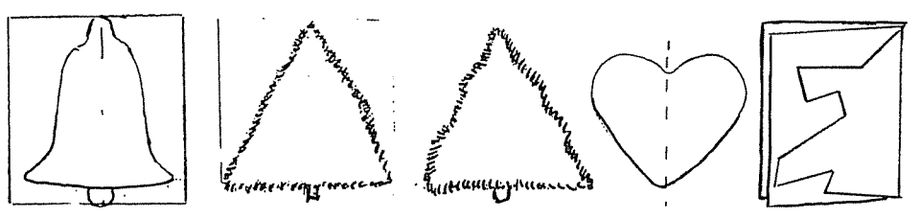
Line symmetry can be defined as repetitions of exactly alike parts on opposite sides of a line. Do children need to know this definition to begin a study of line symmetry? This is not necessary as children will develop the vocabulary and meaning by carrying out different activities and verbalizing their reactions among themselves and with the teacher. Phrases like "the parts are just alike", "exactly the same" or "the shape is flipped over or folded down" or "folding matches the parts" are expressed as children gain experiences that will enable them to understand the concepts. Teachers can introduce into their discussions such terms as "image", "reflection", even "lines of symmetry" or "symmetrical" when the children are ready for them.

> "**Symmetry,** as wide or as narrow as you may define its meaning, is one idea by which humanity through the ages has tried to comprehend and create order, beauty, and perfection." -- Humann Weyl

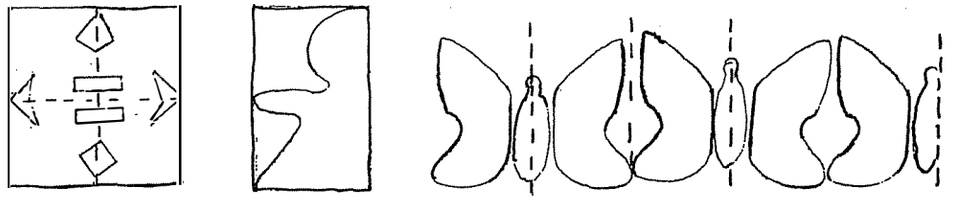
Given the opportunity to create symmetrical figures, children will develop an intuitive notion of the concept. The following activity suggestions have been carried out successfully with children in the third grade. However, activities like these could be readily adapted to a range of primary and junior grade levels.

## ACTIVITY SUGGESTIONS

1. Paper is folded in various ways and shapes are cut or torn from the folded paper. Cuts can be random or drawn before hand. The shape can also be fitted back into the hole by flipping the piece over (Geometry and Visualization).

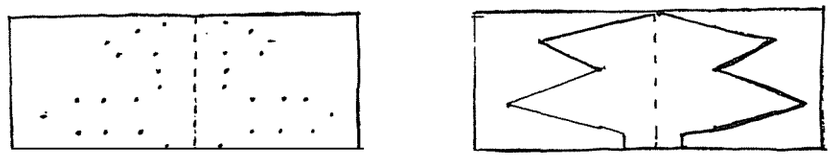


Folding paper into quarters introduces symmetry about two axes, while extensions of cutouts can be made by repeated folding.

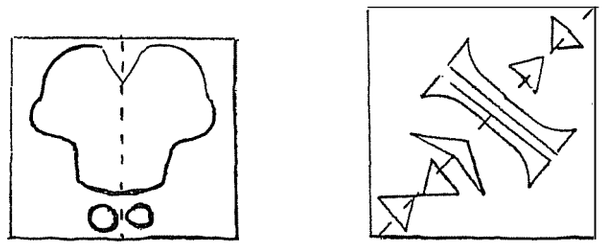


2. Fold a piece of heavy paper in half. Draw a pattern along the fold line. Using a needle or compass point punch holes along the outline to obtain the other half of the pattern. This activity shows students that in a symmetrical shape each point on one side of the fold is matched by a point on the other side which is the same distance from the fold.

3. A similar activity which reinforces this skill is the familiar task of children, a Join-the-dot picture. On a folded piece of paper several dots are drawn and labelled A, B, C... Holes are pricked through both sides of the paper folded together. The holes on the opposite side are labelled similar to the first side. The dots are then joined using a ruler.

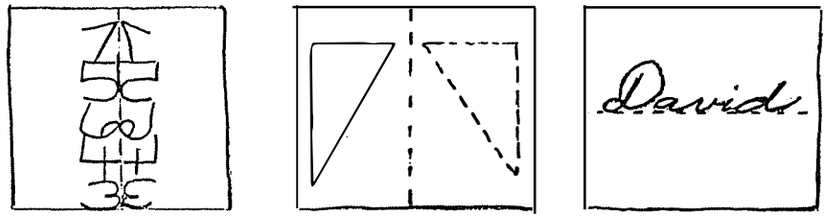


4. Blot prints can be made by applying a blot of paint next to a fold on a piece of paper. Refold the piece of paper and press down firmly. Compare matching parts on each side of the fold.



5. Fold heavy paper in various ways, in half or in quarters, cut out pieces from the folds and unfold. Use pattern as a stencil. Have other students decide how th paper was folded.

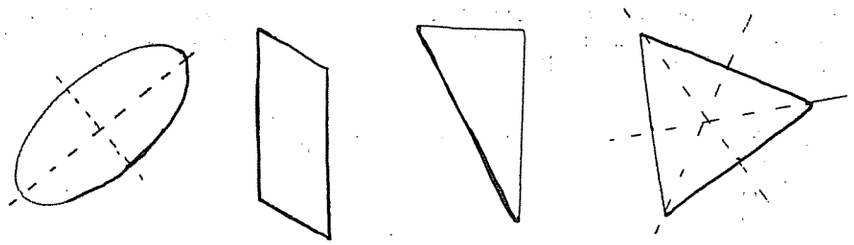
6. Print letters, numbers, shapes or students' names on the fold line of a piece of paper using paint or heavily pressed dark crayon. By pressing the folded paper together, a symmetrical figure can be seen.




7. Carbon paper and tracing paper can also be used.
   (a) A piece of carbon paper is placed face down onto a sheet of paper. Fold both papers so that the carbon is inside. Draw a design on the outside, unfold, remove carbon paper and symmetrical figures are produced on the inside of the paper.

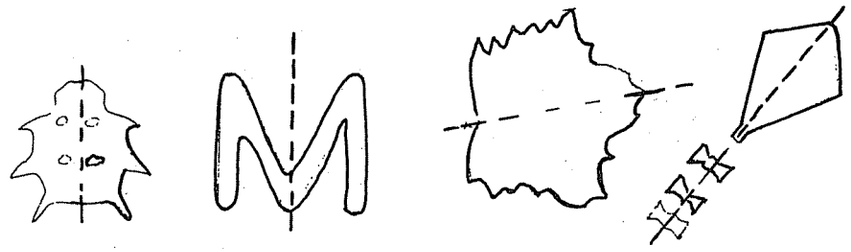
   (b) Trace a figure onto one side of a piece of tracing paper. Fold the tracing paper in half with the figure on the outside. Tracing over the figure produces the image.

8. Find the line of symmetry of regular and irregular polygons by folding, cut-out shapes in different ways to make the two halves fit.

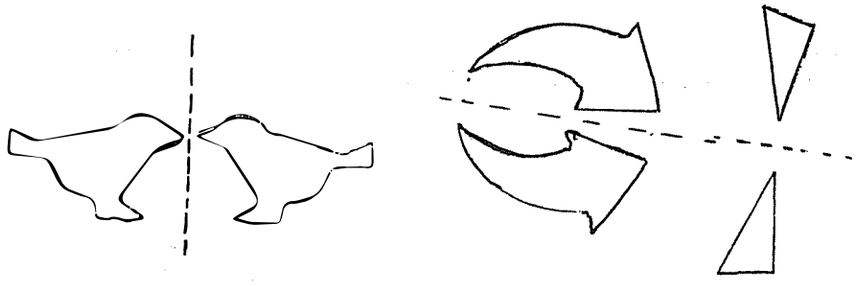


9. Mirrors and transparent pieces of plastic (or the commercial product 'Mira') can be used to find the line of symmetry of polygons as well. The Mira is on the line of symmetry when the image of the part in front of the mirror seems to fall on top of the rest of the figure behind the Mira.

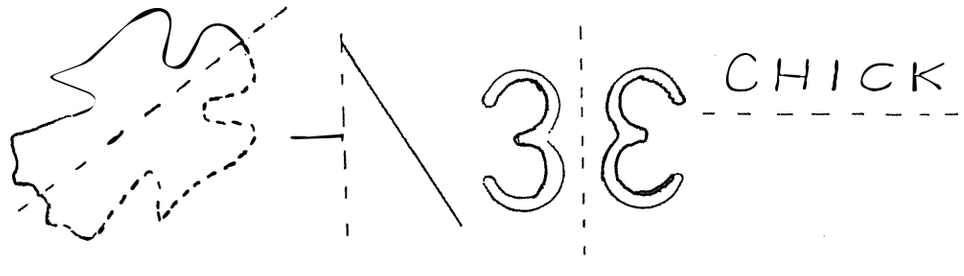
10. A line of symmetry can be drawn through other types of symmetrical figures using a transparent mirror (See Mira Math Activities for Elementary School).



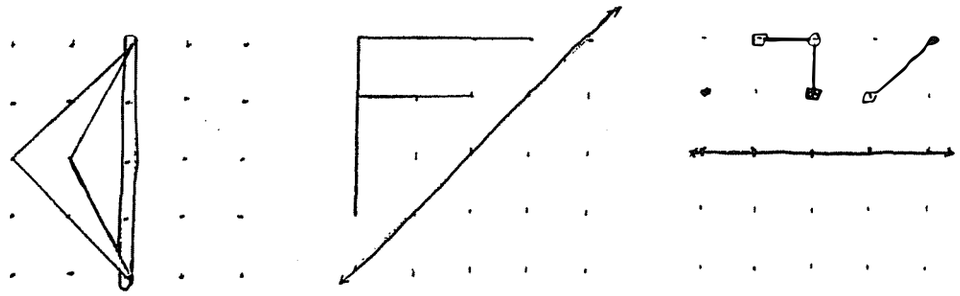
11. The line of symmetry between two figures that reflect each other can be drawn using a Mira.



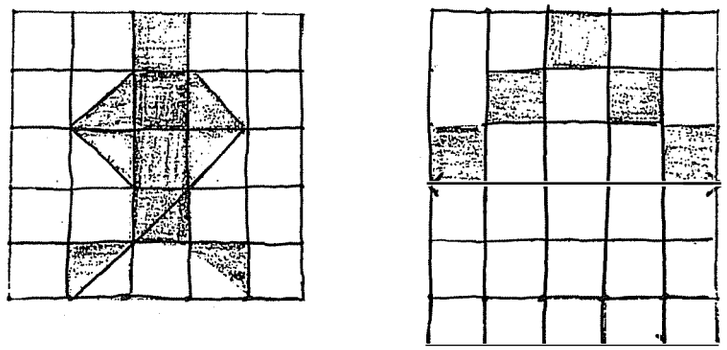
12. A Mira can be placed next to a figure or on the reflection line and the image can be sketched by looking through the plastic. Examples could be tried freehand first.



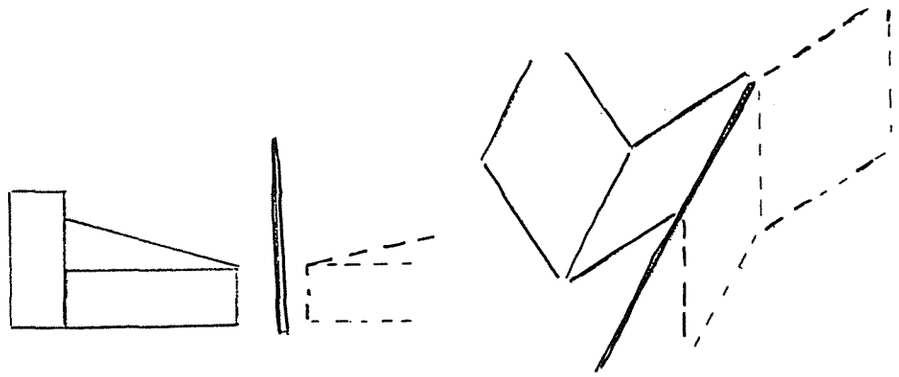
13. Geoboards and dot paper can be used to reflect a figure across a line. A vertical line can be made on the geoboard with a coloured elastic. Simple figures can be reflected across vertical, horizontal and diagonal lines. Check for accuracy using a plastic mirror (see Geoboard Geometry by Margaret Farrell).



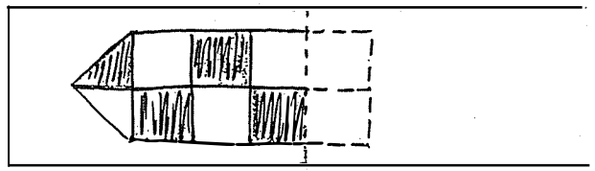
14. Using grid paper, simple designs can be reflected across a centre line.



15. Using different shaped blocks, various patterns and constructions can be built and their mirror image copied. (Symmetry with Pattern Blocks, A).

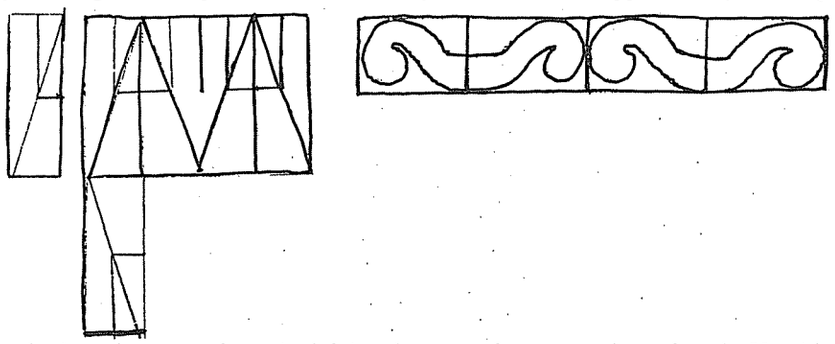


16. Packets of cut-out shapes (i.e. coloured squares and half-squares) can be used to create symmetrical designs. Fold a piece of paper. Reopen, the crease will form the reflection lines. Paste shapes on one side of the line. When the design is completed on one side of the fold, a symmetrical reflection of the design can be pasted on the other side (see John Van de Walle; Charles S. Thompson).



17. Making patterns by flipping pattern cards provides further experience with reflection symmetry.

(a) A line pattern is drawn on both sides of a card. The cards are lined in rows vertically or horizontally, making sure the patterns match. Every second card is flipped over (Martin Q. Johnson).



(b) Any simple shape is cut out from the left hand corner of a square piece of card. Use this card as a template. On a long strip of paper trace around the cut-out shape. Then flip the shape over and trace it again. Continue like this until the pattern is complete. Borders for bulletin boards can be made in this manner. Making patterns leads naturally into rotational symmetry and motion geometry--slides, flips and turns.

18. Symmetry Hunt
After considerable experience with producing symmetrical shapes using paper-folding and mirrors, children are naturally able to transfer the concept to objects in the real world--letters, leaves, floor tiles, insects, decorations and pictures of all sorts.

The study of line symmetry is a part of a number of geometrical experiences and can be extended in many directions. Symmetry activities develop informally the concept of congruence and many properties of familiar figures. They also lead naturally into rotational symmetry and other topics of transformational (motion) geometry.

(Karla Tessaro is an elementary school teacher in Hamilton, Ontario)

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Mathematics Resource Project, **Geometry and Visualization,** Palo Alto: Creative Publications Inc., 1977.

Johnson, Martin L. "Generating Patterns from Transformations", **The Arithmetic Teacher,** (March, 1977), 191-195.

**Mira Math Activities for Elementary School,** Canada: Mira Math Co., 1973.
--Commercially produced material available from Spectrum Educational Supplies Ltd., 8 Denison Street, Markham, Ontario.

Pasternack, Marian; Silvey, Linda. **Pattern Block Activities A.** Palo Alto: Creative Publications, Inc., 1975.

Van de Walle, John and Thompson, Charles S. "Let's Do It, Cut and Paste for Geometric Thinking", **Arithmetic Teacher,** (September, 1984), 8-13.

# AN ANNOTATED BIBLIOGRAPHY IN GEOMETRY: RATIONALE, APPROACHES, SYMMETRY

*by*

*Karla Tessaro*

Alspaugh, Carol Ann, "Kaleidoscopic Geometry", **Readings in Geometry from the Arithmetic Teacher,** NCTM (1970), 30-31. A number of geometrical concepts can be illustrated using a kaleidoscope (two mirrors hinged together and set up at any desired angle on a desk).

Barton, Patricia Cole. **The Effects of a Piagetian Geometry Model on Beginning School Children.** Document Resume (November, 1975). This study was designed to determine if a group of five- and eight-year-old students' understanding of selected geometric ideas paralleled Piaget's topological, projective, and Euclidean thesis. Additional purposes were to investigate which geometric ideas the group learned easiest, and to examine the relationships between geometric understanding and factors related to age, ability and teaching approaches.

Brune, Irvin H. "Geometry in the Grades", **Readings in Geometry from the Arithmetic Teacher,** NCTM (1970), 107-116. Brune's article provides a logical basis for introducing geometry early in the experience of the child, noting the importance of informal geometric readiness to the formal axiomatic study to be undertaken in the secondary school.

Burger, William F. "Geometry", **Arithmetic Teacher,** 32 (February, 1985), 52-57. In this article, the author focuses on how reasoning can be developed through the study of two-dimensional shapes, their properties and the relationships among them. Findings from a research study (Burger et al. 1981)-a number of children's misconceptions and conclusions drawn about their understanding of common geometric figures - is discussed.

Burrows, Charles K; Okey, James R. **The Effects of a Mastery Learning Strategy on Achievement,** National Centre for the Development of Training Materials in Teacher Education, Bloomington, Indiana, (April, 1975). Bloom has argued that most students, not just a few, should be able to do top quality school work if given appropriate instruction. Mastery learning, which includes frequent diagnostic testing followed by remedial instruction, has been proposed as an effective strategy for increasing pupil achievement. In this study, teachers used a mastery learning strategy with elementary school students in a geometry unit.

Del Grande, John J. **Geometry The Agony and the Ecstacy.** The Board of Education for the City of North York, 1979. A case for geometry is built--why geometry must be begun at an early age and why break away from tradition on teaching methods--using the support of other educators and mathematicians. Geometry programs K-13, aims and objectives, historical and contemporary views and approaches on some of the topics elaborated upon.

Del Grande, John. "The Reflection", **Department of Mathematics, The Board of Education for the Borough of North York,** June 1971. This handbook was prepared to provide an introduction to the concept of the reflection transformation. Through the use of concrete material and mirrors, a range of activities to be produced on cards were developed.

Dennis, J. Richard. "Informal Geometry through Symmetry", **Readings in Geometry from the Arithmetic Teacher,** NCTM (1970), 32-35. This article provides a brief development of the concepts of congruence and symmetry of polygons.

Edwards, Ronald R. "Discoveries in Geometry by Folding and Cutting", **Arithmetic Teacher** 24 (March 1977), 196-198. A number of paper-folding activities are described. The activities illustrate geometric concepts, theorems and construction techniques.

Egsgard, John C. "Geometry All Around Us--K-12", **Readings in Geometry from the Arithmetic Teacher,** NCTM (1970), 73-81. This section contains a comprehensive outline of suggested content for elementary school program (chart levels kindergarten to twelve). It concentrates on the study of shape, size, measure and transformation and some of the relationships among these topics.

Fielker, David S. "Strategies for Teaching Geometry to Younger Children", **Educational Studies in Mathematics** 10, (February, 1979), 85-133. Topics discussed include: reasons for strategies; the

use of a hierarchy of spaces from topological to Euclidean in teaching; structural strategies; pedagogical strategies; and mathematical strategies.

Forseth, S. D., Adams, P. A. "Symmetry", Readings in **Geometry from the Arithmetic Teacher,** NCTM (1970), 36-38. This article shows how art activities using concepts of symmetry can be used with almost any grade. Using a stenciled asymmetrical design students construct seven basic strip patterns involving translations, reflections and rotations.

Fuys, David; Welchman Tischler, Rosamond "Mirrors and Symmetry" **Teaching Mathematics in the Elementary School.** Toronto: Little, Brown and Company, (1979), 111-117. In this section, a number of activities explore how mirrors are related to type of geometry known as symmetry. Directions for using a variety of mirror and pattern cards are given. Symmetry of polygons as well as activities involving line symmetry (folding) are explored.

Horak, Willis and Virginia. "Geometry taught here!" **Instructor** (February 1983), 76-79. Geometry should be a frequent and familiar part of the elementary school mathematics curriculum. Exercises to help children master basic terminology and visualize shapes and geometric relationships are described. They range from a "shape bag" which teaches simple identification of figures, to more complex activities involving polyominoes and tangrams.

Hurbert, Elizabeth. "One Point of View. Manipulatives Are Good Mathematics!" **Arithmetic Teacher,** 32 (February, 1985), 4. Manipulatives motivate students to think mathematically. Teacher's concerns that there is not enough time to use manipulatives, that their use is the same as playing games and that they are difficult to manage with large classes, are debated.

Inskeep Jr., James E. "Primary-Grade Instruction in Geometry", **Readings in Geometry from the Arithmetic Teacher,** NCTM (1970), 82-86. A brief resume of reasons why geometry should be taught is given, followed by suggestions and techniques for primary teachers who need direction in implementing a program.

Johnson, Martin L. "Generating Patterns from Transformations", **Arithmetic Teacher** 24, (March, 1977), 191-195. Beginning with a set of design cards and a rectangular grid, students can generate interesting and colourful designs using rotations, reflections and translations.

Kerr Jr., Donald R. "A Case for Geometry: Geometry Is Important. It is There, Teach It". **Arithmetic Teacher** 26 (February, 1979), 14. Gives examples of direct uses of geometry in everyday life and how other areas of study are built on the knowledge of geometry.

Kidder, F. Richard. "Euclidean Transformations: Elementary School Spaceometry", **Arithmetic Teacher** 24 (March, 1977), 201-207. Children's perception and understanding of three-dimensional figures and of operations on them is discussed. Piaget's findings are reviewed and more current research is described. Classroom implications are explored.

Lovell, Kenneth R. "Intellectual growth and understanding mathematics: implications for teaching". **The Arithmetic Teacher** 19 (April, 1972), 277-283. Six points of Piaget's cognitive development system are used to form the basis of educational implications that relate to classroom organization and general teaching approaches--small groups, hands-on activities, social intercourse using verbal language and teacher-directed.

Lindquist, Mary Montgomery, Thomas P. Carpenter, Edward A. Silver and Westina Mathews. "The Third National Mathematics Assessment. Results and Implications for Elementary and Middle Schools". **Arithmetic Teacher,** 31 (December, 1983), 14-19. The results (1983) of students' performance at ages 9, 13 and 17 on whole numbers, fractions, decimals, problem solving and other mathematic concepts and skills are compared to last assessment in 1978. The positive and negative changes found in this assessment are elaborated upon.

Mansfield, Helen. "Projective Geometry in the Elementary School", **Arithmetic Teacher,** (March, 1983), 15-19. This article suggests reasons for including projective geometry in the curriculum and describes some lessons that were taught to a class of ten-year olds. The author uses Piaget's Theory on how children learn geometry to show need to present projective ideas elementary school children.

Mathematics Resource Project. "Planning Instruction in Geometry", **Didactics and Mathematics:**

**The Art and Science of Leructions are said to provide mildly mentally retarded students with concrete visual and artistic notions of geometry.**

Ogletree, Earl J. "Teaching Geometry: An Experiential and Artistic Approach", **Document Resume,** (1981) page 10. The view that geometry should be taught at very grade level is promoted. Primary and elementary school children are though to rarely have any direct experience with geometry, except on an incidental basis. Children are supposed to be able to learn geometry rather easily, so long as the method and content are adapted to their development and learning style. Form drawing, a type of discipline freehand geometry drawing designed by Rudolf Steiner (1861-1925) is promoted.

Pegg, John. "How Children Learn Geometry: The Van Hiele Theory", **The Australian Mathematics Teacher** 41 (July, 1985), 5-8. This article describes some of the aspects of the Van Hiele theory - a hierarchial sequence of five levels of thinking - and some of the implications of the theory for the planning of geometric instruction in the classroom.

Poem "Geometry", **Arithmetic Teacher** (February 1977) page 107.

Rosskopf, Myron F. "Piagetian Research and the school mathematics program", **The Arithmetic Teacher** 19 (April, 1972), 309-313. Mathematics teaching applications of one concept-function of Piagetian research is hinted at for children 3 to 11. Emphasis is placed on the provision of exploratory experiences and the importance of language in formulating concepts.

Robinson, G. Edith, "Mathematics Learning in Early Childhood: Geometry. **NCTM Thirty-seventh Yearbook** (1975), 205-225. The value and techniques of pre-geometry and geometry instruction in primary grades are discussed. Geometric concepts such as open and closed figures, connectedness, similarity, congruence, straightness, and angle are presented through suggested activities, lesson plans, and evaluation procedures.

Robinson, G. Edith, "The Role of Geometry in Elementary School Mathematics", **Readings in Geometry from the Arithmetic Teacher,** NCTM (1970), 92-99. This article presents the worth of geometry and geometric approaches to other topics in mathematics. The ways in which geometry can add new dimensions to better understanding in the areas of: measurement, properties of natural numbers, order properties and the concept of operations is discussed.

Russell, Dorothy and Elaine M. Bologna. "Teaching Geometry with Tangrams". **Arithmetic Teacher** 30 (October 1982), 34-38. Geometry is viewed as the most neglected area of the elementary school mathematics curriculum. Tangram activities provide numerous worthwhile mathematical experiences for children. A method of constructing tangrams through paper folding is followed by suggested spatial visualization, measurement and additional activities. (MP)

Schoedler, James. "A Comparison of the Use of Active Game Learning with a Conventional Teaching Approach in the Development of Concepts in Geometry and Measurement at the Second Grade Level", **School Science and Mathematics,** 81 (May-June, 1981) 365-370. The Houghton-Mifflin Modern School Mathematics Program for Second Grade is compared to the active game learning medium approach (a motivating, concrete, movement approach to teaching). Results showed no difference between the academic and active game groups.

Silverman, Helene. "Geometry in the Primary Grades: Exploring Geometric Ideas in the Primary Grades", **Arithmetic Teacher** 26 (February, 1979), 15-16. Planned experiences for exploration of materials are suggested for introducing geometric concepts. Examples discussed include motion, symmetry and shape.

Skypek, Dora Helen. "Geometric Concepts in Grades 4-6", **Readings in Geometry from the Arithmetic Teacher** NCTM (1970), 100-106. Using some of Piaget's conclusions and ideas, this article deals with "how" children learn geometric and topological concepts. Presented are two of Piaget's hypotheses for teaching of geometry concepts in the uper elementary school: 1. action on objects precedes perception and 2. there is an order to the development of geometric concepts.

Suydam, Marilyn N. "Forming Geometric Concepts", **Arithmetic Teacher,** 33 (October, 1985) 26.

Research is cited which provides evidence that effective instruction in geometry is weak--that it revolves around recognizing shapes and development of vocabulary. Also sources are given that indiacte other ideas that can be developed.

Trafton, Paul R., LeBlanc, John F. "Informal Geometry in Grades K-6", **Geometry in Mathematics Curriculum** (Thirty-sixth Yearbook), NCTM (1973), 11-51. The development of geometry in elementary school curriculum--historical background, European and American from 1800's to 1970. Content of elementary school geometry is discussed, emphasizing the typical program, the development of geometry as represented in texts and summarizing the major topics in the kindergarten through grade 2 and grade 3 through grade 6 program. Three contributions of geometry to the learning of children are presented as justifications for its inclusion in the curriculum. Perspectives on classroom use and the focus for future programs are also included.

Thorton, Carol A. "Rx = Geometry: Perceptual-Motor Help for Many Handicapped Learners" **Arithmetic Teacher** 27 (October, 1979), 24-25. This article explains some of the difficulties children with visual perception problems have, interpreting what they see. Perceptual awareness can be enriched through carefully structured geometric activities and the techniques learned can be applied to other tasks.

Van de Walle, John; Thompson, Charles S. "Let'd Do It, Cut and Paste for Geometry Thinking", **Arithmetic Teacher** 27 (September, 1984), 8-13. Activities involving patterns, enlarging designs, symmetry and tesselations are explained. Each of the activities suggested nothing more than some coloured construction paper cut into simple shapes.

Vigilante, Nicholas J. "Why Circumvent Geometry in the Primary Grades", **Readings in Geometry from the Arithmetic Teacher,** NCTM (1970), 87-91. Vigilante's discussion presents both psychological and pedagogical reasons to teach geometry. The article gives a number of meaningful ways to present geometric patterns to primary children.

Walters, Marion. "An Example of Informal Geometry: Mirror Cards", **Readngs in Geometry from the Arithmetic Teacher,** NCTM (1970), 25-29. This article provides a technique for teaching the understanding of symmetry of Mirror Cards. The basic problem posed by the Mirror Cards is one of matching by means of a mirror, a pattern on one card with a pattern shown on another card. Spatial perceptions and ideas of geometric transformation are by-products of this approach.

Walters, Marion. "Geometry for young students in the 80's" **Arithmetic Teacher** (February, 1981), 45-49. Gives reasons why useful geometric activities should be provided throughout the elementary school curriculum; why it's natural; problem-solving; visualizing; use of concrete material; math confidence and predictions; reasoning; and links to other branches of mathematics.

Weissglass, Julian. "Symmetry", **Exploring Elementary Mathematics.** U.S.A.: W. H. Freeman and Company, (1979), 243-247. Chapter 19 "Symmetry" gives the meaning of the concept and describes several activities using pattern blocks, mirrors and the folding technique that will help children to explore the intuitive concepts of symmetry.

Wheatley, Charlotte L. "Mathematics Experiments: An Alternative Teaching Strategy", **Arithmetic Teacher** 27 (October 1979), 18-23. This article presents a teaching strategy with a rationale, examples and procedures for implementation. A math experiment is a small-group activity whose strategy lies in the learning of problem-solving.

Williams, Elizabeth; Shuard, Hilary. **Primary Mathematics Today New Metric Edition.** London: Longman (1976), 104-105. Several suggestions are given to show the concept of symmetry by making shapes that balance (folding activities).

Williford, Harold. "What does research say about geometry in the elementary school?" **The Arithmetic Teacher** 19 (February, 1972), 97-104. The intent of this review is to determine what research (before 1972) has to say about geometry instruction in the elementary school. The research is classified under the following sections: geometric capabilities of elementary school pupils, instructional variables related to geometry achievement, transfer effects involving geometry, findings concerning teachers and geometry and feasibility studies in geometry.

Woodward, Ernest. ''Geometry with a Mira'', **The Arithmetic Teacher,** (February, 1977), 116-117.
What is a Mira and what is its purpose? This brief article explains how the Mira is a useful device for teaching ideas of symmetry, congruence and reflections and for constructing parallel, perpendicular, and bisecting lines.

## OAME MEMBERSHIP

OAME is a professional organization of close to 1400, involved in all aspects of teaching mathematics-from Post-Secondary to Kindergarten. There are currently eight active chapters. The chapters and their locations are: COMA (Ottawa-Carleton), Golden Section (Hamilton-Niagara), GVMA (Waterloo & Area), NOMA (Northern Ontario), PRMA (Oshawa area), PAME (Peel), QSLMA (Belleville, Kingston, Smiths Falls, etc.), SAME (Scarborough).

Some of OAME's major activities include:
1. Mini-conferences hosted several times a year by the chapters across the province.
2. The Annual Meeting, held in early May: 1986-London; 1987-Scarborough; 1988-Niagara Falls. Two or 3 days of concentrated mathematics interspersed with social activities.
3. Mathematical journals - the **Gazette** and **Abacus -** several times a year.

Membership costs $18 per year, with a special rate of $80 for 5 years.

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# METHOD OF TEACHING AND REVIEWING ROUNDING

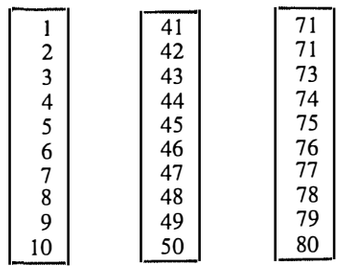
*by*
*Patricia Hutchinson*

Learning to round numbers is an important part of learning to estimate. Students often have a great deal of difficulty mastering the skills involved with rounding because of difficulty in several areas. One area is understanding that they must look at the number in the place before the one they are rounding to, in order to make their decision. Some have difficulty remembering what to do with the numbers in the places to the right of the one they are interested in. Students also have problems understanding that rounding is always the same process no matter which place value they are working with.

The following is one way I found to help overcome these problems. It consists of an initial experience, ways to expand the experience to cover rounding to any place value and a review/drill method.

## INITIAL EXPERIENCE

1. Give each student a piece of graph paper with large blocks.
2. Have the students cut 10 strips of 10 blocks each out of the graph paper.
3. Have the students take one strip and number the blocks from 1 to 10 from top to bottom. Next number the blocks on the second strip from 11 to 20, the third from 21 to 30 and so on until they have numbered to 100.

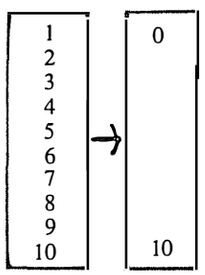








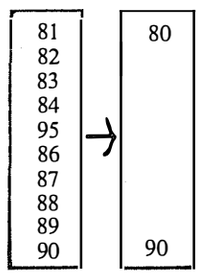

4. Have the students orally count by 10's from 0 to 100 while you write the numbers on the blackboard.
5. Ask the students which number on the blackboard is closed to the first number on the first strip. (zero) Have them turn the strip over and put a zero at the top.
6. Ask which number on the blackboard is closest to the last number on the front of the strip. (10) Have them put it on the back of the strip at the bottom.
7. Repeat steps 5 and 7 for all of the strips.







Front Back







Front Back

8. Have the students take a red pencil and go through the strips and circle all of the numbers that have a number 5 in the ones column. Orally develop the idea that the numbers before the red circled number are closer to the number at the top on the back of the strip and that the numbers

after the red circled one are closer to the last number on the back of the strip.

9. Name a number and have the students locate it on the correct strip. Ask "Is the number before or after the red circled number?"
   e.g. call 34- Students identify it as before 35 and therefore closer to the number on the top of the strip's back. The correct answer is stated as - 34 is almost 30

   Do as many examples (excluding numbers ending in 5) as necessary for the students to be able to call out the correct answer with ease.
10. Once the students have the idea give them a number ending in 5 and have them try to round it. Develop the rule that you wish to use. (If the students understand even and odd numbers then develop the rule of rounding to the even number, if desired). The most common rule is to round up if 5 is the considered number.

**DRILL/REVIEW STEP**

You can continue to work as a large group with the teacher calling out numbers and the students identifying the answers collectively or when called upon. You can also have the students work in pairs or small groups until they are able to round the numbers without turning the strips over.

**WAYS TO EXPAND**

To extend the rounding to the 100's place have the students add a zero to the right of each number on the strips. Discuss what this does to the numbers on the strips. Repeat steps 9 and 10 of the initial experience using numbers with zero in the ones column. Once the students are able to do this quickly, add numbers in the ones column. Ask "What 100's would you round the number 270 to? What would you round 271 or 272 to? Why does the number in the ones column not change your answer?"

The same process can be used for rounding to 1000 or to any other whole place value.

To use it for decimal places have the students place a decimal point in front of the number for rounding to the ones place or a 0.0 for rounding to tenths.

| Rounding to 100's | Rounding to ones | Rounding to tenths |
|---|---|---|
| 610 | 0.610 | .0610 |
| 620 | 0.620 | .0620 |
| 630 | 0.630 | .0630 |
| 640 | 0.640 | .0640 |
| 650 | 0.650 | .0650 |
| 660 | 0.660 | .0660 |
| 670 | 0.670 | .0670 |
| 680 | 0.680 | .0680 |
| 690 | 0.690 | .0690 |
| 700 | 0.700 | .0700 |

(Patricia Hutchinson teaches in Hamilton, Ont.)

# MINIMUM NUMBERS HAVING SPECIFIED NUMBERS OF DIVISORS

*David R. Duncan and Bonnie H. Litwiller*
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*Cedar Falls, Iowa*

On an early morning television newscast an interesting as well as surprising question was posed:

> There is a remarkable number less than 1000 which has exactly 32 divisors. Do you remember what it is?

The question was posed in a strange way. Except for a very small number of people it is doubtful that any of the listeners could pull the answer from their memory. A more reasonable question would have been: Can you find this number? The answer is 840 and the methods for determining this answer will be given subsequently.

This problem gives rise to a more general question:

> Given any natural number m, find the smallest natural number having exactly m natural number divisors.

To solve this we shall develop and make extensive use of a basic number theory principle. Consider any natural number greater than 1. It can be written uniquely as the product of primes; let its prime product representation be $p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots p_n^{a_n}$ where $p_1, p_2, p_3, \cdots, p_n$ are distinct primes. Any divisor of this product can use only the primes $p_1, p_2, p_3, \cdots, p_n$. The exponents on these primes for such a divisor cannot exceed $a_1, a_2, a_3, \cdots, a_n$ respectively. Consequently the selection of a divisor can be accomplished by performing the following series of n tasks.

1) Choose the exponent on $p_1$ (in the divisor). This task can be accomplished in any of $(a_1 + 1)$ ways, since the exponent can be any of the numbers $0_1, 1_1, 2_1, \cdots, a_1$.

2) Choose the exponent on $p_2$. Similarly this task can be accomplished in any of $(a_2 + 1)$ ways.

n) Choose the exponent on $p_n$; this can be accomplished in $a_n + 1$ ways.

By the Fundamental Principle of Counting the number of ways of performing these n tasks is $(a_1 + 1) \cdot (a_2 + 1) \cdot (a_3 + 1) \cdot \ \ldots \cdot (a_n + 1)$. Two special cases would be to select 0 for each exponent (yielding the divisor 1), and selecting $a_1, a_2, a_3, \cdots, a_n$ for the divisor, yielding the original number $p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdot \ldots \cdot p_n^{a_n}$.

We now shall address the given problem of locating a natural number less than 1000 with exactly 32 positive divisors. Let us restate the problem as requiring the smallest natural number with exactly 32 divisors. If that smallest number is less than 1000 our task is accomplished.

First write the number to be found as $2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot 7^{a_4} \cdot \ \ldots \ .$ We must identify $a_1, a_2, a_3, \cdots$ so that $(a_1 + 1) \cdot (a_2 + 1) \cdot (a_3 + 1) \cdot (a_4 + 1) \cdots = 32$. The problem then becomes one of identifying products that lead to 32.

An observation is necessary. Suppose the smallest number with n divisors is $A = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot \ldots \cdot p_n^{a_n}$; then $a_1 > a_2 > a_3 > \cdots a_n$. To justify this observation suppose, for instance, that $a_1 < a_2$. Then $B = 2^{a_2} \cdot 3^{a_1} \cdot 5^{a_3} \cdot \ \ldots \cdot p_n^{a_n}$ would have exactly the same number of divisors A, but $B < A$ since $B = A(\frac{2}{3})^{a_2 - a_1}$. A consequence of this observation is that no prime can be "skipped". For example, the smallest number with n divisors could not have factors of 2 and 5 without also containing factors of 3.

The following are possible factors for the product $(a_1 + 1)(a_2 + 1)(a_3 + 1) \cdot \ldots = 32$:

| Factors of 32 in Decreasing Order | Values of $a_1, a_2, a_3, \cdots$ | $2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdots$ |
|---|---|---|
| 32 | 31 | $2^{31} = 2{,}147{,}483{,}648$ |
| 16, 2 | 15, 1 | $2^{15} \cdot 3^{1} = 98{,}304$ |
| 8, 4 | 7, 3 | $2^{7} \cdot 3^{3} = 3{,}456$ |
| 8, 2, 2 | 7, 1, 1 | $2^{7} \cdot 3^{1} \cdot 5^{1} = 1{,}920$ |
| 4, 4, 2 | 3, 3, 1 | $2^{3} \cdot 3^{3} \cdot 5^{1} = 1{,}080$ |
| 4, 2, 2, 2 | 3, 1, 1, 1 | $2^{3} \cdot 3^{1} \cdot 5^{1} \cdot 7^{1} = 840$ |
| 2, 2, 2, 2, 2 | 1, 1, 1, 1, 1 | $2^{1} \cdot 3^{1} \cdot 5^{1} \cdot 7^{1} \cdot 11^{1} = 2{,}310$ |

Note that the smallest number with 32 divisors is 840. The 32 divisors are

listed as follows:

$2^0 \cdot 3^0 \cdot 5^0 \cdot 7^0 = 1$

$2^1 \cdot 3^0 \cdot 5^0 \cdot 7^0 = 2$

$2^1 \cdot 3^1 \cdot 5^0 \cdot 7^0 = 6$

$2^1 \cdot 3^0 \cdot 5^1 \cdot 7^0 = 10$

$2^1 \cdot 3^0 \cdot 5^0 \cdot 7^1 = 14$

$2^1 \cdot 3^1 \cdot 5^1 \cdot 7^0 = 30$

$2^1 \cdot 3^1 \cdot 5^0 \cdot 7^1 = 42$

$2^1 \cdot 3^0 \cdot 5^1 \cdot 7^1 = 70$

$2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 210$

$2^2 \cdot 3^0 \cdot 5^0 \cdot 7^0 = 4$

$2^2 \cdot 3^1 \cdot 5^0 \cdot 7^0 = 12$

$2^2 \cdot 3^0 \cdot 5^1 \cdot 7^0 = 20$

$2^2 \cdot 3^0 \cdot 5^0 \cdot 7^1 = 28$

$2^3 \cdot 3^1 \cdot 5^1 \cdot 7^0 = 60$

$2^2 \cdot 3^1 \cdot 5^0 \cdot 7^1 = 84$

$2^2 \cdot 3^0 \cdot 5^1 \cdot 7^1 = 140$

$2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 420$

$2^3 \cdot 3^0 \cdot 5^0 \cdot 7^0 = 8$

$2^3 \cdot 3^1 \cdot 5^0 \cdot 7^0 = 24$

$2^3 \cdot 3^0 \cdot 5^1 \cdot 7^0 = 40$

$2^3 \cdot 3^0 \cdot 5^0 \cdot 7^1 = 56$

$2^3 \cdot 3^1 \cdot 5^1 \cdot 7^0 = 120$

$2^3 \cdot 3^1 \cdot 5^0 \cdot 7^1 = 168$

$2^3 \cdot 3^0 \cdot 5^1 \cdot 7^1 = 280$

$2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 840$

As other examples of our method consider:

1) Find the smallest natural number having exactly 31 positive divisors. If the number in question is written as $2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdots$, then $(a_1 + 1) \cdot (a_2 + 1) \cdot (a_3 + 1) \cdots = 31$. Since 31 is a prime, the only possibility is for $a_1 + 1$ to be 31; this forces $a_1$ to be 30. Thus, the smallest natural number with exactly 31 divisors is $2^{30}$. Its divisors are 1, 2, $2^2$, $2^3$, $\cdots$ $2^{30}$. Note that this is not the only number having exactly 31 divisors; $3^{30}$, $5^{30}$, $7^{30}$, $\cdots$ also have exactly 31 divisors. The smallest number, however, is $2^{30}$.

2) Find the smallest natural number having 36 divisors. If the number in question is written as $2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdots$, then $(a_1 + 1) \cdot (a_2 + 1) \cdot (a_3 + 1) \cdots = 36$. The following are possible factors for this product.

| Divisors of 36 in Descending Order | Values of $a_1, a_2, a_3, \cdots$ | $2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdots$ |
|---|---|---|
| 36 | 35 | $2^{35}$ |
| 18, 2 | 17, 1 | $2^{17} \cdot 3^1$ } >10,000 |
| 12, 3 | 11, 2 | $2^{11} \cdot 3^2$ |
| 9, 4 | 8, 3 | $2^8 \cdot 3^3 = 6912$ |
| 9, 2, 2 | 8, 1, 1 | $2^8 \cdot 3^1 \cdot 5^1 = 3840$ |
| 6, 6 | 5, 5 | $2^5 \cdot 3^5 = 7776$ |
| 6, 3, 2 | 5, 2, 1 | $2^5 \cdot 3^2 \cdot 5^1 = 1440$ |
| 3, 3, 2, 2 | 2, 2, 1, 1 | $2^2 \cdot 3^2 \cdot 5^1 \cdot 7^1 = 1260$ |

In this case the last number on the list was the smallest; however, for 32, the solution was not the last number on the list. The numbers having the required number of divisors must be checked individually since no general rule specifies which is the smallest. As this checking is done, some can be easily eliminated without calculating their exact values because of their evidently large size, but there will frequently be several that are reasonably close to each other in size; these must be checked individually.

Table 1 shows the results when our method is applied for other numbers of divisors. The alert reader is encouraged to verify our results by building tables similar to those we used for 31, 32, and 36.

Table 1

| Number of Divisors | Smallest Natural Number Having the Given Number of Divisors |
|---|---|
| 1 | 1 |
| 2 | $2^1 = 2$ |
| 3 | $2^2 = 4$ |
| 4 | $2^1 \cdot 3^1 = 6$ |
| 5 | $2^4 = 16$ |
| 6 | $2^3 \cdot 3^1 = 12$ |
| 7 | $2^6 = 64$ |
| 8 | $2^3 \cdot 3^1 = 24$ |
| 9 | $2^2 \cdot 3^2 = 36$ |
| 10 | $2^4 \cdot 3^1 = 48$ |

| | |
|---|---|
| 11 | $2^{10} = 1024$ |
| 12 | $2^2 \cdot 3^1 \cdot 5^1 = 60$ |
| 13 | $2^{12} = 4096$ |
| 14 | $2^6 \cdot 3^1 = 192$ |
| 15 | $2^4 \cdot 3^2 = 144$ |
| 16 | $2^3 \cdot 3^1 \cdot 5^1 = 120$ |
| 17 | $2^{16} = 65,536$ |
| 18 | $2^2 \cdot 3^2 \cdot 5^1 = 180$ |
| 19 | $2^{18} = 262,144$ |
| 20 | $2^4 \cdot 3^1 \cdot 5^1 = 240$ |
| 21 | $2^6 \cdot 3^2 = 576$ |
| 22 | $2^{10} \cdot 3^1 = 3072$ |
| 23 | $2^{22} = 4,194,304$ |
| 24 | $2^3 \cdot 3^2 \cdot 5^1 = 360$ |
| 25 | $2^4 \cdot 3^4 = 1296$ |
| 26 | $2^{12} \cdot 3^1 = 12,288$ |
| 27 | $2^2 \cdot 3^2 \cdot 5^2 = 900$ |
| 28 | $2^6 \cdot 3^1 \cdot 5^1 = 960$ |
| 29 | $2^{28}$ |
| 30 | $2^4 \cdot 3^2 \cdot 5^1 = 720$ |
| 31 | $2^{30}$ |
| 32 | $2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 840$ |
| 33 | $2^{10} \cdot 3^2 = 9216$ |
| 34 | $2^{16} \cdot 3^1 = 196,608$ |
| 35 | $2^6 \cdot 3^4 = 5184$ |
| 36 | $2^2 \cdot 3^2 \cdot 5^1 \cdot 7^1 = 1260$ |
| 37 | $2^{36}$ |
| 38 | $2^{18} \cdot 3^1 = 786,432$ |
| 39 | $2^{12} \cdot 3^2 = 36,864$ |
| 40 | $2^4 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 1680$ |
| 41 | $2^{40}$ |
| 42 | $2^6 \cdot 3^2 \cdot 5^1 = 2880$ |
| 43 | $2^{42}$ |

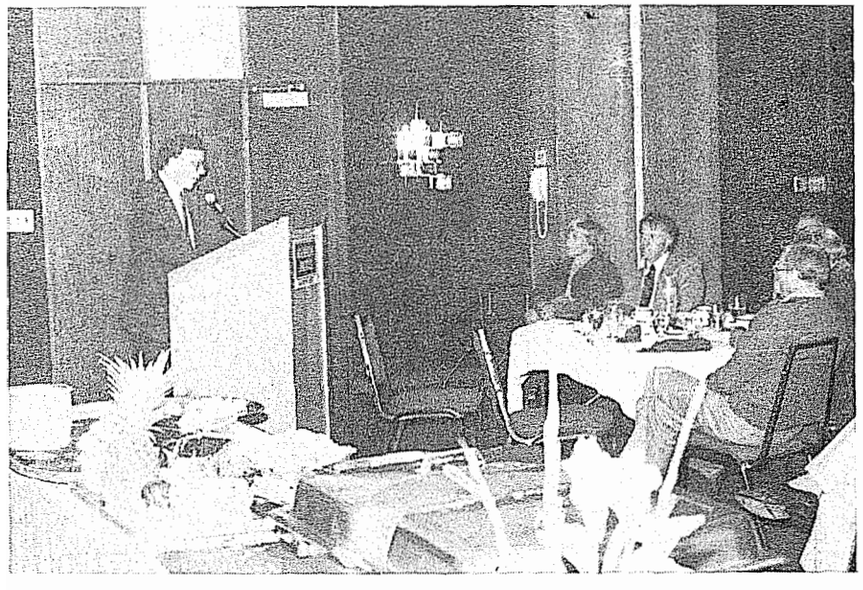
| | |
|---|---|
| 44 | $2^{10} \cdot 3^1 \cdot 5^1 = 15{,}360$ |
| 45 | $2^4 \cdot 3^2 \cdot 5^2 = 3600$ |
| 46 | $2^{22} \cdot 3^1 = 12{,}582{,}912$ |
| 47 | $2^{46}$ |
| 48 | $2^5 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 3360$ |
| 49 | $2^6 \cdot 3^6 = 46{,}656$ |
| 50 | $2^4 \cdot 3^4 \cdot 5^1 = 6480$ |

Challenges for the reader and his/her students:

1) Continue the table to 100 divisors.

2) Given a natural number q, find the number <u>less</u> than q possessing the <u>greatest</u> number of divisors.

(David Duncan and Bonnie Litwiller are Professors of Mathematics at the University of Northern Iowa)



*Moe Liebovitz, after dinner speaker. Leadership Seminar '85*
**Notice the fine spread of food on the buffet*

# AN INTRODUCTION TO THE NATURAL EXPONENTIAL BASE "E"

*by*
*Frank Vitale & Maureen Ricciuto*

We were dissatisfied with the explanation we found in the literature on the natural exponential base "e". We came across several approaches to explain how the natural base "e" may be derived but they lacked clarity and conviction on how exactly the value of "e" results.

**INTRODUCTORY LESSON 1:**

- Students make tables and construct the graphs of $y = 2^x$, $y = 3^x$, $y = 10^x$.

- With straight edge, they construct, by eye, tangents to the curve at $x = -1, 0, 1, 2, 3$.

- They then measure and tabulate the slopes of the tangents constructed.

- Then complete the following table.

| x | $2^x$ | slope of tangent | $\frac{\text{slope of tangent}}{2^x}$ |
|---|---|---|---|
| -1 | | | |
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |

- Hopefully, the last column will yield the constant 0.7 approximately. Students will speculate as to what this constant represents.

- They do the same for $y = 3^x$ and $y = 10^x$.

- Usually, the students comment that there must be an easier way to find the derivative of an exponential function.

- A suggestion might be to try 2 $\boxed{\text{lnx}}$ on their calculators.

- A gasp of surprise will be heard at this time!

CONSOLIDATION: These exponential functions have derivatives which are **proportional** to the value of the function.

- By considering the constants of proportionality for $y = 2^x$ and $y = 3^x$, the students could speculate as to the value of the constants for the functions between $y = 2^x$ and $y = 3^x$. - there must be a function for which the constant is 1.!!

- Have students consider the derivative of $y = 2^x$ from first principles.

e.g. $$\lim_{h\to 0} 2^x = \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h}$$

$$= \lim_{h\to 0} \frac{2^x (2^h - 1)}{h}$$

$$= \lim_{h\to 0} 2^x \cdot \lim_{h\to 0} \frac{2^h - 1}{h}$$

- Students could evaluate $\lim_{h\to 0} \frac{2^h - 1}{h}$ using a calculator.

- Then $y = c^x$ from first principles where $y = c^x$ is the function whose derivative is $c^x$, i.e., the constant is 1.

**LESSON 2:**

1. Development of the Derivative of the Special Exponential Function, such that the derivative of the function is the function itself.

2. The value of "e" as an Infinite Series:

**Proof** of Part (1):

If $y = f(x) = c^x$, $c > 1$, $x \in R$ We want $\frac{d\,c^x}{d\,x} = c^x$

$$\frac{dy}{dx} = \lim_{\Delta x\to o} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{c^{(x+\Delta x)} - c^x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{c^x \cdot c^{\Delta x} - c^x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{c^x ( c^{\Delta x} - 1)}{\Delta x}$$

$$= \lim_{\Delta x\to o} c^x \cdot \lim_{\Delta x\to o} \frac{c^{\Delta x} - 1}{\Delta x}$$

We want the second limit

i.e. $\lim_{\Delta x\to o} \frac{(c^{\Delta x} - 1)}{\Delta x} = \lim_{\Delta x\to o} (1) = 1$

$$\frac{c^{\Delta x} - 1}{\Delta x} = 1$$

$$c^{\Delta x} - 1 = \Delta x$$

$$c^{\Delta x} = 1 + \Delta x$$

$$c = (1 + \Delta x)^{\frac{1}{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \left[(1 + \Delta x)^{\frac{1}{\Delta x}}\right]^{x} \cdot \lim \left[1\right],$$

$$= \lim_{\Delta x \to 0} \left[(1 + \Delta x)^{\frac{1}{\Delta x}}\right]^{x} \cdot \left[1\right]$$

Evaluating $\lim_{\Delta x \to o} (1 + \Delta x)^{\frac{1}{\Delta x}}$ ; use a change of variables as follows

$$\text{let } n = \frac{1}{\Delta x} \quad , \quad \therefore x = \frac{1}{n}$$

and as $\Delta x \to o$, $n \to \infty$

$\lim_{n \to \infty} (1 + \frac{1}{n})^{n}$ , Use the Binomial theorem to expand $(1 + \frac{1}{n})^{n}$ for a large value of n

$(1 + \frac{1}{n})^{n}$ , for large n

$$= \binom{n}{o} 1^{n} \cdot \left(\frac{1}{n}\right)^{o} = \frac{n!}{(n - o)\,!\,o\,!} \cdot 1^{n} \cdot 1 = 1 \cdot 1 \cdot 1 = 1$$

$$+ \qquad\qquad +$$

$$\binom{n}{1} 1^{n-1} \cdot \left(\frac{1}{n}\right)^{1} = \frac{n!}{(n-1)!\,1!} \cdot 1 \cdot \frac{1}{n} = 1$$

$$+ \qquad\qquad +$$

$$\binom{n}{2} 1^{n-2} \left(\frac{1}{n}\right)^{2} = \frac{n!}{(n-2)\,!\,2!} \cdot 1 \cdot \frac{1}{n^2} = \frac{n\,(n-1)}{2!} \cdot \frac{1}{n^2} = \frac{n^2 - n}{2!\,n^2}$$

$$+ \qquad\qquad +$$

$$\binom{n}{3} 1^{n-3} \cdot \left(\frac{1}{n}\right)^{3} = \frac{n!}{(n-3)!\,3!} \cdot 1 \cdot \frac{1}{n^3} = \frac{n\,(n-1)\,(n-2)}{3!\,n^3} = \frac{n^3 - 3n^2 + 2n}{3!\,n^3}$$

$$+ \qquad\qquad +$$

$$\binom{n}{4} 1^{n-4} \left(\frac{1}{n}\right)^{4} = \frac{n!}{(n-4)\,!} \cdot 1 \cdot \frac{1}{n^4} = \frac{n(n-1)\,(n-2)\,(n-3)}{4\,!\,n^4} = \frac{n^4 - 6n^3 + 11n^2 - 6n}{4\,!\,n^4}$$

$$+ \qquad\qquad +$$

$$\vdots \qquad\qquad \vdots$$

$\therefore \lim_{n \to \infty} (1 + \frac{1}{n})^{n}$

$$= \lim_{n\to 0} 1 + \lim_{n\to\infty} \frac{n}{n} + \lim_{n\to\infty} \frac{n^2 - n}{2!\, n^2} + \lim_{n\to\infty} \frac{n^3 - 3n^2 + 2n}{3!\, n^3} + \lim_{n\to\infty} \frac{n^4 - 6n^3 + 11n^2 - 6n}{4!\, n^4} + \ldots$$

$$= 1 + 1 + \lim_{n\to\infty} \frac{\frac{n^2}{n^2} - \frac{n}{n^2}}{2!} + \lim_{n\to\infty} \frac{n3 - 3n^2 + 2n}{\frac{n^3}{3!}} + \lim_{n\to\infty} \frac{\frac{n^4 - 6n^3 + 11n^2 - 6n}{n^4}}{4!} + \ldots$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + . = 1 + 1 + .5 + .16 + .0416 + .0083 + .00138 + .000194 = 2.718228$$

Evaluate the above using calculator for the first 8 terms and we get

$c \doteq 2.718228$, for $c = \lim_{\Delta x \to 0} (1 + \Delta x)^{\frac{1}{\Delta x}} = e$, call it the contant "e"

$$= \left[c\right]^x = c^x$$

$$\therefore \frac{d\, c^x}{d\, x} .1 = c^x .1 \Rightarrow \frac{d\, e^x}{d\, x} = e^x .1,$$ provided $c = 2.7182$ or $c = \lim_{\Delta x \to 0} (1 + \Delta x)^{\frac{1}{\Delta x}} = e$
(Eulers number is called "e")

Euler's Number
(e = 2.718281828459045...)

**LESSON 3:**

Because we were studying the Binomial Theorem in Algebra at the same time as we were introducing the exponential function in Calculus, I gave the following assignment to my Algebra class. Most students were able to complete it and the better students were impressed by the results.

1. Using the Binomial Theorem, expand: (at least 5 terms)

$$(1 + \frac{1}{n})^n =$$

2. Using the fact that the limit of a sum equals the sum of the limits, find: (at least 5 terms)

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n =$$

3. Using a calculator to find the sum of the first 8 terms, estimate the numerical value of the limit in (2) above.

4. Expand: (At least 5 terms)

$$(1 + \frac{1}{n})^{nx} =$$

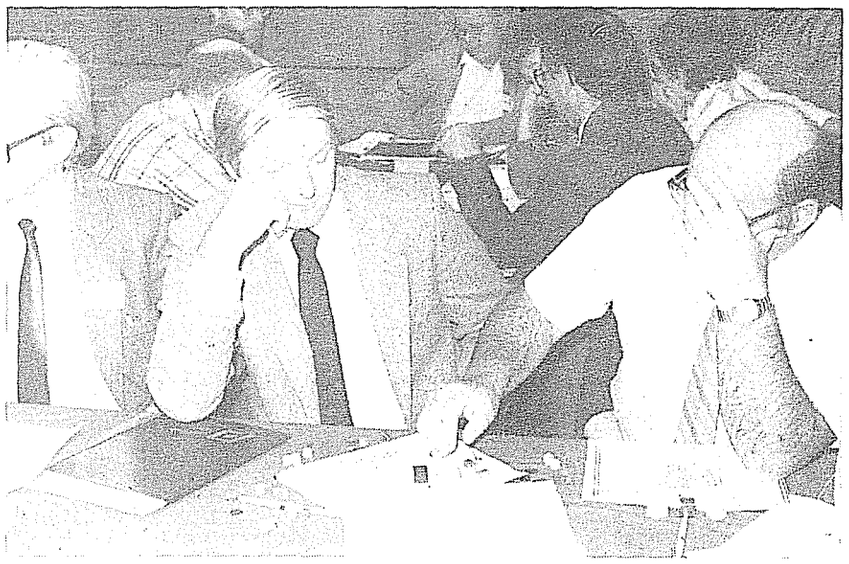
5. Find

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{nx} =$$

6. Find, using the fact that the derivative of the sum equals the sum of the derivatives, (n is constant)

$$\frac{d\left[\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{nx}\right]}{dx} =$$

(Frank Vitale and Maureen Ricciuto teach at Dunbarton High School in Pickering, Ontario)



*...Contemplating the next move.'' Leadership Seminar '85*

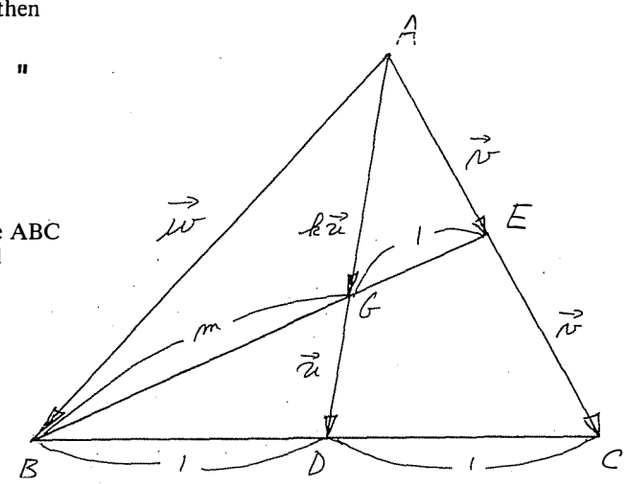
# THE MEDIANS OF A TRIANGLE ARE CONCURRENT!
## Two Additional Vector Proofs

*by*
*Henry Derynck*

Below are shown two different approaches to proving that the medians of a triangle are concurrent.

1. This proof is based on the theorem: "If A,B,C are collinear, if AB : BC = a : b and if 0 is any point not collinear with A,B,C, then

$$\overrightarrow{OB} = \frac{b\overrightarrow{OA} + a\overrightarrow{OC}}{a + b}$$ "




Let medians AD and BE of triangle ABC intersect at G, let AG : GD = k : 1 and let BG : GE = m : 1

Then in triangle ABE

$$k\vec{u} = \frac{\vec{w} + m\vec{v}}{m + 1} \qquad (1)$$

In triangle ABC, $(k + 1)\vec{u} = \dfrac{\vec{w} + 2\vec{v}}{2}$ (2)

From (1) $\vec{w} = k(m + 1)\vec{u} - m\vec{v}$ (3)

From (2) $\vec{w} = 2(k + 1)\vec{u} - 2\vec{v}$ (4)

(3) - (4), $\vec{0} = (km + k - 2k - 2)\vec{u} + (2 - m)\vec{v}$

$\therefore$ $\vec{u}$ and $\vec{v}$ are linearly independent,

$km - k - 2 = 0$ and $2 - m = 0$

Solving we have m = 2 and k = 2

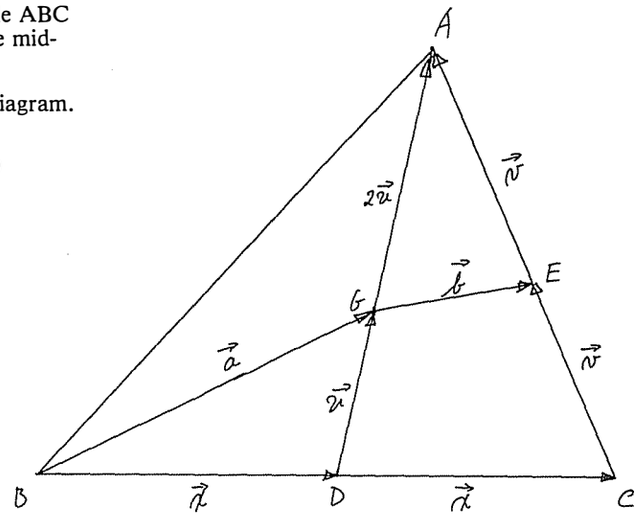
AG : GD = 2 : 1

Suppose medians CF and AD intersect at H. In a similar way it can be proven that AH : HD = 2 : 1. As only one point can divide a line segment internally in a given ration, G and H must be the same point. Therefore the medians of triangle ABC are concurrent.

2. This proof is based on the definition: "If $\vec{BC} = k\vec{AB}$ then A,B,C are collinear points."

Let G be on median AD of triangle ABC such that AG = 2GD and let E be mid-point of AC. Join BG and GE.

Let $\vec{a}, \vec{b}, \vec{x}, \vec{u}, \vec{v}$ be as in the diagram.




Then $\vec{a} = \vec{x} + \vec{u}$ (1)

$\vec{b} = -\vec{u} + \vec{x} + \vec{v}$ (2)

In triangle ADC, $2\vec{v} = -\vec{x} + 3\vec{u}$ or $\vec{v} = -\frac{1}{2}\vec{x} + \frac{3}{2}\vec{u}$

Substitute into equation (2)

$$\vec{b} = -\vec{u} + \vec{x} - \frac{1}{2}\vec{x} + \frac{3}{2}\vec{u}$$

$$\vec{b} = \tfrac{1}{2}\vec{x} + \tfrac{1}{2}\vec{u}$$

$$= \tfrac{1}{2}(\vec{x} + \vec{u})$$

Substitute from equation (1) and we have $\vec{b} = 1/2\ \vec{a}$

∴ B,G,E are collinear. Since AE = EC, BGE is a median. Let F be the mid-point of AB. In a similar manner it can be proven that C,G,F are collinear and hence that CGF is a median.

∴ The medians of triangle ABC are concurrent.

(Henry Derynck teaches at Westgate C.V.I. in Thunder Bay, Ontario)

# "SYNTHETIC MULTIPLICATION"

*by*

*Louis Levine*

While teaching synthetic division to my Grade 12 classes, it occurred to me that synthetic multiplication might exist and have some use for multiplication of a polynomial by a binomial.

Most, if not all, textbooks use synthetic division by x - a and even say that to divide by ax - b one must use ordinary long division. Of course this is not true, since one may use b as the multiplier in the process and remembering to divide each of the quotient coefficients by "a" to get the correct quotient. The remainder is always correct as is.

$\frac{b}{a}$

For example:

division of $12x^4 - 23x^3 + 19x^2 - 24x + 19$ by $3x - 2$

| | | | | | | 2 |
|---|---|---|---|---|---|---|
| | 12 | -23 | 19 | -24 | 19 | 3 |
| | 0 | 8 | -10 | 6 | -12 | |
| ÷ 3 | 12 | -15 | 9 | -18 | 7 | |
| | 4 | -5 | 3 | -6 | | |

∴ quotient is $4x^3 - 5x^2 + 3x - 6$ R. 7.

Reverse this for synthetic multiplication, adding a "0" for a "0" remainder at the end of the row of coefficients. Here's how the procedure would work for example 1:

(a) Place 0 below the first term and multiply each coefficient by 7, placing the product below the **next** coefficient to the right.

(b) To get the coefficients of the product having one degree higher than that of the original polynomial, subtract terms of the second row from those of the first row. The differences create a third row which are the coefficients of the product polynomial.

**EXAMPLE 1**

Multiply $(x - 7)(6x^3 - 5x^2 + 8x - 9)$

| | | | | | 7 |
|---|---|---|---|---|---|
| 6 | -5 | 8 | -9 | 0 | |
| 0 | 42 | -35 | 56 | -63 | |
| 6 | -47 | 43 | -65 | 63 | |

∴ product is $6x^4 - 47x^3 + 43x^2 - 65x + 63$

**EXAMPLE 2**

Multiply $(4x + 3)(8x^5 + 11x^4 - 7x^3 + 5x^2 - 4x + 11)$

∵ multiplier here is - 3, each coefficient is multiplied by 4.

| | | | | | | | - 3 |
|---|---|---|---|---|---|---|---|
| 32 | 44 | -28 | 20 | -16 | 44 | 0 | 4 |
| 0 | -24 | -33 | 21 | -15 | 12 | -33 | |
| 32 | 68 | 5 | -1 | -1 | 32 | 33 | |

$\therefore$ product is $32x^6 + 68x^5 + 5x^4 - x^3 - x^2 + 32x + 33$

Note: Addition of like terms is taken care of automatically.

Synthetic multiplication may also be applied to finding examples of long division in which the dividend has missing terms. In the examples below, the numbers used are arbitrary so that an infinite number of these may be created.

**EXAMPLE 1**

Divisor let us say is $3x - 2$. Required: a quartic dividend $f(x)$ with a zero coefficient for the cubic term.

Let $f(x) = (3x - 2)(ax^3 + bx^2 + 7x - 3)$ and expand by synthetic multiplication.

| | | | | | $\frac{2}{3}$ |
|---|---|---|---|---|---|
| 3a | 3b | 21 | -9 | 0 | |
| 0 | 2a | 2b | 14 | -6 | |
| 3a | 3b-2a | 21-2b | -23 | 6 | |

Here $3b-2a = 0 \therefore b = \frac{2}{3}a$ Now let $a = 6$

$\therefore b = 4$ and we have $f(x) = 18x^4 + 13x^2 - 23x + 6$

**EXAMPLE 2**

Divisor is $5x + 7$ for a quintic dividend $g(x)$ missing the fourth and second degree terms.

Let $g(x) = (5x + 7)(ax^4 + bx^3 + cx^2 + dx - 4)$ and expand

| | | | | | | $\frac{-7}{5}$ |
|---|---|---|---|---|---|---|
| 5a | 5b | 5c | 5d | -20 | 0 | |
| 0 | -7a | -7b | -7c | -7d | 28 | |
| 5a | 5b + 7a | 5c + 7b | 5d + 7c | -20 + 7d | -28 | |

Let $5b + 7a = 0 \therefore b = \frac{-7a}{5}$. For $a = 10$ say $b = -14$.

Also $5d + 7c = 0 \therefore d = \frac{-7c}{5}$ and for $c = -5$ say, $d = 7$.

Thus $g(x) = 50x^5 - 123x^3 + 29x - 28$.

(Louis Levine teaches at Humberside Collegiate Institute in Toronto)



# PROBLEMS

This month's problems come from an NCTM publication entitled **Mathematical Challenges II Plus Six,** which contains 100 problems (and solutions) from algebra, geometry, number theory, probability, and trigonometry, as well as general "puzzle problems." It is from this general category that I've selected this month's problems. Many of you will recognize two of these as almost "classics" of paradox.

1. Two men divided an amount 50-50, yet one got 100 times as much as the other. How can this be?

2. A father left 17 cows to his three sons, with instructions that the first son get one-half, the second one-third, and the youngest one-ninth of the cows. How could they do this without "dividing" any cows? A friend offered to give them a cow, making 18. The first son then got one-half (9), the second one-third (6) and the youngest one-ninth (2). Since $9+6+2=17$ they gave the friend his cow back. How was this possible?

3. Three men stopped overnight at a hotel and decided to share one room, which cost them $10 each. The manager then noticed that the rate for the room should have been $25, so he gave a bellboy $5 to return to the men. The bellboy decided three men could not divide $5 evenly, so he kept $2 for himself and returned $3 to the men. This made the cost of the room $9 each, and the bellboy had $2. But $9+9+9+2-\$29$. What happened to the "missing" dollar?

4. A,B,C, and D are four different weights. When they are placed on a scale:
A and B balance C and D

A and C outweight B and D

c is lighter than D

Arrange the weights in order from heaviest to lightest.

Answers on page 38

## WHAT'S GOING ON...

**Highlights of the upcoming Mathematics Leadership Seminar**
*by*
*Paul Lessard*



ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

ASSOCIATION ONTARIENNE POUR

L'Enseignement des Mathematiques
15th ANNUAL MATHEMATICS LEADERSHIP CONFERENCE

OCTOBER 22 - 24, 1986

**L**EADERSHIP Roles
**E**DUCATORS from Primary, Junior and Intermediate
**A**LTERNATE Teaching Strategies
**D**EVELOPING Novel Teaching Strategies
**E**NRICHING your Math Program
**R**ESOURCES in the 80's
**S**ELECTED Activities from around the Province
**H**OPES for the Future
**I**NDEPTH Study an Organizing Feature of the 2 Day Seminar
**P**ARTICIPATION

Each participant will select 9 sessions from 16 small workshops and 5 featured presentations over these 2 days. This smorgasbord of sessions will be a complimentary blend of hands-on workshops, work session presentations and discussion oriented presentations.

Featured speakers will include:

| | | |
|---|---|---|
| Mr. Frank Clifford | Executive Director | Ministry of Education |
| Doctor Zeo | Author and T.V. Personality | Owl Publications |
| Mr. M. R. Gough | Principal | Hamilton-Wentworth Roman Catholic Separate School Board |
| Mr. Robert Robinson | Consultant | Hamilton Board of Education |
| Mr. G. Knill | Education Officer | Ministry of Education/ Mowat Block |
| Mr. D. Dottori | Education Officer | President/Elect of O.A.M.E. Ministry of Education/North Bay |

Workshops have been selected by the Committee to emphasize the issues of mathematics education in the 80's. The Committee is responding to the notions of accuracy, understanding, and logical thought while continuing to emphasize the importance of a well balanced math program.

A highly motivating feature of this seminar will be the "Puzzle Corner". This "Puzzle Corner" will resember a "Challenge Penny Sale". Participants will place their answers to puzzles in shoe boxes and glass jars, etc. These puzzles will require guessing, estimating, predicting and calculating. There will be a prize table at this site that further establishments this "Penny Sale" flare.

## WORKSHOP LEADERS

**The Junior-Intermediate Team**

### Real Math For Real Kids

| | | |
|---|---|---|
| Jay Symington | Gifted Consultant | Peel Board of Education |
| Ron Sauer | Math Consultant | Waterloo County Board of Education |
| Michael Campanella | Teacher | Hamilton-Wentworth Roman Catholic Separate School Board |
| Jack O'Neill | Enrichment Teacher | Hamilton-Wentworth Roman Catholic Separate School Board |
| Ron Scoins | Professor | University of Waterloo |

**The Primary Team**

### Shared Discovery Mathematics An Inquiry Approach

| | | |
|---|---|---|
| Carolyn McCann | Primary Co-ordinator | Hamilton-Wentworth Roman Catholic Separate School Board |
| Joan Moore | E.C.E. Consultant | H.W.R.C.S.S.B. |
| Judy Adams | Enrichment Teacher | H.W.R.C.S.S.B. |
| Marg Collins | Teacher | H.W.R.C.S.S.B. |

**Your Conference Co-ordinators:** Paul Pogue ..... Barrie
Dino Dottori ..... North Bay
Paul Lessard ..... Hamilton
Morley & Mona MacGregor ..... Sarnia

## ANSWERS TO THIS MONTH'S PROBLEMS

1. One got \$50; to the other \$.50.

2. The three fractions 1/2, 1/3, and 1/9, do not have a sum of one, but a sum of 17/18.

3. The cost established by the manager was \$30-\$5 = \$25. The cost to the men was 3 x \$9 = \$27. The difference of \$2 was what the bellboy kept.

4. A > D > C > B.

# $176 in 86

The registration fee of $176.00 includes:

Shared Accomodation 2 nights
5 Meals over 2 days
2 Complimentary Socials

and
9 Sessions on Mathematics Education

Interested teachers and principals should contact:

Mr. Morley MacGregor
247 Bright Street
Sarnia, Ontario
N7T 4E9

**APPLICATION FORM**
**OAME**
**ANNUAL LEADERSHIP SEMINAR**
at Ramada Inn
5444 Dixie Road,
Mississauga, Ontario
October 22-24, 1986

**REGISTER NOW** **Fee $176.00**

Name .....

Mailing Address .....

.....

School Board .....

.....

Grade taught .....

_____ I desire accommodation Friday evening (at an additional cost)

Fee may be paid in early September but we need your registration form **now.**

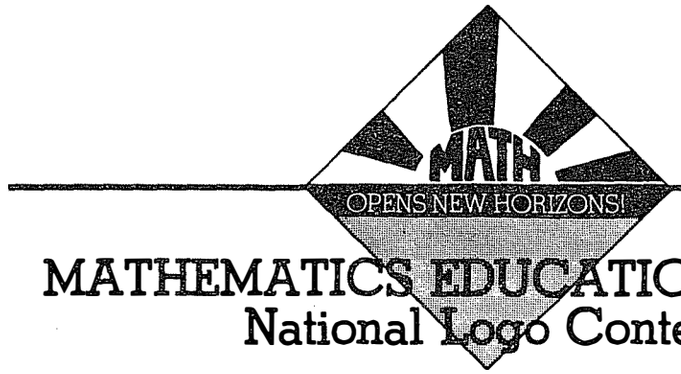
All applications will be confirmed upon receipt of application and/or fee.

OAME phone number (519) 337-8382

See you at the Ramada Airport West on October 22 - 24, 1986

Paul Lessard
President
O.A.M.E.

P.S. Posters are coming to help you promote this Mathematics LEADERSHIP SEMINAR




# MATHEMATICS EDUCATION MONTH
## National Logo Contest

All students in grades K-12 are encouraged to participate

**Guidelines:**

- Designs for use on posters and buttons must be related to mathematics.
- All entries must be originals on paper no larger than 21 cm x 30 cm and become the property of NCTM. Copies or adaptations of other artists' works, cartoons, or logos will not be considered.
- Entries will be judged on the originality of the idea or concept rather than artistic ability.
- Computer-generated entries must be accompanied by an unprotected disk containing the program.
- Entries from any local, state, and provincial logo contests may also be submitted in the national contest.
- Each entrant's name, grade, and school address must be typed or printed legibly on the back of the entry.
- Entries must be mailed to NCTM by a classroom teacher or mathematics supervisor whose name, address, telephone number, and signature, plus member number for NCTM members, appear on the back of the entry as verification of originality.
- A $3 submission fee must accompany each entry.
- All entries and fees must be received by 30 April 1986.

**Prizes:**

All students submitting entries to NCTM will receive a certificate of participation.

One student winner from each state, the District of Columbia, the U.S. Territories, each Canadian province, and one international winner will receive a solar-powered calculator.

One national winner will receive a microcomputer system, and his or her logo will be used in conjunction with Mathematics Education Month 1987.

**Bonus Awards:**

All NCTM members forwarding entries that are chosen as national winners will receive a complimentary one-year extension of their NCTM membership services if their NCTM member number was included on the back of the entry sheet at the time of submission.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1906 Association Drive, Reston, VA 22091 ■ 703/620-9840 ■ CompuServe 75445,1161 ■ Source STJ228 ■ MCI MAIL 266-2653



# COUCIL COMMENT

*by*
*Bob Robinson*

**Editor's Note:** *Council Comment appears in an issue of the* ***Gazette*** *following an OAME/AOEM Council meeting. Your comments are welcome and should be addressed to the editor of the* ***Ontario Mathematics Gazette.***

## NOMINATIONS COMMITTEE

The Nominations Committee (Chairman - Paul Pogue, Dave Davidson, Bob Robinson) have put together an outstanding slate of candidates. Please take a few moments to carefully peruse the list and vitae. Our president-elect, directors and councillors are all elected and not appointed as in some associations. Do both our association and the candidates a service and vote for the candidates of your choice. If you are not a member, consider becoming one so that you can take part in this democratic process.

## CHAPTER REPORTS

### COMA/AMOC

Executive: Chairman: Duff Butterill
Secretary: Davie Hunter
Treasurer: John LeBrun
Chapter Representative: Jerry Lunney
Executive Representatives: Mayer Alvo, Joe Byrnes, Linda Loder, Barry Nicoll, Fran Outerbridge, Richard Rancourt, Ralph Tross

The association has an active executive and a growing membership.

COMA's first activity for the year, held 1985 11 14, featured Dr. Lola May speaking on Problem Solving. At this dinner meeting, Dr. May kept the attention of those present displaying her usual wit and expertise.

On 1986 01 31 COMA hosted a panel discussion dealing with the university perception of the new OAC mathematics courses. Panel members participating in this lively discussion were Dave Alexander, Ministry of Education; John Coleman, Queen's University; Marcel Deruay, Ottawa University; and Kenneth Williams, Carleton University.

Plans are underway for a publisher's book display to be held at the Embassy West Hotel in Ottawa, Tuesday 1986 04 29 from 3:00 p.m. to 7:00 p.m. Thos who attended last year are still talking about the good food and fine display of materials.

### GOLDEN SECTION

Golden Section's efforts are being geared toward the hosting of the '88 Annual OAME/AOEM Conference in Niagara Falls, 1988 05 5-7. The program committee is now in place.

### GVMA

A spring meeting is planned for 1986 04 17 at Eastwood Collegiate in Kitchener. Watch for further information.

### NOMA

Executive: Chairman - George McNabb, Sudbury
Secretary-Treasurer - Bob Tuck, North Bay
Northern Representative - Bill Adamson, Kapuskasing
Western Representative - To be determined

On 1985 11 1-2 NOMA held its annual meeting and professional development workshops in North Bay at Cheppawa Secondary School.

The '86 NOMA "Annual" will be held in Sudbury on 1986 09 26-27. Through the efforts of Mickey Sandblom the former OAME planning committee has been enlisted to plan and run this important professional activity.

*** PAME (Peel Association for Mathematics Education)**
* Council has voted the acceptance of a new chapter. Welcome aboard ...Peel!

Their chapter representatives has not been appointed at this time and an executive is to be formed. A co-operative conference is being planned for 1986 02 21. Over 200 teachers (K-13) have registered.

**PRMA**
PRMA had a very successful meeting on 1985 11 15, with 10 seminars and about 200 people in attendance.

An evening mini-conference is planned for 1986 04 15 with several workshops being organized.

This chapter has awarded one scholarship this year from their scholarship fund.

Medals are also being presented to about 70 students who have shown excellence in Mathematics Competitions from grade 6-13. This has become an annual awards night which will be held on 1986 06 05 this year.

**QSLMA**
The fall meeting was very successful even though the weather did not co-operate. The mathematical complexities of Halley's comet or comet Halley were discussed.

Their spring meeting is planned for 1996 04 16 from 4:00 p.m. to 9:00 p.m. at Queen's University.

Membership in QSLMA has increased.

**SAME**
On 1985 10 22 a dinner meeting was held from 4:00 p.m. to 9:00 p.m. where participants had a choice of 16 workshops (one before and one after dinner) representing all divisions.

Their first afternoon P.D. session was conducted. Woody Sparrow and John Ketelaars demonstrated their newly developed Apple microcomputer software suitable for mathematics from grades 6-10.

Future afternoon P.D. sessions are scheduled for Secondary teachers on Tuesday, 1986 04 08 and primary division teachers on Tuesday, 1986 04 15.

A major Professional Development Day was held on Friday, 1986 02 14 with 18 sessions offered for all levels of instruction including sessions for science teachers at the elementary and secondary levels.

Elections for next year's executive will be held in late April.

## LEADERSHIP SEMINARS

**LEADERSHIP '85**
The 1985 Leadership Seminar went well and was very successful. The participants went away very excitied and enthusiastic to further the teaching of mathematics within our province. It is unfortunate, however, that more people do not take part in this excellent and economical professional activity.

**LEADERSHIP '86**
The 1986 Leadership is being organized and will take place 1986 10 22-24. There has been a change in format this year in that sessions will not take place on Saturday.

The registration fee will be $176.00 in '86 and include shared accommodation, 2 nights, 5 meals over 2 days, 2 complimentary socials, and 9 big sessions on Mathematics Education.

Once again the convenient and deluxe accommodation of the Ramada - Airport West will be utilized.

Watch for forthcoming promotional material or contact Morley MacGregor, 247 Bright Street,

Sarnia, Ontario N7T 4E9.

## ANNUAL MEETINGS

### '85 ANNUAL

Mickey Sandblom presented the OAME '85 (Sudbury) financial statement and Council, once again, expressed their appreciation to Mickey and his committees for conducting an excellent Annual Meeting.

### '86 ANNUAL

The planning is going very well and an exciting "Annual" is anticipated. You should have received your registration forms and supporting information already. It is important to note that if you are planning to stay in residence at the University of Western Ontario, your registration **MUST** be in April 1, 1986 with the accompanying accommodation fee.

### '87 ANNUAL

SAME continues to plan actively for this big event. Every official has been co-operative and eager to host OAME's 1987 Annual. Approximately 50 SAME members indicate a desire to be involved in the planning, organization and management of the convention.

If there are any speakers, topics, or volunteers to suggest ... contact Steve Martin, Conventions Chairman. Also, Steve would like to know if you have any suggestions for a conference logo.

### '88 ANNUAL

The dates are 1988 05 5-7 for the '88 Annual in Niagara Falls. The program committee is now in place.

### OAIP

OAME continues to have input on the Advisory Groups at both the Intermediate and Senior Divisions. The Geometry pool at the Junior Division is ready to go to print but must pass through the subject Advisory Group first.

### NCTM

OAME's delegate at the NCTM Annual in Washington will be Ralph Connelly and the alternate -Dave Davidson.

Notice of a logo contest for Education Month was announced. Submissions should be made directly to NCTM. Awards will be given to each state and province.

NCTM has also announced that they are offering all teachers of mathematics, including elementary teachers, who are in their first year of teaching a 50% reduction of NCTM membership fees until 1986 06 01. Contact NCTM or Ralph Connelly for further information.

## COMMITTEE REPORTS

### MEMBERSHIP

Our membership statistics are quite good. However, we must maintain and even improve our situation. Everyone is encouraged to promote membership in our association.

### PUBLICATIONS

Watch for a change in format for our **Gazette** in the not-too-distant future.

### COMPUTERS

Computer software reviews will be appearing in issues of the **Gazette.** The availability of public domain/OAME software is also being discussed.

### PROBLEM SOLVING

There appears to be widespread misunderstanding by teachers of what problem solving is. This committee is investigating what teachers are actually doing with regard to problem solving at the Junior and Intermediate levels.

**FORMATIVE YEARS**

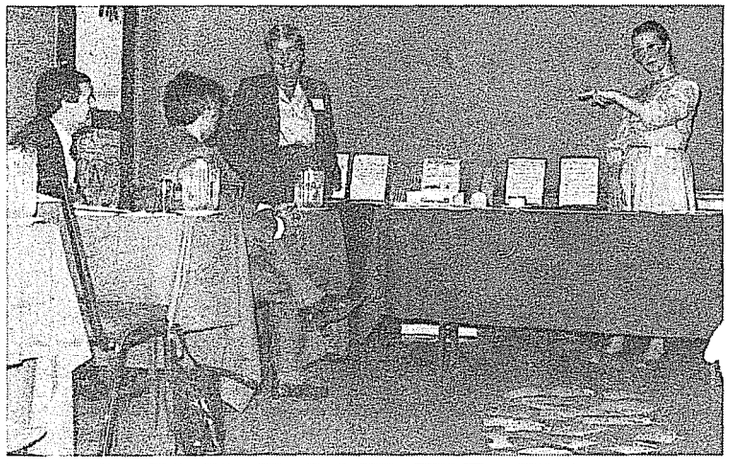
Correspondence has been received from the Ministry. As always, OAME is most interested in providing input.

**WOMEN AND MATHEMATICS**

This committee has seven speakers for the London Conference and is requesting speakers to participate in Leadership '86 and the Scarborough Annual. They also recommend that speakers address equity issues whenever appropriate.

**NEW BUSINESS**

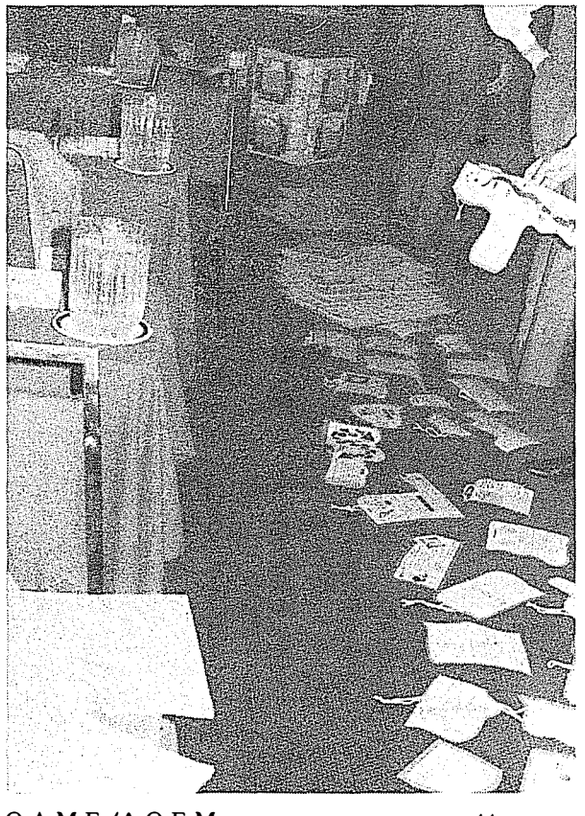
Dino Dottori, our President-elect, has been appointed Superintendent of Education and the North Bay office of the Ministry of Education ... Congratulations Dino!!



*Nanette Deger,*

*"A Tried and True*

*Make and Take...*



*...with plenty activities to work from."*

*Leadership Seminar '85*

## NOTICE TO AUTHORS

Articles submitted for publication should be typewritten or reproduced from type written copy, double-spaced on 8½" x 11" paper with wide margins, and on one side of the paper. Figures and diagrams are to be drawn on a separate sheet of paper in india ink. Authors should take the greatest possible care in preparing the original manuscript. Two copies should be submitted to the editor and one kept by the author as protection against loss. The authors full address and affiliation must appear at the end of the manuscript.

## AVIS AUX AUTEURS

Les articles soumis pour publication doivent être dactylographiés à double interligne et avec de grandes marges au recto de feuilles 8½" x 11". Les figures et les diagrammes doivent être dessinés à l'encre de chine sur des feuilles séparées. Les auteurs sont priés de préparer leur manuscrit avec soin et de soumettre deux copies à l'éditeur, tout en conservant une copie en cas de perte. L'adresse et l'affiliation de l'auteur doivent accompagner le texte.

Editorial Correspondence and Articles:
to -

Ralph Connelly
Brock University
College of Education
St. Catharines, Ont.
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Members of O.A.M.E./A.O.E.M. receive the *Gazette* and *Abacus* free. Single copies $4.00; one year $12.00 in Canada, $14.00 in United States, and $16.00 outside Continental North America.

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