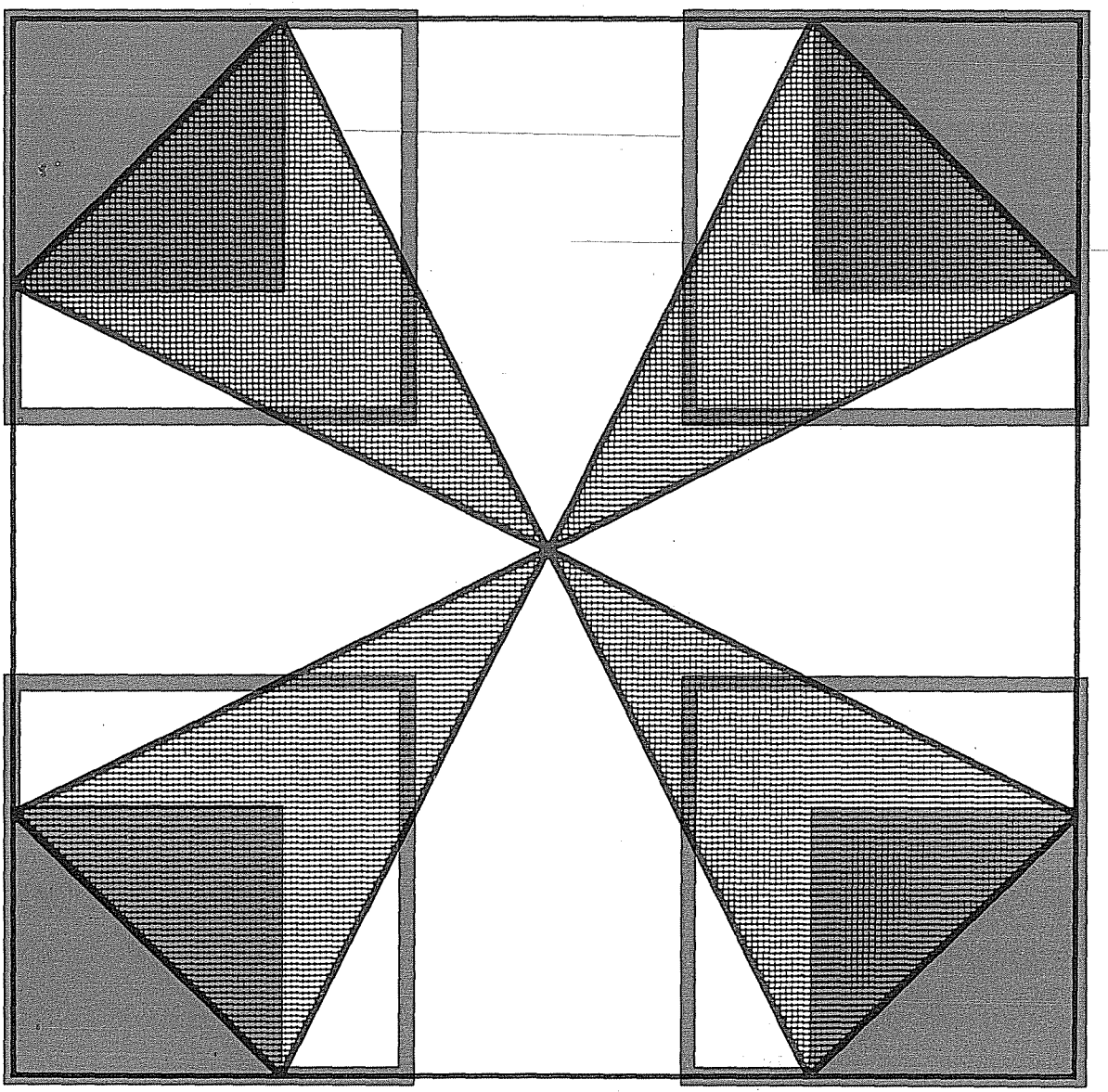
ontario mathematics gazette

VOLUME 27 NUMBER 2 DECEMBER 1988



THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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Deadlines for submission of information to be included in O.M.G. are as follows:

October 15	for the December issue
January 30	for the April issue
June 15	for the September issue

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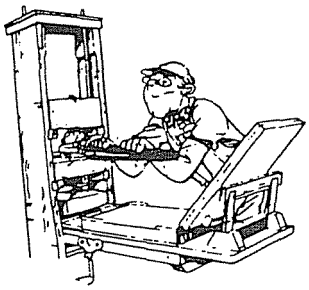
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DECEMBER 1988

December -- half the year is gone. It has been a busy time here. A full teaching load, supervision of the practicum *and* O.M.G., has meant little time to compose this editorial. The Pocket Oxford Dictionary states that an editorial is an article written or sanctioned by the editor. Though my random thoughts may not constitute an article, they are sanctioned.

This period of time, as O.A.M.E. moves to a new era and a new secretary treasurer, is a time for restructuring. Hard thought must be given to many areas. Your new administrator will be attempting to carry out the *policies* which the executive and council of O.A.M.E. put in place. He has new tools to do the job, and there will be new ways. Many things must be considered. Might I suggest a few?

1. Is it time to monitor how well we are meeting the objectives stated in our constitution?

   First objective listed - to aid the professional growth of mathematics educators by proposing, initiating, co-operating in preservice and inservice programs designed to improve mathematical understanding and methods of teaching. Last objective listed - to maintain an active and representative mathematics educator's "voice" in Ontario. With the final report from the Provincial Steering Committee on Teacher Education in abeyance, there is still time for O.A.M.E. to be a part of the decision-making process on teacher induction, the year in schools following the preservice year.

2. From October 19th to the 21st, the popular Leadership Seminar was held. It has been most successful during its 17 years. Many local chapters are also showing leadership by means of a variety of local conferences and written materials, e.g. G.V.M.A. publications. Is it time for a change in the focus of the O.A.M.E. Leadership Seminar? Might I suggest an objective -- to improve the quality of mathematical leadership in schools of Ontario by identifying and developing strong prospective Mathematics Co-ordinators, Mathematics Department Chairpersons and Special Assignment Teachers for the Elementary Schools.

3. At the annual meeting of O.A.M.E., held in Niagara Falls, May 7, 1988, only a few members were present. A financial statement dated April 30, 1988 was available. At the bottom was this sentence "This is not an audited statement". Is it time for the executive and council to be considering regularized accounting procedures with proposed budgets? Fiscal responsibility allows any organization to meet their aims and goals. We have been indeed fortunate to have done so much over the 26 years of our existence.

4. President J. Symington reported that chapters have grown from 8 to 11 with membership topping 1700. He looks forward to an ever-expanding organization and stated that "change can and should be a positive force behind an organization, whose mission is to promote excellence in mathematics education throughout the Province of Ontario". Is it time to examine the organization? With 24 councillors, an executive of 6 and 11 chapter representatives, we have a large council which meets twice a year.

5. Members of O.A.M.E. obtain O.M.G. and have the opportunity to attend conferences at reduced rates. They also have the knowledge that O.A.M.E. positions on mathematics are being represented. Is it time to broaden our horizons?

O.A.M.E. has a long and distinguished history. It started with the Ontario Association of Teachers of Mathematics and Physics section of Ontario Educational Association and continued from 1881 to 1960. The Ontario Mathematics Commission and O.A.T.M. joined to form O.A.M.E. in 1973 and has been working with the same willing administrative workers ever since. The organization has functioned well during the past. We now look forward and must plan for the future. It is good to read in Bob Robinson's column Council Comment, that planning for change has begun.



July 22, 1988

Dear Editor:

As discussions on appropriate evaluation increase across the province, many school boards and individual teachers are searching for meaningful evaluation techniques. Many are caught in the dilemma of choosing between accountability, which is often equated with standardized testing, and activity-approach classrooms. We are witnessing changes in the way teachers interact with children, but few changes in the methods teachers are expected to employ to evaluate children in mathematics.

It would be interesting to publish a series of articles over the next few years to assist teachers in developing innovative and effective approaches to pupil and program evaluation. I hope that this article will encourage discussion and generate other submissions from O.A.M.E. members.

Yours truly,
Susan Stuart
Mathematics Consultant K-8
Sudbury Board of Education
O.A.M.E. Councillor, 1986-

Editors Note: Is this worthy of an issue for O.M.G.? Let us hear from you. In particular, put your thoughts and findings into an article. Susan Stuart has done so. Watch for her article in a future issue.

Office of the President & Vice-Chancellor
Room 113, Stevenson-Lawson Building
The University of Western Ontario
London, Ontario
N6C 5B8

Dear Bob,

I would like to congratulate you and your colleagues on the revival of this important publication and to compliment you on its quality.

Best wishes for continued success with O.M.G.

Sincerely,
K.G. Pedersen
President & Vice-Chancellor

Toronto Board of Education
885 Dundas Street East
Toronto, Ontario
M4M 1R4

Dear Bob,

Congratulations on the new look of the Gazette! I love it! It's a welcome change.

Sincerely,
Lorna Wiggan

The Waterloo Board of Education
51 Ardelt Avenue, Box 68
Kitchener, Ontario
N2G 3X5

Dear Bob,

Excellent September issue! I am very impressed by what you were able to accomplish in such a short time.

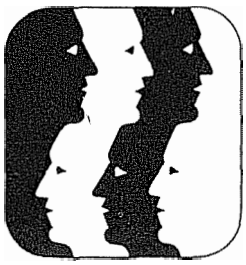
Stu Telfer

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Viewpoint



MATHEMATICS FOR THE GIFTED: ONWARDS OR SIDEWAYS

by
Tom Griffiths
A.B. Lucas Secondary School
London, Ontario N5X 1L8

A few years ago the O.A.M.E. produced a policy paper on the education of mathematically talented students. One of the recommendations in this document was that the enrichment work for these students should be studying the topics from the ministry guidelines, or the standard classwork, in more depth, rather than to continue on with more advanced topics. The recommendation was for horizontal rather than vertical development. The discussion in council on this clause became very heated, with the proponents for the two possible views being very clearly divided with elementary teachers wanting horizontal, and secondary teachers wanting vertical development. The motion for horizontal development passed by one vote.

As a secondary school teacher my personal opinion is that the mathematically talented students should be allowed to develop vertically, as well as horizontally. That is, allow and encourage them to develop mathematically, both by expanding their knowledge and skills at their age level, as well as investigating concepts normally encountered by students older than themselves.

My reasons for this recommendation are several. First, it is the nature of mathematically precocious children that they study far beyond their peers in this area. In extreme cases mathematicians as young as fourteen years old have been known to develop new mathematical ideas. Even very good (but not exceptionally talented) mathematics students work at a pace far beyond their peers. The average child, on encountering equations or word problems for the first time finds these topics difficult to understand, and takes many hours of concentrated study to become barely competent. The talented student takes one reading and is ready and eager to extend the skill and understanding to equations with several variables. The important motivator is that these students want to extend the original ideas into new concepts. Being restricted to the topics studied by their peers is a burden to them and, worst of all, boring. These children develop very rapidly; they can pick up an idea so quickly it is almost unbelievable. Their peers, however, will struggle with the same concept for many hours and never master it.

One meaning of higher mathematical intelligence is that people with such intellect think about and learn mathematics more rapidly than others of the same age with less of this kind of intelligence. This in itself means that these children should be studying work beyond that of their peers. Acceleration beyond the normal ministry curriculum, which is designed for the average child, is the normal pace for the child with higher intelligence. There is, of course, a continuous difference between students, rather that a quantum jump between groups of children, so that individual students have to be catered to individually. This is particularly difficult to do in a class of thirty or so. However, one of the pleasant things about mathematics, as against other subjects, is that it is to some extent a solitary endeavour in which the participant works in the realm of his or her own mind. It is, therefore, possible to give students problems or puzzles to play with by themselves.

The one area I have found that mathematically talented students like above all others is the realm of problem-solving, and the problems need not be only mathematical in nature. They especially love work logic problems such as *Quizzles*. They will, of course, ask questions and require help at times. I find that very often, especially with my more senior and high ability students, I cannot always answer their questions. In such cases both the student and I have to be content with my giving them some possible directions for investigation rather than answers. This is not so frustrating as it may seem. It is permissable for a teacher to say "I don't know. Why don't you look in these directions?" It is a nice surprise for the students to be studying a topic which leads to questions which their teacher cannot answer. Occasionally the students will then come up with some excellent and original

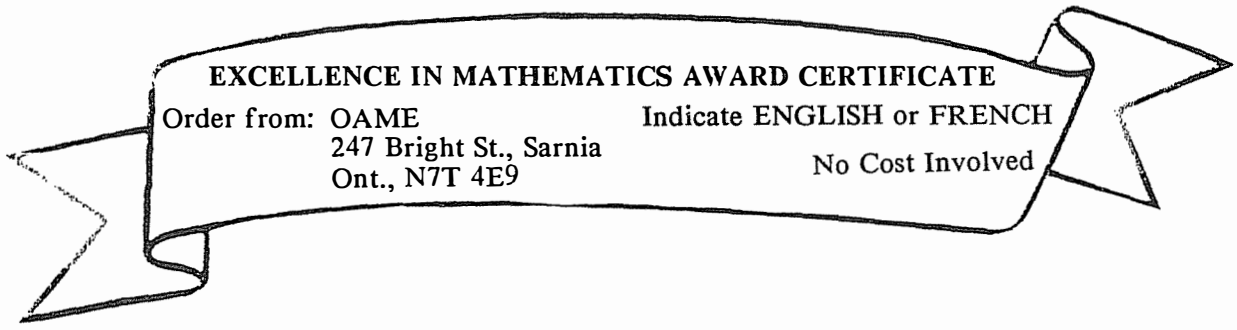
ways of looking at a problem, which is a nice surprise for the teacher. If everything is carefully compartmentalised, as in a text book, this open-endedness cannot exist.

The nature of mathematics is also another reason why the development of these students should be allowed to be vertical rather than horizontal. In history it is possible to study an episode or period from many different aspects: reading, modelmaking, writing, investigating characters and their inter-relationships, investigating the culture of the time, investigating the political and social milieu of the time. I am sure that history teachers would add several other approaches. In English the same breadth of possibilities also applies. A student can read all of an author's work, something I have found a great joy to do, and then the author can be studied in his or her own right. In mathematics, once an idea at the student's level has been appreciated and a few examples, problems and other questions considered, it is time to see where that idea goes. The students not only need to see where the ideas flow, for that is the very nature of mathematics, but, in my experience, they want and demand to follow the ideas further. It just depends on how much I assist them in this activity.

The topics to be studied need not only be in the ministry core material course outlines. Many of the topics normally in university curricula are accessible to much younger able mathematicians. The reason many topics are not in the ministry course outlines is rather that the average sets the pace and there is a finite amount of time available, not that the material is beyond the understanding of students of a particular age. One of the great joys of mathematics is that there are many topics which can be studied with very little knowledge and experience. Some of these topics are Euler's formula for polyhedra and his proof for the Konigsberg Bridge problem, Diophantine equations, modulo arithmetics, and number theory, which yields problems for all age levels. Then there are the logic puzzles and alphanumerics. There is a tremendous wealth of topics which can be studied by young mathematicians if these students are not restricted to horizontal development. I have rarely found that when students submerge themselves in an area of problem-solving they do not encounter concepts that are required for solving the problem which come up further on in the course or a later year's work. I have rarely seen a student get into the study of numerical geometry and not very soon want the tools of trigonometry. Usually I can give the student enough of this topic that they can use it for a tool. Should I say "No, you can't know this until your peers are ready"? I do not suggest that a student be given the next year's text and told to read it. They may be given problems and puzzles of a broad enough nature that they will find something of interest. They will not all be interested in the same topic, so a variety have to be provided. Then let them go without restriction. Let them use calculators, computers, graph paper or whatever they can think of. The results are amazing and delightful.

Are we not holding talented students back if we do not allow them to develop at a natural pace? This natural pace is to a great degree determined by the nature of mathematics, the nature of the student, and their joy of discovering new ideas. Mathematics is one area in which the teachers can allow their students to discover the joy of the world of ideas most easily.

As a final thought I find that I am often criticized for my interest in talented and gifted children, and in my wish to work with them. It is sometimes suggested that I should also work more with the less talented if I am being a real teacher. I do not notice that teachers of the less able are asked to prove their worth by working with gifted students. I cannot help thinking that embedded in this prejudice is the reason why the need for appropriate education for the gifted and talented has not yet been fully accepted.





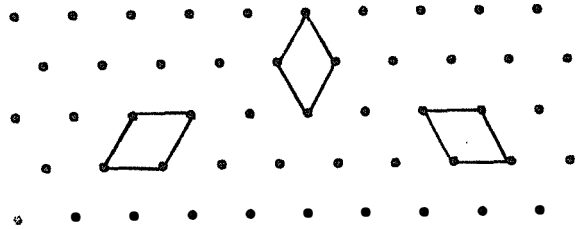
AREA ACTIVITIES ON NON-STANDARD DOT PAPER

by
Bonnie H. Litwiller and David R. Duncan
Professors of Mathematics
University of Northern Iowa
Cedar Falls, Iowa

Mathematics teachers are always on the lookout for situations that can involve their students in exploring mathematical ideas and making conjectures about them. Geometric situations provide a good setting for such explorations and conjectures. In this article we use isometric dot paper as the setting for our work.

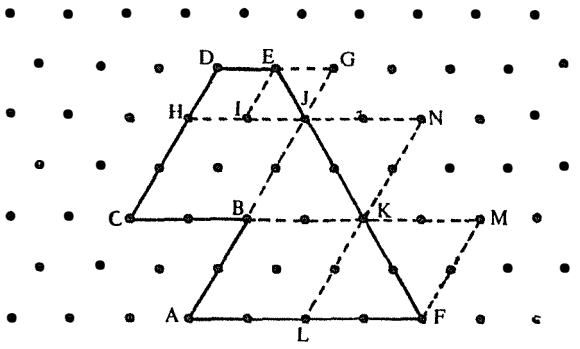
Suppose that we wish to determine the areas of polygons on isometric dot paper using the rhombus as the unit of area. Figure 1 shows three such unit rhombi.

Figure 1



To determine areas of more complicated polygons, they must often be subdivided and analyzed as shown in Figure 2.

Figure 2



The area of polygon ABCDEF may be determined in rhombi units as follows:

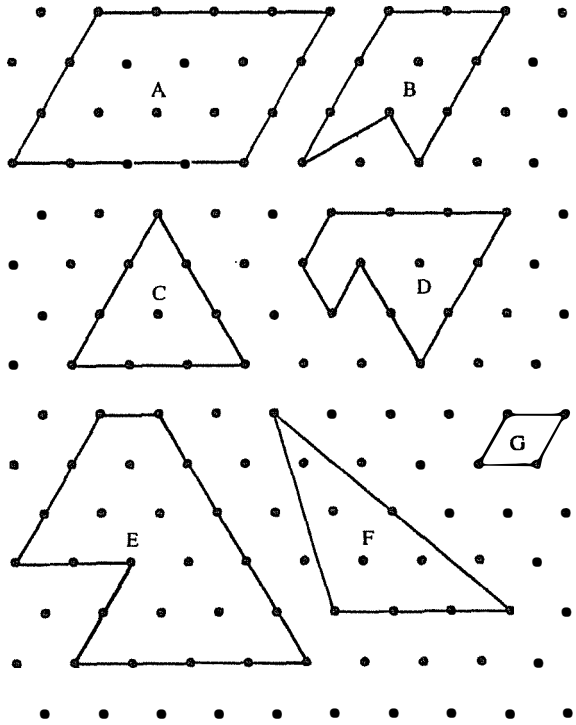
1) Polygon ABKL has area 4.
2) Polygon CHJB has area 4.
3) Polygon DEIH has area 1.
4) ΔEIJ has area equal to $\frac{1}{2}$ the area of polygon EIJG or $\frac{1}{2}$ (1) = $\frac{1}{2}$.
5) ΔBJK has area equal to $\frac{1}{2}$ the area of polygon BJNK or $\frac{1}{2}$ (4) = 2.
6) ΔFKL has area equal to $\frac{1}{2}$ the area of polygon FLKM or $\frac{1}{2}$ (4) = 2.

Thus the total area of polygon ABCDEF is $4 + 4 + 1 + \frac{1}{2} + 2 + 2 = 13\frac{1}{2}$ rhombi units.

A familiar formula for area of simple polygons on square dot paper is Pick's formula: $A = I + \frac{1}{2}B - 1$ where I is the number of points inside the polygon and B is the number of points on the boundary of the polygon. Does this area formula hold on the isometric dot paper?

Figure 3 displays several simple polygons and Table 1 compares the areas of these polygons by counting and by applying Pick's formula.

Figure 3

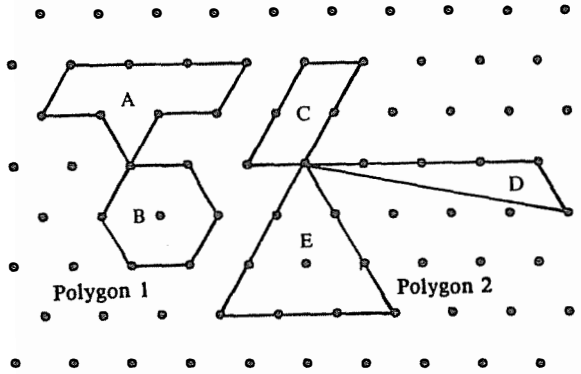


What would happen if non-simple polygons were drawn on the isometric graph paper? Does Pick's formula still hold? Two such polygons appear in Figure 4.

Table 1

Polygon	Area (Counting rhombi)	I	B	Area (Pick's formula)
A	12	6	14	6 + ½ (14) - 1 = 12
B	5	1	10	1 + ½ (10) - 1 = 5
C	4½	1	9	1 + ½ (9) - 1 = 4½
D	5½	1	11	1 + ½ (11) - 1 = 5½
E	13½	6	17	6 + ½ (17) - 1 = 13½
F	6	4	6	4 + ½ (6) - 1 = 6
G	1	0	4	0 + ½ (4) - 1 = 1

Figure 4



The area of polygon 1 may be found by adding the areas of polygons A and B. The area of A is 3½ while the area of B is 3; thus, the area of polygon 1 is 6½ rhombi units.

Applying Pick's formula with I = 1 and B = 14, we obtain $1 + \frac{1}{2}(14) - 1 = 1 + 7 - 1 = 7$. This "Pick area" is ½ larger than it should be.

The area of polygon 2 is found by adding the areas of polygons C, D, and E. These areas are, respectively, 2, 2, and 4½; thus, the area of polygon 2 is 8½ rhombi units. Applying Pick's formula with I = 1 and B = 19 we obtain $1 + \frac{1}{2}(19) - 1 = 9\frac{1}{2}$. This "Pick area" is one larger than it should be.

Conjecture: If a point on the boundary is the corner point of more than one simple polygon, one-half the number of "excess" polygons must be subtracted from the results of Pick's formula. For Polygon 1, one-half must be subtracted and for Polygon 2, one must be subtracted. Specifically, Pick's formula is modified as follows:

$A = I + \frac{1}{2}B - 1 - \Sigma\,[\frac{1}{2}(C-1)]$ where C is the number of simple polygons at any one vertex.

For most vertices C = 1, meaning that the vertex is a "corner point" of only one simple polygon. In these cases C - 1 = 0 and does not contribute to the summation. If C > 1, meaning that a vertex is a corner point of more than one simple polygon, then C - 1 > 0 and does contribute to the summation. Each vertex must be considered separately and their respective contributions summed as indicated in the formula. Observe that for simple polygons C always is 1, thus reducing the modified Pick's formula to the original Pick's formula.

Figures 5 & 6 displays several non-simple polygons, and a discussion of the area of each follows.

Figure 5

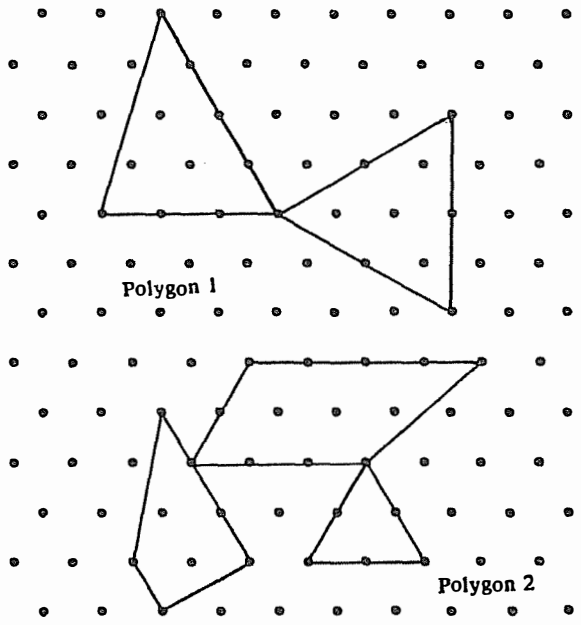
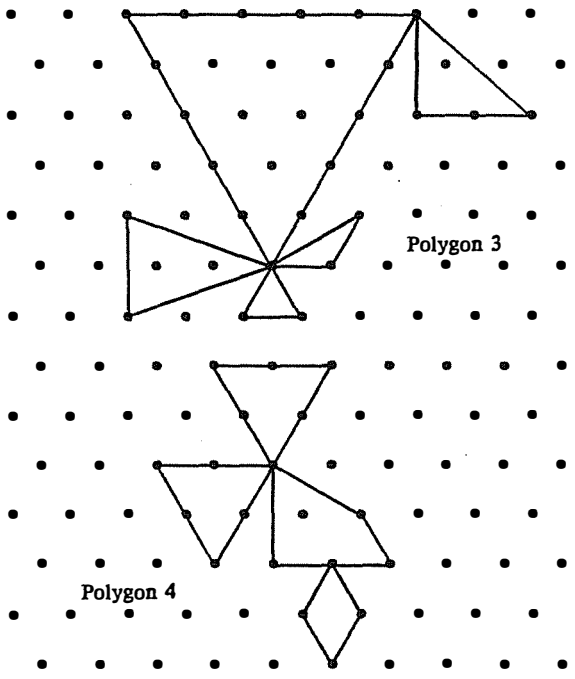


Figure 6



Polygon 1

I = 7

B = 13

C:one value is 2; all the other values are 1 and do not contribute to the sum.

A = 7 + ½ (13) - 1 - ½ (2-1)

= 7 + $\frac{13}{2}$ - 1 - ½

= 12 rhombi units

Polygon 2

I = 5

B = 20

C: 2 and 2

A = 5 + ½ (20) - 1 - ½ [(2-1) + (2-1)]

= 5 + 10 - 1 - ½ (2)

= 13 rhombi units

Polygon 3

I = 9

B = 24

C = 4 and 2

A = 9 + ½ (24) - 1 - ½ [(4-1) + (2-1)]

= 9 + 12 - 1 - ½ (4)

= 18 rhombi units

Polygon 4

I = 1

B = 18

C: 3 and 2

A = 1 + ½ (18) - 1 - ½ [(3-1) + (2-1)]

= 1 + 9 - 1 - ½ (3)

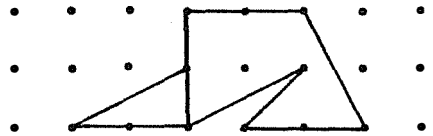
= 7½ rhombi units

For each of the four polygons of Figures 5 & 6 verify the results of the areas found by using the modified Pick's formula. Subdivide the polygons into rhombi and count to check your answers.

Challenges to the reader:

1) Draw other non-simple polygons including more intricate polygons. Does the modified Pick's formula still hold?

2) If non-simple polygons were drawn on square dot paper, does the modified version of Pick's formula hold?



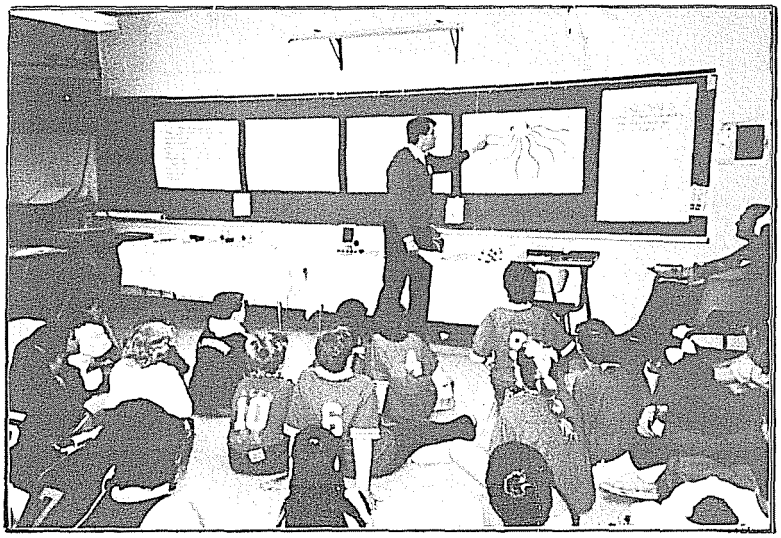
LONDON JUNIOR MATH CHALLENGE

by
Gina MacDonald
Lorne Ave. P.S.
London, Ontario N5W 3K7

Editor's Note: Math Contests. Much good work is being done in local schools. We would like to publish a variety of examples of these attempts to improve feelings about Mathematics, not only the Math Olympiads, Descartes Contests, but also school clubs. Let us know about yours.

How long has it been since you have been to an elementary school math contest? Do you remember the warm-up games, the marching music, the intense discussion with your partner over some of the problems, the magician, the ice-cream, the prizes and trophies? These are not the memories you have of math contests? Then perhaps you need to visit the Junior Math Challenge.

On May 4th, the London Board of Education held its second annual Junior Math Challenge. Twenty schools entered a team of six members each. Each team consists of 2 grade 4's, 2 grade 5's and 2 grade 6's. The teams rotate from Challenge site to Challenge site with the help of older student volunteers. Teachers bringing a team to the Challenge help to man the six sites. This year the emphasis was on Patterns & Sequencing, Mental Math, Geometry, Measurement, Problems Solving and the very popular Puzzle Corner. The first four sites are done individually. Problem Solving is an event for pairs where the two students of the same grade level work together.



The above picture shows teacher, Bob Moore going through the solution to questions in the ever popular Puzzle Corner.

In the Scoring Room, the results from each site are tabulated. The lowest score of each team is eliminated and the other five scores are tallied for a team score. The renowned Puzzle Corner is the only site that does not contribute to an overall team score. Here, the students are challenged to answer any of the wall-mounted puzzles they can by depositing their answer in a bag beside each puzzle. When all the centres have been completed, the teams meet in the Puzzle room where names are randomly selected and if the answer is correct, they choose a prize from an elaborate prize table. While this penny carnival type activity is going on, the score keepers are madly tallying each team score so that first, second, and third place can be announced before everyone leaves.

Peter Menning, a local Magician served as halftime entertainment for the students. The entire Challenge lasted from 1:00 p.m. to 5:30 p.m. on a Wednesday afternoon. There were 120 students from 20 different schools who attended this year's Challenge.

The idea for this innovative math contest came to London from the Hamilton-Wentworth Separate School Board. The concept was presented at an O.A.M.E. conference in the Fall of '86. Close contact and the exchanging of ideas concerning the Challenge has continued between the two Boards. So, if your memories of math contests include pressure, anxiety and uniformity, perhaps you need to attend a Junior Math Challenge and see for yourself how much fun math contests can be.



COCKROACHES ARE HERE TO STAY: PROBLEM SOLVING WILL NOT DIE

by
Ron Reesor
Faculty of Education
The University of Western Ontario
London, Ontario N6G 1G7

For over two decades I have been a successful (I wasn't fired) secondary school mathematics teacher and department head. During the past few years I have become more and more disenchanted with my teaching of the mathematics curriculum and therefore it is with trembling hand and reluctant pen I begin to expose my feelings. It seems to me that the students' classroom experiences are very routine, unimaginative, and filled only with the mechanical manipulation of numbers and shapes. After being in this type of environment for several years, I believe that students develop some very distinct, yet false impressions about mathematics, and how it relates to them and their education. Here are some of them:

a) Every math question has only one right answer.
b) There is only one right way to do a math question.
c) All mathematics questions are found in textbooks.
d) Mathematics is an isolated subject.
e) Mathematics is seldom used in the 'real world'.
f) The students feel it is *unfair* if they are required to write English sentences to explain a concept - "This is Math, not English.".
g) Mathematics deals only with the mechanical manipulations of numbers, letters, and shapes.
h) Problem solving relates to word problems with algorithmic procedures used in their solutions: i.e. from the curriculum guidelines "A strategy is evident immediately. Difficulties in solution are related to one's ability to carry out the strategy correctly." (3).

The following quote sums up my thinking: "Math does make me think of a stainless steel wall--hard, cold, smooth, offering no handhold, all it does is glint back at me. Edge up to it, it doesn't take your shape, it doesn't have any smell, all it does is make your nose cold. I like the shine of it--it does look smart, intelligent in an icy way. But I resent its cold impenetrability, its supercilious glare." (1)

An analogy that I use to describe the situation is comparing the teaching of math to coaching (teaching) a sport. In both instances we teach the skills piece-meal, but here the similarity ends. In the sport the players are central to the game and we let them play the game. To be successful they must use and combine all the skills that were taught. The more they play the game the more confident and proficient they become. Their enjoyment increases and consequently their individual skills also improve. On the other hand in the teaching of mathematics I seldom let the students play the game. Too often I stopped them after the skills had been taught. There are several reasons for this:

a) The curriculum contains too much content and therefore there is not sufficient time to let the students play the game.
b) I am not always sure how to play the game myself.
c) It requires more time and energy to organize and manage the game.
d) The Socratic (expository) style of teaching predominates and therefore the mental training of the players does not allow for an easy transition from skills to the game.
e) The textbook format does not allow the game to be played.

When the Ontario Ministry guidelines were published in 1985, I read with eager anticipation that one of the main thrusts was to be on problem solving. The emphasis was to be on problems where "a strategy is not immediately evident. Difficulties in solution are initially related to choosing an appropriate strategy." (3). This reminded me of "An Agenda for Action" published in 1980 by the National Council of Teachers of Mathematics. One of the recommendations in this document was that problem solving was to be the focus of school mathematics in the 1980s. It is becoming more apparent that what is stated in the documents and what is happening in the classrooms are miles (whoops-kilometres) apart. The authors of the texts have not tied problem solving to the curriculum, but rather have interjected problem solving type questions throughout their books (i.e. they have posed problems that are not related to the topic or unit - just extra questions). The courses of study that I have read either ignore

the topic altogether or approach it the way the authors have done. It would appear that we have not progressed very far - treading water will keep you afloat, but eventually you get tired - swimming is much more fun.

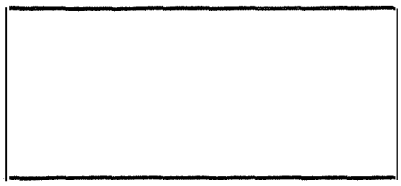
I believe the resolution of my concerns is tied to this problem solving concept. We have to organize the curriculum around problem solving procedures and present a more unifying or holistic approach to the study and learning of mathematics - rather than the bit by bit behavioural approach that exists today. Understanding each particular skill does not necessarily translate into understanding the entire structure (i.e. playing the game).

I reflect back to the many problem solving sessions that I have attended where many interesting and varied problems were presented, but at the end of it all one nagging question always remained - How can I use this in my classroom? The answers to that question were either too vague or not given. My goal in the next few pages is to outline a model that I believe could be used successfully to make problem solving the central focus of the classroom. This model could be used in many subject areas. In the next issue I will use this model to outline a framework for designing a particular unit, and give detailed samples of experiments, and problems for this unit.

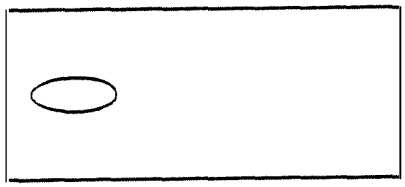
The key elements in the whole process are the teacher and the attitudes, insights, and enthusiasm which that person brings to the classroom. The teacher must enjoy solving problems and constantly be on the lookout for new and interesting problems. One must have a thorough understanding of Polya's phases of the problem-solving process:

- Understanding the problem.
- Developing strategies.
- Solving the problem.
- Looking back.

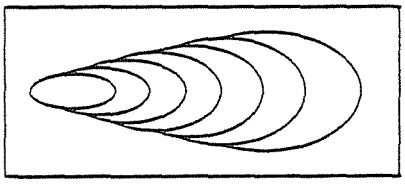
A teacher must try and present the material in as interesting way as possible in order to arouse the intellectual curiosity of the students. The logical, step by step approach used in texts does not offer many opportunities to perk student interest. We must become better story-tellers and learn how to set the scene for our students. This approach will not work on a menu of once or twice a week - it must be the central approach in the classroom. I will represent the classroom atmosphere (teacher created) as a plane (shown as a rectangle).



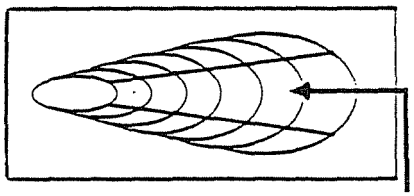
When the student enters this atmosphere, the first week is spent on introducing the problem solving process and giving the student some experience in solving problems. Puzzles, games, and curiosities would be used to establish a different classroom environment. The students should be involved in a great deal of group work, given the opportunity to explain their solutions to the class, and make up problems for each other. The shift is towards a student-centred classroom. The heuristics, and strategies are ingredients that are outlined in many good books (5), (6), (8), (9). In these early stages it is important for the student to have many successes in order to build confidence. The student should also begin to realize that being wrong, making a mistake, and being stuck are integral to the process. This initial exposure will be represented by an elliptical shape in the plane.



After this exposure the students will be confronted each day with a strategy that will further develop their skills and understanding. This could be done as a warm-up (beginning) exercise each day. I will represent the students' development as shown in the following diagram.



The teacher will begin teaching the curriculum using problem solving techniques, after the initial exposure. I will represent the curriculum as being bounded by two straight lines to signify the sequential form that the curriculum has taken.



CURRICULUM

As soon as I drew this diagram the picture of a cockroach flashed through my mind and therefore the name of this model is the C. Roach Model. However negative your feelings are toward these creatures, the representation is there and I'm going to use it. The analogy could very well be that C. Roach (the student) is enticed into the classroom where he is fed and nourished with curriculum served on a problem solving platter. When C. Roach leaves the classroom he takes his problem solving skills (body) along to still another class where he could continue to grow with this process. Real-life problems encountered outside of school will be more easily solved with his body of knowledge about problem solving. His legs (the underpinning of this model), which are not visible, represent the psychological, philosophical bases for delivering the curriculum using a problem solving model. As C. Roach returns daily he becomes more and more comfortable in the classroom and consequently the role of the teacher gradually changes. I have represented the changes by four different divisions in the diagram. These four divisions or levels are taken from the work of Mary Grace Kanatowski (4).

Level One - Model

- students have little understanding of problem solving and would need to be led through the various strategies for solving problems.

Level Two - Crutch

- students may be insecure about independent problem solving.
- need encouragement and help.

Level Three - Problem Provider

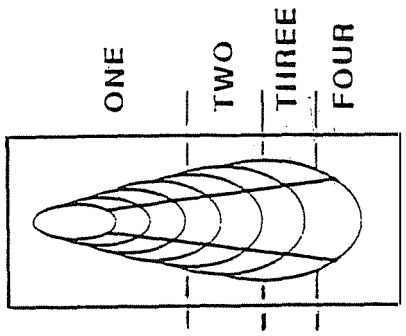
- students feel comfortable with problem solving. They realize that many answers or no answer are acceptable results.

Level Four - Facilitator

- students are now interested in elegancy and are on the lookout for new problems.

I am not claiming that all students will reach all four levels, but it seems reasonable to expect some students would reach Level Four.

The part of the body not included in the curriculum represents extra (optional) assignments, puzzles, games, and some non-routine type problems.



If this model were to be used for several consecutive courses, then the initial exposure time would necessarily shrink and this would allow more time for the content. Claims have been made that after students become familiar with the process they are able to cover more material in a shorter period of time (7). Teaching the content through a problem solving process will initially require more preparation time and careful planning by the teacher. The evaluation of the problem-solving process is one that is often neglected or ignored. Charles & Lester (2), and Schoenfeld (10) have devised similar schema for evaluating the process. A scheme for evaluating the heuristics is:

Heuristic	Marks
1. Understanding the problem	0,1,2,3,4
2. Developing strategies	"
3. Solving the problem	"
4. Looking back	"

Set up your own marking scheme for each heuristic and indicate what each mark will signify. A complete description can be found in the references.

The Ministry of Education for the Province of Ontario has given its teachers a mandate to develop the problem solving process. We, as teachers, have been left on our own to initiate, and implement this new approach in teaching the mathematics curriculum. I have tried to outline a model that could be used in developing problem solving skills in our students. In the next issue I will outline the design of a particular unit using this model.

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Teaching Tips

TEACHING LINEAR RELATIONSHIPS

Course: MAT IA

Objective: Represent Relations by tables, ordered pairs or arrow diagrams

Materials: Counters (bingo markers are good)

This is a puzzle which requires the moving of a counter from one corner of a square to the diagonally opposite corner in the MINIMUM number of moves, only one counter being moved at a time, HORIZONTALLY or VERTICALLY only.

Figure 1

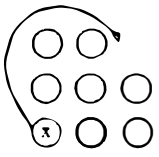
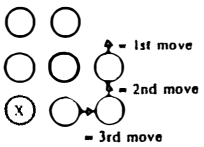


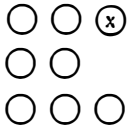
Figure 2



YOU DO NOT HAVE TO MOVE THE COUNTERS AS SHOWN. These have been drawn to show you how to play the game and that you should not move counters diagonally. (Fig.2)

1. Set out the counters as shown in Figure 1 and count the minimum number of moves necessary to get the cross counter up to the space. It does not matter where you leave the final space. Make a note of this minimum number.

Figure 3



Final Arrangement

2. Set out a 4 by 4 square and repeat. Note the minimum number of moves.

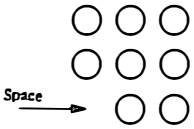
3. Set out a 5 by 5 arrangement, and a 6 by 6 arrangment, and find the relationship between the minimum moves necessary for squares of any size.

4. How many moves necessary for a 10 by 10 sqare?

5. Write the equation for this relationship.

6. Repeat this exercise, but the space must be left in the opposite corner.

Figure 4



7. Investigate rectangles.

Todd Romiens
Assistant Coordinator - Math/CAI
Windsor Board of Education
Windsor, Ontario N9A 6K1

MATHEMATICAL INTUITION AND INSIGHT FOR EVERYONE

by
Chester Carlow
87 Riseborough Avenue
Willowdale, Ontario M2M 2E2

When the great mathematician, Carl Friedrich Gauss, was a boy his school master asked the class to add all the numbers from 1 to 100. He expected everyone to go through the long process of adding them one by one in order. Gauss, however, figured out the answer very quickly by insight and simple mental arithmetic. He realized that the numbers could be paired to give a constant sum as follows:

$$\begin{aligned} 1 + 100 &= 101 \\ 2 + 99 &= 101 \\ &\vdots \\ 50 + 51 &= 101 \end{aligned}$$

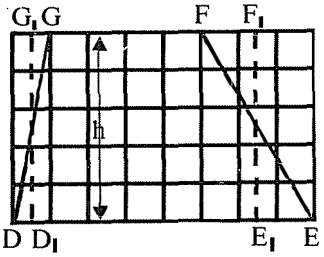
So the answer must be 50 X 101, or 5050.

Young Gauss has founded the sum of an arithmetic series by a general method which had not been taught and would not normally be taught until several years later. What is more, Gauss was undoubtedly being taught in a step-by-step, rule bound approach which makes intuitions and insights virtually impossible. This is not true of all teaching approaches.

To see how intuitions may be shut off or turned on let us consider two different ways of teaching a generalization usually taught around grade 8 -- the formula for finding the area of a trapezoid.

Teaching Approach #1 (Intense focus for short period)

- Sequence learning in earlier grades so that students already know how to find the area of a rectangle.
- Draw a trapezoid, DEFG, on a squared grid paper.



- Draw perpendicular lines through the centres of the two non-parallel sides of the trapezoid to the lower base and the upper base extended. (see figure)
- By counting part areas of the grid squares observe that the area of the trapezoid DEFG equals the area of the rectangle $D_1E_1F_1G_1$.
- By counting the sides and part sides of the grid squares notice that the length of side D_1E_1 of the rectangle is the average of the lengths of the lower and upper bases of the trapezoid.

$$\overline{D_1E_1} = \frac{\overline{DE} + \overline{FG}}{2}$$

- We know how to find the area of the rectangle

$$A\ (D_1E_1F_1G_1) = \overline{D_1E_1} \times h$$

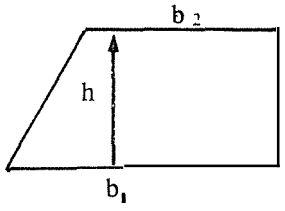
- So the area of the trapezoid must be

$$A\ (DEFG) = \frac{\overline{DE} + \overline{FG}}{2} \times h$$

- In general the area of a trapezoid is

$$A\ (Trap) = h \times \frac{b_1 + b_2}{2}, \text{ or}$$

$$= \frac{h}{2} \times (b_1 + b_2)$$

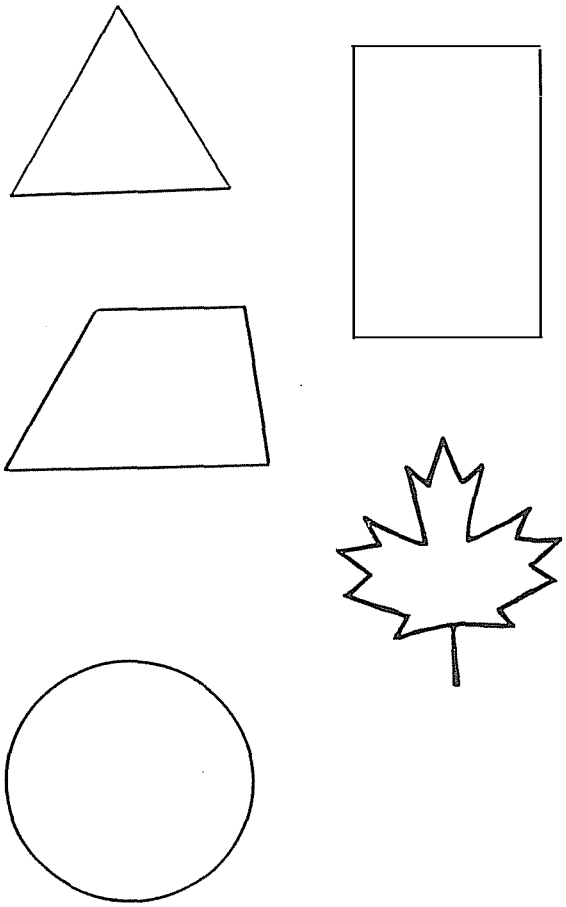


- Find the areas of many trapezoids until the formula is firmly fixed in the minds of the students.

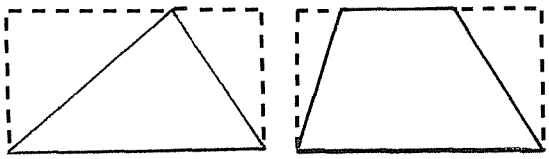
Teaching Approach #2 (Light focus extended over time)

- Have children begin in kindergarten to make simple holistic estimates of more or less, bigger or smaller, and about how many (without partial counting or analyzing the parts).

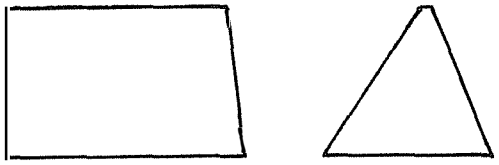
- By grades 3 or 4, through further estimates, have them develop as stable perceptual anchors the areas of 100 cm^2 and 1000 cm^2, no matter what the shapes of the figures or surfaces are which have those areas.



- With student focus on making good estimates, intersperse estimates which will help the students realize intuitively that the area of a triangle is about one half the area of a corresponding rectangle, and the area of a trapezoid is more than half, but less than the whole, area of a corresponding rectangle.



- With students unaware of purposeful teacher planning except to promote good estimators, intersperse other estimates over time to suggest that trapezoids may be "almost rectangles" or "almost triangles".



Once these intuitive, holistic insights have been experienced at the sub-verbal awareness level, students will be ready to clarify and complete their understandings by counting, calculating, working with formulas and other analytic procedures as follows:

- Establish the formula for the area of a trapezoid informally by exercises such as the following:

Find the areas of the following by any method you wish.

1. Trapezoid
   height = 10 cm
   lower base = 10 cm
   upper base = 4 cm

2. Triangle
   height = 10 cm
   lower base = 12 cm
   upper base = 0 cm

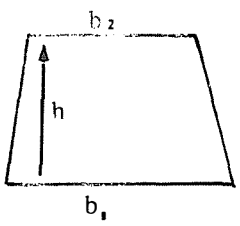
3. Trapezoid
   h = 8 cm
   b_1 = 12 cm
   b_2 = 8 cm

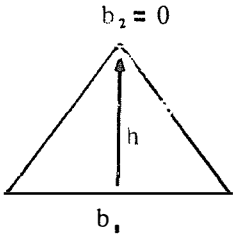
4. Trapezoid
   h = 10 cm
   b_1 = 16 cm
   b_2 = 12 cm

5. Rectangle
   h = 10 cm
   b_1 = 14 cm
   b_2 = 14 cm

6. Trapezoid
   h = 10 cm
   b_1 = b_1 cm
   b_2 = b_2 cm

- Through exercises such as 2 and 5 encourage the insight that the triangle and rectangle are limiting cases of the trapezoid. Therefore their area formulas must be related.

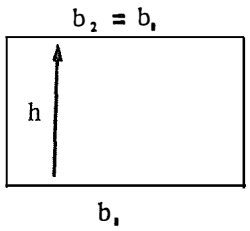




$$A = h \times \frac{b_1 + b_2}{2}$$

$$A = h \times \frac{b_1 + 0}{2}$$

$$= h \times \frac{b_1}{2} \text{ or } \frac{1}{2} b\,h$$



$$A = h \times \frac{b_1 + b_1}{2}$$

$$= h \times b_1 \text{ or } b\,h$$

- Have less practice using the formula. Spend the time saved promoting other intuitions and insights through estimating.

- Prove the formula formally and easily after students have learned to do formal proofs with congruent triangles.

The appropriate grade level for this kind of analysis depends, of course, upon the strength of the intuitive background of each particular group of students. The analysis is so non-threatening when it confirms ones intuitions, however, that it might easily be done prior to grade 8. In fact, one might expect the Ministry of Education to move many topics to lower grades if teaching approach #2 becomes widespread in Ontario.

The second approach may be used to learn about any mathematical fact or relation or generalization that can be clearly expressed with concrete or semi-concrete materials. Thus we can have pre-addition estimating that prepares children for learning the basic addition facts more meaningfully, as well as other kinds of estimating which precede and accompany other learning.

Comparison of the Two Approaches

In the first approach the formula is developed in a single lesson devoted exclusively to that one objective. It is the analytic, step-by-step, algorithmic approach which is widely accepted at the present time as the most effective and most accountable way of teaching mathematics. Unfortunately, however, understandings tend to develop in relative isolation by this approach. also the narrow focus acts as a set of blinders that reduce intuitions and insights to almost zero for most students.

The second approach is a combined linear and non-linear one. The non-linear phase is relational and holistic. It provides the time and opportunity for intuitions and insights to develop. The linear, analytic phase may take place at the informal or the formal level, as in the above example. It makes explicit with language and other symbols what is already partly "felt" at the intuitive level.

It is often assumed that experiences with concrete or semi-concrete materials automatically lead to meaningful learning. The process of confirming generalizations at the concrete level, however, may still leave some generalizations vague and isolated in the students' minds. What makes the second approach potentially more meaningful for all students is that it

1) provides a richerbackground leading to clearer and more stable ideas, and

2) brings out relationships which prevent the isolation.

These, rather than the concrete and semi-concrete experiences, are the essential characteristics of meaningful understanding.

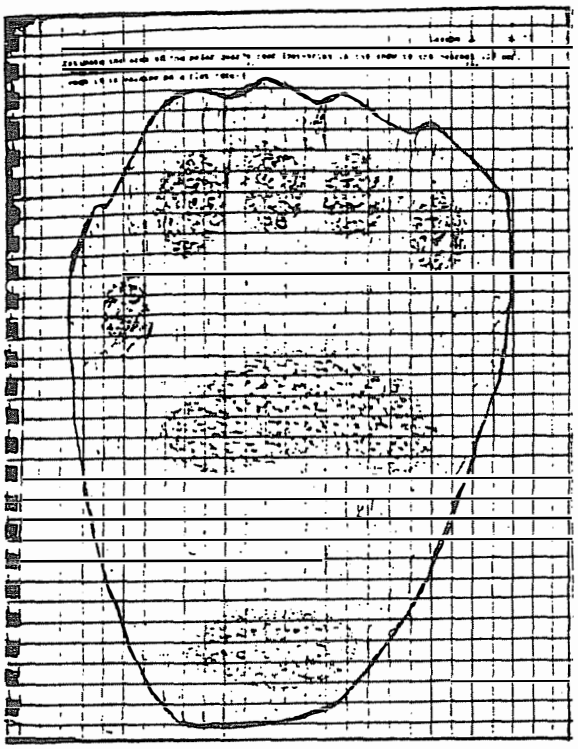
Current Obstacles to Holistic Estimating

There are two major obstacles to the widespread use of holistic estimating to promote intuitions and insights:

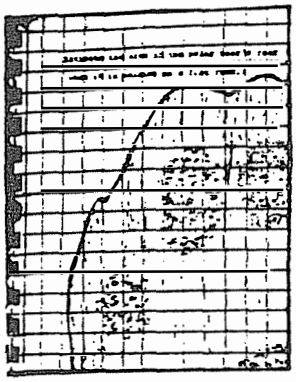
1) teachers are generally committed to the linear, analytic approach, and

2) textbook writers and other learning materials developers generally do not properly distinguish immediate and holistic estimating from mediated, analytic judgements of quantity.

For example, teachers who have spent their entire careers teaching math by the direct, step-by-step, linear approach tend to teach estimating in the same way. This is seen in the comments of a well known Ontario educator when asked to estimate the area of a life sized rear footprint of a polar bear placed upon a 1000 cm^2 page. (See reduced copy.) The educator, without the educated visual perspective that develops with practice in holistic estimating over time, estimated a high 750 cm^2. When told that the actual size of the footprint was only about 525 cm^2, he indicated how a series of related estimates might be sequenced to insure that children might be able to make closer estimates and avoid the occasional experience of failure.



First, he argued, the children might estimate the size of about $\frac{1}{4}$ of the footprint on a 250 cm^2 piece of paper. Next they could estimate the area of about $\frac{1}{2}$ the footprint on a 500 cm^2 piece of paper. Finally they could estimate the area of the whole footprint on the 1000 cm^2 piece of paper.



This step-by-step analysis of parts might indeed promote more accurate estimating. But the potential costs are high. It is but one of the many pitfalls that undermines the students' desire or ability to make accurate holistic estimates. (See Carlow, 1986). And without the grasp of the whole which come in part from such estimates, mathematical intuitions and insights will not occur.

The second major obstacle to promoting holistic estimating at the present time is the widespread tendency among mathematics curriculum developers to label as "estimating" all imprecise judgements of quantity, whether immediate and holistic or mediated and analytic. This makes it difficult to establish the fact that they should be taught in very different ways.

The best method for teaching students to make rough judgements of quantity which are mediated by such activities as rounding off, mental computations, measurement with non-standard units, or analysis of parts appears to be the familiar step-by-step approach enriched with some holistic experiences. The best method of teaching students to make immediate and holistic judgements, on the other hand, has not yet been developed. All that is known is that the linear approach will destroy the intuitive freshness that is desired, and that there are many pitfalls to teaching the topic effectively.

Perhaps a solution to the problem would be to label all mediated judgements of quantity "approximating" and reserve the label "estimating" for immediate and holistic judgements of quantity. This would highlight the difference between the two and promote the further development of instructional strategies to promote holistic judgements. Without the distinction we can look forward to endless confusion and ineffective instruction in this area in the years ahead.

Conclusion

Intuition and insights may be promoted in many teaching contexts other than holistic estimating. All that is needed is an occasional relaxation from the stifling effects of a strong one-dimensional focus on one isolated algorithm after another.

Nevertheless, genuine estimating based upon internalized perceptual anchors, appears to have unique potentials for promoting such intuitions and insights. For example, neither explorations with concrete materials, nor work with computers, nor problem solving is as fundamentally related to the real world, fundamentally relational, and fundamentally holistic as is direct (unmediated) estimating.

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A CARD GAME PARADOX

by
L. Keith Roy
Department of Mathematical Sciences
Lakehead University
Thunder Bay, Ontario P7B 5E1

Three sets of 3 cards are lying face down on a desk. Player A selects one of the sets and player B then selects one of the remaining sets. The players are aware of the values of all cards in each of the 3 sets and may use this knowledge in selecting their respective sets of cards. The players then randomly select one card each from their respective sets of cards. The player who selects the highest card is the winner. Suppose the sets are as follows.

black cards - 3,7,8

white cards - 2,6,10

green cards - 4,5,9

To determine which set of cards has the best chance of winning consider the following tables.

	3	7	8
2	x	x	x
6		x	x
10			

	2	6	10
4		x	x
5		x	x
9			x

	4	5	9
3	x	x	x
7			x
8			x

An x indicates which cards at the top of the table are larger than cards at the side of the table. By counting x's we arrive at a probability of 5/9 that the black cards will beat the white cards, a probability of 5/9 that the white cards will beat the green cards, and a probability of 5/9 that the green cards will beat the black cards.

The three 5/9 probabilities seem paradoxical since their total is greater than 1. However note that of the 27 spaces in the 3 tables, each set of cards is a winner in only 9 spaces. For instance 3,7,8 wins where the x's appear in the first table and where the blanks occur in the last table. It is the three 9/27 probabilities that should add to 1.

Although no set of cards is 'best', a player's best strategy is to allow the opponent the opportunity of selecting a set of cards *first*, after which the player selects the set of cards which has probability 5/9 of defeating the opponent's set. For example, if the first player picks the white set, the second player should pick the black cards. Three questions arise:

Question 1: Can 3 sets of more than 3 cards be found where the common winning probability is greater than 1/2?

3 sets of 4 cards:

black cards - 4,4,4,7

white cards - 1,3,6,8

green cards - 2,5,5,5

	4	4	4	7
1	x	x	x	x
3	x	x	x	x
6				x
8				

	2	5	5	5
4		x	x	x
4		x	x	x
4		x	x	x
7				

	1	3	6	8
2		x	x	x
5			x	x
5			x	x
5			x	x

In the above, the winning probability in each case is 9/16.

3 sets of 5 cards:

black cards - 3,3,5,7,7

white cards - 0,2,4,9,11

green cards - 1,1,6,8,10

	3	3	5	7	7
0	x	x	x	x	x
2	x	x	x	x	x
4			x	x	x
9					
11					

	1	1	6	8	10
3			x	x	x
3			x	x	x
5			x	x	x
7				x	x
7				x	x

	0	2	4	9	11
1		x	x	x	x
1		x	x	x	x
6				x	x
8				x	x
10					x

In the above, the winning probability in each case is 13/25.

3 sets of 6 cards:

black cards - 5,5,5,5,7,11

white cards - 0,2,4,9,12,12

green cards - 1,3,6,8,10,10

	5	5	5	5	7	11
0	x	x	x	x	x	x
2	x	x	x	x	x	x
4	x	x	x	x	x	x
9						x
12						
12						

	1	3	6	8	10	10
5			x	x	x	x
5			x	x	x	x
5			x	x	x	x
5			x	x	x	x
7				x	x	x
11						

	0	2	4	9	12	12
1		x	x	x	x	x
3			x	x	x	x
6				x	x	x
8				x	x	x
10					x	x
10					x	x

In the above, the winning probability in each case is 19/36.

It seems to be a reasonable conjecture that the answer to question 1 is 'yes'.

QUESTION 2: How large can the common winning probability be?

3 sets of 5 cards:

black cards - 4,6,6,6,8

white cards - 1,3,5,10,12

green cards - 0,2,7,9,11

	4	6	6	6	8
1	x	x	x	x	x
3	x	x	x	x	x
5		x	x	x	x
10					
12					

	1	3	5	10	12
0	x	x	x	x	x
2		x	x	x	x
7				x	x
9				x	x
11					x

	0	2	7	9	11
4			x	x	x
6			x	x	x
6			x	x	x
6			x	x	x
8				x	x

In the above, the winning probability in each case is 14/25. In fact, a probability of 15/25 is possible. Consider the following.

black cards - 0,0,3,3,5

white cards - 2,2,2,2,2

green cards - 1,1,1,4,6

	0	0	3	3	5
2			x	x	x
2			x	x	x
2			x	x	x
2			x	x	x
2			x	x	x

	2	2	2	2	2
1	x	x	x	x	x
1	x	x	x	x	x
1	x	x	x	x	x
4					
6					

	1	1	1	4	6
0	x	x	x	x	x
0	x	x	x	x	x
3				x	x
3				x	x
5					x

3 sets of 6 cards:

black cards - 5,5,5,5,5,10

white cards - 0,2,4,7,9,11

green cards - 1,3,6,6,6,8

	5	5	5	5	5	10
0	x	x	x	x	x	x
2	x	x	x	x	x	x
4	x	x	x	x	x	x
7						x
9						x
11						

	0	2	4	7	9	11
1		x	x	x	x	x
3			x	x	x	x
6				x	x	x
6				x	x	x
6				x	x	x
8					x	x

	1	3	6	6	6	8
5			x	x	x	x
5			x	x	x	x
5			x	x	x	x
5			x	x	x	x
5			x	x	x	x
10						

In the above, the winning probability in each case is 20/36. In fact, a probability of 21/36 is possible. Consider the following.

black cards - 3,5,5,5,5,8

white cards - 2,2,2,7,7,7

green cards - 1,4,6,6,6,6

	3	5	5	5	5	8
2	x	x	x	x	x	x
2	x	x	x	x	x	x
2	x	x	x	x	x	x
7						x
7						x
7						x

	1	4	6	6	6	6
3		x	x	x	x	x
5			x	x	x	x
5			x	x	x	x
5			x	x	x	x
5			x	x	x	x
8						

	2	2	2	7	7	7
1	x	x	x	x	x	x
4				x	x	x
6				x	x	x
6				x	x	x
6				x	x	x
6				x	x	x

It is interesting to note that in all the above sets of tables, all diagonal positions are free of an x at least once in the three tables of a set. This condition must be satisfied in order to find sets of cards with common 'paradoxical' probability greater than 1/2. However this condition does not guarantee a solution. Consider the following set of 5X5 tables with 16 x's.

	?	?	?	?	?
?	-	x	x	x	x
?		-	x	x	x
?			x	x	x
?			x	x	x
?			x	x	x

	?	?	?	?	?
?	x	x	x	x	x
?	x	x	x	x	x
?	x	x	x	x	x
?				-	x
?					-

	?	?	?	?	?
?	x	x	x	x	x
?	x	x	x	x	x
?			-	x	x
?				x	x
?				x	x

In the above, all diagonal positions are free of an x at least once. (These positions are marked by a -.) However it is not possible to find 3 sets of numbers which will work for this set of tables.

So the answer to question 2 seems to be that the size of the probability is governed by the conditions just described.

Question 3: Are there 'paradoxical' probabilities associated with 4 sets of cards.

Consider the following example (similar to the famous Effron dice).

black cards - 7,7,7,7,7,7

white cards - 5,5,5,5,10,10

green cards - 2,2,8,8,8,8

blue cards - 3,3,3,9,9,9

	7	7	7	7	7	7
5	x	x	x	x	x	x
5	x	x	x	x	x	x
5	x	x	x	x	x	x
5	x	x	x	x	x	x
10						
10						

	2	2	8	8	8	8
7			x	x	x	x
7			x	x	x	x
7			x	x	x	x
7			x	x	x	x
7			x	x	x	x
7			x	x	x	x

	3	3	3	9	9	9
2	x	x	x	x	x	x
2	x	x	x	x	x	x
8				x	x	x
8				x	x	x
8				x	x	x
8				x	x	x

	5	5	5	5	10	10
3	x	x	x	x	x	x
3	x	x	x	x	x	x
3	x	x	x	x	x	x
9					x	x
9					x	x
9					x	x

In the above, the winning probability in each case is 24/36.

The reader is invited to experiment with 4 or more sets of 7 or more cards.

GOING AFTER THE BIG CHEESE

by
Jeff Martin
Program Advisor of Mathematics
Etobicoke Board of Education
1 Civic Centre Court M9C 2B3

This article originated as the result of numerous discussions at the University of Waterloo during the marking of the annual Descartes Competition. Concerned teachers would ask, "How do you run your Senior Math Club? Do you have weekly meetings? Do you prepare by doing old Descartes' exams?"

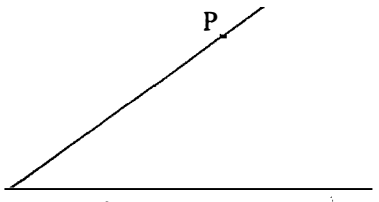
Questions like these and others prompted me to give a talk at the annual OAME conference in Scarborough and hence this article.

In September, I make an announcement that the Senior Math Club will hold its first meeting Friday morning at 7:30 a.m. A light breakfast of juice and donuts is served. If, after the first meeting, I feel the attendance is low, I make personal visits to senior classes and speak to those students who I feel would benefit from attending the Senior Math Club.

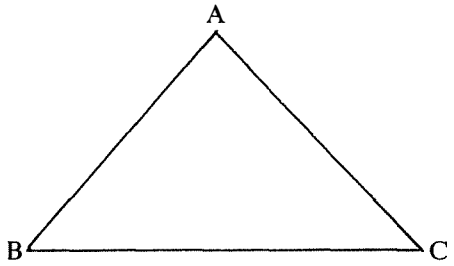
At the first meeting, I'll have a couple of problems on the board (usually geometric since most students right-brained by nature and this forces the student to draw a diagram.)

Here are two such problems:

1. Construct, using straightedge and compasses, a circle that is tangent to both lines and that has the point P as its point of tangency to one of the lines.



2. Construct a line parallel to BC to divide Δ ABC into 2 equal areas. You may use straightedge and compasses only.



I tell the students that while they're 'doing' the problems, they are to consider the thought processes they are using. This is what we call metacognition, 'thinking about your own thinking'. These ideas become the working strategy.

i.e.
- work backwards
- exploit symmetry
- divide into subcases
- modify the problem

At the first meeting, we also discuss the purpose for being there. We want to prepare for the upcoming math contests. We want to learn how to do problems as opposed to exercises. We want to explore the use of different strategies in order to solve these problems. We want to be able to think mathematically. We want to get enjoyment out of solving problems. We want to have FUN.

First of all, I differentiate between exercises and problems. Exercises are what they do each day - these are important because they build up a tool chest of facts. Problems are questions that involve a substantial amount of thinking. They are ones in which the solution is not at first evident. All problems are composed of 3 types of information.

1. Information concerning given data.

2. Information concerning operations that transform one or more expressions into one or more new expressions.

3. Information concerning a goal.

The problems also may be concerned with more than one topic.

i.e. i) sequences and the cosine law
ii) analytic geometry and calculus

The second thing we talk about is the upcoming contests. The early contests such as the AHSME are multiple choice whereas the Descartes is an essay response contest. That is, the solution is not readily evident. These questions require that the students employ one of the aforementioned strategies. Students need lots of practice at solving all types of problems.

It sounds trite, but you become a good problem solver by becoming actively engaged in solving problems. Have a game plan. Read the paper entirely first and use a system of symbols: ✓ (for can do), O (have some idea), X (no idea right now). Plan to spend 15 minutes per question.

How To Get A Club On Its Feet

1. You find someone in the Mathematics Department who loves math, knows the content of all the math courses, likes kids, likes to solve problems, doesn't mind admitting they don't have all the solutions and doesn't mind spending extra time. WOW! does such a person really exist?
2. Begin the club early in September. March is too late.
3. Meet at least once a week for a time period > 1.5 h.
   (We meet Friday morning at 7:30-9:00)
4. Insist the students do preparatory work as well as extra problems. (This is most difficult.)
5. Solicit new members besides the ones who come voluntarily.
6. Elect officers and collect dues, $1.00 per month (buys donuts, juice, periodicals, etc.)
7. Have a few dinner meetings with pizza and a speaker. (The team from Waterloo is available upon request.)
8. Enter 2 or 3 contests - mention others that certain individuals might enter.
9. Insist they keep a separate note book.
10. **HAVE FUN.**
11. Don't pretend to know all the answers. Converse; Don't always give problems to which you know the answer. You can always find somebody with a solution - somewhere. Don't spend months doing old Descartes exams - leave this until March. Spend time on interesting topics not covered in secondary schools (number theory, modular arithmetic, theory of equations, partial fractions, pigeon-hole principle).
12. You might create or pass out a tool chest of basic knowledge. (E. Barbeau's small 1001 problems book has a tool chest in Volume I).
13. Have students set up a Junior Math Olympiad for the grade 8 feeder schools after the Descartes is written.
14. Have interesting problems. (See Hall and Knight, 1960)
15. Don't give all the solutions - rather give hints - discuss heuristics.
16. Let students work in pairs and encourage them to talk out loud.
17. Show them there is more than one solution to problems.
18. Lastly - You learn problem-solving by solving problems! Profound eh? 'Use your head to learn the subject-use the subject to learn to use your head.' (Gudder, 1976).

'The essence of mathematics is not to make simple things complicated, but to make complicated things simple.' Stanley Gudder (1976)

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Gudder, Stanley. A Mathematical Journey. New York: McGraw-Hill, 1976.

Hall, H. S. & Knight, B.A. Higher Algebra. London: Macmillan, 1960.

Schoenfeld, A. H. Mathematical Problem Solving. Toronto: Academic Press, 1985.



1988 POSTER CONTEST WINNERS

Winners of the 1988 OAME poster contest are:

1. $100 prize. Anne Yu, Loretto Abbey, Toronto. The condition for fitting one triangle inside another.

2. $75 prize. Cynthia Ye, St. Joseph's College School, Toronto. Fermat's Last Theorem.

3. $50 prize. Fiona D'Costa, Loretto Abbey, Toronto. Solution of the equation $(x + y + z)^3 = xyz$, $x,y,z \in N$.

4. $25 prize. Dianna Gasborro/Maria Rotundo, Loretto Abbey, Toronto. Covering every square on a chess board with 8 chess pieces.

THINKING THROUGH PROBLEM SOLVING II

by
Barry Onslow and Ron Reesor
Faculty of Education
The University of Western Ontario
London, Ontario N6G 1G7

In the last issue we looked at the rationale for *why* we are writing this column, *what* we hope it will achieve, and *how* the procedures might be implemented. In this issue we will expand on the latter point and suggest how a class may be structured to facilitate the process. The classroom atmosphere should encourage cooperation rather than competition, and become more student centred, not teacher dominated.

Alternative Strategies for Solving the Problem

1. Understanding the problem.

Pairs

Pair off the students and designate one as a *talker* and the other as a *listener*. The talkers describe the problem in their own words, the listeners ask the talkers to clarify unclear statements. This continues until agreement is reached. Finally the teacher randomly selects two students to explain the problem, thus ensuring that everyone is attempting to solve the same problem.

Groups

Divide the class into groups with four students in each group. preferably of mixed ability. Each group should appoint a spokesperson. Time should be given for the groups to discuss the problem and to decide on what is being asked. Finally the teacher randomly picks two spokespersons to explain the problem, thus ensuring that all groups are attempting to solve the same problem.

2. Carrying out the plan.

Continue with the talkers as the problem solvers and process recorders. The talkers describe their thought processes as they attempt to solve the problem.

The partners listen and only interject if there is disagreement or to ask for clarification. These persons should concentrate on what the talkers are saying rather than trying to solve the problem. Thus the listeners become attentive and aware of the problem solving process. This arrangement continues until either the problem is solved or hints are required from the teacher.

Allow time for the groups to discuss the problem, decide on a strategy, and carry out this strategy to solve the problem. The teacher circulates among the groups and only answers questions of clarification. The teacher's presence will help to ensure that all members of the groups are contributing to the solution of the problem. The teacher may have to give hints where an impasse is reached. These hints, however, should not be so obvious as to provide the solution, or take the ownership of the process from the students.

3. Looking Back

Differing strategies will be presented. The teacher acts as a chairperson, managing the discussion, demanding justifications, and seeking generalizations. Finally the teacher will summarize the results.

Talking about ideas often helps students become more precise about what they mean, and thinking aloud can help to clarify difficult concepts. Both students become more aware of the problem solving process.

Spokespersons will report their groups' solutions to the class and when there is disagreement alternative strategies will be examined. The teacher acts as a chairperson, managing the discussion, demanding justifications and seeking generalizations. Although the spokesperson is the main reporter, other students are encouraged to make contributions. Finally the teacher will summarize the results.

NOTE: for the next problem the roles in each pair should be reversed.

DISCOVERING RELATIONSHIPS

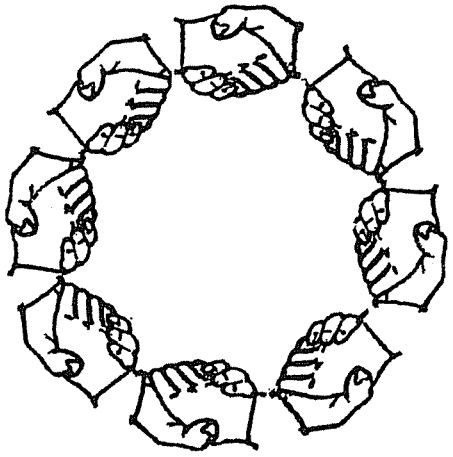
These 3 problems could be tackled sequentially by a class to reveal the similarities. A more profitable procedure, however, might be to divide the class into groups or pairs (see previous page for in depth discussion) with 1/3 of the class answering each question. During the discussion period students would demonstrate solution strategies and the similarities among them will hopefully be brought into focus. These similarities should also be highlighted by the teacher.

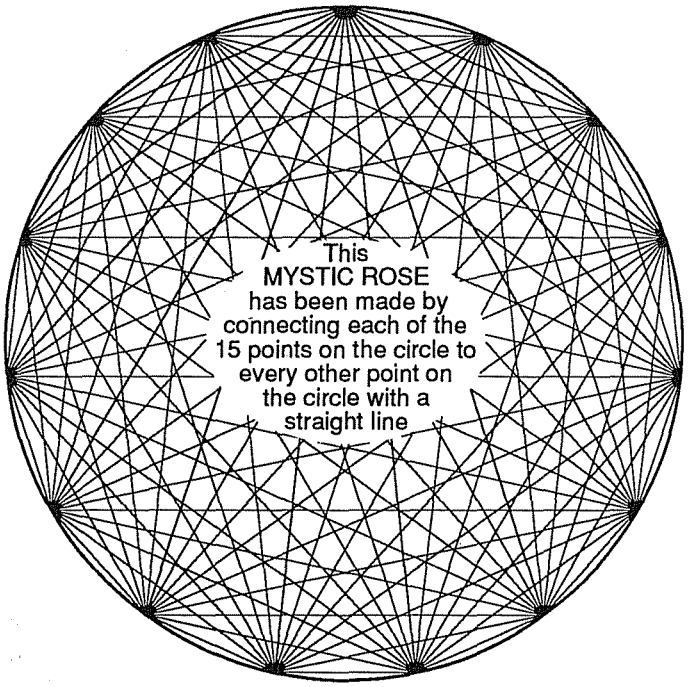
There are 7 people at a party. Everyone shakes hands ONCE with everyone else.
How many handshakes take place?

Solving a specific example.

There are 30 people at a convention. Everyone shakes hands ONCE with everyone else.
How many handshakes take place?

Working towards a generalization





How many straight lines are there when 7 points are connected?

Solving a specific example.

How many straight lines will there be when 30 points are connected?

Working towards a generalization

A round-robin, singles badminton tournament is to be organized. In this type of tournament every person plays against every other person.

How many games will there be if 7 players enter the tournament?

Solving a specific example.

How many games will there be if 30 players enter the tournament?

Working towards a generalization.

1. Understanding the problem.

- Have students read the problem and restate it in their own words.
- Students need to know:
  If A shakes B's hand then B has shaken A's hand
  If A is joined to B then B is joined to A
  If A plays B, then B has played A

2. Developing and Carrying out a plan

Hints: Act it out
Do a simpler problem
Make an organized list
Make a table, chart or diagram
Is there a pattern?

3. Looking Back

Some possible strategies are:

a) Using logical reasoning.

Each of the 7 persons must shake hands with each of the 6 other persons, so 7 x 6. However, this would mean everyone would shake hands twice

$$\therefore \quad \frac{7 \times 6}{2} \text{ or } \frac{n(n-1)}{2}$$

b) Making an organized list and looking for a pattern:

No. of people No. of points No. of players	No. of handshakes No. of lines No. of games
1	0
2	$1 = 1$
3	$3 = 1+2$
4	$6 = 1+2+3$
5	$10 = 1+2+3+4$
6	$15 = 1+2+3+4+5$
7	$21 = 1+2+3+4+5+6$
•	$= (1+6) + (2+5) + (3+4)$
•	$= 7 + 7 + 7$
•	$= 7 \times 3$ or $7 \times \frac{6}{2}$ (one less than the number of handshakes)
n	$= 1+2+3+\ldots+(n-3)+(n-2)+(n-1)$ $= [1+(n-1)] + [2+(n-2)] + [3+(n-3)] + \ldots$ $= n + n + n + \ldots$ $= \frac{n(n-1)}{2}$

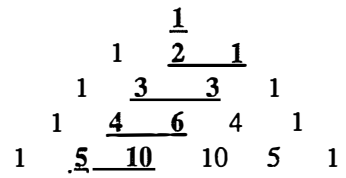
Extension

a) Students could be told of KARL FREDERICK GAUSS who used a similar procedure to answer a question far faster than his teacher thought possible. His teacher had asked the class to add up the first 100 natural numbers.

b) Students studying combinations in discrete mathematics could represent the problem as "how many two-handshake (line, game) combinations can be formed from N people (points, players)?"

$$_nC_2 = \frac{n(n-1)}{2}$$

c) Students studying Pascal's Triangle might recognize that the solution to the three problems can be found here:





IS 70 MINUTES TOO LONG?

by
John C. Egsgard
Patrick Fogarty Secondary School
Orillia, Ontario



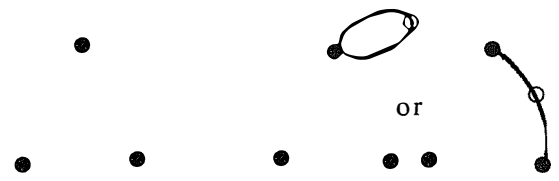
There has been much discussion about the pros and cons of the semestering system. I believe that the mathematics achievement of our students is curtailed by this system; but, that is not the topic for this article. A by-product of the semestering system is that frequently students have finished working before the period ends. Note that I did not say that they have finished their work but that they have ceased doing anything profitable. In such cases I like to have some ideas to present that are mathematical in nature, are interesting, are exciting, and "will *not* appear on the exam."

Some of the things that I present are given in this article. You can find many other similar items in many of our Ontario textbooks and in the various mathematics journal such as the Gazette and the Mathematics Teacher. A number of useful books are listed at the end of the article.

1) The Game of Sprouts

This is a game for two players. Place 3 dots anywhere on a page. Each player draws an arc joining a dot to itself or to another dot. When a player joins two dots that player places another dot on the arc drawn. This provides one extra dot to use in the game. No arcs may cross and no dot may have more than three arcs leaving it. The last player to make a valid move wins.

Figure 1



Start After an initial move

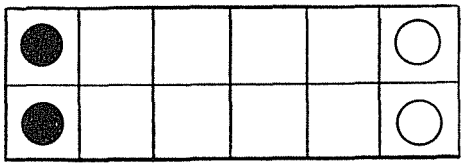
(Reference: Math 12, 3rd Edition, Gage, Page 11)

Editor's Note: Readers wishing a more detailed discussion could also refer to Student Math Notes (NCTM), September 1984.

2) The Game of Counters

Black and white each have two counters. A counter may move any number of squares horizontally, to the right or to the left, but it may not pass over an opposing counter. A player loses if the player is unable to move, that is, if the player is jammed against the end by both of an opponent's men. Black moves first.

Figure 2



(Reference: Math 12, 3rd Edition, Gage, Page 10)

3) The Game of Nim

The game is played by two players. Each starts with a pile of toothpicks, say 12 in each pile. In turn each person takes away some toothpicks, as many as the person wishes provided *all toothpicks come from the same pile.* The loser is the one forced to take the last toothpick.

(Reference: Math 12, 3rd Edition, Gage, Page 6)

4) ababab is divisible by 3,7,13 and 37

Tell the students to write out any six digit number consisting of a two-digit number repeated three times such as 252525. Using a calculator or otherwise have each member of the class divide his or her six digit number by 2,3,4, ... in turn. List the number being tried on the blackboard. Cross the number out if any of the six digit numbers being checked is not divisible by that number. When you come to the numbers 4,6,8 and 9, etc. ask the students why that number does not have to be tried. After obtaining 13 tell the students that one other divisor exists, then give them time to find it.

Challenge them to determine why these numbers work. Since this is supposed to be

fun, I only give them a few minutes to come up with an answer. Someone usually does.

If you multiply 3,7,13, and 37 together you obtain 10101 = k. 25k gives 252525, 82k gives 828282, and ab k gives abababab. QED!

5) Four Number Tricks

A

1. Think of a number	x
2. Add 17	x + 17
3. Double	2x + 34
4. Subtract 4	2x + 30
5. Double again	4x + 60
6. Add 20	4x + 80
7. Divide by 4	x + 20

Mind reader subtracts 20 from final answer to get original number.

B

1. Think of a number	x
2. Add 18	x + 18
3. Multiply by 4	4x + 72
4. Subtract 7	4x + 65
5. Double	8x + 130
6. Subtract 130	8x
7. Divide by 8	x

Mind reader uses the number given.

C

Age and Money Trick

1. Multiply age by 20
2. Add 4
3. Multiply by 5
4. Add change < 100 cents
5. Subtract 20

If result is 1526, age = 15, change = 26 cents.

D

Birthday Trick

1. Write your birth month, 1 to 12
2. Multiply by 5
3. Add 6
4. Multiply by 4
5. Add 9
6. Multiply by 5
7. Add day of month of birthday

Mindreader subtracts 165. If result is 320, birthday is March 20.

6) Max Sobel's Fire and Water Trick

1. Write a 2 digit number between 50 and 99.
2. Add 82.
3. Cross out hundred's digit (it will always be 1) and add this digit to the remaining two digit number.
4. Subtract this result from the original number.
5. Write result on paper and burn paper. Crush ashes on the back of your hand. Dunk hand in water. The result (which is always 17) appears. (NOTE: You must use a piece of soap to mark 17 on the back of your hand before class.)

7) Calendar Tricks

9 date square

1. Enclose a square of 9 dates for any month
2. Add the 9 numbers

Mindreader is given total of 9 dates, mentally divides by 9 and announces the date in the middle of square.

x - 8	x - 7	x - 6
x - 1	x	x + 1
x + 6	x + 7	x + 8

sum = 9x

4 date square

1. Enclose a square of 4 dates
2. Add the 4 numbers

Mindreader is given total of 4 dates, mentally divides by 4 and subtracts 4. Result is first of the 4 dates.

d	d + 1
d + 7	d + 8

sum = 4d + 16
4(d + 4)

16 date square

1. A student encloses a square of 16 dates. Unnoticed by students you observe the dates in the upper left and lower right corner. Add these dates, double, then write your result on paper and place it in an envelope which you seal.

2. With your back to the students give these instructions.

a) Circle any number and delete the

remaining 3 numbers in the same row as the circled number, and delete the remaining 3 numbers in the same column as the circled number.

b) Circle any remaining number and delete the numbers left in the same row and column.

c) Circle any remaining number and delete remaining numbers in same row and same column.

d) A single number remains, circle add circled numbers and open sealed envelope.

Secret: The method of selection forces the student to choose exactly one number from each row and each column. Thus the circled numbers must total $(a + b + c + d) + (1 + 2 + 3) = a + b + c + d + 6$. But $b + c = a + d$, so total $= 2a + 2d + 6 = 2[a + (d + 3)] = 2$[sum of diagonally opposite numbers in the square.]

a	a + 1	a + 2	a + 3
b	b + 1	b + 2	b + 3
c	c + 1	c + 2	c + 3
d	d + 1	d + 2	d + 3

8) Max Sobel's game of 50 - beat the world champion

Max Sobel has many tricks in his book Teaching Mathematics.

The teacher claims to be world champion in the game of 50, and never defeated. The students or teacher starts by giving any number from 1 to 6. Then student and teacher alternate adding any number from 1 to 6 to the previous sum. The first one to reach 50 wins the game.

To win the teacher always adds to end with a sum 1 more than a multiple of 7. The teacher will always win if the teacher selects 1 as the first number and adds the 7-complement of the number the student gives. If a student demands the privilege of going first and begins with one, the teacher refuses to continue thus preserving the undefeated title.

9) Hofstader's Circles

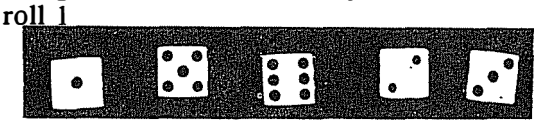
Draw three circles two of which have the same radius. Colour one of the two equal radii circles red and the other yellow. Colour the remaining circle red. Ask the question: "Which circle is different."

▶ For the answer see the end of this article.

10) Petals Around the Rose

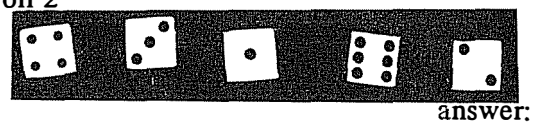
The game is quite simple. A potentate rolls five dice. The potentate knows the secret that gives the answer for any roll of five dice.

roll 1



answer: 6

roll 2



answer: 2

The potentate rolls the 5 dice and informs the observers that the potentate can tell them three things.

1. "The name of the game is Petals Around the Rose, and the name is important."

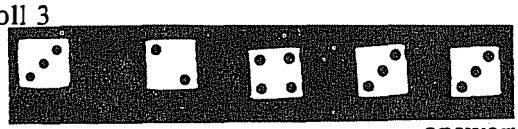
2. "The answer for any roll is an even number."

3. "The answer for this roll is --". (The blank is filled by the number represented by the roll.)

No potentate may ever reveal the method used to obtain the answer in number 3. This is a very important regulation of the society. Indeed, every potential potentate must promise never to reveal this secret before she or he may become a member of the society.

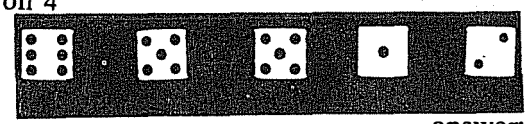
Here are some more examples. Can you figure why the answer is as given?

roll 3



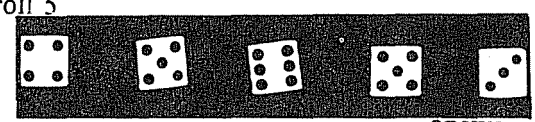
answer: 6

roll 4



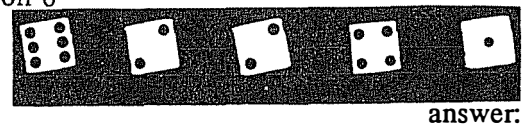
answer: 8

roll 5



answer: 10

roll 6



answer: 0

Do you see a pattern that will allow you to determine the answer? Remember if you do, do not tell anyone else?

The amazing thing about this game is that some people see how to get the answer in only 3 or 4 rolls, while others struggle for days before the truth finally dawns upon them. Even mathematics teachers and high school principals have problems seeing the pattern involved.

Be aware that the game is very addictive. Students (and teachers) may discover that they will have difficulty finding time to eat or do their homework while they are working on their solution to the game.

Once a person can give the answer to six consecutive rolls of the 5 dice that person becomes a potentate and a member of the society. Any potentate may enrol others into the society. You can make up certificates of membership for a school society to be given to those who have become potentates.

(Reference: Finite Mathematics: A Search for Meaning, Gage, page 86)

11)Binary Number Trick

Students think of a number from 1-64 then tell you on which cards (see Fig. 1) they see their number. You add the numbers in the upper left corner of the cards containing the secret number and announce the number.

1	3	5	7	9	11	13	15
17	19	21	23	25	27	29	31
33	35	37	39	41	43	45	47
49	51	53	55	57	59	61	63

2	3	6	7	10	11	14	15
18	19	22	23	26	27	30	31
34	35	38	39	42	43	46	47
50	51	54	55	58	59	62	63

4	5	6	7	12	13	14	15
20	21	22	23	28	29	30	31
36	37	38	39	44	45	46	47
52	53	54	55	60	61	62	63

8	9	10	11	12	13	14	15
24	25	26	27	28	29	30	31
40	41	42	43	44	45	46	47
56	57	58	59	60	61	62	63

16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63

32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63



Editor's Note: For a full explanation of how this trick works see An Introduction to the History of Mathematics (3rd ed.) by Howard Eves, pg. 20.

The answer to number 9 is that the circle that has something in common with each of the other two circles is the different circle, that is, the red circle equal in radius with another circle.

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TEACHER ASSESSMENT PRACTICES IN HIGH SCHOOL CALCULUS: DIFFERENCES IN CLASS TIME, ASSIGNMENT AND TEST CONTENT, AND GRADING PRACTICES

by

Rose E. Traub, Philip Nagy, Katherine MacRury and Roslyn Klaiman
The Ontario Institute for Studies in Education

This is the first half of a short version of a long report under the title "Teacher Assessment Practices in a Senior High School Mathematics Course." The second half, discussing differences in grading of the same set of exam papers, will appear in the next issue. A copy of the long report is available from the authors on request. This research was conducted with the support of the Ontario Ministry of Education, through Transfer Grant 52-1028 to the Ontario Institute for Studies in Education.

Introduction

The purpose of the research was to explore the meaning of grades assigned by different teachers of the same Grade 13 mathematics course, and to formulate possible explanations of any differences in meaning found to exist among the teachers. In Ontario, the high school curriculum is constrained by provincial guidelines, which specify minimum course content and length. Local school boards are responsible for implementing the guidelines in their schools, with differences consequently possible in topic emphasis, grading methods, and quality of achievement expected for a given mark.

The project began in the spring of 1986, with selection of a mathematics course. In making this choice, due regard was given to an ongoing transition from Grade 13 courses to Ontario Academic Courses (OACs). A comparison of Ministry Guidelines for Grade 13 and OAC mathematics courses (Ontario Ministry of Education, 1972, 1985) revealed that the overlap was substantial for Grade 13 and OAC Calculus. Choice of calculus also meant that the results of the survey of calculus examinations by Alexander (1987) would complement and inform the results of this study.

The choice of calculus was made in consultation with a five-member Advisory Committee for the project, consisting of:

- Dr. David Alexander, Faculty of Education, University of Toronto and The Ontario Ministry of Education.
- Dr. Edward Barbeau, Department of Mathematics, University of Toronto.
- Dr. Gila Hanna, Department of Measurement, Evaluation and Computer Applications, The Ontario Institute for Studies in Education.
- Mr. George McNabb, Mathematics Teacher, Sudbury Board of Education, representing The Ontario Association of Mathematics Educators.
- Mr. John Scott, Mathematics Consultant, Toronto Board of Education.

The design of the study called for the recruitment of 20 teachers. Practical considerations limited choice to teachers working in the urban core of Southern Ontario. In the end, the participants were 17 teachers from 17 different schools in 13 different boards. (The teachers were guaranteed anonymity, so their names must remain confidential.) All participants held an undergraduate degree in mathematics, had at least five years experience teaching senior mathematics, and had taught the calculus course at least three times. Some descriptive information on the classes of the 17 teachers is provided in Table 1.

Several types of data were collected. The 17 teachers were asked to:

- complete a log throughout the Spring 1987 semester for one Grade 13 Calculus class, recording the activities undertaken each class period and the time devoted to each activity;
- list on the log the homework and seat-work assigned each day;
- report the criteria used to arrive at student grades for the course, including tests, quizzes, examinations, and other factors (e.g., participation, attendance), along with the relative weights of each; and
- mark a common set of 20 final examination papers obtained from a class not involved in the study.

In the material that follows, the use of classroom time is considered first. Successively thereafter, attention is turned to the content of the teaching and testing, and to testing policy and practice. The exam-marking part of the study is reported next issue.

Table 1

Some Characteristics of the Schools Classes

Teacher Number	School Size[a]	Class Size	Teaching Hours[b]	Number of Tests[c]
1	1400	17	108	10
2	950	30	103	8
3	300	13	107	7
4	1200	15	97	6
5	1100	18	96	6
6	1200	28	105	7
7	900	22	106	6
8	1900	19	105	10
9	1900	16	103	7
10	1000	25	106	10
11	2050	24	105	13
12	1800	31	103	10
13	1500	27	110	9
14	1250	21	105	9
15	1300	13	104	8
16	1400	26	103	3
17	950	27	114	9

[a] Rounded to the nearest 50.

[b] Including time for examinations and tests.

[c] Not including mid-course and final examinations, if either was administered.

Use of Class Time

In their daily logs, most teachers provided descriptions that indicated the kinds, order, duration and focus of the teaching/learning activities undertaken during each class period. Four teachers, however, provided more information about topics covered in a class than about activities undertaken, and a fifth delegated the task of completing the log to different students, thus providing an uneven record. The logs of these five teachers were excluded from further consideration in this part of the study, leaving the data for 12 teachers.

There was considerable variation among the 12 teachers in number of class periods and length of courses. The number of class periods, ranged from 80 to 109 (Mean 86). Total class time for the calculus credit ranged from 96 to 114 hours (median 105). The scheduled length of class periods varied from 40 to 80 minutes, although on a given day the time actually spent in class might have been less than what was scheduled for any number of reasons.

Six categories of class activities were defined from terms used in the logs:

- Administration: taking attendance, making announcements.
- Direct teaching: presentations, demonstrations and discussions focusing on new material.
- Student practice: seat-work and board-work pertaining to new material (including handouts, assignments and orally presented problems considered in class), with opportunity for individualized instruction.
- Homework: tasks assigned for independent completion, either in class time or outside.
- Review: class time used (i) to cover previously learned material, including prerequisite knowledge acquired in other courses (e.g., algebra), and content previously covered in the course, (ii) to prepare for tests and exams, and (iii) to mark or review tests, quizzes and exams.
- Assessment: quizzes, class tests and exams administered in class time.

Analysis of the time spent on each type of activity as a percentage of total time produced the following results: (a) Administration - 0 to 5 percent (median 1%); (b) Direct Teaching - 17 to 52 percent (median 26%); (c) Student Practice - 8 to 47 percent (median 29%); (d) Homework - 11 to 43 percent (median 18%); (e) Review - 4 to 14 percent (median 11%); and (f) Assessment - 8 to 16 (median 10%). Clearly, the teachers differed substantially in their use of class time.

In a search for patterns in these data, coefficients of correlation (over teachers) were computed. A negative correlation was found between total logged time (in hours) and the percentage of time spent on direct instruction. This suggests that teachers with greater amounts of class time tend to do less direct teaching. This interpretation is corroborated by the findings that percentage of time for direct instruction correlated negatively with percentage of time for homework and student practice, whereas the percentages of time for homework and student practice were each positively correlated with total time. The largest percentage of time, overall, was devoted to student practice. Negative correlations between the percentage of time for review and for assessment suggest that teachers who place relatively high emphasis on practice in their teaching of calculus place a relatively low emphasis on review and assessment activities.

Content of Assignments and Tests

The information about content came from the daily logs of homework and seat-work assignments that were maintained by the teachers, and from the quizzes, term tests and exams (plus marking schemes) submitted by the teachers. For this and the remaining parts of the report, information has been included from all 17 teachers.

A comment is in order at the outset of this section lest our results be taken as implicitly critical of the teachers who participated in the study. The Guideline for the Ontario Grade 13 Calculus Course (Ontario Ministry of Education, 1972) mandates broad content areas, but not relative importance. Thus, we were not investigating whether some teachers exercise better or worse judgment as to what should be in the curriculum. Instead, we were investigating differences in the judgments made by qualified and experienced teachers.

A scheme was devised for categorizing the content of the course. The starting point was the 1972 Grade 13 Calculus Guideline (Ontario Ministry of Education, 1972) and the contents of two Ministry approved texts for the course. When a satisfactory version of the category system had been produced, two students, both about to graduate from a B.Sc/B.Ed. program in mathematics education and both experienced in practice teaching the calculus course, reviewed and revised the system, and then applied it to questions on teacher-produced handouts, quizzes, tests and exams.

The category system includes 126 topics. In applying this scheme, the two students achieved an inter-rater agreement of 88%. To simplify reporting, the 126-topic scheme was collapsed into 14 categories; at this level, inter-rater agreement was 97%. The 14 Content Categories were themselves classified into six Content Groups: I - the basic skills of calculus (limits, sequences and series, differentiation, and integration); II - proofs of basic theorems; III - applications of differentiation skills, also referred to as differentiation graphing (slope and equation of a tangent, curve sketching); IV - applications of integration skills, also referred to as integration graphing (area between curves, volume of revolution); V - situational problems (motion problems, related rates, maxima and minima); and VI - optional topics (complex numbers, polar coordinates), optional in that a school might decide to deal with these topics in one of the other senior mathematics courses - Algebra or Relations and Functions.

Content of Assignments

Table 2 is a record of the percentage of homework and seat-work questions assigned during the course by each teacher, the questions having been classified according to the six content groups. In addition, the total number of question on which the percentages are based is given for each teacher. Differences among teachers in total number of questions assigned was great. The median was 1037 questions, but the range was from 460 to 1622 questions.

Several results in Table 2 stand out. First, the emphasis on basic skills (Content Group I) was high for all teachers, ranging from 41% of questions assigned to 76%. Second, the greater the emphasis on basic skills, the more almost everything else was deemphasized. Third, little attention was paid to questions involving proofs and first principles, although increased emphasis on these issues is mandated in the new Ontario curriculum.

Table 2

Percentage of Assigned Questions by Content Group and Total Number of Assigned Questions

Teacher	Content Group[a]						Total
	I	II	III	IV	V	VI	Number
1	49		12	5	13	20	1622
2	56		11	5	25	3	836
3	65		13	5	12	4	1498
4	53		21	9	17		4456
5	74		8	8	6	5	1003
6	64		10	8	18		460
7	66	1	9	4	20		917
8	76	2	11	1	11		1388
9	49		14	11	14	12	1395
10	41		24	15	18	2	1341
11	57	2	17	5	19		967
12	70		10	5	14		1083
13	46		14	11	21	8	970
14	54		19	5	23		1037
15	66		10	12	12		1343
16	50	2	13	5	18	12	850
17	54		18	8	19		1274

Note: The percentages for a row may not sum to 100 due to rounding error.

[a] See text for a description of the content groups.

Content of Tests

All questions used by teachers in quizzes, classroom tests, and exams were categorized. From teacher-supplied marking schemes and weighting systems, the relative (percentage) weight of every question in the calculation of final grades was determined. These relative weights were summed to yield the percentage of marks toward the final grade that were allocated by each teacher to questions in each of the six Content Groups (Table 3). The percentages in Table 3 indicate a relatively heavy emphasis in testing on basic skills (Content Group I). Emphasis on Content Group II (proofs of basic theorems) was relatively low. The teachers varied considerably in the degree to which they emphasized each content group, but this variation is especially noticeable for Group VI (optional topics).

Table 3

Percentage of Test and Examination Marks by Content Group

	Content Group[a]					
Teacher	I	II	III	IV	V	VI
1	35	2	14	12	20	17
2	50	3	13	3	20	11
3	51	7	16	4	15	6
4	33	7	18	12	30	
5	43	2	13	16	20	6
6	50	3	13	13	21	
7	46	2	18	8	26	
8	58	1	18		23	
9	37	3	14	11	22	14
10	33	2	23	12	27	2
11	35	6	27	8	24	
12	48	8	15	8	22	
13	26	4	19	15	28	9
14	33	5	24	7	31	
15	48	6	14	14	18	
16	38	4	15	6	25	13
17	34	10	15	14	25	

Note: The percentages for a row may not sum to 100 due to rounding error.

[a] See text for a description of the content groups.

Effect of exemptions. A study was made of the effects of exemptions from final exams on the content of the assessments of student achievement. Four teachers followed a policy, mandated by the board or the school, of exempting students with a high term mark (typically 65% or more) from the final exam. In one of these four classes, the final marks of the exempted students were based on assessments of substantially different content than the final marks of non-exempted students.

Effect of discarding test results. One teacher divided the semester into five segments, referred to as terms. Each term contained up to six short quizzes and one test. The tests for Terms 3 and 5 were considered to be the mid-course and final exam respectively. Students were allowed to drop the test and quiz results for one of Terms 1, 2, or 4 from the calculation of their final grades. Dropping the test and quiz results for Term 4 produced final marks based on an assessment of somewhat different content than dropping the test and quiz results for either Term 1 or Term 2.

The foregoing results point to problems with the practices of exemptions and selectively discarding test results. The expectation of many consumers of high school grades is that they reflect achievement of the same curriculum. By exempting some students from final exams or discarding some term results from the calculation of final grades, with different results discarded for different students, marks within the same class will reflect achievement of different kinds. Unbeknownst to consumers, differences among such marks are uninterpretable.

Comparing the Contents of Tests and Assignments

A comparison of corresponding percentages in Tables 2 and 3 reveals a general tendency for teachers to do less testing than assigning of content in Group I (basic skills), slightly more testing than assigning of the content in Groups II and III (proofs and differentiation graphing), and considerably more testing than assigning of the content in Groups IV and V (integration graphing and situational problems). At least some of this pattern must be due to differences in the relative size of questions for basic skills compared to that for the other content groups. (Ten differentiation exercises may require less time to complete than one applications problem.) The greater emphasis on basic skills (Group I content) in assignments than in tests may also reflect the belief that practice makes perfect, not the belief that basic skills are especially important. Moreover, the greater emphasis on Group II content in testing than in assignment may mean that proofs are considered important, but are dealt with by class instruction and demonstration rather than by assigned exercises.

Grading Practices

The data collected about testing and grading practices were used to study the grading processes that were used and the actual grades that were assigned.

The Process

The 17 teachers were found to use 22 different grading systems. More than one system was in use by each of the four teachers who followed an exemptions policy - the grading system for a student of these teachers depended on whether the student had been exempted from the final examination. Also, the teacher who set aside some of a student's marks in calculating the final grade employed at least two different systems. (Refer to the description of this method given in the previous section of the report).

Two grading criteria were used almost exclusively: tests, usually administered at the end of units of work, and examinations, administered near the mid-point of the course or the end or both. The number of tests ranged from three to 13, the number of examinations from one to two. For seven of the 17 teachers, tests and exams represented 100% of the students' grade. For the other 10 teachers, the weights for tests ranged from 30% to 80% of the final grade. The additional criteria used by these teachers included quizzes (six teachers, weight ranging from 2% to 20%), assignments (six teachers, weight ranging from 3% to 6%), and a subjective mark for participation (four teachers, weight ranging from 5% to 20%). Six of the teachers gave no mid-course exam, while the four who followed exemption policies had no final exam for the exempted students. The remaining grading systems included both mid-course and final exams. (We use the designations first-half and second-half of the course rather than first-term and second-term to avoid possible confusion over the meaning of term and semester. All our data were collected in semestered schools during the Spring Semester. The break between first-half and second-half of the course occurred about mid-April.)

The teachers combined the different test and examination marks into final grades in several different ways. The marks for a test or exam were either (a) weighted according to the number of marks in each (simple summation) or (b) re-weighted to make the weights of each test or exam equal or (c) re-weighted to reflect the teacher's perception of the relative importance of the topics covered by each test or exam. Similarly, the halves of the semester were weighted either equally or unequally. Five teachers weighted each half equally (at least for some students). The other eleven teachers weighted the first-half of the course less (about 30%) than the second-half (about 70%).

The time spend on testing activities (including exams) varied enormously, ranging from nine hours for one teacher to more than 17 hours for another. On average over the 17 teachers, 8.8 hours (range 3 to 15.2 hours) were spend in writing 8 tests (range 3 to 13 tests), not including mid-course and final examinations. The lengths of tests varied from 25 to 75 minutes. The total number of test questions administered during the semester average 104, and ranged from 53 to 180.

The cycles of teaching and testing throughout the course were examined. Nine of the teachers seemed to have more regular cycles of teaching and testing than the others. All teachers tested at more-or-less regular intervals throughout the first-half of the course, but the testing patterns for eight teachers became erratic in the second half. The teachers who followed more regular teach-test cycles also gave a greater number of tests on average (10 compared to 7).

Another factor that was considered is the number of different topics covered in a test. The taxonomy of 14 content categories was used to describe test coverage. The number of categories covered ranged from one to eight per test over all the tests given by the teachers. The average (for a teacher) of the number of categories per test ranged from 2 to 4.7. All but two teachers gave at least one test covering only one content category. For five teachers, the final exam included content categories that had not been covered in class tests. A possible reason is that the final examinations used by these teachers were set by other calculus teachers. When one teacher in a school sets a common final exam, that individual requires a certain amount of prescience to set questions that the students taught by other teachers have had an opportunity to learn. A class that proceeds more slowly than expected, for example, is likely to be disadvantaged by the exam.

The Grades

The average mid-course and final grades for each class and the difference between the two were calculated. The distinction between mid-course and final grades is important in Ontario for Grade 13 courses offered in the Spring Semester. Early in April, Ontario

universities begin their admissions process. For the courses a student is taking in April, the school submits interim grades (normally the mid-course grades in semestered schools) to the Ontario University Applications Centre, and students receive a conditional acceptance or rejection based on these grades. A concern expressed by several teachers in the study is that students become less motivated to work once the interim grades have been submitted.

Averaged over all 17 teachers, the final grade was 67, six points less than the average mid-course grade of 73. For every teacher, the mean final grade was either lower than or at best equal to the mean mid-course grade. The range of mean final grades was 54 to 75, that of mean mid-course grades, 61 to 79. Although there is a high correlation between the two sets of grades for a teacher, the difference between a teacher's mean mid-course and mean final grades was as much as 15%.

A comparison of term marks with final exam marks turned up two results of interest: In the 13 classes in which all students wrote a final exam, the final exam marks were lower than the term marks by about 12 percentage points (58% compared to 70%). Also, final exams tended to discriminate more than term marks. For the 13 classes with no exemption policy, the standard deviation of final exam marks was about 50% larger than that of term marks.

Despite all the differences found in grading processes and in the grades themselves, no clear indication was found in the data provided by the 17 teacher-participants that the observed differences in grading process were related to the observed differences in grades.

(Editor's note: The concluding section of this report, examining how the same teachers responded to a request to grade a common set of examination papers, will be reproduced next issue. This concluding section of the report will include, as well, a summary of the entire report.)

REFERENCES

Alexander, D. W. (1987). Evaluation practices: Calculus interim report. Toronto, Ontario: Ministry of Education.

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POSTER CONTEST

The Ontario Association for Mathematics Education (OAME) invites students of Ontario Elementary and Secondary schools to submit posters on the topic "Mathematics mind bogglers for the Intermediate Division".

Up to four prizes of $100, $75, $50, $25, will be awarded by the judges on the basis of artisitic merit and representation of the topic.

All posters become the property of the OAME, which may publish and distribute winning entries, giving credit to the students.

The final date for receiving entries will be March 29th, 1989. Winners will be notified by mail prior to the publication of results. Some of the entries will be exhibited at the OAME 1989 annual conference in Ottawa.

Entries should be sent to: Mr. Tom Griffith
c/o A. B. Lucas Secondary School
656 Tennent Avenue
London, Ontario N5X 1L8

OAME OAME OAME OAME OAME OAME OAME OAME OAME OAME OAME OAME OAME

PROBLEMS

by
Ronald Lancaster
Oakville-Trafalgar High School
Oakville, Ontario L6J 3L6

An unlimited amount of gasoline is available at one edge of a desert that is 800 km wide. There is no source of fuel on the desert itself. A truck can carry enough gasoline to go 500 km (this amount will be called one "load") and it can build up its own refuelling stations at any spot along the way. These caches may be any size and you may assume that there is no loss due to evaporation. What is the minimum amount (in loads) of gasoline that the truck will require in order to cross the desert?

SOLUTION AND COMMENTARY

I came across this problem in one of Martin Gardner's many excellent books. (The 2nd Scientific American Book of Mathematical Puzzles, Simon and Schuster, pg. 152) The following solution is essentially the one presented by Gardner.

It is easy to see that the farthest away from the edge of the desert that the truck can get to with one load is 500 km.

How far away can the truck get to with 2 loads? The answer turns out to be 500(1 + 1/3) km. This can be done in four trips by first setting up a cache at a spot 500/3 km from the edge of the desert. The truck begins with a full load, goes to the cache, leaves 1/3 of a load, returns, picks up another full load, arrives at the cache and picks up the cache's 1/3 of a load. It now has a full load with which it can go 500 km. The truck ends up 500(1 + 1/3) km from the edge of the desert.

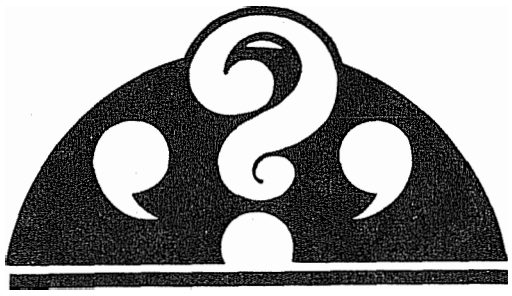
With 3 loads, the farthest away from the edge of the desert that the truck can get to is 500(1 + 1/3 + 1/5) km. This can be done as follows: The first cache is 500/5 km from the edge of the desert. The truck begins with a full load, arrives at the cache, leaves another 3/5 of a load, and returns to the edge of the desert. The truck picks up another full load and arrives at the cache with 4/5 of a load in its tank. This, together with the fuel in the cache makes 2 loads with which the truck can go another 500(1 + 1/3) km as explained above. The truck ends up 500 (1 + 1/3 + 1/5) km away from the edge of the desert. Note that this is equal to 766 and 2/3 km, not quite the required 800 km.

With 4 loads, the farthest away from the edge of the desert that the truck can get to is 500 (1 + 1/3 + 1/5 + 1/7). Note that this is a bit more than the required 800 km.

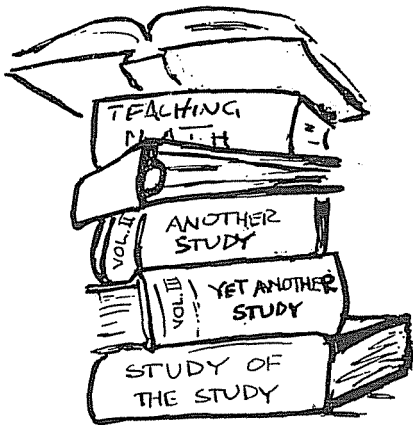
For the truck to end up exactly 800 km from the edge, it is necessary to set up a cache that is 500/15 (i.e. 33 and 1/3 km) from the edge. In 5 trips the truck can build up this cache so that when the truck reached the cache at the end of the seventh trip, the combined fuel of the truck and cache will be 3 loads. Using the procedure outlined above, the truck can go the remaining 766 and 2/3 km. The total amount of gasoline consumed in 3 and 7/15 loads, or a little more than 3.46 loads.

It can be shown that the infinite series 1 + 1/3 + 1/5 + ... diverges and so the truck can cross a desert of any width!

I gave this problem to all my students on the first day of school as part of an assignment. I gave them one week to work on it. I made it clear to them that I was more interested in how they approached the problem than in their actual answer. Most of the students were not used to dealing with an open-ended problem like this, but with some encouragement from me they all made a good effort. The lowest answer I received was 3.8 loads. I am glad that I started off the year with this problem because it set an excellent tone for things to come and it got the students thinking on a different wavelength. This is the type of problem that anyone can work on and my students told me that many of their parents worked on it too.



BOOK REVIEWS



PROBLEMS, PROBLEMS, PROBLEMS: VOLUME 1
Canadian Mathematics Competition: University of Waterloo, Waterloo, 1988.

This useful little booklet is, as the title suggests, a collection of problems chosen from a variety of mathematics contests sponsored by the University of Waterloo. The problems it contains are considered to be suitable for students in grades 9, 10 and 11. Despite the fact that these problems have appeared on contests over the years, the book has some features that make it much more useful than a simple file of old question papers.

For example, the book is divided into topics such as decimals, angles, averages, answer the question asked, counting and challenge problems. This method of organization makes it easier for the teacher to find appropriate problems for either contest preparation or classroom use. Furthermore, within each section there are both multiple choice items and questions requiring a complete solution. The second half of the book contains answers and solutions to all of the more than 300 problems.

This book provides a handy resource for the bookshelf of any secondary mathematics department.

Reviewed by Eric Wood
Division of Curriculum Studies
Faculty of Education
The University of Western Ontario
London, Ontario N6G 1G7

CHILDREN AND NUMBER: DIFFICULTIES IN LEARNING MATHEMATICS
Martin Hughes, Basil Blackwell Inc., 1986.

In the early 70s Martin Hughes became dissatisfied with Piaget's assessment of the mathematical capabilities of young children. Hughes felt that because Piaget did not pay close enough attention to the relationship between the context and the language of his tasks, he had underestimated the abilities of young children. Rather than probe understanding through the classical Piagetian logical tasks, Hughes describes children's thinking about early number through questions closely aligned with those experiences encountered prior to and in the initial stages of schooling.

Although Hughes believed that young children understood a good deal about number, he also recognized that children had great difficulty understanding formal school mathematics and often left school with poor conceptual frameworks. Consequently he undertook to discover what mathematics school children do understand, why so many of them fail to learn formal mathematics successfully and what can be done to assist the learning process. The book is organized accordingly.

His findings are most revealing and demonstrate that young children do have s sound knowledge of number, but not in the formal sense that we expect in school. The following extract is with Amanda, age 3 years 11 months (page 46).

MH:How many is two and one? (Long pause. No response) Well how many bricks is two bricks and one brick?
Amanda:Three
MH:Okay. So how many is two and one?
Amanda:(Pause) Four (hesitantly)?
MH:How many is one brick and one more brick?
Amanda:Two bricks
MH:So how many is one and one?
Amanda:One, maybe.

Hughes indicates that abstract notions such as, "How many is one and two?" have little meaning to young children. Where else, apart from school is a child ever asked to add one and two? One and two what? Such questions are just as bizarre to young children as asking, "Is red bigger than yellow?" In fact, children treated this question seriously, explaining for example, that red was smaller because the red cushion was smaller than the yellow curtain.

School age children of six and seven had just as much difficulty when it came to representing the operations of addition or subtraction for verbal problems. They used ingenious ideas such as soldiers marching from left to right, or pictures of hands removing or adding blocks, but NOT A SINGLE CHILD, from a sample of forty-eight, used the conventional + and - signs even though these were used regularly in school.

Hughes's book emphasizes the gap between *real world math* and *formal school math.* The real world mathematics makes sense to children whereas much formal school mathematics does not. Providing the opportunity for children to make that link must be a key objective of the mathematics lesson. Hughes describes many ways to facilitate this process including games, and Seymour Papert's LOGO and turtle graphics.

This book is well written and full of pertinent examples which illustrate the author's sound knowledge in this field. It is a book not solely for the primary teacher, but for anyone interested in how children can develop solid conceptual understanding rather than becoming frustrated by meaningless role manipulation.

Reviewed by Barry Onslow
The University of Western Ontario
Faculty of Education

KENNETH D. FRYER AWARD

In Memory of the late Dr. K. D. Fryer, OAME in conjunction with the Grand Valley Mathematics Association established the Kenneth D. Fryer Award for Excellence, to be presented annually to a Mathematics Department demonstrating excellence in mathematics education as well as contributing to the overall development of students beyond classroom activities.

At this time OAME is requesting nominations for consideration by the committee for this year's award. It will be presented at the annual conference in Ottawa.

Please complete a copy of the form below and forward it to:

M. B. MacGregor
Sec/Treas. OAME
247 Bright Street
Sarnia, Ontario N7T 4E9

Deadline for nomination submission: March 11, 1989

KENNETH D. FRYER AWARD
NOMINATION FORM
1988-1989

I/WE nominate School: _____
Address: _____
Principal: _____
Math Dept. Head: _____
Nominated By: _____

Please attach a separate sheet to the nomination form outlining the resons for your nomination.

1988 PROBLEM SOLVING RESEARCH AWARD WINNER

Ruth Douglas
Ottawa Board of Education



Editor's Note: This brief outline is included to illustrate some of the problem solving activities that resulted in Ruth's winning first place in the 1988 contest. Thanks to Chester Carlow for bringing this work to our attention. More details about this year's contest follow the report.

Original Situation

The program was developed to meet the needs of students who qualified for an enrichment program at the grade five and six levels. Most were enrolled in French Immersion Programs. They received their basic math instruction in French.

The aims of the program were to:

- complement the work of the classroom teacher's program.
- enrich rather than accelerate.
- incorporate math under the umbrella theme of thinking.

Over a period of time, five distinct math problem solving tools evolved:

- simplify by
  - isolating relevant data
  - drawing a sketch/diagram
- use patterns, lists, sequences
- logic
- manipulation, trial and error
- incubation.

Standard textbook problems in general were quite easy for the gifted students in the study. They were often stated in very concise language and contained clearly delineated problem statements.

What Was Wrong

The simplicity of standard textbook problems made them unsuitable for the enrichment program. Many such problems were so easy that students could solve them "off the top of their heads" without the aid of distinct problem solving tools. However, when problems were selected to be of sufficient difficulty to bring out the usefulness of the problem solving tools, the mechanical accuracy of the students began to slip. This, in turn, led to a loss of confidence in performing operations involving multiplication, short division, decimals and fractions. As they lost confidence in their mechanical ability, students reverted to long paper and pencil solutions (e.g., 5 + 9 + 16 was written in a neat column and carefully added, complete with the carrying crutch). Problem solving lost its glamour as a thinking exercise as attention was directed to the mechanics.

The challenge was to free the children from the burden of the mechanics and so allow them to enjoy and practise the process of problem solving.

Process Used

First it was necessary to find or make up problems suitable for the teaching of each of the problem solving tools. In teaching students to isolate relevant data, for example, the following two rhymes were used along with a number of other problems.

Little Girl Blue

Little girl blue
Come blow your horn.
Six sheep in the meadow
Three cows in the corn.
Now where's the little girl
Who looks after the sheep?
She's in front of the VCR
Fast asleep.

If three sheep were asleep beside her, for how many animals was she responsible?

Little Miss Muffet

Little Miss Muffet
Sat on a Tuffet,
Counting her money one day.
Then came Mrs. Spider
And sat down beside her,
And took half her money away.

"Oh, no! I can't spare it.
I'm willing to share it.
I really don't know what to do.
Please give back a quarter,"
Miss Muffet besought her.
"It means more to me than to you."

"That can't be denied,"
Mrs. Spider replied.
"My web and my flies are quite free.
I'll partly relent.
Here's a dime and a cent,
So now you'll have one dollar three."

Poor Miss Muffet! How much did the spider keep?

"Little Girl Blue" was teacher written while "Little Miss Muffet" was used by permission from Figures Are Fun (Hunter, 1959).

Some suggestions from J. Hope were found useful in relieving the loss of confidence in performing mechanical operations (Hope, 1986). Hope observed that while a calculator holds an obvious advantage over paper and pencil calculation when the calculation task is complex, there is nothing more convenient than mental calculation when the calculation task is relatively simple.

A two-fold approach was taken:

- Isolate problems involving operations at the students' grade level and allow students to use calculators as desired.
- Develop a program of oral arithmetic completely divorced from the problem solving part of the program.

To carry out the oral arithmetic activities, fifteen minutes were isolated from the program each week and strategies were developed to strengthen mechanical weakness. A series of mechanical operations was presented orally and the final answer recorded.

Relevant variables bearing on success were identified as:

- difficulty of operation
- speed of delivery
- number of steps per question
- number of operations per question
- choice of mechanical skill.

Because of the range of mechanical ability each question had a stop midway for those with weaker skills, while the most talented were expected to complete the entire sequence of operations. For example,

23 + 9 + 6 + 13 (for weaker students)
23 + 9 + 6 + 13 + 15 + 23 + 7 (for better students).

Hazekamp's teaching suggestions for questions of the type 38 X 5 or 25 X 64 were incorporated (Hazekamp, 1986). These are more difficult and were attempted only after the first five types had been mastered.

On any day a series of ten questions from any one of six categories provided the basis for the lesson. Each student, in private consultation with the teacher, determined whether to do the full question or the partial one. Performance on the previous week's work was weighted heavily in this decision.

Outcomes

This simple mechanical skills development has resulted in an increase in students' confidence in math, both in the enrichment program and in regular classroom mathematics. It has freed students from the burden of mechanical struggles and allowed them to focus their full attention on the process and techniques of problem solving.

As reported by Hope, Wandt and Brown (1957), it was also found that most everyday calculation problems were solved using mental techniques rather than paper and pencil ones.

Future Studies

It would appear that this program modification not only encourages mental calculation but prepares the students for a life long skill in problem solving.

REFERENCES

Benton, S. (1986). A summary of research on teaching and learning estimation. In H. Schoen and M. Zweng (Eds.) Estimation and Mental Computation. Reston, VA: National Council of Teachers of Mathematics (NCTM).

Hazekamp, D. (1986). Components of mental multiplying. In H. Schoen and M. Zweng (Eds.) Estimation and Mental Computation. Reston, VA: NCTM.

Hope, J. (1986). Mental calculation: anachronism or basic skill? In H. Schoen and M. Zweng (Eds.) Estimation and Mental Computation. Reston, VA: NCTM.

Hunter, J. (1959). Figures are Fun. Copp Clark Co., Toronto (Used by permission).

Reys, B. & Reys, R. (1986). One point of view: mental computation and computational estimation -- their time has come. Arithmetic Teacher 33 (7), 4-5. Reston, VA: NCTM.

ATTENTION ELEMENTARY SCHOOL TEACHERS

AWARDS FOR THE TEACHING OF PROBLEM SOLVING

If you are an elementary school teacher who has developed "neat" and effective ways and means of improving the teaching of problem-solving in your classroom, school, or district, then this announcement is for you.

Chester Carlow, in cooperation with OAME, is sponsoring awards for the improvement of teaching problem-solving in the elementary schools: **$300 for first place, $200 for second place, and $100 for third place.**

WHAT IS REQUIRED:

Teachers are asked to submit a report which details the innovative instructional strategy(ies) which were used, and the results relative to the improvement in students' abilities to solve problems.

Teachers should select an instructional strategy which looks promising in terms of previous theory and research on problem solving. This will promote steady progress in the area.

Bibliography to get started:

Berry, Lois Langfor, **Problem Solving In The Elementary School Curriculum**, ED 253-416, 46 pages

Morrow, Lorna, **The Classification of Mathematics Problems By Young Children: An Exploratory Study**, Doctoral Dissertation, O.I.S.E., 1988

Riley, Mary S. and others, **Development of Children's Problem Solving Ability In Arithmetic**, ED 252-410, 47 pages

Complete information about the structure of the reports can be obtained from:

Ralph Connelly
11 Berkwood Place
Fonthill, Ontario
L0S 1E0
(416) 688 5550

J. Symington
11570 Yonge Street
R.R. # 1
Richmond Hill, Ontario
L4C 1X7
(416) 884 4802

Chester Carlow
87 Risebrough Avenue
Willowdale, Ontario
M2M 2E2
(416) 223 2932

Your completed entry should be sent to Chester Carlow before April 1, 1989.

DON'T KEEP THOSE EXCITING IDEAS A SECRET ANY LONGER, LET US HEAR FROM YOU!

COUNCIL COMMENT



by
Bob Robinson
114 Dromore Cr.
Hamilton, ON L8N 3L1

Comments based on the council Meeting held at the Ramada Inn - Airport West, on Saturday, 1988 10 15.

NCTM Leadership Conference Report

Barry Scully, our NCTM representative, and Arnie Carefoote attended this highly successful conference. Representatives from across Canada shared ideas, concerns, problems and solutions. The participants identified tasks:

a) that might strengthen OAME.
b) that might strengthen the NCTM in Canada.
c) that might strengthen Mathematics education in Ontario.

Many good ideas were presented by the various associations represented. Arnie Carefoote's three-page report is available from Arnie Carefoote, 50 Carslake Cr., Scarborough, Ont., M1J 2A7.

Annual Meetings

1988 Annual (Niagara Falls).Ed Haines made his final report. A job well done by Ed and his colleagues.

1989 Annual (Ottawa, May 11-13, 1989)

A pre-registration package is being prepared for this "Capital Conference". Mini conferences and many exciting social events are being planned during the "Tulip Festival". Don't miss this opportunity to "capitalize" on an unique event.

1990 Annual

Our annual meeting next year will be in Hamilton in May, 1990 (1990 05 10-12). It will take the form of a Canadian Regional Meeting of the NCTM. Interested speakers, or persons you would like to recommend to speak, should be submitted to either of the program chairs (no later than April 10, 1989):

Alex Norrie,	Ralph Connelly
Peel Bd. of Education	11 Berkwood Pl.
5650 Hurontario St.	Fonthill, Ont.
Mississauga, Ont., L5R 1C6	L0S 1E0

Volunteers are required for the following committees: Computer Facility, Educational Materials, Hospitality, Membership, Publicity, Reception and Planned Meal Functions, Registration, Signs and Printing, Student Aides, Supplies/Equipment, and Workshop Support. If you wish to assist in any of these areas, please contact:
George Knill, 164 Hillcrest Ave., Hamilton, ON, L8P 2X4, (416) 525-5788.

1991 Annual

Approval has been granted for PAME (Peel Association for Mathematics Education) to host our annual meeting on 1991 05 9-11.

CHAPTERS

COMA/AMOC

The main project this year is the hosting of our annual meeting in May, 1989. However, COMA will host a mini workshop on Monday, January 23, 1989, at Confederation High School (4:00 p.m.-6:00 p.m.) to be followed by dinner at the Continental Lounge in the Nepean Sportsplex. The speakers will cover all levels of the curriculum with a particular emphasis on Junior-Intermediate Manipulative Skills.

GVMA

The fall conference was held November 10 at Eastwood Collegiate with dinner at the Schwaben Inn.

For information on the publications of GMVA, contact Rachel Hotson, Faculty of Mathematics, Univ. of Waterloo, N2L 3G1.

NOMA

A successful conference was held on September 30-October 1.

NOMA will give financial and logistic support to the Mathematics Olympiad which the Sudbury Board of Education sponsors for Grade 7 and Grade 8 students.

NOMA will sponsor cash awards for top students in zones 62 and 63 of the Canadian Mathematics Competition.

NOMA certificates are being distributed to all northern schools and will be awarded to worthy students for excellence in Mathematics.

Tentative plans have been made to co-sponsor, with the Sudbury Board of Education, a coaches clinic for teachers preparing their students for math competitions.

NWOAME

NWOAME is alive and well. A temporary executive has been formed and election of an executive will be made at an inaugural meeting at Lakehead University. A membership drive is underway and a constitution will be developed.

PAME

PAME is experiencing a period of growth. The annual Mathfest '88 was held October 27. The Peel Math Contest was written on November 16 and involved students of grades 9, 10, and 11. Prizes will be awarded to the top 10 students in the board as well as a trophy to the school with the best team. A Mathematics Day '88 was planned for November 23 in Peel Secondary Schools.

PRMA

PRMA arranged mathematics workshops for November 18, 1988. Members of PRMA are organizing a Durham mathematics contest. The contest involves grade 9, 10 and 11 students in a "Reach for the Top" format.

Plans are underway for their annual mini conference which will be held at General Vanier HS in Oshawa on April 12, 1989.

A Mathematics Awards Nights co-sponsored and organized by PRMA is planned for June 8, 1989 at G.L. Roberts CVI.

QSLMA

A Fall conference/dinner was planned for October 27, 1988, at the Cataraqui Golf Club. A Spring conference is being planned for April, 1989.

SAME

The annual mini conference was held at Maplewood HS on October 18, 1988. Twenty-four speakers were invited. Jan Cornwall was the chair of the organizing committee. A professional activity day is being planned for February 10, 1989 at the Sheraton Parkway.

SWOAME

The Fall math conference was conducted at the University of Windsor on November 18 and 19 1988.

YMA

In October the York Mathematics Association held a highly successful conference.

NCTM

Janet Scully reports that she has placed a consignment order of materials to NCTM in order to provide a display of NCTM materials for chapter conferences. It is hoped that these materials will increase interest in NCTM membership.

COMMITTEE REPORTS

Primary/Junior Committee

Sue Stuart has taken over the chair position for Bob Robinson who has been seconded to the Ministry of Education. A response from the Ministry of Education was received regarding the delay in the translation of support documents and guidelines into the French language. Apparently, an internal Ministry committee has studied this matter and has formulated several recommendations in order to prevent further occurrences.

The provincial review of grade 6 mathematics was discussed. Sue Stuart and Roland Morrow will represent OAME on the Provincial Review consultative Committee.

Membership/Publicity

A letter, along with the OAME membership flyer will be sent to all boards and faculties of education to promote membership.

Publications

The format and structure of the *Gazette* and "Abacus" insert is being discussed. The committee will investigate ways to assist the editors in their tasks.

Intermediate/Senior Committee

The intermediate OAIP material should be completed by December, 1990. The test items are currently being field tested. Teachers may have to change some of their teaching strategies in order to incorporate some of the testing materials such as the use of manipulatives.

The Senior OAIP instruments have been written for all guideline objectives in grades 11 and 12. Final release is not expected until 1991. George McNabb will represent OAME beginning in January, 1990.

The report from OISE, entitled "Teacher Assessment Practices in High School Calculus" was discussed.

The Ministry will continue to support both IBM and ICON. Support for other computers is under consideration.

The committee discussed the concern over the apparent shortage of qualified mathematics teachers and the need for making more teacher training in mathematics available.

Futures

Due to production difficulties the "Survival

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
1988-89 EXECUTIVE & COUNCILLORS

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MEMBERSHIP DUES
TILL DECEMBER 1, 1988

Members of O.M.A.E./A.O.E.M. receive the Gazette and Abacus free. Membership dues for 1 year are $21.00 in Canada, $23.00 in United States and $27.00 outside Canada and United States. Membership dues for 5 years are $95.00 in Canada, $105.00 in United States, $125.00 outside Canada and United States. If paid in **any currency** other than Canadian add $2.00 to each of the above prices. Orders for memberships should be sent to

M. B. MacGregor, O.A.M.E.
247 Bright Street
Sarnia, Ontario N7T 4E9

MEMBERSHIP DUES
AFTER DECEMBER 1, 1988

Membership dues for 1 year are $25.00 in Canada, $27.00 in United States and $30.00 outside of Canada and United States. Membership dues for 5 years are $112.00 in Canada, $122.00 in United States, $140.00 outside Canada and United States.

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Knill, George
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Wiggan, Lorna

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Collins, Eileen
Cornwall, Jan
Davidson, Dave
Griffiths Tom J.
Harvey, M. Elaine
Mitchell, Janet
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Robinson, Robert