Vol. 35 #1, September 1996

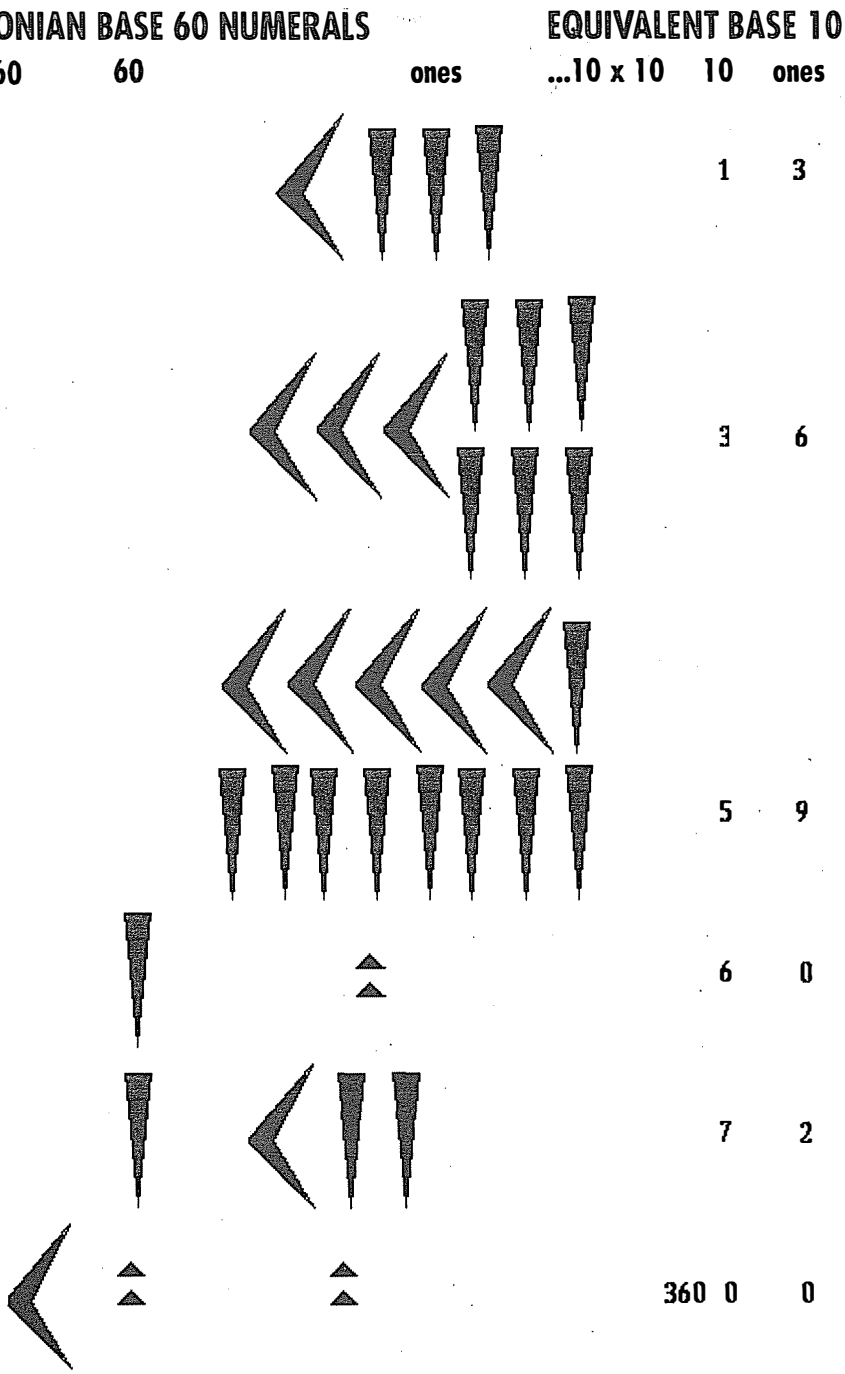
Ontario Mathematics Gazette

IN THIS ISSUE

A New Column!!!
Teaching Ideas from the Editor

Also, don't miss...
Using Writing with Project Work to Learn Calculus

See Inside for More;
Episodes in Mathematics for the Classroom: The Babylonian Numeration System



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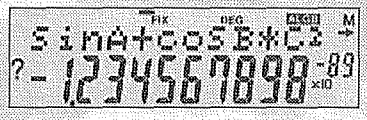
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sin 30 + cos 45 =

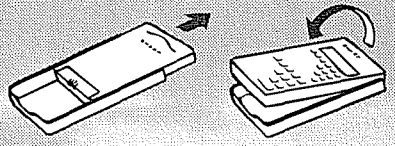
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▲ TABLE OF CONTENTS

Columns

▲ About the *Gazette* 1

▲ Editorial / John Egsgard 3

▲ President's Message / Barry Onslow 4

▲ Letter to the Past-Editor / Ron Lancaster 5

▲ Teaching Ideas from the Editor / John Egsgard 16

▲ About Problem Solving: Fun with Factorials / John Grant McLoughlin 30

▲ Directors' Dialogue / Peter Saarimaki 31

▲ OAME Council . . . . . inside back cover

Articles

▲ Episodes in Mathematics for the Classroom: The Babylonian Numeration System / Medhat Rahim 6

▲ Mathematics Education from a Tutor's Perspective / Frances Schatz 10

▲ Using Writing with Project Work to Learn Calculus / Gord Doctorow / Peter Harrison 16

▲ Using Analytical Geometry to Find All Pythagorean Triples / Kenneth Kam-Chui KO 24

Notices

▲ OAME Excellence in Mathematics Certificates 3

▲ OAME Membership Drive Contest 11

▲ OAME '97 12

▲ Nominations for OAME/AOEM Board of Directors 1997-98 29

▲ Linking Assessment and Instruction in Mathematics: Connecting to the Ontario Provincial Standards Transition Years 33

Advertisements

▲ Sharp Electronics . . . . . inside front cover

▲ GVMA 5

▲ ips Publishing, Inc. 9

▲ Neufeld and Associates 14

▲Texas Instrument . . . . . centre spread

▲ Exclusive 21

▲ McGraw Hill 23

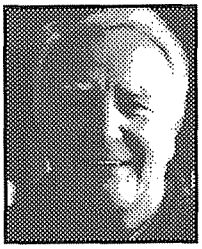
▲ Visual Ed 26

▲ LWCD inc 31

▲ Gage . . . . . back cover

EDITORIAL

JOHN EGSGARD



As I begin my term as editor of the *Ontario Mathematics Gazette* I wish to thank Jack Weiner for the excellent job that he did in his four years as editor. I would ask you to read the *Letter to the Past Editor* to gain an appreciation of what Jack has done.

I have recently retired after teaching secondary school mathematics for 47 years. My years have been happy years and successful years. I hope that the *Gazette* will help all teachers to enjoy the teaching of mathematics as I did.

In most issues of the *Gazette* you will find an article entitled "Teaching Hints from the Editor". It is my hope that teachers at the elementary and secondary level will submit such articles to the *Gazette.* I realize that most of you have little spare time. But I would ask you to use some of this time to share your wonderful teaching ideas with your peers.

Another article has been written by our new *Abacus* editor, Frances Schatz. Frances has spent most of her teaching career tutoring students. Her article will give you some insight into this important work.

At the Eighth International Congress on Mathematics Education in Seville, Spain this past July, I met with about 50 editors of mathematics journals from around the world. We see new challenges in producing our journals because of the possibility of making our work available on the world wide web. Is the OAME ready for this?

John L. Egsgard

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▲ PRESIDENT'S MESSAGE



BARRY ONSLOW

I hope everyone has had a relaxing and enjoyable summer, and is ready for the challenges of a new academic year. When looking towards these challenges I believe it is important for us to keep in mind the mission statement of OAME/AOEM:

- excellence in mathematics education throughout the Province of Ontario;
- excellence for students by striving for a curriculum that will prepare them to meet the mathematical challenges in their lives, by fostering familiarity with the methods of mathematics and an admiration for its principles, and by giving recognition and respect to their own mathematical achievements;
- excellence for educators by promoting their role as facilitators and resource persons, by supporting a variety of teaching methods to meet the varied learning styles and needs of students, by encouraging their students to gain facility in learning both cooperation and independence, and by celebrating the mathematical excellence of both individuals and groups.

The various committees of OAME use these ideals in their efforts to support classroom teachers throughout the province. This year our Special Projects Committee and Specialization Years Committee (no doubt with a new name in '97), will be concentrating on the recommendations for Secondary School Reform. We will be working closely with the Ontario Mathematics Coordinators Association, attempting to make the transition as smooth and advantageous as possible for students and teachers. Neither of the groups want to miss opportunities for improving mathematics education during this period of change. The Ministry of Education and Training (MET) has organized a series of meetings with subject/division associations during 1996-1997. I have attended the last two of these meetings on your behalf, and it is encouraging to see MET consulting with the various provincial associations who have similar concerns to ourselves. One can only hope that this consultative approach will lead to a curriculum that is developed from the bottom up as well as from the top down. Your input is valued, so please contact me or your Chapter Representative to express your opinion. Also, please remember that the Gazette publishes letters to the editor (another means of sharing your thoughts).

During the summer Mickey Sandblom and I had the privilege of representing OAME/AOEM at the Annual NCTM Leadership Conference, this year in Montreal. Many of the provincial organizations favour moving towards a national mathematics curriculum. There is already a western consortium and an eastern consortium. I believe it is more productive for people to work together while attempting to develop a similar wheel. There are far more similarities than differences in our various provincial mathematics curricula. Perhaps, through working together, we can build on each other's strengths while still maintaining those elements of curriculum crucial to the needs of individual provinces. I welcome your opinions regarding this matter.

This year the Ontario Mathematics Olympics will be held in Barrie under the guidance of Bill Morrison, Paul Pogue and the MAC^2 Chapter. The events over the last two years have been sources of inspiration for both the students and teachers who attended. This year we once again have the support of Rockwell International of Canada. We are hoping for even more students to participate in the spring of '97, especially at the Chapter level, so start organizing your Chapter play downs now. You will not be disappointed—just ask the participants from Thunder Bay who won last year's competition.

Remember, your opinions and efforts are valued. We welcome your thoughts on how we can better meet your needs.

A Letter to The Past Editor-Jack Weiner

Dear Jack:

My April 1996 issue of the Ontario Mathematics *Gazette* arrived in the mail this week and I greeted it with mixed feelings. On the one hand I was sad that this was the end of a very special era for the Ontario Association for Mathematics Education. You came on the scene as Editor of the *Gazette* when the OAME was in need of someone with a vision of the type of journal that would satisfy the needs of teachers across Ontario. You took the journal into the age of desktop publishing and you brought major changes to the look and the content. I was President of the OAME when you began your term as Editor and I cherish the time that we were able to spend together discussing the issues of the day and I enjoyed the work that we did on behalf of OAME.

I was also happy that this was your last issue because I know full well that the job that you did was highly time consuming. I know that it was a rarity that anyone ever really said thank you for the job that you were doing and quite frankly I think that very few people realize what it takes to act as Editor for a major publication.

The short thank you in the *Gazette* failed to mention the many other contributions that you have made to Mathematics education in Ontario. For many years you have spoken at conferences all over this Province. You have freely shared your creative ideas with educators and you have helped lead mathematics education into the next century. You have written textbooks, acted as a consultant for an award winning (Gold Medal - 1990 New York Film and TV Festival) television series for TVOntario and you have also given countless hours to the Grand Valley Mathematics Association.

Through all of these commitments over the years you have continued to be one of most popular instructors at the University of Guelph. Your passion for teaching and your high level enthusiasm has had a dramatic effect on the lives of many students.

Congratulations on a job well done Jack!

Sincerely,
Ron Lancaster
Mathematics Teacher and Coordinator
St. Mildred's-Lightbourn School
Past President of the Ontario Association for Mathematics Education

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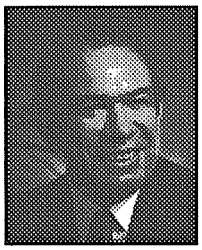
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▲ Episodes in Mathematics for the Classroom: The Babylonian Numeration System

MEDHAT H. RAHIM
FACULTY OF EDUCATION
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THUNDER BAY, ONTARIO, P7B 5E1



Studied at the University of Baghdad, University of London-England, and University of Alberta, Medhat has taught mathematics, statistics, and math methods at University of Alberta, University of Northern Illinois, and University of West-Indies Trinidad. Currently he is an associate professor at Lakehead University, Faculty of Education. His interests include use of manipulatives and technology in teaching and learning mathematics.

The invention of symbols for numerals was a problem for all humankind civilizations throughout history. The Greeks (about 600-300 B.C.) used their 24 alphabets, Alpha, Beta, Gamma, ..., and Omega with three additional invented symbols as numerals to come up with twenty seven symbols used in groups of nines. No specific sign was used to indicate whether a combination of alphabets represented a number or a word. Apostrophes were often used to indicate that the alphabet was to be considered as a numeral (NCTM, 1969, p. 27). They had no symbol for zero nor were they aware of the place-value concept. Although the Greeks were skilled in geometry they seemed hopeless in arithmetic. The Romans (between 500 B.C. and 100 A.D.) were even worse. They used the symbols, I, V, X, L, C, D and M to represent the numbers 1, 5, 10, 50, 100, 500, and 1000 respectively; place-value was not known either. Their combinations of only a small handful of symbols I, V, X, L, C, D and M made it impossible to manipulate numbers. For example,

III times DLXXVIII [3 times 578 in base ten system]

or even a simple sum like

XXIV [twenty four]

plus VIII + [eight]

are somewhat meaningless no matter how you write them.

Much earlier, the Babylonians (about 2000 - 1500 B.C.) presented humankind's first successful invention of a numeration system , the *Babylonian Numeration System*, that was based on the sophisticated mathematical notion of the *Place - Value* where symbols represent different values depending on the *place* they were written.

The Babylonian Numeration System is a base 60 place-value system using only two symbols, a one written as ▼ and a ten written as ❮ (Figure 1).

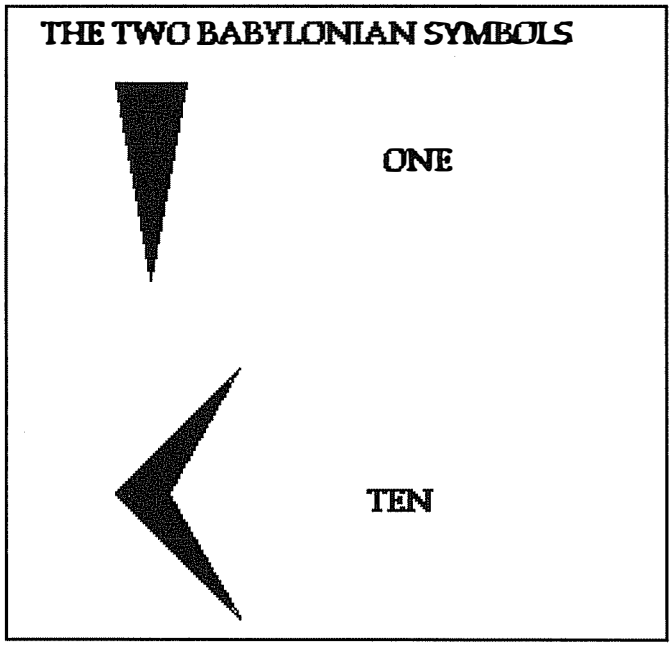


Figure 1

The zero concept was not symbolized in the Babylonian system. However, in writing a number containing zero, a vacant place was left to represent zero. The text of the problem in hand was the only clue for that vacant place. The Babylonian place-value system suffered from the lack of a zero symbol until about 300 B.C. A separate symbol made up of two small triangles arranged one above the other was then used as a *placeholder* to indicate a vacant place (Eves, 1969,

As such, there were 59 'digits' each of which was made of these two symbols, the one ( ) and the ten ( ). Figure 2 below shows some of the 59 'digits'.

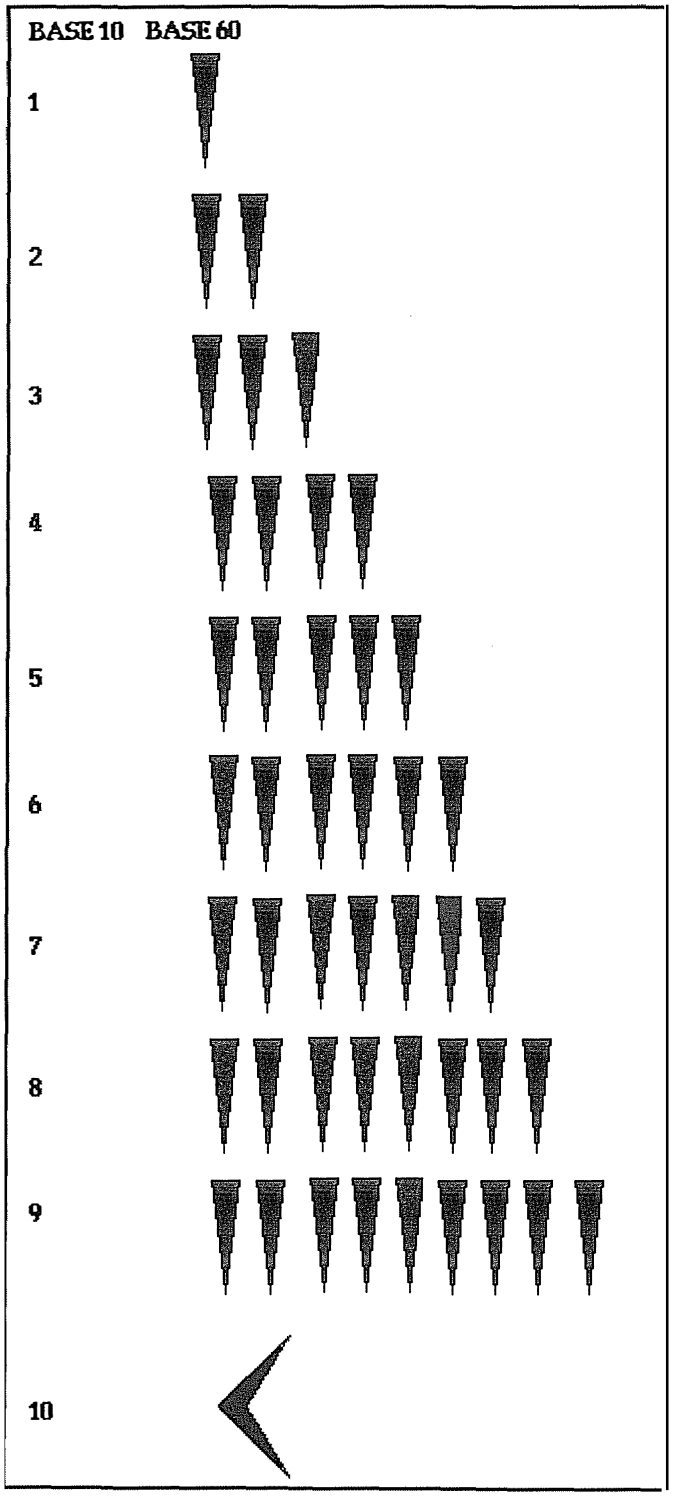


Figure 2 As in the base 10 system ther are 9 non-zero digits (1-9), in the Babylonian base 60 system there are 59 non-zero digits (1-59). Here are some of these Babylonian digits.

The early Babylonians, having little access to suitable stones, used clay tablets as a writing medium. The inscription was pressed into a clay tablet by a stylus specifically made to form wedge-shaped characters known nowadays as cuneiform characters because of their triangular shapes. The tablet was then baked into a hard permanent record. Clay tablets dated from 2000 B.C. to 200 B.C. showed that numbers *less than 60* were expressed by a simple grouping system to base 10 and numbers greater than 60 were expressed according to the place-value concept. It is interesting to note that the Babylonian writing of numerals is often simplified by using a subtractive symbol shown below (Eves, 1969, p. 11 & 15).

Figure 3 shows a contrast between the Babylonian numeration system and our present base-ten (Hindu-Arabic) system.

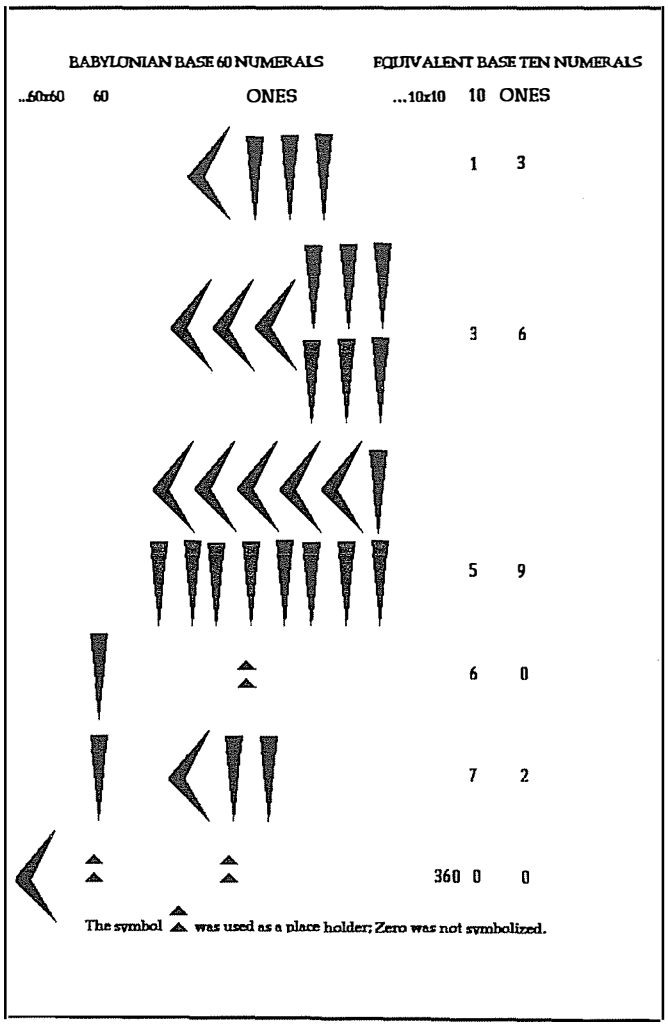


Figure 3

Examples of written numbers, small and large, using the symbols for 1, 10, and subtraction are illustrated in Figure 4.

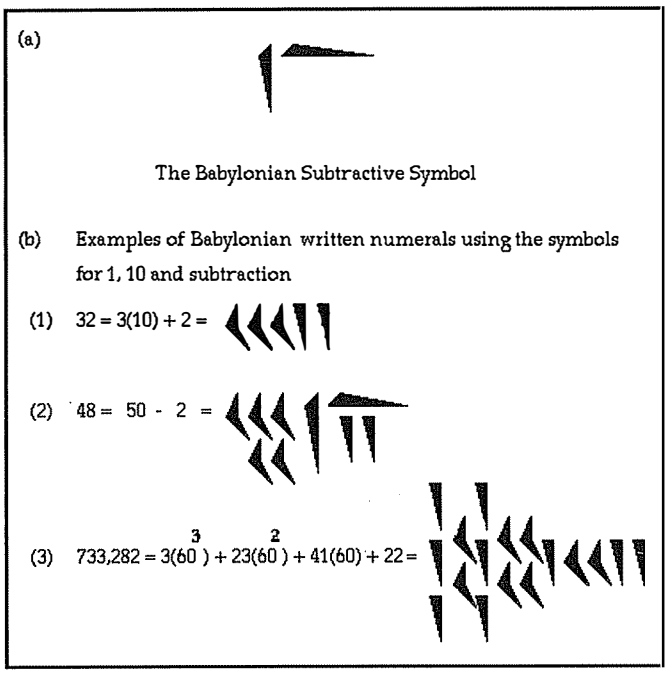


Figure 4

Examples worked out in details:

Change to Babylonian numerals each of the following

(a) 21 (b) 72

(c) 200 (d) 700

(e) 4331 (f) 600

(a) 21 in the Babylonian base 60 system will be expressed as

2 x 10 + 1 since 21 is less than 60. Thus,

21 =



(b) 72 in the Babylonian base 60 system is

1 x 60 + 1 x 10 + 2, thus

72 =



(c) 200 will be expressed in the Babylonian base 60 system as

3 x 60 + 2 x 10, therefore,

200 =



(d) 700 in the Babylonian base 60 system is

11 x 60 + 4 x 10, thus

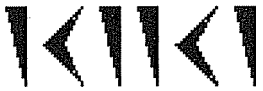
700 =



(e) 4331 in the Babylonian base 60 system is

$1 \times 60^2 + 12 \times 60 + 1 \times 10 + 1$, therefore,

4331 =



(f) 600 in the Babylonian base 60 system is

10 x 60, that is,

600 =



Worksheet

Change to Babylonian numerals each of the following:

(a) 33 (b) 75 (c) 100

(d) 212 (e) 353 (f) 1342

(g) 4362

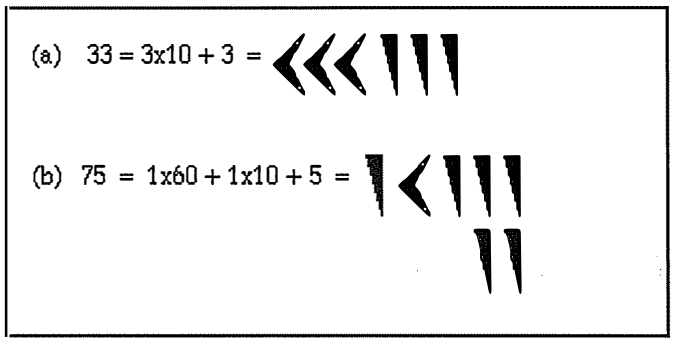


Figure 6 Answers for the worksheet (a-b)

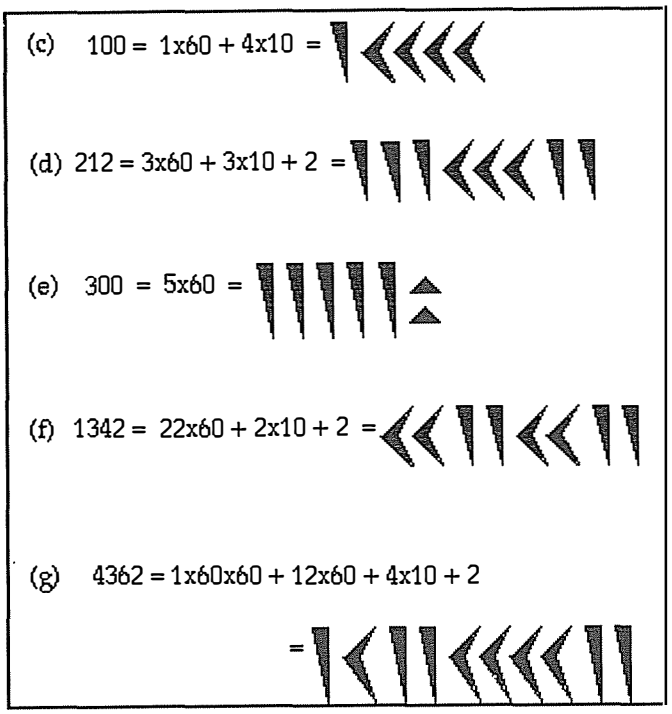


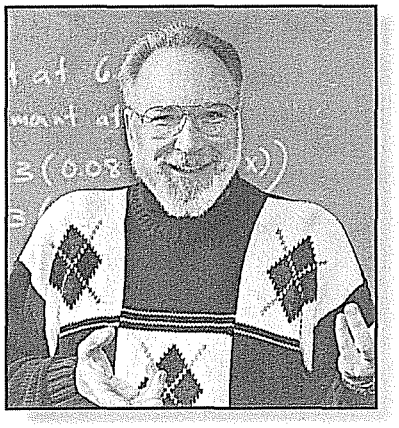
Figure 6 Answers for the worksheet (c-g)

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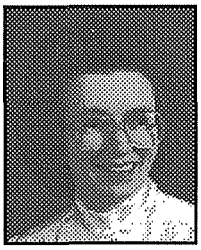
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▲ Mathematics Education from a Tutor's Perspective

FRANCES SCHATZ



As a freelance mathematics teacher and writer Frances Schatz's main interests are tutoring, correspondence teaching, puzzles, and crafts. She teaches transition and senior mathematics and has had craft patterns published.

Private Tutor: My Career Choice

Numbers and patterns have always fascinated me. My earliest childhood experience involved counting on the farm - hay bales in the field, newly hatched chicks, bushels of apples or flower stems for a market bouquet. Counting 3, 6, 9, ... replaced I, 2, 3, ... at an early age. Mother and Dad had, in a pleasant way, helped me develop a strong number sense before I started grade one. Arithmetic in the elementary one-room, one teacher, eight grade school was presented first in workbooks then in texts. Senior students were expected to work cooperatively and to assist in junior grades. Evaluation was based on daily exercises done and short unit tests. In high school I was an average student until grade thirteen provincial exams were written. My excellent exam marks led to mathematics studies at university. Calculus and applied math made all the high school algebra and geometry take on a new meaning for me. Then I knew I wanted to teach mathematics. Although I was a very successful classroom teacher, I discovered my real interest was programs for individuals. When I married my career needed to be home-centred so I could meet the needs of my family and put my creative ability to work; private tutoring was the answer.

Mathematics plays a major role in my 'real world' so my students see and hear about applied algebra and geometry. I design and write crochet and knitting patterns that use geometry shapes and multiples of stitches (sequences and series). I apply transformational geometry to a teddy bear pattern to create a panther from cartoon fame or some other whimsical creature. Craft shows are a wonderful source of applied math items. I share my favourite puzzles with my students. About three months before Christmas I doodle with graph paper, ruler and compasses to design my own greeting card.

My tutorial work not only provides instruction to private students but also includes school liaison with shut-ins and hospital patients as well as teaching by correspondence. In addition, I have learned braille and provide tactiles, tapes or brailled worksheets for visually challenged students.

Tutorial Goals and Expectations

The first contact is initiated by a teacher, counsellor, parent or student. Our first meeting is "let's get acquainted". From the student I gather information about extra-curricular activities, career dreams, personal progress evaluation, health notes, and feelings about school/job. It is important to focus on a student's interests outside of math class so motivational problems for current topic are introduced. Whether a student is struggling to get a course credit or is aiming to earn a scholarship, the tutorial should enhance the classroom lesson. A senior student described the role of teacher/tutor very well by saying:"my teacher is an excellent instructor; however, I need someone to explain certain lesson concepts and I don't wish to be subjectively evaluated by my questions".

Before tutorials start the student also needs to meet the tutor. At the initial meeting I provide a brief resumé, suggest a realistic approach to math help and state my expectations on a brightly coloured page. I highlight the importance of regular attendance in math class with ideas to enhance performance. In addition to keeping up with class work the student is expected to complete a tutorial assignment. All questions must be tried for me to determine understanding of the tutorial and to find any weak links from previous studies. REASONS for incomplete work are accepted in advance of tutorial. EXCUSES (statements made re work or missing books on arrival for tutorial) are not tolerated and will lead to cancellation of tutorial with fee charged. To me success depends on personal initiative and hard work in a disciplined but friendly environment. Student and tutor must work in a teacher-learner partnership to achieve individual excellence and confidence.

Tutorial Log

To identify a student's performance level I review the student's previous tests and/or notebook and occasionally (with student's permission) ask the classroom teacher for input. I keep an envelope file for each student with health, lesson, assignment and observation notes for each tutorial. The student may also be asked to log each math class re topic, homework and personal comment.

My log notes include challenging problems and stories which give an interesting perspective on mathematics teaching.

The Skill-Testing Question: Order of Operations

Evaluate: (-13 + l5)(-8 + ll) ÷ (-12 – 8)(5 – l0)

Just when I believe I have finally mastered the teaching of order of operations, I find out I'm wrong again. This simple looking problem has caused me more anxiety than any other type in mathematics. Mathematics is supposed to have only one answer but what does a tutor do when two answers are correct depending on what course/teacher the student has and when the calculators don't agree.

When I was taught order of operations I learned that any operations indicated by brackets preceded multiplication and division symbols. Then BODMAS came along and it said 'brackets, of(implied multiplication), division and multiplication, then addition and subtraction'. The latest mnemonic is BEDMAS where E stands for exponent calculations. The tutor's challenge: which solution A or B is correct?

A. 2 x 3 ÷ -20 x -5 = l.5

B. (2)(3) ÷ (-20)(-5) = 6 ÷ l00 = 0.06

In test evaluation each one has l:l chance of being correct with incorrect given zero marks. Both are correct - try out the CASIO calculator for one example:

CASIO fx-6300G — Keystroke AC followed by above expression as printed and answer will show 0.06.

Keystroke again with a x sign between each)(and answer is l.5.

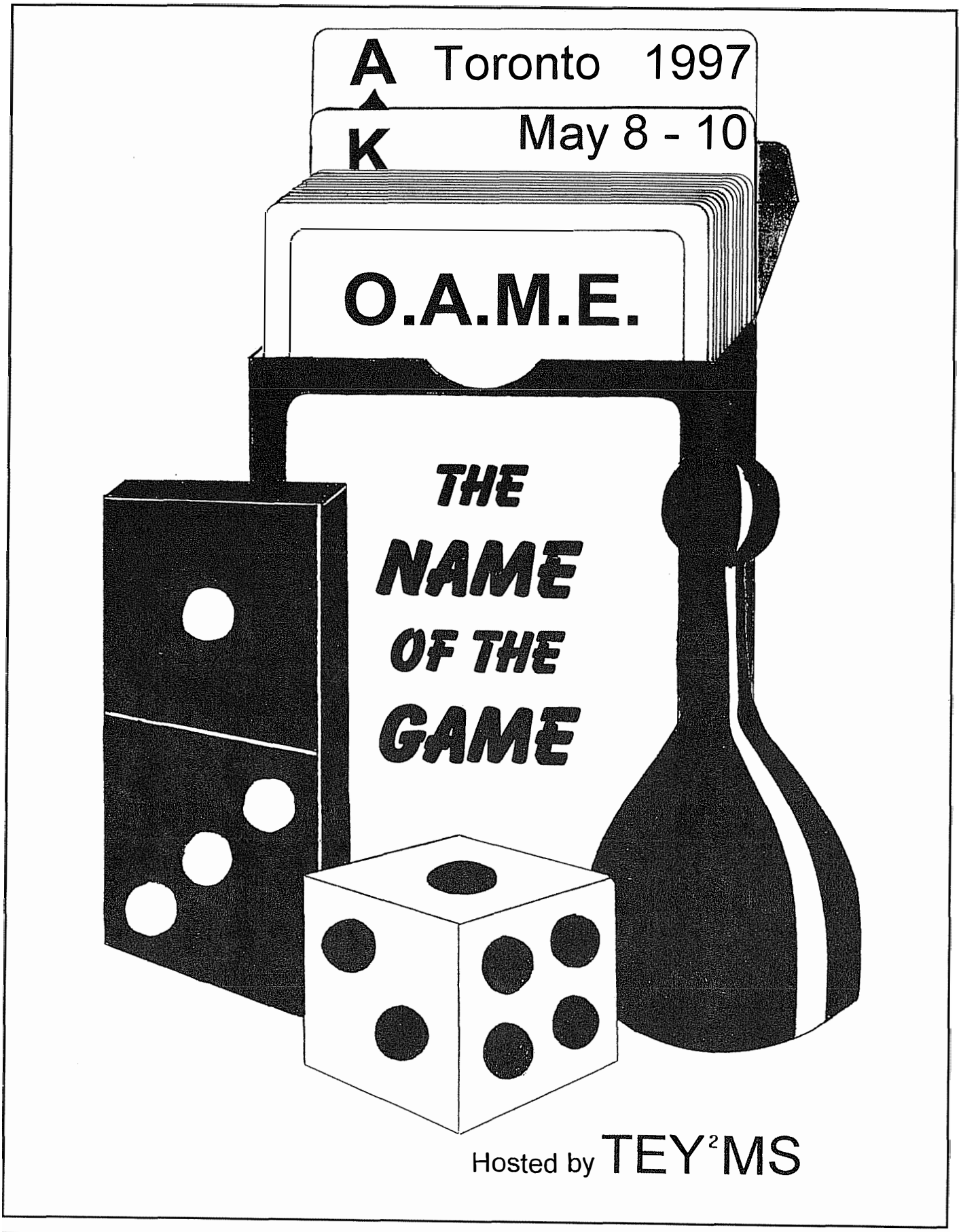
This calculator tells us the given expression is ambiguous.

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This same problem arises in the following:

$\left(1\frac{2}{3}\right) \div \left(\frac{5}{9}\right)\left(\frac{4}{3}\right)$ A. 4 B. $2\frac{1}{4}$

$36 \div 2 \log_6 36$ A. 36 B. 9

We must develop a new mnemonic to deal with the above examples or we need to make all of the above examples unacceptable. It is interesting to note that when I present the original problem in workshops it generates controversy because we teach the method we were taught. Which answer will win the next skill testing question?

Factor Fears

I. Factor: $10x^2 - 11x + 3$

Most of the methods we use for factoring trinomials generate the final answer (5x-3)(2x-1). I realized how vague the instruction 'factor' was when I saw the following solution written by a new Canadian student:

$= 10(x^2 - 1.1x + 0.3)$

$= 10(x^2 - 0.5x - 0.6x + 0.3)$

$= 10[x(x - 0.5) - 0.6(x - 0.5)]$

$= 10(x - 0.5)(x - 0.6)$

This student lost a point out of 2 for bad form because the answer wasn't the implied 'factor over the integers'.

2. Factor fully: $7x^2 - 3y^2x + 3$

This test question is most upsetting to the math anxious student because 'not possible' is not an option yet the expression will not factor. During the test the student asked the teacher if there was a misprint and got the reply, "we've done lots of those in class". There is no room for an incomplete question on a test but there is room for 'factor fully, if possible' which makes a student think and apply skills learned.

Memorized Geometry - No Diagrams

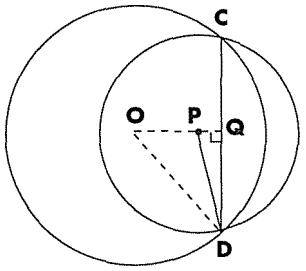
This tops my list of tutorial challenges. Jim, a grade ten mathematics student, realized he was doomed to fail this level again yet he was determined to complete the last mathematics course needed for his diploma. Jim had the best set of geometry notes I have ever seen in the congruency/parallelism topic. He had copious memorized notes about triangle ABC with no diagrams. In the initial interview Jim told me: "When I don't understand a concept, my emotions take control. Fear, depression, anxiety, failure, try to overcome these problems, more failure, and finally twenty years of avoidance". My observations indicated the two major problems that meant trouble in making the diagrams: figure ground perception and fine-motor coordination. Since I am a visual learner in mathematics I had to make major changes to my method of instruction.

Tactile methods included feeling various shapes, using pattern blocks to make designs, folding paper to demonstrate geometric terms and cutting out nets. Then we used ruler and compasses and this proved to be the difficult part because the hand didn't really want to do what the mind desired. Jim heard, understood and wrote good solutions if I verbalized diagram observations. Auditory perception was strong due to extensive use of auditory mode at his workplace where strong listening skills were needed for procedures, schedules and tabled information. I had him make his own cassette tapes of geometry notes and exercises so he could verbalize while writing and listen while reading for study. The neural pathways of feel and hear were assisting vision in overall perception. Jim proved to himself that he could do math and finished grade twelve advanced math with top marks. In fact, he was the first student to point out a second possible solution to a problem in circle geometry.

Which Way Do I Hold These Circles?

Problem: Two circles have radii of 8.0 cm and 11.5 cm. If the circles intersect at C and D so that CD = 14 cm, determine the distance between the centres of the circles.

Jim's Query: Which way do I hold these circles? Jim had cut the two circles out of cardboard and discovered that there were two ways to reach the position where CD = 14 cm. Hold the circles apart then move them together OR start with centres matching and move the smaller circle outward to the required position.



CD = 14 cm
DQ = 7 cm
(chord-centre theorem)
PD = 8.0 cm
OD = 11.5 cm

Figure 1

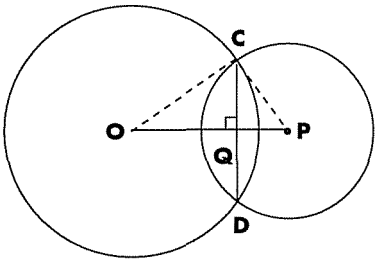
Solution I: (Figure I)

$$PQ = \sqrt{8.0^2 - 7.0^2}$$
$$= 3.873$$

$$OP = \sqrt{11.5^2 - 7.0^2}$$
$$= 9.124$$

Distance between centres

$$OP = 9.124 - 3.873$$
$$= 5.3 \text{ cm (to nearest tenth)}$$



CD = 14 cm
DQ = 7 cm
(chord-centre theorem)
PD = 8.0 cm
OD = 11.5 cm

Figure 2

Solution 2: (Figure 2)

Calculate PQ and OQ as done in solution I.

Distance between centres

$$OP = 9.124 + 3.873$$
$$= 13.0 \text{ cm (to nearest tenth)}$$

What is Math Anxiety?

Math anxiety is a combination of negative, mental, emotional, and/or physical reactions to symbols and logical thought processes brought on by negative experiences. Children and teenagers show anxiety by avoiding assistance, by behaving inappropriately, or by using excuses. Adults may be haunted by memories of past experiences which cause thinking blocks yet they come into the learning situation with lots of self-confidence and self-perception.

The tutor can help a student overcome math anxiety with the following steps:

1. Review placement level. If significant knowledge from a previous level is missing, consider retaking or auditing the weak course. A different level of difficulty may be needed.
2. Homework. Let's find a way to do it comfortably and in a reasonable time frame.
3. Talk out the frustrations. Assist the student in establishing a better communication/working relation with the classroom teacher. The struggling students are hurt the most by remarks made by a teacher out of exasperation re poor test results or perceived lack of effort in class.
4. Teach communication skills. 'I need help with math' is not sufficient if teacher's time has been requested after class. The student needs to know that saying "question #1, p. 149" is not adequate. The tutor can show the way with 'copy the question completely, then do a try or write a question about the problem'.

   The example: $x(w - 2) - y(w - 2) = ?$ is meaningless, unless the complete question includes text instruction such as factor or simplify.
5. Extra time for tests. Students and teachers resist this idea because 'it is not fair to others'. We are not equal in speed of processing information or in writing thoughts, especially in math. Try the idea on several tests to see if extra time does make the test a positive experience. Extra time may only be needed temporarily for confidence.
6. Attention to sensory learning. Observe how ideas are remembered. Recognize how seeing, hearing, feeling, and smelling are involved in thinking and memory. The tutor's thinking/teaching style and tutorial environment can be adapted to meet the

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7. Recreational mathematics. Find joy in problem solving by introducing an interesting puzzle, by using manipulatives, or by discussing a problem from the student's extracurricular interests.

Perspective on Mathematics Education

My perspective on mathematics education looks at various issues in teaching/learning.

Scientific Calculator Allowed at all Times.

We have to trust that the student (7-OAC) has developed a good sense for numbers and understanding of arithmetic operations by the end of grade six. The real world uses calculators and computers. In order to solve problems one has to learn the algorithm for the calculator and that can be as difficult as learning basic arithmetic steps. If work must be shown, then the solution requires a listing of calculator key strokes. Curriculum time should be spent on how the calculator can enhance learning and make mathematics more enjoyable. A history component about pre-calculator methods can add interest. The use of a calculator relieves stress by allowing the student to get on with the task of problem-solving without the roadblock of anxiety-producing calculations. Mastery of scientific calculator leads to graphics calculators and computer mathematics. Let 'evaluate without a calculator' test questions disappear.

Mathematics - Semester or Full Year.

In comparing student's progress in semester versus full year I find students doing a full year mathematics course master more concepts and remember the material over the long term better than their semester friends. I suggest two streams for the current advanced level - ten to OAC. Students could choose between doing one full math credit in one semester or doing two half credits over two semesters. The latter would give a positive learning experience for average and below average students. The half credit would be the same curriculum with more exercises and time for projects to master the topics.

Textbook and Course Outline.

When we read an English or literary work we read/study the book from beginning to end. As a contributing author to text material I know the effort that goes into making a series of chapters that build on skills learned in previous chapters. But I seldom see a mathematics text followed in the order it is written. If the instructor finds the book is poorly organized for class needs then a specific study outline should be provided at the beginning of the course. The tutor encourages a student to look ahead at the next lesson as part of preparation for next class.

Test Preparation and Assessment of Topic Mastery.

Textbook homework follows a definite order and topics are organized. Most students need to learn how to prepare for a test because test questions are permutated. This explains the 'but we didn't do a question like that in the homework'. Often in test evaluation I've seen a blank re a problem yet I can find the required solution method well done in the notebook. Assessment should include homework checks and marked assignments that connect topics within the unit and to previous topics.

Classroom Notemaking.

In many classrooms the student is expected to hear and copy the lesson at the same time. A student understands the lesson in class but finds recall difficult at homework time. Often I have observed that this problem is due to incorrectly written notes. The order may not be what the teacher had on the board. Check the formula students write for length of a line segment in analytic geometry. In writing powers how many times does $49^{\frac{1}{2}}$ turn into $49\frac{1}{2}$? Left-right order can mean 2x written instead of x2.

Overheads.

A few students including myself report headaches and visual strain when overhead projections are used for a length of time in math class. I have determined that the problem is due to the fine motion in the screen projection caused by the fan motor vibrating the projector. Some of us are sensitive to the motion that is not even detectable by most people. Symptoms of the sensitivity may be sleepiness, boredom or inattention and frustration.

In Summary

Past negative math experiences need not continue to burden students. Let's keep working at making mathematics learning/teaching a joy for the teacher, the tutor and most importantly the student.

▲ Teaching Ideas from the Editor

Calculus: a Helpful Mnemonic for Related Rate Problems

GURDSS

G = write down the ***G**iven* ratios

U = write down the ***U**known* rate and **when** it is needed

R = write down the ***R**elationship* among the variables

D = ***D**ifferentiate* R with respect to time, t

S = ***S**ubstitute* in D using G and U

S = ***S**olve* for the unknown in U

Example A and B start from point O. A moves east at 5 km/h while B moves north at 6 km/h. Calculate, to one decimal place, the rate at which they are separating after six minutes.

G: $\frac{dx}{dt} = 5$ km/h, $\frac{dy}{dt} = 6$ km/h

U: $\frac{dz}{dt} = ?$ when t = 6 minutes = 0.1 hours

R: $x^2 + y^2 = z^2$

D: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

S: Subsitute $x = 5(0.1) = 0.5$, $y = 6(0.1) = 0.6$, $z = ?$
Using R to find z: $0.5^2 + 0.6^2 = z^2 \Rightarrow z = \sqrt{0.61}$
Hence $2(0.5)5 + 2(0.6)6 = 2\sqrt{0.61}\left(\frac{dz}{dt}\right)$

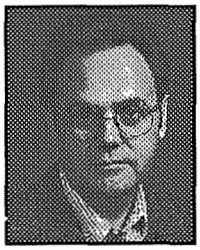
S: $\frac{dz}{dt} = \frac{2(0.5)5 + 2(0.6)6}{2\sqrt{0.61}} \approx 2.8$

▲ Using Writing with Project Work to Learn Calculus

GORD DOCTOROW
AND PETER HARRISON
SCARBOROUGH, ONTARIO



I teach all the math courses (11 to OAC) at ASE1, an alternative secondary school at the Scarborough Board of Education. My Ed. D. (OISE) thesis topic was on using writing to learn high school mathematics.



Until June 1996, I was the head of mathematics at Birchmount Park C.I. with the Scarborough Board of Education. Currently I am a Teacher Specially Assigned to the Program Department of the Scarborough Board with specific responsibilities in Mathematics (7-OAC).

The following is based on a joint presentation that we had made to OAME 95 in Guelph.

Peter Harrison's Introduction

During the spring of 1993. I was teaching OAC calculus using Peter Taylor's textbook (Taylor, 1992) for the first time. I had chosen this book because it emphasizes:

- mathematical modelling
- qualitative analysis of functions, relying upon visual reasoning
- a process-based approach
- mathematical communication.

Many of the problems in this textbook are quite difficult, and require a lot of debate, exploration and discussion. It was necessary to create an atmosphere in my class that was quite different from that in my regular classes, and I found myself confronted with the problem of what would be the most appropriate means with which

to assess the understanding of my students. Traditional pencil-and-paper testing struck me as being completely inappropriate since I couldn't envision a test that would reproduce the conditions of the lessons. Luckily, since Gord was free from teaching that year, we managed to get together to plan an alternative method of assessment. Together, we slowly developed some strategies that incorporated my interests as well as Gord's.

Gord Doctorow's Introduction

In 1992-93, I was on a year's leave from teaching (at ASE1) in order to satisfy residency requirement for the Ed. D. program in Curriculum at OISE. My thesis area was using journal writing to learn mathematics at the high school level by connecting thought and language within the Vygotskian framework. I was fortunate in being able to take on a pilot qualitative research project by working collaboratively with Peter Harrison and his Calculus class. After sitting in on one or two of Peter's classes, engaging in several planning meetings, we settled on combining writing with lengthy project work in the Calculus course. The types of writing that we utilized were:

- prompted, impromptu writing
- dialogue journal writing between students and teachers
- formal writing to produce a project product.

Purpose of the Inquiry

We wanted to explore the following questions in respect to using writing to learn calculus:

1. What does journal writing in the Calculus class reveal about changes in students' attitudes?
2. What cognitive effects, if any, appear to emerge from this activity?
3. What does the formal writing tell us about depth of learning taking place via journal writing activity?

Conceptual Framework

I used the psychological framework of the educational psychologist, Lev Vygotsky (1986). Vygotsky had developed some very important insights into the relationship between language and thought. His work demonstrates that there is a strong and practical link between language development and the development of higher order thinking. He provides criteria for assessing real conceptual growth in the educational context. Furthermore, he argues that there is an important relationship between affects (feelings, emotions) and conceptual development from the standpoint of understanding and fostering real conceptual growth.

Peter Harrison's interests were centred on finding a suitable method of assessment and evaluation to provide a problem-solving focus for the Calculus course. Thus, group project work and a month-long problems assignment were chosen.

Our interests overlapped and dovetailed. My focus was on utilizing informal and formal writing as learning vehicles in the math setting to foster cognitive growth in problem-solving activities. By making journal writing the fundamental vehicle for communication between students and teachers, we felt there would be deeper and longer lasting discussions. We also thought it would cause the students to be more reflective on paper. Moreover, Peter used the vehicle of impromptu writing to get feedback from students to assist him in planning for future projects and to understand how well they were doing at the time.

Structuring the Research Project

The students (initially 35 with three dropping out) were structured into eight project groups (selected by Peter) and were assigned two problems to be solved and written up. The first problem was a difficult, multi-part assignment (Appendix A) that we had anticipated would likely be solved by a very small number of students who would share their solution with other students. The second (Appendix A) was unique to each group. Sharing of ideas and approaches among the students was permitted and encouraged. Each group was required to submit their own collaborative write-up of the project problems.

We told the students that they could ask questions of ourselves about the project only through written dialogue. The writing-to-learn mathematics literature calls this dialogic writing (written student queries and written teacher responses) through journals. Of course, students' written communications about their problem-solving processes provided a useful source of data. In addition, we felt that making the projects sufficiently difficult and long-term (one month) would be an incentive for them to want to communicate with us. For the second project problem, we thought that the individualization of each group's problem would more likely lead the groups to turn to us for assistance, thus resulting in more interactive journal writing.


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TEXAS INSTRUMENTS

Two students were selected for a detailed case study. In addition to the analysis of their writings, I interviewed each of them for approximately an hour at the conclusion of the study to clarify questions and observations that emerged. With the students' permission, Peter and I shared the taped interviews, the students' impromptu and journal writings, and copies of the groups' final project write-ups (along with the evaluation marks and marking scheme).

Collection of the data took place in March and April of 1993. The interview transcripts, student journal writings, and the survey information were used to prepare an interpretive account of cognitive and affective changes in the students attributable to the writing activities.

Prompted, Impromptu Writings

As preparation for the dialogic journal writing associated with the students' projects, we engaged them in four episodes of prompted, impromptu writing. Such writing is characterized as previously unannounced brief, pointed writings that students do in class time. The students were made aware that no formal evaluation would be performed on any of them. Typically, the students were given ten minutes to respond to a set of questions that I formulated after considerable consultation with Peter. The writings were subsequently analyzed for certain kinds of analytic information (explained later in this paper). One writing episode concerned the "water tank problem".

The Water Tank Problem

Appendix A contains a description of a problem using the rates of flow of water into and out of a water tank. This problem is a sample of the sort of problem that the class had been wrestling with. The physical principles in the wording of the problem describe the level of preparation of the students: the idea of the slope of tangent being the instantaneous rate of change and the significance of concavity.

After providing the student project groups with handouts of this problem, we engaged them in the following prompted, impromptu writing exercise placed on an overhead projector.

Prompt 1

Now that you have read the description and requirements of your project assignments, take 5 min. to write your feelings about the assignments.

Here are some typical questions you may want to address:

"What do I understand about this assignment? What don't I understand?"

"What looks familiar? What looks strange?"

"What appeals to me?"

"What gives me bad feelings? Why?"

"How well do I think I'll do on the project?"

"What are my thoughts about getting started?"

"How do you feel about your partners in your assigned group?

(Appendix B contains a list of the other prompts that were used in the study.)

Responding to and Analyzing Students' Writings

Basing ourselves on the writing-to-learn-mathematics literature, we adopted the following ground rules for responding to the students' writings in order to minimize students' inhibitions about writing and maximizing the flow of dialogue:

- No negative comments, only hints, suggestions, references.
- No evaluation marks — commentary only: formative rather than summative.
- Brief responses (not significantly larger than their writing).
- Responses that were partly answering, partly probing, partly reinforcing.

I (Gord) recorded samples of what the students had written — to demonstrate the degree of their willingness to write and to express themselves frankly. I constructed a typology (categorization) of writing: affective comments, meta-affective comments, cognitive comments, and meta-cognitive comments. For each prompted, impromptu set, I selected quotes or paraphrases of the set and placed them in one of the columns headed by one of the four categories in the typology.

Criteria For Each Category in the Typology

Cognitive Comments deal with the kinds of things listed under Bloom's taxonomy for cognitive learnings, e.g. understanding, knowing, applying, synthesizing, analyzing. Meta-Cognitive Comments are reflections on cognitions. "[M]etacognition is a component of critical reflections. . . Critical reflections are reflections removed from the immediacy of particular experiences. Such reflections are thoughts about thoughts directed toward planning, monitoring, reviewing and revising." (Powell and López, 1988: 158).

Affective Comments deal with emotions, feelings, dispositions, i.e. the kinds of things referred to in Bloom's taxonomy of affective learnings. Meta-Affective Comments are reflections on the affective domain. "The second component of critical reflections is what could be termed 'meta-affectivity,' monitoring and controlling one's affective responses to experiences." (Powell and López, 1988: 158).

Sample Analysis of First Prompted, Impromptu Writing Exercise

The Table 1 sample is actually the first of several pages comprising Table 1 in the analysis of the study. It is presented as a sample to give the reader a feel for the way in which the categories were used to analyze actual student responses.

A glance at Table 1: "Re Comments About the Calculus Project" indicates that all four categories were well represented with a substantial showing in the "higher order thinking" categories of meta-affective and meta-cognitive comments. The writings appeared to be very open and very frank. Also present is a sense of rehearsal in the kinds of thinking-out-loud comments that are made with respect to the features of the problems and the kinds of preparations that students expect to engage in. Despite the fact that a large proportion of the students in the class appeared to have English as their second language, there seemed to be a general willingness to write. There were a few comments exuding enthusiasm about doing something different through the project that had a higher cognitive value for them.

The number of students responding was 33. A word count of the submissions yielded the following statistics: 37 words in the smallest submission, 259 in the largest, 105 in the median.

This prompt was the first of four. A similar type of analysis was pursued for the other prompted, impromptu writings.

Case Studies

Other aspects of the analysis of this study included an investigation into

- the dialogic journals used as a means of communication between the project groups and Peter and me
- the formal writeups of the projects.

To facilitate these parts of the analysis, two students – a male and a female – were selected for case studies in order to probe the context and its meanings a little more closely.

Janis (code name), was chosen because of the fact that she was verbally articulate and enjoyed writing — so much so that she was that year's editor of the student newsletter.

Ian (code name), was chosen because of the fact that he had stated through his writing that he felt that he did not benefit from it. In fact, Ian hesitated at first to be a case study participant, citing his view that he did not have as positive an attitude toward it as perhaps some other students in the class. He readily agreed to participate when I explained to him that I had chosen him for that very reason.

Their journals included sketches of the graphs embedded with comments and questions about the problem situation. These occur on one side of the page. On the other side were more elaborated comments and questions addressed to us. My responses were interspersed through their writings. In each case, there was clear manifestation of thinking out loud. In Janis's case, we saw her catching her own errors through her writing and reasoning towards a better solution. In Ian's case, advice was being sought. So, here we had a sense of the students monitoring their thinking, reflecting on the concepts they were grappling with, entering into a dialogue with themselves, their peers, and the teacher.

Even Ian, during an audio-taped interview, admitted that he had found the writing useful. For him, one chief value was that in my and Peter's responses to his written queries, he felt that he had gained much more than if he had consulted with us in an oral mode. The reason he felt this was that he had an enduring record of the discussion that he could review. He told me that sometimes when he received oral responses, he would deceive himself into thinking he had grasped the concepts being explicated only to find later that he could not recall the reasoning. The written dialogue, on the

Table 1 - PROMPT: Comments About the Calculus Project

Affective Comments

"When I first read over the assignment, I felt good about it because I like assignments. I do well on assignments and they help to boost my grades."

"To be honest, I hate the English part of it. I didn't take math to see English."

Comments about which problems seem "pretty understandable" and those which seem difficult and about intention to share with the rest of the group.

"I am not a good writer and my language is not complete yet so I don't know how am I going to manage it."

"I really like it and I think that it a challenge and I am going to try my best and hope that I'll do good...it's going to be fun working on the same questions for a month. I hope that I learn a lot."

"How do I feel about my partners? I think I would work better with some other people. I am not happy but I am okay. I'll do it if I have to."

"This assignment interests me in a way since it is a new thing and also I am curious to get started on this assignment and discussing it with other people...What gives me bad feelings is the fact that comunications are to be written but I also can say that I am not so sure since this is the first time I am doing an assignment like this."

"worrying if I will be able to answer and understand all the questions...I don't really know any of my partners."

Meta-Affective Comments

"I do well on assignments and they help to boost my grades."

"This approach of actually do physical work reminds me of doing labs in science class which I really enjoy."

"I feel that the group I am in should get together after school and work on the project."

"Q #2 I have never seen anything like this before so I am somewhat wary of it. But Mr. Bowie has said he would teach us what we will need to know about. So I guess it won't be too bad."

"The fact that I may not be able to contribute as well as the others in my group gives me bad feelings."

"I think that it is a little strange assignment because I never write down my feelings in my homework. Also, I am very nervous to do the assignment with my partners because of my poor English."

"Since this is new math it is something that is partly or even mostly in the experimental stage. If I need help and none of my friends are able to help me and all the teachers are busy, how am I supposed to know if I am doing things correctly? Also the fact that this is an OAC course and holds a great weight in extra hard work, but I am not complainning."

"Like: - it will be a great project to do if it will well prepare us for University."

Cognitive Comments

"I understand the first part completely and I think by the end of this week I'll know more about question #2 and I'll be able to understand it."

"I am not sure what you want us to do. you asked for a description are you looking for an essay or what?"

"The first question looks familiar but the second looks strange."

"I understand what the questions are asking in theory, but not how to find the answer. I drew the graph but after that I'm lost. The graph looks familiar because it is concave up and decreases."

"I understand this assignment as a group work, the assignment which has been handed out requires the understanding of graphing of slopes, tangents and previous knowledge of work done in class...I don't understand the idea of this assignment. At first when Mr. Bowie is given out the first chapter of calculating, the temperature change and Newton's law of cooling, I thought the work is leading to the chapter of derivatives. So far, none of the questions, require the knowledge of derivatives."

"I understand what is to be done but I do not understand how the graph will look for question 1 (concave down or up or maybe both?)"

"The first questions look very similar to that of #9 in section 1.4."

Meta-Cognitive Comments

"...but what I think is important is the fact that this assignment will actually teach us to think. I really hate working out of a textbook..."

"The first question seems to me much easier than the second one and the second rises my curiosity very strongly."

"What I understand about this assignment is that it is designed to test our approach to these problems by using the ideas that we were taught in previous classes."

" - The problems look like the cooling and heating problems we've done before. But the second question looks a bit strange."

"- Questions are going step by step and each solution is helping the next one."

"Drawing graphs appeal to me; any sort of diagram gives me a better understanding of the question asked."

"The assignment I recieved yesterday is different. This is because it consists of two questions which make me think. They are not easy questions especially the second one but I think they could be solved."

"The second Questions looks familiar to me because it looks like grade 12 stuff.

Eg. horizontal ar vertical asymptote. ."

other hand, facilitated conceptual development when he referred to the writing to clarify his thinking.

Both Ian and Janis had been writers for the formal reports of their group projects and they both scored very high. Janis's writing carried more of the reflective dialogue exhibited in the informal writing. Ian stated that his formal writeup was honed by the practice he had developed in the informal writing.

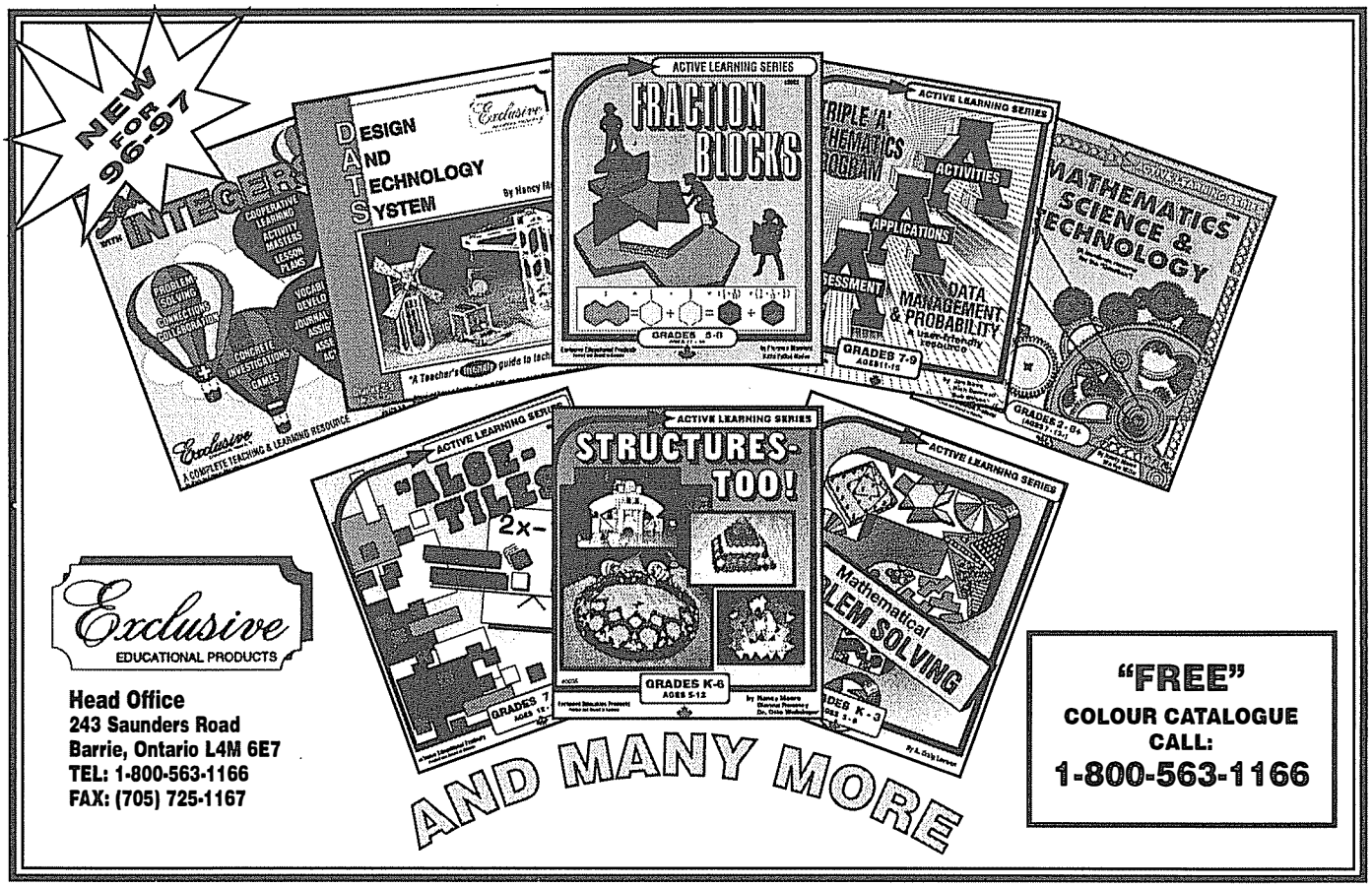
The journal writings had served as the critical vehicle for analysis in the inquiry aside from the case study interviews. The journals were examined for the following effects:

- enhanced student-teacher communication
- increased explicitness on the part of the student being expressed in the student's writing over time
- an improved and positive attitude to math learning by the students
- improved ability by the student to express him/herself cognitively and affectively, and to reflect meta-cognitively and meta-affectively
- increased proficiency on the part of the student to perform the formal write-up of the final project results because the informal, journal writing would act as an effective rehearsing medium.

We felt that our analysis had lent credibility to these claims, but we do not argue the conclusiveness of these claims owing to the fact that we were not able to verify these inferences using proper sampling techniques as a consequence of the qualitative, ethnographic nature of the research. Nonetheless, our research, which was rooted in a real classroom situation, had an important significance in theory making if not theory proving. We were attempting to examine and discover the theoretical elements of writing combined with mathematics learning and teaching.

Benefits From the Perspective of a Practising Teacher

From my (Peter's) point of view the project had achieved some important mathematics teaching/learning goals. The project had promoted clarity of exposition among the students. The writing encouraged a process of polishing a piece of mathematics in a fashion similar



to process writing in honing a piece of problem-solving exposition. It gave an opportunity for the students to demonstrate their thinking process as well as presenting a fine piece of work.

The writing project had produced some very satisfying and surprising results. One such result is the following response to question 1 b) of the "water tank problem." One student (group) wrote:

> *...When do I start the pump?*
>
> *It doesn't matter. If you start the pump when the water level is above 30L, the water flow rate out is greater than the water flow rate in. Thus, the water level will decrease, as long as the rate of water flowing out is greater than the rate of water flowing in — until the rates become equal (i.e. at 30L). At this point, 'equilibrium' is established and the water level stays steady. If you start the pump below 30L, the water flow rate out is less than the water flow rate in. Thus, the water level will rise until the rates become equal at 30L....*

This conceptual exploration was not explicitly required for the solution, but one which clearly emerged from the process-based, dialogic work of the students.

A second surprise occurred in the way in which the students had responded to a second project given to them, which did not require a journal writing or explicitly formulated formal writing component. A large proportion of the students' solutions contained extensive explanatory text with reflective comments reminiscent of the style of the first project. Moreover, the quality of the mathematical work was on a high level.

Concluding Comments

This study was a qualitative study that resulted in a genuine professional collaboration at the level of classroom based research, serving at one and the same time a disciplined inquiry and pedagogic functions.

Peter was able to improve upon his curriculum and teaching methods and to create more writing-based math projects both in Calculus and lower grade courses. Gord eventually adopted the Taylor book and modified the approach employed in the study to include portfolio assessment for his own OAC Calculus class. He has also used prompted, impromptu writing and mathematical autobiographies with his grade 11's and 12's to help him better address the individual needs of his students.

Moreover, Gord was able to reflect on the study to develop further the Vygotskian framework. This led him to develop a set of cognitive indicators for his doctoral thesis study on the relationship between using interactive journal writing and cognitive growth in high school mathematics.

APPENDIX A

Calculus Assignment #1. Group #7.
Due: March 30/1993

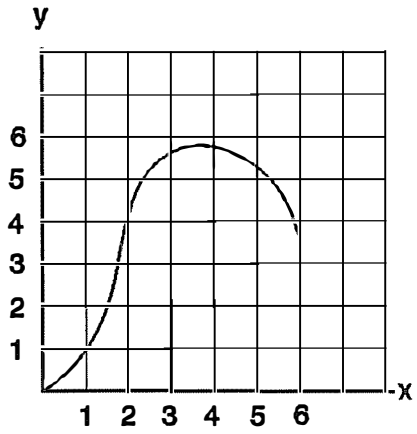
- You are to submit a group report on the following two questions.
- Your report is to be written in a polished, professional fashion.
- One report per group is to be handed in. Each member of your group will be required to submit a confidential note indicating the amount of contribution of each member of your group. Although only one report per group is to be submitted, it is expected that all members will contribute to the best of his or her ability. All members of each group will receive the same mark.
- You may approach either me or Mr. Doctorow for help with this assignment. But all such correspondence will be in written form.
- It is expected that all groups will answer the questions correctly; this assignment is primarily designed to give you the opportunity to present your work in a clear and precise fashion.

1. a) A rectangular tank contains 60 litres of water. If the drain hole in the bottom is opened, the water flows out at a rate which decreases as the water level in the tank goes down. Draw on a piece of graph paper a graph of the volume **V** of water in the tank as a function of time **t**, starting at the moment the drain hole is opened. Assume that the tank becomes empty at t = 8 minutes.

   b) I want to pump water into the tank at a constant rate so that with the drain hole open the water level will stay steady when the tank contains 30 litres. Show how to determine from **your** graph the constant flow rate r into the tank that will make this happen, and measure it as accurately as possible.

   c) Now suppose that the tank contains 60 litres, and at t = 0, the drain hole is uncovered and the pump described in b) above is turned on. Let **W** be the volume of water in the tank at time **t**. On the same set of axes as a), draw the graph of **W** against **t**. Describe as carefully as possible the relationship between the slopes of your two graphs, and illustrate your comments on the picture.

2. The graph of **y** against **x** is given below. Find the value of **x**, $0 \le x \le 6$, which maximizes the expression. Transform the expression beside the graph so that your problem can be interpreted in terms of a family of lines on the graph.



$$\frac{y-3}{x+1}$$

APPENDIX B

List of Other Prompts

Prompt 2

1. Did your test questions meet your expectations? Indicate why or why not.

2. Do you feel that the class discussions and homework adequately prepared you for this test? If so, can you indicate specifically what those things were? If not, can you state what was lacking in your opinion?

3. Do you feel satisfied with your test performance? Why or why not?

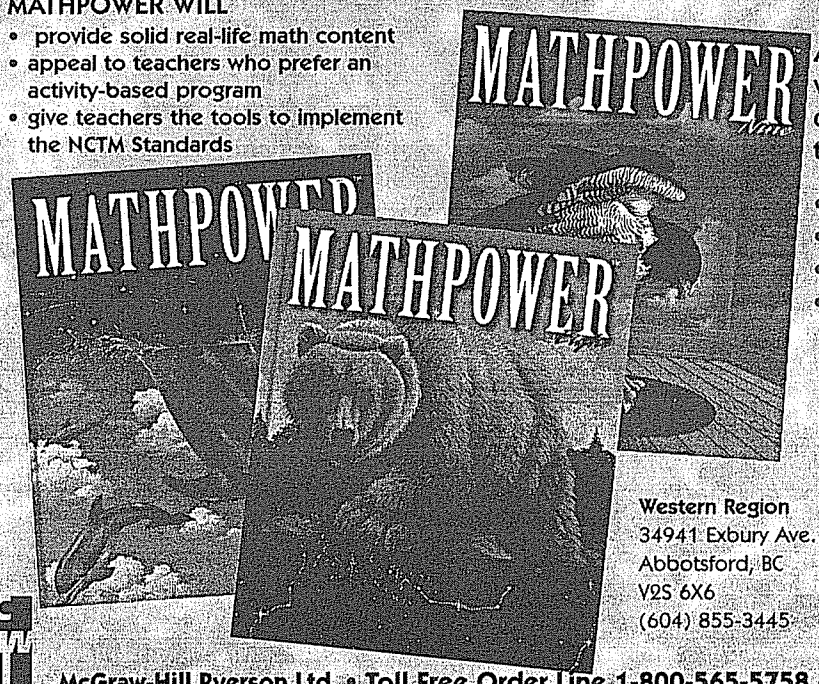
Prompt 3

Today you have been exposed to the idea of constructing the qualitative derivative of a graph if you are given a picture of the original graph.

In the next 5 minutes, write to me about the following things:

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A summary, in your own words, about the idea that you have learned.

An indication of those things you may have questions or incomplete understandings about.

Any questions that have occurred to you that make you curious about this topic.

Prompt 4

Explain how you would draw the graph of the derivative function of this function.

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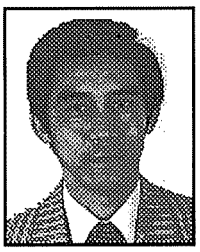
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▲ A Guided Discovery of Pythagorean Triples

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Kenneth Kam-Chui, KO has over twenty years high school teaching experience in mathematics. As a new member of OAME, he is currently teaching at Lester B. Pearson Collegiate Institute. He has a great interst in problem solving.

Introduction

The purpose of this article is to introduce the reader to an analytic geometry method to derive all Pythagorean triples. Pythagorean triples are those positive integral solutions for a, b, c satisfying the so-called Pythagorean equation $a^2 + b^2 = c^2$. Most senior students can easily give some Pythagorean triples, but only very few of them know the general solutions. Many reference books develop the general solution ($p^2 - q^2$, $2pq$, $p^2 + q^2$) using the method of analysis which is far beyond high school students' abilities to understand. The method shown below is accessible by our senior high school students. The following pages include information for teachers and a package for students indicating the process to be followed.

Information for teachers:

Using analytic geometry to find Pythagorean Triples

Grade level: 12-OAC

Materials: students' notes and worksheets

Objectives:

Students will find some Pythagorean triples through finding rational points on a unit circle, construct a table, discover patterns in the table, and find all Pythagorean triples in general form.

Direction:

i) Divide the class into groups of two or three, and let them work together to do these activities. The "student's note" provides information on how to obtain Pythagorean triples. In each activity, the teacher should advise students to check their calculation before moving to the next step.

ii) After completing activity I, students should record their results in the table (activity II). Check the table with the class to insure that all students have the correct values, since the remaining work will base on these data. If students cannot see a pattern to use to complete the table suggest they calculate 2pq, $p^2 + q^2$, and $p^2 - q^2$.

iii) Discuss with the class the answers to the questions in activity III.

iv) Finally, confirm with students that all Pythagorean triples can be derived using this method and highlight that each positive rational slope $m = \frac{q}{p}$ corresponds to a family of Pythagorean triples.

(please refer to the teacher's note at the end of the article.)

For instance, $m = \frac{1}{2}$ corresponds to the family {(3,4,5), (6,8,10), (9,12,15), (12,16,20), ...}; and $m = \frac{1}{3}$ corresponds to the family {(4,3,5), (8,6,10), (12,9,15), (16,12,20), ...}.

Student's Note:

Pythagorean triples are those positive integral solutions for a, b, c satisfying the so-called Pythagorean equation $a^2 + b^2 = c^2$ (1)

The following are some Pythagorean triples:

(a, b, c) = (3, 4, 5), (5, 12, 13), (7, 24, 25), ...

Triples which are proportional to each others are called triples of the same family. So, triples (3,4,5), (6,8,10) and (9,12,15) are triples of the same family.

Divide both sides of equation (1) by c^2. Equation (1) becomes

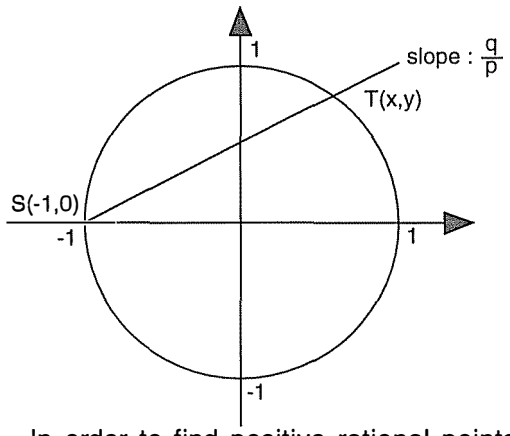
$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \qquad (2)$$

Consequently solving equation (1) for positive integral solutions is equivalent to finding positive rational points on the unit circle:

$$x^2 + y^2 = 1 \quad (3)$$

For example, $(x, y) = \left(\frac{3}{5}, \frac{4}{5}\right)$ is a solution of equation (3) which gives rise to a solution (a,b,c) = (3,4,5). In general (3k,4k,5k) represents a family of solutions, where k is any natural number.

Analytic geometry can be used to find all Pythagorean triples:



In order to find positive rational points on the unit circle, draw a line, with rational slope, through the point S(-1, 0) meeting the unit circle at a point T(x,y), in the first quadrant. In order for the line ST to meet the circle in the first quadrant, the slope of the line ST must lie between 0 and 1. The following example illustrates how to find the point T.

Let the slope of the line ST be a rational number $\frac{2}{3}$.

a. Equation of the unit circle: $x^2 + y^2 = 1$ (3)

b. Equation of the line ST with slope $\frac{2}{3}$ is

$$y = \frac{2}{3}(x + 1) \qquad (4)$$

c. Solve the system of equations (3) & (4):

Substitute (4) into (3)

$$x^2 + \frac{4}{9}(x+1)^2 = 1$$

$$x^2 - 1 + \frac{4}{9}(x+1)^2 = 0$$

$$(x+1)(x-1) + \frac{4}{9}(x+1)^2 = 0$$

$$(x+1)\left((x-1) + \frac{4}{9}(x+1)\right) = 0$$

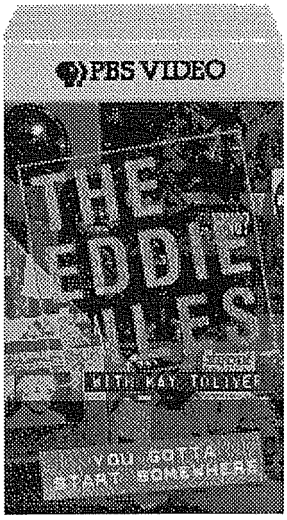
$$(x+1)\left(\frac{13}{9}x - \frac{5}{9}\right) = 0$$

$$(x+1) = 0 \quad \text{or} \quad \left(\frac{13}{9}x - \frac{5}{9}\right) = 0$$

$$x = -1 \quad \text{or} \quad x = \frac{5}{13}$$

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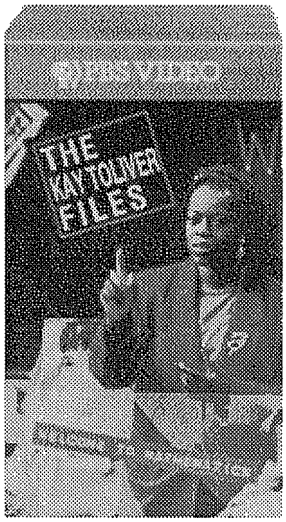
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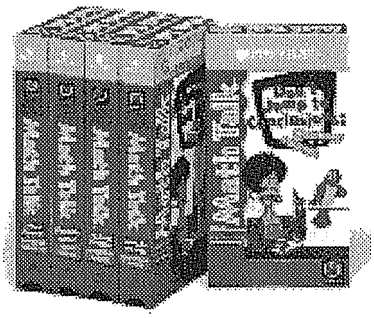
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When $x = -1$, $y = 0$. It is the point S.

When $x = \frac{5}{13}$, $y = \frac{2}{3}\left(\frac{5}{13} + 1\right) = \frac{12}{13}$.

Hence the coordinates of T are $\left(\frac{5}{13}, \frac{12}{13}\right)$.

d. Substitute point T into the equation (3)

$$\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1 \text{ or } 5^2 + 12^2 = 13^2.$$

Hence we obtain a Pythagorean triple (5, 12, 13), or in general (5k, 12k, 13k), where k is any natural number.

Activity I

For each of the following slopes, find the corresponding Pythagorean triple.

(i) slope of ST = $\frac{1}{2}$

a. Equation of the unit circle: $x^2 + y^2 = 1$ (3)

b. Equation of the line ST with slope $\frac{1}{2}$ is? (4)

c. Solve the system of equations (3) & (4):

Hence the coordinates of T are ____.

d. Substitute point T into the equation (3)

Hence we obtain a Pythagorean triple _____.

(ii) slope of ST = $\frac{1}{3}$

a. Equation of the unit circle: $x^2 + y^2 = 1$ (3)

b. Equation of the line ST with slope $\frac{1}{3}$ is? (4)

c. Solve the system of equations (3) & (4):

Hence the coordinates of T are ____.

d. Substitute point T into the equation (3)

Hence we obtain a Pythagorean triple _____.

(iii) slope of ST = $\frac{2}{5}$

a. Equation of the unit circle: $x^2 + y^2 = 1$ (3)

b. Equation of the line ST with slope $\frac{2}{5}$ is? (4)

c. Solve the system of equations (3) & (4):

Hence the coordinates of T are ____.

d. Substitute point T into the equation (3)

Hence we obtain a Pythagorean triple _____.

(iv) slope of ST = $\frac{3}{4}$

a. Equation of the unit circle: $x^2 + y^2 = 1$ (3)

b. Equation of the line ST with slope $\frac{3}{4}$ is? (4)

c. Solve the system of equations (3) & (4)

Hence the coordinates of T are ____.

d. Substitute point T into the equation (3)

Hence we obtain a Pythagorean triple _____.

Activity II

After deriving some Pythagorean triples from the activity I (i)-(iv), record the result in the table: (case 1, 2, 4, 5, 7)

Here slope of ST equals $\frac{q}{p}$, where p, q are positive integers, $p > q$ and p, q have no common factor.

Case	p	q	Slope q/p	Point T (x , y)	Pythagorean triple (a, b, c)
1	2	1	$\frac{1}{2}$		
2	3	1	$\frac{1}{3}$		
3	4	1			
4	3	2	$\frac{2}{3}$	$\frac{5}{13}, \frac{12}{13}$	(5,12,13)
5	5	2	$\frac{2}{5}$		
6	7	2			
7	4	3	$\frac{3}{4}$		
8	5	3			
9	p	q			

Do you observe any patterns? If so, complete the table for the case 3, 6 , 8. If not, solve the corresponding system of equations and then complete the table.

General result:

Suppose the line ST has a slope of $\frac{q}{p}$, where p, q are positive integers with $p > q$ and p, q have no common factor. By using the pattern observed for cases 1 to 8 write the coordinates of the point T in terms of p and q.

Coordinates of T: _____

The corresponding Pythagorean triple: _____.

Complete the table for the case 9.

Activity III

Answer the following questions.

1. Why is the line drawn from the point S(-1,0)?
2. Why must the slope of the line ST have a value between 0 and 1?

3. Find the corresponding Pythagorean triple for each of the following cases:

   (i) p = 5 and q = 2; (ii) p = 7 and q = 4

4. What is the slope of the line ST if the coordinates of T are

   (i) $\left(\frac{5}{13}, \frac{12}{13}\right)$? (ii) $\left(\frac{20}{29}, \frac{21}{29}\right)$?

5. In each of two cases below, is it possible to have a Pythagorean triple with a given value of c?

   If so, find the values of a and b. If not, explain.

   (i) c = 12 (ii) c = 65

6. Find the slope of the line ST corresponding to each of the following Pythagorean triples.

   (i) (3, 4, 5); (ii) (6, 8, 10); (iii) (9, 12, 15).

   What can you infer from the above result?

Teacher's note:

1. Explanation of why the point T must be a rational point on the unit circle.

   The crucial ideas are

   (i) S(-1,0) is a rational point on the unit circle;

   (ii) slope of ST is a rational number.

   a. Equation of the unit circle:

   $$x^2 + y^2 = 1 \quad (3)$$

   b. Equation of the line ST with slope

   $$\frac{q}{p} \text{ is } y = \frac{q}{p}(x+1) \quad (4)$$

   c. Solve the system of equations (3) & (4):

   Substitute (4) into (3)

   $$x^2 + \frac{q^2}{p^2}(x+1)^2 = 1$$

   It is a quadratic equation with rational coefficients. The roots are either both rational numbers or both nonrational conjugates. Since x = -1 (point S) is one rational root, the other root (point T) must be rational. By substitution, we get rational coordinates for the point T.

$$x^2 + \frac{q^2}{p^2}(x+1)^2 = 1$$

$$x^2 - 1 + \frac{q^2}{p^2}(x+1)^2 = 0$$

$$(x+1)\left((x-1) + \frac{q^2}{p^2}(x+1)\right) = 0$$

$$(x+1)\left(\frac{p^2+q^2}{p^2}x - \frac{p^2-q^2}{p^2}\right) = 0$$

$$(x+1) = 0 \text{ or } \left(\frac{p^2+q^2}{p^2}x - \frac{p^2-q^2}{p^2}\right) = 0$$

$$x = -1 \text{ or } x = \frac{p^2-q^2}{p^2+q^2}$$

When x = -1, y = 0. It is the point S.

When

$$x = \frac{p^2-q^2}{p^2+q^2},\ y = \frac{q}{p}\left(\frac{p^2-q^2}{p^2+q^2} + 1\right) = \frac{2pq}{p^2+q^2}.$$

Hence the coordinates of T are

$$\left(\frac{p^2-q^2}{p^2+q^2}, \frac{2pq}{p^2+q^2}\right).$$

d. Substitute point T into the equation (3)

$$\left(\frac{p^2-q^2}{p^2+q^2}\right) + \left(\frac{2pq}{p^2+q^2}\right) = 1$$

or $(p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$.

Hence the Pythagorean triple is

$(p^2 - q^2, 2pq, p^2 + q^2)$, or in general, any natural number triples (a, b, c) which is proportional to

$(p^2 - q^2, 2pq, p^2 + q^2)$.

2. All Pythagorean triples can be derived using the above formulas.

   Let (a,b,c) = (i,j,k) be a Pythagorean triple satisfying the equation $a^2 + b^2 = c^2$.

   Then (x,y) = $\left(\frac{i}{k}, \frac{j}{k}\right)$ is a point with rational

coordinates satisfying the equation of the unit circle $x^2 + y^2 = 1$.

Take $T = \frac{i}{k}, \frac{j}{k}$ and S = (-1,0). We'll claim that ST is a line with rational slope.

$$\text{As the slope of ST} = \frac{\frac{j}{k} - 0}{\frac{i}{k} + 1} = \frac{j}{i + k},$$

therefore the rational slope $\frac{j}{i + k}$ gives rise to the Pythagorean triple (i,j,k).

3. Generalization of this activity:

(A) Find all triples (a, b, c), where a, b, c are natural numbers, satisfying the equation

$a^2 + kb^2 = c^2$ where k is a given non-square number.

(B) Find all 4-tuples (a, b, c, d), where a, b, c, d are natural numbers, satisfying the equation

$$a^2 + b^2 + c^2 = d^2.$$

(C) Find all (n+2)-tuples $(a, b_1, b_2, ..., b_n, c)$,

where $a, b_1, b_2, \ldots, b_n, c$ are all natural numbers, satisfying the equation

$a^2 + b_1^2 + b_2^2 + \ldots + b_n^2 = c^2$ where $n > 1$.

Answers:

(A) The solutions are those triples, (a, b, c) such that

$a : b : c = |p^2 - kq^2| : 2pq : p^2 + kq^2$,

where p, q are any two positive integers.

(B) The solutions are those triples, (a, b, c, d) such that

$a : b : c : d$

$= |p^2 - q^2 - r^2| : 2pq : 2pr : p^2 + q^2 + r^2$,

where p, q, r are any three positive integers

(C) The solutions are those (n+2)-tuples,

$(a, b_1, b_2, ..., b_n, c)$ such that

$a : b_1 : b_2 : \ldots : b_n : c$

$= |p^2 - q_1^2 - q_2^2 - ... - q_n^2| : 2pq_1 : 2pq_2 : ... :$

$2pq_n : p^2 + q_1^2 + q_2^2 + ... + q_n^2$,

where $p, q_1, q_2, \ldots, q_n$ are any (n+1) positive integers.

Nominations for OAME/AOEM Board of Directors 1997-98

Each year, elections are held for President-Elect, 3 Vice Presidents (1 year term) and 6 to 9 Directors (3 year term) of OAME/AOEM. Nominations for these positions are to be submitted by 4:00 p.m., January 15, 1997, to:

Susan Stuart
Past-President, OAME/AOEM
Nipissing University

Include the nominee's qualifications with the Nominations Form. A nominee must be a member of OAME. By March, 1997, the Nomination Committee will publish a slate of candidates, including qualifications, for consideration by the membership. Voting will be by mail and is open to all OAME/AOEM members. The election results will be released at the Annual General Meeting of the OAME/AOEM in Toronto in May, 1997.

Nomination Form

I nominate: _____

For the position(s) of: ☐ President-Elect ☐ Director ☐ Vice-President

Nominee's Address: _____

_____

Telephone: (h) ( )_____ (w) ( )_____

Nominator's Address: _____

_____

Telephone: (h) ( )_____ (w) ( )_____

▲ About Problem Solving: Fun with Factorials

JOHN GRANT MCLOUGHLIN
EDUCATION PROGRAM
OKANAGAN UNIVERSITY COLLEGE



The exclamation mark (!) that denotes factorial may be used because of the rapid manner in which the values of factorials grow. For those who may be unfamiliar with the notation, it suffices to define n! ("n factorial") as follows:

$n! = n(n-1)(n-2)...(3)(2)(1)$

That is, n! equals the product of the positive integers from 1 to n inclusive. For example, 1! = 1, 2!=2, 3! = 6, ..., 7! = 5040, ... A curious fact noted by David Wells (1987) provides a glimpse into the magnitude of the product as n gets larger. The value of 22! has 22 digits in its representation. In fact, n = 1, 22, 23, and 24 each have n digits in their representation of n! Indeed the values ascend sharply.

This column features a selection of problems that involve the special product referred to as factorial. These problems are intended for a wide audience of readers. However, the final probability problem is unlikely to be successfully attempted by those who have not had prior experience with finite mathematics.

The first problem is a personal favourite. The mathematics involved is not complicated. Its beauty lies in the fact that good reasoning can save one from doing many unnecessary calculations, thus, producing an elegant solution.

Find a three digit number xyz such that the number itself equals the sum of the factorials of its respective digits. (ie., 100x + 10y + z = x! + y! +z!)

The next problem (adapted from Booth et al., 1995) is presented as a specific case in which the product of different factorials produces another factorial. Readers are encouraged to find other examples of such products.

Find n given that 3! x 5! x 7! = n!

This type of problem may be adapted into a slightly more challenging form. Rather than combining some know factorials to produce a single larger value, the solver may be challenged to break the larger value down into smaller pieces. The following example is adapted from Wells' (1987) *Rich Source of Numerical Curiousities*.

Find a, b, and c (none of which equal 9), given that 9! = a! x b! x b! x c!

Mathematical contests may feature another type of problem that deals with factorials. The magnitude of large values of n! has been touched upon previously. It is fair to say that it is unreasonable to expect these calculations to be done mechanically. Though it is not unreasonable to determine the number of 0's that will appear at the end of such values. Keep in mind that each factor of 10 in a number provides an additional zero at the end. If one was to break down a large factorial value into primes, there would surely be many more 2's than 5's. Of course, each pairing of a 2 and a 5 reproduces another terminal zero. Therefore, the number of zeros at the end of the value of n! will equal the number of 5's in its prime factorization. A related problem may read:

How many 0's appear at the end of the value 150! (Note: There are more than 30.)

The most common applications of factorials occur in the areas of combinatorics and probability. Here is a problem that will require some algebraic manipulation. THe problem appears in a book by Ross Honsberger (1991), who credits D.J. Smeenk of the Netherlands for initially proposing it in *Crux Mathematicorum*. The published solution in the book is credited to Richard Rhoad of Winnetka, Illinois.

An urn contains w *white balls (w≥3) and* r *red balls. If 3 balls were to be drawn without replacement, the probability that they would all be white is* p. *An extra white ball in the urn would increase the probability by a third of its value. If* r *is as great as these conditions allow, how many red balls are there in the urn?*

Comments on any of these problems are welcomed!

References

Booth, P., Shawyer, B., & Grant Mcloughlin, J. (Eds.), *Shaking Hands in Corner Brook and Other Math Problems for Senior High School Students*, Waterloo: Waterloo Mathematics Foundation, 1995.

Honsberger, R., *More Mathematical Morsels*, Washington: Mathematical Association of America, 1991.

Wells, D., *The Penguin Dictionary of Curious and Interesting Numbers*, London: Penguin Books, 1987.

▲ DIRECTORS' DIALOGUE



PETER SAARIMAKI
COORDINATOR OF MATHEMATICSI
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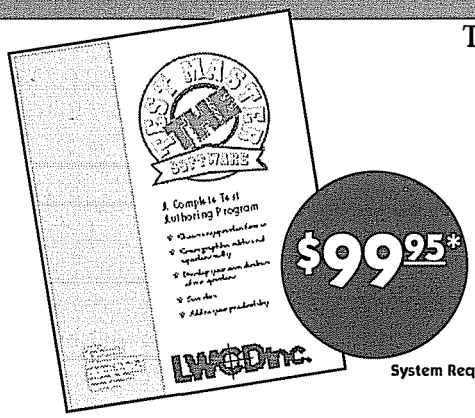
Inroduction

OAME Council meetings are a thing of the past. As of May 1996, OAME became an incorporated body and is now run by a Board of Directors. These notes were taken at the OAME Board of Directors meeting held Saturday June 15, 1996. Comments about the Board and its activities should be directed to the President. Comments about my comments can be sent to the editor.

General

The directors work actually began on the Friday night, when committees met to determine their mandate, direction and budget for the coming year. One committee in fact, started meeting Friday morning to get a head start on the enormous task it has, but more on that later.

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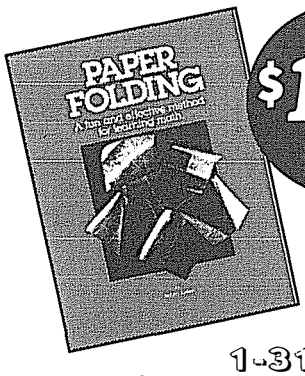
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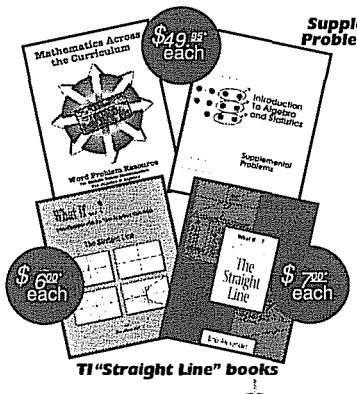
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Saturday morning started at 7:45 a.m. for those wishing to participate in the Math Equity breakfast meeting. The committee's plans will be found later.

Housekeeping

Barry Onslow and the Alexanders covered a range of administrivia, not least of which was the welcoming of new members to the Board, including Judy Crompton as President-elect. Judy is a past President of the Ontario Mathematics Coordinators Association (from the Lincoln Board, around St. Catharines). She has recently spent a year as the national mathematics consultant for a major publisher and will be able to provide a national perspective to the issues concerning OAME.

Also covered were the duties and responsibilities of elected Directors and Chapter Representatives, timing of the Board's activities and any impact on committee work, legitimate expenses, lunch times, and future Board meeting dates (and there are now only two more before the next Board is in place).

Barry, the Prez, shared the Mission Statement, stressing the need for OAME to remain a leader as the province undergoes many possibly catastrophic changes over the next few years. He reminded us that OAME needs to continue it's proactive role with the Ministry of Education and Training, and also with parents, the public and, especially now, with post-secondary education.

Other Business

There were various pieces of correspondence shared relating to awards and recognition of members good deeds. The editors of the *Gazette/Abacus* were retired and introduced, as was appropriate, with more reported in their section of the *Gazette.*

Financial discussions and membership reports seem to get worse all the time as numbers continue to gradually decrease. There was good news in the sales of the Transition Years Assessment booklet, with more to come. A package of resources was again distributed to each director for which members need contact their local chapter.

COMMITTEES

In mid-morning, directors again met by committee to finalize their discussions started Friday night and plans for the up-coming year. Their reports followed lunch.

Constitution and By-Laws

Cleaning up these details is a necessary but thankless task, but we will change that here by expressing our appreciation to Elaine Harvey for her commitment to this long process. By-laws, Terms of Reference, changes due to Incorporation, ... all designed to help the work of the directors proceed smoothly.

Awards

Dave Zimmer reported plans to redesign nomination forms and streamline the process to avoid unnecessary duplication of efforts and submissions.

Math Equity

On-going discussions, professional development projects and Equity Breakfasts will continue to raise the level of awareness of our members.

Publications

Advertisers in the *Gazette* continue and we can all let them know that "we saw it in the *Gazette*". Articles are still needed, for both *Gazette* and *Abacus*, and information on submissions is found inside the front cover. If you are unsure whether your pet project is publishable material, contact one of the editors and they will be glad to advise.

Promotions

New membership forms, a Home and School mailing, pins, T-shirts and golf-shirts, and membership bonuses are all being considered to encourage membership in OAME.

TASK FORCES

Early/Formative Years

The Junior level Assessment booklet is in preparation for fall publication and from all reports will be as well received as the Transition Years version. The committee is also contributing to the Abacus as it has for the past years.

Transition Years

As a follow-up to the Assessment booklet, the committee will produce sample activities for the *Gazette*, and is considering a "challenge exam".

Specialization Years

The planned reform of the secondary program to four years (now deferred until September 1998) and the newly developing University Entrance Exam will keep this committee busy for a while. Members plan to work in conjunction with the OMCA over the school year 1996/7 to draft secondary graduation outcomes and suggested courses or modules, and distribute for comments to secondary teachers and administrators, post-secondary institutions and the Ministry of Education.

OTHER COMMITTEES

Futures

The committee is looking for host chapters for our two major events after the 1998 Annual conference (in North Bay) and the 1997 Ontario Mathematics Olympics (hosted by MAC^2). Support is available for either event in terms of protocol and past practice. In particular, we now have a long term commitment from Rockwell International for the OMO.

'96 conference

Kaye Appleby reported on the very successful conference in London. The delegates and speakers had many compliments and the cheque presented to OAME from the proceeds will go a long way to easing our financial position. A wooden replica of the OAME logo was a very unique speakers gift.

'97 conference

The key-note and feature speakers are being confirmed for "The Name of the Game". Plans are also underway for a celebratory multicultural games night on the Wednesday before the conference, student-designed games and opportunities for the delegates to play some new games.

Leadership '96

Notice of this special program has been sent out with the *Gazette* and to all schools in a general mailing. If you missed out, there is still time to avail yourself of this meeting of future leaders. The timing has been moved to run from Thursday night to Saturday noon, requiring only one time-release day for Boards to cover.

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OMO

Congratulations to the over 100 grade 7 and 8 students who competed in the Second Annual OMO last June 1 in Scarborough, and to the many teachers, coaches, and parents who made it all possible. The winning teams represented Thunder Bay, ISOMA and Scarborough, but the real winners were the students themselves who came away with T-shirts, certificates and new friends.

Special Projects

The major work here will be coordinating the publication of a curriculum discussion document for the "reformed" secondary program in conjunction with the Specialization Years committee and the OMCA. They are also looking to put the last few years of the *Gazette* on CD-ROM.

OTF

OAME, as part of the Council of Secondary Subject Associations, sent a letter to the Ministry recommending that the Secondary School Reform process consider the students, have a more realistic timeline and include meaningful dialogue. Recent events indicate we have been heard.

CHAPTERS

Each chapter puts on some marvellous Professional Development and not all details were available at the June Board meeting. Keep in touch with your local chapter to be aware of their plans, before they happen. These are merely highlights, not compete reports, for which you need to contact your local chapter.

CHAMP

MATHFEST '96 will be held October 3rd. Watch for their next newsletter, including information about a partnership with Rockwell's Milton plant.

COMA

Have you seen the new logo?

GOLDEN SECTION

Celebrating a long history of professional development.

ISOMA

Have you seen their web page yet?

MAC^2

Fall conference at Twin Lakes SS on October 24th.

NWOAME

MATHFEST '96 to be held November 15th.

QSLMA

Watch for the seminar and workshops for elementary teachers with Queens' MST group on October 4 - 5, and then look for the mini-conference October 17th.

Math-TEY^{2}MS

The traditional fall conference will focus on assessment, with a tentative date of October 17th.

Y^{4}MA

Save October 24th for the fall conference at Vaughan SS

Provincial Tests and Ministry stuff

There were several discussions throughout the Board meetings regarding Ministry directions. OAME members are directly involved in Advisory Committees for both the grade 3 and 6 provincial tests. The Grade 3 will be administered to ALL students as a two week unit of study in reading, writing and mathematics in the spring of 1996. The Grade 6 will be a provincial sampling (of about 100 schools) in mathematics only (reading and writing will come later). Contact your chapter rep, or your federation rep for up-to-date information as the year progresses.

As well, several OAME members are part of Ministry writing teams defining the Foundation Requirements for a grade-by-grade look at what the Provincial Standards mean. Look for early versions in the fall.

The other major concern regards the development of appropriate curriculum for the "reformed" secondary program. Barry Onslow has again contacted Ministry and post-secondary representatives to ensure there is a plan for it all, and that secondary teachers can have input.

Many aspects of education and mathematics are changing, but you can have an impact by being informed, involved and working collectively with organizations, such as the OAME. You may send your comments and questions to me at <bb304@freenet.toronto.on.ca

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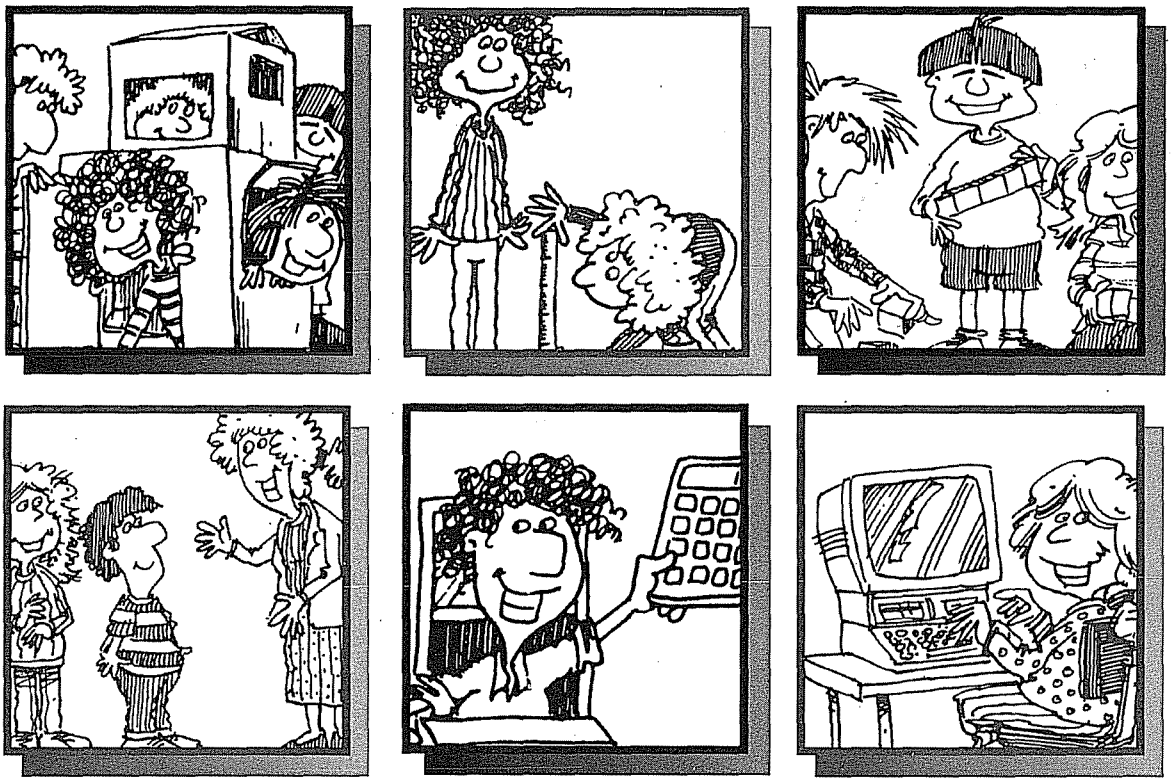
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