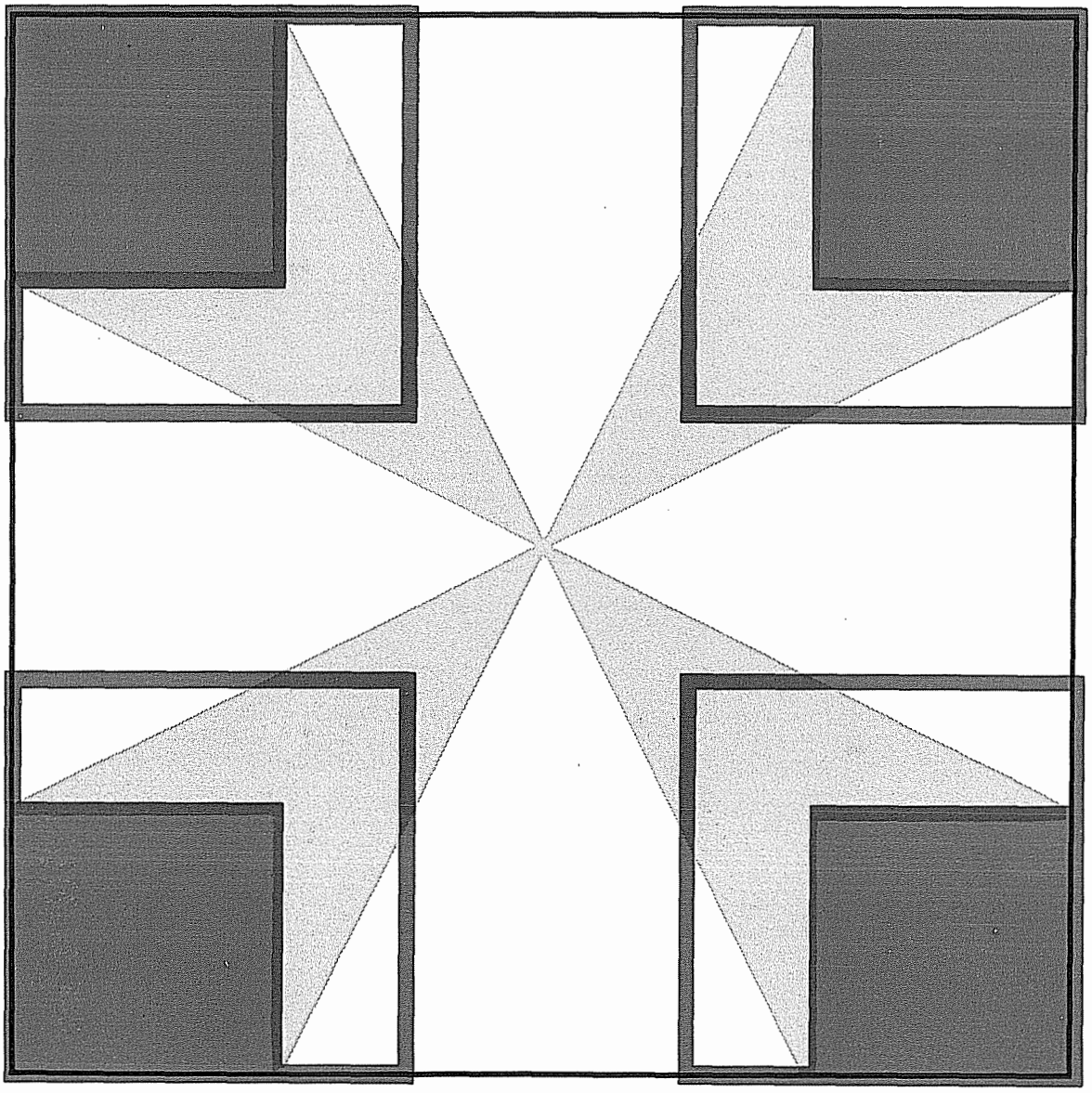
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THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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June 15	for the September issue
September 15	for the December issue
January 15	for the April issue

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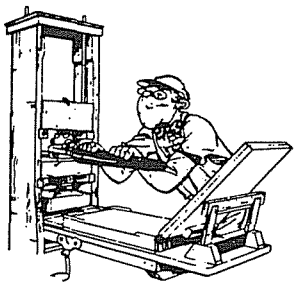
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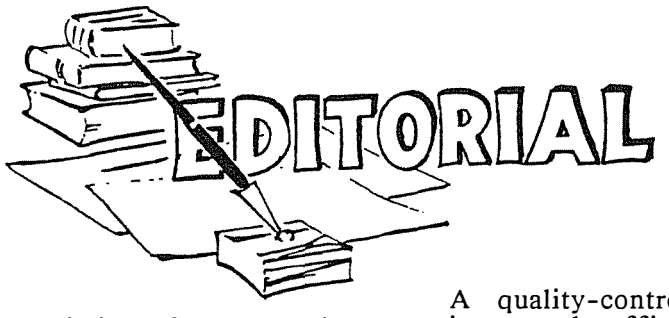
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Anna & Colin Garnhan will be co-editors of the Abacus for 1990/91.
Barry Onslow & Eric Wood will assume co-editorship of the Gazette for 1990/91.
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by
Bob Smith

As the Ontario Association of Mathematics Education and its journal the Ontario Mathematics Gazette (O.M.G.) enters the last decade of the twentieth century, it is suitable that I retire once again from the duties of editor. I do so with full confidence in two colleagues who will continue the work. The September issue with Eric Wood as guest editor and the December issue with Barry Onslow as guest editor have shown you, the reader, the quality of their work. I am sure you are as assured as I am that the journal will continue to offer a wide range of stimulating and informative articles.

If there is to be a general theme for O.M.G. during the years ahead, may I suggest that the N.C.T.M. document Standards be the guide and revision be the theme. This document and the articles generated by it will establish a framework for reform, for the goal is a revision of the mathematics curriculum and its associated evaluation. Such a revision could not be accomplished without a clear set of standards being written. Standards by definition are the specimens or specifications by which the qualities of something may be tested. Such specifications have now been clearly set down and these goals indicate change.

A quality-control change process would improve the efficiency of what is being done without disturbing the basic features of the existing curriculum. Design change fundamentally changes the organization itself. The revision indicated by the Standards document is a design change process - a fundamental restructuring of the mathematics curriculum and in instruction. The N.C.T.M. has created a vision of:

- mathematical power for all in a technological society;
- mathematics as something one does - solve problems, communicate, reason;
- a curriculum for all that includes a broad range of content, a variety of contexts, and deliberate connections;
- the learning of mathematics as an active, constructive process;
- instruction based on real problems
- evaluation as a means of improving instructions, learning, and program.

No revision will occur without you, the teacher being aware of the intent of your organization. Thus, an excerpt from the Standards follows. It highlights some assumptions and standards for each of three levels. It may entice you to obtain a copy of the whole document. Do share the ideas with colleagues as you reflect on the vision. Remember - the ninety's are here and the twenty-first century is not far off.

The Curriculum Standards for Grades K-4.

Underlying Assumptions

The K-4 curriculum should:

a. Be conceptually oriented.
b. Actively involve children in doing mathematics.
c. Emphasize the development of children's mathematical thinking and reasoning abilities.
d. Emphasize applying mathematics.
e. Include a broad range of content.
f. Make full use of calculators

Highlights of the K-4 Standards

1. Shifting of some content to higher grade levels. For example, move place value to Grade 2; limit instruction with whole numbers to no more than 4-digit numbers through Grade 4; delay mastery of basic addition and subtraction facts until Grade 3; delay mastery of basic multiplication and division facts to Grade 4; continue introducing concepts involving fractions and decimals, but delay paper and pencil computations with them until after Grade 4.
2. Computational proficiency still an expectation, but with smaller numbers. For example, restrict divisors to single digit numbers. Do complex computations with a calculator.
3. Greater emphasis on problem solving, mathematical reasoning, measurement, geometry, and estimation.
4. Introduction of new topics such as number and spatial sense, basic ideas of statistics and probability, relations and functions.
5. Heavy initial emphasis on use of physical models.

The Curriculum Standards for Grades 5-8

Underlying Assumptions

The 5-8 curriculum should be designed on the assumption that:

a. Every classroom will have ample sets of manipulative materials and supplies for student use.
b. Appropriate resource material providing problems and ideas for exploration will be available for teacher and student use.
c. A four-function calculator will be available at all times to each grade 5 and 6 student. In grades 7 and 8, a scientific calculator will be available at all times to each student.
d. Every classroom has at least one computer available at all times for demonstrations and student use.
e. All students experience the full range of topics addressed in the Standards.
f. Topics in the individual Standards should be integrated through a selection of rich activities.
g. Students should be actively involved in the learning process, investigating and exploring individually and in groups.
h. Relevant situational contexts should motivate instruction.
i. Instructional activities should take place outside as well as inside the classroom.
j. Mathematical activities should help relate math to other school subjects, such as science and art.

Highlights of the 5-8 Standards

1. Continued de-emphasis from grades K-4 on computational proficiency. In particular, de-emphasize paper and pencil computations with fractions and decimals and increase emphasis on estimation with familiar fractions and their decimal equivalents.
2. Emphasis of methods of computation other than paper and pencil, such as calculators/computers, estimation, and mental arithmetic.
3. Greater emphasis on developing students' ability to choose an appropriate method of computation.
4. Increased emphasis on topics such as probability, statistics, geometry, and measurement.
5. Much greater exposure to data which can be represented graphically and described by using variables, open sentences, equations, and inequalities. Solidify the concept of variable.
6. Emphasis of informal explorations with algebraic ideas that help students gain confidence in their ability to abstract relationships from a context or situation.
7. Development of functional relationships through the generalizations of number patterns.

The Curriculum Standards for Grades 9-12

Underlying Assumptions

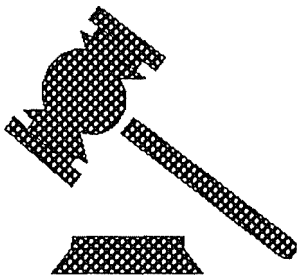
The 9-12 curriculum is based on the assumption that:

a. Students entering grade 9 will have had the mathematical experiences of a broad, rich curriculum as outlined in the K-8 Standards.
b. A level of computational proficiency as specified in the K-8 Standards will be expected of all students. However, no student will be denied access to the study of mathematics in grades 9-12 because of a lack of computational facility.
c. Conceptual and procedural understandings of number, numeration, and operations, and the ability to make estimations and approximations and to judge the reasonableness of results will be strengthened in the context of applications and problem-solving.
d. Scientific calculators will be available to all students at all times.
e. A computer will be available at all times in every classroom for demonstration purposes, and every student will have access to computers for individual and group work.
f. At least three years of mathematical study will be required of all students.
g. These three years of mathematical study will be based on a core curriculum differentiated in terms of depth and breadth of treatment and the nature of applications.
h. Four years of mathematical study will be required of all college-intending students.

Highlights of the 9-12 Standards

1. The same content topics for all students but differentiation within topics. That is, the level of detail of each topic will be different for different ability levels. This implies the elimination of courses such as General Mathematics, Business Mathematics, and Consumer Mathematics.
2. Integration of topics from each major area of mathematics, such as algebra, geometry, trigonometry.
3. Greater emphasis on the following:
   - topics in discrete mathematics
   - probability and statistical models
   - matrices and their applications
   - computer graphing and computer-based methods such as successive approximations and graphing utilities for solving equations and inequalities
   - real world applications and modelling
   - transformational and analytic approaches to geometric concepts
   - structure of number systems
   - recording deductive arguments in sentence or paragraph form
4. De-emphasis (but not elimination) of the following:
   - type word problems
   - simplification of radical expressions
   - factoring to solve equations and to simplify rational expressions
   - operations with rational expressions
   - logarithmic calculations
   - paper/pencil graphing of equations by point plotting
   - geometry from a synthetic viewpoint
   - theorems involving circles
   - two-column proofs
   - verifying trigonometric identities
   - numerical applications of sum, difference, double angle, and half-angle identities
   - contrived triangle applications
   - paper and pencil solutions of trigonometric equations and
   - transforming function equations to standardized form in order to graph them.

PRESIDENT'S MESSAGE



As my term as President draws to a close, I find myself feeling reflective. It has been a very busy year, stressful at times, but also rewarding in ways I can't even yet judge. I have lead a council of 40 enthusiastic, student-centred, and, for the most part, classroom teachers who are leaders in mathematics education. To be involved in OAME means you have made a commitment to the job as a professional. I cannot leave without saying, "Congratulations to all you who have volunteered your time to work at making mathematics education the best it can be. Thank you for the support you have given me. Thank you too, for the support you give the young people in this province every time you stand and advocate for them." Our membership is around the 1800 mark. It pleases me that we have many new members and lots of fresh energy in the new people joining council. Over the next few years, our ranks will swell with rookie teachers. We have a major responsibility to instill in these new teachers a sense of pride in this profession - a feeling of dedication to the cause of education for all the students. Being involved is the first sign of a real professional.

A brief comment about "The Restructuring of Education in Ontario": everywhere schools cater to the University bound student. The person who sees herself not heading to University is a second-class citizen. The counselling departments give their presentations to kids about choices for their futures and all talk tends towards the Ivory Tower. Even Co-op education and Entrepreneurial Studies are paths to take to make another ready for the University challenge. The reality, folks, is that 12% of the students in our midst go to University, 20% go to College, and the majority - a whopping 68% - go to work. A glance at the most recent budget cuts to post secondary education in this province will tell you that the percentages are only going to drop. Our government is not going to subsidize the education of more than 12 percent. The Specialization Years work team has been told to make high school a four year process. They will go to work that much sooner.

Sometimes I have real trouble with the equity issue that the Transition Years is trying to address. If we are suggesting that more kids will have the opportunity to go to University, we are misleading parents. Perhaps we are saying the opportunity to go to University will be open to rich and poor people. I don't see money flowing to the less fortunate. I only see demands on curriculum changes that make the opportunity for the best education (in advanced courses) open to all. I believe this change is essential. What parent would want his or her child in a general-level course? I do not see teachers who care very much about the less fortunate. I sit in staff rooms and hear disdain expressed toward the student in the general level course. Think of those students who are not well supported at home, who show up late or ill-prepared, who are neither well fed, not well dressed, who are slow at getting started. Teachers dismiss such students. They make jokes about kids in the smoking area and turn up their noses. Many do not want to teach general level classes and will not entertain the idea of a basic-level group. Imagine what those comments sound like to the new teacher in our midst. If de-streaming makes it more difficult to see the differences in the kids when they arrive in high school, it will have accomplished something. Schools should adapt to the students who show up, rather than expecting students to adapt to the school. We have to restructure schools so as to meet the needs of the majority, not the needs of 12% of the population.

Mathematics has to change so it works for the people other than those interested in mathematical sciences. We can still teach mathematics, lots of it. But we have to drop our devotion to Greek-based mathematics and move into the twenty-first century. The National Council for Teachers' of Mathematics has produced the Standards Document to provide leadership in understanding the needs for change. The National Research Council has published two books in the last two years on mathematics education, Everybody Counts and Reshaping School Mathematics. They point out that the content we teach is arbitrary and for this day in the information age it is not very useful. So it is time to make some arbitrary choice of another mathematics to teach. We have to demand technology to help us in making the change relevant to the needs of the future workers.

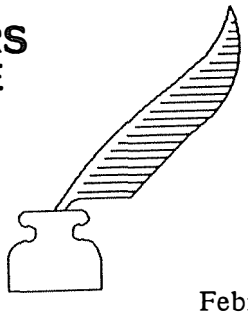
I say to you "Listen to the constructive criticism from colleagues and professionals. Realize that we are important in the education of our young people. We will be able to do the mathematics that swells to fill the gap left by the omission of manipulating algebraic expressions. Please get involved in helping us create new math curricula. The criticisms have been directed towards us. They are telling us that what we are doing is wrong but not what to do to change it. We are good problem-solvers. Let's all get involved in helping solve this problem. Don't be defensive and back off. Only we can make this change effective."

Before I say farewell, let me personally invite you to get involved this year in our OAME Annual and NCTM regional conference in Hamilton. We have a rare opportunity to hear from distant colleagues, their reactions to the criticism directed towards them and us. We'll have an opportunity to gain a more international perspective in conferencing with our partners in mathematics education from all over Canada and the United States.

See you there!

Mary Lou Kestell
President

LETTERS TO THE EDITOR



February 12, 1990

Dear Editor,

As chair of the Intermediate Senior Committee of OAME I am writing an open letter to members and colleagues soliciting responses to the Ministry of Education *Restructuring the Education System Action Plan, 1989/94* with respect to the **Transition Years**, grades 7 to 9. The Intermediate Senior committee is planning to prepare a position paper on the **Transition Years** to be presented to the Ministry of Education during the Consultation Process.

The committee has identified the following 10 topics which need to be addressed by the **Transition Years Working Group**:

1. Mathematics Curriculum Core Content and Changes Necessary 7 - 9 (Destreamed) and the Implications 10 to OAC
2. Teacher In-service and Pre-service Necessary to Implement Change.
3. Student Evaluation in the Transition
4. School Structure and Scheduling at the Elementary/Secondary Interface
5. Diploma Requirements and Compulsory Subjects
6. Staffing Models at the Elementary/ Secondary Interface
7. Policy Development, Ministry of Education Mandate to Boards and Funding
9. Impact on Vocational Programmes, French Immersion and Special Education (Gifted and Modified Basic)
10. Impact on Adult Education. Maturity Evaluation, Entry Into Formal Diploma Requirements, Adult Basic Literacy Educational Continuum

To assist the work of the Intermediate/Senior Committee we ask any person and/or group that would like to address one or more of these issues, or any other issue or concern to please send your comments to:

Ann Jones, Chair
Intermediate/Senior Committee OAME
c/o Ottawa Board of Education
330 Gilmour Street
Ottawa, Ontario
K2P 0P9

Thank you for your assistance.

Ann Jones, Chair,
Intermediate/Senior Committee, OAME

HIGHLIGHTS OF O.A.M.E.

1973 -1988

We have revised the booklet of Highlights of O.A.M.E. 1973 -1984. Anyone wishing this revised copy may request one from us. There is no charge either to the Association or to the members for this booklet.

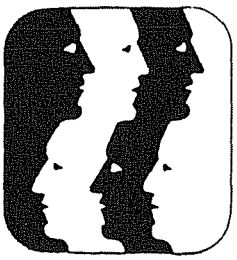
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Corrigendum

The teaching tip by Geoffrey Steel, page 30, in the December issue of *The Gazette* was incorrectly printed. The lines

$$\overrightarrow{OM} \cdot \overrightarrow{ON} = |\overrightarrow{OM}|\,|\overrightarrow{ON}|$$
$$\cos(A - B) = (\cos A, \sin A) \cdot (\cos B, \sin B)$$

should have read: $\overrightarrow{OM} \cdot \overrightarrow{ON} = |\overrightarrow{OM}|\,|\overrightarrow{ON}|\cos(A - B) = (\cos A, \sin A) \cdot (\cos B, \sin B)$
Our apologies to the author and readers.



Viewpoint



NEW MATHEMATICS IS AN ESSENTIAL PART OF NEW TECHNOLOGY

by
W. D. Burgess
Department of Mathematics
University of Ottawa

Every teacher of mathematics knows what reply to expect when his/her profession is mentioned: "I was never any good at math." or "Mathematics was really easy until trigonometry." Equally predictable is the response to the phrase "research in mathematics", it is something like "What is there still to be done?" or "I thought it was all worked out long ago." When this comes from people who have no contact with mathematics (or more exactly those who think they have no contact) it is not surprising. This reaction from university students, even those specializing in mathematics, is startling and disappointing. How is it that students who are quite aware of the rapid advance of sciences such as biology or physics believe that mathematics stopped developing sometime in the distant past?

A student who views mathematics as a subject which may be useful in the lively sciences but which is in itself a relic cannot be convinced that mathematical reasoning (which is after all a technique for establishing the truth of *new* mathematics) is worthwhile. What students call "theory" - by their definition anything with thinking involved - is for them just an old fashioned ritual. This attitude ill prepares high school students for university mathematics courses. It immediately puts them at odds with their university mathematics teachers and contributes to the disastrous results often found in the first year university calculus courses.

There is no simple remedy, and it is **not** suggested that new material be added to the curriculum. There are however two points which teachers of mathematics might emphasize in the context of the present programme: (i) that mathematics is growing and developing at a pace very much like that of the sciences, and (ii) that the profession of being a research mathematician is a lot like that of being any other sort of researcher.

Of course a great deal of mathematical advance comes from within the subject itself. This is somewhat inaccurately called "pure mathematics". The developments in pure mathematics very often get used in science and technology even if the applications were not the motivation. But by its nature, this type of research is very hard to explain to the non-initiated; even mathematicians who work in different areas cannot communicate with each other very well. It is on the practical side where the role of new mathematics is most approachable. This is not to say that we can explain to students the details, but then students do not need to understand recombinant DNA to know that biology is a rapidly evolving subject. So, too, it is not necessary to know what the Fast Fourier Transform is to be aware that it came about quite recently (in the 1960s) and is essential for all sorts of what are now every day things, like digital recordings, compact disks and various ways of filtering noise from signals.

Two very different sorts of practical problems have been tackled (although they are far from a complete solution) by new approaches in statistics, both involving new theory as well as new computational methods. One is the very important question of identifying a source of atmospheric pollution from the complex and varying mix of pollutants and natural substances from many sources. Another is the development of quality control strategies (how to test products efficiently in order to detect subtle production anomalies which are masked

by normal variations) in order to improve the quality of products while not increasing costs by unnecessary testing. New pollution control legislation or a car which works a bit better may not seem to contain new mathematics but they might well do. It would be helpful to our students if they were aware that this is the case.

The Voyager spacecraft is still sending back beautiful pictures and other data from the far reaches of the solar system. The signals broadcast by Voyager are very faint when they reach Earth and yet they can be filtered out from background electronic noise. Students may be startled to hear that the filters involved are not the radio-frequency equivalent of sunglasses but rather mathematical devices of recent invention which, with the essential help of all sorts of wondrous electronic devices and powerful computers, can pick information out of what seems like chaos.

Computerized x-ray tomography (CAT scans) can produce a visual image of what a slice through a human body would look like (a real slice would have brighter colours but would have a less benign effect on the patient). But neither the x-ray machines nor the computers, as marvellous as they are, can produce the picture. Once again an essential part of this device is some lovely mathematics of recent vintage, the discrete Radon transform.

The examples could be multiplied, from aircraft design to robotics. In all of them the end product is a result of advances in chemistry, electronics, software, engineering design, and so on, along with, just as essential as any others, mathematics.

How much new mathematics is being done? We get the same answer as when we ask how much new biology or electrical engineering is being done: "lots" and "more and more". The number of research papers published annually in mathematics is not as high as in the experimental sciences or in technology but Mathematical Reviews (MR), which publishes short reviews of mathematical publications, had 45945 items in 1988. Not all of these represent original research (nor is all the research discussed in MR necessarily brilliant) but most of them do. The above figure is accurate but what follows is guessing based on a bit of sampling in recent issues of MR. About 35 000 were reviews of research articles in specialized journals, and an average article might be 5 pages long (lengths vary from 1 page to a sobering 300 pages), so that 175 000 pages of mathematical research is published in a year - the equivalent of 350 fat books. No one keeps up with all of this; MR arranges the reviews under 94 headings and although a few remarkable mathematicians have something of an overview of the subject, most researchers work in one, two or maybe three of these 94 branches of mathematics.

Mathematicians being in most ways like other people, there are many of these pages which only represent wasted trees. To expect all mathematicians to do profound work is like expecting all musicians to be virtuosos. Yet there are mathematical Mozarts among the Salieris and in all those pages there is an amazing amount which is very clever, a great deal which is extremely useful and some which will change our world.

It is important that students know that mathematics is evolving rapidly, that it is essential for new technology and new knowledge, and that the evolution comes about, after preliminary, guessing, fiddling and trying, by reasoning and proving. What they think of contemptuously as "mere" theory impinges on their everyday lives and on their future. The world of mathematics is vast and exciting; the subject is not easy but some of the key ideas and ways of thinking are accessible to the willing student and indeed are found, in rudimentary form, in the school curriculum. The rigorous thinking which can (and should) be found in elementary mathematics is the same as that used in recent developments and our old friends, algebra, geometry and calculus, are the models for what comes later. The developments of the subject are not in the remote past nor in exotic countries. Important advances are taking place here in Ontario, as elsewhere in Canada. Most of the 94 branches are represented by active research in the universities, government ministries and industrial R and D laboratories in Ontario. Students should be told that our subject is vital, relevant and growing, and even if few of them later become part of it, most of them can have a sense of its worth.

MEANING: THE COMMON ELEMENT IN BOTH READING AND MATHEMATICS

by
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Good readers and good mathematicians, by virtue of their constant search for meaning, have a great deal in common. In an attempt to illustrate this commonality, two models (figures 1 and 2) have been developed and are presented in this paper. Through a comparison of the two models, implications for instruction in mathematics are highlighted.

The first model, entitled Reading for Meaning is best understood by viewing the large lower gear first. This gear represents the process that good readers use to put meaning into print. When good readers read, they are constantly predicting what they will be reading and they are constantly sampling and confirming what they have read. When their predictions are not as expected, they self correct quickly and continue reading. In some cases where predictions are well off the mark, they stop and reprocess what has already been read before continuing to read. In effect, readers continually attempt to construct meaning from the print. Good readers read in such a manner quiet naturally.

This ability to read for meaning is highly dependent on prior knowledge. For convenience, prior knowledge in reading had been divided into graphophonic, syntactic and semantic knowledge. For meaning to take place some knowledge in each area is essential. To make sense of a written passage, the reader must be able to read at least some of the words (graphophonic), must have some knowledge of the way in which the language is used (syntactic) and must have some understanding of the content of the passage being read (semantic knowledge). Although a base level is necessary for each of the three areas, semantic knowledge would appear to be the most important. For example, the reader may be able to read (decode) all the words in a passage, but if the meanings of the words are not known, there will be no way of getting meaning from the passage.

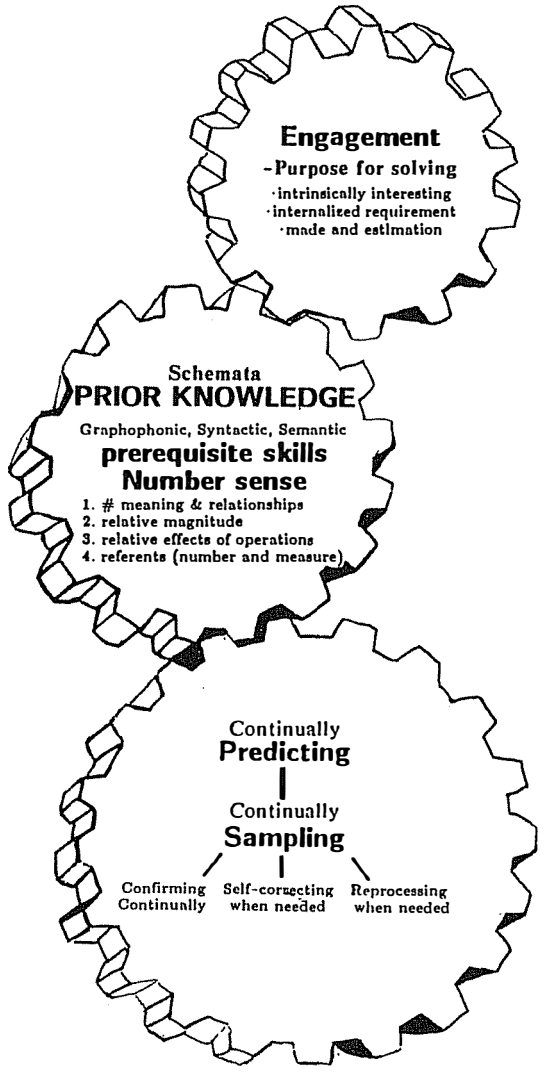


Figure 1. Reading For Meaning: A student is reading for meaning when he is engaged in the material he is reading and when he uses prior knowledge (graphophonic, syntactic, semantic) to help him process (predict, sample, confirm, self-correct, reprocess) what he is reading.

In order to read for meaning, students must have enough prior knowledge to predict what they are going to read. If students do not have the prior knowledge to make these predictions they will not be able to evaluate their

sampling. In turn, they will not be able to self correct small errors which may distort the meaning and they will not have any reason to reprocess material regardless of whether it makes sense or not. Although prior knowledge is essential to reading for meaning, it is not sufficient to merely posses such knowledge.

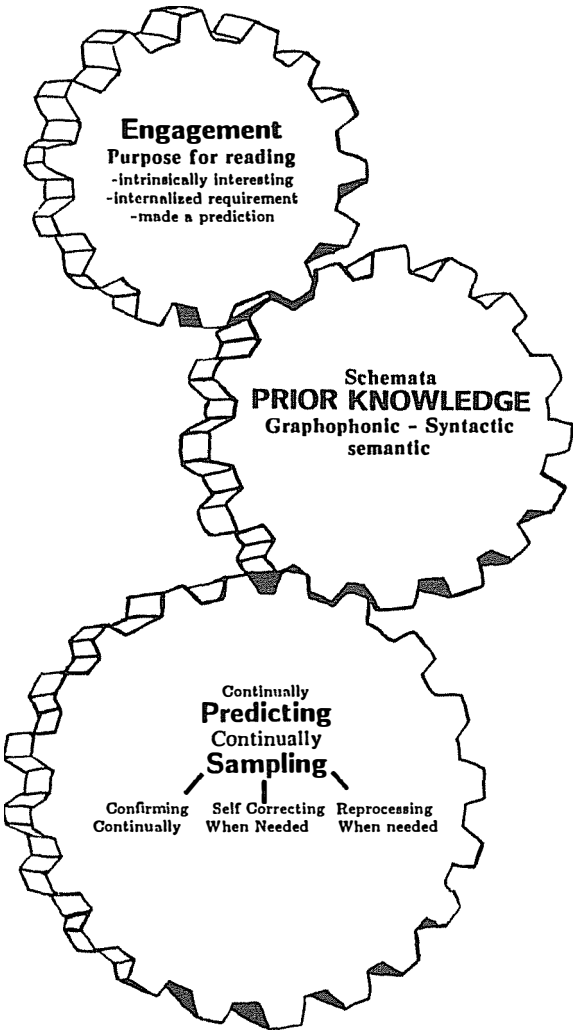


Figure 2. Solving with meaning: A students is solving with meaning when he is engaged in the problem and when he is using prior knopwledge (especially number sense) to help him solve the problem.

The reader must also be engaged in the reading in order to put the prior knowledge to use.To review the process of reading for meaning, as a whole, it can be seen in figure one that the small gear at the top represents engagement. This gear is critical because when it is engaged, the second gear, prior knowledge, is brought into play. In turn, this prior knowledge provides the framework or basis for the third gear, representing the predicting and sampling for meaning process, to come into play.

By examining figure two, it can be seen that a parallel between reading and mathematics has been made. Like reading for meaning, solving with meaning is dependent on the learner being engaged in the question being solved and it is dependent on the learner having the prior knowledge necessary to make sense of the question being answered. Although mathematics requires the same graphophonic, syntactic and semantic prior knowledge as in reading, certain prerequisite skills and number sense are also needed. For example, if a student was working on the question 17 X 5 it would be useful to know that 20 X 5 equalled 100 so that the student could predict that the answer would be near 100. With this prediction in mind, the answer calculated for 17 X 5 could be sampled. If the answer was 85 it could be concluded (confirmed) that the answer was reasonable because it was close to the prediction. If the answer was 8.5, however, the student would suspect something was wrong and would be prompted to check over the work and probably self correct the error. If a similar type process was followed in a problem solving context or within the context of a long algorithm, an unexpected answer might force the student to go back and reprocess much of the work that was done.

In both reading for meaning and solving with meaning, meaning is constructed when the student is engaged in the reading or solving to the extent that prior knowledge is brought to bear on the material being read or solved. Within these similar models, large differences exist when prior knowledge is examined. The prerequisite skills needed in mathematics could very easily be paralleled to the graphophonic, syntactic and semantic knowledge employed in reading for meaning. At the semantic level, common sense in reading could be compared to number sense in mathematics and at the syntactic level, word sentences could be compared to number sentences and algorithms. The grammar of language could also be loosely compared to the place value system in

mathematics. At the graphophonic level, numbers and words could be considered the basic units of meaning. This comparison could be made by equating basic number facts to words.

If the latter comparison is accepted, it can be seen that both mathematics and reading have important basic units. Many argue that these basic units must be automatic before higher learning can take place. For reading, this view has come under fire in recent years as a result of the movement towards the top down approach to reading.

In mathematics, however, most people would agree that basic number fact knowledge, if not essential, is a least a highly important aid in acquiring higher mathematics skills. Skemp elaborates on this point,

> ...it is indeed, essential that the elementary processes become automatic, thus freeing our attention to concentrate on the new ideas which are being learnt...
>
> (Skemp, 1971, p.83)

Although the reading for meaning and solving with meaning processes have parallels and appear to be much the same in terms of their overall functioning, the gear representing prior knowledge can be seen to be more critical in mathematics than in reading. One of the main reasons for this is the hierarchical nature of mathematics. For example, in mathematics the following two principles must be taken into account:

1. Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples.

2. Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner.

(Skemp, 1971, p.30)

Reading is also dependent on a hierarchy of skills (i.e. letters, sounds, words, sentences) but this would not appear to be as critical as in mathematics. For example, a person must be able to recognize some works before a sentence can be understood but does not have to recognize all the words in the sentence to understand the sentence. In reading, the component parts are not as essential as the component parts in mathematics. This is because of the abstractness and compactness of mathematical language where every number and symbol is critical to understanding.
When completing a mathematics question, accuracy is critical at all stages in that if even a small error is made at one point in a question the resulting answer will likely be wrong even if the rest of the question has been done correctly. The compactness of mathematics statements demand accuracy. For example, in the question $26 \div 5$, one hundred percent accuracy is necessary when interpreting the $\div$ sign. Although it may be acceptable to estimate the answer, it is not acceptable to guess what the $\div$ sign means. Unlike reading, where many redundancies exist, there is very little extraneous material in mathematical statements (excluding word problems).

In addition to demanding a high level of accuracy, the compactness of mathematical statements also demand a high level of abstraction. Using the example $26 \div 5$ once again, this statement is very abstract and does not bear any resemblance to any concrete example of 26 items being divided into 5 equal groups. When more complicated mathematical statements in algebra and calculus are encountered the level of abstraction is even more pronounced.

Complicating the mathematics picture is the issues of directionality. Students are taught to work from top to bottom and from left to right. Even in mathematics workbooks, students are encouraged to follow this pattern. When working with the algorithms, themselves, however, no simple rules of direction exist. If students learn algorithms by rote it is very hard for them to decide which direction to proceed in when doing algorithms.

In reading, unlike mathematics, it is possible to misunderstand or not know parts of a passage but through the use of contextual clues still get adequate comprehension from the passage as a whole. Reading has a built-in self correcting mechanism that improves as the student practices. The more students read the better readers they become. Even the reader who struggles initially will continue to

improve as more is read. Unfortunately, the same is not true of mathematics. Practice does not make perfect when incorrect procedures are practised. In fact, practising incorrect procedures often leads to extreme frustration.

In many cases, this frustration is heightened because of misconceptions a student may have. For example, in subtraction, if a student believes the lower number should be subtracted from the higher number regardless of placement, that student will continually make mistakes in subtraction regardless of the amount of practice. If the student is being taught subtraction with regrouping for the first time and the misconception is rectified little damage will be done but if the next topic is started and the misconception is not detected, considerable damage is imminent. When students employ an incorrect subtraction method in word problems they will be puzzled as to why they continually get some questions correct and some incorrect despite using the same procedure. If left undetected when the students reach division, the puzzlement grows even further. The students may find that although they have mastered the division algorithm perfectly and perform it the same way every time, they continue to get a great number of answers wrong (i.e. where regrouping is involved). Students who are unaware of their misconception often come to the conclusion that they are not good at mathematics or that mathematics is just too mysterious a subject to understand.



When the prior knowledge required for reading and mathematics is examined it becomes apparent that a great deal of the prior knowledge needed for reading is acquired naturally and continually, whereas much of the prior knowledge for mathematics is only acquired naturally in the early years. The significance of this is that much of the prior knowledge in reading is not dependent on reading instruction. Knowledge is developed everyday outside of the school and in every classroom inside the school. Virtually every experience a student has represents a contribution to his prior knowledge.

The experiences that go into the prior knowledge bank for reading are also potentially prior knowledge for mathematics, but most of these experiences are never realized. One reason for this is that people don't think in mathematical terms so they don't relate their experiences to mathematics. This includes teachers who comfortably promote language across the curriculum but fail to see math in any context other than math class. More significantly, many of the experiences which could potentially help to develop a person's knowledge bank or schematas cannot be integrated because the prerequisite concepts are not available to allow this integration.

If this is viewed in terms of figure two, the prior knowledge gear becomes inoperative when students reach the stage where the material they are being taught is no longer within their conceptual understanding. They are no longer able to use prior knowledge to put meaning into their work so they become forced to rely on rote learning. When this gear becomes inoperative no amount of energy coming from the engagement gear will promote solving with meaning until the pre-requisites are understood. The more the student tries, them more the wheels spin and frustration results.

For many students, this stage is reached at some point in high school and, despite graduating, they feel that mathematics is a very mysterious subject that they never really understood. For those students who reach this stage earlier, the consequences are likely to be more severe. When this stage is reached in elementary school, the result is often anxiety towards the avoidance of mathematics. Another necessary and critical part of the prior knowledge required for solving with meaning is number sense. Thompson and Rathmell speaking for the editorial panel of the February 1989 *Arithmetic Teacher* state that:

> The Editorial Panel believes that good number sense involves the development of understanding of the following: **(1)** number meanings and relationships, **(2)** the relative magnitude of numbers, **(3)** the relative effects of operations on

> numbers, and (4) referents for quantities and measures as numbers used in everyday situations. (p.2)

Good mathematicians use this number sense naturally just as good readers use common sense naturally.

The importance of number sense should be made explicit to both mathematics instructors and students. The reader not familiar with mathematics instruction might be inclined to say that children already use number sense because they automatically look to see if their answers make sense. Unfortunately, the author's own experience and the results of the Fourth NAEP (Carpenter et al., 1988) suggest that many students calculate mathematics questions in a total vacuum. Number sense, the mathematical equivalent of common sense seems to be excluded from much of the mathematics. Students often view mathematics questions as being divorced from reality and as a result do not use logic in mathematics. To illustrate this point, a grade 9 student gave the answer of 18 for the question 31 - 29 and then read the question and the answer aloud with confidence. When asked if this was true, the student seemed to turn on common sense. The student was then surprised at the answer given. Students should be taught number sense on an ongoing basis to build their framework (schemata) for integrating incoming information. If students are using number sense and answers are calculated that do not make sense they will stop and reprocess.

The importance of number sense cannot be overemphasized because it provides a frame of reference for students. For example, if students are working with fractions they will have a great deal of difficulty checking the reasonableness of answers for the question 1/3 X 1/4 if they do not expect the answer for this question to be smaller than 1/4.

It is also essential that number sense be applied. Often, students who possess number sense or who have prior knowledge in reading fail to apply their knowledge. This happens when students are not sufficiently engaged in the task and as a result they do not realize that the information being processed makes little sense. In terms of figures 1 and 2, the first gear is not sufficiently engaged to put the prior knowledge into play and, therefore, no framework is present for reading or solving with meaning to occur.

Both reading and mathematics instructors have devised ways to help engage students in their work. In reading, students have been encouraged to predict and ask questions before and during the act of reading while in mathematics students have been encouraged to estimate answers before calculating them. Predicting and estimating are both useful tools for engaging students because they give the students purpose. In mathematics, this is important because students are more accurate when they calculate with a purpose (Hope, J., 1989. p.12). In the situation described earlier where the grade 9 student gave the answer 18 for the question 31 - 29, the answer would not likely have been given if the student had been asked to estimate first.

Throughout this paper, an attempt has been made to illustrate that both reading for meaning and solving with meaning are dependent on students being engaged in their work to the extent that they are able to use their prior knowledge to help them put meaning into their work. Noting that prior knowledge in mathematics appears to be more critical than prior knowledge in reading, it would seem imperative that mathematics instructors take seriously the fact that new concepts are understood. Skemp elaborates on this point when he states that:

> ...in the building up of the structure of successive abstractions, if a particular level is imperfectly understood, everything from then on is in peril. This dependency is probably greater in mathematics than in any other subject.
>
> (Skemp, 1971, p.33)

Because prior knowledge in mathematics is largely acquired in the mathematics classroom, it is possible that the acquisition of prior knowledge in mathematics would stop if the students were being taught beyond their conceptual level. In reading, this would be an impossibility because prior knowledge is continually being acquired by students in almost every facet of their life. Mathematics instruction becomes far more critical at the earlier stages than reading instruction because there are few second chances in math, whereas, there are countless second chances in reading. This is best illustrated by the fact

that most students enjoy mathematics before entering school (Ginsburg, 1977, p.54; National Research Council, 1989, p.43) yet vast numbers of intelligent students grow to fear mathematics and never learn to master more than a rudimentary knowledge of it.

The implications for teaching mathematics would seem to be clear. Firstly, if concepts are taught before their prerequisites are mastered, it is likely that students will attempt to cope by using rote learning. Although this might work for a while, it will eventually fail and the chances are high that the students will develop a lifelong fear of mathematics. To prevent this from happening, students must be taught at the appropriate level with the appropriate materials for the appropriate length of time. Secondly, number sense should be developed and the necessity of its use made explicit. If mathematics is taught in a more meaningful manner and if the cloak of mystery is removed, it is likely that more students will succeed in mathematics and more will come to see it as an enjoyable form of learning.

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A **Perfect** Number Puzzle
by John Grant M^{c}Loughlin

1	2	3
4		
5		

Across:

1. This 3 digit number is both a perfect square and a perfect cube.
4. Different even numbers listed in decending order.
5. A perfect square.

Down:

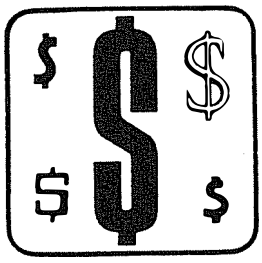
1. Different digits in ascending order.
2. A multiple of 19.
3. Each digit is a perfect square.

Answers on page **40**



A STOCK MARKET PROJECT

by
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This article offers a stock market unit which has been tested successfully with grade 8's as well as grade 9 enriched, advanced and general level classes. It includes the objectives of the unit; an outline; instruction sheets suitable for distribution to student; enough background information about the stock market to get a teacher with no previous knowledge in this area started; possible extensions to the unit; and a list of related activities. The Ministry of Education Intermediate/Senior Curriculum Guidelines for Mathematics have as one of their aims that instruction in this area should "develop an appreciation for the place of mathematics and its widespread application in our culture" (p.28). In selecting material to present to our classes we made sure that "the applications chosen should . . . relate to the interests of the students" (p. 19). One of the powerful motivators for the youth of today is MONEY.

The basic idea of the unit is to teach the students a core of information about the organization of the stock market, and then allow them to carry out transactions as if they were investing money in the Toronto Stock Exchange. They then follow its progress over a two to four week period. Getting started usually requires 3 or 4 full fifty minute classes and then a daily 5-10 minute maintenance period for subsequent days.

The original idea for the development of the project was found in *Foundations of Mathematics for Today and Tomorrow*, by D. Dottori, G. Knill and R. McVean. The chapter involving manipulation of rational numbers contains a series of questions based on stock market examples. The explanations available here and information found in any daily newspaper are sufficient background to get started.

Unit Objectives

Many different objectives can quite naturally be incorporated into the project. They might include the following:

- practice of fundamental math skills involving fractions and decimals
- basic knowledge of stock market operation and strategies
- risk-free increased insight into money management techniques
- opportunity to devise and test hypotheses in "real world" situations
- enjoyment!

Introduction of the Unit

To start the unit, we began by providing the students with the resources and understanding to explore and engage in Stock Market investment strategies. The first day we explained the basic organization of the stock market (private companies encourage the public to invest in their company by purchasing shares in the company, which may increase in value and yield profitable returns), some basic terminology needed to discuss Stock Market procedures (see table 1), and the strategy of buyers (to buy stock when it is priced low and sell it when the price rises).

To further develop these ideas, we encouraged students to bring in newspapers from home to view the listings given in the business section. To proceed with the project, they must understand the information given by the newspaper and be able to extract the data required to make educated decisions about which stocks to invest in.

Using the Sample Day provided (see table 2), questions were asked to review the terminology and test understanding of the goals of investment (a collection of possible

GENERAL TERMS

STOCK - The name of the company which has shares available for sale and purchase on the market.

SHARES - The actual sheets of paper which represent the proof of investment which are exchanged with each sale.

BUYER - The person who purchases shares in a company on the market.

SELLER - The person who starts off holding the shares and then sells them on the market.

PRICE PER SHARE (P.P.S.) - The cost of buying one share of a given stock.

IF STOCK TRADED

VOLUME - The number of shares that were bought and sold during the period discussed (usually given daily).

HIGH - The highest price per share the stock was sold at during the period discussed (day, month, year).

LOW - The lowest price paid per share for the stock during the period discussed.

CLOSE - The price paid per share before the market closed.

CHANGE - The price change per share from the previous period's close to the present period's close (usually daily).

IF STOCK NOT TRADED

BID - The highest price offered by a potential buyer for one share of the stock.

ASK - The lowest price requested by a potential seller for one share of the stock. (Must necessarily be higher than BID).

CLOSE - The price per share paid during the latest trading of the stock (this occurred on previous day).

Table 1

questions is shown in table 3). We found the concepts involved to be generally quite elementary and the majority of the students grasped the strategy quickly. For basic or general level classes, it was helpful to let the students work in groups of 2 or 3 to help them articulate some of the ideas orally as well as develop and analyze strategies together.

Once the students have a solid understanding of the terminology, the lay-out of the newspaper and the goals of the exercise, they are ready to be taught to work with the transaction sheets to record the purchases and sales of stocks.

Monitoring Progress of Stocks

This portion, involving the evaluation of price changes in the market as well as the purchasing and selling of stocks, represents the bulk of the project. Once the basics of this section have been taught to the students, most can work quite independently monitoring the activities in the market and determining a strategy to respond to the changes.

For the first two days of this phase, the students need substantial supervision and guidance to master the procedures used for the recording of market activity, but then class time required for maintenance declined dramatically, until only a 5-10 minute period was required daily for the students to keep track of their progress and modify the stocks that they own. Throughout this section it was helpful to encourage students to bring in newspapers for perusal. We found that it was only necessary to have a dozen or so newspapers available, since sharing a paper among several students can promote interesting discussions of strategies and add interest to the material.

Each student wrote an entry on the tally sheet for each day that purchasing and selling is allowed (a model sheet with sample entries is provided in table 4). Depending on the timetable in the school and the level of responsibility of the students in the class, the teacher may wish to encourage students to update their records each day that the Stock Market is open, or only on the days that the class meets. We found it worked best to have trading only on the days that the class was held.

STOCK EXCHANGE SUMMARY FOR SAMPLE DAY

STOCK	VOLUME	HIGH/BID	LOW/ASK	CLOSE	CHANGE
A	1000	40	30	40	+5
B	200	300	300	300	0
C	400	$21	$19	$19	-3
D	nt	250	300	270	
E	4000	150	140	145	0
F	nt	50	60	60	
G	3100	460	385	440	+30

Table 2.

The entry for each day has two portions: (1) An evaluation stage, when the current value of the stocks held is measured (a convenient convention for deciding which price to use is the closing price, since it is usually an average value of the stock for that day); and (2) the transaction stage, when stocks can be kept, sold and purchased. The tally sheet has a number of checks and balances that assist in eliminating computational errors, since it requires that each student monitor the net worth of his/her stocks at several points throughout the daily entry. The net value of a student's portfolio can only change from day to day, but not within a given day.

The first day is unique, since all the students start the project with the same amount of "cash" ($200.00 seemed to be a manageable amount), and they do not need to evaluate the present value of any stocks. The step-by-step instructions given to the students to follow is provided in table 5. In order for students to be able to practise all of the options on the second day, students were encourage to purchase shares from at least three different companies. This allows greater flexibility on subsequent days. One common error was to forget that all stock prices are given in cents unless a dollar sign ($) is shown.

On the second day, the students were guided through each of the options available with the stock of their choice. It is a valuable exercise to discuss which stocks should be kept, which might be sold and how to select new stocks. Since there is no "right" time to make a move on the Stock Market in absolute terms, it provides a forum for giving students initial direction on how to approach the goal of maximizing the net worth of the stocks held and also discuss alternate strategies. They gained experience with the entries required to keep and sell some of the stock, as well as to

SAMPLE QUESTIONS
(BASED ON SAMPLE DAY)

1. Which is the most expensive stock at the present time?
   (C - it is in dollars. The rest are in cents)
2. Which is the least expensive stock at the present time?
   (A)
3. How much would it cost to buy each of the following: (do not consider any stock broker's commission)

a) 20 shares of stock B at close price?
   (20 X 300 cents = 6000 cents = $60)

b) 10 shares of stock G at the low price?
   (10 X 385 cents = $38.50)

c) 20 shares of stock C at the high price?
   (20 X $21 = $420)

4. Which stock showed the greatest gain over the last day?
   (G - Price rose 30 cents/share)
5. Which stock showed the greatest loss over the last day?
   (C - Price dropped $3/share)
6. Explain what the "4000" means under stock E?
   (4000 shares were sold during the day)
7. Explain what the "30" means under stock A?
   (The day's lowest price was $.30/share)
8. Explain what the "250" means under stock D?
   (A buyer's highest offer was $2.50/share)

Table 3.

SAMPLE ENTRY SHEET

DATE	STOCKS KEPT NAME	#xPPS	VALUE	STOCKS SOLD NAME	#xPPS	VALUE	STOCKS BOUGHT NAME	3xPPS	VALUE	TOTAL/VALUE OF CASH on hand	STOCKS HELD	NET TOTAL
Day 1										$200		$200.00
							A	20x490	$98.00	$102		
							B	200x30	$60.00	$ 42		
							C	100x40	$40.00	$ 2	$198.00	$200.00
Day 2	A	20x500	$100.00									
	B	200x27	$ 54.00							$ 2	$194.00	$196.00
	C	100x40	$ 40.00									
	A	20x500	$100.00							$ 29		
	B	100x27	$ 27.00	B	100x27	$27.00				$ 69		
				C	100x40	$40.00						
							D	100x65	$65.00	$ 4	$192.00	$196.00
Day 3	A	20x___										
	B	100x___								$ 4		
	D	100x___										

Table 4.

buy shares of additional stock. A copy of the handout given to students detailing the evaluation stage and steps for the transaction stage for all subsequent days is given in table 6. Once the project was under way, the teacher

DAY 1 INSTRUCTIONS FOR TALLY SHEET

EVALUATION STAGE

1. Record $200 in "Cash" column and "Net Total", before drawing a single line across page.

TRANSACTION STAGE

2. Choose a stock listed in the Toronto Stock Exchange listings that you are interested in purchasing.

3. Buy as many shares as you wish, and record this information under the "Stocks Bought" heading.

4. Calculate the price of the shares you wish to purchase. i.e. (# x P.P.S.) = (number of shares) x (price per share).

5. Subtract the price of the shares from the cash column, and record your new "Cash" total.

6. If you have sufficient cash left and wish to purchase more stocks, repeat steps 2 to 5 as many times as desired.

7. Once you have purchased all the stocks you wish, calculate the following values for the summary line:

"Total in Stocks" (add all of the entries in the stocks bought value column.)

"Total in Cash" (the last value in the "Cash" column)

"Net Total" (add "Total in Stocks" and "Total in Cash")

N.B. The net total must add to $200.

8. Draw a double line across the page.

Table 5.

was required to provide fewer step-by-step directions, and could move around the classroom making sure that all students were following the procedures outlined and providing encouragement and praise for the successes found either in increases in net worth or well presented work. Since there is generally a great deal of luck involved in the success of a student's investment strategies, we felt that there should be no penalty for selecting stocks that decline in value, but encouragement was given to those students who successfully managed to increase the value of their portfolios. Once we were satisfied that particular students had masteredthe tally procedure, they were encouraged to find creative ways of improving the look of the tally sheet or refining the recording procedures. Several creative suggestions were made to provide additional checks and make record keeping more efficient. Table 4 incorporates some of the ideas the students developed themselves.

The project continued until we saw that a portion of the class had started to hold onto the stocks that they own, and the number of new purchases declined. To conclude the project, the class was informed of the date of the final day of the "Class Stock Market", when all stocks had to be sold. (We found a good length for the project was between 2 to 4 weeks.) This was done several days in advance to allow for last minute planning. Once the students had tabulated the final value of their portfolios, they were given a few days to piece together the project before handing it in for evaluation. An interesting addition

might be for the students to do a guided self-evaluation of their stock market purchasing strategy and their interest in the project.

Possible Extensions to the Project

Through discussion with other teachers, the following suggestions would make the project more challenging and incorporate some additional mathematical concepts:

- using statistical graphing techniques (such as line or bar graph) to follow the progress of stock prices during the project period
- adding columns to the tally sheet to include the calculation of percentage increase or decrease of stock prices
- having students calculate the amount and percent of profit on each day of transactions
- adding transaction fees (either fixed or stock price dependent)

Related Activities to Supplement the Unit

If you wish to enhance the basic stock market project outlined, there are a number of related activities and exercises which might serve to provide a greater depth of understanding about the concepts involved.

1. Visit a stock broker, or the Toronto Stock Exchange.
2. Invite a guest speaker to the class: a stock broker, a mutual fund representative, or a knowledgeable parent who invests in the stock market.
3. Students could present orally, or in project form, the solutions to any one of the following problems associated with stock market investment:
   a) How should a person decide whether or not to invest in stocks?
   b) What information about a company should you have before you invest in its stock?

SUBSEQUENT DAYS INSTRUCTIONS FOR TALLY SHEET

EVALUATION

1. Enter into "Stock Kept" column all stocks you presently own (i.e. those appearing in last day's "Stocks Kept" and "Stocks Bought" columns)
2. Calculate their new values by multiplying the number of shares owned by the updated price per share found in the newspaper. Record amount under heading "Value".
3. Total all entries under "Value" heading and record this value under "Total in Stocks", and transfer the cash down from the end of the previous day and record this in the "Total in Cash" column.
4. Calculate "Net Total" for day (add "Total in Stock" to "Total in Cash", then draw a single line below this statement.)

TRANSACTION STAGE

5. Now you are free to carry out the transactions of the day. Use a separate line for each stock bought or sold. Remember you must deal with each stock you own.

Follow the appropriate instructions below:

To keep some shares:

A. Fill in the "Stocks Kept" column with the number of shares you wish to keep.
B. Calculate the "Value" of the desired shares, i.e. (# of shares) x (price per share)

To sell some shares:

A. Decide on the number of shares to be sold (Remember, the number of shares kept plus the number sold must equal the total owned at the start of the day.)
B. Enter the number and the price per share in the column marked "# x p.p.s." of "Stocks Sold".
C. Calculate the value of "Stocks Sold" and record it.
D. Update "Cash" column by adding amount of money earned through the sale to the last amount in the "Cash" column.

To buy some shares:

A. Select a stock from the current newspaper.
B. Decide on the number of shares to be purchased and calculate the "Value" using the "# x p.p.s." entry.
C. Update the "Cash" column by subtracting the amount in the "Value" column from the previous "Cash" balance.

6. For the summary line, calculate the "Total in Stock" amount by adding all entries in the "Value" column of "Stock Kept" and "Stock Bought" below the single line.
7. Copy the "Cash" value to the summary line.
8. Calculate the "Net Total" by adding the two "Total" columns. N.B. this must be the same value as calculated during the evaluation stage.
9. Draw a double line below the summary line.

Table 6.

c) What is the job of a stock broker?
d) How many stocks make up the T.S.E. Industrial Index? What are they? What is the role of the index?
e) How many stocks are used to compose the Dow Jones Index? What are they? What is the role of the index?
f) Investigate Mutual Funds. What are the advantages and disadvantages of investing in the stock market in this manner? On the basis of these findings, who should invest in these type of stocks?

4. Use one of the many stock market games that are available through most school boards.

Conclusions

This Stock Market Project has the potential to provide students at the intermediate level with an opportunity to broaden their understanding and to see elementary mathematics used in a real world application. It also provides a forum for practising necessary mathematics skills in an interesting manner. The vast majority of students at all levels involved in the classes used to develop and test this unit found it informative, challenging and rewarding. We feel that teaching students through practical applications which they find interesting and relevant to their lives is one of our major goals as educators. This stock market project might be a valuable supplement to the intermediate teacher's resources.

Teaching Tips

EVALUATION TECHNIQUES

Three of the teaching ideas and techniques I have found to be most useful additions to my repertoire in the last few years are:

☞ Give a short test or quiz after every exercise, class work or homework. Mark it as the students complete it. Make sure the faster ones have more work to entertain them. Return the test and take it up immediately. Record the marks, and count them as part of the term mark. This gives students immediate feed-back on their knowledge and understanding. It prepares them for the kind of questions to be asked on a major test to be given later on the same topic. It gives them a chance to remedy deficiencies in their knowledge before such a test. It gives the teacher lots of marks for assessment.

☞ Record marks on a computer. The difference in the computation time preparing for reports is incredible. The flexibility of weighting different aspects of the evaluation techniques is a delight.

☞ In order to prepare for anecdotal reports, I keep a record of my reactions to my students work as I mark the two tests prior to the report. This gives me much more confidence that my comment on the report makes sense, especially earlier in the year when I do not know the students well.

Suggested by Tom Griffiths
A. B. Lucas Secondary School
London, Ontario

THREE GEOMETRIC ILLUSTRATIONS OF THE LIMIT CONCEPT

by
Edward J. Barbeau
University College
University of Toronto

In calculus, limits are used to ferret out information not directly obtainable because of some degeneracy. For example, the slope of a tangent is found as the limit of slopes of secants. One might help students get a better concept of "going to the limit" by using geometrical examples. This note will provide three of these.

1. Collapsing an isosceles triangle

BC is the fixed base of an isosceles triangle ABC with angles B and C equal. The point P is the intersection of the side AC with the bisector of angle B. Suppose the altitude of the triangle from A tends to zero, with the result that A tends towards the midpoint M of BC. Does P have a limiting position? If so, what is it?

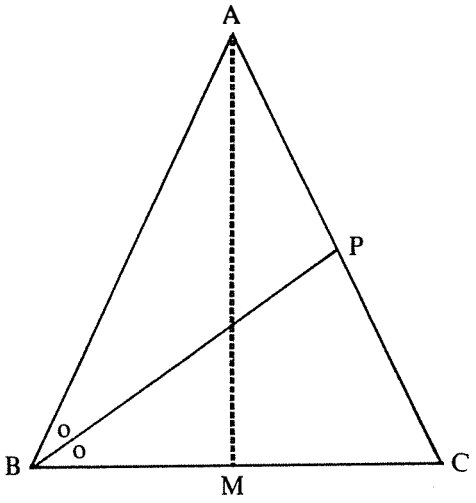


Figure 1

An analytic geometry treatment is possible. However, a synthetic (Euclidean) approach is more striking in that it highlights a relation stable over the variation of A which allows us to readily identify the limiting point. First, we need the result that, since BP bisects ∠ABC, AB : AP = BC : CP. To see this, suppose that D is the point on BP produced such that ∠CDP = ∠CPD.

Then triangles BAP and BCD are similar, with the result that AB : AP = BC : CP. Rewrite this proportion as:

$$BA : BC = AP : PC \qquad ...(1)$$

As A approaches M, P is brought ever closer to the side BC. The left side of (1) approaches the ratio BM : BC, or 1 : 2. Accordingly, AP : PC also approaches 1 : 2, with the consequence that P gets closer and closer to the point Q on BC for which

$$MQ : QC = 1 : 2.$$

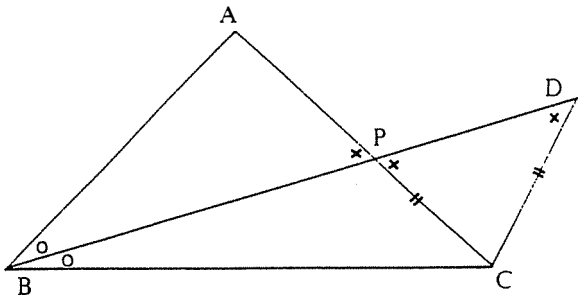


Figure 2.

Thus, we can see that, although the bisector of B and the side AC collapse into the same line as A approaches M, the point P of intersection has a limiting position Q which can be determined from the stable relationship (1). Indeed, Q is on BC and BQ = 2 QC.

In a class, the students might be invited to guess what the answer is, and then investigate the situation by means of diagrams, or, if available, computer graphics.

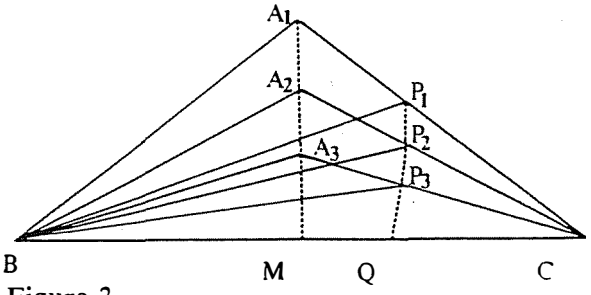


Figure 3.

2. Shrinking a semicircle

Let V be a fixed semicircle with diameter OP and U a variable semicircle with centre O and diameter collinear with OP. The tangent to V through O meets the semicircle U at A. The semicircles U and V intersect at B. The line through AB intersects OP produced at C. What happens to the point C as the radius of U shrinks to zero?

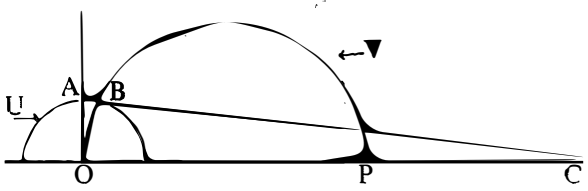


Figure 4.

Your students may have trouble guessing the answer, which is by no means intuitively obvious. While the result can be obtained by some straightforward analytic geometry with a little surd manipulation, there is an elegant synthetic treatment.

Let E be the reflected image of A in the axis OP, so that AE is a diameter of a circle with centre O.

Then $\angle ABE = 90°$. $\therefore \angle EBC = 90° = \angle OBP$ (subtended by diameter OP). It follows that $\angle OBE = \angle PBC$.

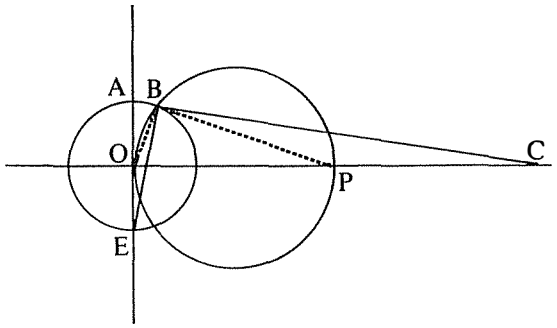


Figure 5.

Also $\angle AEB = 90° - \angle EAB = \angle OCA$. Since triangle BOE is isosceles, so is triangle BPC, with the result that BP = PC. As the radius of U tends to zero, the point B tends to O, so that the length of BP tends to the length of OP. Therefore, the point C must approach a point Q on OP produced for which OP = PQ. Thus, the limiting position of C is twice as far from O as P.

3. Coaxial rays rebounding from a spherical mirror

A ray of light parallel to the axis of a parabolic mirror will always reflect through a single point (the focus) on the axis. This is not true for a spherical mirror. However, there is a rule of thumb familiar to your physics students, that for a spherical mirror, a ray parallel and relatively close to the axis will reflect through that point on the axis halfway between the centre and the mirror, which becomes an approximate *focus*.

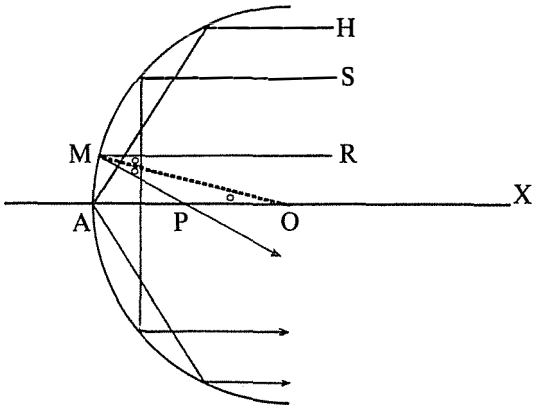


Figure 6.

In the figure, we show a typical ray from R hitting the mirror at M and reflecting across the axis AX at P. Note that, by a principle in optics, the angle of incidence equals the angle of reflection. If the ray comes in along the axis itself, then the reflected ray and axis coincide and the intersection point of ray and axis degenerate. However, as R is taken closer and closer to the axis, it is easy to see that the limiting position of P is midway between the centre O of the circle and the point A where circle and axis meet. Simply note that:

$\angle MOA = \angle RMO$ (parallel lines)
and $\angle RMO = \angle PMO$ (reflection),
so that MP = PO.

As a project, students might be encouraged to examine how the position of P varies with the distance of R from the axis.
Trigonometrically, this can be expressed by the equation

$$OP = \tfrac{1}{2}(MO) \sec \theta$$

where θ is $\angle MOP$. Two configurations of special interest occur when the ray comes in along the side of an inscribed square and the side of an inscribed regular hexagon of the circle.

The limit concept is not easy for students to grasp, and it is well worth classroom time to explore it through numerous examples. While the pocket calculator makes numerical investigations easy, the foregoing cases indicate that geometry too can help clarify ideas.



ARE YOU AN "OLD" MATHEMATICS TEACHER?
(OR A DESCENDENT OF ONE)

Joan Routledge (416-727-4167) and Jack LeSage (705-728-9958) have been asked by the Ontario Association for Mathematics Education to produce a History of Mathematics Associations in Ontario to celebrate 100 years.

If you have any information, convention programmes, pictures, minutes, vignettes, etc involving any past Mathematics organizations in Ontario (math sections of OEA, OATMP, OATM, OMC, OCI, etc) please write:

Jack LeSage



O.A.M.E./A.O.E.M

APPLICATIONS OF MECHANICS TO GEOMETRY

by
John J. Del Grande
Ontario Institute for Studies in Education.
Toronto, Ontario

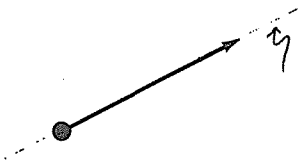


(Condensed version of a talk "With Love From Russia" given at the O.A.M.E., Ottawa, 1989).

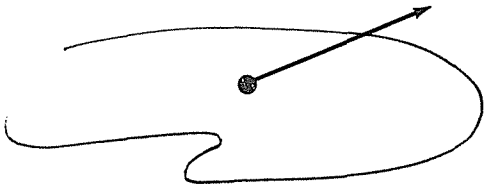
It is customary to find mathematical applications to problems in science but in a rather surprising and interesting approach the Russian author B. Yu Kogan illustrated a novel way of applying some principles of physics to prove some theorems in geometry. These principles are readily accessible to most students taking physics and can arise in the study of vectors in mathematics.

ASSUMPTIONS

1. A **force** can be represented by a vector which describes its magnitude, direction and point of application. The line along which the force acts is called its **line of action.**



2. A **rigid body** is one that keeps its size and shape when a force is applied to it.



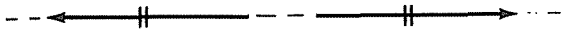
3. A collection of forces is called a **system of forces.** A system of forces is in **equilibrium** if no motion is caused when the system of forces is applied to a body at rest.

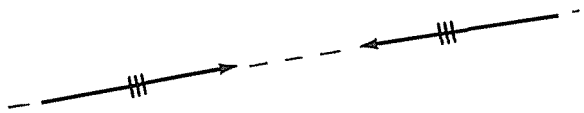
4. Two systems of forces are **equivalent** if they cause the same motion when applied to a rigid body. A system of forces acting on a rigid body can be replaced by an equivalent system without altering a proof or argument.

5. A **resultant** of a system of forces is a single force that is equivalent to the system.

RULES

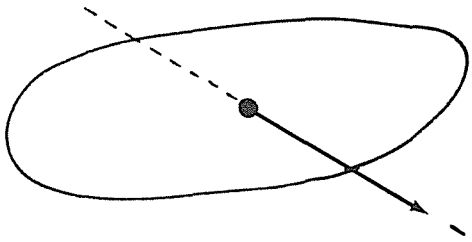
1. **Two forces are in equilibrium** if and only if they have the same magnitude, opposite direction and have the same line of action.





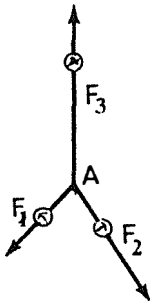
2. A force acting on a rigid body can be arbitrarily shifted along its line of action. That is, if $\mathbf{F}_1$ and $\mathbf{F}_2$ have the same magnitude, direction and act along the same line then they are **equivalent.**

This implies that if a force acts on a rigid body the point of application is not important - the force can act at any point on the line of action. Such forces are said to be represented by a **sliding vector.**



3. The resultant of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at a point is the diagonal vector of a parallelogram having $\mathbf{F}_1$ and $\mathbf{F}_2$ as adjacent sides. This property can be illustrated by the following experiment.

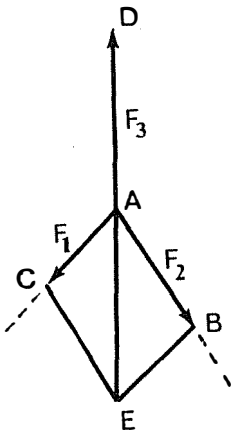
Three strings are attached to a point A and forces $\mathbf{F_1}$, $\mathbf{F_2}$, and $\mathbf{F_3}$ are applied to the string. The magnitudes of the forces are recorded when the system is in equilibrium.



A drawing is made showing the point A and the line of action of the three forces.

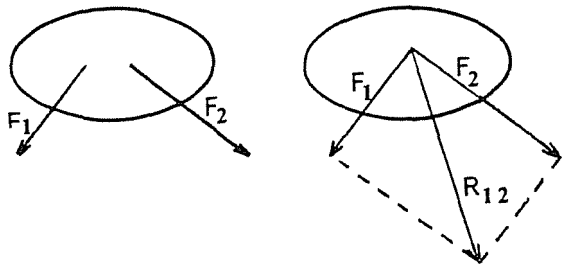
Let AC represent $\mathbf{F_1}$ in direction and magnitude, AB represent $\mathbf{F_2}$ in direction and magnitude, AD represent $\mathbf{F_3}$ in direction and magnitude.

Complete the parallelogram ACEB. It is found that AE = AD and A, E and D are collinear.

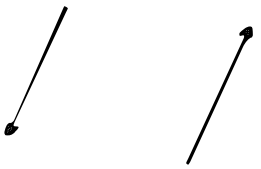


Since the system is in equilibrium the force represented by AE is equal and opposite to the force represented by AD.

Since $\mathbf{F_1}$ and $\mathbf{F_2}$ balance $\mathbf{F_3}$, the force represented by AE balances $\mathbf{F_3}$. The force represented by AE is equivalent to the two forces $\mathbf{F_1}$ and $\mathbf{F_2}$ combined. We call the force represented by AE, $\mathbf{R_{12}}$, - the resultant of $\mathbf{F_1}$ and $\mathbf{F_2}$. $\mathbf{R_{12}}$ is formed by drawing the diagonal of a parallelogram with $\mathbf{F_1}$ and $\mathbf{F_2}$ as adjacent sides. Two forces not acting at a point have a resultant if their lines of action intersect.

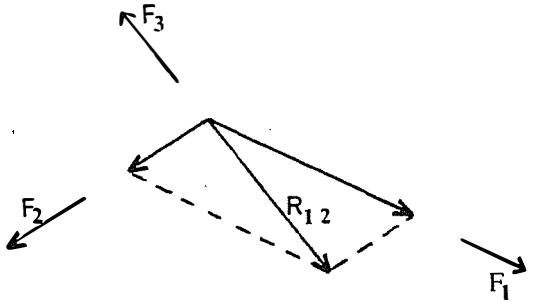


Note: not every system has a resultant. For example two non-collinear forces equal in magnitude, opposite in direction and parallel form a **couple** and do not have a resultant.



4. If we add or remove a system of forces which are in equilibrium to or from another system of forces, the resulting system will be equivalent to the original one.
Thus a collection of systems of forces, each in equilibrium, is a system of forces in equilibrium.

5. If three nonparallel forces acting on a rigid body are in equilibrium, then their lines of action are concurrent.

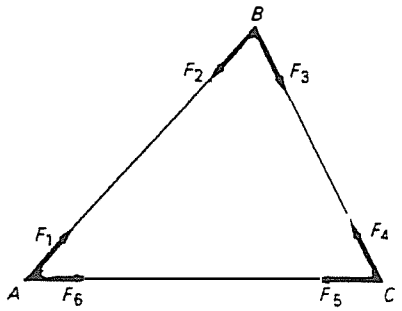


For example consider three forces F_1, F_2, and F_3 in equilibrium then the resultant of F_1 and F_2, say R_{12}, can replace F_1 and F_2. Thus, R_{12}, F_1 and F_2 pass through a common point O. But R_{12} and F_3 are now in equilibrium and occurs only if R_{12} and F_3 are equal, opposite and collinear. Thus F_3 passes through O the point of intersection of F_1 and F_2. Thus, F_1, F_2 and F_3 are concurrent.

THEOREM

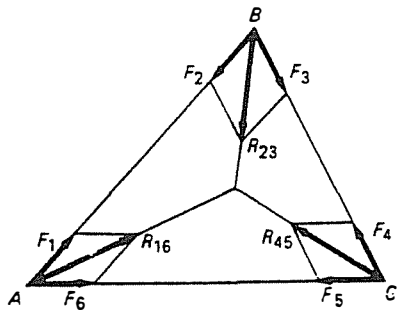
The bisector of the interior angles of a triangle are concurrent.

Let F_1, F_2, F_3, F_4, F_5, and F_6 be six forces, equal in magnitude acting along the sides of ABC as shown.



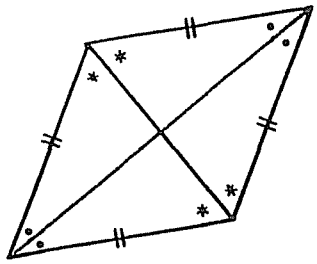
Since F_1 and F_2 are collinear and $F_1 = -F_2$
$\therefore$ F_1 and F_2 are in equilibrium.
Similarly F_3 and F_4 are in equilibrium, and F_5 and F_6 are in equilibrium.

The collection of systems (F_1,F_2), (F_3,F_4) and (F_5,F_6) is in equilibrium
Replace F_1 and F_6 with their resultant R_{16}.
Replace F_2 and F_3 with their resultant R_{23}.
Replace F_4 and F_5 with their resultant R_{45}.
$\therefore$ R_{16}, R_{23} and R_{45}, are in equilibrium and their lines of action are concurrent.



Since $|F_1| = |F_6|$, R_{16} is the diagonal of a rhombus with F_1 and F_6 as adjacent sides.

However, the diagonals of a rhombus are angle bisectors of the rhombus.



$\therefore$ the diagonal, R_{16}, of the rhombus is an angle bisector of $\angle BAC$.

Similarly, R_{23} and R_{45} are angle bisectors.
$\therefore$ the bisectors of the angles of $\triangle ABC$ are concurrent.

THEOREM

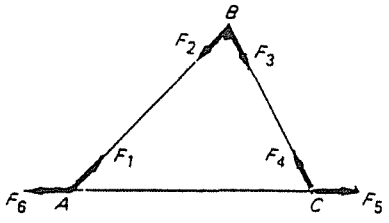
The bisectors of two exterior angles and an interior angle of a triangle are concurrent.

Select 6 forces in equilibrium so that the resultants of the forces in pairs act along the bisectors of the required angles.
Let F_1, F_2, F_3, F_4, F_5, and F_6 be the six forces where

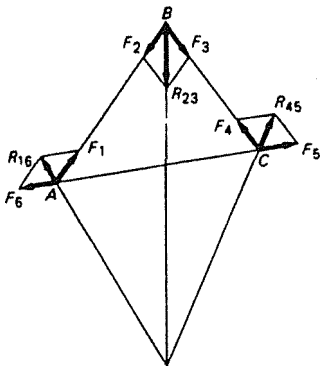
$$|F_1| = |F_2| = |F_3| = |F_4| = |F_5| = |F_6|$$

and

$$F_1 = -F_2,$$
$$F_3 = -F_4,$$
$$F_5 = -F_6$$



Use the following diagram to complete the proof.



THEOREM

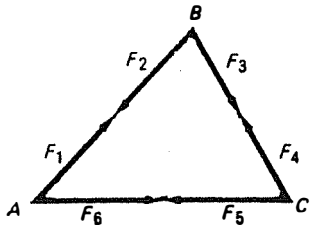
The medians of a triangle are concurrent.

Select 6 forces in equilibrium so that the resultants of the forces in pairs act along the medians of the triangle.

Let the forces have magnitude equal to one-half the length of the corresponding side of the triangle.

Therefore:

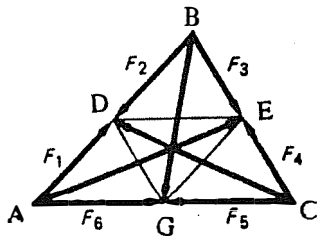
$|\mathbf{F}_1| = |\mathbf{F}_2|$ and $\mathbf{F}_1 = -\mathbf{F}_2$
$|\mathbf{F}_3| = |\mathbf{F}_4|$ $\mathbf{F}_3 = -\mathbf{F}_4$
$|\mathbf{F}_5| = |\mathbf{F}_6|$ $\mathbf{F}_5 = -\mathbf{F}_6$



D,E and G are the midpoints of the sides. ADEG is a parallelogram. The resultant $\mathbf{R}_{16}$ of $\mathbf{F}_1$ and $\mathbf{F}_6$ is a diagonal of the parallelogram and acts along the median of the triangle.
Similarly $\mathbf{R}_{23}$ and $\mathbf{R}_{45}$ are medians.

Since $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_6$ are in equilibrium, then $\mathbf{R}_{16}$, $\mathbf{R}_{23}$ and $\mathbf{R}_{45}$ are in equilibrium.

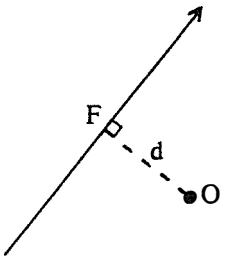
Therefore the lines of action of $\mathbf{R}_{16}$, $\mathbf{R}_{23}$ and $\mathbf{R}_{45}$ are concurrent.
Therefore the medians of triangle ABC are concurrent.



DEFINITIONS

The moment of a force about a point O is the measure of the turning effect or torque of the force about O.

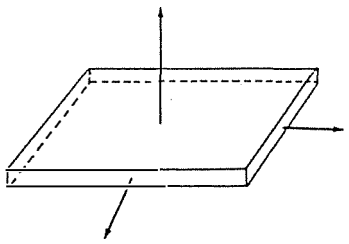
The magnitude of the moment of a force **F** about a point O is $|\mathbf{F}|d$ where d is the perpendicular distance from O to the line of action of **F**.



Moments can produce **clockwise** or **counter-clockwise** turning effects. For equilibrium, the sum of the clockwise moments equals the sum of the counterclockwise moments.

PROPERTIES OF PRESSURE ON THE FACES OF A VESSEL CONTAINING GAS UNDER PRESSURE

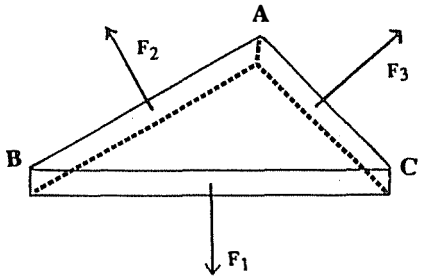
If a thin rectilinear container is filled with a gas under pressure:
- a total pressure on a face is proportional to the area of the face
- the total pressure acts perpendicular to the face
- the total pressure acts at the geometric centre of the face



THEOREM

The perpendicular bisectors of the sides of a triangle are concurrent.

Consider a very thin triangular container filled with gas under pressure.



Let the total pressures on the faces be the forces $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{F_3}$ acting at the centres of their respective faces.

For a very thin container $\mathbf{F_1}$ acts at the midpoint of BC and is perpendicular to it. The line of action of $\mathbf{F_1}$ is along the perpendicular bisector of BC.
Similarly the line of action of $\mathbf{F_2}$ is along the perpendicular bisector of AB, and the line of action of $\mathbf{F_3}$ is along the perpendicular bisector of AC.

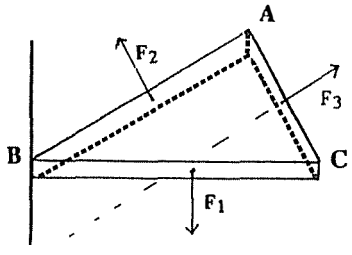
Since the container is in equilibrium under the action of these three forces, then the lines of action of $\mathbf{F_1}$, $\mathbf{F_2}$ and **F3** are concurrent.

The perpendicular bisectors of the sides of a triangle are concurrent.

THEOREM

The Pythagorean theorem.

Consider a thin triangular container filled with gas under pressure. Let the top face of the container be $\triangle ABC$ with a right angle at A.



Let the total pressure on the faces be the forces $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{F_3}$ which act at the centres of their respective faces and at right angles to BC, AB, and AC respectively.

If the height of the container is h, we have

$\mathbf{F_1} = k\ BC,\ \ \mathbf{F_2} = k\ AB,\ \ \mathbf{F_3} = k\ AC$

Take moments about a line perpendicular to the plane in which triangle ABC lies.
$\mathbf{F_1}$ acts at a distance $\frac{1}{2}BC$ from the line
$\mathbf{F_2}$ acts at a distance $\frac{1}{2}AB$ from the line
Since $\mathbf{F_3}$ is perpendicular to AC and $\angle BAC = 90°$
$\mathbf{F_3}$ acts at a distance $\frac{1}{2}AC$ from the line.

$\therefore \mathbf{F_1} \times \frac{1}{2}BC = \mathbf{F_2} \times \frac{1}{2}AB + \mathbf{F_3} \times \frac{1}{2}AC$
$\therefore kBC \times \frac{1}{2}BC = kAB \times \frac{1}{2}AB + kAC \times \frac{1}{2}AC$
$\therefore BC^2 = AB^2 + AC^2$

The author, Kogan, points out that careful investigation reveals that none of the proofs involve circular reasoning. For example, in the theorem of Pythagoras only the law of moments is assumed which in no way relies on the Pythagorean relationship.

The book contains much more than illustrated in this article. For example, the concepts of potential energy and centre of gravity are used to develop formulae for the volume of solids that are usually derived in calculus.

Reference:

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MATHEMATICS ACHIEVEMENT OF BOYS AND GIRLS: AN INTERNATIONAL PERSPECTIVE

by
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Editor's note: This article originally appeared in Mathematics 4 Girls, published by the Ontario Education Research Council in 1988. It is reprinted here with permission.

In the past two decades researchers have shown considerable interest in the relationship between the gender and the mathematics achievement of children in the upper grades of elementary school. Many studies have shown that, by 13 years of age, boys are significantly superior to girls in both their mathematical performance and their attitude towards mathematics.[1] The male advantage is especially pronounced among high-scoring, exceptionally-gifted children, with boys outnumbering girls 13 to 1.[2] However, an examination of research carried out in nine countries indicated that gender-related differences in achievement vary considerably both within and among countries.[3]

Information on gender differences in mathematics achievement is now available from a large-scale study conducted by the International Association for the Evaluation of Educational Achievement (IEA). This research, called the Second International Mathematics Study, used a very large, stratified, random sample of schoolchildren in twenty participating "countries". As elementary education is a provincial concern in Canada, two Canadian provinces, Ontario and British Columbia, were designated "countries" for the purpose of the research. Table 1 lists the "countries" and the numbers of boys and girls involved in the project.

Children from North America, Europe, Africa and Asia were included in the sample. It can be seen from the geographical distribution of the research subjects that the language of instruction and school organization varied considerably. However, the children were comparable in terms of age. The grade in which the modal age was 13 years was chosen in each country. In the United States and Canada, boys and girls in grade 8 participated in this research.

The IEA survey sought detailed information from each of the participating countries on three interrelated aspects of mathematics teaching: the intended curriculum, the implemented curriculum, and the attained curriculum. A brief description of each follows: (1) The intended curriculum is that

Participants in the International Study

Country	Girls	Boys
Belgium:		
Flemish	718	652
French	874	1001
Canada:		
B. C.	1080	1066
Ontario	2401	2421
England	1338	1247
Finland	2091	2303
France	4629	3586
Hong Kong	2726	2822
Hungary	908	844
Isreal	1651	1711
Japan	3924	4167
Luxembourg	1016	989
Netherlands	2667	2769
New Zealand	2574	2629
Nigeria	389	1040
Scotland	627	729
Swaziland	485	414
Sweden	1660	1830
Thailand	1833	1988
U.S.A.	3452	3202
Total	**37043**	**37410**
Percentage	**49.8%**	**50.2%**

Table 1

reflected in curriculum guides, course outlines, syllabi and textbooks adopted by the educational system. In most countries,

nationally defined curricula emanate from a ministry of education or similar national body. In other countries, such as the United States of America, intended goals or specifications of curricular content are often developed by state departments of education or local school districts. (2) The implemented curriculum is that actually taught in the classroom. Clearly, the teachers' selection of topics or patterns of emphasis may not be consistent with the intended curriculum. To identify the implemented curriculum, questionnaires were completed by individual classroom teachers. This highly specific information on curriculum coverage and instructional strategies permitted a comprehensive characterization of what mathematics was taught and how it was taught. (3) The attained curriculum is a measure of what pupils have learned. This article is mainly concerned with the attained curriculum as its focus is on gender differences in mathematics achievement. Extensive achievement tests were designed to measure pupils' knowledge and skills in five areas of mathematics - arithmetic, algebra, geometry, measurement and statistics.

Tests were designed having a number of items in each of these mathematical areas; for example, there were 18 items on statistics. The scoring procedure was to code each pupil's response to each item as "correct", "incorrect", or "omitted". The percentage of correct responses was considered to be the mark earned by the pupil on the subtest. In order to compare the success of the boys with that of the girls, the average mark (arithmetic mean) was computed for each sex on each of the five subtests. Table 2 shows the average percentages of correct responses obtained by boys and girls in each of the five mathematical areas tested. It also indicates the variability (as shown by the standard deviation).

Achievement on Subtests

Mathematical Area	Girls		Boys	
	Mean	S.D.	Mean	S.D.
arithmetic	50	15	50	13
algebra	45	13	45	12
geometry	42	15	44	15
statistics	55	17	55	16
measurement	49	20	51	18

Table 2.

As can be seen from the figures given in Table 2, in the subtests of arithmetic, algebra and statistics, girls and boys did not differ in their average marks. In two other subtests, geometry and measurement, the average marks for boys were slightly higher. The standard deviations shown in the table indicate slightly more variability among girls than among boys on most of the subtests. The fact that the gender difference in mathematics achievement among thirteen-year-old pupils in a very large sample involving twenty countries is very small is a major finding of this research.
Statistical analysis was undertaken to identify particular test items which had a gender difference in response. In the arithmetic subtest, it was found that half the items were more likely to be answered correctly by boys and the other half were more likely to be answered correctly by girls. Similar findings arose from the analysis of the algebra and statistics items. In geometry, boys were more likely to be successful on three-quarters of the items and girls were more likely to be successful on the other quarter. In measurement, boys were more successful on 70 percent of the items and girls on the other 30 percent.

These findings led to an investigation of the "difficulty" level of the items that showed sex differences. The term "index of difficulty" is an empirically determined value. It is defined to be the percentage of pupils who gave the correct response to the item. On the arithmetic subtest, it was found that the items which were more likely to be answered correctly by boys tended to be the more difficult items (index of difficulty below 35 percent) while the items which were more likely to be answered correctly by girls tended to be the less difficult ones (index of difficulty above 65 percent). On the algebra subtest, the levels of difficulty of items more likely to be answered correctly by boys were spread over the entire range of item difficulty, as were those of items more likely to be answered correctly by girls. Girls tended to do better than boys on the easier items in the statistics subtests, the range for the index of difficulty being from 45 to 85. The range of the index of difficulty for the items on which boys were more successful was from 22 to 60. In measurement, there was a clear pattern of greater success rates for boys on the more difficult items.

Multivariate Analysis of Variance

Source of variation	Degrees of freedom	algebra (30)	arithmetic (46)	statistics (18)	measurement (24)	geometry (38)
sex	1	1.39	.17	.15	8.75	12.33*
country	19	20.28*	24.76*	25.62*	11.86*	22.94*
country x sex	19	9.81*	10.18*	6.65*	10.20*	8.03*

Table 3.

Differences Within and Among Countries

A statistical procedure call "multivariate analysis of variance" was used to see if the overall gender differences were consistent from country to country. A summary of the results is shown in Table 3. In parenthesis below the name of each subtest is given the number of items on that subtest. The level of significance used was .01. The asterisks indicate values that are statistically significant.

It will be noted that only in the case of measurement and geometry, are the gender differences greater than could be expected by chance. The are no significant differences between boys and girls in their marks in algebra, arithmetic and statistics. However, on every one of the subtests, there are statistically significant differences among countries. The multivariate test of the country-by-sex interaction is a test of equality of sex differences across countries. It is significant for all five subtests and thus indicates that gender differences are not consistent among countries.

Mean Percentages on the Algebra Subtest

Country		Girls	Boys
Belgium:			
	Flemish	61	56
	French	57	53
Canada:			
	B.C.	52	51
	Ontario	44	44
England		43	42
Finland		46	41
France		51	53
Hong Kong		44	44
Hungary		51	47
Israel		48	50
Japan		61	60
Luxembourg		36	38
Netherlands		49	50
New Zealand		39	38
Nigeria		30	34
Scotland		45	44
Swaziland		26	26
Sweden		33	32
Thailand		39	37
U.S.A.		43	42

Table 4.

In order to get an impression of the differences among the countries as compared to the differences between the sexes, the reader is invited to examine Table 4. This table shows the average (arithmetic mean) percentages of correct responses on the algebra subtest for boys and girls in each of the participating countries.

It will be noticed that the variation among countries is the percentage of correct responses to items in the algebra subtest (from 26 to 61 in the case of girls) is far greater than the sex differences within any country. In some countries, there is no sex difference in mean percentage of correct responses, in some countries girls have a higher achievement than boys, and in other countries boys have a higher achievement than girls. The superiority of one sex over the other in algebra achievement is not consistent from country to country.

When these differences were tested for statistical significance, country by country, it was found there were no significant gender differences in algebra scores in eleven countries, girls were found to be superior to boys in five countries, and boys were found to be superior to girls in four countries. Similar test were carried out for each of the five subtests in each of the twenty countries. The findings are summarized in Table 5. *It should*

Gender Differences by Country in the Five Subtests of Mathematical Achievement

	ALGEBRA	ARITHMETIC	STATISTICS	MEASUREMENT	GEOMETRY
NO DIFFERENCE:	British Columbia England Hong Kong Japan Luxembourg New Zealand Ontario Scotland Swaziland Sweden U.S.A.	British Columbia England Finland Hong Kong Hungary Japan Ontario Scotland Swaziland Sweden U.S.A.	Belgium: Flemish French British Columbia England Hong Kong Hungary Japan Luxembourg New Zealand Nigeria Ontario Scotland Swaziland Sweden Thailand U.S.A.	Belgium: Flemish French British Columbia England Finland Hong Kong Hungary Japan Luxembourg Scotland Swaziland Sweden Thailand	Belgium: Flemish French England Finland Hungary Japan Scotland Swaziland Sweden Thailand
GIRLS SUPERIOR	Belgium: Flemish French Finland Hungary Thailand	Belgium: Flemish French Thailand	Finland		
BOYS SUPERIOR	France Israel Netherlands Nigeria	France Israel Luxembourg New Zealand Netherlands Nigeria	France Israel Netherlands	France Israel Netherlands New Zealand Nigeria Ontario U.S.A.	British Columbia France Hong Kong Israel Luxembourg Netherlands New Zealand Nigeria Ontario U.S.A.

Table 5.

be noted that the word "significant" is used in the statistical sense when describing a gender difference. It indicted that the difference is greater than would be expected just by chance but does not imply that the difference is large enough to be important pedagogically. (As can be observed in Table 4, the differences between the algebra achievement of girls and boys in any one country is not very great).

An examination of Table 5 will show that for each of the subtests, at least half the countries showed no statistical superiority of one gender over the other. On each subtest, more countries had statistically higher mean scores for boys than had statistically higher mean scores for girls. In no country did girls exceed boys in either measurement or geometry.

The purpose of this analysis of a large-scale, international study of mathematics was to determine if, among thirteen-year-olds, there are important gender differences in achievement. The major findings are that gender differences vary from country to country and that they are very small. Since it is very unlikely that biological differences between the sexes vary from one country to another, the data from the Second International Mathematics Study tend to contradict those theories that attempt to explain male superiority in mathematics on the

basis of biological differences - hormones[4], genes[5] or brain organization.[6]

It is interesting to find that a male advantage in mathematics is non-existent among thirteen-year-olds in many countries. Where such an advantage is observed, it is most likely to be in geometry or measurement, the mathematical areas which were least taught in class (as the data on the implemented curriculum showed). It appears that girls and boys assimilate the subject matter taught in class equally well - arithmetic, algebra and statistics. Thus, it seems reasonable to conclude that such gender differences as are found among thirteen-year-olds are due to out-of-class experiences and to psychosocial processes rather than biological differences.[7]

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Teaching Tips

ZÉRO - UNE NOTION PRIMAIRE

A quel numéro est-ce que tu commences à compter?
A zéro, bien sur.
A quel numéro est-ce qu'on commence lorsqu'on compte pàr deux?
A zéro, c'est un nombre pair.

Pourquoi est-ce si important d'enseigner à l'élève une compréhension de zéro? C'est une notion qui comprit toute une gamme de faites réeles. On peut dire que bien de chose commence par rien: le volume d'un ballon, l'age, le temps. Ce n'est pas une notion difficile. Quand c'est enseignée avec de la motivation la notion de zéro donne aux élève une appréhension du concept.

Maintenant, on compte: **0, 1, 2, 3!**

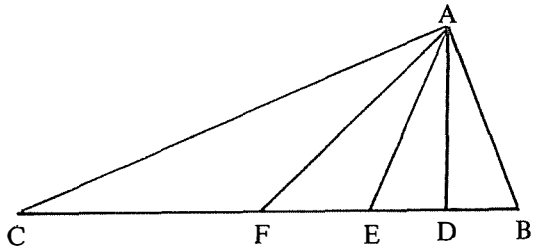
Submitted by Amy Parker
93 Arthur St. N.
Guelph, Ontario.

TEN SOLUTIONS TO A VERY SPECIAL GEOMETRY PROBLEM

Edited by
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Oakville-Trafalgar High School
Oakville, Ontario L6J 3L6

In the April 1989 issue of The Gazette the following problem appeared in my Problems Column.

In triangle ABC the altitude AD, the angle bisector AE, and the median AF divide ∠BAC into four equal parts. Prove that ∠BAC = 90°.

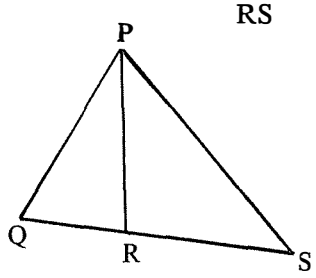


The response to this problem was tremendous and I heard from over two dozen teachers from all over Ontario. I ended up receiving ten different solutions and they are reprinted here for you and your students to look over.

I encourage you to include this problem in your Grade 12 advanced math class. Try to get your fellow staff members to do the same. Post all ten solutions or have students present them in class. The ten solutions involve a wide range of mathematics and there is an opportunity here for students to see "the big picture." Enjoy and employ!

NOTE: Several proofs involve the Angle Bisector Theorem (ABT). For those not familiar with this, a statement of the theorem is given below.

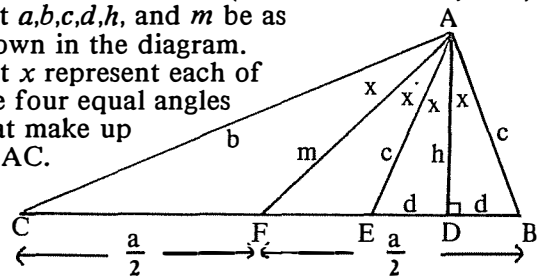
If ∠QPR = ∠SPR, then $\frac{QR}{RS} = \frac{PQ}{PS}$



A proof of this is left for the reader. The proof involves extending QP to T so that PT = PS. This would be an excellent problem to give to Grade 12 students.

Solution 1 Dr. Ian McGee
University of Waterloo

CF = FB (given)
AE = AB (△AED ≡ △ABD, SAS)
Let *a,b,c,d,h,* and *m* be as shown in the diagram.
Let *x* represent each of the four equal angles that make up ∠BAC.



$\frac{b}{c} = \frac{a - 2d}{2d}$ (ABT, AE in △ABC) (1)

$\frac{b}{c} = \frac{\frac{1}{2}a}{\frac{1}{2}a - d}$ (ABT, AF in △AEC) (2)

$\frac{m}{h} = \frac{\frac{1}{2}a - 2d}{d}$ (ABT, AE in △ADF) (3)

Equating (1) and (2) gives

$\therefore \frac{a - 2d}{2d} = \frac{\frac{1}{2}a}{\frac{1}{2}a - d}$

$\therefore \left(\frac{a}{d}\right)^2 - 8\left(\frac{a}{d}\right) + 8 = 0$

$\therefore = 4 + 2\sqrt{2}$ (4)

Other root is negative and thus not possible since $\frac{a}{d} > 0$.

Substituting (4) into (3) gives

$\frac{m}{h} = \frac{1}{2}\left(\frac{a}{d}\right) - 2$

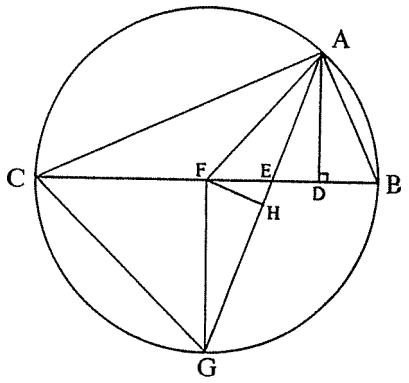
$= \frac{1}{2}(4 + 2\sqrt{2}) - 2$

$= \sqrt{2}$

In a right angled triangle if the ratio of the hypotenuse to a leg is equal to $\sqrt{2}$, then the legs are equal, making the triangle isosceles with angles of 45°, 45°, and 90° (this can be shown without trigonometry). Applying this to △AFD gives that ∠FAD = 45°. Thus $2x = 45°$ and so $4x = 90°$. Hence ∠BAC = 90° as required.

Solution 2 — **Dr. Joe Csima**
McMaster University

Let G be the point of intersection between AE extended and the circumscribed circle of $\triangle ABC$. Let H be the midpoint of chord AG. Join F to H, C to G, B to G, and F to G.



BG = CG	($\angle BAG = \angle CAG$, equal angles subtend equal chords)
BF = CF	(given)
$\angle FBG = \angle FCG$	(ITT)
$\therefore \triangle BFG \equiv \triangle CFG$	(SAS)
$\therefore$ GF $\perp$ BC	($\equiv\triangle$s, BFC is a straight line)

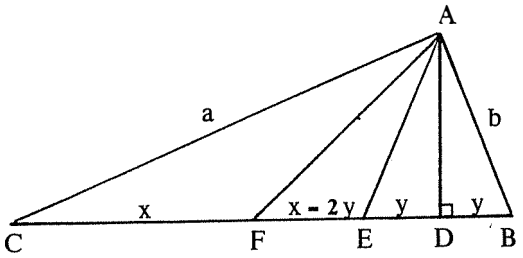
Thus FG is the perpendicular bisector of chord BC.

AD $\perp$ BC	(given)
$\therefore$ GF $\parallel$ AD	(PLT, C)
$\therefore \angle FGA = \angle DAG$	(PLT, Z)
$\angle GAF = \angle DAG$	(given)
$\therefore \angle GAF = \angle FGA$	(transitivity)
$\therefore$ AF = GF	(ITT)
AH = GH	(given)
$\therefore \triangle FAH \equiv \triangle FGH$	(SAS)
$\therefore$ FH $\perp$ AG	($\triangle$s $\equiv$, AHG is a straight line)

Thus FH is the perpendicular bisector of chord AG. Therefore the centre of the circle lies on FH and FG (both perpendicular bisectors of chords) so F is the centre of the circle. This makes BC a diameter so $\angle BAC = 90°$ as required (inscribed angle in a semicircle).

Solution 3 — **Dave Cattani**
Hammarskjold H.S.
Thunder Bay, Ontario

Let a = AC, b = AB, x = CF = BF, and y = ED, Since $\triangle ADE \equiv \triangle ADB$ (SAS) then y = BD.
Thus FE = $x - 2y$ (see the diagram below)



$$\frac{x}{x - 2y} = \frac{a}{b} \qquad \text{(ABT, AF in } \triangle ACE\text{)} \quad (1)$$

$$\frac{2x - 2y}{2x} = \frac{a}{b} \qquad \text{(ABT, AE in } \triangle ACB\text{)} \quad (2)$$

Equating (1) and (2) gives

$$\frac{x}{x - 2y} = \frac{2x - 2y}{2x}$$

$$\therefore 2y^2 - 4xy + x^2 = 0$$

$$\therefore \quad y = \frac{2 - \sqrt{2}}{2}\,x \qquad (3)$$

The other solution is not possible since $y < x$.
Substituting (3) into (1) gives

$$\frac{x}{x - 2\left(\frac{2 - \sqrt{2}}{2}\right)x} = \frac{a}{b}$$

$$\therefore b = (\sqrt{2} - 1)a \qquad (4)$$

Applying the Pythagorean Theorem to $\triangle ACD$ and $\triangle ABD$ gives

$$a^2 = h^2 + (2x - y)^2 \qquad (5)$$
$$b^2 = h^2 + y^2 \qquad (6)$$

Subtracting (6) from (5) gives

$$\begin{aligned} a^2 - b^2 &= (2x - y)^2 - y^2 \\ &= 4x\,(x - y) \\ &= 4x\left(x - \frac{2 - \sqrt{2}}{2}\,x\right) \\ &= 2\sqrt{2}\,x^2 \end{aligned} \qquad (7)$$

Substituting (4) into (7) gives

$$a^2 - (\sqrt{2} - 1)^2 a^2 = 2\sqrt{2}x^2$$

$\therefore \quad (2\sqrt{2} - 2)a^2 = 2\sqrt{2}x^2$

$$\therefore \quad a^2 = \frac{\sqrt{2}}{\sqrt{2} - 1}x^2$$

$$= (2 + \sqrt{2})x^2 \qquad (8)$$

Squaring both sides of (4) gives

$b^2 = (\sqrt{2} - 1)^2 a^2$

$= (3 - 2\sqrt{2})(2 + \sqrt{2})x^2$ using (8)

$= (2 - \sqrt{2})x^2$ (9)

Adding (8) to (9) gives

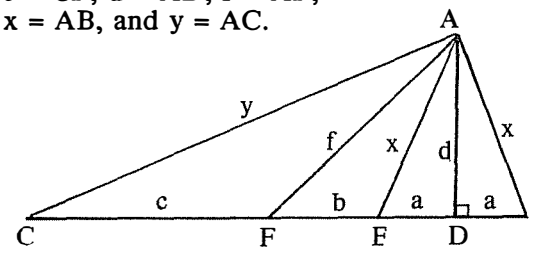
$a^2 + b^2 = (2 + \sqrt{2})x^2 + (2 - \sqrt{2})x^2$

$= 4x^2$

Thus $AC^2 + AB^2 = BC^2$ and so $\triangle ABC$ is right-angled. Since BC is the hypotenuse, then $\angle BAC = 90°$ as required.

Solution 4 **Rob Smythe**
Oakville Trafalgar H.S.
Oakville, Ontario

Let a = DB, b = FE,
c = CF, d = AD, f = AF,
x = AB, and y = AC.



CF = BF (given)

$\therefore$ c = 2a + b

ED = BD (given)

AD = AD (common)

$\angle EDA = \angle BDA$ (given)

$\therefore \triangle ADA \equiv \triangle BDA$ (SAS)

$\therefore$ a = DE and x = AE

$$\frac{y}{x} = \frac{c}{b} \qquad \text{(ABT, FA in } \triangle CAE)$$

$$= \frac{2a + b}{b} \qquad (1)$$

$$\frac{y}{x} = \frac{c + b}{2a} \qquad \text{(ABT, AE in } \triangle CAB)$$

$$= \frac{a + b}{a} \qquad (2)$$

Equating (1) and (2) gives

$$\frac{2a + b}{b} = \frac{a + b}{a}$$

$$\therefore \quad b = \sqrt{2}a \qquad (3)$$

Thus $c = 2a + b$

$$= (2 + \sqrt{2})a \qquad (4)$$

Squaring both sides of (4) (the reason for this will be clear later) gives

$$c^2 = 2(3 + 2\sqrt{2})a^2 \qquad (5)$$

$$\frac{f}{d} = \frac{b}{a} \qquad \text{(ABT, AE in } \triangle FAD)$$

$$= \sqrt{2} \qquad \text{(using (3))}$$

$\therefore f = \sqrt{2}d$

$$\therefore f^2 = 2d^2 \qquad (6)$$

$$\frac{CD}{AD} = \frac{AD}{DB} \qquad \text{(MPT, AD} \perp \text{BC)}$$

$$\therefore \frac{c + b + a}{d} = \frac{d}{a}$$

$$\therefore d^2 = (3 + 2\sqrt{2})a^2 \qquad (7)$$

Substituting (7) into (6) gives

$f^2 = 2d^2$

$= 2(3 + 2\sqrt{2})a^2$ (8)

Equating (8) with (5) gives

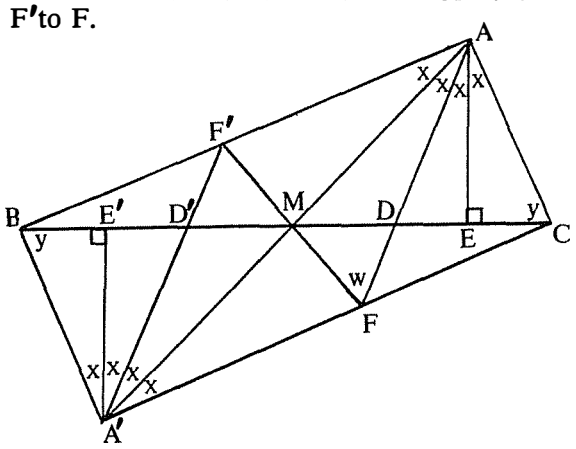
$f^2 = c^2$

$\therefore f = c$

Thus AF = BF = CF and so BDC being a straight line segment is the diameter of a circle whose centre is F. Therefore $\angle BAC = 90°$ as required (inscribed angle in a semicircle).

Solution 5 **Bill Bisset**
Earl Haig S.S.
North York, Ontario

Apply a half-turn about M. Extend A′D′ to F′ on BA and extend AD to F on A′C. Join F′ to F.



Let x be each of the four equal angles that make up ∠BAC. Let y = ∠ACE. Note that $x + y = 90°$. Let w = ∠MFA.

∠A′BE′ = ∠ACE (by construction)

∴ BA′ ∥ CA (PLT - Z)

∠A′CB = ∠ABC (by construction)
∴ A′C ∥ AB (PLT - Z)
∴ AFA′F′ is a parallelogram.

A′M = AM (in any parallelogram, diagonals bisect each other)
∠MA′F = ∠MAF (given)
∴ A′F = AF (ITT)
∴ △A′MF ≡ △AMF (SAS)
∴ ∠A′MF = ∠AMF (△s ≡)
∴ ∠FMA = 90° (A′MA is straight line)

∴ x + w = 90° (sum of the angles in a triangle equals 180°)

∴ w = y (x + y = 90°)

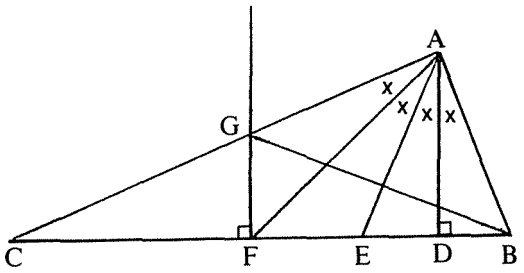
∴ ∠MFA = ∠MYA
∴ AMFC is a cyclic quadrilateral

∴ ∠FMA + ∠FCA = 180°
∴ 90° + ∠FCA = 180°
∴ ∠FCA = 90°

∠BAC + ∠FCA = 180° (PLT, C)
∴ ∠BAC + 90° = 180°
∴ ∠BAC = 90° as required.

Solution 6 **Eric Wood**
U. of Western Ontario

Let x be each of the four equal angles that make up ∠BAC. Construct a perpendicular to BC at F meeting AC at G. Join G to B.



GF ∥ AD (PLT, C)
∴ ∠GFA = ∠DAF (PLT, Z)
= 2x
∠CGF = ∠GFA + ∠GAF (EAT)
= 2x + x
= 3x
CF = BF (given)
GF = GF (common)
∠CFG = ∠BFG (by construction)
∴ △CFG ≡ △BFG (SAS)
∴ ∠BGF = ∠CGF (△s ≡)
= 3x
∠FAB = 3x (given)
∴ ∠BGF = ∠FAB
∴ GABF is a cyclic quadrilateral (two equal inscribed angles subtended by same chord)

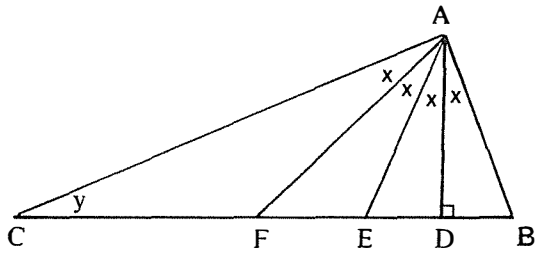
∴ ∠BFG + ∠BAC = 180° (opposite angles in a cyclic quadrilateral are supplementary)

∴ 90° + ∠BAC = 180°
∴ ∠BAC = 90° as required.

Solution 7 **A Trigonometry Solution**

Dave Cattani
Hammarskjold High School
Thunder Bay, Ontario

Let x = ∠CAF = ∠FAE = ∠EAD = ∠DAB
and y = ∠ACF



$\angle AFE = x + y$ (EAT)
$\angle AED = 2x + y$ (EAT)
$\triangle ADE \equiv \triangle ADB$ (ASA)
$\therefore \angle AED = \angle ABD = 2x + y$

Using the sine law:

In △ACF: In △AFB:

$\frac{\sin x}{CF} = \frac{\sin y}{AF}$ $\frac{\sin 3x}{FB} = \frac{\sin(2x + y)}{AF}$

$\therefore \frac{AF}{CF} = \frac{\sin y}{\sin x}$ $\therefore \frac{AF}{FB} = \frac{\sin(2x + y)}{\sin 3x}$

$CB = FB \therefore \frac{AF}{CF} = \frac{AF}{FB}$

$\therefore \frac{\sin y}{\sin x} = \frac{\sin(2x + y)}{\sin 3x}$

$\sin y \sin 3x = \sin x \sin(2x + y)$
$\sin y (\sin 2x \cos x + \cos 2x \sin x) =$
$\quad \sin x (\sin 2x \cos y + \cos 2x \sin y)$
$\sin 2x (\sin y \cos x - \sin x \cos y) = 0$
$\sin 2x \sin(y - x) = 0$
$\therefore 2x = 0$ or $2x = 180°$
or $y - x = 0°$ or $y - x = 180°$

The only case that yields a triangle is $y - x = 0°$, so $y = x$.

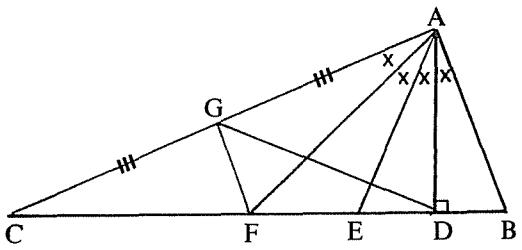
In △ACD, $y + 3x + 90° = 180°$
$x + 3x + 90° = 180°$
$4x = 90°$

Thus ∠BAC = 90° as required.

Solution 8 **Jeri Lunney**

St. Pauls High School
Nepean, Ontario

Construct FG such that CG = AG.
Join G to D. Let x be each of the four equal angles that make up ∠BAC.



∠ADC = 90° (given)

∴ A,D, and C lie on the circle with diameter AC and centre G.

∴ GD = AG (equal radii)

∴ ∠GDA = ∠GAD (ITT)
= 3x

$\frac{CG}{GA} = \frac{CF}{FB}$ (given)

∴ GF ∥ AB (side-splitting theorem)

∴ ∠GFA = ∠FAB (PLT, Z)
= 3x

∴ ∠GFA = ∠GDA

∴ AGFD is a cyclic quadrilateral.
(equal angles subtended by the same segment AG)

∴ ∠AGF + ∠ADF = 180°

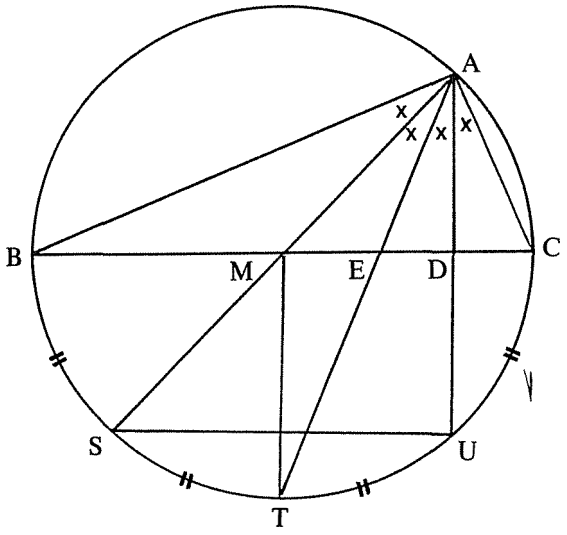
∴ ∠AGF + 90° = 180°

∴ ∠AGF = 90°

∴ ∠BAC = 90° as required (PLT, C)

Solution 9 **Dr. Ross Honsberger**
University of Waterloo

Let x be each of the four equal angles that make up ∠BAC. Extend AM, AE, and AD to meet the circumcircle at S,T, and U as shown in the diagram. Join S to U.



∠SAB = ∠UAC (given)
∴ arc SB = arc UC
∴ SU ∥ BC

AD ⊥ BC (given)
∴ AD ⊥ SU
∴ ∠SUA = 90°
∴ AS is a diameter of the circumcircle.

Join M to T

∠TAB = ∠TAC (given)
∴ arc TSB = arc TUC
∴ ∠TMB = ∠TMC
∴ ∠TMB = 90° (BMC is a straight line)

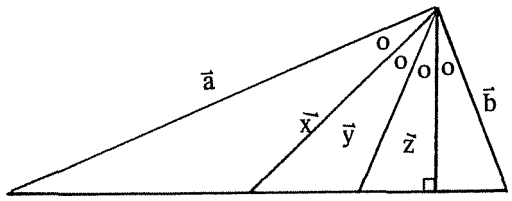
Therefore TM is the perpendicular bisector of chord BC. Thus the centre of the circle lies on TM. But the centre also lies on diameter AMS. Therefore M is the centre and so BC is a diameter. Hence ∠BAC = 90° as required (inscribed angle in a semicircle).

Solution 10 **Dr. Tom Davison**
McMaster University

[In the following solution, Vectors will be indicated by bold face type.]

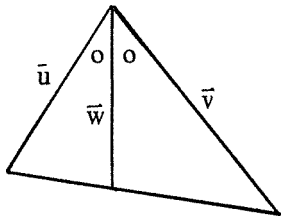
Given: All the indicated angles are equal, there is a right angle as shown and
$\mathbf{x} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

Prove: $\mathbf{a} \cdot \mathbf{b} = 0$



The main idea is to express the concept of equal angles vectorially.

Vector version of Angle Bisector Theorem

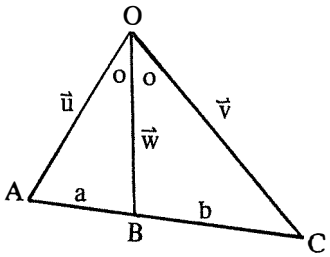


If the angle between **u** and **w** is equal to the angle between **w** and **v** then

$$\mathbf{w} = \frac{|\mathbf{v}|}{|\mathbf{v}| + |\mathbf{u}|}\,\mathbf{u} + \frac{|\mathbf{u}|}{|\mathbf{u}| + |\mathbf{v}|}\,\mathbf{v}$$

Proof of Vector Version of Angle Bisector Theorem

Let A,B, and C be as shown and let a:b be the ratio in which B divides AC.



$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB}$$

$$= \mathbf{OA} + \frac{a}{a+b}\mathbf{BC}$$

$$= \mathbf{OA} + \frac{a}{a+b}(\mathbf{OC} - \mathbf{OA})$$

$$= \frac{b}{a+b}\mathbf{OA} + \frac{a}{a+b}\mathbf{OC}$$

$$= \frac{1}{\frac{a}{b}+1}\mathbf{OA} + \frac{\frac{a}{b}}{\frac{a}{b}+1}\mathbf{OC}$$

$$= \frac{|\mathbf{v}|}{|\mathbf{v}|+|\mathbf{u}|}\mathbf{u} + \frac{|\mathbf{u}|}{|\mathbf{u}|+|\mathbf{v}|}\mathbf{v}$$

NOTE: Using the non-vector version of the Angle Bisector Theorem it follows that

$$\frac{a}{b} = \frac{|\mathbf{u}|}{|\mathbf{v}|}$$

Many thanks to Neil McLaren from Oakville Trafalgar High School for the above proof.

Proof of $\mathbf{a} \cdot \mathbf{b} = 0$

Note that $|\mathbf{y}| = |\mathbf{b}|$ (congruent triangles)

The Angle Bisector Theorem (vector version) gives that

$$\mathbf{y} = \frac{|\mathbf{b}|}{|\mathbf{b}| + |\mathbf{a}|}\mathbf{a} + \frac{|\mathbf{a}|}{|\mathbf{a}| + |\mathbf{b}|}\mathbf{b} \qquad (1)$$

since the angle between $\mathbf{a}$ and $\mathbf{y}$ equals the angle between $\mathbf{y}$ and $\mathbf{b}$.

To abbreviate some algebra set $t = |\mathbf{a}| + |\mathbf{b}|$
Using this notation (1) becomes

$$t\mathbf{y} = |\mathbf{b}|\mathbf{a} + |\mathbf{a}|\mathbf{b} \qquad (2)$$

Since $|\mathbf{y}| = |\mathbf{b}|$ and $\mathbf{y} \cdot \mathbf{y} = \mathbf{b} \cdot \mathbf{b}$, squaring both sides of (2) gives

$$t^2 |\mathbf{b}|^2 = |\mathbf{b}|^2 |\mathbf{a}|^2 + 2|\mathbf{b}||\mathbf{a}|\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2 |\mathbf{b}|^2$$

$$\therefore t^2 |\mathbf{b}|^2 = 2|\mathbf{a}|^2 |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\mathbf{a} \cdot \mathbf{b} \qquad (3)$$

The Angle Bisector Theorem (vector version) also gives that

$$(|\mathbf{y}| + |\mathbf{a}|)\mathbf{x} = |\mathbf{y}|\mathbf{a} + |\mathbf{a}|\mathbf{y}$$

$$\therefore \quad t\mathbf{x} = |\mathbf{b}|\mathbf{a} + |\mathbf{a}|\mathbf{y}$$

Thus $t^2 \mathbf{x} = t|\mathbf{b}|\mathbf{a} + |\mathbf{a}|t\mathbf{y}$

$$\therefore \quad t^2 \mathbf{x} = t|\mathbf{b}|\mathbf{a} + |\mathbf{a}|(|\mathbf{b}|\mathbf{a} + |\mathbf{a}|\mathbf{b}) \qquad (4)$$

But $\mathbf{x} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

Since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, we may equate coefficients of $\mathbf{b}$ in (4) to obtain

$$\tfrac{1}{2}t^2 = |\mathbf{a}|^2 \quad \text{so } t^2 = 2|\mathbf{a}|^2 \qquad (5)$$

Substituting (5) into (3) gives

$$2|\mathbf{a}|^2 |\mathbf{b}|^2 = 2|\mathbf{a}|^2 |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\mathbf{a} \cdot \mathbf{b}$$

Thus $\mathbf{a} \cdot \mathbf{b} = 0$ as required.

ON CHERCHE DES MANUSCRITS

Nous avons besoin de vos idées!
On encourage les lecteurs à soumettre des manuscrits qui examinent les techniques dans la classe ou des questions de l'enseignement de mathématiques (écrits en français) pour la critique.

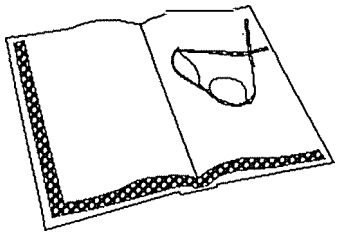
Les instructions pour les auteurs peuvent être trouvées au début de la revue.

ON CHERCHE DES CRITIQUES

Afin de faciliter la politique de la gazette à propos de la publication d'articles en anglais et en français, il nous faut des critiques pour les articles français. Veuillez envoyer votre nom à:

The Editor
The Ontario Mathematics Gazette
c/o University of Western Ontario
Faculty of Education
London, Ontario N6G 1G7

BOOK REVIEWS



Mathematics in the Primary[1] School.
Richard Skemp.
Routledge: London, UK. 1989.
ISBN 0-415-2519-2

Richard Skemp has been recognised for several decades as being a leading international mathematics educator. In *Mathematics in the Primary School* he relates a sound theoretical framework to practical activities which address most mathematics topics taught in the primary and junior divisions.

The book is divided into two sections. Part A describes Skemp's theoretical framework and offers a rationale for why so many students have difficulty understanding the abstract nature of mathematics and constructing sound conceptual structures.

In one example he explains that communicating a mathematics concept by giving a definition is analogous to teaching the meaning of the word 'red' through the definition: red is the colour we experience from light in the region of 0.6 microns. There is little benefit to be gained from such a definition. Instead we need to provide real life experiences such as pointing to red clothing, red flowers etc. From these experiences children construct for themselves a true understanding of red. It is only after lower order abstract concepts such as *red* have been constructed that it is possible to construct higher order abstract notions such as *colour* or *attribute*. There is a hierarchical structure in which a thorough understanding of lower order abstract notions is a prerequisite for the introduction of higher order abstract ideas.

Skemp believes mathematics is the most abstract subject young children encounter, and for understanding a higher order concept such as place value, children must first of all understand lower order abstract notions such as the natural numbers, order, counting, unit objects, sets of objects, sets as single entities, sets of sets, numerals and the distinction between these and numbers, numeration, and bases for numeration. It is not surprising that young children have difficulty conceptualizing such an abstract idea as *place value* or that they need many different experiences linking the symbolic mathematical code to real world experiences.

Skemp has always advocated that the child be an active participant in the learning process and the second part of his book offers a collection of activities[2] to assist the child to construct sound conceptual structures. He advises that throughout these activities the teacher should **observe and listen, reflect, discuss.** The teacher cannot successfully transmit knowledge. However, through the provision of well structured activities and appropriate questions at opportune times, children are able to construct concepts to share with their teacher and peers. When Skemp presented some of his activities in Canada, one participating teacher remarked, "It's as though the children's thinking was out on the table."

Learning is a shared experience: between children and between children and teacher. Everyone continues to learn from each other. As Skemp so aptly states in his summary:

Those who learn from one
who is still learning
drink from a running stream.
Those who learn from one
who has ceased to learn
drink from a stagnant pond.

Skemp's materials are a most valuable resource for the elementary teacher, if not as a primary reference then as supplementary resources.

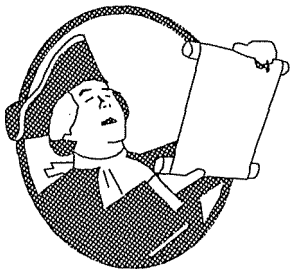
1. In the U.K. primary spans grades JK - 5.
2. Two later books, Structured Activities for Primary Mathematics, Volumes 1 and 2 contain Skemp's complete set of mathematical activities for the primary and junior divisions.

Reviewed by Barry Onslow
Division of Curriculum Studies
Faculty of Education
University of Western Ontario

Puzzle Solution:

(from page 13)

7	2	9
8	6	4
9	6	1



COUNCIL COMMENT

by
Bob Robinson

These comments are based on the Council Meeting held at the Ramada Inn, Airport West, on February 10, 1990.

ANNUAL MEETING (HAMILTON) MAY 9-12, 1990

The programmes and promotional brochure have been mailed. The theme "A Free Trade of Ideas" is very appropriate as there is a superb blend of Canadian and American speakers. A gala banquet is planned. This NCTM Canadian Regional Conference hosted by OAME promises to be one of the most worthwhile mathematics education conferences with an international flavour to be held in Ontario in years. If you wish to receive a program booklet contact Harry Traini, Mathematics Co-ordinator, Hamilton Board of Education.

ANNUAL MEETING (ETOBICOKE) MAY 9-11, 1991

The 1991 annual is being organized by CHAMP and will be held at the Constellation Hotel in Etobicoke. They are planning to present a program which will meet the needs and interests of all educators. Marg Warren, program chair, is seeking recommendations for speakers. To receive a speaker recommendation form, contact Marg Warren, Applewood Heights S.S., 945 Bloor Street East, Mississauga, Ontario, L4Y 2M8.

CONSTITUTION

A revised constitution will be presented for ratification to OAME members at its Annual General Meeting in Hamilton on Friday, May 11, 1991 at 4:15 p.m. in the Heritage Room of the Sheraton Hamilton.

Nominations

The following persons have been nominated and the ballots were mailed on Monday, February 19, 1990.
President-elect: Ann Jones, Roland Morrow. **Director**: David Handley, Ron Lancaster, Jeri Lunncy, Susan Stuart. **Councillor**: Linda Colgan, John Cumming, Carol Gorewich, Olive Fullerton, Myrna Ingalls, Peter Lenczweski, George McNabb, Lorna Morrow, Amy Parker, Patricia Rae, Jan Scully, Jack Weiner, Lorna Wiggan, David Zimmer.

Conference Guidelines

Arnie Carefoote reported that his committee is assembling a comprehensive and much needed resource binder to assist those planning future conferences.

Leadership Seminar

The annual leadership seminar will be held November 14-16, 1990 and the Airport Ramada Inn in Mississauga. A draft program agenda has been proposed. For more information, contact Mary Lou Kestell.

LIFE MEMBERSHIP

This year's recipient of life membership in the OAME will be announced at the gala banquet to be held at the Hamilton Conference on Friday evening in the ballroom of the Sheraton Hamilton Hotel at 7:00 p.m.

NCTM

New NCTM publication brochures are available either from Jan and Barry Scully or NCTM. Any person wishing to advertise a chapter's mathematics mini-conference in an issue of the Arithmetic Teacher may do so providing 3 months notice is given.

CHAPTER REPORTS

CHAMP

Winners of the CHAMP mathematics contest will be recognized at the Spring Event "Investigate the Mysteries of Math" to be held at Our Lady of Mount Carmel on March 29, 1990 between 4:00 p.m. - 6:00 p.m. Speakers will be Trevor Brown, Stewart Cravern and Ron Lancaster. Mathfest was well attended and the publishers display was great.

COMA/AMOC

1990 got off to a roaring start for COMA with its now annual January Mini-Conference. The conference attracted 238 participants, 70 of whom stayed for dinner. Overall a tremendous success for this drab time of year.

COMA is always trying to get new members for its parent organization, OAME. Any OAME member is automatically a member of COMA.

After a one year break for the OAME '89 Ottawa Conference, the annual Publisher's Display is back. It will be held in the Embassy West Hotel, Carling Ave. Ottawa, on Monday April 23, 1990, from 3:00 until 6:00 p.m. The publishers will be asked to concentrate on manipulatives and computer software. Attendance is free to all COMA and OAME members. Planning has also begun on the fall workshop/conference.

Golden Section

The annual mini-conference will be held at Brock University on Tuesday, April 10, from 4:00 to 6:30 p.m. The theme will be "Manipulation and Active Learning". Supper at the Pond Inlet will follow the workshops. Jeff Martin will be the keynote speaker.
Judy Crompton has taken over the job of coordinating the Chapter's activities following the retirement of Warren Sebben.

GVMA

GVMA held its annual fall meeting on Thursday, November 9, 1989 at EastWood Collegiate in Kitchener. There were eight general interest sessions plus Dinner Time Math and an excellent after dinner speaker, Dr. Victor Rausch, whose specialty is hynotherapy. There were in excess of 300 people in attendance.
GVMA has published three booklets of Sample Final Exams with Solutions and Marking Schemes, one each for OAC Calculus, OAC Algebra and Geometry, and OAC Finite Mathematics. To purchase copies contact Bonnie Findlay, 519-885-1211.
Plans are underway for a spring meeting - April 5, 1990.

NOMA

NOMA experimented by moving the Fall conference to Thursday evening and all-day Friday, October 29/30 to coincide with a local P.A. day for the Sudbury boards of education (usually Friday evening and Saturday morning). Attendance improved dramatically, especially for Sudbury teachers, but dropped for out-of-towners. Our planning committee recieved many fine compliments about a well-organized conference.
For our 1990 conference we are planning to follow a similar format, with some adjustments:

- session pre-registration to prevent over-crowding by planning repeat sessions
- move keynote speakers to Friday morning
- plan earlier to give better advance publicity
- lobby with school boards to have even more teachers freed to attend math sessions
- direct mail to schools and individual NOMA members
- tentative date October 25/26, 1990

Other activities in progress:

- Printing of "NOMA Excellence in Mathematics" certificates for free distribution to schools (in limited numbers).
- Sponsorship of cash awards to Zone 62 and 63 winners of Canadian Mathematics Competition - grade 9 to OAC level.
- Planning to publish short newsletters for direct mail to individual members and school principals / mathematics heads. This will include NOMA news, reports about OAME and provincials concerns.
- Computerization of membership mailing list to enable direct mail of newsletters, conference publicity, ...

NWOAME

NWOAME has begun collecting materials for the September edition of Abacus. We have chosen a theme and articles are being sought from both Elementary and Secondary schools across NWO.
NWOAME and Lakehead University will jointly host a Math Awards Day at Lakehead University on May 18, 1990. They plan to recognize students from across NWO who have excelled in this year's various Math Contests.
"NWOAME '90" will be held on Friday, September 28 at Confederation College in Thunder Bay. The format will be the same as that of our first Conference. We welcome suggestions for speakers, topics, interest items.

PRMA

Several successful workshops were held on Durham's Boardwide Professional development Day November 17, 1989. P.R.M.A. will again sponsor the Durham Mathematics League Competition in May 1990.
This year's Mini-Conference will be held on Thursday, April 5th at General Vanier High school in Oshawa. Speakers are currently being arranged.
The Mathematics Awards Night will again be organized by PRMA and held on June 7th at G.L. Roberts C.V.I.

QSLMA

Since their president, David Wing has retired from teaching, he has also retired as President of Q.S.L.M.A. Vice-President, Brian Gallagher will fill in for the remainder of the term. Brian is the Head of Mathematics at Gananoque Secondary School.
The plans for the Spring Conference are well under way. As usual, the Conference will provide professional development seminars and a banquet. A new added feature will be an after dinner dance, with music provided by a live band. The date for this gala occasion is April 6, 1990. The location is The Royal Military College, Kingston.
The joint venture last Fall with C.O.M.A. was so successful, that we have agreed to do it again. The joint conference will be held at the Opinicon Lodge, (Chaffeys Locks), in early October as last year.

SAME

Plans are underway for "Springfest" at Maplewood High School on April 4, 1990 at 4:00 p.m. - 6:00 p.m.
Professor Ed. Barbeau will be our featured speaker for the senior-intermediate divisions. Jan Clemens and Denise White will do a "Make and Take" for the Primary-Junior Division. Special treats will be prepared by Maplewood Food Services. Remember the annual October Conference, also at Maplewood High School on October 16. More information will follow.

SWOAME

The Fall Conference "Using the Calculator and Computer in the Mathematics Classroom." was held in Windsor on December 1 and 2 and was very successful.
An evening "DINNERLUDE" is planned for May 3 in Windsor. Sessions for elementary and secondary levels will take place followed by dinner and a guest speaker.

Y4MA

The mini-conference, held November 15, 1989, had a superb turnout of 280 people. There were excellent speakers and thanks are extended to the councillors who assisted. Each participant received a folder which contained a writing pad, pencil, speaker list, evaluation form and a volunteer sheet for committees.

The "Spring Annual" will be held on May 31, 1990 at the Thornhill Community Centre. The speaker will be Ron Lancaster who will speak on "Math Magic".
The Y^4MA would like any information on constitutions, newsletters, and membership promotion ideas.

COMMITTEE REPORTS

Women and Mathematics

The annual breakfast will be on Friday May 11, 1990 from 7:15 a.m. - 8:15 a.m. in the Webster Room of the Hamilton Convention Centre. Sylvia Gold is the featured speaker. Men are most welcome to attend. If fund-raising efforts are successful, there should be no admission charge.

Membership

The membership brochure will be updated and become more generic in nature. A major project this year is to target contacts within the Faculties of Education.

Publications

The editors of the Gazette and Abacus were commended for their new format and excellent articles. New departments such as "Mathematics in the News" and "Issues in Mathematics" are being considered. The "Problem Solving" section is receiving many and varied solutions from readers. Well done!

Features

The committee has been in contact with the Operations Research Association of Ontario with the intention of utilizing resource people for co-op programs for senior students at the advanced level; at mathematics conferences; or in high schools. The emphasis would be on mathematics in the real world. The possibility of a student conference is being explored. The committee is updating candidate criteria for the awards and prizes given by OAME.

Primary/Junior

The committee is anxiously awaiting the consultation papers regarding the Early Years, Formative Years, and Transition years.

Intermediate/Senior

The ministry of Education action plan was discussed with particular interest on the Transition Years, Specialization Years, and Technical Education. Requests will be made to become involved as a member of the reaction teams. The committee has raised 9 issues of concern regarding the Transition Years for which response will be requested through this issue of the Gazette. The provincial reviews were discussed also. The teacher qualification survey has had about a 50% return to this date. Look for a report in the near future.

THANK YOU!

President Mary Lou Kestell thanked council for the support she has received and passed on best wishes to the councillors and executive members who are leaving council and not seeking re-election. Among those departing are former president Paul Lessard and immediate past president J. Symington.

FUTURE COUNCIL MEETING SCHEDULE 1990-91

Annual Meeting at Sheraton Hamilton Hotel, May 11, 1990; 4:15 - 4:45 p.m.

EXECUTIVE	COUNCIL
June 8-9	September 22
November 16	January 12
February 1	March 30
April 12	



The Membership and Publicity Committee has a deal for you.

I ♥ Math buttons promote the appreciation of mathematics among students and teachers. They are available in French and English and 40¢ each or $18.00 for 50.

Order from Michael Sandblom

Make cheque payable to O.A.M.E.



Hosted by:
Ontario Association for Mathematics Education

Write: Harry Triani, Hamilton Board of Education,

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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Past Pres.	Symington, J
Pres.	Kestell, Mary Lou
Pres-Elect	Sandblom, Michael
Director	Lunney, Jeri
Director	Morrow, Roland
Director	Weiner, Jack

MEMBERSHIP DUES

Subsciptions to the *Gazette* alone are no longer available; however, members of O.A.M.E./A.O.E.M. receive the *Gazette* and *Abacus* free. Membership dues for 1 year are $25.00 in Canada, $27.00 in United States and $30 outside Canada and United States Membership dues for 5 years are $112.00 in Canada, $122.00 in United States, $140.00 outside of Canada and United States. Orders for memberships should be sent to:

Secretary-Treasurers

Don and Carol Attridge, O.A.M.E.
1 Southdale Dr.
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