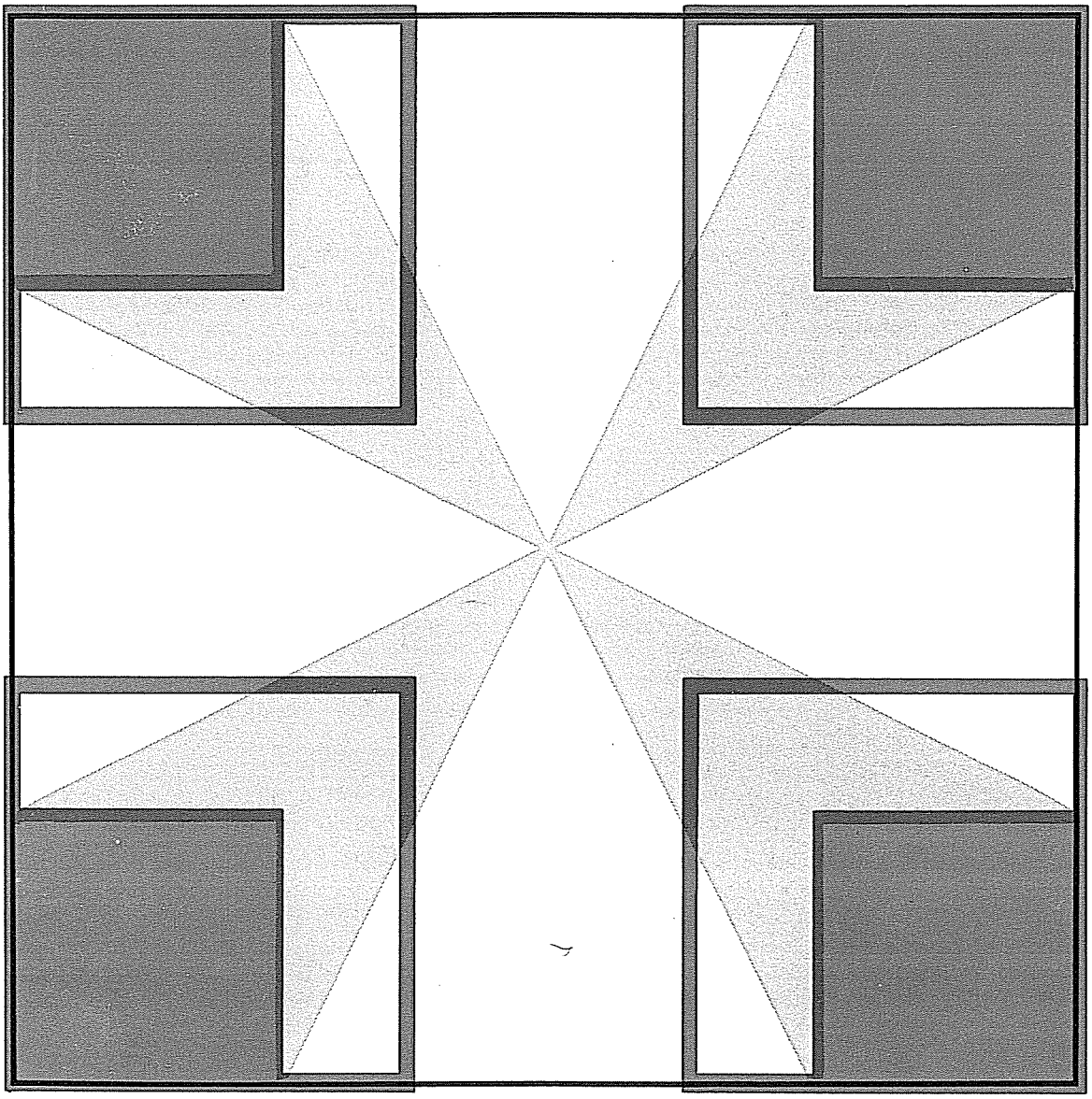
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Volume 30 Number 1 September 1991



THE ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
L'ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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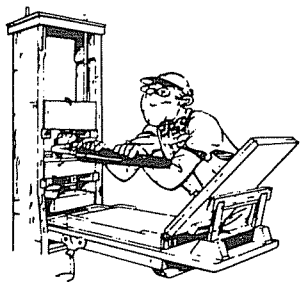
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Barry Onslow & Eric Wood are
co-editors of the Gazette for 1991/92.

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by
Barry Onslow

As we move towards the 21st century important decisions have to be made, not only about what we teach in mathematics, but also about how we teach the subject. Rather than react to the Ministry of Education's future decisions for mathematics, the Ontario Association for Mathematics Education, in co-operation with the Ontario Mathematics Co-ordinators' Association, want to take a pro-active stance in guiding Ontario's future course in mathematics. The two groups have therefore prepared a consultation paper entitled *Ontario Focus on Renewal of Mathematics* included in this issue in place of the *Abacus*. You are strongly encouraged to respond. A form is provided in the centre of each booklet. Ann Jones, our new president, describes the intentions of the document in her *President's Message*. Please read it carefully and participate. We have the opportunity to make a difference, but if we ignore this opportunity, we only have ourselves to blame if the province's mathematics program heads in a different direction to that desired by the membership.

The articles in this issue offer suggestions for the future direction of our mathematics classes. Roy Blomstrom's viewpoint presents an English teacher's perspective of a problem familiar to many of us -- the rigidity of the false, naive beliefs held by our pupils. It is refreshing to see how the beliefs are *played out* in a non-mathematics class and Ron's expertise in creating disequilibrium is a technique many of us could use in the classroom. How to help children overcome their disequilibrium and construct correct conceptual schemas is a challenge still being addressed by the math education community.

The first in a series of quotations from the *Curriculum and Evaluation Standards for School Mathematics* is interpreted by Alex Norrie. In subsequent issues, Alex will highlight key elements from the *Standards*, which are assisting our vision for renewal of mathematics in this province. The importance of the quote is reflected in several of the articles featured in this issue.

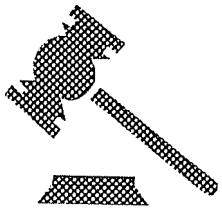
Hilde Howden emphasizes the significance of visualization when learning early numeration concepts; Janice Schwartzman and Harris Shultz demonstrate that visualization can assist older pupils' understanding when *summing the cubes*; and Gurcharan Singh Bhalla offers a Hindu geometrical justification of formula for the sum of a geometric series. In our multicultural classrooms it is crucial that students realize the invention of mathematics does not belong solely to Western Europeans.

Alex's chosen quotation is again highlighted by John Clark's descriptive article (the second in the series) about the Toronto Board's holistic evaluation procedure for assessing such topics as problem posing, problem solving, investigations, real-world applications and communications, all based on what Toronto pupils *can* do. For mathematics assessment John turns to the Arts, where qualitative, rather than quantitative, aspects are considered. It is an interesting comparison and deserves further examination as we hopefully move away from the traditional standardized skill tests.

Robert Sovchik re-examines problem solving in the elementary school and submits several interesting variations of typical process problems. David Duncan and Bonnie Litwiller provide an intriguing *real-world* problem for OAC finite students, and Ron Lancaster offers further evidence of his creativity by demonstrating how mathematics can be integrated with community promotions to stimulate pupil and parent interest. The editorial board looks forward to receiving future articles in which authors discuss aspects of their mathematics courses taught using a problem solving/posing approach.

Next year's OAME/AOEM conference will be organized by the Y^4MA chapter. Consequently the colours of York University have been selected for this year's front cover. The editors await your reactions as we work to provide a journal which is provocative, informative, pragmatic, and can help guide the future direction of mathematics in Ontario.

PRESIDENT'S MESSAGE



Included in this issue of *Ontario Mathematics Gazette* is a copy of *Ontario Focus on Renewal of Mathematics Education: A Consultation Paper* developed jointly by L'Association Ontarienne Pour L'Enseignement des Mathematiques/Ontario Association for Mathematics Education (OAME) and L'Association Ontarienne des Coordonnateurs de Mathematiques/ Ontario Mathematics Co-ordinators' Association (OMCA). This paper is the beginning of a process of change for mathematics education in the Early, Formative, Transition and Specialization Years with respect to our beliefs, teaching methodologies and strategies, curriculum content and teacher education.

The OAME and the OMCA have also sent to the Ministry of Education a proposal which asks that OAME and OMCA, in partnership with the Ministry, become actively involved in the renewal of education in Ontario in general and mathematics education in particular. We want to join in partnership with the Ministry of Education to accomplish the following five objectives:

Consultation and Planning

- Circulate Ontario Focus on Renewal of Mathematics Education: A Consultation Paper to the appropriate educators, community members, business and industry representatives and school board trustees. Based on the responses received, to prepare and validate an Ontario Mathematics Education Curriculum Plan, Early, Formative, Transition and Specialization Years.

Benchmarks

- Develop benchmarks for Mathematics Education in coordination and consultation with the Ministry of Education evaluation initiatives which will support the new initiatives in mathematics education.

Delivery

- Develop teaching strategies, methodologies for the delivering of the new and renewed mathematics curriculum including a holistic approach to mathematics supporting technological applications and the integration of mathematics across the curriculum.

Support Documents

- Develop curriculum ideas, materials, computer applications, etc., in support of the new and renewed curriculum, methodologies and strategies.

Teacher Education

- Prepare teacher in-service and pre-service strategies and methodologies in support of the implementation of the new and renewed curriculum, methodologies, strategies, and application of the benchmarks.

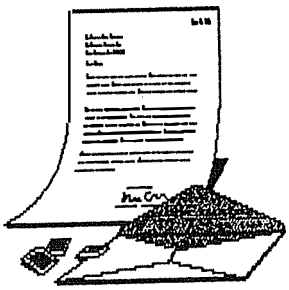
The time-line for our proposal is from June, 1991, with the circulation of the *Ontario Focus on Renewal of Mathematics Education: A Consultation Paper* to June, 1991, the beginning of the implementation.

To accomplish the five objectives of the project a steering committee and work team would be established, coordinated by OAME and OMCA and the Ministry of Education. Included in our proposal, OAME and OMCA are asking for funding from the Ministry of Education to hire appropriate staff including Project Co-Coordinators who would be mathematics educators, one with Early and Formative Years experience and one with Transition and Specialization Years experience, a research assistant and a secretary. The budget request includes monies for special contracts and other costs associated with the project.

We are awaiting a response from the Ministry of Education to our proposal and are optimistic that the Ministry will want direct assistance from OAME/OMCA. The OAME and OMCA can provide contacts, expertise and credibility inherent in a collective membership of mathematics educators who are classroom teachers of students in the Early, Formative, Transition and Specialization years, curriculum administrators, and university and community college teachers and Ministry of Education officials.

This is a very ambitious project but the OAME, in cooperation with the OMCA, look forward to assuming a leadership role in this change process. The beginning of the process is our request for responses to the OAME/ OMCA consultation paper enclosed. We hope that you will become actively involved in the process individually, through chapter organizations, or interest groups. We look forward to your input and continued interest as we move into this most important era in mathematics education.

M. Ann Jones, President



LETTERS TO THE EDITOR

York Mills Collegiate
Don Mills, ON.
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Dear Editor:

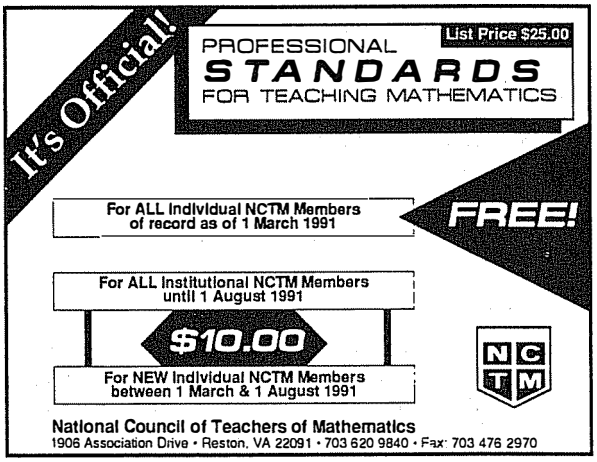
While enrolments at post-secondary institutions remained stable in the United States, a healthy 45 percent increase was seen in the number of undergraduate mathematics degrees granted from 1980 to 1985. But at Potsdam College of the State University of New York, the figure was an astonishing 152 percent increase. Of those who graduated with honours, over 40 percent were mathematics majors. Moreover, at Potsdam about 60 percent of mathematics degrees were also awarded to women, compared with the national United States average of 43 percent. The high rate of women in mathematics courses at Potsdam can't be attributed to female role models simply because, of the fifteen faculty members in the mathematics department, only one is female.

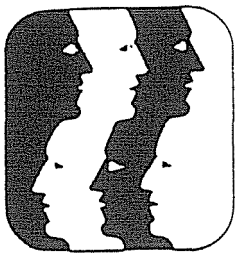
Another remarkable fact concerning this sleepy-hollow college is that its students are primarily from a lower-middle-class background , often from small farming communities, and usually they are the first in their family to attend a post-secondary institution.

What's the Potsdam secret? It was simply the caring attitude the mathematics faculty displayed towards their students and the way that sincere approach affected learning. One major observation was that the faculty did not complain about their students. In most staff lounges one usually hears teachers complaining about their "moronic" students and the impossibility of getting across a mathematics concept. During lecturer breaks at Potsdam, professors always commented on how their classes mastered a difficult concept that particular day. The Potsdam model stressed that students should not learn mathematics on their own. Contrary to popular opinion, mathematics should not be learned in isolation. The Potsdam system uses a tutorial model wherein older students tutor younger ones. This system could easily be implemented in any high school. Another interesting facet of the Potsdam model is the use of space. At Potsdam, everything that lives and breathes mathematics is on one floor. A single area is devoted to mathematics; there students and professors meet to discuss problems informally. Perhaps the idea is something to ponder when designing an efficient secondary school.

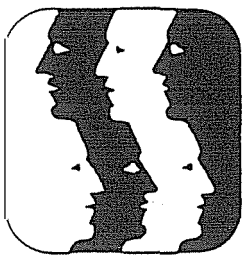
Although no teaching technique for producing good mathematics students is sure-fire, the Potsdam approach emphasizing a healthy classroom environment deserves a closer look. Too often mathematics has been presented in a "solution prize" way. One shouldn't study mathematics simply to get into a university, as most high school students believe. All the hype concerning the mathematics mystique must disappear if the high drop rate in this subject is to be reduced. Mathematics is a tool that trains one to think creatively. A good course for a mathematics student should also be a good course for any student. The exclusivity must wear off.

Michael Zichowski.





Viewpoint



VENEER

by
Roy Blomstrom
Assistant Head of English
Sir Winston Churchill High School

Many years ago, I used to teach a Grade 13 English course that included a large number of science-fiction works. One of the lessons I taught concerned the difference between "soft" and "hard" science-fiction. The former stresses feelings and downplays science. The latter tries to remain true to most of the laws of science and mathematics -- though, occasionally, as in the case of faster-than-light travel, it errs on the side of romance.

Finding a good example of a law that placed limits on what the hardcore writer of science fiction was allowed to do, was difficult. Then, one day, I came across an article by Isaac Asimov called "Just Right" in *The Solar System and Back*, a collection of some of his essays from *The Magazine of Fantasy and Science Fiction.* In the article Asimov bemoaned (Asimov is extremely good at bemoaning, by the way) "infantile science fiction" and went on to explain the square-cube law -- the law which states that for any object that increases in size uniformly, the surface area increases as the square of the linear measurement (the length of the edge of a cube, for example) and the volume increases as the cube of the measurement.

The day after reading the article, I tried to use the material in a lesson. What happened in the course of that lesson completely changed the way I thought about education.

I began with a quick definition of hardcore science fiction and then drew a cube on the chalkboard. Beside the first cube I drew a second one, its sides twice the length of those of the smaller one.

"Here are two cubes," I said. "The smaller one has one-inch sides and the larger one has two-inch sides. How many little ones do you think would fit inside the bigger one?

There were many puzzled faces. This was, after all, an English class, not a math class. "Four," someone finally suggested. "No, eight," someone else said. And then inspiration struck me.

"Let's vote on it," I said.

We voted, and while the majority felt that the answer was, indeed, eight, there were some who stuck by four. "We have a problem, here," I said. "So, I'm going to tell you what the answer is. The catch is that I'll either be telling you the truth, or else I'll be lying. The answer is eight. Now, how many of you believe me?"

There was no change in the vote. Those who had gone with four stuck by it, determinedly.

"All right," I said. "The answer really is eight. I'm not fooling you. And I think I can prove it to you two different ways."

The first proof consisted of calculating the volume of both cubes. The second cube had eight times the volume of the first. Then, I

drew the bottom layer of small cubes inside the drawing of the larger cube. Even with my poor drawing it was obvious that one could fit four small cubes on the bottom and, by inference, four more on top of the first layer.

"Anyone still think the answer is four?" No one did.

"Good! Now, if the larger cube weighs 160 pounds, what does the smaller one weigh?" Some of the students reached for their calculators. "Twenty, " someone said. "Why?" I asked. "Because the smaller one is one-eighth the size of the bigger one."

Ah, sweet logic! The triumph of science and mathematics over magic.

Then, in the best theatrical tradition, I frowned and drew two human figure on the chalkboard. One was twice as tall as the other. I tried my best to keep the proportions equivalent.

"This is Fred and this is Jim," I began. "Fred and Jim are twins, identical in every respect except that Fred is twice as big as Jim. He is twice as tall, twice as broad, and so on. Jim is perfectly proportioned. The two would always be mistaken for each other were it not for the fact that Jim is only three feet tall while his brother is six feet tall." I marked their heights on the board.

Time for the question. "If Fred weighs 160 pounds, what does Jim weigh? Decide, and then we'll vote on it."

We voted. About two-thirds of the class felt that Jim weighed eighty pounds. A third opted for forty pounds. No one wanted twenty pounds -- the correct answer.

"Oh dear," I said, putting as much theatre into my face and voice as I could manage without losing all my dignity. "None of you got the right answer. It's twenty pounds. Of course, I could be lying to you as I tell you this. But, I think you can see that there's no real difference between this problem and the last one, can't you? See, I even kept the weights the same. The little cube looks exactly like the bigger one just as Jim looks exactly the same as Fred. In both cases, the bigger one is twice the size of the smaller one in all dimensions. The answer has to be twenty pounds. Now, how many of you agree?"

Not a single student moved from the original answer. I had convinced no one. I had managed, however, to forget the original intent of the lesson. This was getting interesting.

"Maybe," I ventured, "you're thinking that things like lungs and bones and such affect the weight somehow. Tell you what -- let's pretend that both of them are solid wooden statues."

We voted on the weight, again. No change.
"Okay," I said and sighed heavily. "What's the problem? Why can't I convince you?"
"People aren't cubes, " one of my more vocal students said. "They don't have regular shapes."
"Does that make a difference?" I asked.
"Yes."
"How?"
"I don't know."
"Oh. Maybe I can convince you if we do a demonstration. I need a volunteer, though -- someone about six feet tall."

I had the volunteer come to the front of the room and stand facing the class. "Imagine that this is Fred," I said, "and that he weighs 160 pounds. I have a chainsaw -- still warm from a horror movie --, and I'm going to cut him from top to bottom right through the middle." I made the appropriate chainsaw noises and cutting gestures. "Now, I'm going to push this half way, and leave him standing on one leg. How much will the standing half weigh?"
"Eighty pounds," came the answer.
"Ah, ha!" I chortled in my best mad scientist voice. "And I've only cut one dimension in half. Now, I make him half as tall." More chainsaw noises. "After I push off the top half, how much does the rest weigh?"
"Forty pounds."
"Or less," someone added.
"And I still have one more cut to make." I pointed out. "I have to reduce his size in all

three dimensions. When I make that last cut his weight will be down to twenty pounds. . . unless, of course, I've been pulling the wool over your eyes. Let's vote on it.

"How many of you now think that I've been telling you the truth -- that is, that the weight really is twenty pounds?"

This time, two-thirds of the class were on my side. One-third were still not convinced.

"Some of you still are not convinced. What's the problem?"

"Sir," one of the unconvinced said, hand raised hesitantly, "kids that are three feet tall don't weigh twenty pounds. They'd be dead if that's all they weighed. They weigh a lot more than that."

"Oh, Oh, dear," I said. "You could be right. Someone who's on my side now, explain about the three-foot-tall kids." No one came to my aid. "Well," I said, "I guess it's time to confess. I've been tricking you. The answer really is forty pounds, not twenty." I paused and looked at them before continuing. "How many of you can see, now, that the real answer is forty?"

Two-thirds of the class now agreed that forty was the answer. A few among them looked startled at the quick change of mind of some of their peers. Some even commented out loud on it. "It's not forty, it's twenty. He's lying."

"They're right," I said. "The real answer is twenty. And as for the three-foot-tall kids, they're not proportioned the same as adults. They have little chests, big heads, short arms and legs. If they looked like adults, though, they would weigh twenty pounds."

It was time for another twist. I paused and looked at the cube diagrams on the chalkboard. "Say, can someone tell me the surface area -- the total area of the outside -- of the small cube?" The students were now watching each other. They were most reluctant to hazard a guess. "Well," I prompted, "what's the area of one side of the cube?"

"One square inch," came a reluctant reply.
"And if there are six sides?"
"Six square inches."
"What about the larger cube? Let's start with one side."
"Four square inches."
"For a total of what, since there are six sides?"
"Twenty-four square inches."

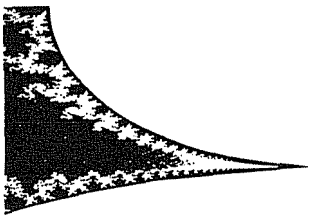
"Wait a minute," I said. "If the big one has an outside surface area of twenty-four square inches, that's only four times larger than the area of the small cube. I thought the small cube was one-eighth the size of the big one." I didn't wait for anyone to volunteer an analysis.

"Let's do one final vote. Make up your minds once and for all. What do you think Jim weighs? Is it eighty, forty, or twenty pounds?"

The result of the final polling? A perfect split -- one-third of the class opted for each of the three weights.

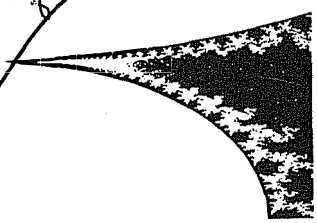
I have used this lesson many times since. The results have rarely been much different than they were that first time. I am no longer surprised when I hear of students making decisions that seem to fly completely in the face of logic and reason. I am no longer as confident that students understand what I am teaching them -- even when all the heads are nodding up and down. Nor am I any longer one of those who thinks that in our modern country it would be impossible for a megalomaniac to gain control of a well-educated populace.

I see the square-cube law everywhere: in the horoscopes of the newspaper, in the up-raised fists of the chanting marchers, and I hear it in the droning rumble of the spaceship on the silver screen -- that spaceship that travels where sound cannot, and still makes noise. No wonder the media can give equal credence to the pronouncements of physicists and psychics, astronomers and astrologers, physicians and quacks. Witchcraft is alive and well and living in the square-cube law. Bemoan that, Isaac.



WHAT IS A FRACTAL?

by
Hubert J. Ludwig
Department of Mathematical Sciences
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The concept of a fractal is a relatively new development in mathematics. The identification of fractal geometry as a field of study occurred around 1973. In the course of his work, Benoit B. Mandelbrot began to see connections between mathematical concepts and tools, as well as physical phenomena which had not been previously noted. Many aspects of the work of Mandelbrot involve the examination of objects and phenomena, which are already familiar to mathematicians and/or scientists, from a new standpoint. Shapes which had been invented to provide counter-examples in mathematics and which many mathematicians felt were "monsters" have been found to fit nicely into fractal geometry. Fractals are also created as the result of many natural processes. In *The Beauty of Fractals*, Peitgen and Richter comment on the occurrence of fractals in nature.

> Fractals are all around us in the shape of a mountain range or in the windings of a coast-line. Like cloud formations and flickering fires some fractals undergo never-ending changes while other, like trees or our own vascular systems, retain the structure they acquired in their development.

The fractal concept has fascinated many individuals. It has turned out to be especially intriguing for people who do not have backgrounds in mathematics or science. Pictures of fractals or representations of naturally occurring objects which have been produced by fractal techniques often catch their eye. One individual (Der-Hovanessian) was even inspired to write a poem contrasting Euclid and Mandelbrot.

What, then, is fractal geometry? The geometer L.M. Blumenthal is reputed to have observed that geometry is what geometers do when geometers do mathematics. In this vein we could say that fractal geometry is the study of fractals. This leads immediately to the question, what is a "fractal"? In his writings on fractals Mandlebrot observes that he would prefer to leave the concept undefined and simply allow those who are interested in fractals to discover what they are by working with them. (Mandelbrot has observed that one must not define fractal too soon.) He further observes that mathematicians prefer to have definitions at the beginning, so that when a term is used there will be agreement concerning that which is being discussed. The formal definition which Mandelbrot provided states that a fractal is a set whose Hausdorf-Besikovitch dimension strictly exceeds its topological dimension. Computing the Hausdorf-Besikovitch dimension of a set requires advanced topological techniques and is not suitable for an introduction of fractals. Topological dimensions are the usual 0, 1, 2, 3, etc., with which we are familiar from our early introduction to, and subsequent work with, Euclidean geometry. A broken line curve in a plane would have topological dimension 1 (since it is topologically equivalent to a line) but as a fractal its dimension might be a number between 1 and 2 (thus having the dimension as a fractal exceed its dimension in the usual sense). Differences in fractal dimension express differences in a non-topological aspect of form, which Mandelbrot calls fractal form.

As the study of fractal geometry has developed, it has been observed that there are objects which should be fractals which do not satisfy the formal definition proposed by Mandelbrot. Other definitions have been proposed which lead to the same problem. At least one mathematician has suggested that

perhaps at this point in the development of fractal geometry, it might be well to emulate the manner in which biologists treat the concept of life. There is no formal definition of life, but rather there is what mathematicians would call a set of sufficient, but not necessary, conditions which a form might satisfy in order to be called living. For example, if a form possesses sentience, most would agree that it is living, but a form can be classified as living and yet not possess sentience. Thus we would have a list of sufficient, but not necessary, properties which a set might have in order to be classified as a fractal. In this way it would be possible to say that the dimension property stated by Mandelbrot guarantees that a set is a fractal, but that it is possible for fractals to fail to have the dimension property. If this viewpoint is eventually accepted, we would essentially be back where Mandelbrot initially wanted us to be with reference to the definition of a fractal.

In the following, fractal geometry is introduced in the spirit which Mandelbrot encouraged both by looking at some fractals and noting some of the properties and characteristics which make them fractals, and by contrasting fractal geometry and Euclidean geometry. This is done so that the reader can begin to develop a feeling for the points of view taken in fractal geometry and begin to understand how a fractal geometer must think about and look at objects.

As an example of older ways of looking at nature, consider the words of Galileo.

> Philosophy is written in this grand book - I mean the Universe - which stands continuously open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.

Then consider Mandelbrot's observation in the preface to *The Fractal Geometry of Nature.*

> Why is geometry often described as "cold" and "dry"? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

The following statements illustrate the contrast between what is observed in Euclidean geometry and what is observed in fractal geometry. If we abstract the statements about fractals, we would then have the beginnings of a set of sufficient conditions for a set to be classified as a fractal.

1. Euclidean shapes do not yield any more detail on smaller scales; fractal shapes yield infinite detail at all scales, no matter how small they are.

2. In fractal geometry magnified subsets look like (or appear to be identical to) the whole and to each other. In Euclidean geometry shapes (in general) become ever smoother upon magnification.

3. Euclidean geometry is static; fractal geometry is dynamic.

4. Euclidean shapes are, in general, generated by procedures which terminate (drawing lines, circles, graphs, etc.); fractals are generated by recursive procedures.

The phenomenon referred to in statement 2 is usually called "scaling." If you have ever seen a sequence on television or in the movies in which the point of view begins from a great altitude and then moves closer to the surface of the earth you will recall that coastal features, for example, seem to repeat the same basic shape as one moves toward them. This is an example of the scaling phenomenon.

Another example of the scaling phenomenon is given by the story of two individuals who were stranded on a desert island and decided to

alleviate the boredom by producing a map of the island. Each prepared his own map and measured distances by pacing. The diagram on the left in Figure 1 represents the map prepared by the taller person and the diagram on the right represents the map prepared by the shorter.

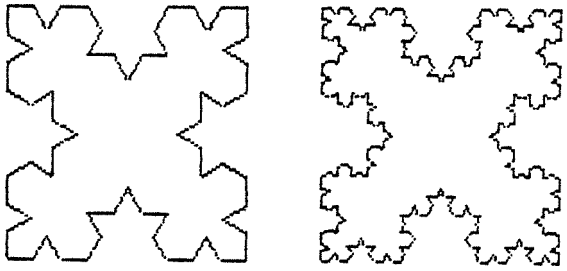


Figure 1.

The map prepared by the shorter person shows the same sort of detail as does the map of the taller, but it shows more since there were coastal features which the taller could step over but which were of such length that the shorter had to pace around them, and hence found it necessary to show them as details on his map.

Another aspect of this phenomenon is demonstrated by situations in which two nations will share a border but will declare different official lengths for the common border. Mandelbrot discussed such occurrences in one of his early works when he asked "How Long Is the Coast of Britain?" The answer is that it depends on the length of the measuring device used. The length that you obtain for a stretch of the bank of a stream using a ten-foot pole as your measuring instrument will differ from the result you obtain by utilizing a yardstick.

Mandelbrot has observed that the construction of fractals is unavoidable if you are given a sequence of operations. In this spirit, consider the following shape.

If we look at Figure 2 from the standpoint of Euclidean geometry, we see four segments, all having the same length. We can measure the angles and find one whose measure is 60° and



Figure 2.

two whose measures are 120°. This is the static aspect of Euclidean geometry. The figure is there; we can proceed to describe it. In fractal geometry, we take a dynamic viewpoint. We consider that we began with a segment, tri-sected it, removed the middle third, and then replaced it by the two segments (each one-third the length of the original segment) which meet at a 60° angle. In other words, we begin with a segment and then replace it by four segments, each having length one-third the length of the original segment, and oriented as indicated. This then is the first step in a process which is to be repeated over and over on each segment which is produced, eventually resulting in the production of a fractal. The second stage of the process yields the shape on the left in Figure 3 (there are, at this point, twelve segments in the shape), and the third stage of the process yields the shape on the right in Figure 3 (now there are 48 segments in the shape).

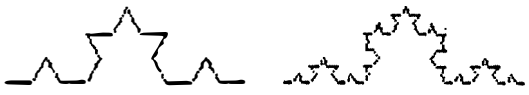


Figure 3.

If we begin with an equilateral triangle and apply the iterative process described above to each side of the triangle then the third application of the procedure produces the result shown in Figure 4. (In all cases we need

to keep in mind that the shape shown is not the fractal but is simply one of the stages in the development of the fractal. The fractal shape is the one which results when the process is iterated an infinite number of times.) This original shape which was shown in Figure 2 is called the generator for the fractal.

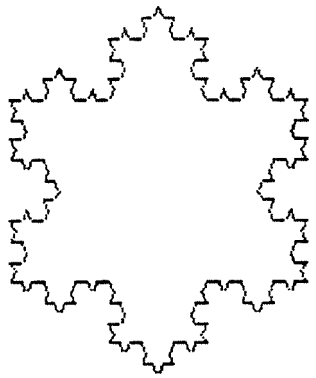


Figure 4.

The fractal in Figure 4 is one of the most familiar fractals and is called the snowflake curve. This curve was invented by Helge von Koch as an example of a continuous function which is nowhere differentiable but, since it is the result of an infinite sequence of operations, it is a fractal curve. Another interesting mathematical aspect of this curve is that it has infinite length (the length is multiplied by four-thirds at each step in the process) but bounds a finite area (since one can easily construct a circle whose interior will contain the region bounded by the curve.)

Now consider the generator shown in Figure 5. Each segment is $(\sqrt{3})^{-1}$ times the length of the original segment, the middle segment is perpendicular to the original segment and has a common midpoint with it.

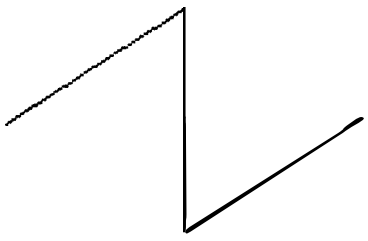


Figure 5.

Figure 6 shows the results of the second, third, fifth, and eighth iterations of the process. Note that a region of the plane seems to be becoming "filled in." This process is related to "plane filling" curves which were developed by Peano long before Mandelbrot invented fractal geometry. This is another example of a "monster" which fits nicely into fractal geometry, since it is produced by an iterative process.

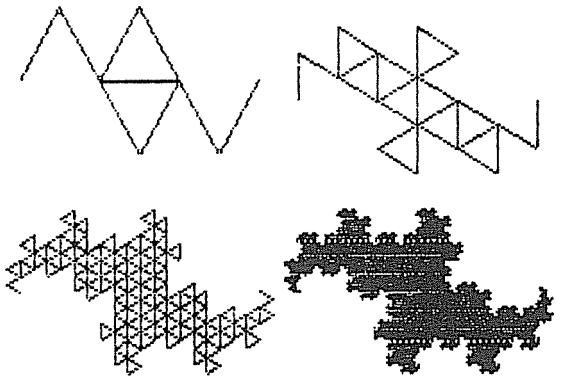


Figure 6.

In the generation of the snowflake, we produced a fractal shape by "adding to" the original shape. It is also possible to produce a fractal by "taking away" from the original shape. For example, one could begin with a triangular region and then remove the region bounded by the medial triangle from the region. This then is the process which is to be repeated over and over again on each new triangular region which results. The ensuing fractal is called a Sierpinski triangle. The figure which results when the process is applied to an isosceles right triangle is shown in Figure 7.

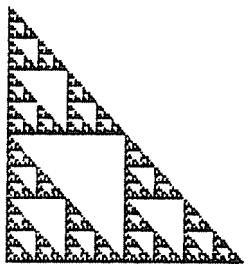


Figure 7.

Figure 8 provides another look at a Sierpinski triangle derived from an isosceles right triangle. The region represented on the left is the unit square having one vertex at the origin and sides along the positive halves of the coordinate axes. The region represented on the right is a portion of the same Sierpinski triangle, but now the square represented is 10^{-35}units on a side. Contrast this with the behaviour of the computer graph of a function in the coordinate plane when a zooming process is utilized. The graph of the function will, in general, become smoother and smoother and tend to look like a straight line. For the fractal shape we note that a magnification of 10^{35}still reveals the same type of structure as does a magnification of 1.

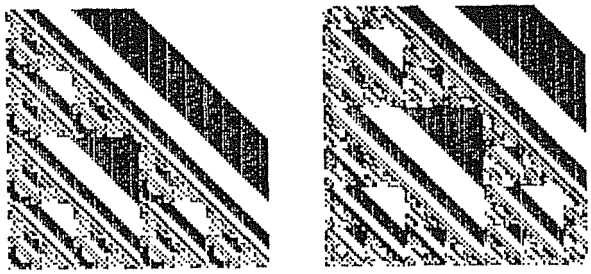


Figure 8.

If you want to look at a figure or a naturally occurring feature and determine whether or not it is a fractal, you can get a good feeling for the answer to the question by considering whether or not it has the scaling property, or whether or not it is possible to produce the shape by the iteration of some process.

Here we have an intuitive approach to fractal geometry and the identification of fractals.

If you would like to investigate the fractal concept and its history further, the book by Gleick provides an excellent history of the development of the topic which was written for the non technically oriented person. The article by Barcellos will provide an introduction to the determination of fractal dimensions.

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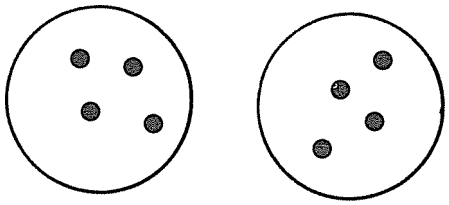
THE ROLE OF VISUALIZING IN LEARNING MATHEMATICS

by
Hilde Howden
Albuquerque, New Mexico

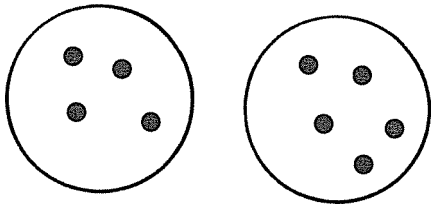


Josh must have been watching for me. He took my hand and led me to a spot on the floor where he had begun to organize a set of dominoes. He was busy explaining how to make a "number train" before any of his first-grade classmates realized I was there. They were all huddled in pairs concentrating on making their own trains.

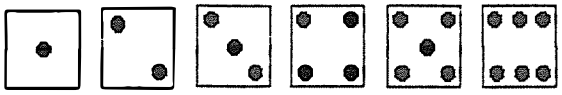
After reading an article about the value of having children visualize numbers, Josh's teacher had invited me to work with her class on some appropriate activities. She was concerned about the way the children performed number recognition exercises in their workbooks--the kind that require children to draw a connecting line between a set of objects and the numeral that identifies the number of objects in the set. What bothered her particularly was the way many of the children pointed to each pictured object and counted, often missing a picture or counting one more than once. They followed the same procedure, even when two succeeding sets contained the same number of objects in slightly different orientation,



or when one more object is added to a given arrangement.



She believed that if the objects were arranged in some consistent pattern the children were involved in choosing, they would recognize their number without counting. Before I arrived, the children had compared a variety of dot patterns representing 1 through 6, and voted on the arrangements they preferred. The patterns used on dice and dominoes were their favourites.



I asked Josh whether he had voted for these patterns. "Sure", he replied. "Why?" I asked. Josh thought a moment and said, "Because they make the most sense." When I asked him what he meant, Josh answered, almost disdainfully, "Can't you see?" My silence must have made him believe that I couldn't see what he meant, so he picked up some nearby counters, placed two of them like this ⁘, and said, "That's two. See, one, two." as he pointed to first the one counter and then the other. "Can you see that?" Aligning another counter between the two already there ⁘, he continued, "And one more is three." Thinking that my silence meant that I still didn't understand, Josh said, "See, it's just like four and five. You start with four and put in one more," as he illustrated with counters.

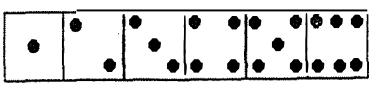


When I asked how he knew there were four to start with, he patiently explained that there are four corners and there's a counter in each

corner, so there must be four counters. "But what if you don't have counters, and you just see the picture?" I asked. "Well, if the picture has something in every corner, you know there are four somethings, because there are four corners" was his quick reply.

Intrigued by his reasoning, I pursued the questioning by asking, "If I say 'five', how do you know how many counters to put down? Do you count one, two, three, four, five?" Without hesitation, Josh said, "I guess I could, but it's faster to see four corners and one in the middle." "But if I just say five and don't put down any counters, how can you see them?" I asked. "I see them in my head," was Josh's explanation.

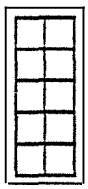
To practice recognition of the selected dot arrangements, the children were working in pairs to find dominoes with successive numbers of dots and aligning them to make number trains.



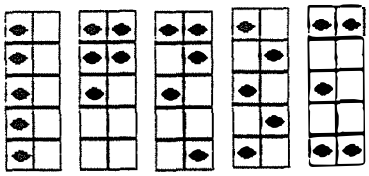
Then, with their eyes closed so that they could "see the dots in their heads," the children placed imaginary dots in the air as their teacher called out a number.

I suggested that we get into groups of three and play a game. In each group, the children took turns tossing a die. With each toss, the other two children tried to be the first to tell what number was tossed. In addition to identifying the number of dots, the respondent described the reasoning he/she used. The first few rounds went rather slowly. As the children gained confidence, explanations such as, "Two and one more, four and two more, five take away one," could be heard all around the room. The children were reasoning and discovering composition and decomposition techniques on their own.

That evening, I made up a set of tens frames from half-inch hardware cloth--one for each child and several for myself. The ones for the children were mounted on tag board. Mine were glued to heavy gauge acetate for use on an overhead projector.

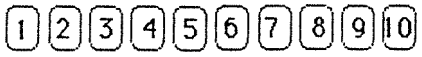


The next day, after playing with their tens frames and some pinto beans for a few minutes, the children were asked to count out five beans to place in their tens frame in any way they wished. Several children volunteered to reproduce their arrangements in the frames on the overhead projector for everyone to see before we voted on which arrangement we liked best.



Before voting, we talked about how the arrangements were alike and how they were different, which made an attractive pattern, which showed us that five is an odd number, and which most quickly showed us that there were five beans in the frame. The arrangement with all the beans in a single column was the overwhelming favourite because it showed that half the frame was full; that meant there were five beans. Tim, who objected to the vote because he liked his arrangement better, was convinced when his best friend, Noah, told him to close his eyes and see both arrangements in his head.

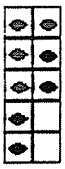
For practice in writing the numerals from 1 to 10, and as preparation for what turned out to be the children's favourite way of responding, each child made a set of numeral cards.



A tens frame containing from zero to ten beans was displayed on the overhead projector for approximately three seconds. Each student decided how many beans were displayed and held up the appropriate card. Before going on to the next display, children explained how they decided how many beans were shown, and they verified each other's strategies on their own tens frames. For example, most of the children reported that they saw this display as five take away one, or four beans.

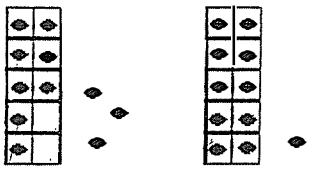


However, there was a mixed reaction to the following display.



Some children saw it as five and three more, while others saw it as ten take away two.

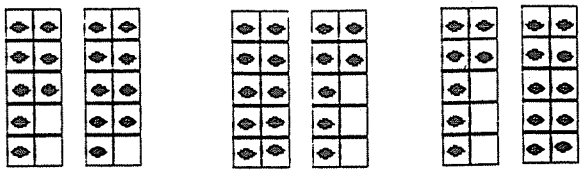
The next time I visited the class about a week later, the children were using similar visualization skills to help generate strategies for finding sums greater than ten. In explaining how they determined the number of beans in a display such as the following, most of the children described moving beans to fill the frame and then counting any extras.



8 + 3 = 10 + 1

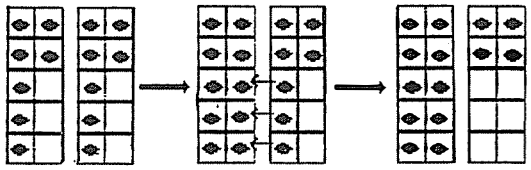
When I commented that I hadn't seen anyone moving beans, the children explained that you pretend to move them. Josh, who had taken me on as his personal responsibility, said, "Oh, you know. It's like seeing them move in your head."

They used the same strategy when all the beans were shown in frames. Some children preferred to always fill the first frame while others said it was easier to fill the frame with the fewest "empties."



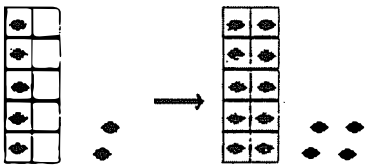
8 + 9 10 + 7 7 + 10

Doubles and near-doubles were their favourite combinations because they were "easiest to see in your head." Sharing strategies gave the children confidence in their own ability and convinced them that there are many correct ways to view and solve any problem. For example, when shown the expression 7+7 on the overhead projector, almost every child in the class quickly held up the 1 and 4 numeral cards in the correct order. However, their descriptions of what they saw in their heads varied. Some saw two tens frames, each containing seven beans, which they moved as follows:

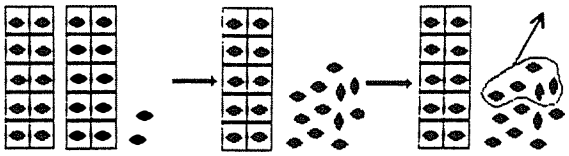


Some said they didn't bother with two frames. Instead they mentally placed five of the first seven beans in a frame and left the other two "loose." Then they placed five beans from the second addend into the same frame to fill it and had two more loose beans. That made a

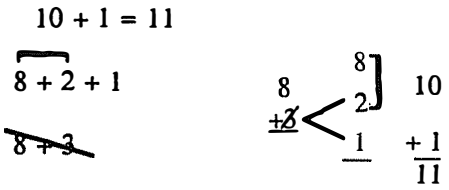
full frame and four loose beans, or 14 altogether.



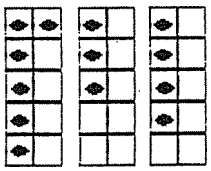
Based on how they visualized operations, children invented their own vocabulary. When working with tens frames to model subtraction, they talked about "dumping a frame." For example, to model 22 take away 5, they "dumped a frame to have enough loose beans."



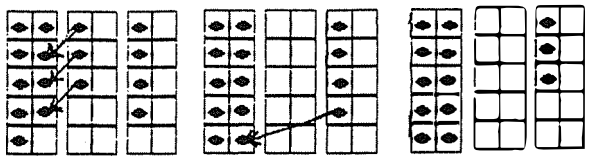
The teacher reported that the carryover to written work was remarkable. Regardless of whether the addends were written horizontally or vertically, the children immediately looked for combinations of ten. This was obvious when children were asked to "talk" the problem as they did it. They would invariably say something like, "Eight and two is ten, and one more in eleven," as they added 8 and 3. Some children recorded this procedure on their papers.



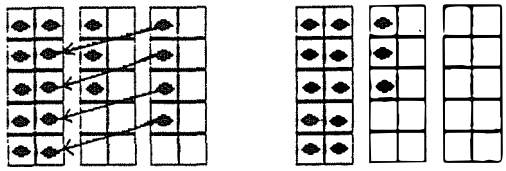
The tens frame display of three addends, such as 6 + 3 + 4, generated a lot of good reasoning.



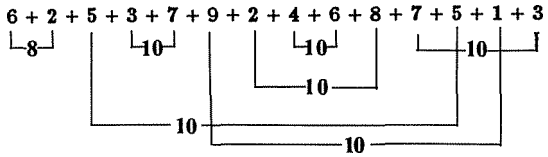
Some of the children said they moved beans twice in their heads, and used tens frames to illustrate what they visualized.



Others said they moved beans only once, because they saw that the beans from the third from would fill the first frame.



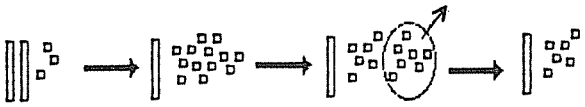
When one of the class leaders, who had originally visualized two operations in his head, said, "Gee, that's neat. Does it work for four numbers?" everyone began looking for the neatest way to add a column of numbers. It became a class challenge to write problems that had lots of "full frames." Each day for several weeks, children took turns writing problems such as the following on the chalkboard.



The rest of the class, working in cooperative groups, tried to find unique ways of grouping addends to give "full frames."

When base ten blocks were introduced to more easily model three-digit numbers, the teacher noticed that most children moved easily between the blocks and the frames models, sometimes even combining the two. A discussion she overheard between two children who were using base ten materials to model 23 - 7 convinced her that children will invent their own strategies if allowed to. They represented 23 as two tens rods and three ones blocks, made a

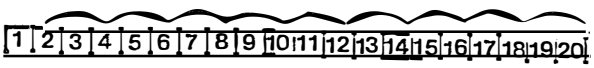
fair exchange for one of the tens rods, took away seven ones blocks, and decided that the difference could be written as 16.



However, as one of them recorded the problem, using the subtraction algorithm, he said, "That's not right. We can't have more than nine in a column, and when I make a fair exchange, I get 13 in this column. I get 113 and that's not the same as 23." To which his partner responded, "That's only a pretend number. It's like dumping a full frame of beans, so you have loose ones you can take away."

Throughout the school year, the children suggested a variety of models that they could then picture in their minds for each new concept that was introduced across the curriculum. Two particular mathematical examples were of special interest to me since they illustrated how the children applied visualization to problem solving and making connections, both of which are NCTM standards.

When skip counting was introduced, one group got out a meter tape and a toy frog, which they hopped from one skip-counting number to the next. Another group numbered all the floor tiles next to the wall in the hall. It extended the length of one classroom section of the U-shaped building, made a turn at the principal's office, passed the library and cafeteria, and down the full length of the other classroom section. Then they literally stepped off various skip-counting numbers.



A third group that had been previously told to be very quiet when they worked in the hall near the principal's office decided to make a hundreds square of tiles in the hall just outside their own classroom door. After outlining the square with masking tape and numbering the first 20 tiles, they discovered that "only a giant" could step from tile 10 to tile 12. They were about to take a vote on scrapping the hundreds square idea when one of the group members said, "I think we can solve the problem by putting tile 11 below tile 10. The resulting array met everyone's satisfaction, and they named it their zigzag hundreds square.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

1	2	3	4	5	6	7	8	9	10
20	19	18	17	16	15	14	13	12	11
21	22	23	24	25	26	27	28	29	30
40	39	38	37	36	35	34	33	32	31
41	42	43	44	45	46	47	48	49	50
60	59	58	57	56	55	54	53	52	51
61	62	63	64	65	66	67	68	69	70
80	79	78	77	76	75	74	73	72	71
81	82	83	84	85	86	87	88	89	90
10	99	98	97	96	95	94	93	92	91

Some time later, when the class was discussing how to estimate the sum of 19 and 53, several of the children were not quite sure that 53 was closer to 50 than to 60. The teacher asked whether anyone had a mind picture that showed 53 was closer to 50 than to 60. The

shortest girl in the class said, "Close your eyes and think about the skip-counting tiles in the hall. I can jump from 53 back to 50, but there's no way I can jump from 53 to 60. That's too far for me."

During the school year, the students in this first-grade class validated many of the current research findings concerning visualization.

- Visualization helps students to develop a very personalized understanding of mathematics.

- Visualization helps students to invent their own strategies.

- Students who visualize are generally better problem solvers than students who don't visualize.

They also reinforced my own gut feelings about visualization experienced while teaching mathematics to students of all ages.

- Visualization, or imagery, is a way of thinking.

- Visualization is a global skill, it is not limited to mathematics.

- Visualization is a developed skill. It is not a talent we're born with, although some individuals may be more predisposed to it than others.

- Visualization is not developed in isolation; we need something to visualize--an object, an idea, a relationship, a concept.

- Visualization requires making connections. To form a mental picture of an abstract idea requires imbuing the idea with some kind of physical properties, that is, forming a physical model of the idea.

I was fortunate to have grown up in a three-generation household. My grandparents and father told us lots of stories of their childhood, and my mother read to us almost every evening. My sister, brother, and I all remember visualizing every detail of these stories. We did the same thing as we listened to the Singing Lady, Jack Armstrong, and Steamboat Bill on radio. There wasn't any TV back then to provide the pictures, so we manufactured them for ourselves. Over the years, I've found the same tendency to think in pictures in many Native American children, whose culture is steeped in the story-telling tradition. In today's society, most children are deprived of such opportunities to invent their own pictures.

Mathematics educators are leading the effort to teach visualization. In a recent speech, Dr. Lynn Arthur Steen, past president of the Mathematical Association of America and member of the Committee on the Mathematical Sciences in the Year 2000, identified visualization as a necessary component of the mathematics curriculum at all levels, from pre-school through graduate school.

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MATHEMATICAL PERFORMANCE AND HOLISTIC EVALUATION IN THE TORONTO BOARD OF EDUCATION'S BENCHMARKS

by
John L. Clark
Toronto Board of Education.
Toronto, Ontario

The developmental phase of the Toronto Board of Education's Benchmarks was described in the December 1990 issue of the *Ontario Mathematics Gazette*. In this paper, the notion of performance in mathematics will be explored incorporating references to performance in the arts, and the use of holistic evaluation of student performance in the Toronto Benchmarks will be illustrated.

NEED FOR CHANGE

Educators interested in mathematics assessment these days use terms such as performance assessment, authentic measurement, and alignment. Generally, these terms describe new methods for evaluating student achievement which better reflect what is valued in the learning of mathematics. Most educators today believe that traditional forms of assessment have placed too much emphasis on algorithmic thinking and on paper and pencil methods, and these have had the overall effect of separating evaluation from learning. Educators have also recognized that evaluation is a driving force in how and what teachers teach, and ultimately it shapes the beliefs of teachers and students about mathematics itself.

While it is generally accepted that traditional methods have been adequate for the purpose of evaluating factual knowledge and algorithmic thinking, there has been intense debate whether they are adequate for the purpose of evaluating what the NCTM *Standards* (1989) calls mathematical power, and others have called mathematical habits of mind. This includes the ability to formulate as well as solve mathematical problems, to carry out an investigation, to make and test conjectures, to apply mathematics in the ways it is used outside the classroom, and to reason and communicate both in mathematical language/ notation and in everyday language.

Many educators believe that experiential approaches and the use of manipulative materials should also be reflected in assessment, and they seek better methods for determining students' understanding of key concepts. They believe that the assessment of mathematical knowledge should include the students' attitudes such as their like or dislike of mathematics, and their mathematical dispositions such as confidence, curiosity and willingness.

Consequently, in mathematics education today, there is a major shift from traditional methods of assessment to new and improved methods. This shift is characterized by a greater effort to evaluate students' ability to use mathematics with power, and to apply it in the ways it is used both by mathematicians and by others in real applications. The overall effect of new approaches is to align testing with teaching, and evaluating with learning. New approaches also produce information which is qualitative as well as quantitative, and is more descriptive of students' broader mathematical abilities.

PERFORMANCE IN MATHEMATICS

To understand what is meant by performance in mathematics it is helpful to think about the arts where the concept of performance is more widely used. For example, in a dramatic, musical, or athletic performance we observe the integration of a wide variety of cognitive and physical skills, attitudes and dispositions. The performances are authentic in the sense that what is truly valued in each artistic endeavour, either as an amateur or professional, is at the heart of the performance. As we watch a performance in the arts we take into account many elements, and we are more likely to consider them qualitatively than quantitatively, and simultaneously as a whole rather then analytically as parts. We think

about what we like or do not like about the performance and what we feel are its strengths and weaknesses.

In an address at a recent international conference on assessment in mathematics, Tom Romberg of the University of Wisconsin - Madison made the following comparison between evaluation in figure skating and in mathematics education.

Figure Skating	Mathematics Education
compulsory figures	short answers
short program	performance
long program	projects and extended work

Although the comparison is problematic in some respects, it brings out many interesting issues for consideration. For example, it is significant to note that in figure skating at the world level of competition, officials are doing away with compulsory figures. The parallel of this decision for assessment in mathematics education is obvious!

Like the short program in skating, the notion of performance in mathematics is one in which a task is fairly well defined and requires the integration of many important concepts, skills, procedures, and dispositions. The task is authentic because it involves the use of what is truly valued in mathematical activity and applications. Like the long program in skating, projects and extended work in mathematics are more open-ended, require more time, and involve many decisions. It is interesting to note that in the United Kingdom the requirements in mathematics for the GCSE (General Certificate of Secondary Education) have been reformed to include oral and practical work, and teacher-assessed extended projects of an investigational nature.

Extensive use was made of performance assessment in the Toronto Benchmarks. Students were required to formulate and solve problems using a wide variety of manipulative materials, and the majority of their performances were videotaped. The next section will describe how the performances were evaluated.

HOLISTIC EVALUATION IN THE TORONTO BENCHMARKS

Traditional methods for scoring which focus mainly on correctness, and the use of part marks and percentages, do not transfer easily to performance or authentic assessment in mathematics. Different methods are needed. In the Toronto Benchmarks, we combined holistic scoring with observation of actual student performances recorded on videotape and in written form, and found this combination to be a powerful method for evaluating mathematical performance. Previous educational use of holistic assessment has been mainly in the rating of students' writing. We believe that the holistic approach warrants further exploration and expanded use in the evaluating mathematical performance. Following are illustrations of holistic assessment in a print Benchmark from the Grade 6 library and a video Benchmark from the Grade 8 library. Analytic scoring was also used in Benchmarks but the majority of videotaped performances and some of the printed work were evaluated holistically.

Write A Story: A Print Benchmark Applying Mathematics to Everyday Problems. (M6-17)

Write A Story is the title of a grade 6 print Benchmark which was scored holistically. In this activity students were evaluated on their ability to use mathematical concepts and arithmetic operations with understanding, to apply mathematics to the solution of everyday practical problems, and to create a mathematical word problem. The students were shown three pictures (a Winter Olympics, six dogs, and a child), and asked to choose one. They were then asked to write a mathematical problem based on the picture, and told that they would not have to solve the problem.

The resulting written problems given by a sample of students were rated holistically by teachers who worked closely with the Benchmark Team. Each problem was rated at one of five levels using holistic scoring criteria which were drawn out of a close study and comparison of the students' problems. It is important to note that the holistic scoring criteria developed in the Toronto Benchmarks were not

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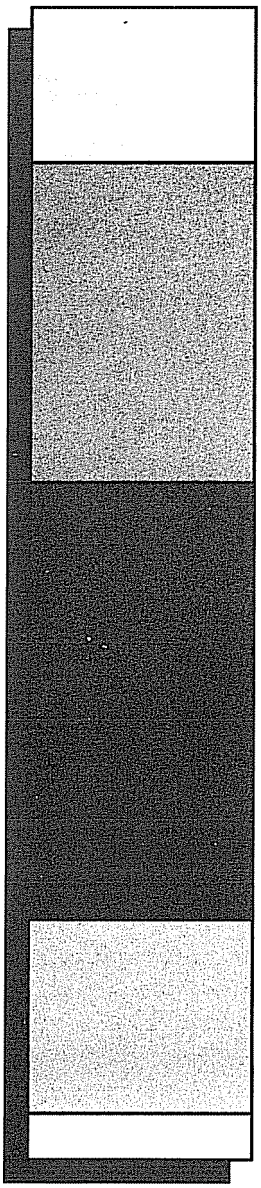
STANDARDS
OF
STUDENT
ACHIEVEMENT

TORONTO
BOARD
OF
EDUCATION

M6 17 **Grade 6 Mathematics**
Print

WRITE A STORY

RANGE OF RESPONSES



HOLISTIC SCORING CRITERIA

LEVEL FIVE
The student writes a multi-step story problem that involves more than one mathematical operation. The story problem is logical, creative, realistically relates to the picture and allows a numerical solution. If units (e.g. dollars) are worked into the story, they are appropriately chosen and used. The student's control of standard written English does not interfere with reading the problem.

LEVEL FOUR
The student may write a multi-step story problem with one-digit numbers or a one-step story problem with multi-digit numbers. The student relates the story to the picture. Elements of the picture may be used in original ways. The student's control of standard written English does not interfere with reading the problem.

LEVEL THREE
The student writes a one-step story problem which can be solved easily. One-digit numbers are usually used. The student may present the information and various elements of the problem without properly stating a mathematical problem. The student's control of standard written English may interfere with reading the problem.

LEVEL TWO
The student writes a very simple story with an attempt to incorporate a mathematical problem. The story may be garbled, units may be confused and numbers may be incorporated illogically. The student's control of standard written English may seriously interfere with reading the problem.

LEVEL ONE
Very limited response or no response.

Figure 1.

determined in advance but were based on a study of actual student performances. In this sense Toronto Benchmarks describe what our students *can* do on a variety of tasks. The holistic criteria and the percentage of students who scored at each level on this Benchmark are provided in Figure 1. The entire Benchmark (M6-17) includes the following elements: a statement of Ministry and Board objectives, a description of the activity, five levels of performance with holistic scoring criteria for each level, a graph indicating the percentage of students who were scored at each level, samples of students' work at each level, and other findings. Figure 2 illustrates the samples of student work at each level.

The Benchmarks are not tests. *Write a Story* should be seen as an activity which is an integral part of teachers' ongoing classroom programs in which students formulate problems frequently for one another to solve. Problems can be based on a variety of stimuli such as newspaper clippings or field trips, and not just on pictures. They can be written by students working individually or in groups. They can be written on chart paper and displayed on the classroom bulletin board, or can be recorded on cards and filed in a box. Students can choose their best problems and include them in a folder along with other written work which is shown to parents. Teachers can conference with individual students, as they do in a language program, and use the holistic criteria to make suggestions for how problems can be made more interesting or challenging. Teachers can rate individual student problems using the Benchmarks's holistic criteria and sample problems as reference points.

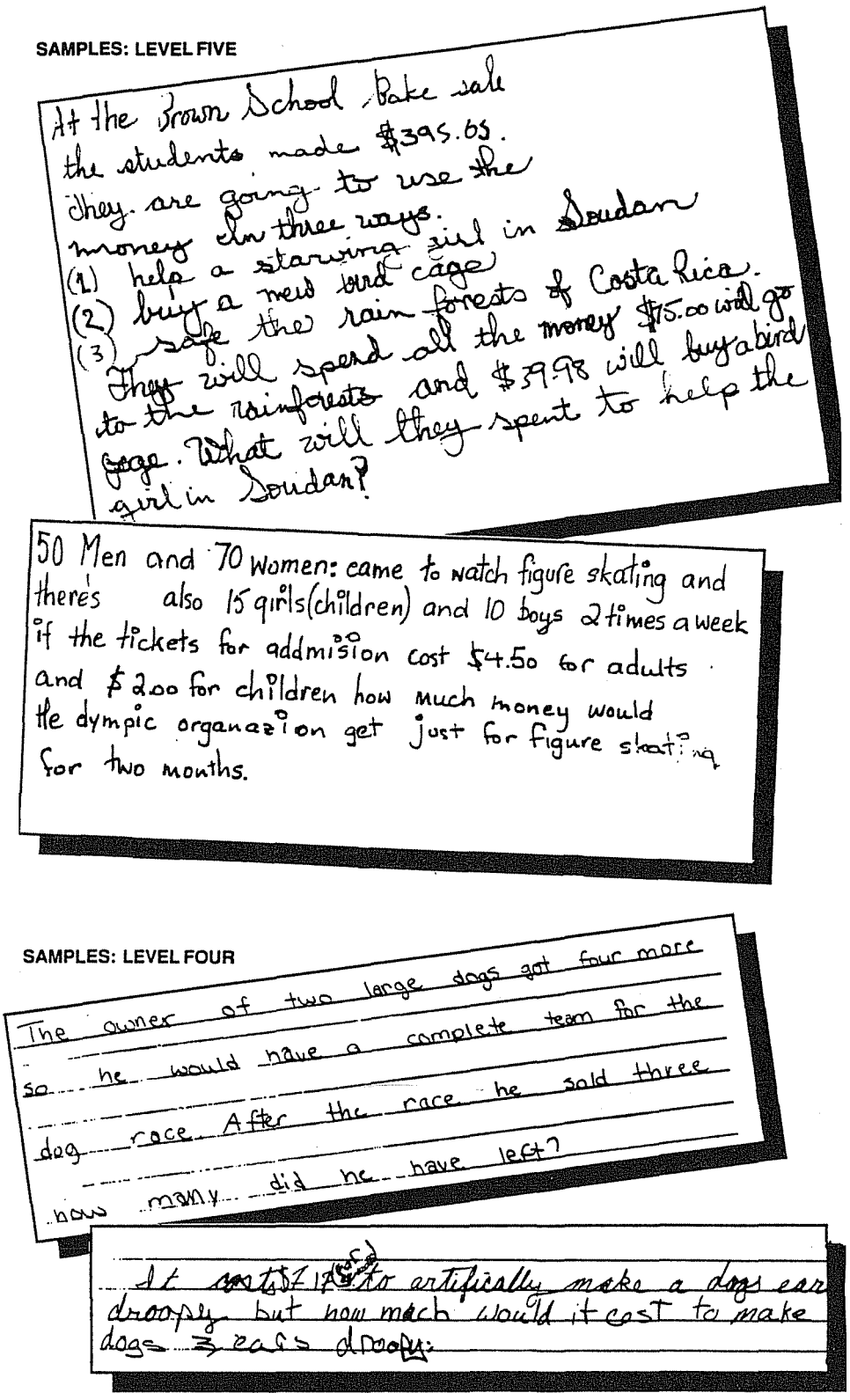
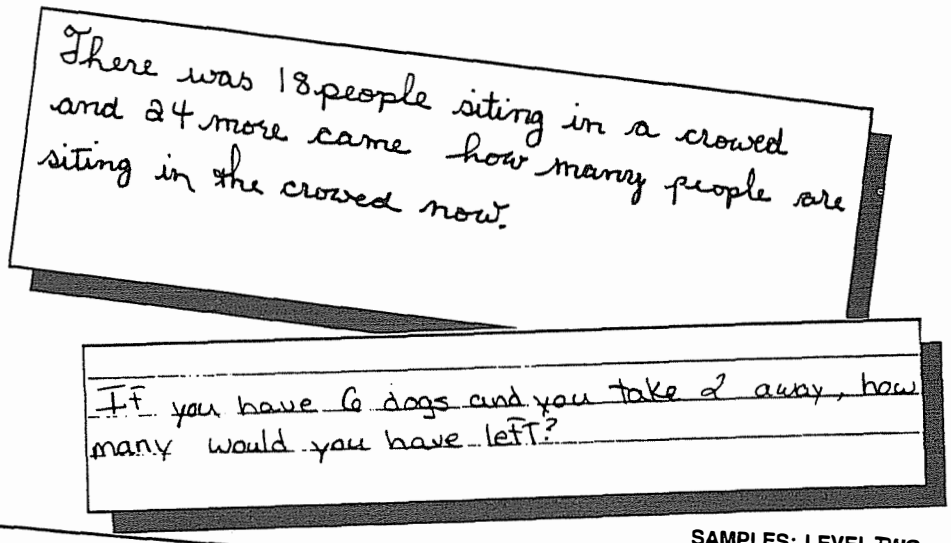
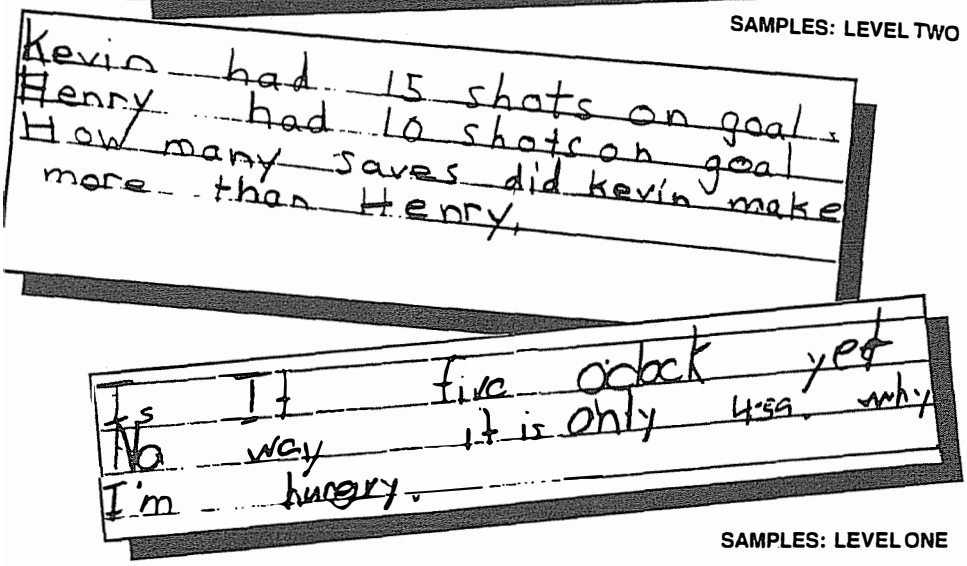


Figure 2.

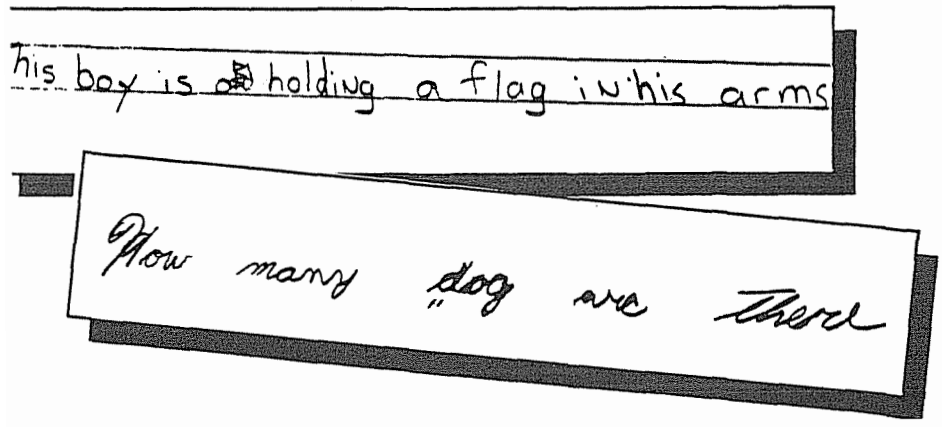
SAMPLES: LEVEL THREE



SAMPLES: LEVEL TWO



SAMPLES: LEVEL ONE



Going On A Hike: A Video Benchmark Applying Rate & Ratio (M8-6)

Going On A Hike is the title of a grade 8 video Benchmark which was scored holistically. In this activity students were evaluated on their ability to apply a scale ratio to a practical problem, and to use relationships involving time, distance, and speed. The students were first shown a map of Toronto and asked to locate their home and school. They were then asked to work out the time required to walk along a path from one point on the map (High Park) to a second point (Ashbridge's Bay). The materials provided were a calculator, ruler, string, compasses, measuring tape, and masking tape. If the students did not know how to proceed, they were prompted to find the distance using the scale on the map. If necessary, they were also prompted by being given a walking speed. After solving the problem they were asked to explain the problem and their solutions.

As with *Write A Story*, the holistic criteria for *Going On A Hike* were not determined in advance, and in this case were drawn out of observation of the students' videotaped performances rather than their written work. The holistic criteria and percentages of students are provided in Figure 3. The percentages are as they were found; Benchmark data have not been adjusted to a normal distribution (Bell curve). The final Benchmark videotape shows samples of students' performances at the top three levels and there is an unrated performance at the end of the tape which the viewer is

invited to rate using the criteria and sample performances.

Once teachers become familiar with the holistic scoring criteria and videotaped student performances (or samples of written work) for a Benchmark, they can use the information as reference points for observing and evaluating their own students as they engage with similar activities in the ongoing classroom program. The criteria form a basis for evaluating a student's progress over a period of time.

STRENGTHS OF HOLISTIC EVALUATION

Here is a summary of the strengths which we believe exist for the approach taken with holistic evaluation of mathematical performance in the Toronto Board's Benchmarks.

- Observation of students in active learning situations combined with holistic evaluation produces a powerful method for assessing higher order thinking and a complex set of mathematical abilities. Holistic evaluation integrates process, content and affect in the same assessment, and a wide variety of skills are taken into account simultaneously in a meaningful context rather than in isolation.

B|E|N|C|H|M|A|R|K|S

STANDARDS OF STUDENT ACHIEVEMENT

TORONTO BOARD OF EDUCATION

6

Grade 8 Mathematics Video

GOING ON A HIKE

RANGE OF RESPONSES

HOLISTIC SCORING CRITERIA



13%

LEVEL FIVE
The student measures the route accurately making good use of the available materials. The student uses the scale on the map correctly to find the actual distance. The student understands the need for the missing information of a walking speed and may ask the evaluator for it, or the student may provide it with justification from personal experience. The student gives a clear explanation of the problem and its solution, demonstrating an understanding of key concepts and mathematical operations, and obtains a reasonable answer for the time. The student's work is accurate and units are used consistently and appropriately. The student takes ownership of the problem and is confident throughout.

20%

LEVEL FOUR
The student encounters some minor difficulties in solving the problem which are the result of inaccurate measurements or calculations, or a confusion in the use of the scale. The methods used are generally correct. The student may realize the final result is not quite right but may not be able to pinpoint the difficulty. The student may seek assistance from the evaluator.

30%

LEVEL THREE
The student may experience confusion and may be helped towards the goal. The student may need to be given the prompt for walking speed or may draw on personal experiences unsuccessfully. The student experiences considerable difficulty in using the scale. There may be some confusion involving the relationships among time, distance and speed. The student may make a rough measurement of the distance ignoring details of the route. The student gives an explanation of the problem and its solution that demonstrates a poor understanding of the key concepts and mathematical operations.

22%

LEVEL TWO
The student lacks confidence, is confused and needs a great deal of help. The student makes rough measurements ignoring the details of the route or uses tools inappropriately. The student does not have a good grasp of the relationships among time, distance and speed, and does not know how to use the scale on the map. The student is unaware that the answer given for the time may be quite unreasonable.

15%

LEVEL ONE
The student may guess at the answer for the time. The student has no ideas for how to solve the problem and is totally dependent on the evaluator. The student may give a very limited response or no response.

Figure 3.

- The language of the holistic criteria can be used to communicate the strengths of a student's performance both to parents and to the student. Teachers can use the criteria and samples of students' work in a Benchmark as reference point to make judgements of their own students' progress and report this to both the parent and students.

- Students themselves can benefit from an understanding of the holistic criteria and student demonstrations at various levels contained in a Benchmark. They can see videotaped performances and samples of students' written work, and begin to understand what is important in learning mathematics, and what is required to be academically successful.

- The holistic criteria and samples of students' work for each Benchmark activity describe a range of problem solving skills from a low to high level of development which teachers can use to assist individual students or groups of

students. Teachers can observe and judge the approximate level of a student's performance in a task and then design classroom activities which will support the growth of the student to higher levels of performance.

▸ The holistic criteria for one Benchmark activity can be used generically for a set of related activities. For example, criteria and sample problems for the activity of writing a mathematical problem based on a picture can be used with pictures which are different from those used in the Benchmark, or the Benchmark can be adapted to similar activities involving other materials and approaches.

▸ Holistic evaluation is generally more efficient than analytic scoring (right/wrong, part marks). It can take less time and produces information which is more meaningful to teachers, students, parents and administrators. Because the information provided is more descriptive of students' progress, it can be used both for making modifications to classroom programs and improving communication with parents.

SUMMARY

A rationale for a shift to new methods of assessment was presented at the beginning of this paper. Many jurisdictions are exploring the use of performance or authentic assessment which integrates important mathematical process components with content. Interesting insights into performance assessment in mathematics can be gained by an analysis of performance assessment in the arts. The use of holistic evaluation in performance assessment in Toronto Benchmarks was illustrated using a print and video Benchmark, and the strengths of the holistic approach were summarized.

The members of the Toronto Benchmark Team hope that others who are working in the area of performance assessment will share their work. We feel that we have made a good start in a new direction but our understanding of the philosophy and methods of new forms of assessment is still evolving. We believe that as all of us explore improved methods for assessing mathematical performance, we can learn from one another's experiences and progress towards more appropriate and equitable forms of assessment for our programs and students.

The members of the Toronto Board of Education's elementary school Benchmark team were Freda Appleyard, Peter Billingsley, John Clark, Margaret Evans, Dr. Sylvia Larter, Phoebe Rankin and Donald Rutledge. John Clark was the Coordinator of Mathematics.

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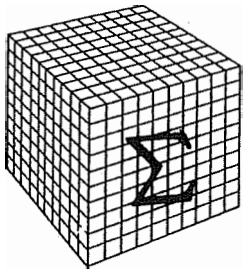
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SUMMING THE CUBES

by
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University High School
and
Harris S. Shultz
Department of Mathematics
California State University

The formula

$$(1) \qquad 1^3+2^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

is used in calculus to compute areas under $y = x^3$ without using the Fundamental Theorem of Calculus. Typically, (1) is proved by induction, a method with which relatively few students are comfortable. In this article we show how this formula can be derived by counting in two different ways the number of rectangles in an n by n grid. We demonstrate both ways by using an 8×8 grid (Figure 1) and then generalizing to an $n \times n$ grid.

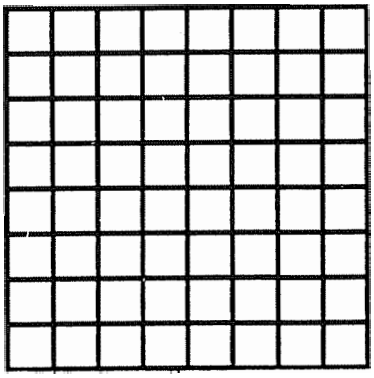


Figure 1.

A rectangle is uniquely determined by two vertical lines and two horizontal lines. For example, the two vertical lines and two horizontal lines shown in Figure 2 uniquely determine the 5 by 2 rectangle illustrated.

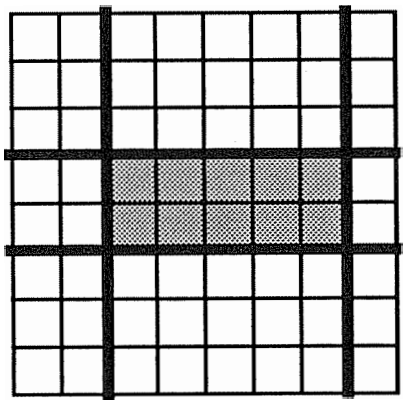


Figure 2.

The two vertical lines can be chosen in

$$\binom{9}{2} = \frac{(8\times9)}{2} \text{ ways.}$$

and the two horizontal lines can be chosen in

$$\binom{9}{2} = \frac{(8\times9)}{2} \text{ ways.}$$

Therefore, the number of rectangles in an 8 by 8 grid is

$$\left[\frac{(8\times9)}{2}\right]^2$$

Let us count the number of rectangles in a different way. The table in Figure 3 counts the number of rectangles that have minimum side 3, that is, rectangles having width 3 and height at least 3 or height 3 and width at least 3.

Width	Height	No.of Rectangles
3	8	1x6
3	7	2x6
3	6	3x6
3	5	4x6
3	4	5x6
3	3	6x6
8	3	1x6
7	3	2x6
6	3	3x6
5	3	4x6
4	3	5x6

Figure 3.

The total number of rectangles that have minimum side 3 is therefore:

$$6[1+2+3+4+5+6] + 6[1+2+3+4+5]$$

$$= 6\left[\frac{6\times7}{2} + \frac{5\times6}{2}\right]$$

$$= 6^2\left[\frac{5+7}{2}\right]$$

$$= 6^3$$

We leave it to the reader to generalize the above by verifying that, for $k = 1,2,\ldots,8$, the number of rectangles having minimum side k is $(9-k)^3$. Since each rectangle has minimum side k, for some $k = 1,2,\ldots,$ or 8, it follows that the number of rectangles in an 8 by 8 grid is

$$1^3 + 2^3 + \ldots + 8^3$$

Consequently,

$$1^3+2^3+4^3+5^3+6^3+7^3+8^3 = \left[\frac{8\times9}{2}\right]^2.$$

This demonstration can be generalized to an n by n grid. The number of rectangles, counting one way, is equal to

$$\left[\frac{n(n+1)}{2}\right]^2,$$

and, counting another way, is equal to

$$1^3 + 2^3 + \cdots + n^3.$$

This provides proof of (1).

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HINDUS' GEOMETRICAL JUSTIFICATION OF THE FORMULA FOR THE SUM OF A GEOMETRIC SERIES

by
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The Hindus' geometrical justification of the formula for the sum of a geometric series is an example of their use of geometry to justify algebraic relations. In the Vedic period (800 B.C. to 200 A.D.), Hindus had employed algebra to support rules for geometric constructions, transformations and combinations of rectilinear figures. The post Vedic period, however, witnessed the reversal of the roles of Hindu geometry and algebra. In this period, algebra leaned heavily on geometry for demonstration and justification of its results.

Hindu scholars were conversant with the algebraic relationship:

$$\frac{1}{r-1}T_{n+1} = S_n + \frac{1}{r-1}T_1 \qquad \text{(A)}$$

where T_1 = first term, r = common ratio, n = number of terms, $T_{n+1} = (n+1)th$ term, and S_n = sum of n terms of a geometric series.

From (A), S_n can be easily derived by a simple algebraic manipulation:

$$S_n = \frac{1}{r-1}T_{n+1} - \frac{1}{r-1}T_1$$
$$= \frac{1}{r-1}(T_{n+1} - T_1)$$

The Hindu geometric demonstration for relation (A) followed a set of simple steps:

Let T_{n+1} = area of rectangle ABCD (Figure 1)

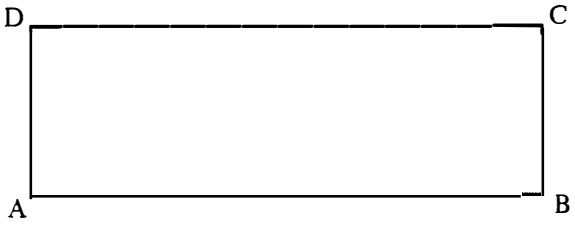


Figure 1.

Divide rectangle ABCD into $(r-1)$ equal rectangles - AEFD, EGHF, GIJH... (Figure 2)

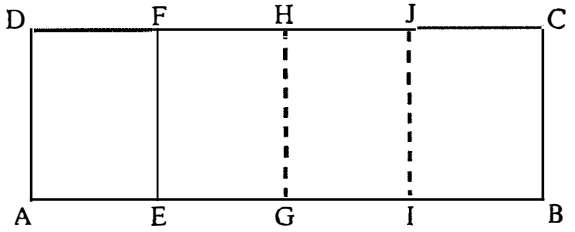


Figure 2.

$$\text{rectangle } AEFD = \frac{1}{r-1}(\text{rectangle } ABCD)$$
$$= \frac{1}{r-1}T_{n+1} \qquad (1)$$

Again divide rectangle AEFD into (r) equal rectangles - APRD, PSTR, SXYT... (Figure 3)

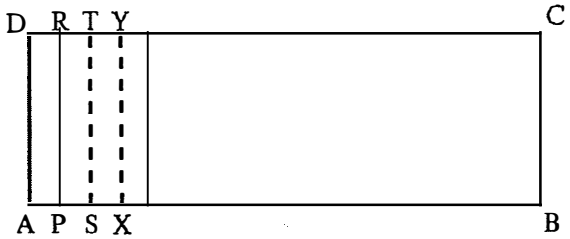


Figure 3.

rectangle $APRD = \frac{1}{r}$(rectangle $AEFD$)

$$= \frac{1}{r}\left(\frac{1}{r-1} T_{n+1}\right) \text{ From (1)}$$

$$= \frac{1}{r-1}\left(\frac{1}{r} T_{n+1}\right)$$

$$= \frac{1}{r-1}(T_n) \qquad (2)$$

rectangle $PEFR = (r-1)$(rectangle $APRD$)

$$= (r-1)\left(\frac{1}{r-1} T_n\right) \quad \text{From (2)}$$

$$= T_n \qquad (3)$$

From figure 3, we have:

rectangle $AEFD$ = rectangle $APRD$ + rectangle $PEFR$ (4)

From (1), (2), (3), (4), we have:

$$\frac{1}{r-1} T_{n+1} = \frac{1}{r-1} T_n + T_n$$

$$= T_n + \frac{1}{r-1} T_n$$

Similarly, we can establish that:

$$\frac{1}{r-1} T_n = T_{n-1} + \frac{1}{r-1} T_{n-1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{1}{r-1} T_3 = T_2 + \frac{1}{r-1} T_2$$

$$\frac{1}{r-1} T_2 = T_1 + \frac{1}{r-1} T_1$$

Adding column-wise, we have:

$$\frac{1}{r-1} T_{n+1} + \frac{1}{r-1} T_n + \ldots + \frac{1}{r-1} T_3 + \frac{1}{r-1} T_2$$

$$= (T_n + T_{n-1} + \ldots + T_3 + T_2 + T_1)$$

$$+ \frac{1}{r-1} T_n + \frac{1}{r-1} T_{n-1} + \ldots + \frac{1}{r-1} T_3$$

$$+ \frac{1}{r-1} T_2 + \frac{1}{r-1} T_1$$

Cancelling equal terms, we get:

$$\frac{1}{r-1} T_{n+1} = (T_n + T_{n-1} + \ldots + T_3 + T_2 + T_1)$$

$$+ \frac{1}{r-1} T_1$$

$$\frac{1}{r-1} T_{n+1} = S_n + \frac{1}{r-1} T_1$$

Which concludes the geometrical proof of the formula for the sum of a geometric series.

It is conjectured that Hindu scholars may not have demonstrated the formula for a general case. But they had justified it for particular cases. An extant manuscript, Kriyakramakani, includes geometrical justification of the formula for the sum of a geometric series where $r = 4$.

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DISTRIBUTING EXCEPTIONAL CASES AMONG THREE SETS

by
David R. Duncan and Bonnie H. Litwiller
Professors of Mathematics
University of Northern Iowa
Cedar Falls, Iowa



Mathematics teachers are always looking for situations that may provide probability exercises for classroom use. One that occurs naturally involves the distribution of exceptional cases among sets; we shall use three sets in our example.

Suppose that there are 75 new cars which are to be consigned randomly to three auto dealers in groups of 25 each. Among the 75 new cars are four "lemons" which will likely cause the receiving dealers significant problems. How likely is it that these four "lemons" will be distributed "evenly" among the three dealers?

Recall that the number of ways in which a set of n elements can be partitioned into r subsets having, respectively, n_1, n_2, n_3,...,n_r elements each ($n_1 + n_2 + n_3 + ... + n_r = n$) is

$$\binom{n}{n_1,\ n_2,\ n_3,\ \ldots,\ n_r} = \frac{n!}{n_1!\ n_2!\ n_3!\ \ldots\ n_r!}.$$

Consequently, the number of ways in which 75 new cars can be consigned to three subsets of 25 each is

$$\binom{75}{25,\ 25,\ 25} = \frac{75!}{25!\ 25!\ 25!}$$

To justify this formula in this particular example, we note that the 3-way assignment can be regarded as the performance of 2 consecutive tasks:

Task 1: Choose 25 of the original 75 cars for Dealer 1 -

$$\binom{75}{25} \text{ or } \frac{75!}{(25!)(50!)} \text{ possibilities.}$$

Task 2: Choose 25 of the remaining 50 cars for Dealer 2 -

$$\binom{50}{25} \text{ or } \frac{50!}{(25!)(25!)} \text{ possibilities.}$$

The number of ways of performing the two tasks consecutively is:

$$\binom{75}{25} \times \binom{50}{25} = \frac{75!}{(25!)(50!)} \times \frac{50!}{(25!)(25!)} = \frac{75!}{(25!)(25!)(25!)}.$$

Table 1 reports the possible consignments of the four lemons among the three dealers together with the probability for each consignment.

Let us regard the 2-1-1, 1-2-1, and 1-1-2 distributions as "even". One of these distributions will occur with a probability of .46278 or 46.3%. In just over half of the cases (53.7%) and "uneven" distribution occurs. An uneven distribution is more likely than an even one.

Now let us consider the situation in which there are 5 lemons among the 75 new cars.

Table 2 reports the probability that these lemons are distributed in various ways among the three dealers. The calculations are organized the same as those of Table 1. For instance the probability of a 2-3-0 distribution is:

$$\frac{\binom{5}{2,\ 3,\ 0} \cdot \binom{70}{23,\ 22,\ 25}}{\binom{75}{25,\ 25,\ 25,}}.$$

Consignment of lemons	Probability of this consignment	
	Consignment of the lemons ↘ · Consignment of the non-lemons ↙	
4-0-0	$\dfrac{\binom{4}{4,0,0} \cdot \binom{71}{21,25,25}}{\binom{75}{25,25,25}}$	≈ .01041
3-1-0	$\dfrac{\binom{4}{3,1,0} \cdot \binom{71}{22,24,25}}{\binom{75}{25,25,25}}$	≈ .04731
3-0-1	(Same calculation as for 3-1-0)	≈ .04731
2-2-0	$\dfrac{\binom{4}{2,2,0} \cdot \binom{71}{23,23,25}}{\binom{75}{25,25,25}}$	≈ .07405
2-1-1	$\dfrac{\binom{4}{2,1,1} \cdot \binom{71}{23,24,25}}{\binom{75}{25,25,25}}$	≈ .15426
2-0-2	(Same calculation as for 2-2-0)	≈ .07405
1-3-0		≈ .04731
1-2-1		≈ .15426
1-1-2		≈ .15426
1-0-3		≈ .04731
0-4-0	(Same calculation as for 4-0-0)	≈ .01041
0-3-1		≈ .04731
0-2-2		≈ .07405
0-1-3		≈ .04731
0-0-4		≈ .01041

Table 1.

We shall display in Table 2 only the final numerical approximations.

Consignment of Lemons	Probability of this Consignment
5-0-0	.00308
4-1-0	.01832
4-0-1	.01832
3-2-0	.03998
3-1-1	.08329
3-0-2	.03998
2-3-0	.03998
2-2-1	.13036
2-1-2	.13036
2-0-3	.03998
1-4-0	.01832
1-3-1	.08329
1-2-2	.13036
1-1-3	.08329
1-0-4	.01832
0-5-0	.00308
0-4-1	.01832
0-3-2	.03998
0-2-3	.03998
0-1-4	.01832
0-0-5	.00308

Table 2.

Let us regard only 2-2-1, 2-1-2, and 1-2-2 as having an even distribution. In every other case there is a difference of two or more between the numbers of lemons in at least two of the dealers' consignments. The probability of an even distribution is approximately 39.1%. In just over 60% of the cases the distribution is uneven.

Challenges to the readers and their students:

1 What would happen if there were six lemons among the 75 cars?

2 Suppose that the number of new cars was a multiple of four and there were varying numbers of lemons. What would happen if the new cars were to be consigned equally to four dealers?

3 Compose a computer program to compute these answers.

4 Find other examples of probability from real world situations.

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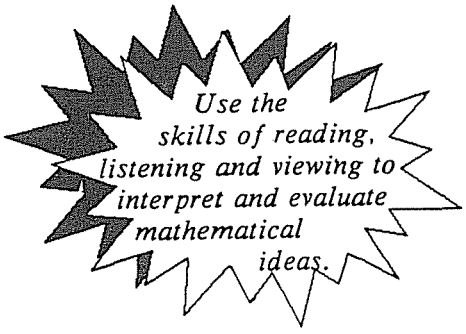
Each televised lesson runs 15 minutes. Topics covered include buying and selling merchandise; markups and discounts; gross pay for employees paid by the hour. for piecework, and on commission; filling out tax return forms; simple and compound interest rates; calculating car expenses, and using graphs in business contexts.

The TV programs are accompanied by a teacher's guide written by Ron Lancaster that is available through TVOntario's Client Services.

Media contact:
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INTERPRETING THE STANDARDS

(This feature is designed to help contextualize a number of recommendations from the NCTM curriculum and evaluation standards for school mathematics - we invite your comments.)

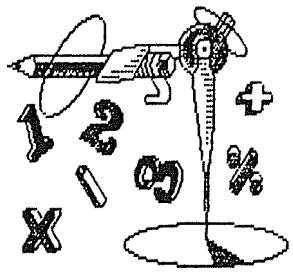


The ability to read, write, listen, think creatively and communicate about problems is an important basic skill. Communication of mathematical ideas and the understanding of mathematics begins with experiences that encourage children to have something to say. The expression of something to say requires use of symbols, sounds, gestures, movement, pictures, signs, letters or words.

Children can best achieve the skill of express-ing personal experiences when they are en-couraged to represent ideas for themselves and others. Sometimes a picture or diagram is the best method of communicating information. Viewing and interpreting graphs is a good example. Sometimes writing about a concept consolidates thinking. You would understand what a cube was if you could write in your own words a short paragraph telling someone what a cube looks like. For this you would need some language of mathematics and this is often obtained by reading about mathematics. The language of mathematics can also be found by listening to discussions about mathematical experience. How many? Is it longer? How do they find the magic number anyway?

Quotes interpreted by Alex Norrie
Peel Board of Education

ELEMENTARY SCHOOL PROBLEMS AND SELECTED SOLUTION STRATEGIES



by
Robert J. Sovchik
Faculty of Education
University of Akron
Akron, Ohio

The recently published NCTM *Curriculum and Evaluation Standards for School Mathematics* provides a clear direction for teaching problem solving in the elementary school. The importance of this goal cannot be understated. In fact, numerous council reports such as the 1980 *Agenda for Action* indicated the importance of teaching problem solving. In today's information-based society, problem solving is, indeed, the primary goal of mathematics instruction.

For quite some time, emphasis has been placed on teaching strategies, heuristics, which enable students to solve a wide variety of problems. The Soviet studies in the psychology of learning and teaching mathematics (SMSG, 1975) as well as the work of George Polya (1962) were indicative of this trend. The NCTM *Standards* amplify this point of view by stating:

> A major goal of problem solving instruction is to enable children to develop and apply strategies to solve problems. Strategies include using manipulative materials, using trial and error, making an organized list or table, drawing a diagram, looking for patterns, and acting out a problem.
>
> (NCTM, 1989, p.24)

In this article, several problems which highlight some of these strategies will be presented and solved.

Before we look at specific strategies, a *big picture* comment is in order. Today, many curriculum guides advocate a solution strategy model which traces its roots to George Polya's work. For instance, students are advised to do the following:

1) understand the problem
2) develop a plan of solution
3) carry out the plan
4) look back over the solution process

Some of the specific strategies that follow are helpful in all phases of this procedure, particularly during the *develop a plan of solution* stage. These strategies are the following:

1) Select appropriate notation
2) Make a drawing, figure, or graph
3) Identify wanted, given, and needed information
4) Restate the problem
5) Write an open sentence
6) Draw from your cognitive background
7) Construct a table
8) Guess and check
9) Systematize
10) Make a simpler problem
11) Construct a physical model
12) Work backwards
13) Generalize
14) Check the solution
15) Find another result
16) Study the solution process

The problems that follow are considered *process* or nonroutine problems suitable for elementary school students. Although story problems are amenable to many of the same strategies, they are all too often overemphasized in an elementary school mathematics program. Children need to see and solve a wide variety of problems in order to refine their problem solving skills.

SAMPLE PROBLEMS

In this section, some non-traditional problems and the strategies that they emphasize are developed. These problems may prove useful as a basis for teaching specific solution strategies. Gifted students may find these problems particularly enjoyable and may yield solution strategies different from those anticipated. Also, it is difficult, perhaps impossible, to say that a problem exhibits only one type of solution strategy. Therefore, combinations of strategies appear in the solutions.

The first problem that is presented is a good problem for illustrating a variety of solution strategies. Although this problem has been analyzed elsewhere (Leblanc, 1977), its representation of several solution strategies merits another look.

There are eight people at a party. If each person shakes hands with the other guests, how many handshakes will there be?

There are several ways to approach this problem. Making a table is a good place to start.

People	2	3	4	5	6	7	8	
Handshakes	1	3						

By utilizing a table, a simpler related problem can be utilized to build up systematic information. For example, three people at a party could shake hands in the following manner:



The diagram yields three handshakes. Continuing, four people can be diagrammed as follows:



thus yielding six handshakes. Our table now looks like this:

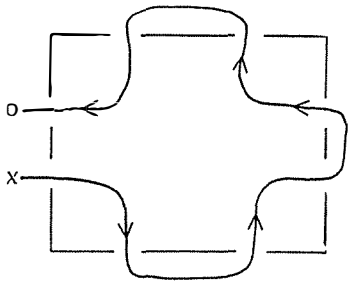
People	2	3	4	5	6	7	8	
Handshakes	1	3	6	10				

We can draw a diagram for five people and obtain ten handshakes. At this point, a generalization can be made. The handshake data contains triangular numbers, exhibiting differences of 2, 3 and 4. Recognition of this generalization can yield a continuation of this pattern to yield 15, 21, and 28 handshakes, respectively. The completed table now looks like this:

People	2	3	4	5	6	7	8	
Handshakes	1	3	6	10	15	21	28	

Again, this problem offers several solution strategies. A teacher can see whether students exhibit the strategy of making a table, drawing a diagram, working a simpler problem, or systematizing and generalizing. Thus, a structure for teaching these strategies is observed. Obviously, other procedures may be utilized to solve this problem, but awareness of strategies can help guide a child's solution procedures.

Another problem solving strategy is emphasized by the following problem.



A person starts at X and enters a funhouse. The person pays $1.00 to get in and loses half of the money in her possession while she is in the funhouse. She then pays $1.00 to get out. She journeys to the next entrances and pays $1.00 to get in, loses half of her money, and pays $1.00 to get out. This action is repeated until she exits and has zero amount of money. How much money did she start with?

This problem emphasizes the strategy of looking backwards. Trying to start at the entrance, or beginning, can make this a very difficult problem. On the other hand, working backwards and adding one, doubling, and adding one again yields the following sequence of numbers: 1,2,3,4,8,9,10,20,21,22,44,45. A trial example points out that if the person has $3.00 before entering the last door, then a payment of a dollar would leave $2.00 remaining, half of which is $1.00. The last dollar is spent exiting the final door. In short, wherever we halved, we now double and wherever we subtracted $1.00 we now add $1.00.

Another problem which emphasizes the working backwards strategy is the following:

Sue baked some cookies. She put one-half of them away for the next day. Then she divided the remaining cookies evenly among her three sisters so that each received four. How many cookies did Sue bake?

These two problems may be used when teaching problem solving in order to determine whether students discover the strategy of working backwards. They may also serve as classroom experiences designed to teach this strategy.

Here is another problem which illustrates some of the strategies that have been discussed:

How many people would learn of a rumour in 70 minutes if at the onset, each person who hears it tells four new people in ten minutes, but then tells no one else?

Several strategies can be useful here. Particular emphasis on working a simpler problem and making a table are important aids. For example, in ten minutes, five people will be aware of the rumour. In 20 minutes, 21 will be aware. Continuing in this manner yields a pattern:
(4 x 5) + 1 or 21
(4 x 21) + 1 or 85
(85 x 4) + 1 or 341
(341 x 4) + 1 or 1 365

Time	People aware of rumour
0	1
10	5
20	21
30	85
40	341
50	1365
60	5461
70	21845

Thus, in 70 minutes, 21 845 people will hear the rumour. Try to solve this problem by drawing diagrams.

Another example of the strategy of making a table and looking for a pattern is the following:

If 865 people are enrolled in a single elimination tennis tournament, how may matches are needed to determine a winner?

People	1	2	3	4	5	6	7	8
Matches	0	1	2	3	4	5	6	7

A different type of problem solving strategy is presented by the next problem.

How many different ways can you express 100 using exactly six nines?

The problem is a very good creative exercise. It demonstrates the strategies of finding another way to solve the problem and finding another solution. Here are a few solutions:

$$100 = 99 + \frac{99}{99}$$

$$100 = (9 \times 9) + (9 + 9) + \frac{9}{9}$$

$$100 = \frac{999 - 99}{9}$$

Another example of these strategies is the problem of using exactly four fours to express the number two. Also, the creative problem *How many ways can you use a barometer to measure the height of a building?* seems to

demonstrate the necessity of finding alternative ways to solve problems as well as alternative solutions.

In summary, problem solving strategies for elementary students need to reflect a broader interpretation of problem solving than a narrow word-problem approach. This article has attempted to provide sample problems which emphasize the following strategies: constructing a table, making a simpler problem, working backwards, finding another way to solve the problem, finding another solution, and, perhaps, implicit in at least the construction of a table strategy, is the related strategy of generalizing a pattern. These problems may serve as source material for diagnosing the problem solving repertoire of elementary students, especially in the identification of gifted elementary students.

Also, problems such as these may be used to teach specific strategies. It should be repeated that these strategies are only suggested guidelines; students may exhibit unique strategies or combinations of the previously mentioned strategies.

A file of additional sample problems closes this article.

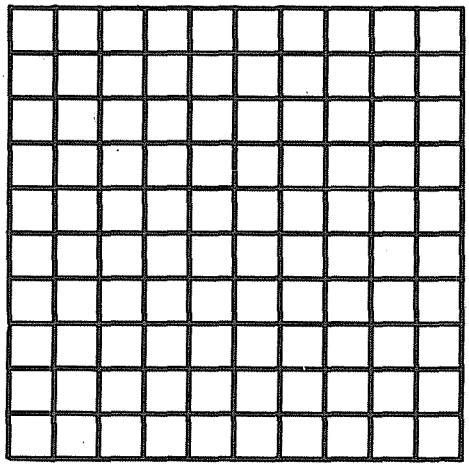
SUPPLEMENTARY PROBLEMS

1) Constructing a Table

   a) If parking tickets are $2.00, how much will three tickets, five tickets, or ten tickets cost?
   b) How many ways can you make 25c in change?
   c) How many line segments can you draw connecting ten points on a circle?

2) Making A Simpler Problem

   a) If the perimeter of a rectangle is 108 cm and one side is 36 cm, what are the lengths of the other sides?
   b) How many square are here?



3) Working Backwards

   a) If two whole numbers have a sum of 18 and a product of 45, what are the numbers?

4) Finding Another Way To Solve The Problem And Finding Another Result

   a) How much does a million dollars weigh?
   b) In how many ways can you use a rectangular box?

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by
Ron Lancaster
St. Mildred's-Lightbourn School

Last year Little Caesars Pizza ran a promotion called CRAZY EIGHTS in which several items were sold at prices that involved only the number 8. For example, two small pizzas with 8 toppings could be purchased for $8.88. I put together the following assignment for students in my Grade 11 Mathematics class and they enjoyed working on the problems. Last week I handed out a similar set of problems at our Grade 8 Parent's night. This time the problems all involved the number 7, to tie in with Little Caesars' current Lucky Sevens promotion. I have encouraged parents to work on the problems with their child and then to send me their answers. I plan on marking their work and returning it with some solutions and comments, along with a coupon from a local Little Caesars outlet. This "backdoor" approach to stimulating parental interest might just work for your school.

1. Using exactly four 8s, write all of the numbers from 1 to 100. You may use the operations of addition, subtraction, multiplication, and division, as well as juxtaposition, (putting two or more digits side by side as in 88) exponents, radicals, factorials, and decimal points. If your students end up with some numbers that they are having trouble getting, you might allow them to use the greatest integer function. Several examples have been given below to help get you started.

$$3 = \frac{88}{8} - 8$$

$$89 = 88 + 8^{8-8}$$

$$100 = \left(\frac{8}{.8}\right)\left(\frac{8}{.8}\right)$$

2. How many zeroes are there at the end of each of the following numbers?

a) 8!
b) 88!
c) 888!
d) 8888!

Depending on the grade level of your students, you may have to define what the factorial symbol ! means. One way of doing this is to give your students a few examples like the ones given below and let them go from there.

5! = (5)(4)(3)(2)(1) = 120
11!= (11)(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)
= 39 916 800

3. Find all of the different ways in which 8 queens can be placed on a chessboard so that no queen attacks another.

4. The number 24 678 051 is quite interesting because

$2^8+4^8+6^8+7^8+8^8+0^8+5^8+1^8=24\ 678\ 051$

There are two other eight digit numbers that are equal to the sum of the eighth powers of their digits. Find these numbers.

5. If all numbers were written in English alphabetical order, what would be the first number in the sequence? What are the next three numbers? Are there similar numbers for other languages?

6. In the early 1960s there was an American musical group that recorded under the name Little Caesar and The Romans. Who was Little Caesar? What songs were released by this group? What are the members of this group doing today? What other group had a name that started off with Little Caesar? What songs did they record and who were the members of this group?

SOLUTIONS AND COMMENTS

1. My students were able to obtain an answer for most of the numbers from 1 to 100. While several numbers remained elusive, they were able to obtain multiple solutions for many of the numbers. You might want to have your students enter their answers on several large sheets of paper posted on a bulletin board. To save space at this time I won't run my students' answers, but feel free to send me what your students come up with. Perhaps in another column I can run some results.

Readers may recognize this problem as a variation of the famous four 4s problem. A great deal has been written about this problem and I have listed several references below. I have used the first reference for finding articles on a wide variety of topics from recreational mathematics. The four 4s problem is referred to in Volume 2. The second reference is classic Gardner material.

A Bibliography of Recreational Mathematics
Volumes 1, 2, and 3
William L. Schaaf
Available from The N.C.T.M.

The Numerology of Dr. Matrix (Pages 51-54)
Martin Gardner
Simon and Schuster

2. The critical thing to observe here is that the number of zeroes at the end is equal to the number of pairs of 2s and 5s that can be formed from the prime factorization of the given number. Since there are more 2s than 5s for all factorials, the number of pairs can be found by determining the number of 5s that occur in the prime factorization. If you find this confusing, consider the prime factorization of 88! given below.

$$2^{85}3^{13}5^{6}7^{3}11^{2}13^{2}17^{1}19^{1}23^{1}$$

Note that it is possible for form 6 pairs of 2s and 5s and so 27! ends in 6 zeroes. In case you are interested, the value 27! is given below.

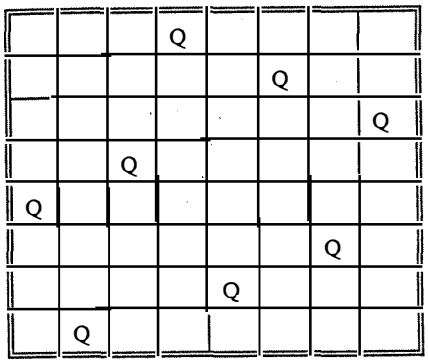
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Incidentally, the above results were obtained using MATHEMATICA. Before you get worried, your students can do this problem without finding the prime factorizations. Here is a neat way of obtaining the number 6: 5 goes into 27, 5 times; 25 goes into 27, 1 time; 5 + 1 = 6. Let's try this for 888! 5 goes into 888, 177 times; 25 goes into 888, 35 times; 125 goes into 888, 7 times; 625 goes into 888 once: 177 + 35 + 7 + 1 = 220. Therefore there are 220 zeroes at the end of 888!. Do you understand why this works? If not, talk it over with the people around you.

For information on factorials I highly recommend that you consult the following book. An entire chapter is devoted to factorial oddities.

Mathematical Magic Show (Pages 50-65)
Martin Gardner
Available from the M.A.A.

3. There are 12 fundamental ways of arranging the eight queens so that no queen attacks another. Each of the twelve basic solutions can be rotated and reflected to give seven other patterns, except for one of them, which because of its symmetry, gives only three other patterns. Therefore, there are 92 solutions altogether. One solution is given below.



This is a very famous problem and hundreds of articles have been written about it. Two excellent references are given below.

Mathematics on Vacation (pp. 34-54)
Joseph Madachy
Thomas Nelson and Sons Ltd.

The Unexpected Hanging (pp. 186-197)
Martin Gardner
Simon and Schuster

4. The other two numbers are 24 678 050 and 88 5593 477. My students found the first one, but they were not able to find the second one. Even with the use of a computer, this is a tough problem. I like to use results like these on my tests as simply a question where students are asked to work out the value of the expression. When I take up the test I say "Hey! Did you notice anything special about the result?" I then suggest that they look for other similar numbers, with exponents other than 8 too. Here is an example where the exponents are all 4s.

$$9^4 + 4^4 + 7^4 + 4^4 = 9\,474$$

For more information (and I mean **more**) consult the following books.

Mathematics on Vacation (pp. 144-177)
Joseph Madachy
Thomas Nelson and Sons Ltd.

The Penguin Dictionary of Curious and Interesting Numbers.
David Wells
Penguin Books

5. I will leave this problem for you and your students to work on. Please let me know what you come up with, especially with other languages. For more number trivia, consult the Penguin Dictionary mentioned above.

6. David Johnson (Little Caesar) was lead singer for this group. Other members of this group were Johnny Simmons, Early Harris, Carl Burnett, and Leroy Sanders. In 1961 they signed with Robert Keene's Del Fi records (the same company Ritchie Valens recorded for) and in April of that year they recorded their biggest hit "Those Oldies But Goodies". I have no idea what members of this group are doing today. The other group that had a similar name was Little Caesar and the Consuls.

The last questions may not suit your personality or style, but don't be too quick to pass it by. This question gives the students a chance to do some research and quite a few of their parents might enjoy working on it too. I would even suggest that you get a copy of their hit song and play it in class. **Stardust Records** on Ottawa Street in Hamilton has a tremendous stock of music from the 1950s and 1960s. Call Sharon at (416) 545-4011 for directions or more information about other songs that you could tie in with your courses.

SOLUTIONS TO THE ALPHAMETICS
(from the December 1990 issue of *The Gazette*)

1. JRM = 104 THE = 273
   FUN = 586 ONE = 963
1 and 2 are interchangeable, as are 0 and 7

2. Minimal number of summands is 96.
   HELL = 2099 HEAVEN = 201504
The maximal number of summands also has a unique solution and I will leave it to you to find the solution.

3. OK = 45 NO = 84 JOKE = 9453
   DO = 24 NOT = 847 SMOKE = 10 453

4. SMOKE = 32497 MAKES = 28973
   ME = 27 CHOKE = 61497

5. HATE = 2683 TO = 84 DO = 94
   NEED = 7339 TO = 84 SPEED = 10339
   2 and 7 are interchangeable.

6. A = 6 TREAT = 75867
   CARTER = 365 785 PEANUT = 486 127
   BUTTER = 927785
   3 and 4 are interchangeable

7. EASY = 9458 AS = 45 ONE = 729
   TWO = 167 THREE = 10399
   2 and 6 are interchangeable.

8. NUDE = 8290NOT = 857RUDE = 3290
   NOR = 853CRUDE = 13 290

9. WWGR = 1134WWOO = 1166
   YYBB = 2200WOBB = 1600
   OWBB = 6100

10. COME = 1859 ONE = 809
    COME = 1859 ALL = 433
    HAVE = 2469 A = 4 BALL = 7433
0 and 6 are interchangeable.

11. Unfortunately, there are two solutions.

    MEET = 2990 A = 6 PEANUT = 196 780
    FARMER = 364 294 MR = 24
    CARTER = 564094
1 and 3 are interchangeable.

    MEET = 6990 A = 2 PEANUT = 192 340
    FARMER = 528698 MR = 68
    CARTER = 728098

12. There are 13 279 summands and $$ = 77 and DOLLARS = 1022483

13. OH = 62 SO = 36 MUCH = 4702
    LUCK = 5709 $ = 1 FLUSH = 85 732

14. This alphametic has many solutions. I meant to include the following condition that appeared with my original version:

What is the largest RAT that I could have seen?

    WAS = 159 IT = 32 A = 5
    RAT = 752 I = 3 SAW = 951

15. Z = 7 SLEEPY = 163324

16. POOPER = 833850
    SCOOPER = 2633850

ON CHERCHE DES CRITIQUES

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SEVENTH INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION

The 7th International Congress on Mathematical Education (ICME-7) will be held in the city of Québec (Canada) from August 16 to 23, 1992. It will be the seventh in a series of congresses of the International Commission on Mathematical Instruction (ICMI), following those of Lyons (1969), Exeter (1972), Karlsruhe (1976), Berkeley (1980), Adelaide (1984) and Budapest (1988).

In an effort to meet the diverse needs and interests of the 3000-3500 expected participants, the program will cover all of the major areas of mathematics education at the elementary, secondary and post-secondary levels. Activities will include lectures, working groups, topic groups, workshops, short communications, posters, project presentations, and films, as well as exhibitions of textbooks, software, and other types of materials. Here are a few examples of themes that will be discussed during the congress:

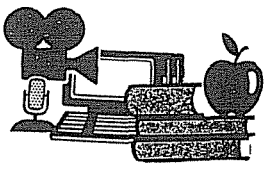
Improving students' attitudes and motivation
Mathematics for early school leavers
Innovative assessment of students in mathematics
Students' misconceptions and inconsistencies of thought
The impact of calculators on the elementary school curriculum
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Modeling activities in the classroom
Students' difficulties in calculus
Undergraduate mathematics for different groups of students
Pre-service and in-service teacher education
Methodologies for research in mathematics education

Founded in 1608 by Samuel de Champlain, the city of Québec, capital of the Canadian province of the same name, is the cradle of French civilization in North America. Because of its unique charm, its historical past and its exceptional location dominating the majestic Saint-Laurent river, Québec is a privileged place which attracts tourists from all over the world. The ICME-7 congress will take place on the campus of Université Laval, which offers facilities and services making it a most convenient place to hold such an international event.

The Second Annoncement of the congress will be published in1991 and will contain detailed information about the program, as well as forms for registration, accommodation, and submission of short communications or posters. In order to receive it, please write to: **Congrès ICME-7 Congress, Université Laval, Québec QC, Canada, G1K 7P4**, or communicate with the secretariat of the congress by phone: (418) 656-7592, or by fax: (418) 656-2000, or by electronic mail: ICME-7@LAVALVM1.BITNET

REVIEWS

Editor:
George Gadanidis



Polynomials by E.J. Barbeau
Published by Springer-Verlagt

The purpose of this book is to provide an in-depth study for students and teachers who wish to broaden their knowledge of mathematics by studying a specific topic. This is in addition to the curricula work in which many topics are surveyed, but to no great depth. It is also a possible alternative to extension through problem solving and contests or a possible study for students who have fast tracked through high school mathematics and require more work in their last year.

The book is intended to start from the high school material in solving polynomial equations, and extend to a full study of polynomials, including both theoretical, skill and problem solving development, as well as giving a historical background. The topics studied include combinatorics, geometry and number theory where they are relevant to the study of polynomials. Background knowledge is needed in manipulating algebraic expressions, solving linear and quadratic equations, both simple and simultaneous. Some knowledge of trigonometry, exponential functions and logs is required, and a background in Calculus is useful but not necessary. Opportunity to use the materials is offered in the form of practice exercises, exploration topics and problems from journals and contest papers.

One very useful feature of this book work is that each chapter has a set of hints for the exercises. Also answers and full solutions are given. The completeness of these can be judged by noting that they take up 149 pages. The book also contains suggestions for further reading from texts and journals, a glossary of terms, and an index.

The book is an extremely comprehensive study of polynomials, and would be an excellent challenge for exceptional students or teachers wishing to review or complete their knowledge of this topic. All in all this is a well written, complete and self-contained work.

Reviewed by: T. J. Griffiths

Marilyn Burns. *Mathematics: for middle school. Videotapes for 6-8 Staff Development.* New Rochelle, NY: Cuisenaire.

For those of us who are uncomfortable with traditional "Chalk and Talk" methods of mathematics education, Burns' three videos, *Mathematics for Middle School*, offer a refreshing alternative. Produced by Cuisenaire Co. of America, in 1989, this three-part series focuses on the roles of student, teacher, and mathematics content. Accompanying discussion guides can be used to construct in-service workshops devoted to staff development. We were intrigued by the classroom teaching examples where students were self-reliant, the teacher was not the sole dispenser of knowledge, and concepts tended to be integrated with one another and with shared experiences.

In many ways, Burns' approach is consistent with NCTM's five general goals for mathematics students: (1) that they learn to value mathematics, (2) that they become confident in their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) that they learn to reason mathematically." (NCTM, 1989). Implicit in these goals are that students should be exposed to a variety of interrelated experiences; and "encouraged to explore, to guess and even to make and correct errors so that they gain confidence" (NCTM, 1989). Thus reading, writing, discussion, conjecture, and argument are part of students' tasks.

The *Ontario Ministry of Education Guidelines* (1985) also support this view. The classroom should provide "many opportunities for students to experience success and positive reinforcement". There should be an "acceptance of alternative solutions", and mathematical concepts and generalizations should be related to applications familiar to the learner (ibid). In order to achieve this, a number of strategies are recommended: individual projects, groups, and pairing of students, such that a wide range of interests and skills are spanned. In addition, students are specifically encouraged to verbalize their understanding (ibid). Given the daily demands of the mathematics classroom, the above goals may appear to be a tall order for anyone to fill. Burns' approach is one solution worth consideration.

PART I: The Role of the Student illustrates the importance of student communication about mathematics. With the use of many active learning situations, it exemplifies the benefits of

having students explain the reasons behind their thinking.

The video depicts a co-operative learning, hands-on approach. In their groups, students are given opportunities to work with several manipulatives. The first lesson considered makes use of Cuisenaire Rods (coloured rods of varying lengths) to help students learn ratio and proportion. Students are given several tasks to perform in groups and are instructed to brainstorm with fellow members to develop solutions to the problems. After the groups have been given enough time to develop an understanding, they engage in a whole-class discussion to explain and defend group solutions. The teacher listens to each explanation attentively, sometimes having students clarify their statements, but rarely correcting any false ideas. The videotape continues in this manner making use of manipulatives: Colour Tiles to study perimeter and probability, and Pattern Blocks and Hinged Mirrors to study angles.

Overall, Burns promotes a holistic approach to learning. The tasks that students are asked to undertake always involve many aspects of the mathematics' curriculum and provide for the development of good problem solving skills. She stresses the importance of having students explain what they are thinking, verbally and in writing. When students discuss their reasoning they are engaged in clarifying their own ideas. They have the chance to extend their thinking, and solidify their understanding. As they do this, other students in the class also benefit by having their own ideas confirmed or by hearing another approach to solving the problem. Teachers can benefit from listening to students' explanations as well. Paying close attention to students' responses can allow a teacher to asses what the students do and do not understand.

The second video, PART II: The Role of the Teacher, and its accompanying discussion guide focus on how a teacher can make the learning of mathematics meaningful to students by having them become more actively involved in the learning process. The approach is to first allow students to engage in problem solving activities related to the topics so that they will build their own understanding. Ten essential characteristics for employing problem solving in the classroom are outlined in the discussion guide and the video demonstrates all of these. However, it is the picture of students working cooperatively in small groups while using manipulatives that really stands out. In the lesson on fractions, the students were given Cuisenaire rods and asked to work out relationships between the various rods and discuss their ideas. This was followed by a full class dialogue in which groups presented their solutions to each other. Although some students were initially confused, the teacher did not jump in with the right answer. Misconceptions and errors are accepted as part of the learning process. By the end of the lesson, however, the students were able to see where they had gone wrong by listening to the explanations supplied by the other groups.

All too often one hears the cry "having groups do problem solving sounds good, but how would I ever incorporate it into a mathematics classroom?". This video and its discussion guide answer that question.

PART III: The Role of Content demonstrates how to approach mathematics concepts in a variety of ways and how to integrate different areas of the mathematics curriculum. Five lessons focus on six different mathematical ideas: percent, geometry, logical reasoning, algebra, and arithmetic skills. In all lessons, the primary emphasis is on improving the students' thinking and reasoning skills.

The first two lessons show different approaches to teaching percent. The first is in a real-world context and the second uses an investigation of area. When a discrepancy in thinking among students occurs, they are given a chance to think about it further so that students unravel contradictions for themselves. The third lesson integrates geometric investigation and exploration of fractional relationships through Tangrams. Students again come to their own conclusions about concepts of area and fractions through group interaction and class presentation. Lesson four gives students an opportunity to continue logical reasoning skills as they figure out what assortment of colour tiles fit a set of clues. Finally, concepts of arithmetic, geometry, and algebra are integrated in the fifth lesson. Here students work on their own reasoning skills by determining how many squares there are around borders, first in groups and then through class discussion.

These techniques are essential for rich, problem solving lessons focussing on developing students' understanding and appreciation of mathematics.

Reviewed by: *Amy Beck, Michael D'Agostino, John Kennedy, Mark Rand.*
Pemstep 1990-91
Faculty of Education. U.W.O.

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