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ONTARIO MATHEMATICS GAZETTE

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E D I T O R I A L

In this issue, I feel like discoursing on a rather unusual topic, namely, GRAMMAR. You may wonder how this is connected with mathematics, but I hope to show you that there is a very pertinent connection.

Some years ago, I taught Portuguese in the University of Toronto. At that time, we were at the very height of the influence of the "progressive educators" in the Province of Ontario. A secondary-school inspector could pontificate to a group of teachers "I hope that I never catch any teacher in any of my schools teaching grammar". Well, he probably didn't! At least, I found that students beginning first year Portuguese did know the meaning of the words "noun" and "verb"; they had heard of the word "pronoun", but weren't sure what it meant; and they could not recall ever encountering the words "adjective", "adverb", "preposition", "conjunction".

This ignorance of the basic building blocks of English prose does not seem to have decreased over the years. Recently I have had occasion to attempt to tutor a student in University mathematics. He is really quite a gifted student, but is having considerable difficulties which are due to a poor background in English (gained in the educational system of one of our large cities). These difficulties stem from weaknesses in "Reading, Writing, Spelling, Grammar". And most of them boil down to the shocking fact that he nowhere encountered "sounds" or phonetics in his primary and secondary school education. A word like "indubitable" is very difficult for him, since he has to try to memorize the spelling of the word, rather than sound it out as a collection of syllables. While he has a good memory, it is too much to ask any student to memorize a dictionary without any basic principles; so I got an old Ontario Speller of 1900 vintage and started him in on this Speller, beginning at Grade I. He was quite astonished to find the words that students in Primary School encountered in those days; and it did him a great deal of good. Needless to say, his difficulties in reading and his slowness in writing were also based, to quite an extent, on his hopeless attempt to memorize each word as a basic unit rather than recognize phonemes as units.

The other difficulty which this same student has was the inability to construct meaningful sentences. He could understand mathematics, but he could not express himself. He could not get his thoughts into words. I quickly found that a great deal of this difficulty was explained by his lack of any knowledge of the parts of speech. We do not expect our students to do mathematics without learning $9 + 7 = 16$ (although apparently there have been progressive educators who have felt that this knowledge would appear in children's minds by osmosis without any effort or memorization); similarly, we can not expect students to form clear and correct statements if they do not know the basic units of sentence structure.

Now who is responsible for the lack of grammatical knowledge which this student shows. Probably he is himself responsible

up to a point. But the main responsibility rests upon a group of politicians: the various men who, during the last twenty-five years, have held the portfolio of Minister of Education for the Province of Ontario. These men have not been educators, and they have allowed this sort of situation to develop. Now a great reform is needed; we must have more work on sentence structure in the primary schools and even in the secondary schools. Do you remember the old "Entrance Examinations" from Grade 8 (Senior Fourth then)? How many students have even seen the word "parse" today? Do you remember when we had to "pick out the subordinate clauses and state their kind and relation"; how many students today could do that? Or indeed, how many would know what a clause is? Yet if you do not have a clear idea as to what a clause is, how can you use one in building up a sentence?

I have laid blame on the Ministers of Education for the de-emphasis on grammar and sentence structure (I realize that there is an improvement at present, although we still have a long way to go). But this blame must be shared. Secondary school teachers, through their Federation, and University professors, through their Universities, could have protested against unsound tendencies. Their sins have been sins of omission rather than commission; however, they have been silent when they should have spoken. They should have been more active in curriculum development and in the prevention of curriculum decay.

And now for the moral of the editorial as regards mathematics. We are stressing the teaching of mathematical concepts and ideas more; this is excellent. But let us not forget that mathematical skills must continue to be stressed. Understanding is a splendid thing, but the ability to do mathematics is also essential. Already more students probably fail in University mathematics from inability to do simple things, like finding least common denominators, than from any other single cause. Let us have both concepts and skills.

Many of you will be able to attend the O.E.A. at Easter and hear Professor Beberman speak on one of the several approaches to mathematics currently being used in the United States. Try to hear him, but remember that he would undoubtedly be one of the first to point out that many other approaches are also being tried. As Professor Hlavaty pointed out last year, there is no one of these approaches which will be the "future programme". The task of the educators of this province, at all levels, is to be aware of what is being done, and then make sure that the Ontario programme avoids extremism and excess; that it uses all the good individual ideas from the various trial programmes that are suggested; that it also retains all the good ideas of present programmes; that tried and true material be kept, whether it is labelled "new" or "old", whether it is labelled "modern" or "traditional". Labels do not matter; sound mathematics does, and it is what all of us are aiming at.

Ralph G. Stanton
Editor-in-Chief

A METHOD OF FINDING INTEGRAL SOLUTIONS OF THE EQUATION $x^2 + y^2 + z^2 = w^2$

Edward Evans, Guelph

In Professor K. O. May's Elementary Analysis (No. 12, p.582, 1956 edition), there is the following problem.
"Find the equations of the planes passing through (10,2,6) and (8,6,1) and tangent to a sphere of radius 5 with centre at the origin."

The solution of this problem involved a surd, $\sqrt{3411}$. It occurred to me to try to find a similar problem involving integral coefficients. I used a sphere of radius 6, and a known point (4,2,-4) on it. The tangent plane at this point is $2x + y - 2z = 18$. Then, taking two other points at random on this tangent plane, provided they are not collinear with (4,2,-4), I can get the other tangent plane through these two points. For example, taking (9,8,-4) and (-7,4,-14), I get as the two tangent planes

$$2x + y - 2z = 18, \quad 38x + 109y + 70z = -810.$$

In verifying, I found that

$$(2)^2 + (1)^2 + (-2)^2 + (-2)^2 = (18/6)^2 = (3)^2,$$

and $(38)^2 + (109)^2 + (70)^2 = (-810/6)^2 = (135)^2$.

By varying the points I was able to get many such cases.

A simplified general form, using a sphere of radius r tangent to the plane $z = -r$ at the point $(0,0,-r)$, and finding the other tangent plane through $(a,b,-r)$ and $(c,d,-r)$, where $ad - bc \neq 0$ led, as was pointed out to me by Professor Pounder, to the identity

$$A^2 + B^2 + \left(\frac{A^2+B^2}{4n} - n\right)^2 = \left(\frac{A^2+B^2}{4n} + n\right)^2,$$

where $A = (a-c)$, $B = (b-d)$, $D = (ad-bc)$, and $D/2r = n$.

Letting $n = 1$,

$$A^2 + B^2 + \left(\frac{A^2+B^2}{4} - 1\right)^2 = \left(\frac{A^2+B^2}{4} + 1\right)^2.$$

Message from the President of the Mathematics and Physics Section of the O.E.A.

Today the fastest-growing and most radically changing of all sciences is mathematics. It is the only branch of learning in which all the major theories of 2000 years ago are still valid, yet never before has there been such a flood of fresh ideas. New developments have been extensive; new concepts have been revolutionary. Indeed mathematics today is an entirely different discipline from what it was at the turn of the century.

One inevitable result of this explosive development of mathematics has been the creation of new subject matter. Such fields as mathematical logic, probability, statistical inference, topology, and abstract algebra, are largely or wholly the products of recent mathematical research.

Another result of the recent growth of mathematics has been the reorganization, extension, and transformation of parts of older mathematics. For example, earlier mathematicians often looked upon algebra from a manipulative point of view. Skill in performing operations within the system was the goal rather than the understanding of the properties of the system. It was more important to know that $(+5)(-3) = (-15)$ than to know the reason why this was so. In other words, rote was emphasized in place of understanding. The contempory point of view, while not ignoring the manipulative skills necessary for efficient mathematical thought, puts the chief emphasis on the structure or pattern of the system and upon deductive thinking.

The difference in point of view between the older approach and the modern one is well put by W.W. Sawyer in his book, Prelude to Mathematics:

> "The mathematician of older times asked, 'Can I find a trick to solve this problem?' If he could not find a trick today, he looked for one tomorrow. But ... we no longer assume that a trick need exist at all. We ask rather, 'Is there any reason to suppose that this problem can be solved with the means that we have at hand? Can it be broken up into simpler problems? What is it that makes a problem solvable, and how can we test for solvability?' We try to discover the nature of the problem that we are dealing with."

The fact that mathematics can now be taught with greater understanding, the fact that the mathematics of the past fifty years is playing an increasing role in our society, is now being recognized in Ontario. It is to be hoped that

more and more teachers will recognize the great advantage that this approach has for the majority of students and will take the necessary steps to see that their students do not lose out.

Two items on our Easter Programme are of special interest to teachers interested in learning more about the new approach. Our main speaker, Dr. Max Beberman, has been a leader in the United States in teaching mathematics with understanding. One of the seminars on Tuesday afternoon will be a report of the use of the new approach with Grade 7 students.

I hope that I shall see you at the Easter Meeting.

Father John Egsgard, C.S.B.

Ontario Association of Teachers of

Mathematics and Physics

Easter Convention

Programme

Ontario College of Education,
Room 374 and Adjacent Rooms.

Tuesday, April 16, 1963.

12:45 p.m.	Registration and Examination of Book Displays.	
1:15 p.m.	Minutes; Financial Statement; President's Remarks.	
1:30 p.m.	Dr. Max BEBERMAN:	Positive and Negative Numbers and How to Get Them from the Real World.
2:30 p.m.	Dr. Norman MILLER:	How to Enrich the Mathematics Programme for the Better-than-average Pupil.
3:15 p.m.	Report on N.C.T.M. Coffee Break	
3:45 p.m.	Seminars:	I - The Grade VII Experiment in Toronto Schools on the New Approach to Mathematics. - A Panel Discussion - Chairman: Mr. Wynn Bates (Toronto Math. Consultant)
		II - The Geometry Part of the Experimental Course in Grade XI Mathematics. - Chairman: Mr. A. Lebow

(Central S.S. - Hamilton)
- Speakers: One or more of the authors of the experimental Gr. XI text.

III - Suggestions for the Teaching of Euclidean Geometry.
- Chairman: Mr. Lloyd Auckland (St. Thomas C.I.)
- Speaker: Mr. David Palmer (Sarnia Central C.I.)

IV - Graphing Through the Grades.
- Chairman: Mrs. Helen Hay (Queen Elizabeth C.V.I.)
- Speaker: Dr. Douglas Crawford (Queen's University).

4:45 p.m. Adjournment

8:00 - 10:00 p.m. Reception (see below)

Wednesday, April 17, 1963.

9:00 a.m. Examination of Book Displays. Report of Nominating Committee.

9:15 a.m. Mr. James MACLACHLAN: Mathematical Implications of the New Grade XI Physics Course.

10:00 a.m. Dr. Max BEBERMAN: Functions and Geometry.

11:00 a.m. Prof. Bruce MACLEAN: Programmed Learning and its Implications for Teachers of Mathematics.

11:45 a.m. Introduction of New Executive. New Business. Adjournment for Luncheon.

12:30 p.m. Luncheon.

RECEPTION FOR CONVENTION SPEAKERS

The executive and councillors of the Mathematics and Physics Section of the Ontario Educational Association are arranging a reception at which members, their wives and husbands, may meet the speakers at the Easter Meeting.

Time: Tuesday, April 16th, 8 p.m. to 10 p.m.

Place: Sixth Floor - Education Centre, 155 College Street, Toronto.

Parking: In-door garage - McCaul Street, opposite the Education Centre - no charge.

Come, meet the speakers, and chat with your colleagues!

Refreshments will be served.

Luncheon

Members attending the Luncheon last year spoke so well of the tasty meal and the convenience of the new location that we felt that it was the general wish to return to the same spot again:

BLOOR STREET UNITED CHURCH

- North side of Bloor St., directly across from O.C.E.

Tickets will be on sale on Tuesday afternoon and on Wednesday morning.

THE CONVENTION SPEAKERS

1. Dr. Max Beberman.

Dr. Beberman was born in New York City, where he received his early education. He graduated in Mathematics from the College of the City of New York, and received his doctorate in Education from Columbia Teachers' College.

He is a speaker whose experience in the teaching of Mathematics and Science covers all grades in Public and High Schools to University entrance. His teaching career has taken him as far north as Nome, Alaska, and as far south as Florida State University. He has also held teaching posts in New York City, in Columbia Teachers' College, and in the University of Illinois High School. If he needs further recommendation, there is the fact that he is the father of seven children.

At present, as well as teaching in the High School at the University of Illinois, Dr. Beberman is Professor of Education at the University, and is director of the Committee on School Mathematics, a project co-sponsored by the University and various foundations.

Dr. Beberman is an enthusiastic and gifted speaker. If your teaching needs a spring tonic, his addresses will supply it.

2. Dr. Norman Miller.

Dr. Miller received his Secondary School Education in Aylmer Collegiate Institute, Elgin County, and his University education at Queen's and Harvard, finishing with a doctorate in Mathematics.

His teaching career, which began at Seaforth Collegiate Institute, was interrupted by the first world war during which time Dr. Miller served overseas. In 1919 he taught for a short time in Khaki College, England, but Queen's University knew him and called him to its Department of Mathematics

in 1919. Here for forty-two years Dr. Miller taught and inspired students. During these years he was co-author of textbooks in Mathematics for secondary schools and kept in close touch with Secondary School Teachers. Of the Easter meeting of the O.E.A., he writes: "Attending the O.E.A. Easter meeting is very much like going to church. If you make it a habit, then you don't have to make up your mind each time whether to go or not."

Many of us have been fortunate enough to work under Dr. Miller during his long and helpful career or to meet him and his charming wife who is also a graduate in Mathematics. We have heard how graciously and effectively he is working with gifted students in Mathematics from the schools in Kingston, and we are anxious to know how he is meeting the practical needs of these young people. To those of you who do not know him, we say: "Come to the convention and hear a dedicated teacher whose charm, wit, and practical good sense will impress you".

3. Bruce MacLean.

Bruce MacLean is a graduate of Oakwood Collegiate Institute, Toronto, and of McMaster University, Hamilton. Upon graduating in Mathematics and Physics, he began postgraduate work at McMaster and a teaching career in North Toronto Collegiate Institute. During the war, he joined the Air Force as a Radar Officer and later went on to assist the Royal Canadian Navy in its Radar Training Plan. After the war, he finished his postgraduate work, and collaborated with other naval officers in writing for the Navy its current textbook on Radio and Radar Systems.

His career as a teacher and author has continued. For fourteen years he was with University of Toronto Schools, and at present is Associate Professor of Methods in Mathematics at the Ontario College of Education. He is also a member of a well-known team of writers which has provided textbooks in Mathematics for High Schools.

Those of us who know Bruce MacLean expect a stimulating and frank message. To those of us who do not know him may, we say this: "If you feel it's been pounded into you when he finishes, remember he's a born teacher, and he can't do it any other way.

4. James MacLachlan.

James MacLachlan graduated in Engineering Physics at the University of Toronto and began his career with the Defence Research Board. Fortunately for our profession, he began teaching in 1953; at present, he is head of the Science Department at W. A. Porter Collegiate Institute in Scarborough.

His activities are not confined to the classroom. The summer of 1958 found him in Boston working with the Physical

Science Study Committee on the development of a new Physics course. Since then he has served on the Physics Curriculum Committee of the Ontario Department of Education, and has been busy helping to prepare a textbook for Grade Eleven Physics. In addition, he is an active member of the Canadian Association of Physicists.

James MacLachlan is well qualified to speak to us, and teachers of Mathematics or Physics should find helpful suggestions in his address.

SEMINARS

The introduction of Seminars at the conference last year was favourably received and comments to the effect that they should become a regular part of the programme were made frequently. Mr. Garth Kaye has acted as co-ordinator of committees arranging the seminars for the coming meeting. These committees have planned seminars based on topics rather than grade level. As usual, the topics will be opened up by the leaders, and then a period will be devoted to group discussion. Pick your group in advance and come prepared to add your bit to the discussion if you care to, or just relax and enjoy the thoughts of your friends.

PARKING

Please note that the University of Toronto now has an attendant on each of its parking lots. A charge, on an hourly basis, is made for parking on these lots, including the one at O.C.E. The main streets in the district have parking meters and most of the side streets have "One hour parking" signs.

We feel that you should know about this in advance so that you will be prepared; the hourly charge is small (15 cents), but, if you prefer, the T.T.C. provides fine service to and from the building.

THE SPHERICAL MIRROR

Douglas E. Scovell
Malvern C.I., Toronto

In the recent article in the "Ontario Mathematics Gazette" entitled "On the Mathematics Curriculum of the High School", a plea is made for a greater correlation between science and mathematics at the high school level. The following problem might serve as an instructive application of analytic geometry

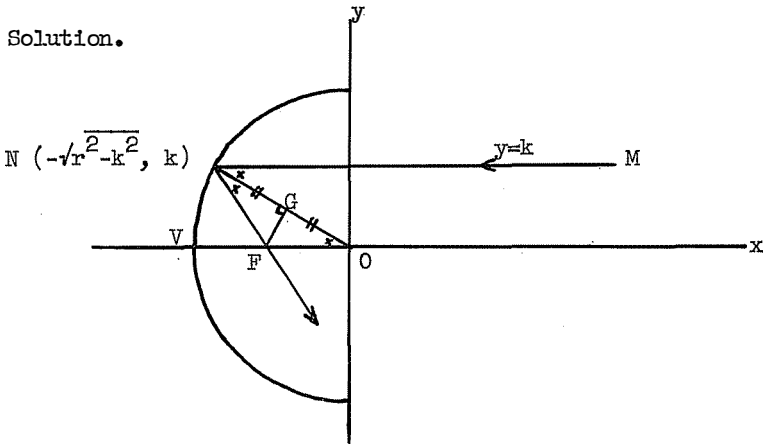
to a problem in physics.

In a study of spherical mirrors high school texts usually state that experiment shows that a ray of light travelling toward a spherical mirror parallel to its axis will be brought to a focus at a point on the axis approximately half way between the vertex of the mirror and the centre of curvature. This feature of the spherical mirror is responsible for its ability to form images of real objects. The closer the impinging ray is to the axis, the better will be the approximation. Practically, this means that spherical mirrors are usually made quite shallow.

Problem: Assuming only the law of reflection - the angles of incidence and reflection, measured with respect to the normal, are equal - and geometric properties of the circle, apply analytic geometry to a plane section of a spherical mirror to deduce the above physical phenomenon.

The reader might like to try this problem himself before looking at the following

Solution.



Let a plane section of the mirror be represented by the circle whose equation is $x^2 + y^2 = r^2$.

Let MN be a ray of incoming light which is || to the x-axis and at a distance of k units above the x-axis. The equation of MN is $y = k$.

The coordinates of N are $(-\sqrt{r^2-k^2}, k)$.

The normal to N is NO, the radius to N. From the law of reflection, $\angle MNO = \angle ONF$. Since MN || FO, $\angle NOF = \angle ONF$; hence $\triangle ONF$ is isosceles. FG, the right-bisector of NO passes the rough F, the point where the reflected ray cuts the axis of the mirror (x-axis). We require the abscissa of the point F.

The coordinates of G are easily seen to be $(-\frac{1}{2}\sqrt{r^2-k^2}, \frac{1}{2}k)$. The equation of GF is

$$y - \tfrac{1}{2}k = (\sqrt{r^2-k^2}/k)(x + \tfrac{1}{2}\sqrt{r^2-k^2}).$$

When $y = 0$, we get

$$-\tfrac{1}{2}k = \sqrt{r^2-k^2}\,(x + \frac{\sqrt{r^2-k^2}}{2})/k;$$

thus

$$x = \frac{-r^2}{2\sqrt{r^2-k^2}} = \frac{r}{2\sqrt{1-(\frac{k}{r})^2}}.$$

Note that this result is dependent on k, the distance of the light ray from the axis. When k is small relative to r, $x \doteq -\frac{1}{2}r$; this demonstrates that the focal point is mid-way between the centre of curvature O and the vertex V, when the incoming light-rays are close to the axis.

GRADE 13 PROBLEMS, 1962

2. Given $n + r$ letters, of which n are A's and r are B's, show that the number of different sequences that be formed from the A's and B's if each sequence must contain all n A's, is ${}_{n+r+1}C_r$.

Solution 1. The statement may be proved inductively. Let $N = n+r$. The theorem is true for $N=1$; if $n=1$, $r=0$ there is one sequence, and if $n=0$, $r=1$ there are two sequences (we must include the sequence with no elements).
Assume the theorem is true for $N=k-1$ (i.e., for all n and r such that $n+r = k-1$). Choose any r and n such that $n+r=k$. The corresponding sequences are of two types, those beginning with an A and those beginning with a B. The sequences beginning with an A are in one-to-one correspondence with the sequences of n-1 A's and r B's (there are ${}_kC_r$ such); similarly the sequences beginning with a B are in one-to-one correspondence with the sequence of

n A's and r-1 B's (there are ${}_kC_{r-1}$ such). The number of sequences with n A's and r B's is thus given by

$$
\begin{aligned}
{}_kC_r + {}_kC_{r-1} &= \frac{k!}{r!(k-r)!} + \frac{k!}{(r-1)!(k-r+1)!} \\
&= \frac{(k+1)!}{r!(k-r+1)!} \\
&= {}_{k+1}C_r \,.
\end{aligned}
$$

Solution 2. The sequence will contain n A's and m B's where m ranges from 0 to r. The number of ways of arranging n+m objects of which n are A's and m are B's is

$$\frac{(n+m)!}{m!\,n!} \,.$$

So the answer is

$$\sum_{m=0}^{r} \frac{(n+m)!}{m!\,n!} = \sum_{m=0}^{r} {}_{n+m}C_m \,.$$

If r = 0, it is clear that

$$(*) \qquad \sum_{m=0}^{r} {}_{n+m}C_m = {}_{n+r+1}C_r \,.$$

Now we use induction. Let k be a value of r for which (*) is true. Then

$$\sum_{m=0}^{k} {}_{n+m}C_m = {}_{n+k+1}C_k \,.$$

Try the value k+1. Then

$$
\begin{aligned}
\sum_{m=0}^{k+1} {}_{n+m}C_m &= \sum_{m=0}^{k} {}_{n+m}C_m + {}_{k+1+n}C_{k+1} \\
&= {}_{n+k+1}C_k + {}_{n+k+1}C_{k+1} \\
&= {}_{n+k+2}C_{k+1} \,.
\end{aligned}
$$

This result has the required form; so the result holds, by induction.

4. It is estimated that a certain mine will yield a net annual income of \$15,000 for the next 25 years, at the end of which time it will be exhausted. Calculate the present value of the mine if the purchaser expects a return of 9% per annum on his investment, and can accumulate a replacement fund at 4% compounded annually.

Let P be the present value of the mine. At the end of 25 years the investor expects a return of $V=P(1.09)^{25}$. On the other hand, the proceeds from the mine and the replacement fund will be given by

$$V = 15{,}000\,(1 + 1.04 + (1.04)^2 + \ldots + (1.04)^{24})$$

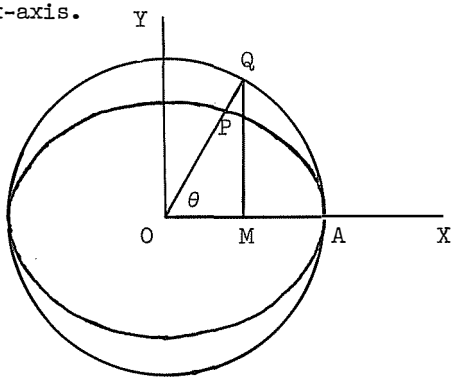
$$= 15{,}000\,\left(\frac{(1.04)^{25}-1}{.04}\right).$$

Equating these two expressions for V we find $V = \$72{,}443.79$.

6. (a) Prove that the coordinates of any point $P(x,y)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be represented by $x = a\cos\theta$, $y = b\sin\theta$. The parameter θ is known as the eccentric angle of the point (x,y); interpret θ geometrically.

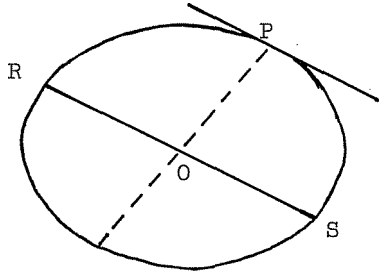
(b) Prove that the eccentric angles of point on the ellipse situated at the ends of conjugate diameters differ by a right angle.

(a) Let Q be the point on the circle $x^2+y^2 = a^2$ such that OQ makes an angle θ with the positive x-axis, and let M be the foot of the perpendicular from Q into the x-axis.



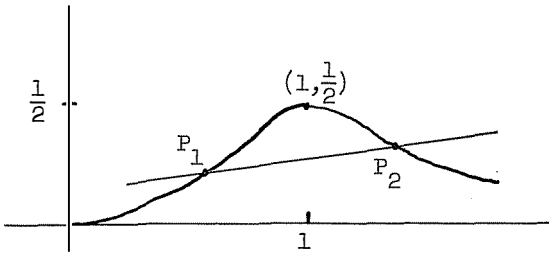
Then $OM = OQ\cos\theta = a\cos\theta$, and QM cuts the given ellipse in the point P. Also $PM = \frac{b}{a}\sqrt{a^2-OM^2} = b\sin\theta$.

(b) The tangent at P to the ellipse has slope $-(b\cot\theta)/a$. It is easy to verify that the line through



O parallel to this tangent cuts the ellipse at the points R[a $\cos(\theta+90^{\circ})$, b $\sin(\theta+90^{\circ})$], and S[a $\cos(\theta-90^{\circ})$, b $\sin(\theta-90^{\circ})$]. Statement (b) is immediate, since R and S are the ends of the diameter conjugate to the diameter through P.

8. Find, as a function of x, the slope of the tangent to the curve $y = \frac{\sqrt{x}}{1+x}$ at any point P(x,y) on the curve.



Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be points on the curve, so that

$$y_1 = \frac{\sqrt{x_1}}{1+x_1} \quad \text{and} \quad y_2 = \frac{\sqrt{x_2}}{1+x_2} .$$

The slope of P_1P_2 is given by

$$\frac{y_1-y_2}{x_1-x_2} = \frac{\frac{\sqrt{x_1}}{1+x_1} - \frac{\sqrt{x_2}}{1+x_2}}{x_1-x_2} = \frac{1 - \sqrt{x_1x_2}}{(1+x_1)(1+x_2)(\sqrt{x_1}+\sqrt{x_2})}$$

The slope of the tangent at $P_1(x_1,y_1)$ is the limit of the slope of the secant P_1P_2 as $P_2 \longrightarrow P_1$ along the curve.

Setting $x_2 = x_1$ and $y_2 = y_1$ in the expression for the slope of the secant P_1P_2 we have the slope of the tangent at P_1, namely,

$$\frac{1-x_1}{2(1+x_1)^2\sqrt{x_1}} .$$

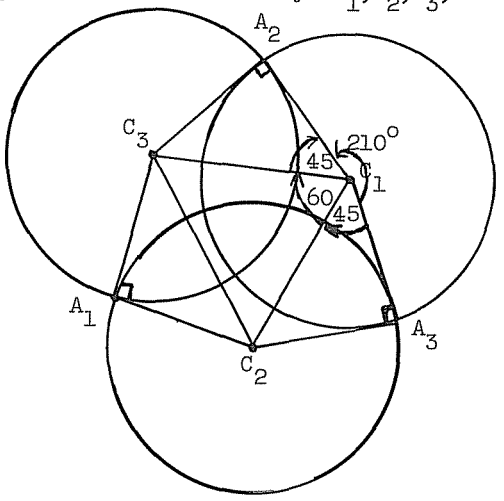
The slope of the tangent at $P(x,y)$ then is

$$\frac{1-x}{2(1+x)^2\sqrt{x}} .$$

The problem can also be solved using Calculus; the slope of the tangent is given by the derivative $\frac{dy}{dx}$.

10. Three equal circles, with radii r units, are so placed in a plane that each cuts both the others orthogonally. Find the area of that part of the plane which is covered by at least one of the circles.

Let the centres be denoted by C_1, C_2, C_3, and the boundary



points of intersection by A_1, A_2, A_3 as shown in the diagram. Consider $\triangle C_1A_2C_3$; $C_1A_2 = C_3A_2 = r$ and the orthogonality implies that A_2 is a right angle. Thus $C_1C_3 = \sqrt{2}\, r$ and the area of $\triangle C_1A_2C_3$ is $r^2/2$. Similarly $C_1C_2 = C_2C_3 = \sqrt{2}\, r$. Thus triangle $C_1C_2C_3$ is equilateral with area $\frac{\sqrt{3}}{2} r^2$.

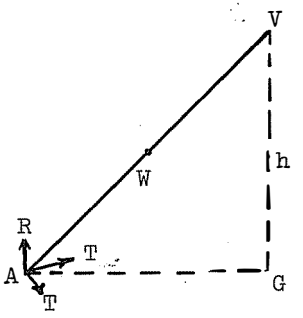
Finally consider the region bounded by the first circle and the radii C_1A_2 and C_1A_3. The angle $A_3C_1A_2 = 45^\circ + 60^\circ + 45^\circ = 150^\circ$ so that the angle within the area is 210° and the area is $\frac{210}{360}\pi r^2 = \frac{7}{12}\pi r^2$. The two similarly situated regions have the same area by symmetry.

Thus the total area A is given by

$$A = 3(\frac{r^2}{2}) + \frac{\sqrt{3}}{2}r^2 + 3(\frac{7}{12}\pi r^2) = (\frac{3}{2} + \frac{\sqrt{3}}{2} + \frac{7}{4}\pi)r^2.$$

12. A tripod is formed by three uniform rods of equal length, each weighing w pounds. The rods are loosely joined at the top, which is h feet above a smooth floor on which the feet rest. The feet are held in position at the vertices of an equilateral triangle of side d feet by a loop of string stretched around them on the floor. Show that, if the tension in the string is T pounds, then $T = \frac{wd}{6h}$.

Let V denote the top; A,B,C, the vertices of the equilateral triangle; G the foot of the perpendicular from V to the floor. Because of symmetry it is sufficient to consider the system composed of one of the rods and parts of the string as shown in the diagram. The weight of the rod w is downward and acts through the centre of the rod. Since the floor is smooth, the reaction at A is upward. Thus $R = w$.



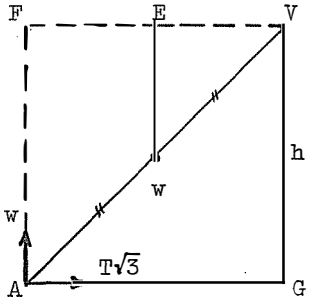
To find T take moments about a line through V perpendicular to the plane VAG. Notice that the two tensions in the string are equivalent to a single force of magnitude $T\sqrt{3}$

along AG. We then have, using the notation of the second diagram,

$$T\sqrt{3}\,h + w\,EV = w\,FV;$$

hence

$$T = \frac{w\,FE}{\sqrt{3}\,h} = \frac{w\,AG}{2\sqrt{3}\,h}.$$



Since G is the centroid of the triangle ABC of side d, a simple computation shows $AG = d/\sqrt{3}$ which yields the desired formula.

BOOK REVIEWS

Essentials of Mathematics.

by Russel V. Person, John Wiley and Sons Inc., New York - London, 1961. 646 pages.

The book has five major sub-divisions: arithmetic, geometry, algebra, logarithms, and trigonometry. The author develops his subject by a discussion of the method of solution along with several examples of type problems. Each chapter is well supplied with problem exercises which stress the applications of mathematics to problems in technology.

Most of the present course of study for Grade XI and Grade XII is in the book; however, it does not include a section on the Slide Rule, and the development of statics is confined to vectors.

By using simple wording and a liberal number of clear and detailed diagrams, the author has produced a text which an average technical student should have no difficulty in reading.

Although this book does not contain all the material outlined in the course of study for Grades XI and XII, it is admirably suited as a reference text for both students and teachers of technical mathematics.

D. Walsh, H.B. Beal Technical-Commercial High School, London.

Fundamentals of Freshman Mathematics.

by C.B. Allendoerfer and C.O. Oakley, McGraw-Hill Book Company Inc., 475 pages (Price $7.10).

The authors have written this text to further the cause of modern mathematics in high school and college curricula. In 1955, the same company published their Principles of Mathematics. In publishing this new text, they have implemented recommendations of various mathematical committees and their main purpose is "to bridge the gap between Intermèdiate Algebra and Analytic Geometry and Calculus".

The text begins with a discussion of the nature of abstract reasoning and mathematical proof followed by a thorough treatment of the number system. The next eight chapters deal with algebra, which ranges from middle school work to university work, arranged in a very interesting order. Simultaneous equations are treated in detail and the algebra of matrices is introduced and applied to the solution of simultaneous equations. The following is a partial list of topics for individual chapters: Sets and Equations, Inequalities, Functions and Relations, Algebraic Functions, Logarithmic and Exponential Functions. The Trigonometry is divided into two sections, computational and analytic. Another chapter deals with Analytic Geometry. The remainder of the text, 71 pages, deals with limits and calculus.

There is no lack of problems in the text and answers are given for the odd-numbered problems. Sample solutions for problems are given throughout. The average high school student would probably experience difficulty in reading the text. It would be a useful reference text for any high school teacher of mathematics or science as a source for material to enrich the better-than-average class.

B. Jackson, H.B. Beal Technical-Commerical High School, London.

Elements of Linear Algebra.

by Lowell J. Paige and J. Dean Swift, Ginn and Company, Toronto 1961, xvi + 348 pages, $7.00.

This book is intended for a one-semester course in matrix algebra for sophomores and juniors in mathematics, physics, and engineering. The first six chapters start with a brief survey of sets, real numbers, fields, and mappings. A careful discussion of the algebra and geometry of three-dimensional vectors then leads to a thorough treatment of finite-dimensional vectors and linear equations. This is followed by a discussion of Euclidean n-space and determinants. The sixth chapter, on linear transformations and matrices, treats the elementary algebra of matrices and establishes the connection between matrices and linear operators.

Chapter Seven is on the concept of rank, groups, and the orthogonal group. Chapter Eight, on bilinear and quadratic forms, goes as far as Sylvester's law of Inertia and a geometrical interpretation of positive definite forms as metrics. Chapter Nine introduces the complex field (including a proof that it is essentially the only two-dimensional division algebra over the real field), and treats the algebra of polynomials. Chapter Ten deals with the characteristic values and vectors of symmetric matrices under the orthogonal group over the real field, and the application of this idea to the geometrical study of quadratic forms under orthogonal congruence. The corresponding theory for hermitian matrices under the unitary group over the complex field is then sketched. Normal matrices are not treated and thus no analysis is given of the structure of orthogonal matrices. Finally Chapter Eleven deals fully with the rational and classical canonical forms for matrices under similarity.

The treatment, especially in the first half, is very slow-paced, with careful explanations of all difficult points and a lavish use of numerical examples to illustrate theorems, definitions, and also the crucial steps in a proof. While this is carried to great lengths in the opening chapters, it eventually becomes more moderate and ultimately (in the last chapter) probably insufficient. There are numerous examples and problems, the great majority of which are of an easy routine character or slightly extend the theory in the text. The accuracy of the proof-reading, the enunciations of definitions and theorems, and the validity of the proofs leave little to be desired.

One logical flaw occurs in the chapter on polynomials. It is necessary to show that the mapping $p(x) \longrightarrow p(a)$ from the ring of polynomials in an indeterminate x with coefficients in a field F, into the field F, is a homomorphism, in order to validate the author's proof of the remainder theorem.

It is a pity that block matrices are not discussed in the section on matrix algebra, because their sudden introduction in connection with canonical forms may prove disconcerting to the reader.

The authors feel that the first six chapters and the equivalent of one other chapter would make a suitable one-semester course. Such a course would, in my opinion, be rather dull, since most of the interesting material is in the second part of the book. It all depends on how much of the introductory material (analytic geometry, determinants and so on) is being treated in other courses, and how far you think you can go with sophomores.

B. Abrahamson, University of Toronto.

THE HERON FORMULA FOR THE AREA OF A TRIANGLE

The Heron Formula for the area of a triangle when given three sides a,b,c, is usually deduced, in Grade 13 Trigonometry, from the half-angle formulae. The following note shows how this formula can be developed independently, without assuming these formulae. We write:

$$\Delta = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} ab [(1 + \cos C)(1 - \cos C)]^{\frac{1}{2}}$$

$$= \frac{1}{2} ab \left[1 + \frac{a^2 + b^2 - c^2}{2ab}\right]^{\frac{1}{2}} \left[1 - \frac{a^2 + b^2 - c^2}{2ab}\right]^{\frac{1}{2}}$$

$$= \frac{1}{4} [(a^2 + 2ab + b^2) - c^2]^{\frac{1}{2}} [c^2 - (a^2 - 2ab + b^2)]^{\frac{1}{2}}$$

$$= \frac{1}{4} [(a + b - c)(a + b + c)(c + a - b)(c - a + b)]^{\frac{1}{2}}$$

Now let $a + b + c = 2s$; we immediately obtain

$$\Delta = \frac{1}{4} [(2s - 2c)(2s)(2s - 2b)(2s - 2a)]^{\frac{1}{2}}$$

$$= [s(s-a)(s-b)(s-c)]^{\frac{1}{2}}.$$

Sin nx, Cos nx, Tan nx

The periodicity of the trigonometric functions is probably the most mathematically significant feature of trigonometry. A function f is said to be periodic if there is a constant p such that, for all values of x, $f(x) = f(x + p)$. The smallest positive value of p for which this is true is the period. The sine function, for example, satisfies the equation $\sin x = \sin(x + 2\pi)$. Thus the values of the sine of an angle recur in intervals of 2π radians. This recur-

rence does not take place in smaller intervals, and therefore 2π is the period of the function.

In drawing the graph of a trigonometric function, advantage should be taken of its periodic nature. If the graph is obtained for an interval equal to the period, then this part of the graph can be reproduced in other intervals to the right and the left. Since the graph of cosine x is exactly the same as that of sine x, but with a shift of $\pi/2$ radians in virtue of the identity $\cos x = \sin(x + \frac{1}{2}\pi)$, the period of the cosine function will also be 2π radians. However, $\tan x = \tan(x + \pi)$; consequently, the period of the tangent function is π radians.

Now what is the period of $\sin 2x$, or more generally, of $\sin nx$? Every time there is a change of 2π in the angle, the value of sine x recurs. As x varies from 0 to $2\pi/n$, nx varies from 0 to 2π so that the period of $\sin nx$ is $2\pi/n$. Thus the period of $\sin 2x$ is π, while the period of $\sin x/5$ is 10π. In $\sin nx$, if n is a constant greater than one, there is an accordian-like effect, with the curve being squeezed in so that the peaks are closer together (cf. Figure 1).

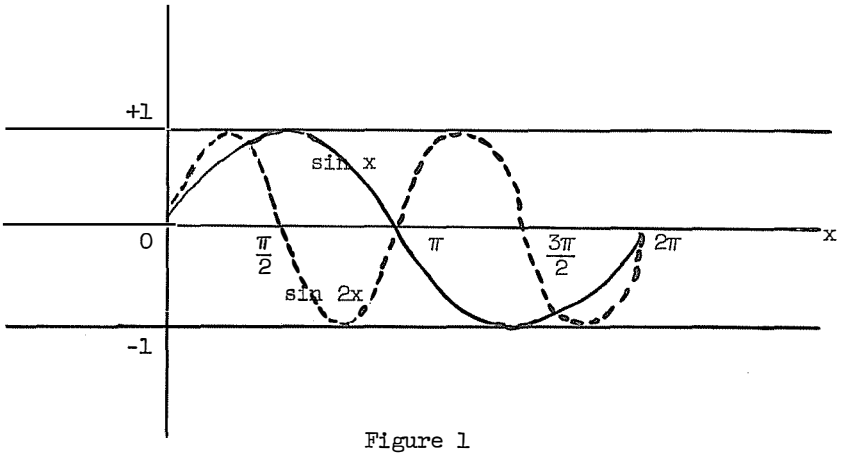


Figure 1

In Figure 1, we have drawn two periods of $\sin 2x$, but only one of $\sin x$. The period of $\sin x/2$ is 4π; hence the effect of multiplying x by a fraction less than one is to stretch the curve.

If $y = a \sin x$, the value of y oscillates between +a and -a. We say in this case that the amplitude of the curve is a. Thus the amplitude of $3 \sin x$ and $3 \sin 2x$ is 3. Similarly the amplitude of $4 \cos x$ and $4 \cos 2x$ is 4, but the period of the first curve is 2π radians while that of the second is π radians.

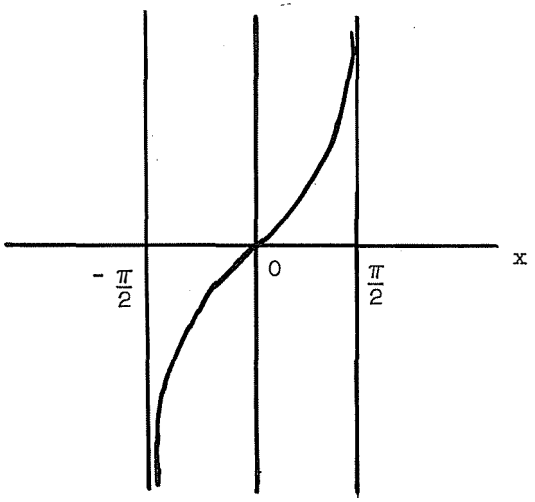


Figure 2

The graph of tan x goes through all possible values in the interval $-\pi/2 < x < +\pi/2$; so the period of the tangent function is π (cf. Figure 2). Similarly the period of tan 2x would be $\pi/2$ (cf. Figure 3).

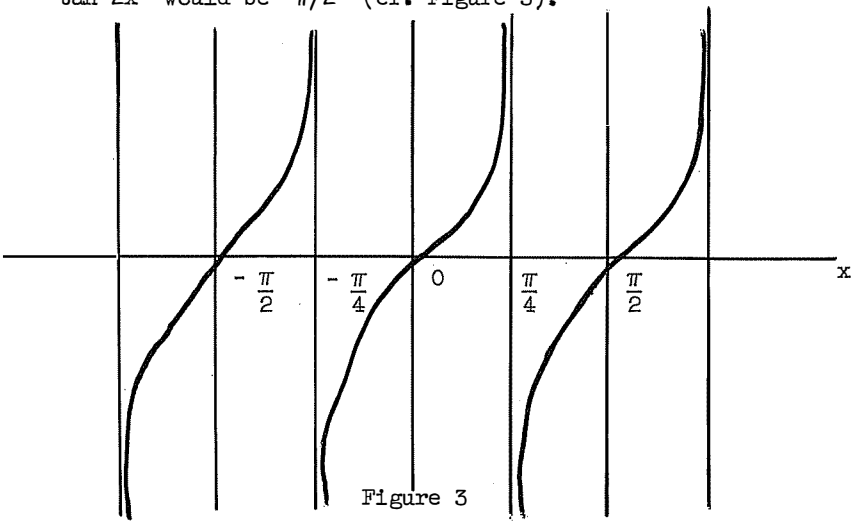


Figure 3

We should also mention that the graph of the curve $y = \sin(nx + c)$ crosses the axis at the point $(-c/n,0)$; the graph will then have its period in the interval from $-c/n$ to $-c/n + 2\pi/n$. If there is an amplitude factor a, we get the curve $y = a \sin(nx + c)$ which oscillates between $y = -a$ and $y = +a$, with period $2\pi/n$. This situation is illustrated in Figure 4 for the curve $y = 2 \sin(3x + \pi/6)$. Similar considerations hold for the other trigonometric curves.

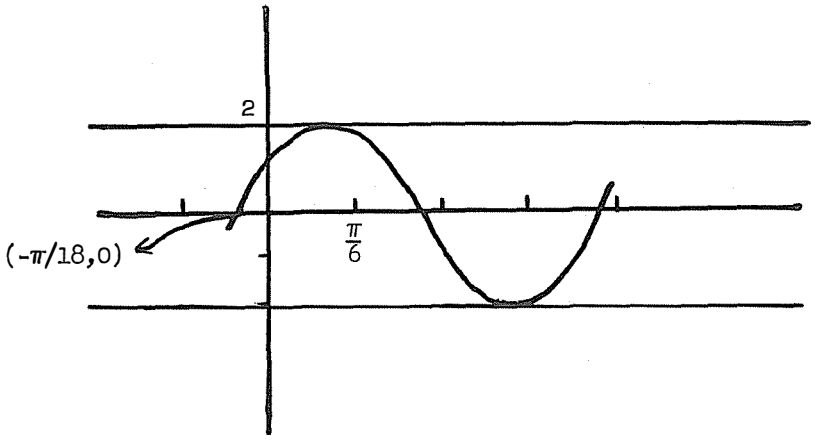


Figure 4

Finally, we should note that sometimes a more complicated curve can be drawn by adding ordinates of standard curves. In Figure 5, we have drawn the curves $y = 3 \sin x$ and $y = 4 \sin 2x$; then we have combined their ordinates by addition to give the compound curve $y = 3 \sin x + 4 \sin 2x$. The period of this compound curve is 2π, since $\sin x$ repeats after 2π units, and $\sin 2x$ repeats after π units.

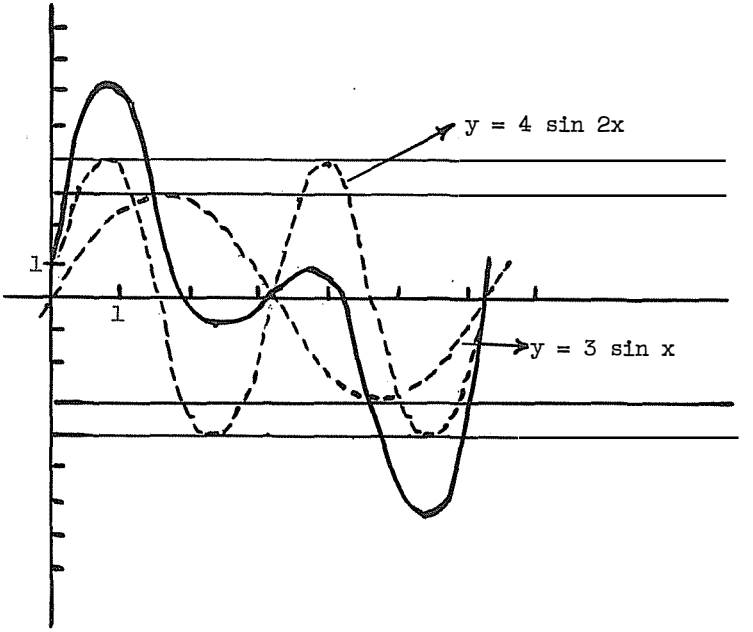


Figure 5

The chapter on the graphs of the trigonometric functions in Hall and Knight's Elementary Trigonometry is also recommended for reading by most students.

Exercises.

1. Find the period of each of the following functions, giving the amplitude of the sine and cosine graphs:

   (a) $\cos 3x$ (b) $\cos x/3$ (c) $\tan 5x$

   (d) $2 \sin 4x$ (e) $3 \cos \pi x$ (f) $\sin 25\pi x$

   (g) $10 \sin 120\pi x$

2. Sketch the graph of the following curves:

   (a) $y = 3 \sin x$ (e) $y = \tan x/2$

   (b) $y = 3 \cos x$ (f) $y = 5 \sin 4x$

   (c) $y = 2 \sin 2x$ (g) $y = 3 \cos x/3$

   (d) $y = 4 \sin x/2$

3. Sketch the graphs of the following curves:

   (a) $y = \sin (2x + \pi/3)$ (c) $y = 2 \sin 3x + \cos 2x$

   (b) $y = 3 \cos (2x + \pi/4)$ (d) $y = \sin 2x + 3 \cos 4x$

BIBLIOGRAPHIC MATERIAL

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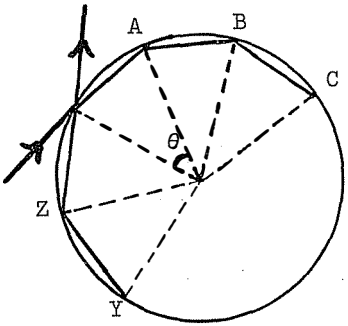
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PROBLEMS SOLUTIONS

Problem 2. (Proposed by Norman Gillespie) A ray of light is sent through a small hole into a hollow sphere where it is reflected in an equatorial plane. The reflected rays are equal chords of the equatorial circle, each subtending an angle θ at the centre of the circle. What is the condition satisfied by θ if the ray eventually emerges from the hole through which it entered?

Solution by M.C. England. Suppose that n reflections take place within the sphere at points A,B,C, ..., Y,Z. Suppose also that the beam travels N times around the interior of the sphere before emerging. From geometry and the laws of



reflection, we see that the equal chords subtend $n+1$ equal angles of size θ at the centre of the circle. Hence

$$(n+1)\theta = N(2\pi), \quad \theta = 2N\pi/(n+1).$$

Note that $0 < \theta < \pi$; so $0 < 2N < n+1$.

Since $N/(n+1)$ is the quotient of two integers, it is a rational number. Hence θ must be a rational multiple of π.

Conversely, let θ be a rational multiple of π, say $\theta = a\pi/b$, where a and b are positive integers. We suppose that a and b have no common factor. If a is even, we write $a = 2N$, $b = n+1$, and see that the number of reflections is $b-1$; the number of times the beam travels around the circle is $\frac{1}{2}a$. On the other hand, if a is odd, we write $\theta = 2a\pi/2b$, and see that the beam travels a times around the circle while undergoing $2b-1$ reflections.

Thus we see that the rationality of θ/π is a necessary and sufficient condition for the emergence of the beam.

Other solutions from: Norman Gillespie.

Problem 3. (Proposed by R.A. German) A tower of height h stands on a level plane. Points A and B lie in a straight line through the foot of the tower. AB subtends an angle α at the top of the tower and an angle 2α at a point exactly $h/4$ units up the tower. Express AB in terms of α and h.

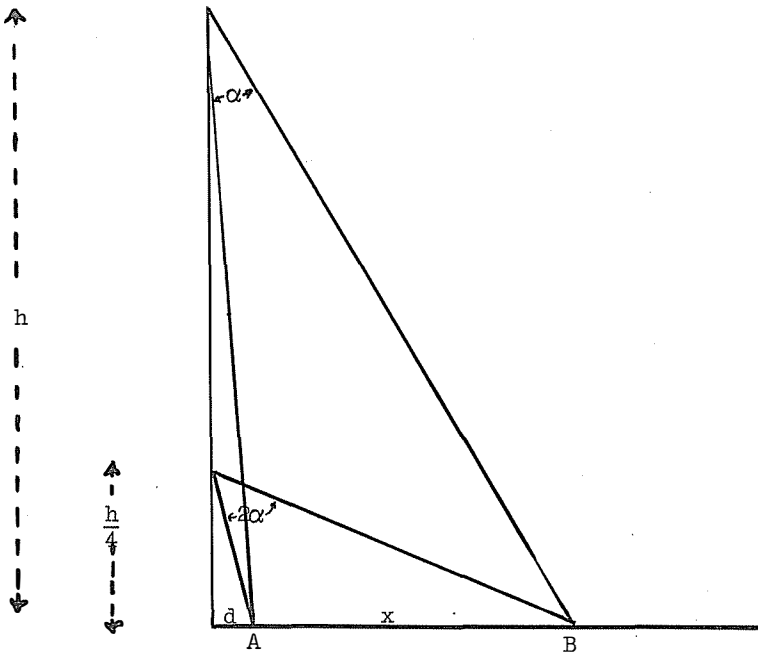
Solution by F.C. Asbury. In the accompanying diagram,

$$\alpha = \tan^{-1}\frac{h}{d} - \tan^{-1}\frac{h}{d+x}$$

$$2\alpha = \tan^{-1}\frac{h}{4d} - \tan^{-1}\frac{h}{4(d+x)}$$

Hence
$$\tan\alpha = \frac{\frac{h}{d} - \frac{h}{d+x}}{1 + \frac{h^2}{d(d+x)}} = \frac{hx}{d^2 + dx + h^2} \qquad (1)$$

$$\tan 2\alpha = \frac{\frac{h}{4d} - \frac{h}{4(d+x)}}{1 + \frac{h^2}{16d(d+x)}} = \frac{4hx}{16d^2 + 16dx + h^2} \qquad (2)$$



From (1) $d^2 + dx = hx \cot \alpha - h^2$

(2) $16d^2 + 16dx = 4hx \cot 2\alpha - h^2$

Hence $16hx \cot \alpha - 16h^2 = 4hx \cot 2\alpha - h^2$

i.e. $x(16h \cot \alpha - 4h \cot 2\alpha) = 15h^2$

and $\frac{x}{h} = \frac{15}{16 \cot \alpha - 4 \cot 2\alpha}$ where $\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha}$.

Substituting this last, and simplifying, we get

$$AB = x = \frac{15h \cot \alpha}{14 \cot^2 \alpha + 2} = \frac{15h \tan \alpha}{14 + 2 \tan^2 \alpha}$$

In the diagram, as drawn to scale,

$h = 100$ mm

$d = 6$ mm (obtained from equation (1), where)

$\alpha = 26^\circ 34'$ so that $\tan \alpha = .5$

$AB = 52$ mm.

Other solutions by: M.C. England, M. Gupta, R.A. German.

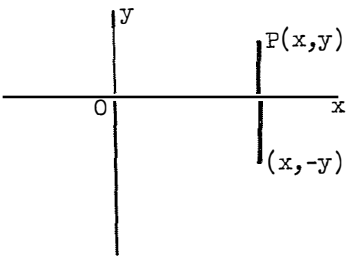
Editor's Note: If the value $h/4$ in the problem is replaced by h/u, where u is a known numerical value, then it is easy to show that the result is

$$x = \frac{2h(u^2-1)\cot\alpha}{u[(2u-1)\cot^2\alpha + 1]} = \frac{2h(u^2-1)\tan\alpha}{u[\tan^2\alpha + (2u-1)]} .$$

THE SYMMETRY LAWS FOR CARTESIAN CO-ORDINATES

Quite often useful information about a graph can be obtained by noticing whether or not it is symmetrical about the axes, about the origin, or about the 45° line. The present note gives a brief discussion of tests for symmetry.

1. Symmetry about the x-axis.

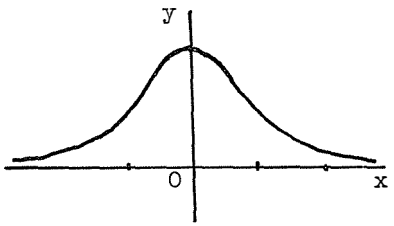


It is clear from a diagram that, if $P(x,y)$ is a general point, then its mirror image in the x-axis is the point $(x,-y)$. So a curve will be symmetrical about the x-axis if and only if it has the property that $(x,-y)$ lies on the curve whenever (x,y) lies on the curve. We can formally state this result as

Rule 1: A curve is symmetrical about the x-axis if and only if the equation of the curve remains unaltered when y is replaced by $-y$.

2. Symmetry about the y-axis. Exactly the same sort of argument as led to Rule 1 will produce

Rule 2: A curve is symmetrical about the y-axis if and only if the equation of the curve remains unaltered when x is replaced by $-x$.



Example 1. The witch, given by

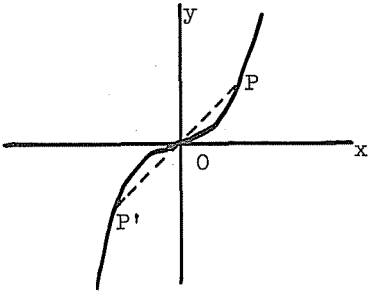
$$y = \frac{1}{1 + x^2} ,$$

is symmetrical about the y-axis.

3. Symmetry about the origin. We say that a curve is symmetrical about the origin if, whenever the point (x,y) is on the curve, the point $(-x,-y)$ also lies upon the curve. In short, to any point P on the curve corresponds

a point P' situated so that P'OP is a straight line bisected at O. We thus have

Rule 3: A curve is symmetrical about the origin if and only if the equation of the curve remains unaltered when x is replaced by -x and y is replaced by -y.

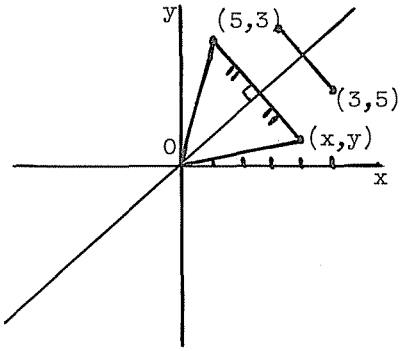


Example 2. The "cubical parabola" given by the equation

$$y = x^3$$

is an illustration of a curve which is symmetrical about the origin.

4. Symmetry in the line $y = x$.



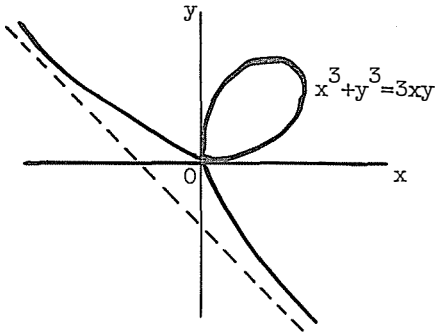
If we consider the line $y = x$, an elementary consideration of congruent triangles shows us that the mirror image of the point (3,5) is the point (5,3); in general, the image of the point (x,y) is the point (y,x). We thus obtain:

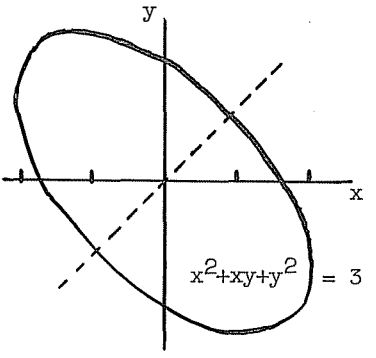
Rule 4: A curve is symmetrical in the 45° line if and only if the equation of the curve remains unaltered when x and y are interchanged.

Example 3. The folium of Descartes, which was also used in the last issue of the Gazette as an illustration, is an example of a curve which is symmetrical about the 45° line. Other examples are the circle

$$x^2 + y^2 = 25$$

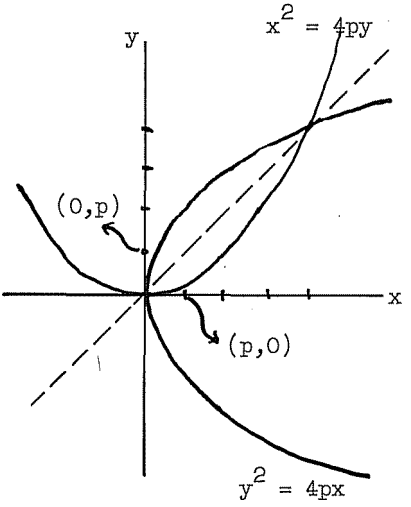
and the ellipse $x^2 + xy + y^2 = 3$.





The most useful applications of the Symmetry Rules are related to Rule 4.

Example 4. The student studies the properties of the parabola $y^2 = 4px$ in great detail; it has axis along the x-axis and focus at $(p,0)$. The reflected image of this parabola in the 45^o line will then be the parabola $x^2 = 4py$. This parabola will have its axis along the y-axis and its focus at the point $(0,p)$. Consequently, it is not necessary to study parabolas with their axes along the y-axis at all.

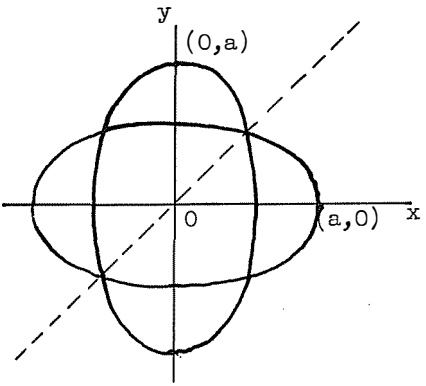


Example 5. The student also studies in detail the properties of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with major axis along the x-axis, foci at the points $(\pm\sqrt{a^2-b^2}, 0)$, vertices at $(\pm a,0)$.
It immediately follows that an ellipse with major axis along the y-axis will have equation

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$



It vertices will be at $(0,\pm a)$ and its foci at $(0, \pm\sqrt{a^2-b^2})$. All its properties will be "reflections" of those of the first ellipse.

Exercise.

Develop a condition for symmetry about the line $x = 1$.
