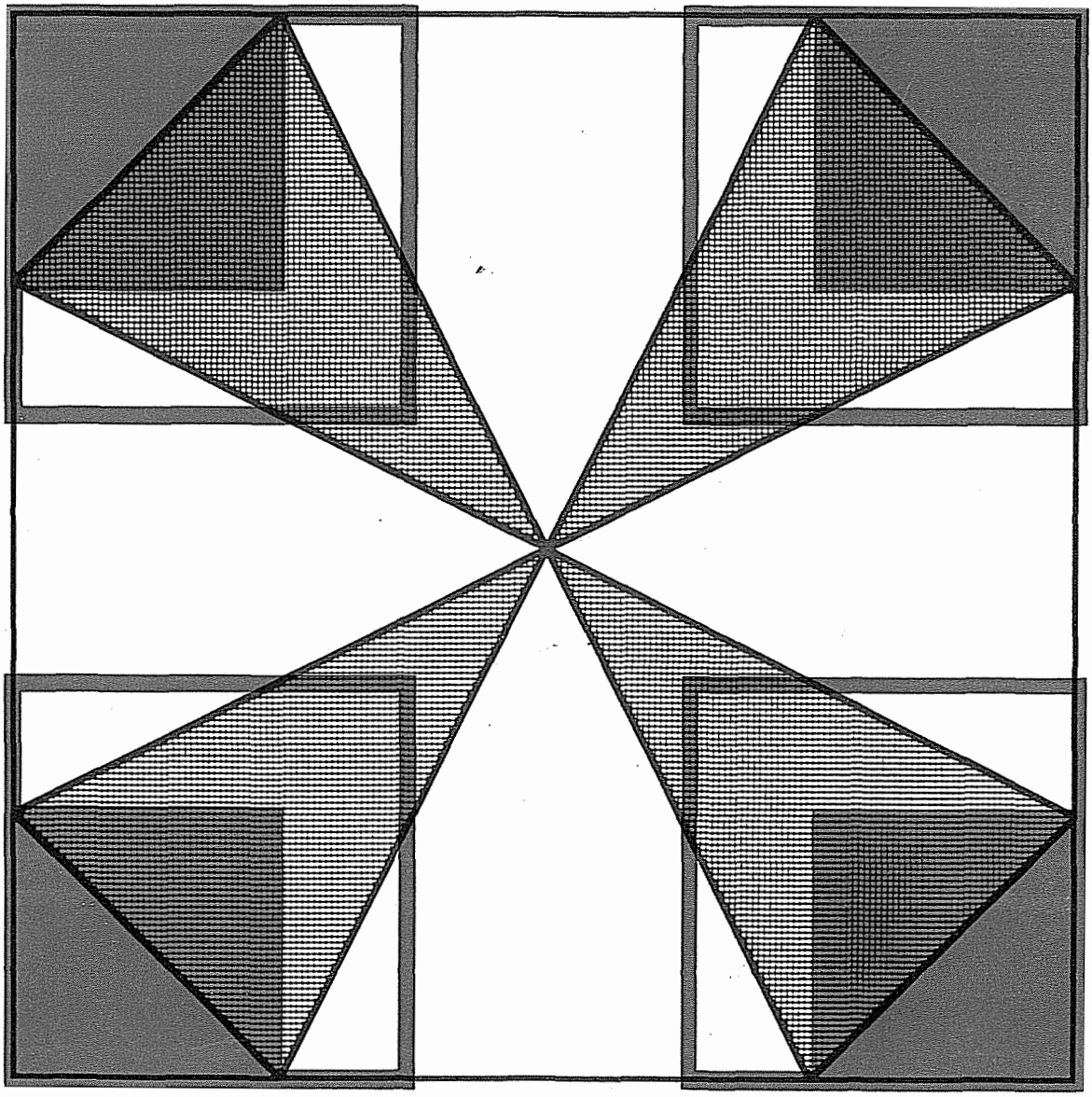
ontario mathematics gazette

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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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Deadlines for submission of information to be included in O.M.G. are as follows:

October 15	for the December issue
January 30	for the April issue
June 15	for the September issue

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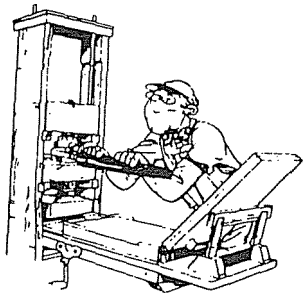
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September - the beginning of a new school year -new pupils, new problems, new resolutions. In March of 1981, as retiring editor, I wrote about problem solving. My problem to-day, as I return as your editor, is how to make the Gazette more attractive and more relevant to teachers in the classroom - a better journal. Some moves in this direction will be attempted during the coming year.

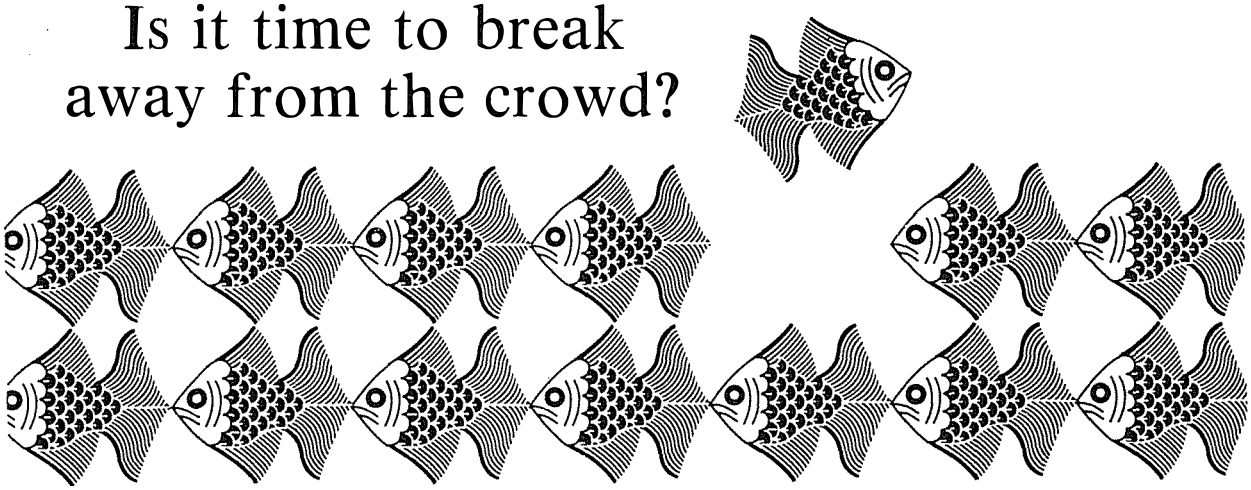
I believe that O.M.G. can serve as a forum for discussion about the teaching of Mathematics, a source of helpful ideas as well as carry information about your organization. To make this belief a reality, your editors have made some changes which we hope will bring forth a better journal. Changes worth attempting are:

1. A policy of blind reviews will be adopted.
2. Galley proofs of manuscripts will be returned to authors for proof reading.
3. We are looking for an associate editor to deal with French articles, because we would hope to publish a French article in each issue. Suggest a qualified person.
4. A column titled "Viewpoint" will be introduced which we hope will generate discussion, debate & letters to the editor.
5. A teaching ideas file will be started in this issue called Teaching Tips.
6. Most obvious will be the new cover design and split page text.

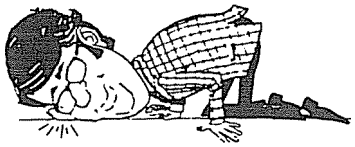
The colours this year and each succeeding year are for a reason. Suggest the reason.

We, the editors have resolved to improve the quality of the Ontario Mathematics Gazette. As you meet your new classes, I am sure you too have resolved to do better teaching. Let us hope that our resolutions are achieved.

Is it time to break away from the crowd?



Send us your teaching ideas.
They should be sharable.
We would like to start
Teaching Tips
(see pages 8, 13, 35 for an example)



LETTERS TO THE EDITOR

by
Bob Smith

183 Ridout Street S
London, Ontario
N6C 3X8
December 8, 1987

Dear Editor:

I subscribe to the Ontario Math Gazette and enjoy the various issues.

I wondered if you might solve a problem for me and officially solve my quest.

In a mathematical sentence is it proper to include dollar signs as in the following example.

A = \$5.98 + \$2.99 + \$3.52
or
A = 5.98 + 2.99 + 3.52
or
whatever

Opening and closing statements would be used as well.

Hope to hear from you soon.

Yours truly

Paul A. Leinweber

Dear Paul:

Your question has to do with form. In a mathematical sentence, the dollar signs would not be included. If younger children are writing their thoughts down to solve an addition question, I would not be concerned, but be satisfied about their thinking. If a senior grade was writing this as a mathematical statement, I would make them aware of the difference.

Bob Smith
Editor

Alberta
June 1, 1988

Dear Editor:

I have been fortunate to attend the O.A.M.E. annual meetings both in Scarborough last year and in Niagara Falls this year. Let me congratulate you on the excellence of the work done at both. The O.A.M.E. Annual Conferences are the best that I've ever attended.

An Alberta Teacher

O.A.M.E. can be proud of the quality of its conferences. When we sometimes get "down" on ourselves for the things we could be doing better, it's good to remember also those things we are doing extremely well.

Ralph Connelly
Former Editor

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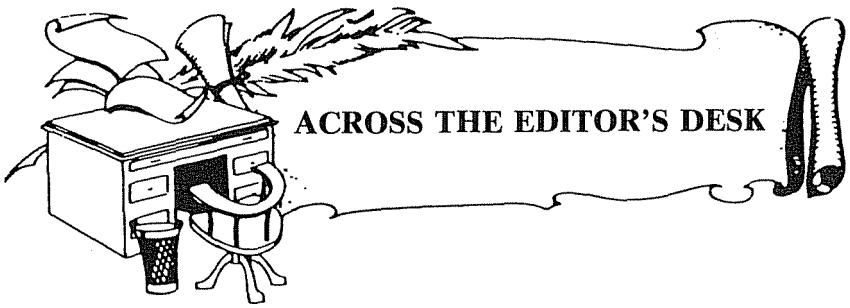
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Bob Smith

This period of change over (March-Sept.) has not been an easy one. I appreciate the efforts made by the acting editor, Dr. Ralph Connelly, to make an easy transition. He was able to pick up the reins after the untimely death of Clifford Gravelle and has passed on an organized backlog of material. With the patience of our contributors and the hard work of my two associates, Eric and Barry, I feel now we have an efficient, functioning organization. But do not hesitate to write if something has been missed during this period of change.

Let me take a moment to introduce the two associate editors.

Barry Onslow has been an educator for the past twenty years, teaching both elementary and secondary school mathematics. He has acted as consultant for the elementary school Mathematics Laboratory Project in Calgary, designed on Piagetian principles, and developed an assessment instrument used to evaluate grade 7 and 8 student performance for The Calgary Mathematics Project, 1980. From 1983 to 1985, Dr. Onslow was a member of the Diagnostic Mathematics Project, Shell Centre for Mathematics Education, University of Nottingham, England. His doctoral research concentrated on the design and development of suitable teaching strategies, and their transferability to the regular classroom. He has presented keynote addresses, concerning children's understanding of mathematics, to convention audiences in Vancouver and Toronto, and has written articles and presented lectures at mathematics education conferences, on the same topic.

Eric Wood has been an educator for the past eighteen years. In addition to teaching secondary school mathematics, he has served as the mathematics consultant for the London Board of Education, where his primary duties revolved around the development and support of elementary teachers. He has published articles in the *Mathematics Teacher,* the *Ontario Mathematics Gazette,* and has articles forthcoming in both the *Arithmetic Teacher* and *For the Learning of Mathematics.* He is presently completing his doctorate at Michigan State University where he is specializing in the field of teacher preparation and staff development.

Both are colleagues in the mathematics department, Faculty of Education, University of Western Ontario.

Eric will be a guest editor for the September, 1989 issue and would encourage articles for that issue. Theme will be *Relations, Patterns and Functions.*

Barry will be guest editor for the December, 1989 issue and would encourage articles for that issue. Theme will be *Statistics & Probability.*



Barry Onslow



Eric Wood

PRESIDENT'S MESSAGE

by
J. Symington

This will be an exciting year for O.A.M.E. One in which this organization will face many changes with many more yet to come.

I must first congratulate Don Attridge on his appointment as Secretary-Treasurer of O.A.M.E. Don officially assumes his new role in Ottawa at the May Annual Conference 1989. However, I suspect we will be seeing Don at council meetings long before that. Don has been a strong supporter of O.A.M.E. throughout the years. He was the first "Past President" of O.A.M.E. and last year he was the recipient of the "Life Membership" award. I'm sure all of the O.A.M.E. membership joins me in wishing Don all the best with this future position.

To-date, we have the strongest membership representation ever. This past year brought forth over 1700 members to our organization. As well, the number of chapters has grown from 8 to 11. Our chapter representation now spans the province from Thunder Bay to Ottawa and from Kingston to Niagara Falls. At the annual meeting in Niagara Falls, the membership welcomed these three new chapters to our organization: York4 Mathematics Association; South Western Ontario Mathematics Association; North Western Ontario Mathematics Association. The opportunity for our memberships to grow is there, let's continue to spread "the word."

If you attended the '88 Annual Conference in Niagara Falls, I am sure you will join me in congratulating the entire organizing committee for the excellent conference. If you missed that conference, then plan to attend the '89 Annual Conference in Ottawa. The plans for this conference are well underway and looks like it will be another huge success.

This year we must thank Bob Smith for taking on the job as Editor of the Gazette. Bob has been the Editor of the Gazette in the past and brings much experience to this job. Thank you Bob for your assistance and commitment to mathematics education.

Of those deserving recognition and thanks at this time is the now "past President" Dr. Ralph Connelly. Ralph faced a challenging year as President of this organization as well as being involved with the organization of the Annual Conference in Niagara Falls. Ralph's hard work and commitment to O.A.M.E. helped make his year a successful one. The one last job Ralph has is to organize this year's Leadership Seminar, October 19, 20, 21, 1988. The program has been finalized and appears to be the best yet. The workshops span all four divisions which makes this seminar a must for those attending fall conferences.

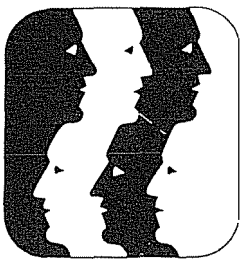
Change can and should be a positive force behind an organization especially for one exhibiting such potential growth as O.A.M.E. Our mission has been, and will continue to be, "to promote excellence in mathematics education throughout the Province of Ontario."

1988 PROBLEM SOLVING RESEARCH AWARD WINNERS

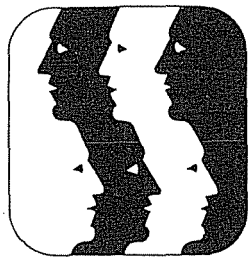
First Prize Winner - Ruth Douglas of Ottawa
Second Prize Winner - Trudy McKeown of Kingston
No Third Prize awarded.
Format and criteria for the awards as well as an example will be printed in the December issue of O.M.G. Watch for the award-winning report.

NOTICE TO OAME MEMBERS

New Certificates will be issued for the year 1990/91.
For those members who have Certificates with all the designated areas for seals filled, we ask that you place them at some place to your liking on the certificate for the years 88/89 and 89/90.



Viewpoint



NEW GOALS OF MATHEMATICS INSTRUCTION FOR ELEMENTARY SCHOOLS

by
Bill Newton

Formerly
Ministry of Education
Willowdale, Ontario

Now
Ministry of Education
Melbourne, Australia

This quick sketch canvasses some of the emerging emphases in mathematics instruction that could be of interest in elementary school programs. The paper is designed to give a flavor of areas likely to become increasingly important to competent, critical, numerate citizens. The paper also makes a quick stab at areas in the mathematics course which are "dead as dodos". It begins with a comment on the political context in which such changes are likely to occur.

THE POLITICAL CONTEXT IS NOT FAVOURABLE TO A CHILD-CENTRED ACTIVE-LEARNING PEDAGOGY

We are in the midst of a period of intense educational reform but the focus of that reform is reactionary, not progressive. Politicians are not listening to educators. They are listening to parents and other education stakeholders. So, despite a decade in which our knowledge gains on children's learning has been enormous, especially in reading, writing and metacognitive skills, the political push for change reflects concerns which are basic skills oriented, and which are tuned to technocratic-efficiency indicators such as accountability, "benchmarks", testing, school improvement, excellence, and so on. These are legacies of such fear-mongering reports as "A Nation at Risk" and Carnegies' "A Nation Prepared". Such is the mileau in which educators throughout the world must press their case.

Constance Kamii, probably the best known of elementary educators in the USA has commented that it's not easy preserving our professional gains, and that too often, the dilemma is resolved in favor of the political rather than the pedagogical.

All of this points in the direction of the obvious: pedagagy must itself be political. To review "The Formative Years" as though we, and especially teachers, live our lives in a political/philosophical vacuum, would be naive. Teachers will especially feel the weight of parents' concerns.

There is an answer to the dilemma, probably more than one. But a simple answer is - Education. There is simply no countervailing information reaching parents that could explain what the pedagogical alternatives are.

Any review should include as part of its implementation plan, proposals for workshops for parents to explain any changes that are being made and why they are being made. Until that happens, parents will stay with what they know - their own school experience -and teachers will bear the brunt of their not unreasonably reactionary concerns. All of which mitigates against significant change flowing from any review of "The Formative Years".

THE MATHEMATICS OUR STUDENTS NEED

Our students need more than skills. Add, subtract, long division, ad nauseum. It's dinosauric. They need the ability to think and calculate on their feet. In particular, there are five or six areas which should be emphasized.

1. **Quantitative Thinking** e.g.
   - an "at-homeness" with number and applying number to real situations
   - facility with estimation and mental computation
   - non-formal written computation (there are a lot of informal methods which children will invent)
   - practice in developing one's own assumptions in problem solving
   - handling real data, personal data
   - exploring real data, making sense of it. eg. designing a way of keeping track of a season's sports results in hockey. Making inferences from such data.

2. **Number Sense** e.g.
   The key element of "number sense" is the capacity to estimate - number, quantity (distance, time, mass, volume, ...) e.g. How far away is that house? How fast do I walk, run? How many people are here? How did you estimate it? Estimating the results of computation. Having a sense of accuracy. Being a good guesser and estimator in number, measurement and location.

   The context for "number sense" is verbal, intuitive, discursive, interest and reality based, problematic as distinct from right or wrong, and importantly, outside the classroom for much of it.

3. **Mental Computation**
   - informal strategies designed to get exact answers;

     e.g. $56 + 29 \rightarrow 55 + 30$
     $75 - 59 \rightarrow 76 - 60$

   - informal strategies designed to obtain estimates eg round off and work with "leading digit": $541 \div 33$ becomes $500 \div 30$
   - strategies for round-off and approximation to give a "feel" for number

e.g. $\frac{25}{32} + \frac{1}{4} = \frac{3}{4} + \frac{1}{4} = 1$

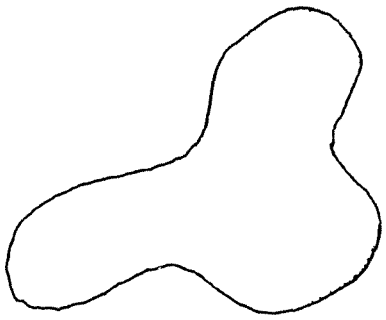
$2.32 + 12\ 7/8 = 2 + 13 = 15$

"Messy" numbers can be rounded-off. Estimation becomes simple.

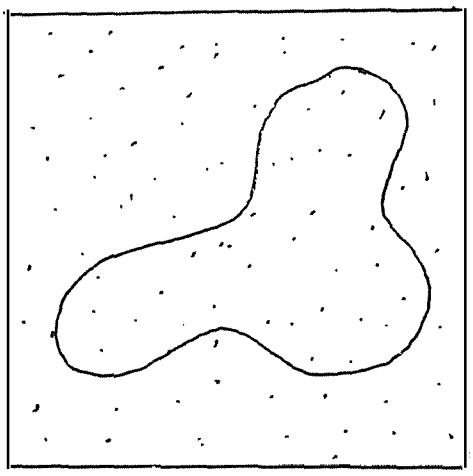
4. **Guestimating and Problem-Making**
Students design their own interesting problems and devise strategies to arrive at a guestimate, e.g., a coal train rolls by. How much coal? Need to guess number of coal trucks, column of truck (length, width, height). How many metric tonnes? What are the quantitative problems which interest students? Framing a problem so that it becomes solvable.

5. **Elementary Data Analysis, Probability and Statistics**
Yes, at elementary level. Personal data, (Heights, weights ...), percentages (informal sense of %), graphs, newspaper tables and graphs, and their interpretation. Computer simulation, random events, e.g.

(a) Find the area of:



(b) Surround the figure with a regular figure, say a square, whose area is known



Area of Square is 4 units

(c) "Bombard" area with random points, and count the proportion of points (p) in the irregular figure. Then area of irregular figure = p x 4.

A totally new view of area calculation brought about by new technology and high speed computer simulation and calculation. Other computer simulation experiments involving random events like coin and die tossing, prediction, expectation.

6. **Intuitive Geometry and Spatial Awareness** e.g.
Map reading, street directories, orienteering, directions. Body sense of N.S.E.W. and their relationship to phenomena known to the child. e.g. our street runs E-W. My school is due West from home.

Greater emphasis on the practical, reality-based geometry of location and position in space. Far less emphasis in elementary schools on the formal geometry of shape, including isometries and the geometry of transformation.

Note that each of the above areas emphasises mathematics which is practical, personal, reality-based, meaningful, useful. By emphasizing these aspects of mathematics, we imbed the "raw" mathematical content in contexts which are meaningful to children, and which support their natural attempts to make sense of things. Teachers too, will feel less alienated from mathematics with this approach. Many elementary teachers feel anxious about teaching a mathematics which is so clearly unrelated to their own knowledge, interests and experiences.

SUPERCEDED GOALS OF MATHEMATICS INSTRUCTION

1. **Long Division**
A process seemingly designed to be made obsolete by the calculator.

Long division with two and more digit divisors can be replaced by the calculator, estimation, round-off and informal strategies involving, pattern in numbers. e.g. 384 divided by 26.9 is approximately 400 divided by 25 = 16 which compares well with the calculator result 14.3.

2. **Long Multiplication**
The formal long multiplication process for "multiplies" like 352 x 29 can now be dropped. Use informal methods or estimation combined with use of the calculator.

3. **Equal Additions Subtraction** ("Borrow and Pay Back")
Like long division and long multiplication, this is a hangover from an uncomprehending, digit-manipulating past. We're not training ciphering clerks any more.

Again, use informal methods, patterns in number,

estimation, calculator, ...

4. **Sets and Set Language Terminology**
   These topics are inappropriate for children at these levels. They do not provide the generalizations and understandings that were hoped for.

5. **Formal Geometry**
   Formal geometry of shape, including jargon-ridden terminology (isometries, translations, reflections, ...) should be expunged forthwith. The geometrics of (i) shape and (ii) location should be dealt with informally, practically and linked to real-world applications. Formal geometry at the P-J level is alienating for teachers and students alike and counter-productive to the goal of teaching geometry in elementary schools.

GOALS OF MATHEMATICS INSTRUCTIONS THAT WILL NEVER CHANGE

1. Knowledge and understanding of the arithmetical operation (addition, subtraction, multiplication and division) and their inter-relationships.

2. Knowledge and understanding of the basic number facts and
   (i) their extension to larger numbers e.g. $3 \times 5 = 15$, $3 \times 50 = 150$, $30 \times 50 = 1500$. Critical for developing estimation skills.
   (ii) their inter-relationships $3 \times 5 = 15$, $15 \div 5 = 3$, $1/3$ of $15 = 5$, ...

3. Place value: key concept for structure of number, operations, mental computation, estimation. Key to flexibility in use of number
   e.g. $78 = 70 + 8 = 60 + 18 = 50 + 28 =$
   Place value concepts are especially useful in mental computation.

4. Pattern in number, as with extended number facts (above) and other number patterns; especially tables, number charts, geometric patterns, patterns in nature.

A FINAL COMMENT

Our focus then is to develop and extend the quantitative aspects of children's lives, to give them skills and a number sense to deal with those aspects.

Today's students will enter a world of employment in which the formal mathematics requirments for individuals will be diminished! Think how little mathematics is required even today. The trend will continue. As citizens though, they will need to understand mathematics (and science) not just for jobs, but for living. We need a population that is not merely literate or numerate, but one which is *conceptually* literate and thoughtful. One which understands what the options are, what the debate means, and isn't "snowed" by experts showering them with graphs, charts, tables and percentages.

A tall order eh?

Well, that's the direction we should be heading.

Teaching Tips

EXPLAINING TRANSFORMATIONS TO THE QUADRATIC FUNCTION

When studying the quadratic function $y = (x - p)^2 + q$ students typically graph things such as $y = (x - 2)^2$ and $y = x^2 - 2$ to establish the effect of these changes on the graph of $y = x^2$. Unfortunately there is a troublesome inconsistency: The graph of $y = (x - 2)^2$ is moved two units to the right rather than left as expected, while the graph of $y = x^2 + 2$ is moved up two units just as one would expect!

A good explanation for this apparent problem is to rewrite an equation as $y = (x - 3)^2 + 5$ in the form $y - 5 = (x - 3)^2$ and then observe that *it is not the curve that moves but the reference system or the axes.* Consequently if the axes move five units down and three units to the left, the curve *appears* to move five units up and three units right. The vertex would then be (3, 5) which is the correct result. This explanation has the advantage of being consistent and not requiring any detailed knowledge of transformation equations.

Eric Wood

MAGIC SQUARES FOR THE CLASSROOM

by
William A. Ewbank
Associate Professor of Mathematics Education
Taylor University
Upland, Indiana 46989

Teachers through the eighth grade feel the need, from time to time, to give their classes practice in addition, and all textbooks have pages of drill problems for this purpose. But it makes more sense if this review can be combined with a glimpse into the interesting patterns of the number system. Magic squares have supposedly been known to man for over 4000 years (Ball and Coxeter, 1974), and their study gives much scope for mental arithmetic, as well as promoting concentration and problem solving. This article will suggest ways in which magic squares can be introduced in the early grades, with increasing challenges at intermediate and middle levels.

In essence a magic square is an array of numbers in which each row, each column, and each main diagonal add up to a "magic constant." The simplest version is the 3 x 3 (known as a magic square of order 3), using the numbers 1-9 -- the so-called regular magic square:

8	1	6
3	5	7
4	9	2

There is only one solution to the regular order 3 magic square, not counting rotations and reflections as different. Below are examples of variants:

4	3	8
9	5	1
2	7	6

8	3	4
1	5	9
6	7	2

Younger children should solve magic squares with manipulatives, namely, small numeral cards. They can make these themselves. First graders will need a string of beads or equivalent to aid in the addition, and second graders should use a number line and/or beads.

I would not ask students of any age to solve the above regular order 3 magic square without giving at least one hint -- 5 has to go in the centre. Younger students would need another hint -- the even numbers go in the corners. Advanced students will be able to deduce that the magic constant must be 15, and some might even be able to analyze the situation and set up the magic square without any hints.

Next, we could consider whether the arrangements would still be "magic" if we were to:

add a given number to each number:
$(a = b \rightarrow a + n = b + n)$

multiply each number by a given number:
$a + b = c + d$

$\rightarrow n(a + b) = n(c + d)$

$\rightarrow na + nb = nc + nd$

The answer is Yes in each case, for the reason given in the above theorems. Can this treatment be combined? Yes. Can you see how the following transformation has been effected?

8	1	6
3	5	7
4	9	2

→

17	3	13
7	11	15
9	19	5

The rule here was: $a \rightarrow 2a + 1$.

Classes of third to fifth graders could be assigned different rules, e.g. $3a + 1$, $2a - 1$. etc., and instructed to check that the new arrays formed are truly "magic."

Older students who have been introduced to prime numbers may notice that the second magic square above is all primes except for two numbers (9 and 15). Can you have a 3 x 3 magic square using only primes? There is only one even prime number (2), and so no other even numbers are candidates. But for this reason, 2 could not be included, because some lines of three numbers would end up even:

even + odd + odd = even.

And the rest would add to odd:

odd + odd + odd = odd.

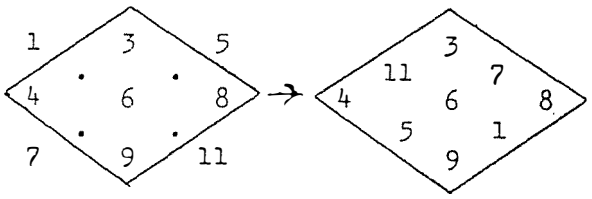
H.E. Dudeney devised a magic square using the number 1, and eight prime numbers less than 100. This is an advanced problem, which requires a knowledge of what sets of numbers qualify for inclusion in any magic square. This question will be considered next.

Magic squares of order *n* can only be made from *n*

arithmetical progressions (AP) of numbers, the difference between the first numbers in each AP being constant. A simple example will make this clear:

1	3	5
4	6	8
7	9	11

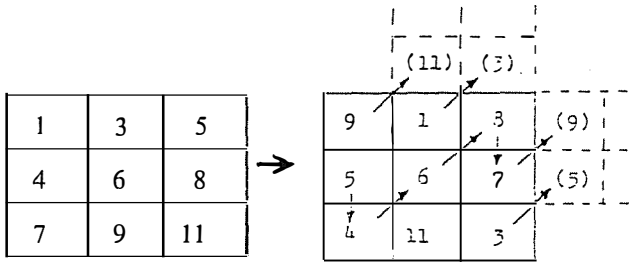
1, 3, 5 . . ., 4, 6, 8 . . ., 7, 9, 11 . . ., are all APs with common difference (CD) 2. Vertically there is a CD of 3, so the vertical sequences are also APs. There are various ways of getting this array into its "magic" form. One that will appeal to children is the "jumping method." A diamond is inscribed around the middle five numbers. Into the spaces (marked by dots) are "jumped" the four corner numbers. Diagrammatically:



4	11	3
5	6	7
9	1	8

Each outside number has "jumped across" the middle 6 into the vacant space. The magic constant is 18.

Another method for arranging the magic square is to place the numbers into the nine cells in a systematic way. Diagrammatically:



(treated as the sequence 1, 3, 5, 4, 6, 8, 7, 9, 11)

The rules are:

1) start in the top middle cell,

2) go N.E. When you "jump out" of the square, go down (or across) to the corresponding cell in the square, and continue N.E.

3) if a cell is occupied, drop down to the one below.

4) if you land up in the top right hand corner, drop down one cell.

The magic square above is a reflection of the one achieved by the "jumping method."

Before extending the methods used above, we will examine the magic square of order 4. The regular sequence 1-16 written in a 4 x 4 array in order is easily changed into a magic square by switching the order of the two main diagonals. Diagrammatically.:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Not only does each row, each column, and each main diagonal total 34, but so do the four corner numbers, the four middle numbers, and certain others. e.g. (2 + 8 + 15 + 9) and (3 + 12 + 14 + 5). This is only one of 880 different arrangements of 1-16 into a magic square (Moran, 1981).

Some questions can now be investigated --

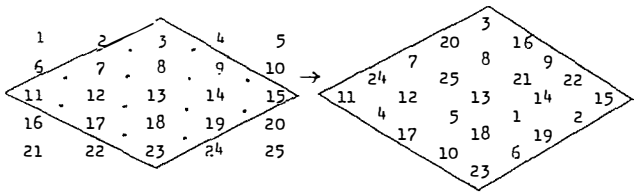
1) Does a 4 x 4 magic square remain magic if a constant number is added to or subtracted from each number?
2) -- or if each number is multiplied by a constant?
3) Can a series of four APs, with a CD vertically, be made into a magic square by the switching of diagonals?

All of these investigations, which should lead to positive results, give a great deal of practice in addition. If practice in multiplication is also needed, Madachy gives this 6 x 6 magic square, which also has the remarkable property that the inner 4 x 4 square has a magic *product* of 401 393 664 in all rows, columns, and main diagonals (Madachy, 1966):

223	283	200	322	163	164
177	408	336	244	12	178
228	122	24	306	448	227
258	36	488	112	204	257
308	224	102	48	366	307
161	282	205	323	162	222

Generally the construction of even-order magic squares is much more difficult than that of odd-order ones.

Now we will see how the methods used above for constructing the 3 x 3 magic squares are used for other odd-order magic squares. Older students will enjoy adapting the "jumping method" to 5 x 5, which is shown diagrammatically:



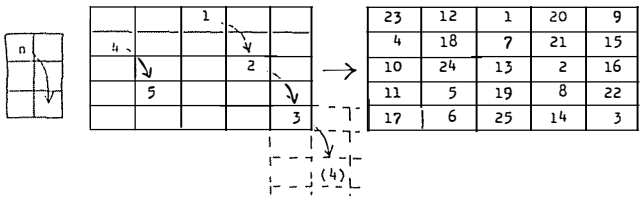
11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

It will be noted that with the 3 x 3, one number at each corner had to "jump" inside the diamond. With the 5 x 5, it is three at each corner. The number to be "jumped" in each corner is a function of the order of the magic square, as indicated in the table:

order of magic square	number at each corner to be jumped
3	1
5	3
7	6
9	10
•	•
2n + 1	½n(n + 1)

The second method, starting at top center and going N.E., uses the rules outlined above, and works for all odd-order magic squares. There are numerous other methods, leading to over 275 million different regular 5 x 5 magic squares (Moran, 1981), but one other will be given -- the "knight's move."

In the "knight's move" method, which itself has many variations, you start in the middle and move to empty cells by a chess knight's move. Diagrammatically:



23	12	1	20	9
4	18	7	21	15
10	24	13	2	16
11	5	19	8	22
17	6	25	14	3

Students could apply these methods to magic squares of their choice, using a series of APs, and could check them. A good challenge would be for students to work in pairs (for checking each other's results), and design a 5 x 5 magic square along the above lines, and then interchange three numbers, rewriting the result as a puzzle to be solved by their classmates. An example of such a "distorted" magic square is given below:

22	34	19	33	20
22	24	36	23	25
27	24	26	21	25
27	29	16	28	30
32	28	31	18	30

In this puzzle, two rows, two columns, and the two diagonals are unaltered, and the remainder will not add to the magic constant. The puzzle is to restore the magic square by locating the numbers that have been misplaced.

Another problem would be to recreate a magic square whose numbers had been completely scrambled, as below.

18	40	30	24	28
27	15	31	32	26
34	33	23	43	25
37	27	35	21	29
31	19	22	36	39

To solve this, the numbers would have to be rearranged into five APs, and the magic square constructed by one of the given methods.

The notes here are just a scratching of the surface of the topic of magic squares, a topic that leads to many avenues of intellectual pursuit, sometimes with much complexity. The puzzle literature, some of it listed in the bibliography, is replete with these mathematical curiosities. Students not considered gifted mathematicians can have fun and much computational practice at the same time. The teacher need not be burdened with a heavy load of problems to correct, because magic squares are largely self-checking. The teacher can have fun too!

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AN APPLICATION OF POLYNOMIALS USING A PROBLEM-SOLVING SETTING

by
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The following activity can be used in virtually any advanced level secondary school mathematics course, but has been written with the intent that it would be used in either grade 9 or 10. It is intended to reinforce the algebraic manipulation of polynomial expressions. In completing this activity, students will also develop a better understanding for what is meant by a decimal number, and will further develop their problem-solving skills.

It is not necessary to get involved in very complex mathematics to demonstrate that the subject has an inherent beauty all of its own. Simple number patterns can form the basis of some very interesting activities.

When we write the decimal number *247* we realize that this means "2 groups of *one hundred* plus 4 groups of *ten* plus 7 *ones*". A shorter way of saying this is $247 = 2 \times 100 + 4 \times 10 + 7 \times 1$. In order that no confusion arises, the following notation is introduced:

$$[abc] = 100a + 10b + c$$

Thus,
$$[348] = 100(3) + 10(4) + 8.$$
$$[57] = 100(0) + 10(5) + 7$$
$$= 10(5) + 7.$$

After polynomial expressions have been presented to the class, the teacher might want his/her students to be able to expand, simplify, and manipulate algebraic expressions which involve simple polynomials. One way to do this in an innovative way is as follows.

At the beginning of the lesson, the teacher claims to be able to mentally calculate the square of any two-digit number having a units digit equal to 5 faster than any student. Students are invited to submit such numbers, and the teacher responds immediately by writing the result on the boards. For example,

$15 \times 15 = 225$

$25 \times 25 = 625$

$35 \times 35 = 1225$

and so on.

After a few such computations, some students will begin searching for patterns in the answers. A discussion will result, with most students being able to conclude that the digits in the *ten's* and *one's* column of the answer must be *2* and *5* respectively. What is not readily apparent is how to complete the calculation.

It is now appropriate for the teacher to proceed to illustrate algebraically how these mental calculations can be done.

The problem in its most general setting is to find the square of [a5]. Here is the simple algebraic solution.

$$[a5][a5] = (10a + 5)(10a + 5)$$
$$= 100a^2 + 100a + 25$$
$$= 100a(a + 1) + 25.$$

(This solution reinforces several skills, including finding the product of two binomials, collecting like terms, and finding a common factor).

From the answer above, it is clear that the right-most two digits are 25. Furthermore, the weight in the *hundred's* column is $a(a + 1)$.

Thus, upon comparing 25 x 25 with [a5][a5], we see that a = 2. The right-most two digits in the answer are 25, and the left-most two digits are 2 x 3 = 6. Hence 25 x 25 = 625. (The teacher should ask the class to verify this result.)

It is interesting to see what happens when $a(a + 1)$ exceeds 10. Ask the students to mentally calculate 35 x 35 using this technique. Most students will see that in this case, a = 3, and thus a(a + 1) = 12. A brief discussion amongst some of the students might follow, with the conclusion being that when this situation arises, it is necessary to perform a *carry* into the next column to the left. Thus *12* groups of *one hundred* can be restated as *1* group of *one thousand* and 2 groups of *one hundred*. The computation is shown below.

$$35 \times 35 = 100(12) + 25$$
$$= 1200 + 25$$
$$= 1225.$$

Once this example is completed, have the students use the rule to find the product of the remaining squares of the form [a5]. They will be fascinated. Moreover, they will truly be proud of the fact that they used some simple algebra to arrive at the rule.

As a result of this presentation, some interesting extensions which may form the basis of further activities or investigations will follow. It is interesting to have students investigate whether or not it is possible to find rules which will allow one to *easily* compute products of the form

1. [a1][a1], [a2][a2], ... [a9][a9]
2. [a1][a1][a1], [a2][a2][a2], ... [a9][a9][a9]
3. [a(a + 1)1][a(a + 1)1]

(For example, 341 x 341 = ? Notice here that the

hundred's digit and the *ten's* digit are consecutive.)

Such investigations may be carried out by having the students work in groups. One method of dividing the class into groups for this investigation is described as follows.

Prior to breaking the class into groups, the teacher should decide on the number of investigations to be undertaken. For example, suppose that it is desired to investigate whether or not it is possible to find rules for *easily* computing products of the form [a1][a1][a1], ... [a5][a5][a5]. The class would then be divided into *home* groups, with five students per group. (If the number of the students in the class is not divisible by five, the teacher may decide to have one or two groups with six students.) The *home* group structure would appear as follows, assuming four groups of five students.

GROUP 1					GROUP 2				
A1	B1	C1	D1	E1	A2	B2	C2	D2	E2
GROUP 3					**GROUP 4**				
A3	B3	C3	D3	E3	A4	B4	A4	D4	E4

The students within each group would then decide amongst themselves who would be responsible for each of the five investigations. Once this has been decided, the students from each group responsible for investigating the same product would form a subgroup and begin their investigation. For example, the *A's* might be responsible to investigate the product *[a1][a1][a1].* The subgroup structure would be as follows.

SUBGROUP A				SUBGROUP B			
A1	A2	A3	A4	B1	B2	B3	B4
SUBGROUP C				**SUBGROUP D**			
C1	C2	C3	C4	D1	D2	D3	D4
SUBGROUP E							
E1	E2	E3	E4				

Once each subgroup has completed their investigation, the students return to their home groups. Each member in the home group is then responsible to present his/her findings to the other members of the group.

Teaching Tips

ASSISTING THE CONCEPTUAL UNDERSTANDING OF ADDITION AND SUBTRACTION OF INTEGERS

Helium:
A positive quantity.





Sandbags:
A negative quantity.



Use a vertical rather than a horizontal number line.

Concrete Representations		Abstract Concept
Add helium, the balloon rises.	⟷	Adding a positive quantity produces a larger answer.
Subtract helium, the balloon falls.	⟷	Subtracting a positive quantity produces a smaller answer.
Add sandbags, the balloon falls.	⟷	Adding a negative quantity produces a smaller answer.
Subtract sandbags, the balloon rises.	⟷	Subtracting a negative quantity produces a larger answer.

* * *

Suggested by Paul Godward
Student 1987/88, Faculty of Education, U.W.O.

COMPUTING DIFFERENCE TABLES

by
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While teaching a grade 10 computer class about the history of computers, a topic they generally find boring, I stumbled across a topic they found rather fun. We were discussing Charles Babbage's *Difference Engine.* The students had just finished reading about its ability to calculate differences between numbers in a column and then print those differences to form a new column. We started to calculate some *Difference Tables* and soon discovered some interesting number patterns emerging. Since we were using exponents in our calculations, the task soon became tedious. We were spending more time calculating the tables than looking for the number patterns. This, of course, was a perfect opportunity to utilize a modern computer. I generalized the algorithm and produced the computer program which follows. It produces *Difference Tables* in a matter of seconds, allowing students to spend time searching for number patterns rather than performing tedious calculations.

Basically a *Difference Table* consists of the following. The first column (the x column) consists of numbers which have a common difference. The program allows the user to define the common difference between these numbers, the starting value for the column and how many values will appear in the column. The second column is made up of the value of the first column raised to an exponent (the up-arrow on the table headings means "to the power of"). The program asks the user to define the value of the exponent for the table.

The third column is really the first difference column, called D1. It is made up of the differences in the values of the previous column. For example, the first entry in column D1 will be made in row two and consist of the difference in the entries in row one and row two of the previous column. The entry in row three of column D1 will consist of the difference in row two and row three of the previous column and so on. The subsequent columns are calculated the same way (i.e. from the closest previous column). This process continues until all of the entries of a difference column become the same.

On the set of tables I've included here, a couple of patterns stand out. Generally, for x to the power n, the differences in the table become the same in the nth difference column (i.e. Dn). This will always be true no matter how the original x column is set up. The other pattern of interest on this set of tables is that the value contained in the last column of each table is given by the formula $d^n n!$, where d is the difference between the values in the x column, and n is the exponent of the table.

The program was written on a Commodore 128, but it can be easily translated to other types of computers. Lines 320 and 370 are where the actual entries of the table are calculated. It is necessary to add .5 to the entry and then strip the decimal part from the calculation using the BASIC integer function in order to prevent rounding errors. Line 440 translates the numeric values of the table into string values so that the spacing in the printed tables is more easily controlled. This program is extremely versatile and is capable of producing endless types of *Difference Tables* to be searched for number patterns.

```
100 REM DIFFERENCE TABLES
110 REM JIM O'KEEFE
120 REM ST. PETER'S HIGH SCHOOL
130 REM PETERBOROUGH, ONTARIO
140 :
150 REM READ IN PARAMETERS FROM THE USER
160 DIM A(50,50)
170 INPUT"POWER OF X";POWER
180 INPUT"START VALUE FOR X";XSTART
190 INPUT"DIFFERENCE BETWEEN VALUES OF X";D1
200 INPUT"HOW MANY VALUES FOR X";XNUMBER
210 :
220 REM SET UP TABLE HEADINGS
230 PRINT" X";"      ";"X ↑";POWER;
240 FOR I = 1 TO POWER
250       PRINT "      D";STR$(I);"      ";
260 NEXT
270 PRINT
280 :
290 REM CALCULATE VALUES FOR THE TABLE
300 FOR I = 1 TO XNUMBER
310      A(I,1) = XSTART
320      A(I,2) = INT(XSTART↑POWER + .5)
330      XSTART = XSTART + D1
340 NEXT
350 FOR COL = 3 TO (POWER + 2)
360      FOR I = (COL-1) TO XNUMBER
370           A(I,COL) = INT(A(I,COL-1)-A(I-1,COL-1) + .5)
380      NEXT
390 NEXT
400 :
410 REM LINE UP COLUMNS FOR OUTPUTING
420 FOR I = 1 TO XNUMBER
430      FOR J = 1 TO (POWER + 2)
440           A$ = STR$(A(I,J))
450           IF A(I,J) = 0 AND J > 2 THEN A$ = ""
460           FOR K = 1 TO (8-LEN( A$))
470                A$ = A$ + "  "
480           NEXT
490           PRINT A$;
500      NEXT
510      PRINT
520 NEXT
530 END
```

Figure 1. Difference between X values is 1
Starting Value of X is 1

X	X ↑ 2	D 1	D 2
1	1		
2	4	3	
3	9	5	2
4	16	7	2

X	X ↑ 3	D 1	D 2	D 3
1	1			
2	8	7		
3	27	19	12	
4	64	37	18	6
5	125	61	24	6

X	X ↑ 4	D 1	D 2	D 3	D 4
1	1				
2	16	15			
3	81	65	50		
4	256	175	110	60	
5	625	369	194	84	24
6	1296	671	302	108	24

X	X ↑ 5	D 1	D 2	D 3	D 4	D 5
1	1					
2	32	31				
3	243	211	180			
4	1024	781	570	390		
5	3125	2101	1320	750	360	
6	7776	4651	2550	1230	480	120
7	16807	9031	4380	1830	600	120

X	X ↑ 6	D 1	D 2	D 3	D 4	D 5	D 6
1	1						
2	64	63					
3	729	665	602				
4	4096	3367	2702	2100			
5	15625	11529	8162	5460	3360		
6	46656	31031	19502	11340	5880	2520	
7	117649	70993	39962	20460	9120	3240	720
8	262144	144495	73502	33540	13080	3960	720

Continued on page 16

ON CHERCHE DES MANUSCRITS

Nous avons besoin de vos idées!
On encourage les lecteurs à soumettre des manuscrits qui examinent les techniques dans la classe ou des questions de l'enseignement de mathématiques (écrits en français) pour la critique.

Les instructions pour les auteurs peuvent être trouvées au début de la revue.

Figure 2. Difference between X values is 2. Starting Value of X is 0

x	x ↑ 2	D 1	D 2				
0	0						
2	4	4					
4	16	12	8				
6	36	20	8				

x	x ↑ 3	D 1	D 2	D 3			
0	0						
2	8	8					
4	64	56	48				
6	216	152	96	48			
8	512	296	144	48			

x	x ↑ 4	D 1	D 2	D 3	D 4		
0	0						
2	16	16					
4	256	240	224				
6	1296	1040	800	576			
8	4096	2800	1760	960	384		
10	10000	5904	3104	1344	384		

x	x ↑ 5	D 1	D 2	D 3	D 4	D 5	
0	0						
2	32	32					
4	1024	992	960				
6	7776	6752	5760	4800			
8	32768	24992	18240	12480	7680		
10	100000	67232	42240	24000	11520	3840	
12	248832	148832	81600	39360	15360	3840	

x	x ↑	D 1	D 2	D 3	D 4	D 5	D 6
0	0						
2	64	64					
4	4096	4032	3968				
6	46656	42560	38528	34560			
8	262144	215488	172928	134400	99840		
10	1000000	737856	522368	349440	215040	115200	
12	2985984	1985984	1248128	725760	376320	161280	46080
14	7529536	4543552	2557568	1309440	583680	207360	46080

Figure 3. Difference between X values is 3. Starting Value of X is 0

x	x ↑ 2	D 1	D 2				
0	0						
3	9	9					
6	36	27	18				
9	81	45	18				

x	x ↑ 3	D 1	D 2	D 3			
0	0						
3	27	27					
6	216	189	162				
7	729	513	324	162			
12	1728	999	486	162			

x	x ↑ 4	D 1	D 2	D 3	D 4		
0	0						
3	81	81					
6	1296	1215	1134				
9	6561	5265	4050	2916			
12	20736	14175	8910	4860	1944		
15	50625	29889	15714	6804	1944		

x	x ↑ 5	D 1	D 2	D 3	D 4	D 5	
0	0						
3	243	243					
6	7776	7533	7290				
8	59049	51273	43740	36450			
12	248832	189783	138510	94770	58320		
15	759375	510543	320760	182250	87480	29160	
18	1889568	1130193	619650	298890	116640	29160	

LES MATHEMATIQUES ET LA METEOROLOGIE

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Une des ironies de l'enseignement des mathématiques est que les problèmes de la vie réelle, bien qu'intéressants, sont souvent trop difficiles pour les élèves à l'école secondaire. Par conséquence, les problèmes qui paraissent dans les manuels sont parfois up peu forcés. C'est lors de mon séjour à l'office national de la météorologie que j'ai trouvé plusieurs applications de trigonométrie qui sont à la fois intéressantes et aussi éducatives aux élèves à l'école secondaire. Les problèmes exigent quelques connaissances de bases mais dans la plupart des cas les élèves auront déjà entendu parler des idées présentées à cause des bulletins météorolgiques à la télé chaque soir. En tout cas, ces idées suscitent des discussions intéressantes entre le professeur et les élèves. Ils peuvent donc profiter de la discussion du problème avant même de le résoudre en se servant d'arithmétique.

Que répondront les élèves à la question "Quelle est la hauteur du plafond du nuage orageux?" Une réponse typique peut-être sera "Qui donc s'en fiche?" Et bien, on peut répondre les pilotes s'en fichent puisque les nuages orageux sont extrêmement dangereux aux avions. Les grands vents (de haute vélocité) dans le nuage et dans les environs peuvent déchirer les ailes des avions. De plus les éclairs sont toujours un danger. Malheureusement, les pilotes doivent suivre leurs horaires. Souvent le vol alentour des tempêtes prend beaucoup plus de temps que le survol même des nuages orageux. Pour cette raison la réponse à la question "Quelle est la hauteur des nuages orageux" est très importante pour les pilotes. Car avec cette information ils peuvent ensuite décider si leurs avions peuvent survoler la tempête en sureté.

L'élève demandera peut-êntre comment le pilote a connaissance des tempêtes dans son trajectoire. La réponse est l'existence de radar météorologique. Un transmitteur sur le sol envoie un signal de radio qui bondit sur les goulettes dans le nuage et puis est renvoyé à l'antenne radar. La réception d'un signal produit une région noire sur l'écran. Une photo typique vue sur l'écran de radar de la région du lac Huron et du lac Erie parait à la figure 1. Des exemples

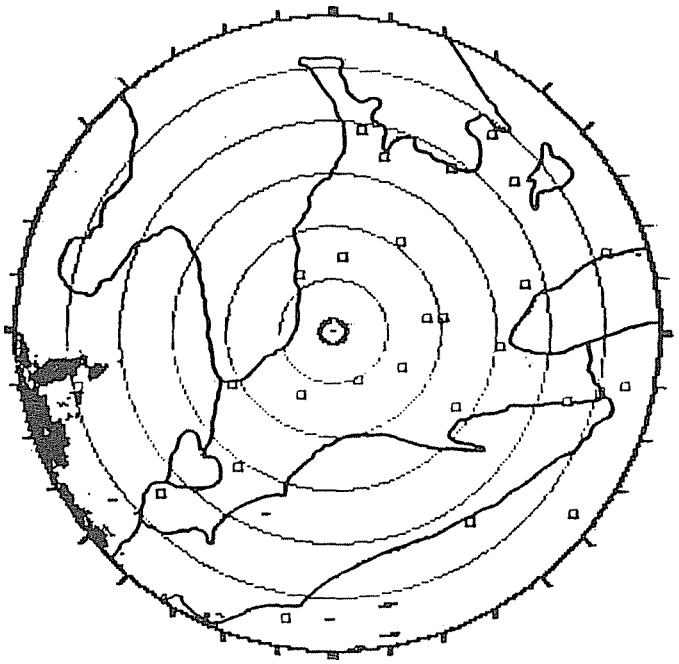


Figure 1 - Image de Radar Typique

pour votre région peuvent être procurés de votre office national de la météorologie si vous voulez montrez à vos élèves des exemples authentiques. La région noire à l'extrême gauche représente la bande de pluie qui arrive dans la région de l'ouest. La direction et la vitesse du mouvement sont établies en comparant les images de radar prises à tous les dix minutes. De cette façon, le pilote peut voir les régions difficiles avant de s'envoler. De plus, l'antenne de radar peut être incliné électroniquement par up petit contrôle dans le bureau météorologique. Désormais on arrive à un angle où le signal de radar manque complètement le nuage donc il n'y a pas de reflet. Quand ceci arrive la région noire disparait de l'écran.

Les cercles sur l'écran de radar (à une distance de 40 km dans cette exemple) révèle à l'observateur la distance du plafond du nuage et l'angle de l'antenne révèle l'angle d'inclination. Disons que l'angle de l'antenne est 4° et la distance des cercles est 80 km, nous pouvons alors calculer la hauteur du nuage comme si dessous (voir figure 2).

$$\frac{x}{80} = \tan 4°$$

$$x = 80 \tan 4°$$

$$x = 5.6 \text{ (km)}$$

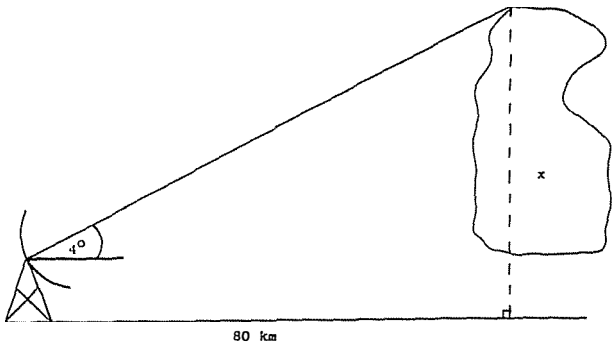


Figure 2 - Se Servant de Radar Pour Mesurer la Hauteur des Nuages

Des tables donnant la relation entre l'angle d'élévation, la distance des cercles et la hauteur des nuages étaient employées pour faciliter les calculations. Aujourd'hui les calculations sont probablement faites par l'ordinateur.

Puisqu'un jet peut voler à un altitude de 10 km, il sera peut-être une bonne idée de survoler la tempête. Les pilotes posent habituellement cette sorte de question aux météorologues.

Une deuxiéme application comprend la calculation de la hauteur du plafond du nuage le soir. Le plafond du nuage est la hauteur la plus basse où le nuage est solide. Encore c'est important de comprendre que c'est un vrai probléme puisque même les pilotes les plus expérimentés ne sont pas permis de s'envoler si la plafond du nuage est trop bas. L'aéroport de notre région exige un plafond de 61 m mais cette valeur peut varier selon la région géographique et les régulations fédéraux.

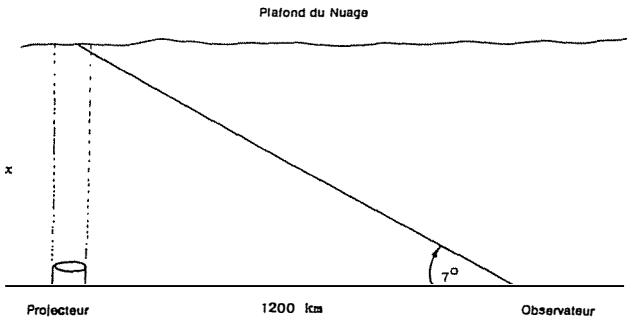


Figure 3 - Comment Mesurer la Hauteur du Plafond des Nuages Quand C'est Nuit

La situation à un bureau météorologique typique est illustrée à la figure 3. Un projecteur, à une distance fixée de la station d'observation, est allumée. Le projecteur illumine le bas du nuage et ceci est clairement visible le soir. L'angle d'élévation du projecteur est alors mesuré par l'observateur.

Si l'observateur est à 1200 m du projecteur et l'angle d'élévation est 7° on peut voir que

$$\frac{x}{1200} = \tan 7°$$

$$x = 147 \text{ (m)}$$

Si la visibilité est bonne (c'est-à-dire plus de 0.8 km) l'avion peut s'envoler sans danger.

Dans les deux cas on ne se soucie pas de la hauteur de l'observateur puisque en comparison à l'exactitude des autres mesures l'erreur est négligeable.

Ça serait un peu étrange si un éléve ne demandait comment les plafonds de nuages sont mesurés durant la lumiére du jour.

En générale, les observateurs de météorologie estiment la hauteur du plafond mais quand le plafond est trop bas et qu'il y a question de sécurité, une méthode alterne est utilisée. Un ballon rouge est gonflé de hélé jusqu'à ce qu'il commence à lever un poids fixé. A ce point, le ballon a une vitesse de montage de 55 m/min. Le ballon est donc lâché et l'on mesure le temps qu'il prend à disparaître dans le bas du nuage. La hauteur dans ce cas est simplement le multiple de la mesure du temps pris à disparaître et la vitesse de montage.

On rapelle les mots de Mark Twain: "Tout le monde parle de la métérologie, mais personne ne fait aucune chose pour la changer." Enfin vous pourrez fair quelque chose en mathématique au sujet de la météorologie même si vous êtes incapable de la changer.

ON THE TEACHING OF THE QUADRATIC FORMULA

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We would like to share with the reader an actual experience that took place in our Methods of Teaching Mathematics course. Upon completion of the class, all the future teachers felt a valuable lesson had been learned. We will present a dialogue similar to that which actually occured as the first three methods of deriving the quadratic formula are presented. In order to save space, the final method of derivation will not use the dialogue form.

TEACHER: How many of you (student teachers) know the derivation of the quadratic formula?

Only two of the fourteen students raise their hands.

TEACHER: Okay. As this is a truly important concept in high school math, you will now learn it. Before you understand the derivation of the quadratic formula and before you attempt to teach it, the complete square identify must be investigated.

$(a + b)^2 = a^2 + 2ab + b^2$

This should be shown by carrying out the indicated multiplication and then shown geometrically by the following diagram (stress should be placed on the middle term of 2 ab).

$(a + b)(a + b) = a^2 + ab + ba + b^2$
$= a^2 + 2ab + b^2$

	a	b
a	a^2	ab
b	ba	b^2

$= a^2 + 2ab + b^2$

Several examples are to be presented to insure the concept is understood.

$x^2 + 6x + 9 = (WHAT)^2$, $WHAT = x + 3$

$a^2 + 5a + \frac{25}{4} = (WHAT)^2$, $WHAT = a + \frac{5}{2}$

$x^2 + 3x +$ (WHAT) WHAT is needed to complete square?, (9/4)

TEACHER: Now let us consider the general form of a quadratic equation:

$ax^2 + bx + c = 0$

What should we do first?

CATHY: Well, we can dividie both sides of the equation by a and the equation becomes:

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

TEACHER: Good start Cathy, but what happens if $a = 0$?

SUE: In that case you can't divide by a. If $a = 0$, then we don't have a quadratic equation and our equation can be solved simply.

$bx + c = 0$ and $x = -c/b$

TEACHER: Great! But why do we have to divide by a? Why can't we multiply by a instead?

BILL: You mean we can derive the quadratic formula by multiplying by a? We all thought that the only way to derive it was to divide by a and then complete the square.

TEACHER: Unfortunatley, most books only show that one method but there are several methods that may be employed. My plan for today is to show you four methods of deriving the quadratic formula. As I see it, you have all been exposed to only the method that Bill has mentioned. Let's continue with a quick review of this method. What is the next step?

ELLEN: Subtract c/a from both sides.

$x^2 + \frac{b}{a}x = -\frac{c}{a}$

TEACHER: Now we can compare the left hand side of the equation with the identity for (a + b) squared, we need 2x times (something) plus (something) squared. To find this something we may proceed as follows:

$x^2 + \frac{b}{a}x = -c/a$

$\frac{b}{a}x = 2x \cdot (WHAT)$

(WHAT) must be $\frac{b}{2a}$ because

$2x \cdot \frac{b}{2a} = \frac{xb}{a} = \frac{b}{a}x$

CHARLIE: This is a piece of cake now. We can add (b/2a) squared to both sides and we end up with:

$$x^2 + 2x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

TEACHER: What now?

JUAN: The left side looks like a square so we can write the equation as:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Now, all we have to do is solve this equation for x and we are done.

TEACHER: Let's do it!

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

or

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

SUE: But how do we do it by multiplying by a? We must start out with an equation looking like:

$$a^2x + abx + ac = 0$$

LINDA: I think I see what is going on. First, we subtract ac from both sides. The next step is to group ax together.

TEACHER: Terrific, Linda. Our equation now looks like:

$$(ax)^2 + (ax)b = -ac$$

We can concentrate on the middle term. ax times b must be equal to 2 times ax times (what)?

$$(ax)b = 2(ax)(\text{WHAT})$$

CHARLIE: (What will be b/2. Then we have to add (b/2) squared to both sides, and we end up with the same type of equation as we had in the first derivation.

TEACHER: So we can solve for x in the same manner and come up with the exact same formula, as it should be.

$$(ax)^2 + 2ax\,\frac{b}{2} + \frac{b^2}{4} = -ac + \frac{b^2}{4}$$

$$(ax + \frac{b}{2})^2 = \frac{b^2 - 4ac}{4}, \quad ax + \frac{b}{2} = \pm\frac{\sqrt{b^2 - 4ac}}{4}$$

$$ax = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4ac}}{4}, \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

TEACHER: Do you know how the Ancient Hindus solved the quadratic equations?

No response.

TEACHER: I will show you. It is very similar to what we have already done. First, c is subtracted from both sides of the general equation. The next step is to multiply both sides by 4a. This is very easy to remember since the quadratic formula has a term of 4ac.

$$ax^2 + bx = c$$

$$4a^2x^2 + 4abx = -4ac$$

$$(2ax)^2 + 2(2ax)b = -4ac$$

As we can see, at this point all that is needed is the addition of b squared to both sides. This method is appealing since it eliminates working with fractions until the last step.

$$(2ax)^2 + 2\,(2ax)b + b^2 = -4ac + b^2$$

$$(2ax + b)^2 = b^2 - 4ac$$

$$2ax + b = \pm\sqrt{b^2 - 4ac}\,; \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The final method of derivation will be presented without the use of the dialogue format in the interest of keeping this article to a reasonable length.

Before the fourth derivation is introduced, it is necessary to show the connection between (a+b), (a-b), and ab in the following identity:

$$(a - b)^2 = (a + b)^2 - 4ab$$

To form a quadratic equation with roots *p* and *q*, we may set (x-p) times (x-q) equal to zero. Since the general form of a quadratic equation is also equal to zero, we may divide both sides of this equation by *a* and equate this new equation to our equation with roots *p* and *q*.

$$(x - p)(x - q) = 0\,, \quad ax^2 + bx + c = 0$$

$$(x - p)(x - q) = x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Performing the indicated multiplication leaves us with the resulting equation:

$$x^2 - (p + q)x + pq = x^2 + \frac{b}{a}x + \frac{c}{a}$$

This equation must be true for all $x \varepsilon R$. For this to be so, the two quadratic expressions must be identical, and the coefficient of like terms must be equal.

Comparing the like terms of this equation, we may conclude that

$$-(p + q) = \frac{b}{a} \text{ or } p + q = -\frac{b}{a}$$

$$pq = c/a$$

At this point, we may make use of the previously discussed identity involving (a + b), (a - b), and ab. By addition and subtraction, we are now able to solve for p and q the roots of our quadratic equation.

$$(p - q)^2 = (p + q)^2 - 4pq$$

$$(p - q)^2 = \left(-\frac{b}{a}\right)^2 - \frac{4c}{a} = \frac{b^2}{a^2} - \frac{4c}{a}$$

$$(p - q) = \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{a^2}}$$

$$p + q = -\frac{b}{a}$$

$$2p = -\frac{b}{a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{a^2}}$$

$$p = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$2q = -\frac{b}{a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{a^2}}$$

$$q = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

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ON CHERCHE DES CRITIQUES

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MATHEMATICAL PASSIONS

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> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the trappings of painting or music, yet sublimely pure, and capable of stern perfection such as only the greatest art can show.

This quotation from Bertrand Russell speaks to the detachment, objectivity, and perfection of mathematics. Russell, of course, is referring to the product rather than the process of mathematical creation. The mathematicians who have created this great subject, however, are often passionate, far from detached, and at times not as objective as one might wish - not only in their daily lives but also in their attitudes towards mathematics and in their dealings with fellow mathematicians. It is the purpose of this article to record a number of especially forceful or strongly felt expressions by mathematicians about their subject, their work, and the work of their colleagues. These will be taken from various historical periods and from several areas of mathematics. This human dimension of the mathematical enterprise is often missing from the teaching of mathematics. Throughout its history mathematics has been held in high esteem by philosophers and it has been an important source of various philosophical theories. The following are two well-known quotations from Plato, perhaps the greatest philosopher of ancient Greece, which attest to the importance he attached to mathematics:

> No one ignorant of geometry shall enter here.

This is said to have been the inscription above the entrance to Plato's Academy, which resembled in many ways a modern university.

> He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side.

This refers to the discovery by the Pythagoreans (about 100 years before Plato's time) of what we would express today as the irrationality of $\sqrt{2}$. The Pythagoreans had previously limited the concept of number to positive integers, and had thought that all mathematical relationships could be expressed in terms of positive integers or their ratios. The discovery, therefore, that the ratio of the diagonal to the side of a square cannot be so expressed, was a great shock to them. It engendered a crisis in Greek mathematics (at least from our perspective) and profoundly influenced its subsequent development (see [2]).

We are not used to thinking in terms of crises and revolutions in mathematics, but the discovery of incommensurability was a veritable revolution in mathematical thought in ancient Greece. We now make a great leap of over 2000 years - into the 19th century - to touch on another revolution in mathematics. (There were several in the 19th century). Its roots lie again in ancient Greece - in Euclid's formulation of deductive geometry. As is called for in any deductive theory, Euclid begins his development of geometry with axioms (postulates). He postulates five axioms, of which the fifth, the so-called parallel postulate, was contentious from the start. The postulate states (in its equivalent modern formulation) that through a point not on a given line there exists only one line parallel to the given line and coplanar with it. This is not as self-evident as Euclid's other four postulates (why can one not have two parallels through the point to the given line?), and the task, undertaken by many of the prominent mathematicians during the next two thousand years, was to derive the parallel postulate as a *theorem* from Euclid's other four postulates. All attempts proved unsuccessful - so much so that the well-known mathematician and philosopher D'Alembert (18th century) referred to this state of affairs as "the scandal of geometry."

The depth of feeling that attempts to prove the parallel postulate engendered can be gauged from the letter that Wolfgang Bolyai wrote in the 1820's to his son Janos about his work on the parallel postulate:

> You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone ... I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified ... to mankind. I accomplished monstrous, enormous labors; my creations are far better than those of others and yet I have not achieved complete satisfaction ... I turned back when I saw that no man can reach the bottom of the

night. I turned back unconsolded, pitying myself and all mankind. I admit that I expect little from the deviation of your lines. It seems to me that I have been in these regions, that I have travelled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time. I thoughtlessly risked my life and happiness ...

The young Bolyai, who was also working on the same problem, was, however, not deterred. He responded to his father as follows:

> It is now my definite plan to publish a work on parallels ... I have discovered such wonderful things that I was amazed ... out of nothing I have created a strange new universe. All that I have sent you previously is like a house of cards compared to a tower. I am no less convinced that these discoveries will bring me honor than I would be if they were completed.

The "strange new universe" that Janos Bolyai was creating was non-Euclidean geometry. This work vindicated Euclid, for it implied that a proof of the parallel postulate is *impossible* and that stating it as an axiom was therefore the correct approach (see [3], [5]). This, as we mentioned, had a revolutionary impact on mathematics and beyond. In particular, it disentrenched mathematics from proclaiming absolute truths. It introduced relativity into mathematics (see [4]).

Another idea which has aroused emotions and controversy in mathematics is the concept of infinity. As Hilbert put it:

> The infinite: no other question has ever move so profoundly the spirit of man.

Already in ancient Greece the philosopher Zeno had confounded fellow philosophers and mathematicians with his four famous paradoxes concerning motion and the question of the infinite divisibility of space and time.

Galileo tried to deal with the infinite in his "Two New Sciences." He observed the existence of a one-to-one correspondence between the positive integers and their squares. However, he did not draw the groundbreaking consequence which Cantor drew 250 years later. Instead he wrote:

> This is one of the difficulties which arise when we attempt with our finite minds to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this, I think, is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to the other.

The great Gauss, often referred to as the Prince of Mathematics, writing to his friend Schumacher in 1831, echoes similar sentiments (although in a different context):

> As to your proof, I must protest most vehemently against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a figure of speech.... No contradictions will arise as long as Finite Man does not mistake the infinite for something fixed....

Of course, it was precisely as something fixed, as a completed entity, that Cantor regarded the infinite - as against the potential infinite of the calculus - when he began making his fundamental breakthroughs in the 1870s (see [2], [5]). He agonized for years over this break with tradition:

> It is traditional to regard the infinite as the indefinitely growing or in the closely related form of a convergent sequence, which it acquired during the 17th century. As against this I conceive the infinite in the definite form of something consummated, something capable not only of mathematical formulation, but of definition by number. This conception of the infinite is opposed to traditions which have grown dear to me, and it is much against my own will that I have been forced to accept this view. But many years of scientific speculation and trial point to these conclusions as to a logical necessity, and for this reason I am confident that no valid objections could be raised which I would not be in a position to meet.

Objections *were* raised. This is not too surprising since Cantor's work was revolutionary. However, like all revolutionary ideas, it was opposed with emotion, much of it bitter. Witness Poincare, one of the greatest mathematicians of the turn of the century:

> Later generations will regard the Mengenlehre [set theory] as a disease from which one has recovered.

Kroneck, who was among the most vehement opponents of Cantor's ideas, and who with his incessant criticism is said to have driven Cantor into an asylum, wrote to Lindemann in 1882 after the latter had proved the transcendentality of pi:

> Of what use is your beautiful investigation regarding pi? Why study such problems, since irrational numbers are nonexistent?

Kronecker was a precursor of the intuitionist-constructivist philosophy of mathematics (it has its adherents even today) which rejects anything that cannot be constructed in a finite number of steps, especially the infinite as a completed entity (see [2], [6]). Now, the irrationals require for their definition nonconstructive methods and the use of infinite collections - this, then, is the context for Kronecker's statement.

Of course, there were also many positive reactions to Cantor's work. Hilbert's comment especially must have been soothing to Cantor's ears and mind:

> No one shall expel us from the paradise which Cantor has created for us.

It is noteworthy that the two greatest mathematicians of the turn of the century - Poincare and Hilbert - had such diametrically opposite views of the infinite. Far from a "disease," Cantor's [infinite] set theory proved, not long after its creation, to be of fundamental importance in many areas of mathematics. At the same time, it did not quite turn out to be a "paradise", beset as it was by serious problems - the paradoxes of set theory (see [5]).

The calculus - and its extension, analysis - were also not spared emotional outbursts. Newton and Leibniz are nowadays considered the coinventors of the calculus, but this was not the view in their time. Leibniz was accused of plagiarism by some English mathematicians; Continental mathematicians, especially the Bernoulli brothers, came vigorously to his defence. The bitterness of national feeling reached such a point that in 1726, a decade after Leibniz had died, Newton deleted from the third edition of his *Principia* all reference to the fact that the Leibniz had developed methods in the calculus similar to his own. (He had mentioned Leibniz' work in the first two editions.) As a consequence of this disgraceful dispute over priority, the English and Continental mathematicians ceased to exchange ideas for most of the 18th century. Since Leibniz' analytical methods proved to be much more effective than Newton's geometrical methods (which he used in his major work *Principia*), mathematics developed during the 18th century much more rapidly and extensively in Europe than in England.

Although the calculus as developed by Newton and Leibniz was instrumental in solving some of the major scientific problems in the 18th century, it was not based on firm logical foundations - in fact the very concept of limit, which is basic to a rigorous foundation of the calculus, was missing (see [1]). This aroused considerable criticism in the 18th century. One of the most forceful among the various attacks on the foundations of the calculus came from Bishop George Berkeley, a well-known 18th century philosopher. It appeared in 1734, in a work entitled *The Analyst,* bearing a long-winded subtitle:

> Or a Discourse to an Infidel Mathematician Wherein it is Examined Whether the Object, Principles, and Inferences of the Modern Analysis are More Distinctly Conceived, or More Evidently Deduced, than Religious Mysteries and Points of Faith. "First Cast the Beam Out of Thine Own Eye; and Then Shalt Thou See Clearly to Cast Out the Mote Out of Thy Brother's Eye."

The "Infidel Mathematician" apparently was Edmond Halley, a friend of Newton. Berkeley had been angered on having a sick friend refuse spiritual consolation because Halley had convinced the friend of the untenable nature of Christian doctrine.

Newton's calculus of fluxions (the differential calculus) seemed to Berkeley to be only a lucky compensation of errors:

> ...by virtue of a twofold mistake you arrive, though not at science, at the truth.

The rates of change (derivatives), Berkeley says, are but

> the ghosts of departed quantities. Certainly ... he who can digest a second or third fluxion [derivative], a second or third difference, need not, methinks, be squeamish about my point of Divinity.

Berkeley's criticism of the calculus was well justified. In the absence of the concept of limit, the computation of derivatives was heuristic at best, often involving, in fact, a "compensation of errors": for example, the x would be cancelled from numerator and denominator of a rational expression (thus assuming it is nonzero) and subsequently "neglected" (thus, in effect, being set equal to zero). (See [1] for details.) Berkeley's motive was, however, likely as much as defense of theology against the growing threat posed to it by the mechanism and determinism of the 18th century as it was a rebuke for the weak foundations of the subject (see [5]).

Most of the mathematicians of the 18th century did not heed Berkeley's admonitions. The 17th century had bequeathed to them a marvelous tool - the calculus - and they exploited it to solve major scientific problems, and advanced the power of the calculus to create what turned out to be major branches of mathematics: infinite series, ordinary and partial differential equations, differential geometry, the calculus of variations.

A critical period of rigorous thinking and reevaluation of the foundations of analysis set in at the beginning of the 19th century. One of its principal exponents was the young Norwegian mathematician Niels Henrik Abel. Commenting on 18th-century work on series in which infinite series were formally manipulated (like finite algebraic sums) without regard to questions of convergence (see [1], Abel says:

> Divergent series are the work of the devil. [They only cause] calamities and paradoxicalities.

The concept of function and the type of functions studied had undergone a considerable evolution since Johann Bernoulli had formally introduced his definition of a function in the early 18th century. In the second half of the 19th century Riemann provided an example of an integrable function with infinitely many discontinuities, and Weierstrass gave his famous example of a continuous, nowhere-differentiable function (see [1]). (Throughout much of the 19th century continuous functions were thought to be differentiable except possibly at a finite number of points!) Examples of other "pathological" functions soon followed. Not everyone was pleased with these development, as the following two quotations from Hermite (in 1893) and Poincare (in 1899), respectively, indicate:

I turn away with fright and horror from this lamen-

table evil of continuous functions without derivatives.

Logic sometimes makes monsters. For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions.

Algebra, too, must not be left out of the picture. An important branch of algebra in the 19th century was the Theory of Invariants. It had its origins in number theory and projective geometry (where the concept of invariance plays an important role), but in the second half of the 19th century it was studied from an algebraic point of view. As Sylvester said in 1864:

As all roads lead to Rome so I find in my own case at least that all algebraic inquiries, sooner or later, end at the Capital of modern algebra over whose shining portal is inscribed the Theory of Invariants.

Very briefly, the subject calls for the determination of algebraic expressions which remain invariant under a change of coordinates.

P. Gordan was one well-known mathematician who devoted most of his mathematical life to the study of the theory of invariants. His results were rather specialized and the proofs called for long and difficult computations. He was working on computing invariants in special cases when Hilbert proved a general theorem about invariants (the so-called Finite Basis Theorem) which took care, in one swoop, of all the special results that Gordan had been working on for years. (Although the notion of invariance is fundamental in mathematics, the algebraic theory of invariants, so vigorously pursued in the 19th century, is nowadays remembered only in history books; Hilbert's result is said to have "killed" it.) Hilbert's proof, however, was existential (i.e. it showed the existence of a certain "finite basis" without showing how to compute it) - a type of proof not very common at that time. Seeing this proof Gordan retored:

This is not mathematics, it is theology.

We take non-constructive existence proofs for granted. (For example, we prove the existence of solutions of polynomial equations and differential equations without knowing what the actual solutions are.) Such proofs, however, were rarely encountered before the 19th century, and gained far from universal acceptance during that century. Moreover, the intuitionist -constructivist movement fashioned in the 20th century disallowed such proofs (see [6]). Gordan's declaration must thus be seen against this background. Furthermore, Gordan later "recanted" and said:

I have convinced myself that theology also has its advantages.

It is perhaps fitting to close with the following statement, which encompasses all the mathematical fields mentioned in this paper. It is made by the famous American mathematician B. Peirce (1809-1880), in a lecture to his class:

Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth.

Peirce was referring to the identiry $e^{\pi i} + 1 = 0$. Although easily derivable from the formula $e^{i\Theta} = \cos\Theta + i\sin\Theta$, it is a truly wondrous and awe-inspiring identity, combining the five most important constants in mathematics: 0, 1, i, π, e, arising in arithmetic (0 & 1), algebra (i), geometry (π) and analysis (e), and the most fundamental operation (+) and relation (=) in a single formula!

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PROBLEM-SOLVING-DEVELOPMENT, IMPLEMENTATION, EVALUATION AND BEYOND!

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Peel Board of Education
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Recent studies, completed by educators such as Dossey, Charles, Lester, LeBlanc and Kilpatrick, have recognized a need for a focus on teaching problem solving in the elementary school program. Indeed, the National Council of Teachers of Mathematics declared 'problem solving' as the major focus for the 1980's. Mathematicians such as Hilton (1985)[1] have verbalized that this is just a recognition of the obvious and he claims that you really cannot do any mathematics without solving problems. The result of all this dialogue is a clear curriculum planning direction. Mathematics educators must teach students the thinking processes for obtaining solutions to problems.

The direction toward teaching students to become better problem solvers may be clear to mathematics educators. However, it is not clear what should be done to move 124 elementary school programs, 2600 teachers and 50,000 students in a new direction which recognizes problem solving as a life skill. An attempt will be made here to outline the development, implementation and evaluation of problem solving objectives within a large public school board.

DEVELOPMENT

At the Peel Board of Education, we began by making a major revision of our Mathematics Core Curriculum. First, we defined 'problem solving'! Problem solving became, "What You Do When You Don't Know What To Do". Lester (1982) said the same thing in this manner:

1. You want or need to find a solution.
2. You have no readily available procedure for finding the solution.
3. You must make an attempt to find a solution.

Particularly in grades 4 to 8, acceptance of this problem solving definition led us in the direction of changing curriculum objectives. Further reflection on the work of Charles (1984) narrowed the change to include two categories of curriculum objectives related to problem solving.

The first set of objectives were labelled 'skill applications'. These address an instructional mode similar to material in traditional textbook word problems. Students learn something, they recognize examples and they apply their knowledge. Here are two examples of skill application objectives:

Grade 4 - skill application (SA)
- identify when to apply addition and subtraction in a given situation.

Grade 5 - skill application (SA)
- create a situation which demonstrates an understanding of when to use multiplication of whole numbers.

It was considered absolutely crucial to include both an "identifying" and a "creating" objective for most major skills or operations.

A translation process for skill application was determined. Teachers were asked to keep this translation process in front of the students all the time by means of a skill applications guide (see figure 1).

Figure 1

SKILL APPLICATIONS IN TRANSLATION
1. Read the problem.
2. Write a number expression (translation).
3. Do the mathematics (operations).
4. Check your answer.

The second newly designed set of curriculum objectives was labelled problem solving. Each of these contained a strategy which must be taught to students. For example:

Grade 6 - problem solving (PS)
- use 'working backwards' as a strategy for solving problems.

Suydam (1982) summarized recent research on problem solving that suggested the conscious teaching of problem solving strategies. Problem solving that demands flexibility and acquisition of a repertoire of strategies is one way to help students attain flexibility. To enhance the process of solving problems teachers also constantly displayed the translation model shown in figure 2.

Figure 2

PROBLEM SOLVING IN TRANSLATION
1. Understand the problem (a block).
2. Develop a plan (strategy).
3. Implement your plan (solution).
4. Think back (check or extend).

The final step - 'Think Back' - was implemented not simply as a check for a correct answer but a way of reviewing the entire process and solution; looking for new content or an extension which utilizes what the solving of the problem has taught you.

IMPLEMENTATION

New problem solving and skill application objectives were introduced into Peel mathematics programs in the school-year 1985-86. Mathematics resource teachers offered planning and implementation assistance and teachers participated in workshops based upon methodology for teaching students to be better problem solvers.

Several large schools recognized problem solving as an 'across the curriculum' endeavour. These schools encouraged the translation model (understand, plan, solve, think) across a variety of subject areas including Family Studies and Industrial Arts.

The positive comments from teachers, principals and resource staff indicated that implementation was proceeding. However, an informal interview survey of teachers by the mathematics staff determined that there were implentation needs that were not being addressed. It became evident that we needed data on student achievement in problem solving as a baseline for making further program or implementation decisions.

EVALUATION

How do you evaluate problem solving? To assess the effectiveness of efforts in Peel, a program evaluation plan based on the idea of holistic scoring mentioned by Charles, was developed. It looked like this:

U - Understanding the problem
- 0 - no understanding, complete misinterpretation
- 1 - partially misinterpreted
- 2 - complete understanding

S - Strategy or planning
- 0 - no strategy or inappropriate plan
- 1 - partially correct procedure,
  - part of data interpreted correctly
- 2 - a plan which could lead to a correct solution if no processing errors are made

A - Answer to the problem
- 0 - no answer or a wrong answer from an inappropriate plan
- 1 - copying error; computational error; answer labelled incorrectly
- 2 - correct solution

A team of resource teachers prepared 2 problem solving tests (5 items) and 2 skill application tests (5 items) for each level, grades 4, 5 and 6. The skill application objectives covered the operations of addition, subtraction, multiplication and division. Problem solving objectives evaluated at each level were:

Grade 4 - Problem Solving
- PS 6 - Trial and Error
- PS 4 - Manipulation
- PS 5 - Diagramming

Grade 5 - Problem Solving
- PS 6 - Tables and Charts
- PS 4 - Manipulation
- PS 5 - Patterns

Grade 6 - Problem Solving
- PS 6 - Working Backwards
- PS 4 - Manipulation
- PS 5 - Elimination

About 200 students per grade level were included in the sample and the results were recorded by two independent scorers and checked by a third scorer.

The overall results were presented in a report to the Peel Board of Education (1986). This report outlined major requirements for facilitating implementation. Teachers required more inservice, particularly relating to how to evaluate students. Also more teaching resources for problem solving strategies had to be made available.

BEYOND THE EVALUATION

To get teachers and student's to focus on process and strategies, it was recommended that the results of problem solving activities be recorded in a separate place away from the regular daily mathematics notes. A problem solving folder or a portfolio containing a cumulative list of strategies was suggested. Do not hide the strategies!

In addition to evaluation of pupils recorded work we trained teachers in observation and the use of an observation checklist (figure 3).

Figure 3

PROBLEM SOLVING: EVALUATION

Name: *Shelley T.* Term 1 *Sept.-Nov. 28*

OBSERVATION CHECKLIST

WHEN FACED WITH A PROBLEM:	*FREQUENTLY*	*SOMETIMES*	*RARELY*
1. shows willingness to try	✓✓✓✓	✓✓	✓
2. shows self-confidence	✓✓	✓✓✓✓	
3. uses a systematic plan	✓✓✓✓	✓✓	
4. chooses a good strategy	✓	✓✓✓	✓✓
5. finds a correct solution	✓	✓✓✓✓✓	
6. works well in small groups		✓✓✓	✓✓✓

SPECIFIC OBSERVATIONS:
9/15 argued very rudely in small group (illogical argument)
9/29 suggested 2 good strategies for cannibal/boat problem
10/18 contributed well in full class "brainstorming" on matchstick problems
11/3 excellent solution for clock problem

CHECKLIST AND OBSERVATIONS ARE BASED ON:
1. *daily observations (schedule 2-3 students/day)*
2. *individual discussions (as often as possible)*
3. *problem solving workbook (weekly observation)*
4. *small and large group discussions.*

SOURCE: Arithmetic Teacher, Jan. 1983, p. 6, R. Charles

Teachers found this checklist to be a concrete model for describing and recording pupil performance.

RESOURCE DEVELOPMENT

Although there are many published resources recommended to Peel teachers, it was clear that specific resources which address our objectives were required. A writing team designed seven teaching

modules: simplification, manipulation, diagramming, logical reasoning, working backwards, patterning and organized lists.

Each module was designed in three parts. Instructional notes (figure 4) presented teachers with methodology and problems for introducing the problem solving strategies.

Figure 4

INSTRUCTIONAL NOTES:

STAGE ONE:
- present Problem #1 to students
- have them complete it (most students will probably use a random list)
- discuss with students
- examine and compare random listing to systematic listing
- discuss which listing sytle assures that *no* possibilities will be forgotten?

PROBLEM #1

How many 3 digit numbers can you make with the digits 1, 2 and 6? Digits may be repeated.

ANSWER

126	666	111	222
162	662	116	221
216	626	161	212
261	266	611	122
621	661	112	226
612	616	121	262
	166	211	622

(27)

- present Problem #2 to students
- encourage students to use systematic listing
- discuss with class what symbols or representations did you use to represent the children?
- what pattern resulted from your lists?
- describe the organization you used
- can you predict how many different arrangements there would be if Mr. Kannen had 2 children? 3 children?

PROBLEM #2

Mr. Kannen is a very fussy photographer. He has already line up his 5 children in several different ways. He is still *not* satisfied. How many different arrangements are possible?

This was followed by a series of worksheet blackline masters (figure 5) which enabled students to practise using a strategy which has been taught.

Figure 5

WORKSHEET #1

1. Billy has won 4 flashes, 1 for math, 1 for school teams, 1 for citizenship and 1 for honours. What different ways can be arrange them on the arm of his coat.

2. The nursery school teacher told the children to put 2 plastic animals together in the farm at one time. They could use the monkey, the lion, the camel, the elephant and the seal. Show the different animal combinations that the children could find?

3. Show an organization you would use to find how many different ways you could use the following coins to total 37¢.

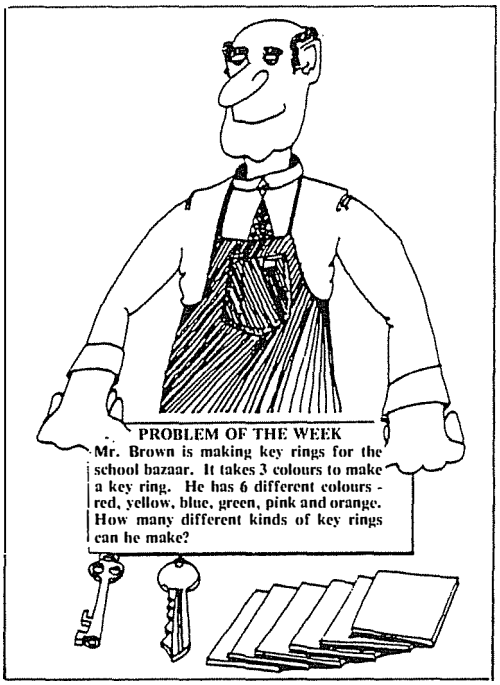
   penny nickel dime quarter

4. A restaurant was planning the food needed for the next day. They needed to know how many combinations were possible. A person's meal was "free" if they choose a combination which had not been chosed before. What are the possible combinations from this menu.

   Choose 1 selection from each course:
   Course 1 - soup, salad
   Course 2 - steak, fish
   Course 3 - pie, cake
   Course 4 - tea, coffee

Finally, ideas such as 'problem of the week' (figure 6) were introduced so that strategies could be reviewed.

Figure 6



CONCLUSION

In the Peel Board of Education, we have developed, implemented and evaluated problem solving objectives. The data analysis led us beyond the mere statement of objectives to ways of effective implementation. Future evaluation will tell us if Peel students are better problem solvers.

REFERENCES

1. Comments attributed to Dr. Peter Hilton, a world renowned mathematician, who shared his views with Peel teachers in an 1985 seminar.
2. Charles R. and Lester F., *A Evaluation of a Process-Oriented instructional Program in Mathematical Problem Solving in Grades 5 and 7,* Journal of Research in Mathematics Education, Volume 15, 1984.
3. Suydam Marilyn N., *Update on Research on Problem Solving: Implications for Classroom Teaching,* Arithmetic Teacher, Volume 29, 1982.
4. Norrie A. L., *An Informal Problem Solving Study, Grade 4, 5 and 6,* Peel Board of Education, 1986.
5. Charles R., *Evaluation and Problem Solving,* Arithmetic Teacher, Volume 30, 1983.

UPCOMING DATES

October 12-14/88	Pittsburg, Penn.	Regional Conference
October 19-21/88	Toronto, Ontario	O.A.M.E. Leadership Course
November 2-5/88	Dekalb, Illinois	P.M.E. - North America
December 1-3/88	Boston Mass.	Regional Conference
February 9-11/89	College Stations, Texas	R.C.D.P.M. Annual Meeting
March 16-18/89	Grand Rapids, Mich.	Regional Conference
April 1989	Ottawa, Ontario	O.A.M.E. Annual Meeting

ANNUAL N.C.T.M. MEETINGS

April 12-15/89	Orlando, Florida	67th Annual Meeting
April 18-21/90	Salt Lake City, Utah	68th Annual Meeting
April 17-20/91	New Orleans, Louisianna	69th Annual Meeting
April 1-4/92	Nashville, Tennessee	70th Annual Meeting
August 16-22/92	International Congress for Mathematics Education Quebec City, Quebec	7th Annual Meeting

THINKING THROUGH PROBLEM-SOLVING

by
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This column is intended to introduce a new section to the Gazette which hopefully will complement other problem-solving sections and cater to a wider audience. In order to assess the usefulness of this column your comments/criticisms are solicited (see the form below).

WHY: Perhaps our first task should be to explain our rationale for writing such a column. According to the Ministry guidelines "developing the ability to solve problems is a major goal of mathematics education". We believe that the sharing of some interesting problems, which can be extended over a wide range of grades and ability levels, will assist in the accomplishment of this goal.

WHAT: The types of problems presented in this column will be non-routine so that for the majority of students there is no readily available procedure for finding a solution. Each problem can be solved at a concrete level by an average junior student and yet the generalization should challenge the more capable senior students. The intent is that problem-solving generates analytic, critical thinking from the majority of our students, not just a select few.

HOW: These problems can be presented in many different forms. One suggested model:
Monday - introduce the problem (5 min)
Wednesday - take up the problem in groups (15 min)
Friday - presentation of solutions by the students (15 min)

This allows time for the students to reflect and think about the problem.

Whichever teaching strategy is adopted there must be three important stages.

(i) **Understanding the problem:**
Students are asked to explain the problem in their own words and a consensus reached as to the intent of the problem.

(ii) **Developing and carrying out the plan:**
Problem solving provides an ideal opportunity for cooperative learning. Relevant discussion provides the means for students to develop rational arguments and also to recognize the strengths and deficiencies in other peoples' contributions. In using groups of 3-4 students a different spokesperson may be chosen each week to explain the group's solutions. This helps to maximize student involvement. Hints can be given to groups who are struggling but suggestions should be such that students feel a sense of accomplishment upon arriving at a solution.

(iii) **Looking back:**
This is the most crucial stage. Solutions should be presented by the students using transparencies, large sheets of paper, or models. Creativity should be rewarded and assessment based on process rather than product (see Norrie, this issue).

Allowing time to reflect on answers and justify conclusions encourages a deeper awareness of the mathematical structure underlying a problem rather than simply seeking an algorithm to generate the correct answer. This provides time for the teacher to reflect carefully upon what the students really *do* know, rather than what it is often *assumed* they know.

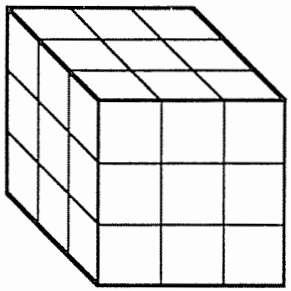
Reaction Form

Grade Levels Taught_____

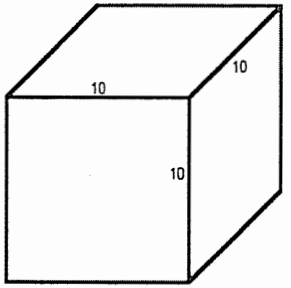
1. Did you use the problem?
2. Were the suggestions useful? Why? Why Not?
3. Should this column be continued? If so, what recommendations would you suggest?

Send responses to the authors.

The Problem of the Painted Cubes



The starting problem will depend on the grade or ability level.



Grade 4.
solving a specific example

Grade 12.
working towards a generalization

The cube is painted purple. It is then cut into 1 X 1 X 1 cubes.

How many cubes have: 0 purple faces
1 purple face
2 purple faces etc.

1. Understanding the problem

- Have students read the problem, and restate it in their own words.
- Students need to know: the properties of a cube, that only the outside is painted, and that this is done before cutting.

MODELS HELP
e. g. Diene's base ten blocks

2. Developing and Carrying out a plan

Hints: Do a simpler problem.
Build a model.
Locate the cubes with 3 faces painted purple.
Locate the cubes with 2 faces painted purple etc.
Make a table.
Is there a pattern?

???

3. Looking Back

Two solutions that have been provided for the generalization are shown below. The first is by Chris Paine, a bright 11 year old.

A. Making a table and looking for a pattern

painted faces	$(3\text{ cm})^3$	$(4\text{ cm})^3$	$(5\text{ cm})^3$	$(6\text{ cm})^3$	.....	$(n\text{ cm})^3$
0	1	8	27	64	.....	$(n-2)^3$
1	6	24	54	96	.....	$(n-2)^2 \times 6$
2	12	24	36	48	.....	$(n-2) \times 12$
3	8	8	8	8	.....	8

B. Generalizing from a model

On any single cube maximum number of painted faces is 3.
Cubes with 3 painted faces are on the corners of the large cube.
There are eight corners on any cube.
Cubes with 2 painted faces are on the edges of the large cube.
There are (n - 2) of these on each edge, so $12(n-2)$.
Cubes with 1 painted face are inside each face of the large cube.
There are $(n-2)^2$ of these on each face, so $6(n-2)^2$.
Cubes with 0 painted faces make up the cube within the large cube.
There are $(n-2)^3$ of these.

Verification:
Students with an algebraic background could show
$n^3 = (n-2)^3 + 6(n-2)^2 + 12(n-2) + 8$

Extension:
A rectangular prism of various dimensions could be examined.



PROBLEMS

by
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1. a) Find the vertex of the parabola $y = x^2 + kx + 2$

   b) In part a) you should have found the vertex to be the point $(-\frac{k}{2}, -\frac{k^2}{4} + 2)$. Use this result to complete the following chart.

parabola	vertex
$y = x^2 - 3x + 2$	
$y = x^2 - 2x + 2$	
$y = x^2 - x + 2$	
$y = x^2 + 2$	
$y = x^2 + x + 2$	
$y = x^2 + 2x + 2$	
$y = x^2 + 3x + 2$	

   c) On the same set of axes, plot all of the points in the right hand column of the chart in part b). What do you notice about these points? Can you prove your observation?

2. Sketch the graph whose equation is given by:
   $(x - 1)(x - 5) + (y - 2)(y - 6) = 0$

3. The diameter of a circle has endpoints A(3, 4) and B(7, 10). Show that the equation of this circle is:
   $(x - 3)(x - 7) + (y - 4)(y - 10) = 0$

   HINT: Let P(x, y) be any point on the circle. Note that $\angle APB = 90°$ and therefore $m_1 m_2 = -1$, where m_1 is the slope of PA and m_2 is the slope of PB.

SOLUTIONS AND COMMENTARY

1. I gave this problem to my Grade 11 class on one of their unit tests and they found it to be quite interesting. For part a) they had to complete the square and for part b) they just had to substitute numbers into a given formula. Note that a student who could not get the answer to part a), still had a chance at getting parts b) and c) done. I occasionally give questions like this. In part c) they found that the points seem to lie on a parabola.

   Quite a few of them were able to prove that the points do in fact lie on the parabola $y = -x^2 + 2$. I had not done any questions similar to this one and I was impressed with how they were able to sort things out.

2. This question was also on one of my Grade 11 unit tests this past semester. It appeared near the start of the test. We had just finished up the section on circles and my students had not seen anything like this equation whatsoever. Some of them left the question and went on, others expanded, collected like terms and obtained:

   $x^2 - 6x + 5 + y^2 - 8y + 12 = 0$

   Most of these students recognized that this was a circle. By completing the square twice, they obtained: $(x - 3)^2 + (y - 4)^2 = 8$ They were able to graph this quite easily.

3. This question appeared near the end of the same test that question 2 was on. When some of the students read through this, they went back to the start of the test and immediately got the problem discussed above (question 2). Most of them, given the hint, were able to work this question through. The details are given below:

   $m_1 m_2 = -1$

   $\frac{y - 4}{x - 3} \cdot \frac{y - 10}{x - 7} = -1$

   $(y - 4)(y - 10) = -(x - 3)(x - 7)$

   $(x - 3)(x - 7) + (y - 4)(y - 10) = 0$

When I took up the test I spent some time discussing this unusual form for the equation of a circle. One student pointed out that another diameter of the circle could be obtained by using the points C(3, 10) and D(7, 4).

INTERVIEW

Taped Talks with Bob Smith



. . . Walter Szetela

Walter Szetela has been a member of the Faculty of Education, U.B.C. since 1970. It was my pleasure to meet him again at the R.C.D.P.M. conference in New Orleans in February. I had been following the work he was doing on Problem Solving, so asked him to tell us about it.

RS: Walter, tell us a bit about your background.

WS: I was a secondary school mathematics teacher for 13 years and have been teaching mathematics education at UBC for 18 years. Instead of elaborating on that, let me tell you instead about my first vivid mathematical experience (Let me assure you it wasn't at all indecent.). My tenth grade teacher had set up a triangle and a line parallel to the base. She then proved that the sum of the angles of the triangle was a straight angle. True for any triangle! I was dumbfounded. After a succession of dull, obvious theorems like the diagonals of a rectangle being equal, this theorem was a revelation. The proof was simple, elegant, convincing. I never suspected that fat and skinny triangles and everything in between all had the same angle measure. After that, mathematics was never going to be dull again.

RS: That's an amazing recollection. I wonder how many others in mathematics can pinpoint the moment of conception of their discovery of the beauty of mathematics.

WS: It wasn't always thus. I worried a whole summer vacation after grade 7 because I knew that I'd be going to grade 8 where there'd be lots of impossible long division.

RS: There must have been lots more in between, but recently you've been following an interest in problem solving. Where has this led you?

WS: Let me back up and relate how I came into problem solving. I had been doing a considerable amount of study with calculators as a benefit to mathematics learning. I had great hopes that calculators would change the face of mathematics instruction. It never came to pass. However, working with calculators, it was very natural to construct interesting problems using real world data. Freed from the chains of computation, students could concentrate on the thinking necessary to solve a problem. One spring on Saltspring Island, using a single calculator, a class of grade 8 students solved a variety of outdoor problems, for example the height of a tree using the shadow method. There were tall and short students, but with the calculator they found the ratios were all the same and of course they obtained approximately equal answers for the tree height. Not long after, problems became the primary interest.

RS: How did you manifest this interest in problem solving?

WS: In several ways. I was especially interested in helping to promote instruction of general strategies for solving process problems. Doug Super and I conducted a study of problem solving and calculators involving 24 grade 7 classes. David Robitaille initiated a course in problem solving for our elementary education students taking a mathematics concentration, and I jumped at the chance to teach it. School districts were ready to adopt better methods of instruction in problem solving, and I was happy to help. Houghton Mifflin started a series of problem-solving booklets, and again I leaped to the challenge of writing interesting problems in such a booklet as well as the problem-solving strand in several of its textbooks.

RS: It sounds as if you took to problem solving with a passion. Is the novelty and zeal wearing thinner now?

WS: Not at all. It's just the opposite. It's been almost too much fun. I thoroughly enjoy the entertainment of weird, wacky, and wonderful solutions. My more recent special interest in the evaluation of solutions to problems probably developed gradually, yet I think I can even pinpoint the actual moment when this interest burst into bloom. I was examining solutions to a problem I had given to grade 5 pupils. The task was to find the average of three test scores of 81, 79, and 83. One pupil showed no work but simply wrote "B-." I was amused and delighted with the answer. It was quite contrary to normal expectations, but it made a lot of sense.

RS: It was a "wrong" answer in terms of the teacher's expectations. How do you think teachers react to such answers?

WS: It made me wonder if teachers in general would react negatively to such "wrong" answers. Are significant numbers of students who interpret problems in unexpected but logical ways penalized? It suddenly became very important to me that we adopt a more open, flexible position on how we grade students

in their attempts to solve problems. The many other solutions which I tried to evaluate made me aware of how difficult it is to be fair and consistent in grading students' problem solutions. I saw also how much could be learned about students' strengths and weaknesses if teachers would be able to take time to follow students' processes in solving problems and how much more important those processes were than the answers.

RS: How have teachers reacted to the challenge of problem solving?

WS: Almost without exception, very positively. Often, before a workshop a worried teacher has told me, "I was never good in math." I try to put everybody at ease with the remark that there are no wrong answers in problem solving. Nearly always that teacher will surprise herself with a clever solution of a problem I present. It's such a pleasure to see the fear give way to satisfaction and confidence.

RS: Do you recall any specific examples?

WS: I always get a variety of solutions to the problem of how many line segments are formed by joining all possible pairs of vertexes in a decagon. Of course that's just the handshake problem in disguise. Of special delight to me was a kindergarden teacher who was always so ingenious in transforming a problem designed for intermediate grades so that her young children could try it. One problem involved the number of 4-digit numerals which could be formed by using the numerals 1, 2, 3 and 4 exactly once each. Her children couldn't write numerals, but she gave them a structurally identical problem using colour cubes in four different colors, and some children solved it.

RS: Do you think that students are better problem solvers today as a result of the great emphasis on problem solving in recent years?

WS: Certainly some of them are, and I've seen enough sparkling solutions to believe it. Unfortunately, the majority of students are still in the Paleozoic Age when it comes to problem solving. To my dismay I found grade 5 students having difficulty with problems which I didn't consider to be true problems. For example, in one problem, students were asked to find the number of books where there were 26 math books in each of 32 cartons and 24 math books in each of 28 cartons. Forty percent of the students simply added the four numerals.

RS: Why do you think so many students are still having difficulty solving problems?

WS: There is still too much emphasis on rules and algorithms at the expense of time that could be spent exploring, thinking, sharing ideas. As a result students treat problems in much the same way as they treat algorithmic exercises. They think all problems are solved by computation, but the computations aren't specified. When computations are sufficient, they're usually satisfied to guess what operation to use instead of making an effort to think about relationships in the problem.

RS: Do you think that as teachers increase their awareness of the importance of problem solving and give greater attention to such problem-solving plans as the Polya four-stage plan, that the teaching of problem solving will improve?

WS: Such a plan is not enought if it isn't fully developed. The first stage, *understanding the problem* is so important, yet it is often bypassed or given short shrift. There has to be whole class discussion of a variety of problem situations to get students in the habit of forming good representations of problems. Incorrect interpretations of problem situations must be treated respectfully and discussed sufficiently for students to see *why* the intepretation may be wrong. So much can be learned from students' discussion of misconceptions if the teacher can resist the temptation to explain or correct that misconception her/himself. Teachers may avoid such discussion because they feel there isn't enough time or because they feel uncomfortable in the less structured discussion situation. I think those discussions are absolutely essential. Problem solving can't be taught well by reliance on teacher's model problem demonstrations for passive students.

RS: You seem to be saying that teachers have to be more willing to sacrifice some of the control that they command with demonstrations followed by practice. Won't teachers be less comfortable by opening up class discussion of problem situations where you can't predict the flow and time of discussion?

WS: It's true that there is such risk, but potentially it has great rewards. My brief experiences with lessons on problem solving in grades 5 and 7 have all been very successful although I admit that the situation of a visitor is not so realistic as that of the teachers who have the class all day all year. Nevertheless, I'll never forget a particular experience I had in a problem-solving lesson with grade 5 class.

RS: I take it you had been invited as a guest lecturer.

WS: The teacher had invited me to give a demonstration lesson. I was leading the class in a discussion of a fencing problem with various approaches invited and offered. They enjoyed the chance to suggest their own ideas. We were having a great time listening to and evaluating suggestions. Later in the day when I returned to the classroom, about half the class got up to show me their additional solutions. It was hard to believe young students could be that interested, but it certainly left me with very positive feelings that when students feel free to express their thoughts and ideas about problem solving situations, they'll come up with some interesting strategies which they enjoy sharing.

RS: When you were on leave, you also observed students solving problems in some European coun-

tries. Did you observe any major differences in the ways that problem solving is taught as compared to Canada?

WS: What I saw may not be typical. There was no evidence of the teaching of my favorite strategy, guessing and testing. In Poland where more mathematics is taught earlier, algebra was used instead of guessing, but success in problem solving was no better than in Canada.

RS: Isn't algebra a lot more efficient than using guessing and testing?

WS: At first, guessing and testing seems crude and unsophisticated. When a student solves by guessing and testing, he/she usually gets a much better feel for the problem situation than a direct attempt to use an equation. The guessing and testing eventually leads to an equation, and the student has a much better understanding of the situation. For example, solving a mixture problem by guessing and testing provides meaningful numerical relationships that helps a student to make sense of the problem.

RS: Did you observe any classes engaged in problem solving in Europe?

WS: I saw only two classes in Germany and two more in France which were a joy to observe. In one class in Lyon the teacher began with a problem situation and asked the students to formulate the question. After discussion of the question she distributed transparencies on which the pupils were asked to write solutions. After a few minutes the pupils took turns displaying their solutions on the overhead projector. It certainly helped the class to see different strategies and to follow the explanation given by each student. In at least one series of French mathematics textbooks a "situation problem" is provided for discussion and motivation for teaching skills and concepts in each unit.

RS: It's a hard road to success in problem solving, but also a very important and interesting one. Thank you for sharing your experiences and ideas with us.

WS: Thank you, Bob. It was my pleasure.

Teaching Tips

? MISCONCEPTIONS ?

Ask Gr. 5, 6, 7 or 8 students these questions:
a) Which is the biggest -
0.73 0.6 0.487 ?
b) How can you tell?

In small groups, discuss which answer is correct.
A consensus should be reached and justifications recorded

Conduct a class debate. Groups with opposing views should present their justifications. It is your responsibility to manage the discussion, maintaining an atmosphere of mutual respect. Leading questions sometimes need to be asked. Justifications are required. The correct conclusion should be summarized at the end of the debate.

One answer, *0.487,* may not be surprising - children simply ignore the decimal point.

It is more surprising that *0.6* is often chosen. Children argue that tenths are larger than hundredths or thousandths, so 0.6 is bigger (6 *tenths* are bigger than 73 *hundredths*).

Base ten blocks (using the centimetre cubes as thousandths) assist the understanding.

Children are more likely to overcome misconceptions if they are actively involved rather than being passive receivers of knowledge.

Barry Onslow

LEARNING MATH WITH LOGO (Apple and C-64)
Rudy V. Neufeld. Logo Publications, London, 1986.

The author believes that LOGO clearly shows the relationship between mathematics and spatial ability. Therefore, with LOGO in the mathematics classroom, an abstract concept takes on concrete meaning. In *Learning Math with LOGO,* he provides some useful and exciting mathematical lessons and activities.

Chapter One presents the Floor Robot to the student. This robot is a key component in the development of the LOGOLAND since the robot will do only what it is asked to do. In Chapter Two, the student is introduced to robot movement (this robot is a turtle) and uses the LOGO commands to discover mathematical relationships. The remaining chapters contain additional activities, discovery lessons and projects to enhance the student's foundation in mathematics. Many of these activities are designed as activity cards which can be recreated by the teacher for use in the classroom.

This book was designed for students in Grades 4 through 7, however, many of the transformation activities would be beneficial to Grades 8-10 students. The author's premise that the activities found in this book could be completed using various methods encourages this extension into the Intermediate level. Other mathematics topics include investigations of the square, rectangle, and triangle, parallel lines, circles, pi and polygons. Many of these activities involve the concept of a placeholder or variable.

The final chapter focuses on the use of LOGO as a problem solving tool. Although problem solving is emphasized throughout the book, additional routines are demonstrated in this chapter to increase the contents of the student's "toolbox" of useful routines.

Many books have been written about LOGO. Most have examples and activities which can be reproduced by the student on the computer. However, this text offers many activities which should be done away from the computer. The student should think through what has to be done (the floor robot exercises encourage this activity) before the student commands the turtle on the computer. These activities also encourage discussion among students about why something happened on the screen. The progression activities and projects make this a valuable resource for teachers of Junior and Intermediate students.

Reviewed by Douglas McDougall
Mathematics Consultant
The Waterloo County Board of Education
Kitchener, Ontario N2G 3X5

LEARNING MATH WITH LOGO
Teacher Resource
Rudy V. Neufeld. Logo Publications, London, 1986.

The teacher resource to the student text contains chapter objectives, pre-activity lessons, lessons, guided discovery activities and extended learning activities. These activities parallel those found in the student text. Every activity in the student text is referenced and discussed in this resource guide.

Loading instructions are given for TERRAPIN LOGO for the Commodore 64 and APPLE computers and for APPLE LOGO on the APPLE computer. Other references are made to these two computers and LOGO versions throughout the book. Mr. Neufeld seems to have anticipated many of the questions that novice and experienced LOGO users ask and has answered them in a very readable form.

Reviewed by Douglas McDougall
Mathematics Consultant
The Waterloo County Board of Education
Kitchener, Ontario N2G 3X5

TEACHING PROBLEM SOLVING: WHAT, WHY AND HOW.
Randall Charles and Frank Lester. Dale Seymour Publications, Palo Alto, CA., 1982.

Although this book has been in print for several years, the authors' lucid explanations of problem solving techniques make it a leader in its field and therefore worthy of note to those who are interested in the topic.

Part One of the book examines the WHAT and WHY of problem solving. What is a problem? What mental processes are involved in successful problem solving? What makes a problem difficult to solve? Why is it important for students to become better problem solvers? These are but a few questions candidly answered by the authors. Although many different types of problems are described, the major emphasis is devoted to process problems in which students have to employ strategies such as looking for a pattern or relationship, making a table or using logical reasoning, rather than using algorithms. The key to a successful problem solving program is to provide the opportunity for students to use analytic thinking.

Part Two examines the HOW: How much time should be devoted to problem solving? How should problem solving experiences be sequenced? How should pupil evaluation be handled? How can instruction be adjusted to meet the needs of high and low achievers? The authors describe several plans which can be employed by teachers who wish to follow their

approach. However, these plans are flexible enough for individuals to modify so as to meet their own needs.

The role of the teacher and student are examined at three key stages: before, during and after problem solving. Specific teaching suggestions and rationales are provided for each phase. The importance of allowing students time to demonstrate and discuss the various possible strategies for any one problem is emphasized and practical recommendations offered.

The evaluation of process problems is important. If students are to spend considerable time at this endeavour, then there must be an accountable assessment process. The authors offer some suggestions, but the section could have been expanded, and alternatives examined in greater detail.

Although the book has valuable advice for all educators, it is perhaps most beneficial for those who are teaching at the primary, junior and intermediate grade levels. It is written in such a style that teachers who do not have a strong mathematics background will feel comfortable attempting many of the authors' suggestions. Consequently, if it is not purchased for an individual's library, which I certainly recommend, its brevity (56 pages) and clarity clearly make it a suitable library addition for any staff room where there is interest in the challenge of creating a problem solving environment in which mathematics becomes relevant, useful and intellectually stimulating.

Reviewed by Barry Onslow
Faculty of Education
University of Western Ontario

Information for Authors

Some things that reviewers consider when reviewing manuscripts are:

1. Does the paper have a clear introduction and purpose?
2. Did you find it interesting?
3. Would you use the ideas in this article in your own teaching?
4. Does the paper explain any mathematical content sufficiently well that the average reader would understand it?
5. Are diagrams or figures, clear, well drawn and camera ready?
6. Is there a clear link between the mathematics and the pedagogy? Does the writer show the classroom application of the ideas in the article as well as the ideas themselves?
7. Is the paper written in a form suitable for publication in an educational journal?

Reviewers and editors are prepared to make small changes in copy as part of their function; however, articles that need significant amounts of editorial work will be sent back to the author for revision, even if the overall tone and/or content is felt to be suitable for the Gazette.

The Membership and Publicity Committee has a deal for you.

I ♥ Math buttons promote the appreciation of mathematics among students and teachers. They are available in French and English at 40¢ each or $18.00 for 50.

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GIRLS AND MATHEMATICS CONTESTS: PART III

by
Bev Marshman and Mary Thompson

***Editor's Note:** This is the third of three articles submitted by the Women and Math Committee.*

In Part II, we promised to discuss some suggestions for improving the participation and performance of girls (and others!) in mathematics contests. In order to gain some insight into the evolution of the contest scene since the days of our youth, we decided to interview some of our first and second year students. We issued a call in a few handy calculus sections for volunteers to come and talk to us about their contest experiences, good or bad. In order to control to some extent the composition of the sample, we told the classes we hoped to obtain substantial numbers from both sexes. In the end we interviewed thirteen female and twenty-one male students.

The interviews were loosely structured on a set of common questions, and our questioning techniques evolved considerably during the two week interview period. Because of this informality and the voluntary nature of the sample participation, our findings cannot be taken as scientifically valid support for generalizations about the UW bound secondary school population. For example, as would be expected, the contest participation rates in our sample were extremely high: thirty of the thirty-four students had written the Descartes contest; six of the females and seventeen of the males had written at least two other UW math contests; eight of the females and fourteen of the males had written other math, physics or chemistry contests (contests originating in Windsor University, Algoma University, the University of Manitoba, Carleton University; the AHSME, AMIE, Atlantic-Pacific contests, Mathletics; Olympiads; Santa Clara and ECIS competitions in Europe; Sir Isaac Newton physics contests, Chem 13). Although our results must over-proportionately reflect the experience of those to whom the contests were important, we find them interesting and suggestive.

1. **How did your school identify people to write contests?**

   In most cases the teachers announced the contests and issued a general invitation to sign up, perhaps encouraging some of the students individually. Some schools had enriched classes of whom most or all members would participate. There were also a few students in the sample, male and female, who themselves took the initiative to find out about the contests and organize participation. The numbers of participants varied greatly among the schools represented, from 1 or 2 students to 1 or 2 classrooms full.

2. **How did your school prepare people for the contests? Did you do any preparation on your own?**

   There were widely varying reports of the school's role in preparation. In order of degree of involvement, responses to "How did your school prepare ..." ranged through "not at all"; handed out old contests (and sometimes solutions); had a few sessions just before the contest; formed clubs, groups or classes with weekly meetings where old contests were done and extra material taken up. A few of the students reported no preparation on their own, but most said they worked on old contests, or did "homework" for their group sessions when these were held. A few reviewed parts of the regular curriculum or studied extra material for the Descartes contest. We noted no great difference between males and females in type of preparation, but some students felt that in their schools the ones who prepared most keenly were usually males.

3. **Did you feel well prepared?**

   Most students (rightly) took this question to apply mainly to the Descartes contest. Of those who wrote the Descartes, 5 females said "yes", 3 said "no" and 3 gave a response in between; 11 males said "yes", 4 said "reasonably", 2 said "no" and 2 gave a response in between "reasonably" and "no". Because coding the responses this way was a matter of judgement and because the sample is non-random we can make no generalization with confidence. However, we succumb to the temptation of adding that, while most students seemed realistic about their preparedness, proportionately more of the females felt they were less prepared than they would like to have been. Unfavourable timing of course taking (e.g. postponement of Algebra until the last term) was cited as a factor by both sexes.

4. **Did you have a strategy for writing the contests?**

   Many of the students appeared to have been coached in strategy to some extent. For multiple choice contests the usual reasoning-cum-elimination ploy was cited often. One student (male) made the following rule: when completely in the dark, choose (B)!

   For written answer contests the most-cited strategy was to go through the questions one by one, but not to spend more time on a question than was productive. A number of interesting individual strategies were given: one (male) student noted that he frequently tried to work 'backwards' from a likely-looking answer when stuck; several students (1 female, 3 male) mentioned seeking out a "recognizable' or 'comfortable' question first, in order to get the 'juices flowing'; one (male) student said he deliberately started with the problems most heavily-weighted in terms of marks; another (male) student would read through the entire paper first, jotting down ideas on each question. Students of both sexes mentioned (either implicitly or explicitly) limits on the amount of time they allowed them-

selves to spend on a problem; one (male) student used a 10 minute limit, and then let his 'subconscious' keep working on the problem while he went on. One (female) student had quite a well-developed strategy for each problem: 'look at it, play with it, draw pictures, wait for inspiration, keep confident, and say to yourself, "Of course I can do this!" '.

5. **Were you personally encouraged by teachers to write? Did the encouragement or lack of encouragement make a difference to you?**

The vast majority (all but 4 (1 female, 3 males)) of the students we interviewed reported receiving personal encouragement from teachers; one recalled his grade 8 teacher as being the most influential and that she stimulated the interest of many of her pupils with puzzles. In some cases (2 males and 2 females), teacher encouragement was deemed essential to their participation in contests, while in others its effect was moderately positive (8 males and 4 females), or not at all important (8 males and 6 females). These figures do not point to any significant difference between the sexes in the effects of teacher encouragement. However, it should also be emphasized that no conclusion about the magnitude (absolute or relative) of such effects can be drawn, since our sample consisted principally of participants, and we have virtually no data on the relationships of non-participants with their teachers.

Some specific positive effects of teacher encouragement were mentioned: it overcame an initial reluctance to participate; it gave the students more confidence and less anxiety about the outcome; it made them want to do better; it involved practical forms of encouragement such as tips on how to do certain problems and individual help. Only one student made a negative comment, that the teacher involved with contest preparation seemed primarily interested in exceptionally talented students; several others said they thought the contest sessions brought about closer relationships with their teachers and this was something they valued. And then there was the (female) student who said that without the teachers' encouragement, she would "never have taken the contests seriously" because "Math had always been a joke [to her] because it was so easy!"

While we did not specifically ask about the home environment, four of the female students reported strong parental support: "My parents always told me I have the brains to do whatever I want"; "my father is a physicist and always encouraged me with puzzles and games". In two cases, older siblings also provided positive role models.

6. **What, if any, were the opinions of your peers regarding your participation? Did they affect you? How?**

This was by far the most complex of the questions we asked, and also the one for which our tactics evolved most as the interviews progressed. In several instances, a student's initial comments seemed to indicate no negative peer influences, but as we dug a little deeper, some interesting features of these rather subtle interactions emerged.

First, some general comments. Just as the number of students who participated in contests varied from a very few to entire classes, so also did the nature of the peer groups described, from virtually non-existent, to highly supportive, tight-knit groups that had been together for many years, bound by common academic (and other) interests. Most often, negative effects came from outside the immediate peer group, from 'other' students. The male-female mix within the 'contest participant peer groups' ranged from 50-50 down to 90-10, but several students commented that it varied noticeably from year to year in their schools.

The two benefits of peer groups most often cited were 'collaboration' (4 females and 8 males), and 'friendly competition' (2 females and 2 males). It was clear that working in a friendly, supportive group environment was highly preferable to many students, rather in contradiction to the stereotype of the brainy, 'ivory tower' contest devotee. (Of course, our sample may well have had a gregarious bias.) Interestingly, the two girls who enjoyed competition were both at the top of their class and were quite comfortable with the idea that others viewed them as "the person to beat".

The remaining students mostly felt quite neutral about the influence of their peers, although two of these, while they thought themselves to be immune, felt that other students were negatively influenced. Two students reported feeling uncomfortable about being seen as 'a contest winner', even though both did ultimately participate. Females seemed more sensitive to peer reaction than males on the whole. There were four girls who reported "ribbing" by other students, although all cited it as friendly, and all but one seemed to have developed a "dish it right back" attitude. Even the one who chose to remain quiet about her interest in mathematical contests in order to avoid the reaction of her peers stated that most regarded it as "funny" or a "harmless eccentricity", and that the boys in her peer group seemed anxious to help her. (On the other hand, a fifth girl said the boys treated her as a 'little sister' at first, then drew away when she did very well on the first test in physics.) Three students did report *actively* helping or encouraging others to come out and participate in contests; then there was the (female) student who dealt with the razzing from outside her peer group by 'helping all the jocks with their math!'

7. **Do you feel writing the contest(s) helped prepare you for university?**

More than half the students answered yes (7 females and 11 males); two (1 male, 1 female) felt it depended on how well you prepared for the con-

tests. They perceived the contests to be helpful preparation because they keep people really interested in mathematics, especially those for whom the ordinary curriculum is not particularly challenging; they help develop thinking and problem-solving skills; they provide insight into what type of material will be involved in University-level mathematics, and hence allow a smoother transition; they give the opportunity to experience problems that cannot be seen through immediately, to learn to get 'over the hump' of keeping calm until you figure it out; and stimulate interest, helping the student to 'aim higher'.

Other Findings

A further question asked was whether the students had written the contests mainly for "internal" reasons (e.g. pleasure, challenge, personal satisfaction) or "external" reasons (e.g. pressure from teachers/parents, possible financial and other awards). By far the majority of them felt that their reasons were primarily "internal". It was striking that for many the enjoyment did not depend crucially on their performance. The most commonly cited external reason was, in connection with the Descartes contest, the intention to apply to the University of Waterloo.

However, there were a few students who had not enjoyed the contests, the main complaints being that the problems were too much harder than, or too unlike, usual exam questions, and that the time limit pressures were not conducive to thought. One (female) student had felt "hassled" by encouragement to write the Descartes. She thought (and we tend to agree) that her vision of the contests as rather frustrating ordeals was shared by many students, particularly those who, though desirous of an education in the mathematical sciences, are not interested in mathematics for its own sake. (This response reminds us to emphasize the following distinction: equal opportunity for both sexes to achieve quantitative competence is surely an important societal goal; but a push for equal participation in math contests might be seen by some as bordering on infringement of personal liberty!)

In doing the interviews we gained a new appreciation of the difficulties for students (and teachers) in schools where the number potentially interested in contests was small. We found ourselves feeling distinct admiration for those students who had essentially "gone it alone", and for those who had undertaken to organize participation and preparation of themselves and their friends.

Suggestions

Some have suggested that if girls are not keen to enter math contests (or to prepare for them) it must be the nature of the contests which is at fault. Perhaps the wording or the types of questions somehow seem to put them in the masculine domain. Perhaps too many of the questions appear so difficult at first sight that the initial panic is hard to quell. The pressures of time and competition may produce more negative than positive feelings in many students who could perform perfectly well, even creatively, on regular examinations, students who might even enjoy working on the problems in leisure hours if no one were watching!

We would endorse the importance of research into the possibility that contests-as-exams may be innately discriminatory. For example, there is evidence[1] that S.A.T. scores in the U.S. underpredict the performance of women as college freshmen and therefore that heavy reliance on S.A.T. scores in admission and scholarship decisions may discriminate against women. Feminist groups are rightly insisting that this phenomenon be faced and dealt with. It could well be that there are changes which could be made in the format and style of mathematics contests which might make them more pleasurable and "fairer" experiences for students who are now less than enthusiastic participants. Although it is likely that most ways of making contests more attractive would also increase the already mind-boggling task of those who have to evaluate the papers, there might be feasible ways of allowing longer time limits, providing more easy questions to calm the nerves and set the creative juices flowing, or allowing for team collaboration.

In contemplating changes to the contests, however, we should not lose sight of the benefits the contests in their current form appear to provide to both sexes. It was clear to us that, depending on the degree to which they are taken seriously, they constitute recreation, challenge, stimulus to intellectual development and character moulding. These effects are felt even by those whose performance is cheerfully "also ran". For the most talented students, the challenge provided is almost uniquely available in the contests, and the fact that they are "divorced from the mundane", as one student put it, is of great importance.

The other obvious place for improvement or change is that of the general educational environment, including cirriculum. Such changes may be necessary quite early on, even in public schools; there is some evidence[2] that brain development proceeds at dissimilar rates for girls and boys. It is possible that grades 5-8 form a critical period for girls, and that "real" math problems should be introduced even before long-division is completely mastered - after all, many famous mathematicians cannot do long division!

At every level, discovery and challenge questions should be *assigned* or at least be offered as 'bonus' questions. (One of our colleagues has a favourite old adage that students learn a fixed fraction f of what they are expected to, so one way to increase total learning is to simply increase your expectations! And, there is certainly ample evidence in studies of the underprivileged that students do indeed tend to live up to our expectations of them.)

Another technique which ought to help build confidence and develop the capacity for analysis is role reversal, which we first saw convincingly described in a presentation by Elaine Harvey at the Sudbury OAME

meeting, 1985: Here the students are encouraged to develop problems and puzzles for others to solve. This, too, is ideally begun before the students have come to feel *entitled* to their more conventional role.

Finally, it seems important to promote development of the type of supportive, friendly groups described by some of our interviewees: these may flourish best with the active participation of highly concerned teachers. While most of the females involved in our study were fairly comfortable about perceiving themselves as mathematically inclined, this is not true within the general school population. We sense that interested girls will continue to need more incentive, approval, and reassurance that they are not making spectacles/fools/unattractive objects of themselves by committing themselves to this kind of effort. They can receive much of this from within a not-too-large group of peers, started as early as possible. Once they are committed, their experience will be much enhanced by *training* (within their groups) -- in the spirit, of course, appropriate to a recreational or leisure activity.

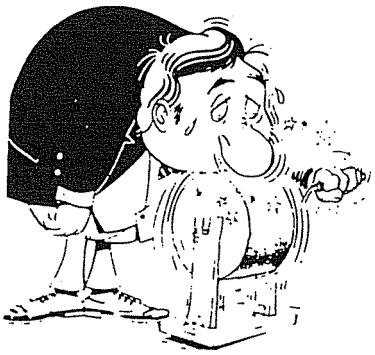
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1. Association for Women in Mathematics Newsletter, Vol. 17, No. 4 (July-August 1987), pp. 18-19.
2. Toepfer, C.F. (1982) Curriculum design and neuropsychological development. *Journal of Research and Development in Education, 15* (3), 1-11.

CALL FOR ARTICLES

The theme for the September, 1989 issue will be
RELATIONS, PATTERNS & FUNCTIONS.
Authors are requested to submit articles
(both short & long, practical and theoretical)
for all ages and ability levels.

The theme for the December, 1989 issue will be
STATISTICS & PROBABILITY.
Authors are requested to submit articles
(both short & long, practical and theoretical)
for all ages and ability levels.



COUNCIL COMMENT

by
Bob Robinson
Bd of Education
100 Main St. W.
Hamilton, Ontario L8N 3L1

Comments are based on the Council Meeting held at the Ramada Inn - Airport West on Saturday, 1988 06 04.

MEMERSHIP FEE

The membership fee will be $25.00 beginning 1988 12 01. All other rates will be adjusted accordingly.

APPOINTMENT OF SECRETARY-TREASURER

Don Attridge has been appointed the new Secretary-Treasurer of OAME. His duties will commence in May 1989 at the Ottawa Annual Meeting.

COUNCIL MEMBERSHIP

The possible reduction of council members is being investigated. Presently there are 40 councillors. Eleven chapters are included which cover the regions of the province. Ralph Connelly is co-ordinating this effort.

NCTM REPRESENTATIVE

Janet and Barry Scully have been appointed NCTM representatives.

17TH ANNUAL LEADERSHIP SEMINAR

This successful and popular seminar is being offered once again this year at the Ramada Inn - Airport West in Mississauga on 1988 10, 19-21. The theme this year is "Setting Standards For School Mathematics." For more information, contact Morely MacGregor at the Sarnia office or Ralph Connelly at Brock University.

ANNUAL MEETINGS

1988 Annual (Niagara Falls)

Well done! A highly successful conference and the weather was perfect. Approximately 1300 people attended. The publisher contributions were sincerely appreciated. The banquet and dance were well attended and the mini-courses were quite successful. A more detailed financial statement will be presented to council at our next meeting.

1989 (Ottawa)

"A Capital Conference" will be held at Ottawa University in May, 1989. Suggestions for program should be sent to Jeri Lunney, 39 Beaumaris Dr., Nepean, Ontario, K2H 7K5. Here is your chance to participate in this exciting "annual" in Ottawa.

1990 (Hamilton)

George Knill and Bob Robinson are the co-chairs of this NCTM Regional Meeting hosted by OAME which will be held in May of 1990. The committee chairs are in the process of being selected. If you wish to assist on any committee, contact George or Bob.

1991 (?)

Arn Carefoote is chairing a site selection committee for the annual meeting in 1991 and beyond. Suggestions should be directed to Arn.

ONTARIO MATHEMATICS CO-ORDINATORS ASSOCIATION

The OMCA liaison, Alex Norrie, reported that 2 profitable days were spent prior to the Niagara Falls Annual in examining the NCTM Standards Working Draft. Watch for their forthcoming document.

CHAPTERS

COMA/AMOC (CARLTON-OTTAWA MATHEMATICS ASSOCIATION/ASSOCIATION MATHEMATIQUE D'OTTAWA CARLTON)

A very successful 4th Annual publisher's display at the Embassy West Hotel in Ottawa. Thirteen publishers participated and 90 people attended. There was a very positive response. Because OAME '89 is being held in Ottawa, the publisher's display will be incorporated into the conference.

A mini-workshop and dinner is being planned for January but much of COMA's effort is being directed toward the '89 Annual - "A Capital Conference".

A new executive has been elected and a new constitution endorsed by the membership.

COMA EXECUTIVE

Past President	Jeri Lunney
President	Dave Hunter
Vice-President	John Lebrun

Secretary	Del Smith
Chapter Rep. to OAME	Peter Lenczewski
Treasurer	Fran Outerbridge

GOLDEN SECTION (HAMILTON AND NIAGARA PENINSULA)

The report centered around the Niagara Falls Annual discussed earlier in "Council Comment".

GVMA (GRAND VALLEY MATHEMATICS ASSOCIATION)

The spring conference took place on 1988 04 14 when 175 teachers attended. Don Fraser was the keynote speaker. His presentation was recorded for later viewing on T.V. Ontario.

Two more after-school conferences are being planned for 1988-89. The fall program will be Thursday, 1988 11 10 at Eastwood Collegiate in Kitchener. For more information, contact Dean Murray, Milton DHS, 396 Williams Ave., Milton, Ontario L9T 2G4.

NOMA (NORTHERN ONTARIO MATHEMATICS ASSOCIATION)

The "fall classic" is being held on Friday, 1988 09 30 and Saturday 1988 10 01 at La Salle Secondary School in Sudbury.

A workshop for teachers on how to recruit, motivate, and prepare students for mathematics competitions is being planned. Support has been received from school superintendents who will grant release time for the teachers. NOMA will cover other costs and provide the workshop leaders.

Student excellence awards are being discussed but not finalized yet. Watch for further information.

NWOMA (NORTHWESTERN ONTARIO MATHEMATICS ASSOCIATION)

This fledgling chapter is in its early stages. An executive is being formed. Support for this new chapter in the Thunder Bay area is being solicited.

PAME (PEEL ASSOCIATION FOR MATHEMATICS EDUCATION)

The December Math contest attracted 16 schools with 750 students. Business supplied the awards.

An elementary problem-solving kit is being developed.

PINE RIDGE MATH ASSOCIATION (OSHAWA/PETERBOROUGH AREA)

A new executive has been elected.

The Pine Ridge Mathematics Conference was held last April with 165 people attending.

A Math awards night is being planned when 100 students will receive medals for a variety of mathematics contests.

QSLMA (QUINTE ST. LAWRENCE MATHEMATICS ASSOCIATION)

We are all saddened by the death of Robin Hood, a former councillor, who passed away. Robin was a teacher with the separate school board for Frontenac Lennox-Addington.

SAME (SCARBOROUGH ASSOCIATION FOR MATHEMATICS EDUCATION)

A new executive is to be elected. A publisher's display and wine and cheese social was reasonably successfully.

SWOAME (SOUTHWESTERN ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION)

The fall conference will be held at the University of Windsor on 1988 11, 18-19. A total of 16 sessions for elementary and secondary teachers will be presented. The keynote speaker is Dr. Eugene Smith from Wayne State University and a past president of the NCTM.

YORK MATHEMATICS ASSOCIATION

An organizing committee is planning the first fall mini-conference to be held on Wednesday, 1988 10 05 at Lewis S. Beattie Secondary School in North York. The theme will be problem solving.

A second evening conference will be held 1989 02 22.

COMMITTEE REPORTS

PRIMARY/JUNIOR COMMITTEE

Roland Morrow	Executive Liaison
Bob Robinson	Chairperson

Shirley Fairfield
Sue Stuart
Paul Lessard
Jan Cornwall
Lorna Wiggan
Olive Fullerton

The winners of the problem solving project are Ruth Douglas (Ottawa) and Trudy McEwen (Kingston). Their submissions will appear in a forthcoming issue of the *Gazette*.

We are still concerned about the delay in the translation of mathematics support documents and guidelines into the French language. This will be pursued.

A letter will be sent to OISE requesting full-time staff members be appointed to provide leadership in mathematics education and supervise graduate work in mathematics education.

We are awaiting the OMCA Standards document for our reaction.

We will have further dialogue pertaining to the Radwinski Report; Junior Review for Grade 6 Mathematics; and "Benchmarks".

PUBLICATIONS COMMITTEE

Ralph Connelly	Executive Liaison
Jack Weiner	Chairperson

Ron Lancaster
Marilyn Pearl
Stu Telfer
Ron Ennis

We welcome Bob Smith as our interim editor of the *Gazette* and wish him well.

Ron Ripley will continue as the editor of the *Abacus.*

The committee will investigate how to obtain feedback regarding how the membership feel about our publication. Please contact Jack Weiner if you have any concerns or comments.

We encourage individuals to write articles for our *Gazette.* Please send any article, no matter how short or long, to our editor, Bob Smith.

INTERMEDIATE/SENIOR COMMITTEE

Arm Carefoote	Executive Liaison
Enzo Carli	Chairperson

Ed Harris
George McNabb
Mary Lynn Jeffries
Mickey Sandblom
Jerry Malony
Ann Jones

The OAIP Grade 11 is almost complete and is going to be field tested. The Grade 12 school should be ready in December. There is some concern about the format of the final problem... whether it will be a paper or disk. If it is on disk, what hardware will be used?

The NCTM Standards document was discussed and are awaiting the OMCA Standards document for reaction.

Further dialogue will transpire regarding the Radwinski Report and the apparent shortage of qualified mathematics teachers teaching mathematics courses at the Secondary level because of the effect of seniority lists in boards with declining enrolments.

MEMBERSHIP/PUBLICITY COMMITTEE

Dave Handley	Executive Liaison
Eileen Collins	Chairperson

Liz Henry
Todd Romiens
Jan Mitchell
Steve Martin

The "I Love Math" buttons have been quite a success and an ad will appear in the *Gazette.* For more information, contact the Sarnia office.

The committee will update and redesign the membership brochure.

The concept of life membership in OAME will be investigated.

FUTURES COMMITTEE

Mary Lou Kestell	Executive Liaison
Jeri Lunney	Chairperson

Dave Davidson
George knill
Peter Lenczewski
Paul Polle
Tom Griffiths
Elaine Harvey

The poster winners were announced. Look for further information in the *Gazette.*

A policies and procedures manual is being prepared for the new Secretary-Treasurer to assist him in his work.

WOMEN AND MATH COMMITTEE

J. Symington	Executive Liaison
Lorna Wiggan	Chairperson

Elaine Harvey
Olive Fullerton

This committee will work with COMA to present a set of sessions at the Ottawa Annual. They will pursue obtaining suitable candidates for council.

FUTURE DATES

EXECUTIVE	**COUNCIL**
1988 11 25	1988 10 15
1989 04	1989 02 11

MCCARTNEY SCHOLARSHIP

Mona & Morley MacGregor wish to announce the 1988 winner of the MacCartney Scholarship to be
PATRICK SURRY
of A.B. Lucas Secondary School, London, Ont.

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
1988-89 EXECUTIVE & COUNCILLORS

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Pres.	Symington, J.
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Director	Carefoote, Arnold
Director	Handley, David
Director	Morrow, Roland

MEMBERSHIP DUES
TILL DECEMBER 1, 1988

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247 Bright Street
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Knill, George
Lunney, Jeri
Martin, Steve
Pogue, Paul
Stuart, Susan
Weiner, Jack

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Fullerton, Olive
Jefferies, Mary Lynn
Jones, Ann
Lessard, Paul
McNabb, George
Rogers, Dr. Pat
Sandblom, Michael
Wiggan, Loma

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Collins, Eileen
Cornwall, Jan
Davidson, Dave
Griffiths, Tom J.
Harvey, M. Elaine
Mitchell, Janet
Lancaster, Ronald
Robinson, Robert