Vol. 35 #4, June 1997

# Ontario Mathematics Gazette

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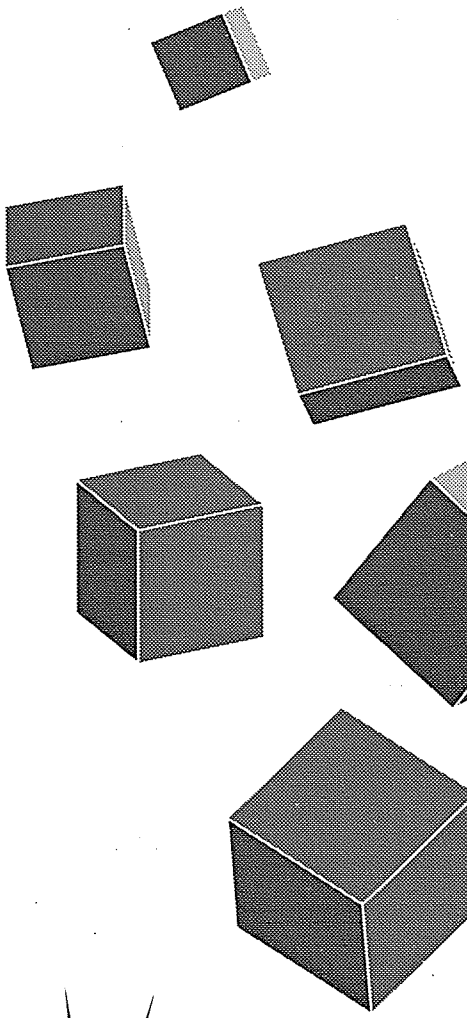
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Technology, Science and Mathematics versus Mathematics, Science & Technology Integration

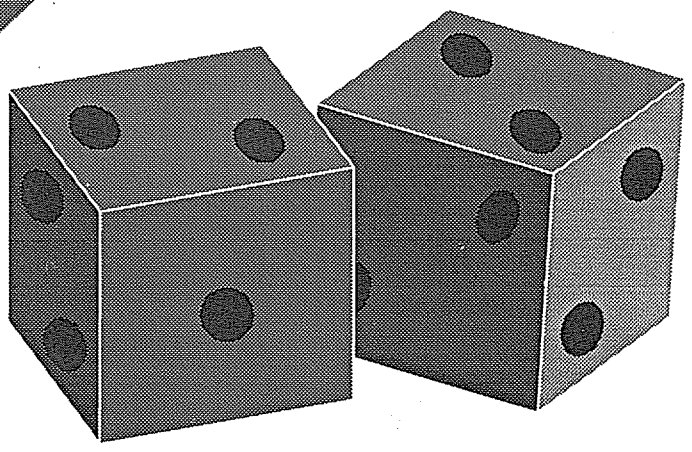
Triangular Triples

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MEET OAME'S INCOMING PRESIDENT JUDY CROMPTON page 4



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Final Exam

1. State the Domain of $y = \frac{x-10}{\sqrt{x^2-16}}$.

2. Which relation is the inverse o

3. Use a half-angle identity to find

a) $\frac{1}{2}\sqrt{2-\sqrt{2}}$ b) $\frac{-1}{2}\sqrt{2-\sqrt{2}}$

4. If $f(x) = 2x-5$ and $g(x) = \frac{2}{x+3}$

a) $\frac{-(5x+11)}{x+3}$ b) $\frac{1}{x-4}$

5. If $\sin\theta = -\frac{1}{2}$ and $\theta$ is in qu

a) $\frac{-\sqrt{3}}{3}$ b) $\frac{\sqrt{3}}{3}$

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Final Exam

1. State the Domain of y=x-10 / √x²-16

2. Which relation is the inverse of $y=(x-2)^3$ ?

a) $y=\sqrt[3]{x-2}$ b) $y=\sqrt[3]{x}-2$ c) $y=\sqrt[3]{x+2}$ d) $\sqrt[3]{x}+2$

3. Use a half-angle ider
cos 67.5°:

a) 1/2√2-√2 b)

4. If f(x)=2x-5 and g(x)

a) -(5x+11) / x+3 b)

5. If sin θ=-1/2 and θ i

a) -√3 / 3 b)

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a) 4x²-2x / x+3 b)

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**FINAL EXAM**

1. State the Domain of $y = \frac{x-10}{\sqrt{x^2-16}}$.

2. Which relation is the inverse of $y=(x-2)^3$?

a) $y=\sqrt[3]{x-2}$ b) $y=\sqrt[3]{x}-2$ c) $y=\sqrt[3]{x+2}$ d) $\sqrt[3]{x}+2$

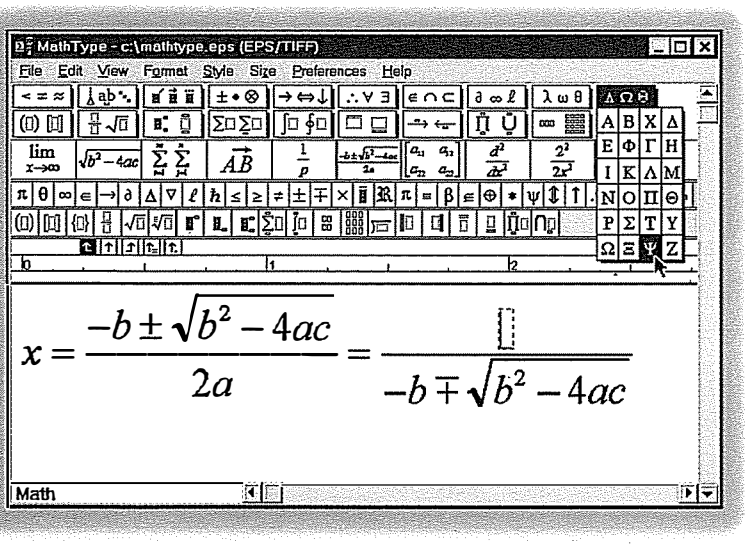
3. Use a half-angle identity to find the exact value of cos 67.5°:

a) $\frac{1}{2}\sqrt{2-\sqrt{2}}$ b) $\frac{-1}{2}\sqrt{2-\sqrt{2}}$ c) $\frac{-1}{2}\sqrt{2+\sqrt{2}}$ d) $\sqrt{3+2\sqrt{2}}$

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Your article should be typewritten or word-processed, double-spaced on one side of 21.6 cm x 27.9 cm paper with 3 cm margins. Diagrams and figures should be drawn by computer, if possible, or neatly drawn in black ink in camera-ready form. Please submit four copies to the editor and keep one as protection against loss. Submissions are blind-reviewed, so please send your full address and affiliation on a page separate from the manuscript. Computer files of text, and graphics where possible, will be requested later if your article is accepted.

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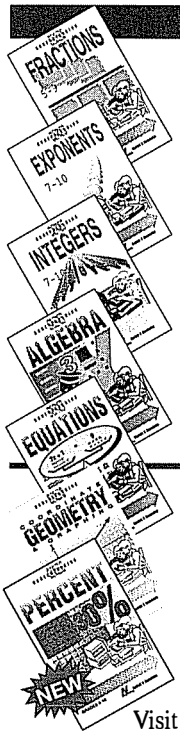

# ▲ Editorial



Students and mathematics are what teaching mathematics is all about. Through the *Gazette* the OAME is doing its part in helping teachers in the classroom strive to become better teachers of mathematics. What is a good teacher of mathematics? Good teachers of mathematics are ones who use their knowledge of mathematics as well as their love and respect for their students to lead these students to an enjoyment of the study of mathematics. Remember a student with a gift for mathematics can learn from any mathematics teacher, good or bad, but every student can be successful when taught by a good mathematic teacher.

This issue contains articles with practical ideas to help teachers in both elementary and secondary classrooms. Look carefully at our first submission on *"Teaching Hints from the Our Readers"* from Jack Lesage. I hope that I receive many more article like this one. Surprising as it may seem, ideas that you have, that you think every one knows, will be helpful to many of our readers. Please share your gems with us.


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# ▲ Outgoing President's Message



BARRY ONSLOW

Where has the year gone? It just doesn't seem like 366 days since I wrote the Incoming President's Message, but I guess that is what happens when one reaches fifty!!

It was marvellous news to hear that our editor, John Egsgard, has won the Mathematics Education Trust Lifetime Achievement Medal from the National Council of Teachers of Mathematics (NCTM). Only three of these highly renowned awards are presented each year. John was presented with his medal at the NCTM annual meeting in Minneapolis this past April 19th. The letter from NCTM stated, "This prestigious award is based upon the many accolades from your peers and your impressive career. Your commitment to excellence in mathematics education, over the years, has been a valued asset to your students and fellow colleagues alike. We at NCTM are extremely proud to be associated with such caliber and level of leadership, service and teaching." Well done, John, and very well deserved.

As we move towards our new mathematics curriculum for grades 10-12 (or is it 9-12?), it is reassuring to know that Mary Lou Kestell and Margaret Warren are members of the Expert Panel for Mathematics (Mary Lou being selected to chair the group). Mary Lou and Marg deserve our gratitude. Both have worked tirelessly to frame (and reframe) a vision for mathematics in our High Schools. They have shared this vision with members of the Board of Directors over the last year and held presentations at our Annual Conference in Toronto. These presentations allowed those attending to become involved in the future directions of secondary mathematics.

I would also like to acknowledge all of the TEY$^2$MS, OAME 97 Conference Planning Committee for organizing an excellent selection of activities (both academic and social) in and around the Sheraton Centre. It is truly amazing that we have so many talented teachers in our province who are consistently willing to share their time and expertise. I thank all of you for making a difference in the lives of Ontario teachers, and ultimately in the lives of Ontario's students. Those people who are so quick to criticize teachers should attend one of our conferences.

Congratulations also to all of our 1997 OAME Award winners, who were presented with their awards at the Annual Conference. Elaine Harvey, recently retired vice-president, and charter member of OAME, is our newest *Life Member*. Mary Storey, who has been a lead teacher in the Junior Division since 1979, won the *Morley MacGregor Award* for demonstrating excellence in elementary mathematics teaching. Ron Lancaster, likely the highest profile mathematics teacher in Ontario, and a well known educator both nationally and internationally, won the *Don Attridge Award* for exceptional and creative mathematics teaching in the secondary school. Charles Cipolla, President and Chief Executive Officer of Rockwell Automation Canada Inc., was presented with the *Mona MacGregor Award* in appreciation of Rockwell's continued support for mathematics in general and the Ontario Mathematics Olympics in particular. Ken McDonald, Head of Mathematics at Bluevale Collegiate Institute, received the *K. D. Fryer Award* on behalf of his department. This annual award recognizes a mathematics department which fosters collegiality, team work, and excellent classroom teaching, as well as demonstrating leadership in the mathematics education community. More will be written about our 1997 Award winners, together with photographs, in the September edition of the Gazette.

The next event on our calendar is the Ontario Mathematics Olympics (OMO), June 7th, in Barrie. We have had more students than ever before participating in our regional play downs, and there are many excellent grade seven and eight students travelling to Barrie to challenge themselves and others in solving the intriguing problems developed by Bill Morrison, Jack LeSage, Paul Pogue and their committees. If you can attend a regional or provincial OMO, I believe you will be amazed by the creativity and ingenuity of these young minds in action.

In closing, I would like to express my sincerest appreciation to the Executive, Board, and so many other members for their continued efforts working to improve the teaching and learning of mathematics in Ontario. Whenever I needed support, people were there to help. Consequently, my year as president was an extremely enjoyable one that I will remember fondly. To those not returning to the Board next year — Elaine Harvey (vice-president, and newest life member), Mickey Sandblom (our NCTM representative, and former President), Lynda Colgan (OMO II organizer), Vince Delisi (our computer expert, and inspirational colleague), Gareth Haddock (our first Web-Site Manager) and Ron Lewis (our fractal man and sage counsel) — you will all be missed. Each one of you has worked vigorously to improve mathematics education in our province. We hope that you will always feel part of our math education family. Do keep in touch and thank you for your dedication.

To Judy Compton and her new executive, I wish you all the very best. Judy's energy and enthusiasm will inspire and invigorate all of us.

# ▲ Incoming President's Message



**JUDY CROMPTON**

Well, the date is April 20, Sunday afternoon. As usual, I am at my desk. Today I have two tests to set and another one to mark. Midterm reports just went in. Two nights a week for the past month have been Descartes prep classes. Next week is the county mathematics olympics, followed by the OAME chapter olympics the week after. Parents' night next week also. And a heads' meeting. Oh ... and cafeteria duty all week. (ick)

Sound familiar?

Teaching is a very busy activity. It takes every ounce (gram?) of a teacher's energy to keep up with student needs and to keep up with the work load.

And that's for the current program and environment.

But what about the future? What about the myriad issues swirling around mathematics education at all levels? Issues such as:

- What is the place of technology in the curriculum? What are the implications of technology?
- Do we need to maintain skill levels traditionally expected? Which skills should be kept and which abandoned?
- How do we create and run a program that focuses on reasoning and problem solving, while still developing skills considered essential?
- How do we link learning in mathematics education with authentic assessment?
- How do we relate to the province-wide testing that has begun in Ontario?

These questions cut across grades and divisions. They are questions that we must answer within the next few years. They are questions being addressed by the Ministry of Education Expert Panel in its deliberations regarding the future of secondary school mathematics.

It's time for all of us to begin considering them. We are very busy, but we must find the energy to think about what we are currently doing and why. Each and every one of us must begin to critically assess what we are teaching in light of the questions listed above.

There is no question that mathematics education has begun to undergo a significant change, a process that will continue over the next several years. How can you become a part of the change process? How can you cope with new circumstances and new expectations brought about by the change?

You have made an excellent start by joining OAME and by reading the *Gazette.* Continue by becoming involved in your local OAME chapter, by attending the workshops that it puts on. Attend the 1997 OAME Leadership Conference, "Teaching and Learning through Rich Learning Tasks", being held November 13-15. Talk with your colleagues about the mathematics program; encourage them to look closely at what they are doing and why.

Change can be good ... when we are prepared for it and when we understand it. The first step to understanding is questioning. The next step is action.

I would like to hear from you as you begin your critical thinking process. My FAX number and E-mail address are on the back page of the Gazette. Let's share what we're thinking.

# ▲ Letter to the Editor

Dear Editor

With reference to your *"Teaching Ideas from the Editor"* in the December, 1996 issue of the *Gazette.* I would like to make a couple of remarks.

It seems to me the reason for factoring these trinomials is to solve equations (or to get the zeroes of the trinomial function). You must have noticed that once you have found d and e such that de = ac and d + e = b the roots are

$$x_1 = \frac{d}{-a} \quad \text{and} \quad x_2 = \frac{e}{-a}$$

and you really have to go no further. It is still true if $a = \pm 1$.

With regards.

Louis Levine

# THE MANIPULATIVE CONNECTION: BASE TEN BLOCKS, TWO COLOUR COUNTERS, ALGEBRA TILES

**LAURA SGROI, ED.D**
**STATE UNIVERSITY OF NEW YORK AT NEW PLATZ.**
**RICHARD SGROI, PH.D.**
**NEWBURGH ENLARGED CITY SCHOOL DISTRICT**
**VAILS GATE MAGNET SCHOOL**

*Laura Sgori is an associate professor of elementary mathematics education at SUNY New Platz. She is also a consultant to school districts and an author of elementary education textbooks.*

*Richard Sgori is the K-12 Mathematics Director for the Newburgh School District in Newburgh, New York. He has co-authored numerous secondary school mathematics and college-level teacher preparation textbooks.*

There are many ways connections can be explored and discovered in a spiraling curriculum. Some common connections that students can experience include:

- interdisciplinary units that assist students in making connections between mathematics and other areas of study;
- integrated mathematics courses that present mathematical topics as they connect to one another rather than as isolated components of the discipline of mathematics;
- topical applications that give students the opportunity to connect mathematical ideas and concepts with real world problems.

The authors of the NCTM **STANDARDS** (1989) state that "... persistent attention to recognizing and drawing connections among topics will instill in students an expectation that the ideas they learn are useful in solving other problems and exploring other mathematical concepts" (p.85). It is the purpose of this article to explore the power of Base Ten Blocks, Two Colour Counters, and Algebra Tiles to illustrate the connections among basic operations on whole numbers, integers, and variables.

## Introduction to the Manipulatives

The three manipulatives listed above share many common characteristics, yet also retain their unique identity. Base Ten Blocks are a proportional place value material that can be used to model the place values of units, tens, hundreds and thousands. (Figure 1). Ones are represented by "units", the tens block is commonly known as a "long", the hundreds block is known as a "flat", and the thousands block is called a "cube."

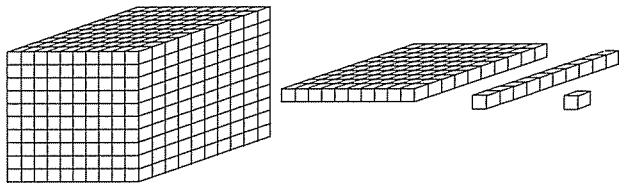


*Figure 1*

Numbers can be modeled using the blocks. Figure 2 shows the representation of the number 1213 with student base ten blocks. A version for the overhead projector is also available, with the notable weakness that it lacks a transparent thousands cube.

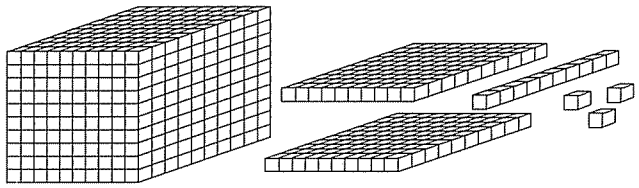


*Figure 2*

Two Colour Counters, or Integer Chips, can be used to model positive and negative numbers. The student version (as opposed to the overhead version) is a chip that is coloured red on one side and yellow on the other. When the yellow side of a chip is facing up, it is considered to have a value of +1, and when the red side shows, it is considered to have a value of -1. The overhead version simply consists of transparent red and yellow chips. When modeling integer operations, it is often useful to form a zero pair. A zero pair consists of a yellow chip paired with a red chip (+1 combined with -1)

which therefore has a value of zero. The presence of zero pairs does not change the value of a model. Figure 3 shows 3 different models of +5 using two, one, and no zero pairs.

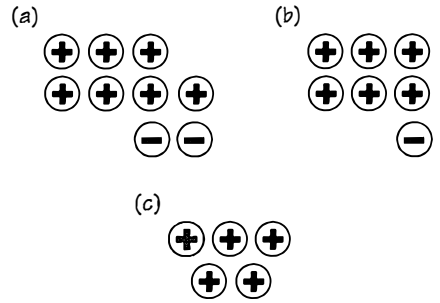
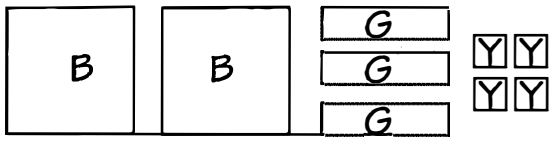



*Figure 3*

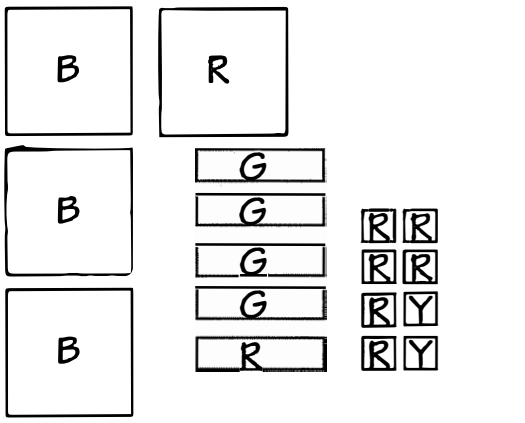
In their simplest form, Algebra Tiles can be used to represent numbers, variables, algebraic expressions, equations and inequalities. The smallest unit is the yellow/red ones tile. Like the two colour counters, the ones tile has a yellow side (representing +1) and a red side (representing -1). The green/red algebra tile represents +x on the green side and -x on the red side. There is purposefully no whole number equivalence between the ones tile and the x tile in order to suggest to students that the x tile can take on any value. (This is in contrast with Base Ten Blocks where one long is equivalent to ten ones.) The blue/red algebra tile represents $+x^2$ on the blue side and $-x^2$ on the red side. Figure 4 illustrates the use of algebra tiles to model $2x^2 + 3x - 4$.



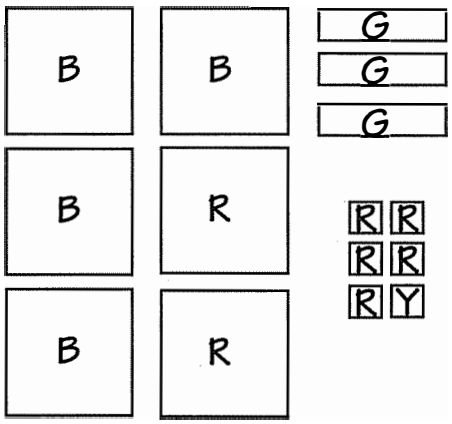

$x^2 + x^2 + 3x + 4$

*Figure 4*

As with two colour counters, zero pairs can be formed using algebra tiles. Figure 5 illustrates two different models of $2x^2 + 3x - 4$ using various zero pairs of algebra tiles.




$3x^2 + (-x^2) + 4x + (-x) + (-6) + 2$




$4x^2 + (-2x^2) + 3x + (-5) + 1$

*Figure 5*

## Beginning to Make the Connection: Divisibility

Divisibility is an often-studied area of the curriculum. This topic is rich in the potential to connect numerical processes with algebraic processes through the use of manipulatives. Two divisibility rules will be explored here, the Rule of Nine and the Rule of Eleven. Their treatment is predicated on the assumption that students are familiar with and understand the concept that if each addend of a sum is divisible by x, then the sum itself is divisible by x. After the method of exploration for the Rule of Nine has been demonstrated to the class, students should form small groups of three or four to discover the Rule of Eleven.

It is well known that a number is divisible by 9 if the sum of its digits is divisible by nine. We will use base ten blocks to examine why this is so. Figure 6 illustrates the number 3429 using base ten blocks.

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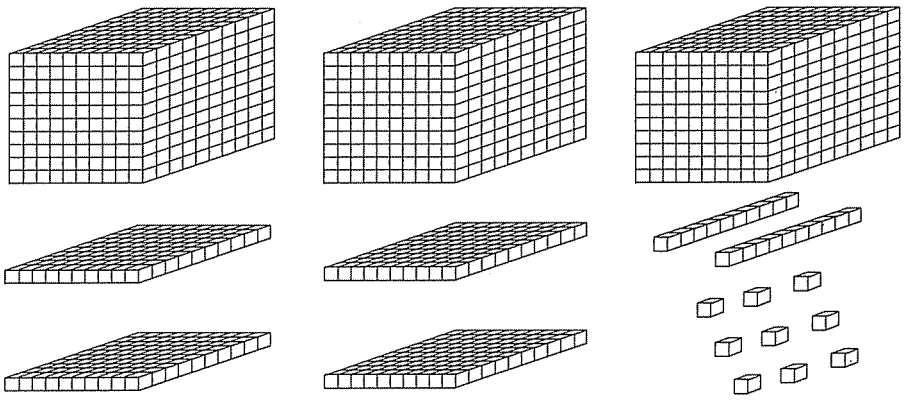


*Figure 6*

The multiple of 9 that is closest to 1000 (a cube) is 999. Introducing zero pairs, 1000 can be represented as 1000 - 1 + 1. The negative one can be represented by a unit cube of contrasting color. (For consistency in this article, we will mark the negative unit cubes with the letter R for red.) The multiple of 9 that is closest to 100 (a flat) is 99. We can again use the concept of a zero pair to represent each one hundred as the sum of 99 and one, since 100 can be expressed as (100 -1 +1). The multiple of 9 that is closest to ten (a long) is 9 itself. Following the same reasoning, we can once again use a zero pair to represent each ten as the sum of nine and one (Figure 7).

Since the groups of 999, 99 and 9 are clearly divisible by nine, it remains to be shown that the sum of the remaining units is also divisible by 9. Notice that this sum is formed by the three units from the cubes, the four units from the flats, the two units from the longs, and the nine remaining units. Since this sum, 3 + 4 + 2 + 9, is divisible by 9, the entire number, 3429, is also divisible by 9. Notice that this sum, 3 + 4 + 2 + 9, defines the sum of the digits of the number. This can be generalized for any four digit number and represented algebraically in the following way.

Let ABCD be any four digit number.

**Then ABCD = 1000A + 100B + 10C + D**

**= (1000 -1 +1)A + (100 -1 +1)B + (10 - 1 + 1)C + D**

**= (999 + 1)A + (99 + 1)B + (9 + 1)C + D**

**= 999A + A + 99B + B + 9C + C + D**

**= 999A + 99B + 9C + A + B + C + D**

This same reasoning can be applied to any multi-digit number.

With the Rule of Nine as a basis, students should now be given Base Ten Blocks and some unit cubes of a contrasting colour representing negative units to attempt to determine the Rule of Eleven. The rule for divisibility by eleven can be explored in much the same manner as

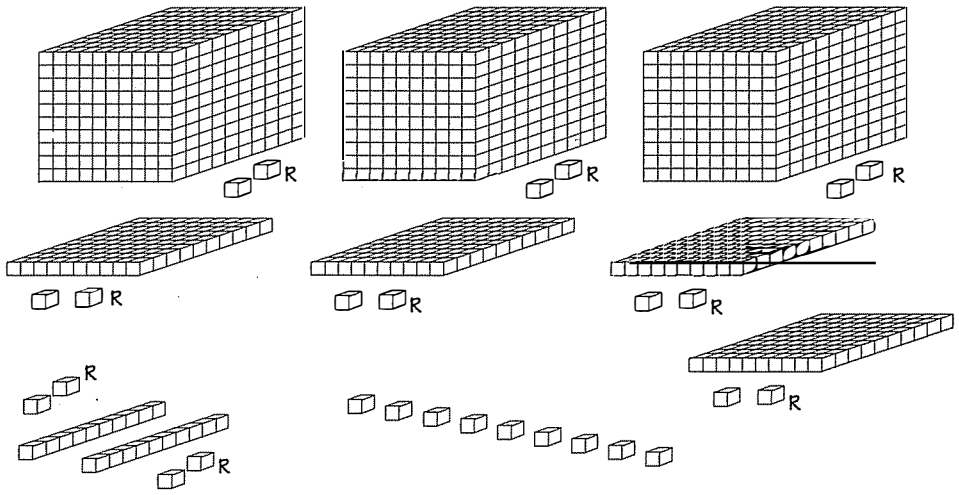



*Figure 7*

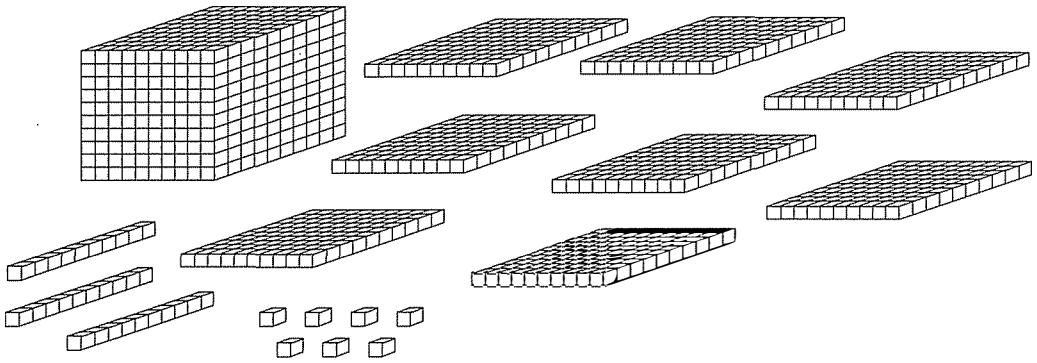


*Figure 8*

the Rule of Nine. We will use 1837 to generate and test the rule. This number can be modeled using base ten blocks and unit cubes of a contrasting colour as follows

Instruct students to look for the multiple of eleven that is nearest to 1000, the multiple of eleven that is nearest to 100, and the multiple of eleven that is nearest to 10. Figure 9 shows the use of zero pairs to represent 1837 in a useful way.

Notice that once the multiples of 11 are set aside, we are left with positive and negative units. If the sum of these integers is divisible by 11, the number 1837 is divisible by 11. Notice that $-1 + 8 - 3 + 7 = 11$. Since 11 is divisible by 11, so too is 1837. Algebraically, this situation can be generalized for any four digit number as follows:

$$\begin{aligned} WXYZ &= 1000W + 100X + 10Y + Z \\ &= (1000 + 1 - 1)W + (100 - 1 + 1)X + (10 + 1 - 1)Y + Z \\ &= (1001 - 1)W + (99 + 1)X + (11 - 1)Y + Z \\ &= 1001W - W + 99X + X + 11Y - Y + Z \\ &= 1001W + 99X + 11Y - W + X - Y + Z \end{aligned}$$

Students should recognize that beginning with the ones digit and moving right to left, the digits are alternately combined as positives and negatives. If the sum of these integers is divisible by eleven, then the entire number is divisible by 11.

## Continuing to Make the Connection: Operations on Integers

Modeling operations on integers is a crucial step in the process of understanding and picturing such operations. There are so many rules that govern these operations that it is not surprising students find them confusing. Giving students the opportunity to work with Two Colour Counters (Integer Chips) frees them from the rule-bound operations. It should be noted here that while the chip models are excellent devices for representing addition, subtraction, and multiplication of integers, we have found that they are awkward when modeling division. We prefer, rather, to treat division as the inverse operation of multiplication.

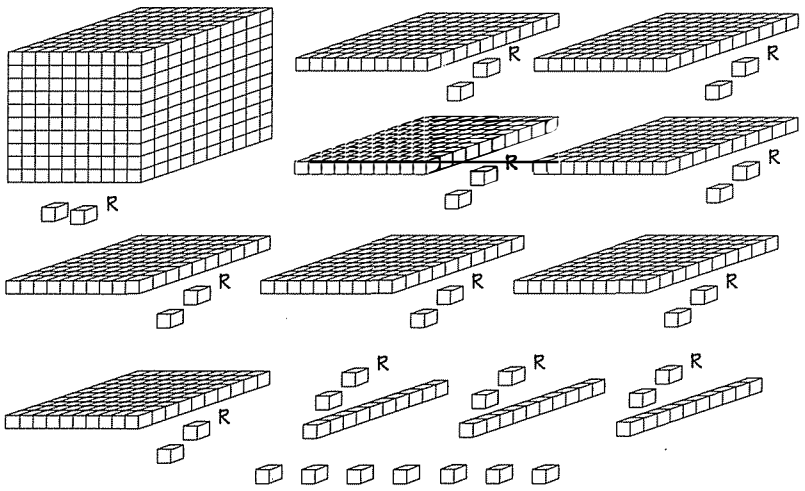



*Figure 9*

### a) Integer Addition

When using Base Ten Blocks to model whole number addition, the sum is represented by the value of the base ten blocks. This may or may not correspond to the actual number of blocks present. For example, using Base Ten Blocks to model the sum of 23 and 34 results in 12 blocks (5 longs and 7 units) representing the sum of 57. This emphasis on the value of the chips as opposed to the number of chips is also the case when using the chip model of integer addition. You will recall that the addition of any number of zero pairs does not change the value of a number. Examine the following illustrations of the use of Two Colour Counters to model integer addition.

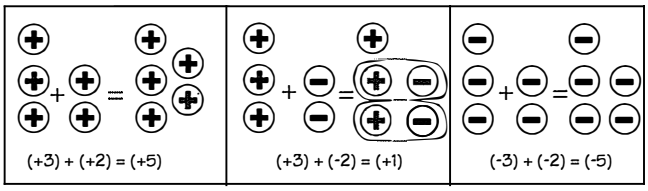



*Figure 10*

### b) Integer Subtraction

In this model, subtraction is interpreted as removing chips from a single set and examining what remains. At certain times, it is necessary to introduce zero pairs into the set in order that the removal becomes possible. Examine the following illustrations.

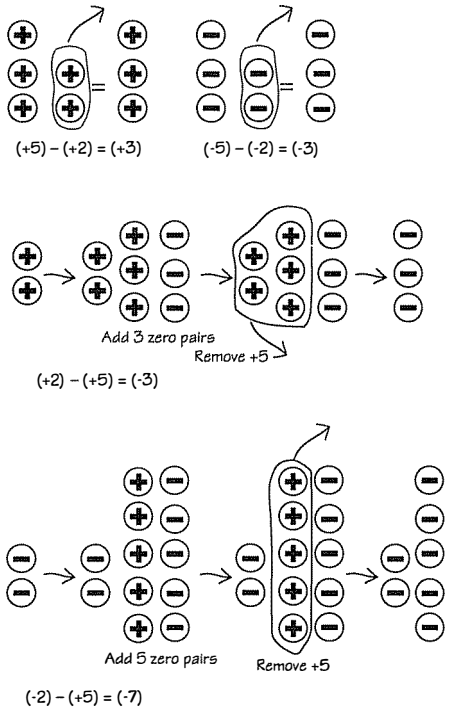



*Figure 11*

### c) Integer Multiplication

Begin with a zero value set. When the first factor is positive, that is interpreted as putting groups of chips into the set. When the first factor is negative, that is interpreted as removing groups of chips from the set. Here are two examples where the first factor is positive.

(+3) x (+2) can be interpreted as "put three groups of two positive chips into a zero value set." Figure 12 illustrates this process.

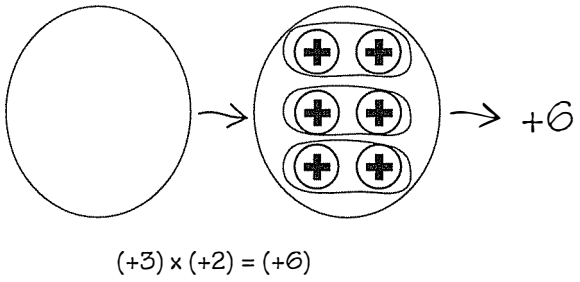



*Figure 12*

(+3) x (-2) can be interpreted as "put three groups of two negative chips into a zero value set." Figure 13 illustrates this process.

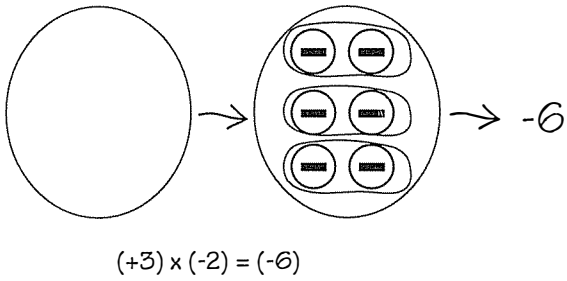



*Figure 13*

**Here are two examples where the first factor is negative.**

(-3) x (+2) can be interpreted as "remove three groups, each containing two positive chips from a zero value set." We wish to remove three groups of two positive chips. There are no positive chips in the set. However, we can put three groups of two positive chips in the set, provided that each positive chip is paired with a negative chip, in order not to change the zero value of the set. Now it is possible to remove three groups of two positive chips from the zero value set. This leaves us with three groups of two negative chips, or six negative chips, leading us to see that the product of -3 and +2 is -6.

Figure 14 illustrates this process.

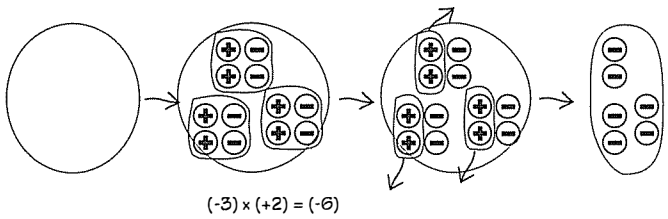



*Figure 14*

(-3) x (-2) can be interpreted as "remove three groups, each containing two negative chips from a zero value set." We wish to remove three groups of two negative chips. There are no negative chips in the set. However, we can put three groups of two negative chips in the set, provided that each negative chip is paired with a positive chip, in order not to change the zero value of the set. Now it is possible to remove three groups of two negative chips from the zero value set. This leaves us with three groups of two positive chips, or six positive chips, leading us to see that the product of -3 and -2 is +6. Figure 15 illustrates this process.

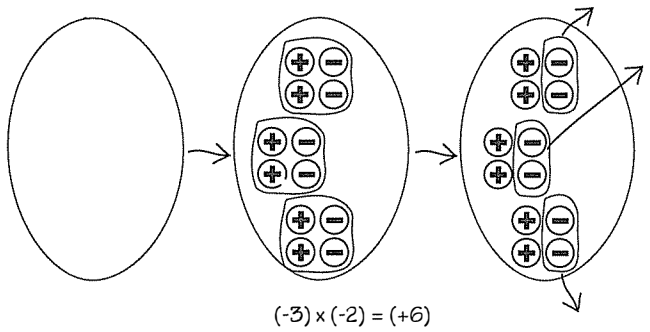



*Figure 15*

## Completing the Connections: Algebra Models

The purpose of this section is not to teach the use of Algebra Tiles. That information can be obtained in *Algebra Tiles for the Overhead Projector* by Hilde Howden (1985, Cuisenaire Company of America). Rather, it is our goal to point out the connections that can be made between Two Colour Counters and Algebra Tiles for modeling addition and subtraction, and between Base Ten Blocks and Algebra Tiles for modeling multiplication. The goal is for students to look back and make connections between arithmetic models and algebraic models.

To find the sum of $-2x^2 + 5x - 4$ and $x^2 - 3x + 1$, each polynomial would be modeled as illustrated in Figure 16. Like terms are combined and zero pairs are removed resulting in a sum of $-x^2 + 2x - 3$.

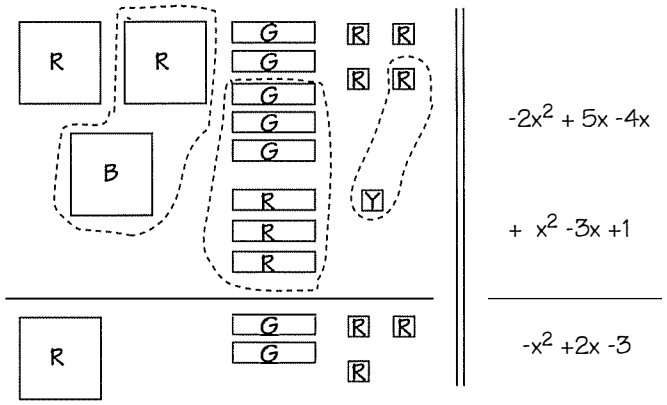



*Figure 16*

To find the difference of $-2x^2 + 5x - 4$ and $x^2 - 3x + 1$, we use the "take away" model from integer subtraction and interpret the process as follows.

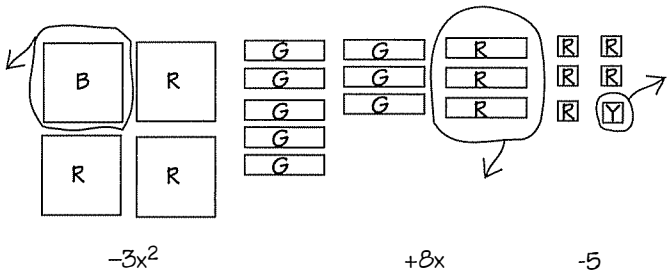



*Figure 17*

- From $-2x^2$ take away $x^2$. To do so, it is necessary to introduce a zero pair in the form of $x^2$ and $-x^2$.
- From $5x$ take away $-3x$. To do so, it is necessary to introduce three zero pairs in the form of $x$ and $-x$.
- From $-4$ take away $1$. To do so, it is necessary to introduce a zero pair in the form of $+1$ and $-1$.

It is now possible to "remove" $x^2 - 3x + 1$ from $-2x^2 + 5x - 4$, which leaves $-3x^2 + 8x - 5$.

The above process is more understandable for students who have had the experience of using zero pairs to operate on integers. The formation of algebraic zero pairs is not entirely new to them since they are able to make the connection between the arithmetic and algebraic models.

It is also possible to make connections between models of arithmetic multiplication and algebraic multiplication. An area model of whole number multiplication is often illustrated for elementary school

students using Base Ten Blocks in the following manner. Figure 18 illustrates the area model for 13 times 17. As you can see, a rectangle is formed using the dimensions of 13 and 17. The interior is then filled with the **fewest** possible number of Base Ten Blocks.

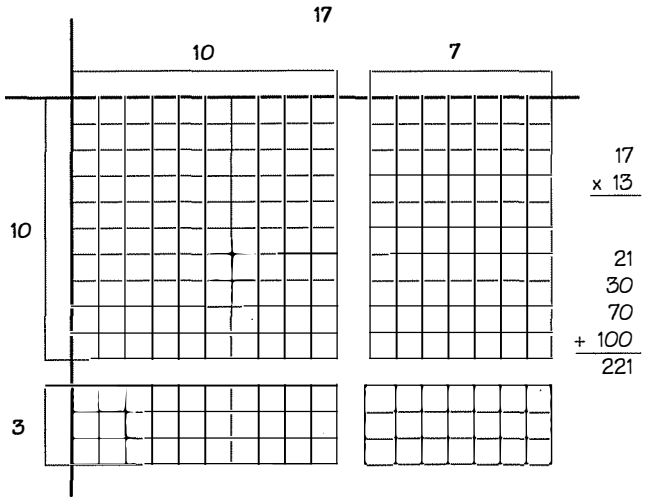



*Figure 18*

Notice that four sections are formed. The sum of the areas of the four sections yields the product of the two numbers. It is important to show students how the Base Ten Blocks model the Distributive Property and mirror the process incorporated in the partial products algorithm.

A purposeful connection can be made between the arithmetic modeling of whole number multiplication using Base Ten Blocks and the algebraic modeling of binomial multiplication using algebra tiles. To find the product of (x + 3) and (2x +1), a rectangle is formed with (x + 3) and (2x +1) as its dimensions. The interior of the rectangle is filled with the **fewest** possible number of Algebra Tiles, as shown in Figure 19.

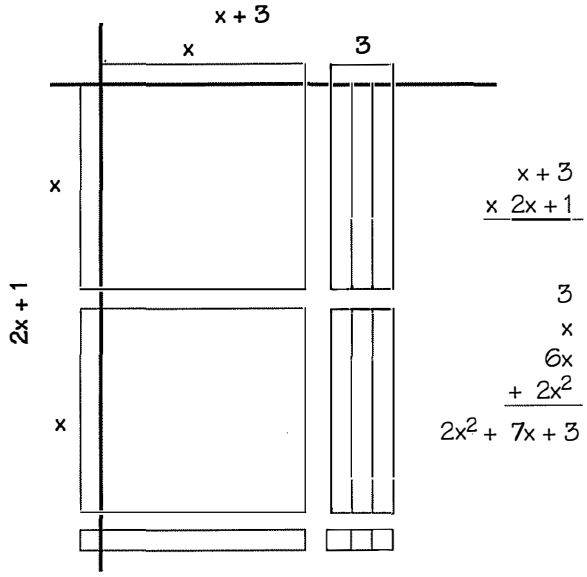



*Figure 19*

As in the Base Ten Model of whole number multiplication, four sections are formed. The sum of the algebraic representations of the areas of these sections yields the product of the two binomials. Notice how the expanded form of the sum mimics the distributive property used for standard multiplication of binomials. This recognition will be important when algebra tiles are used to model factorization.

## Summary

This article has attempted to make three important points. First, one of the "big ideas" of school mathematics is that arithmetic and algebraic expressions can be represented in a variety of equivalent fashions. The precise representation chosen is defined by the context. Second, representing abstract concepts in tangible form can help students "see" their meaning and structure. Last, making explicit connections between both the form and substance of elementary school mathematics and secondary school mathematics can help students connect the concept of number with the concept of variables. Base Ten Blocks, Two Colour Counters, and Algebra Tiles were chosen because of their ability to illustrate these connections.


Howden, Hilde, (1985), *Algebra Tiles for the Overhead Projector.* New York, Cuisenaire Company of America.

National Council of Teachers of Mathematics, (1989), *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA, NCTM.

Sgroi, R.J. and Laura Shannon Sgroi, (1993), *Mathematics For Elementary School Teachers.* Boston, MA, PWS-Kent Publishing Company.



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# ▲ Teaching Ideas from the Editor

## Algebra and Geometry: A Condition for Collinear Points

**Idea:** Given any three points P, Q, R it is always true that .

$$\overrightarrow{OQ} = m\overrightarrow{OP} + k\overrightarrow{OR}$$

Is there anything special about m and k if the three points are collinear?

**Condition:** if three points P, Q, and R are collinear, then real numbers m and k exist such that

$\overrightarrow{OQ} = m\overrightarrow{OP} + k\overrightarrow{OR}$ where m + k = 1,and O is any point. (1)

**Proof:** Since three points P, Q, and R are always coplanar then real numbers m and k exist such that

$\overrightarrow{OQ} = m\overrightarrow{OP} + k\overrightarrow{OR}$ where O is any point in the plane.

Since the points P, Q, and R are collinear, a real number x exists such that

$$\overrightarrow{PQ} = x\overrightarrow{PR}$$

$$\text{Thus, } \overrightarrow{OQ} - \overrightarrow{OP} = x(\overrightarrow{OR} - \overrightarrow{OP})$$

$$\text{Hence, } \overrightarrow{OQ} = (1-x)\overrightarrow{OP} + x\overrightarrow{OR} \qquad (2)$$

Comparing (1) and (2) shows that m = 1 - x, and k = x.

Thus, m + k = 1 - x + x = 1. QED


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# ▲ Teaching Hints From Our Readers

## Introducing Mathematical Induction Via a Game


JACK LESAGE

*Jack has retired and lives in Barrie where he continues to find outlets for his desire to teach.*


I found that many of my students had difficulty with the subtleties of Mathematical Induction.

I used the following game to convince them that two conditions have to be met in order to establish that a conjecture is correct. (This idea was given to me by Al Margison, then of Elmvale DHS, now co-owner of the Admiralty Motel which overlooks the Inner Harbour in Victoria, BC.)

I give each pair of students a game sheet, a set of instructions and a "die" (A simple "die" can be made by writing the numbers 1 to 6 on the 6 sides of a hexagonal pencil.) and have them play the game without telling them why, or what the topic for the day is.

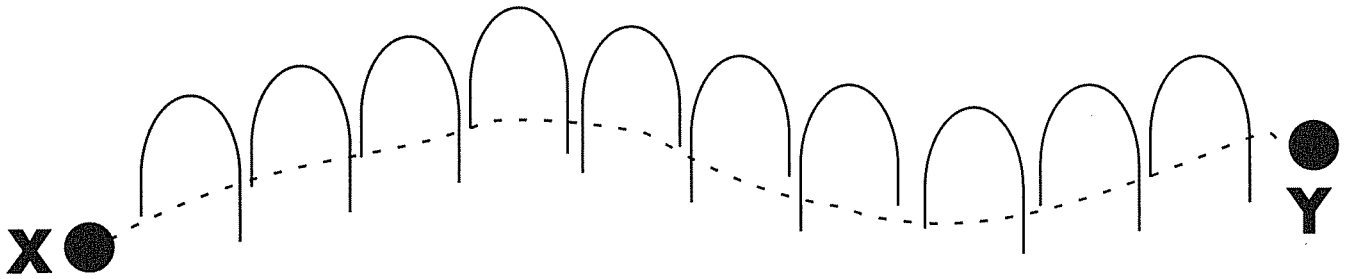
After 10 or 15 min I have them stop and we then discuss why Player B has a huge advantage over Player A. (Player A has to roll a two or a four 12 times while Player B has to roll a three only once.)

But we agree that Rule 6 alone is not enough. It guarantees that, if you get through any hoop, then you can get through the next but it does not guarantee that you can get through the first hoop. So, in order to guarantee a win, Player B must both roll a 3 (get started) and use Rule 6 (If you can go through one hoop then you can go through the next one.)

I then give them my little pitch on the differences between deductive proofs and conjectures arising out of inductive reasoning (the study of patterns). Then, I make the point that once we have a conjecture we would like to prove or disprove it. One of the strategies that can be used to do this is Mathematical Induction.

## CRAZY CROQUET

1. The aim of this game is to move your marker from pole X, through all the hoops, to pole Y.
2. Flip a coin to decide who is Player A and who is Player B.
3. Player A rolls a die first and the players alternate.
4. Player A is allowed to move through a hoop whenever he/she rolls a 2 or a 4.
5. Player B is allowed to move through a hoop whenever she/he rolls a 3.
6. Clearly A has an advantage so, to compensate for this; whenever B goes through a hoop (s)he is allowed to go through the next hoop.
7. Play on!




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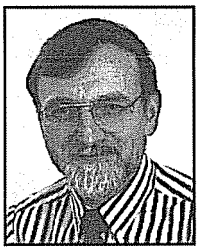
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# ▲ OPTIMIZING FOR TOMORROW

RICHARD DEWEY



*Richard Dewey teaches all levels of math at the Individual Education Centre (IndEC South) of the Peel Board of Education. A teacher for over 25 years, Richard has contributed both as a conference speaker and a writer during that time. He is constantly exploring new ways to make mathematics interesting for all students.*

*What follows is three activity based units on geometry which could be used anywhere from grade 8 to 11. The units may be reproduced for classroom use.*

Anyone who has taught grade 12 math or OAC Calculus knows the difficulty of getting students to understand how to develop a strategy for solving problems which require them to maximize or minimize a value. Depending on where it is applied, the technique may utilize the analysis of a quadratic function or the development of a function that can be analysed through differential calculus. The difficulty occurs when students try to learn a process without having first developed a thorough understanding of what is happening. They need to see how the variation of one of the aspects of the problem produces a measured change that increases then decreases, producing a maximum value. They must have an opportunity to **play** with the concepts in an experimental setting.

The difficulty in doing this kind of activity at the grade 12 or OAC level is that the teachers are already under severe pressures to finish a prescribed amount of work in a short period of time. On the other hand, there is both time and opportunity for teachers to fit this type of unit into the grade 8 or 9 curriculum. It could be applied to that segment of the course which is designated for the study of area and volume. Instead of giving the student a large number of questions where they have to find the areas of figures, they could be given the opportunity to explore patterns of a little more complex nature. This type of exercise will further develop their higher order thinking skills. The best part is that all this can be done without the use of algebra.

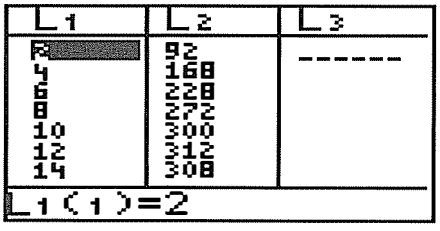
## First, a few notes to the teachers about the units:

### Old MacDonald's farm:

Students may either work individually or in small groups. You may wish to introduce the unit by having your students gather information about Emu ranching and discuss the nutritional value of the meat produced. The chart found in the unit can be enlarged as a reproduced copy and distributed as worksheets to the students. You should work with the students to discuss the values found in the first line of the chart. Let the students fill out the second line and confirm their answers before proceeding. While the students are completing the chart, you can take the opportunity to move around the room to answer questions and evaluate their performance. After all the students are finished you might have the students transfer their data to an overhead transparency or the board. Have them discuss the values and discover that the maximum occurs somewhere between the lengths of 10 and 14 m. You might encourage the students to take a guess at a more exact answer.

If your school has a graphic calculator with an overhead projector like the TI-82 or TI-83, this information can be entered into a chart using the edit, list function under the stat key. Place the length values in $L_1$ and the areas in $L_2$.

The resulting data will look like this:




Using Stat Plot as a line plot of $L_1$ and $L_2$ setting $X_{min} = 0$, $X_{max} = 24$, $Y_{min} = 100$ and $Y_{max} = 320$ will produce a line graph like this:



Alternatively, if the technology is available, the data can easily be keyed into a spreadsheet and displayed as a line graph. Otherwise, students might draw a graph on grid paper.

The second table should start at 10.25 m and increase in increments of 0.25 m to a maximum of 13 m. This will show a maximum area of 312.5 $m^2$ which occurs at a width of 12.5 m. The students should discuss how to determine the fixed cost of the pen as a group activity. Explain that each gate requires one metre of fencing for its construction producing an additional cost per gate of \$49.85 - \$26.91. This should produce a total fixed cost of \$2759.82 and a **minimum** cost per square metre of \$8.83.

If this exercise is used with a grade 12 class, the students will be able to create an area function $A(x)= -2x^2 + 50x$ and determine the maximum value using an algebraic process like completing the square.

**Building a better box:**

You can reduce the time required for this activity by preparing the cut-out graph paper in advance. The student's estimate should show that the maximum volume occurs between cuts of 2 and 4 cm and the maximum volume/surface area should occur between 3 and 5 cm. The students can write the volume of each box on its bottom surface, each box can then be held together with cellophane tape and lined up so the students can see the pattern of changing volume. The students should discuss the impracticality of refining their measurements to 10ths on the physical model. This will help them realize the value of using a mathematical model in its place.

Once again, a graphic calculator or a computer can be used to make the process of determining the optimal size to maximize the volume or the volume/surface area ratio more efficient.

When doing their second try you should suggest that the measures used for the cuts should range between 2 and 5 cm. Their maximum volume should occur at about 3.3 cm and the maximum volume/surface area should occur between 4.1 and 4.2 cm. The exact maximum volume occurs at $3\frac{1}{3}$ cm and the maximum volume/surface area occurs at approximately 4.142 cm. You may wish to discuss why there is a difference between the cut sizes for the maximum volume and maximum volume/surface area ratio.

The questions at the end of the unit can be discussed as a group activity or assigned as individual work. The last question can be used to generate an interesting bulletin board display.

**Minimizing the cost of making a box:**

Carefully review the steps used in getting the values for the first line in the chart. You may wish to review the process of rounding to the nearest 100th. If students ask how the graph was produced, you may wish to discuss how variables are used in generating the equation $y = x^2 + (4x)(1000 \div x^2)$, where x represents the length of a side of the square. After the students have found the dimensions of the box with the most efficient shape you may wish to discuss why the box makers rarely use this shape. This may lead into a discussion of the complexity of how these decisions are made in industry.

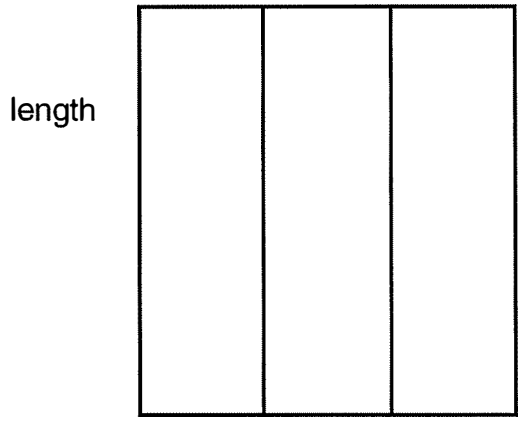
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In today's competitive job market the person who is going to get the job must be able to do the most for his or her employer for the best price. In this investigation you will create mathematical models to optimize real life situations.

## Activity #1. Old Mac Donald's Farm

Old MacDonald has decided to upgrade his farm and change his livestock to raise Emus. His daughter Amy is going to help him plan changes to his farm to make it more efficient. Old Mac has 100 metres of fencing to build a pen for the Emu families. It is a good idea to separate the males, females, and offspring so he must use some of the fencing for dividers.

His planned pen will look something like this:



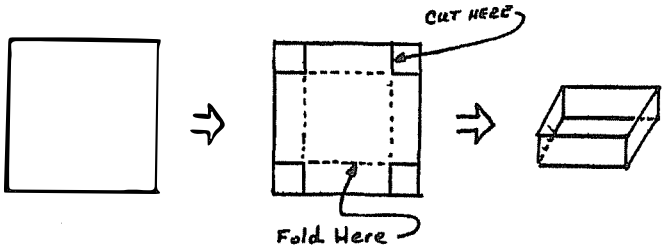

Use a chart like the one below to discover the dimensions necessary to produce the maximum area.

| Length | Total Lengths | Amount Remaining For Width | Width | Area |
|---|---|---|---|---|
| 2 m | 8 m | 92 m | 46 m | 92 |
| 4 m | | | | |
| 6 m | | | | |
| 8 m | | | | |
| 10 m | | | | |
| 12 m | | | | |
| 14 m | | | | |
| 16 m | | | | |
| 18 m | | | | |
| 20 m | | | | |
| 22 m | | | | |
| 24 m | | | | |

1. Between what two numbers will you find the maximum area?
2. Make another table, starting at the smaller of these two numbers, using step increments of 0.25 m (or 25 cm).
3. What length will now produce the maximum area?
4. If each adult emu requires approximately $17m^2$ of area and the young emus only require $7m^2$ and if each pair of emu's average three offsprings, approximately how many emus could farmer Mac hold in his pen?
5. If the fencing, including the posts, cost $26.91 per running metre, and each section requires a one metre gate to the outside at a cost of $49.85, what is the cheapest cost per square metre for the pen?

## Activity #2. Building a Better Box

Let's assume that you work for the Bargain Box Company. Your job is to design an open topped box that can be constructed from a 20 cm by 20 cm piece of cardboard. The machine used to cut the box is able to cut various size squares from the corners but it is only adjustable in steps of one centimetre. The following diagram demonstrates how it is to be constructed.




Use a piece of graph paper (one with centimeter squares is best) to form a square 20 cm by 20 cm. Cut a 1 cm square from each of the four corners as is shown in the diagram. Fold up each side of the square to form an open topped box. Note that the height of the box is 1 cm while the length and width of the square base are each 20 – 2(1) = 18 cm. The dimensions of this box as well as its volume and surface area and the ratio of these two are given in the first row of the table below.

Complete the next line of the table for a cut of 2 cm using the formulas for the volume of a rectangular box and the surface area of an open topped box. (Ask your teacher for help, if you do not know these formulas). Round off the last column, the ratio of the volume to surface area, correct to two decimal places. You may wish to construct this box too. If you do write its volume and surface area on the box.

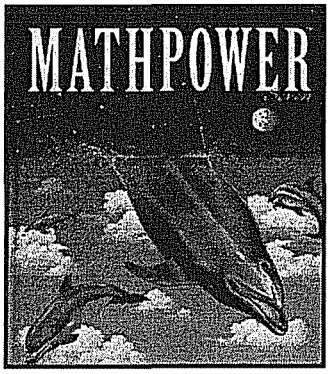
Continue as in the previous paragraph until you have finished the table. Finally, answer the 5 questions you will find after the table.

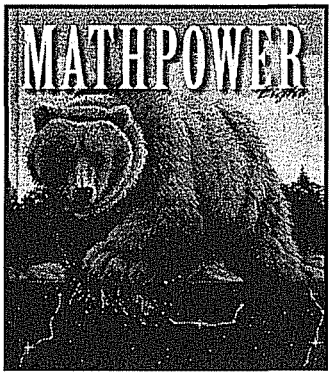
| Size of cut | Height | Length | Width | Volume | Surface Area | Vol./ S.Area |
|---|---|---|---|---|---|---|
| 1 cm | 1 | 18 | 18 | 324 | 396 | 0.82 |
| 2 cm | | | | | | |
| 3 cm | | | | | | |
| 4 cm | | | | | | |
| 5 cm | | | | | | |
| 6 cm | | | | | | |
| 7 cm | | | | | | |
| 8 cm | | | | | | |
| 9 cm | | | | | | |

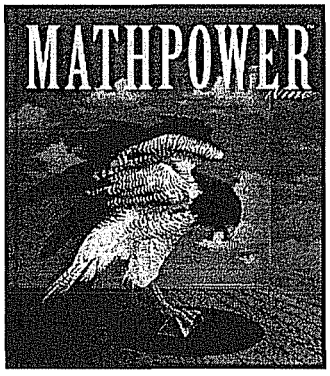
1. Approximately what size cut produces the greatest volume?
2. Approximately what size cut produces the greatest volume for the cardboard used?
3. If the Bargain Box Company could recycle their cut pieces, which of these two sizes do you think would be best?
4. A graph of the volume (vertical axis) and the size of the cut used (horizontal axis) for any size cut is shown below. Using this graph, approximately what size cut will produce the maximum volume? (answer to the nearest 10th)

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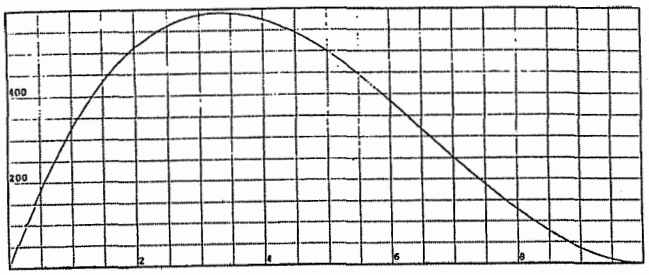
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5. Bring in a cereal box from home. Carefully take it apart and lay it out flat. Design an experiment to test to see if this shape will produce the maximum volume. If it is not the optimal shape, discuss why the company may have decided to use a less efficient shape for their product.

## Activity #3. Minimizing the cost of making a box

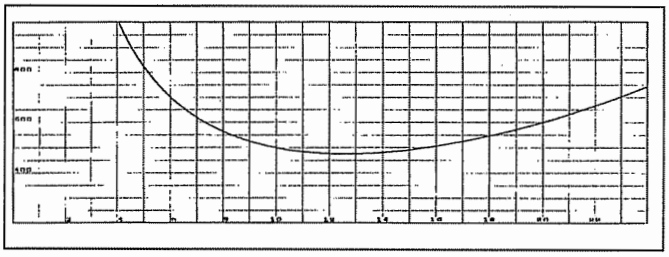
In this investigation you will look at the question of how to determine the best shape and dimensions of a container when a specific volume is required. To make the question a little simpler you will again assume that you are using an open topped rectangular box with a square base. The volume is to be fixed at 1000 cubic cm (1 litre liquid measure). Your objective is to discover what the dimensions of the square base and the height of the box must be in order to minimize the amount of material required to make the box. Thus the problem is to find the dimensions so that the area of the surface of the box will be as small as possible.

To begin the investigation, sketch a diagram representing the box showing clearly the length and width of the base (these are equal, why?) as well as the height of the box. Recall that the volume of the box is the area of the base times the height. In analysing this problem you will once again construct a table. Begin with a length of the base equal to 2 cm. Since the volume is the height times the area of the base, the height can be found by dividing the volume of 1000 $cm^3$ by the area of the base. The answers for a length of 2 cm are given in the first line of the table.

Complete the next line of the table using a length of 4 cm for the square base. Round all entries correct to two decimal places.) Finally, answer the 5 questions you will find after the table.

| Length | Area of Base | Height | Area of Sides | Total Area Surface |
|---|---|---|---|---|
| 2 cm | 4 $cm^2$ | 250 cm | 2000 $cm^2$ | 2004 $cm^2$ |
| 4 cm | | | | |
| 6 cm | | | | |
| 8 cm | | | | |
| 10 cm | | | | |
| 12 cm | | | | |
| 14 cm | | | | |
| 16 cm | | | | |
| 18 cm | | | | |
| 20 cm | | | | |
| 22 cm | | | | |
| 24 cm | | | | |

1. Between what two values can you be sure that the minimum occurred?
2. How could you modify the chart to determine the dimensions for the minimum surface area to the nearest mm (10th of a cm)?
3. Make a new chart to determine the dimensions to the nearest mm that will produce the optimum surface area. *(hint: choose values between 12 and 13 cm)*
4. If the cost of the cardboard for constructing the box is 0.1¢ per $cm^2$, what is the minimum cost of producing the box?
5. Use the graph below to verify your answer to question 3.



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# Using Factoring To Invent Six-Sided Dice With Equally Probable Outcomes

L. KEITH ROY AND GERRY VERVOORT



*Keith Roy (standing) is Professor and Chair of Mathematics and Statistics at Lakehead University in Thunder Bay. Gerry Vervoort (sitting) holds a joint appointment as Professor of Education and Mathematical Sciences at the same university. Both have been OAME members for over twenty years.*

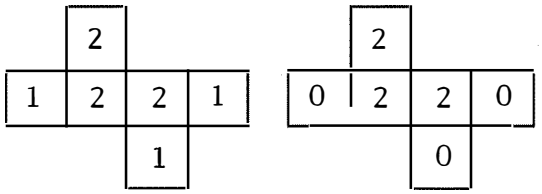
Teachers of the OAC Finite Mathematics course often use gambling games involving dice to illustrate the concept of probability. Using a pair of ordinary dice with the numerals 1 to 6, the outcomes (totals from 2 to 12) are not equally probable.

| + | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Of the 36 possible outcomes of a roll, the number 7 appears six times, that is, has probability $\frac{6}{36}$, while other numbers have a smaller probability.

This type of example becomes more interesting if the following questions are also considered. Is it possible to change the numerals on the dice so that the outcomes are equally probable? If so, is there a technique or procedure which one could follow in order to construct such dice? Is there more than one solution?

As an example consider the following dice:




The outcomes are

| + | 0 | 2 | 0 | 2 | 0 | 2 |
|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 1 | 3 | 1 | 3 |
| 2 | 2 | 4 | 2 | 4 | 2 | 4 |
| 1 | 1 | 3 | 1 | 3 | 1 | 3 |
| 2 | 2 | 4 | 2 | 4 | 2 | 4 |
| 1 | 1 | 3 | 1 | 3 | 1 | 3 |
| 2 | 2 | 4 | 2 | 4 | 2 | 4 |

and each of the four outcomes occurs nine times. Hence each outcome has probability $\frac{9}{36}$. So dice with equally likely outcomes are possible.

The key to constructing such dice is to first note that 1x6 and 2x3 are the only ways to factor six, the number of faces on each die, and then to compare these factors to the factors of N where $\frac{N}{36}$ is the probability of each equally likely outcome. Factorizations of N involving numbers other than 1, 2, 3, or 6 will not be useful. The useful factors 1, 2, 3 or 6 indicate that the numerals on a particular die should appear once only, twice, three times, or six times. Also, if T is the total number of outcomes, then NxT must equal 36.

In the above example where the four outcomes 1,2,3,4 each has probability $\frac{9}{36}$, it follows that T=4 and N = 9. The only ways to factor 9 are 1 x 9 and 3 x 3. The numbers 1 and 9 will not be useful since 1 x 9 involves a number other than 1, 2, 3, or 6. The 3 x 3 factorization indicates that any number on either die should appear three times. Since a total of 1 will appear only if one die has a 0 and the other has a 1, it follows that one die

# OAME/AOEM Awards Nomination Process

## Deadlines

Nominations for awards to be presented at the Annual Conference must be received by January 31 of the calendar year in which the conference takes place.

Nominations received after January 31 will be considered for awards to be presented during the next three calendar years. A nomination will be allowed to stand for three consecutive years. After that time, the nomination must be renewed. A standing nomination may be amended at any time by the original nominator. The nominator need only inform the chair of the Awards Committee in writing of the proposed changes to the nomination form.

## The OAME/AOEM Nomination Form

Write a letter supporting the nomination of your candidate for the indicated award. Provide specific and detailed information about the nominee's activities as they relate to the following areas of consideration;

- demonstrations of excellence and creativity in mathematics education.
- specific contributions to mathematics education in the school, region, province or nation.
- demonstrates leadership in mathematics education in the school, region, province or nation.
- support of and contribution to the activities of OAME/AOEM.
- contribution to the development of students both within and outside of the classroom.

Please include any additional information that you feel may be relevant to the nomination, but which may not be specifically mentioned above.

## The Nomination Cover Page

Complete the Nomination Cover Page and attach it to your letter of recommendation for the nominee.

***Incomplete or inaccurate information may cause a nomination to be withheld form consideration.***

## OAME/AOEM LIFE MEMBERSHIP AWARD

The Life Membership Award is given to recognize a person who has contributed in a significant was to OAME/AOEM; demonstrated outstanding leadership in mathematics education; and accumulated 10 or more years of membership in OAME/AOEM.

## KENNETH D. FRYER AWARD

In Memory of the late Kenneth D. Fryer, OAME/AOEM, in conjunction with the Grand Valley Mathematics Association, established the Kenneth D. Fryer Award. The award is to be given annually to recognize a mathematics department which fosters collegiality, team work, and excellent classroom teaching; contributes to the overall development of students; and demonstrates leadership in the mathematics education community.

## MONA MACGREGOR AWARD

In recognition of the contributions made by Mona MacGregor, former OAME/AOEM Co-Secretary-Treasurer, OAME/AOEM established the Mona MacGregor Award. This is to recognize an individual who has made significant contribution to OAME/AOEM, but who is not necessarily an educator or a member of OAME/AOEM.

## MORLEY MACGREGOR AWARD

In Memory of the late Morley MacGregor, former OAME/AOEM Secretary-Treasurer, OAME/AOEM established the Morley MacGregor Award. The award is to be given annually to recognize an elementary teacher who has demonstrated excellence in mathematics education, as well as contributing to the overall development of students beyond classroom activities.

## DON ATTRIDGE AWARD

In recognition of the contributions made by Don Attridge, a founding member, former president and the previous Executive Director of OAME/AOEM, OAME/AOEM established the Don Attridge Award in the spring of 1995. The award is presented annually to recognize an exceptional and creative secondary school mathematics teacher who enhances the teaching of mathematics in Ontario.

(Copy this form and attach it to the nomination letter)

## AWARD NOMINATION COVER PAGE

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must have three 0's and the other die must have three 1's. Using three 2's on the die with the 1's produces 9 totals of 2 + 0 = 2. Using three 2's on the die with the 0's produces 9 totals of 2 + 1 = 3 and 9 totals of 2 + 2 = 4.

The only possible values for N are the factors of 36: 1, 2, 3, 4, 6, 9, 12, 18, and 36. Using these values, the above technique will generate ten pairs of six-sided dice with equally likely outcomes. The results are summarized in the following table.

| N | Outcomes | Useful Factors Of N | Frequency Of Numbers On Dice | Dice Faces |
|---|---|---|---|---|
| 1 | 1 to 36 | 1x1 | once on each die | 1-2-3-4-5-6 |
| | | | 0-6-12-18-24-30 | |
| 2 | 1 to 18 | 1x2 | once on one die | 1-2-3-4-5-6 |
| | | | twice on the other | 0-6-12-0-6-12 |
| 3 | 1 to 12 | 1x3 | once on one die | 1-2-3-4-5-6 |
| | | | 3 times on the other | 0-6-0-6-0-6 |
| 4 | 1 to 9 | 2x2 | twice on one die | 1-2-3-1-2-3 |
| | | | twice on the other | 0-3-6-0-3-6 |
| 6 | 1 to 6 | 1x6 | once on one die | 1-2-3-4-5-6 |
| | | | 6 times on the other | 0-0-0-0-0-0 |
| | | 2x3 | twice on one die | 1-2-3-1-2-3 |
| | | | 3 times on the other | 0-3-0-3-0-3 |
| 9 | 1 to 4 | 3x3 | 3 times on one die | 1-2-1-2-1-2 |
| | | | 3 times on the other | 0-2-0-2-0-2 |
| 12 | 1 to 3 | 2x6 | twice on one die | 0-1-2-0-1-2 |
| | | | 6 times on the other | 1-1-1-1-1-1 |
| 18 | 1 and 2 | 3x6 | 3 times on one die | 0-1-0-1-0-1 |
| | | | 6 times on the other | 1-1-1-1-1-1 |
| 36 | 1 only | 6x6 | 6 times on one die | 0-0-0-0-0-0 |
| | | | 6 times on the other | 1-1-1-1-1-1 |


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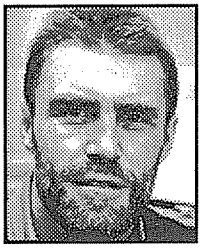
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# ▲ Technology, Science, and Mathematics versus Mathematics, Science, and Technology Integration

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**FACULTY OF EDUCATION**
**QUEEN'S UNIVERSITY, KINGSTON, ONTARIO**



*Geoff Roulet teaches pre-service and M.Ed. courses in Mathematics Education. His research interests include the development of learning environments and activities in which students can discover their own mathematics.*

*The Common Curriculum* (Ontario Ministry of Education and Training, 1995) makes repeated calls for a program that promotes integrated learning and encourages active inquiry. A curriculum that illustrates the connections between disciplines is a worthy goal and linking mathematics to other subjects could help make the discipline appear less abstract and provide student motivation. But not all forms of integration lead to positive results.

## Curriculum and the History of Mathematics

Many integration projects appear to follow Fogarty's (1991) "Sequenced Model" or what the Ministry of Education and Training labels "Parallel Content". In this style of integration, the timing of topics is arranged so that skills and concepts from one discipline are developed in preparation for their use in another. This approach to the linking of subjects is encouraged by the words of the Ministry of Education and Training (1995) when introducing its "Focus on Integrated Learning". "The ability to see links among different areas of learning will enable students to use knowledge and skills developed in one field to learn in another" (p. 10). *The Common Curriculum* goes on to make it quite clear that mathematics may be treated in this fashion.

> *An integrated approach to teaching and learning must include instruction in the fundamental knowledge and skills of particular subjects - **mathematics for example** [bold added]. Integrated contexts can clarify and reinforce students' understanding of basic concepts by stressing their applications. (p. 33)*

Curricula developed with the above statements as guiding principles are likely to present mathematics as a collection of absolute rules and procedures that somehow exist independent of humans and prior to any needs. This is the image of mathematics presented by the traditional textbook approach where pages developing algorithms precede sections illustrating "problems" to which these recently learned formal processes may be applied.

Unfortunately, for much of the general public and many teachers without strong mathematics backgrounds the title of the third broad program area identified by *The Common Curriculum,* "Mathematics, Science, and Technology", represents a natural sequence. They see mathematics as a set of tools one employs to develop scientific theories that are then applied in the design of new and better technologies. This view runs counter to the historical development of human knowledge and does little to promote the study of mathematics through the active inquiry suggested by *The Common Curriculum.*

There is anthropological evidence that human technology or the use of tools pre-dated any development of science or mathematics. In fact,

> *It was the success of the simplest tools that started the whole trend of human evolution and led to the civilizations of today. ... Man began when populations of apes, about a million years ago, started the bipedal, tool-using way of life that gave rise to the man-apes of the genus Australopithecus. (Washburn, 1974, p. 11)*

The evidence does not suggest that these early humans, *Homo faber,* developed their tools through the application of pre-existing scientific or mathematical knowledge. Cultures can make significant technological advances without employing formal theories of mathematics or science. China has a long and rich technological history, with levers in the form of stone spades being employed in agriculture in the years 7000-

6000 BC, but, "the ideas of experimentation (scientific method) and the mathematization of hypotheses were European trends and absent in China" (Needham, 1986, p. 7). The first recorded evidence of a scientific understanding of the "Law of the Lever" appears to have come from Archimedes (287-212 BC) who is reputed to have claimed, "Give me a place to stand and I will move the Earth". His science of the lever was expressed in language and it took another three centuries before Heron (50 AD) provided a mathematical treatment of the subject showing how the force applied and the distance moved vary inversely (de Camp, 1960).

Thus the sequence is often not mathematics, science, and technology but the reverse. Puzzling about why a particular technological solution works, can lead to the generation of scientific theory and attempts to refine that theory may lead to the development of mathematics. Technology and science can provide the questions and contexts for investigations in mathematics. What is required is Fogarty's (1991) "Integrated Model" or what is called "Cross-curricular Connections" in *The Common Curriculum.* Here the concepts and skills of multiple subject areas are addressed through the exploration of an issue or problem. For mathematics that issue or problem may have technological or scientific origins. If a problems-first approach is employed, students may come to see that mathematics is a language developed by humans to facilitate fuller descriptions and analyses of natural and social phenomena. Mathematics is a collaboratively constructed human system and not a set of magic rules and formulas that exist somewhere out in space.

Of course the knowledge flow is not strictly in one direction. A mathematical model of a scientific theory or technological system casts new light on the original problem. By manipulating the model we can identify outcomes to expect when parameters are altered. Such information can suggest new experiments that further explore the scientific theory or changes that could improve the technological system. Thus the new mathematics often serves to do more than solve the original motivating problem and can lead to new questions concerning the phenomenon under study. A problems-first curriculum, beginning in technology or science, can illustrate this creative cycle and show the power of mathematics to go beyond the immediately obvious.

At the Faculty of Education, Queen's University, we have been working on developing units that integrate technology, science, and mathematics and promote the development of mathematical skills and concepts through the addressing of problems. Two examples that develop Transition Years outcomes are briefly described below.

## Probabilistic and Algebraic Modelling of Science Phenomena

*The Common Curriculum* lists the ability to use the particle theory of matter to explain the properties of materials as an outcome for the end of Grade 9. This model of matter can be employed to picture the dissolving of a solid in a liquid as resulting from the break down of the solute into its basic particles and these becoming dispersed in the spaces between the solvent particles. When some solids (potassium permanganate) are dissolved in water we also observe diffusion, as a change in colour spreads out from the original location of the solid. Why do the solute particles move through the solvent?

A first explanation is usually based upon the idea that the liquid solvent particles are in motion and bump into the solute particles pushing them through the solution. But, since the solvent particle motions are random, it can be argued that the direction of bumps on any single solute particle should also be randomly distributed. This suggests that a particle should make little progress but wander around in a limited region near the original location of the solid. The exploration of this apparent conflict in the theory leads to an investigation of chance and probability.

If the path of a solute particle is regarded as the result of motions along three axes we can reduce the problem to a single dimension, and examine the particle's progress along one line. Since, with this restriction, there are only two possible directions for bumps, left or right, this situation may be simulated by flipping a coin (Lewis, 1990).

Students conduct coin tossing experiments and class data is collected. By taking heads as a bump to the right and tails to the left, pupils record the absolute distance from the origin their solute particle would be after 10, 25, 50 and 100 collisions. A search for patterns should result in the observation that the distance travelled (d) is roughly proportional to the square root of the number of

coin tosses or bumps (n). The table in Figure 1 shows the mean distances obtained by one class of 22 students. The individual 100 toss samples may be combined and computer coin toss simulations may be used to extend this data to include larger numbers of particle collisions.

**Distance Travelled by Solute Particle From Coin Toss Simulation**

| Number of Tosses ($n$) | Class Mean Distance ($d$) |
|---|---|
| 10 | 2.5 |
| 25 | 4.1 |
| 50 | 5.8 |
| 100 | 6.5 |

*Figure 1*

Using graphing calculators or a computer graphing utility and a guess-and-check approach, the class can plot curves of $d = kn^{\frac{1}{2}}$ for a variety of k's and find the function that most closely models the data. Figure 2 shows the output from the computer utility, *Data Analysis* (NCTM, 1988), for the simulation results presented in Figure 1. Here the students found their best fit to come with the function $d = 0.75n^{\frac{1}{2}}$.

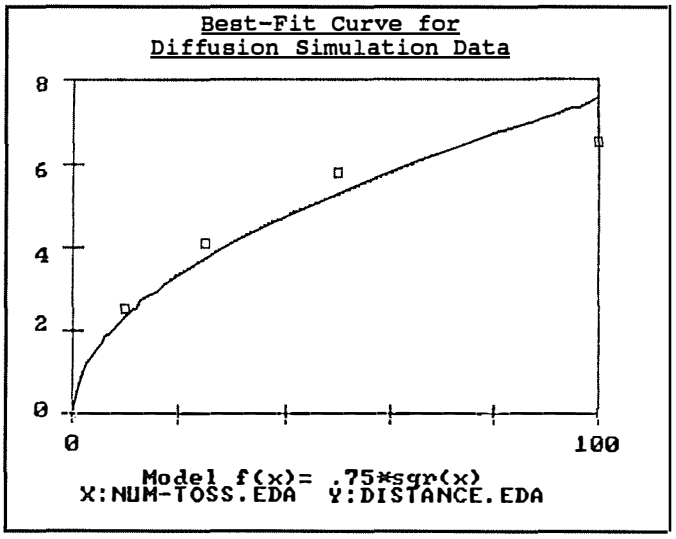



*Figure 2*

The contribution of mathematics to the development of scientific theory is revealed to students when they interpret their model and develop new experiments suggested by the refined picture. Since the number of collisions per second should be approximately constant, the simulation and mathematical model predict that the solute particles, on average, should progress away from the solid, but that the pace of this progress should diminish with elapsed time. Students gather data to test this prediction by, at regular time intervals, measuring the radius of the disc of colour as it expands through a solution. Again employing a graphing tool, students can check to see if the distance versus time plot follows the square root relationship predicted. In fact, if care is taken, the fit is quite good as can be seen in Figure 3, which shows the data and curve obtained by one pair of students. Here the students found that distance, $d$, (measured in grid spaces on a piece of graph paper under the beaker) was proportional to the square root of time, $t$, (measured in minutes) with the equation, $d = 2.65t^{\frac{1}{2}}$ giving the best fit.

**Diffusion Experiment Data**

| Time (minutes) | Distance (Radius) (grid marks) |
|---|---|
| 0 | 0.0 |
| 1 | 2.4 |
| 2 | 3.5 |
| 3 | 4.5 |
| 4 | 5.5 |
| 5 | 6.3 |

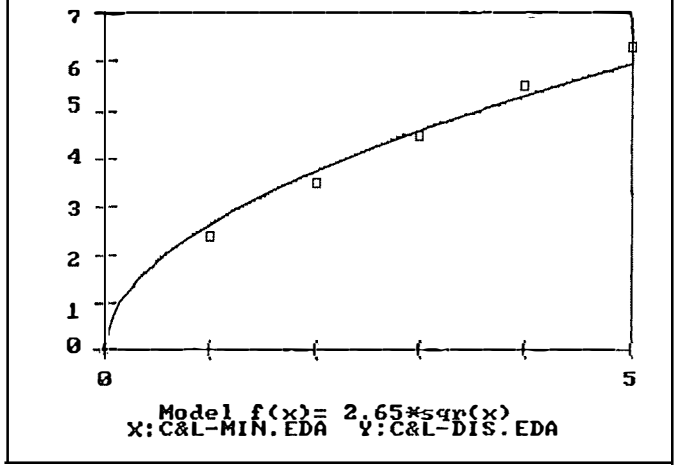



*Figure 3*

While addressing outcomes related to probability, algebra, co-ordinate geometry, and square roots this activity shows students that mathematics can be developed to provide models for scientific theories and that these mathematical models in turn suggest new experiments and changes in scientific theory.

## Geometry, Ratios and Mechanical Linkages

Projects that address Transition Years outcomes concerning the design and construction of products and

structures can present students with many problems that motivate investigation of traditional science and mathematics topics. In one such project we have assigned pupils the task of designing and making an animated toy, a puppet depicting an animal or human with limbs, eyes, mouth, etc. that move whenever an input lever is pushed or pulled. A major portion of this activity is the design and construction of linkages that connect the input lever to the movable limbs. This task involves measurement, ratio, proportion, and geometry; topics that are addressed by *The Common Curriculum* outcomes for the end of Grade 9.

As students experience problems translating the linear motion of the input push and pull into circular motion around the fulcrum of a lever there are many opportunities to explore geometry. In particular, a concrete introduction to the idea of a locus of points is frequently provided when students attempt to move two arms or legs using the linkage shown in Figure 4. Our experience is that pupils miss many of the underlying mathematical ideas unless the teacher interrupts the class at strategic moments and initiates discussions concerning such problems that have developed out of the pupils' work. The context of the technology project provides motivation for the students during these mini-lessons and shows that geometry is not, in its origins, an abstract mental exercise but arises from observation of the world around us.

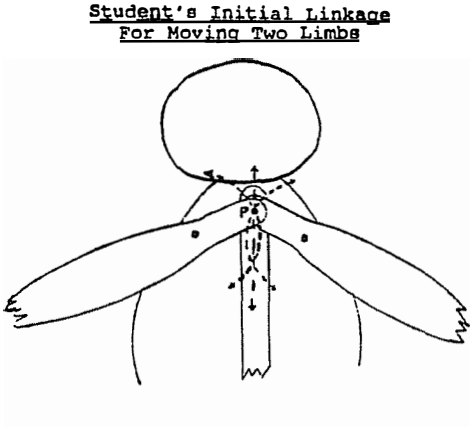



*Figure 4*

A brief focused student exploration reveals that the conditions on point P (see Figure 4) are demanding that it follow three conflicting loci: the arcs bending to the right and left and a straight line along the path of the input lever. Students readily identify the fixed radii of the lever arms as the determining factors of the two arc paths. Through trial-and-error constructions and discussions with work partners the pupils are usually able to come up with mechanisms that produce variable radii. Common student solutions such as cutting a slot along the length of a lever or providing a folding linkage are shown in Figure 5.

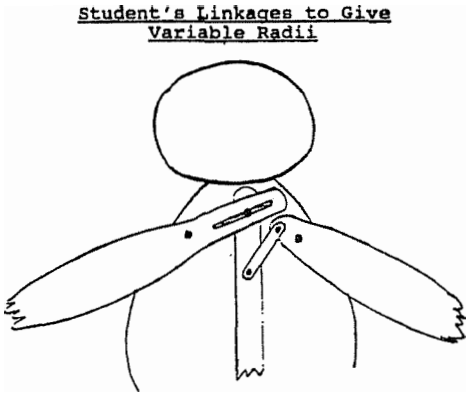



*Figure 5*

As the students work on the task of having their puppets' arms and legs move through appropriate arcs they confront the problem of selecting suitable lengths for input and output arms of levers. Focused experiments at this point can lead to the class identifying the equality between the ratios of lever arms and arc lengths. Work within the context of their puppet projects provides further practice with the proportion concept. This idea is further developed when dealing with the problem of creating a linkage that does not require excessive force to operate.

Explorations in science and mathematics arise naturally and students have concrete examples from which to build more general concepts. Mathematics is constructed in response to a need.

## An Invitation

Our experiences so far with the activities described above suggest significant potential. We at the Faculty of Education, Queen's University, are interested in collecting data on larger scale implementations and making any improvements indicated by this experience. If you would like to use either unit with your classes we would be happy to supply you with more complete information. In exchange we would request a report on the outcomes of the activity and any modifications that you may suggest. If time permitted we might be able to join you while your students were participating in the explorations. If interested please contact Geoffrey Roulet at the Faculty of Education, Queen's University.

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*I would like to thank Joan McDuff and Malcolm Welch of Queen's University for their help in the development of the student activities reported in this paper. I would also like to thank the two anonymous reviewers of the original draft of this paper for their helpful suggestions.*


# ▲ TRIANGULAR TRIPLES


**HELMUT PADDAGS**
**EAST YORK ALTERNATIVE HIGH SCHOOL**
**TORONTO, ONTARIO**




*Helmut is currently teaching at East York Alternative High School where he is using investigations and computers to encourage students to do mathematics.*

In the June 96 issue of the Ontario Mathematics Gazette, J. Grant McLoughlin wrote ABOUT PROBLEM SOLVING: THE IMPORTANCE OF CONTEXT and suggested that triangular triples required further investigation. Triangular triples are the subset of integer triples which can be arranged to form a perimeter of a triangle. The students in my multi-level grade 11 class and I set out to predict the total number of distinct (non-congruent) triangles of integer sides that can be constructed for any given integer perimeter. We had investigated individual cases and found this problem to be a challenging extension. With much effort we were able to derive a solution by experimenting with tooth picks, displaying data on spreadsheets, graphing various relationships and applying arithmetic sequences and series. What patterns could be found in this table of triangles generated by the class?

**perimeters**

3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,

**triangles**

1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 8, 7, 10,

We struggled with several hypotheses before we noticed that perimeters 3 and 6 only made one triangle as did perimeters 5 and 8. Perimeters 7 and 10 made two distinct triangles each. The relationship that we discovered saved us considerable work as we could now concentrate on even numbered perimeters. By recording the results of the triples and organizing them systematically we found that we could dispense with the toothpicks and proceed more efficiently with pencil and paper alone.

The sequence of triangles continued for even number perimeters.

**perimeters**

20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48

**triangles**

8, 10, 12, 14, 16, 19, 21, 24, 27, 30, 33, 37, 40, 44, 48

## Conjecture 1

The number of triangles for any odd number perimeter is the same as the number of triangles for the perimeter which is three more.

We entered our data on spreadsheets and tried to look for patterns by comparing the two columns in various ways and graphing the results. We isolated the following pattern and by trial and error came up with a function for predicting the number of triangles.

| | | | | | | |
|---|---|---|---|---|---|---|
| perimeter | 12, | 24, | 36, | 48, | 60, | ... |
| triangles | 3, | 12, | 27, | 48, | 75, | ... |
| triangles/perimeter | 0.25, | 0.5, | 0.75, | 1, | 1.25, | ... |

For successive multiples of 12 in the perimeter, the ratio of the triangles to the perimeter increased by 0.25.

## Conjecture 2

If t is the number of triangles and p is any perimeter which is a multiple of 12, then

$$t = 3\left(\frac{p}{12}\right)^2$$

Unfortunately this conjecture did not work for the other even numbers that were not multiples of twelve. We decided to investigate the differences in the sequence of the number of triangles.

| perimeter | 12 | 24 | 36 | 48 |
|---|---|---|---|---|
| triangles | 3 | 12 | 27 | 48 |
| series | 3 | 3+9 | 3+9+15 | 3+9+15+21 |

The formula for arithmetic series,

$$t = \frac{n}{2}\{2a + (n-1)d\}$$

gives the number of triangles, t. Here a = 3 and d = 6, then $t = 3n^2$ as by conjecture 2. We extended this approach to other perimeters that were 12 apart.

| perimeter | 16 | 28 | 40 | 52 |
|---|---|---|---|---|
| triangles | 5 | 16 | 33 | 56 |
| series | 5 | 5+11 | 5+11+17 | 5+11+17+23 |

## Conjecture 3

Consider $\frac{p}{12}$ where p is an even perimeter, n is an integer and r < 12 is the remainder. Here r = 4, a = 5 and d = 6, and the arithmetic series yields the number of triangles, $t = 3n^2 + 2n$.

The perimeters with remainder 8 on division by 12 frustrated us for a time.

| perimeter | 8 | 20 | 32 | 44 |
|---|---|---|---|---|
| triangles | 1 | 8 | 21 | 40 |
| series | 1 | 1+7 | 1+7+13 | 1+7+13+19 |

## Conjecture 4

When we replaced n with n +1 in the formula for the arithmetic series we obtained the desired function. Here r = 8, a = 1, d = 6, and n is replaced by n + 1, then the arithmetic series yields the number of triangles, $t = 3n^2 + 4n + 1$.

When the perimeter was divided by 12 and had a remainder of 2, 6, or 10 our arithmetic series approach no longer worked. We returned to our pencil and paper calculations and entered our results on a new spreadsheet and looked for further insights. What do you notice about the two parts of the series?

Here the remainder is 2.

| perimeter | 14 | 26 | 38 | 50 |
|---|---|---|---|---|
| triangles | 4 | 14 | 29 | 52 |
| series | (1)+3 | (1+7)+6 | (1+7+13)+9 | (1+7+13+19)+12 |

Here the remainder is 6.

| perimeter | 18 | 30 | 42 | 54 |
|---|---|---|---|---|
| triangles | 7 | 19 | 37 | 61 |
| series | (3)+4 | (3+9)+7 | (3+9+15)+10 | (3+9+15+21)+13 |

Here the remainder is 10.

| perimeter | 22 | 34 | 46 | 58 |
|---|---|---|---|---|
| triangles | 10 | 24 | 44 | 70 |
| series | (5)+5 | (5+11)+8 | (5+11+17)+11 | (5+11+17+23)+14 |

## Conjecture 5

When the remainder is 2, 6 or 10 the series is complicated by an additional term which is generated by an arithmetic sequence, whose first term b is 3, 4 or 5, depending on the remainder. The common difference e is always 3. The number of triangles is generated by the equation,

$$t = \frac{n}{2}\{2a + (n-1)d\} + \{b + (n-1)e\}$$

When r = 2, a = 1, d = 6, b = 3 and e = 3, then $t = 3n^2 + n$.

When r = 6, a = 3, d = 6, b = 4 and e = 3, then $t = 3n^2 + 3n + 1$.

When r = 10, a = 5, d = 6, b = 5 and e = 3, then $t = 3n^2 + 5n + 2$.

We were satisfied that we had found every case. We tested the entire set of equations on the even perimeters from 50 to 60 and our results matched our pencil and paper calculations. Could these six equations be simplified or consolidated?

## Conjecture 6

We noticed that the coefficient of the linear term of the quadratic was always the remainder r divided by 2. If p is even, $\frac{p}{12} = n + r$, and r < 12 then:

$$t = \frac{n}{2}\{2a + (n-1)d\} + \{b + (n-1)e\}$$

$t = 3n^2 + \frac{r}{2}n$, when r is 0, 2 or 4;

$t = 3n^2 + \frac{r}{2}n + 1$, when r is 6 or 8;

and $t = 3n^2 + \frac{r}{2}n + 2$, when r is 10.

We were pleased that we had managed to reduce the six equations to three and also order them by the size of the remainder. By applying Conjecture 1, add 3, and Conjecture 6 we were able to predict the number of triangles for odd numbered perimeters as well.

## Conclusion

With an inquisitive mind set, experimental data, spreadsheets, knowledge of arithmetic sequences and series and perseverance, we discovered some significant ways to use our mathematics to solve a challenging problem. Many times we were convinced that we had the formula firmly in our grasp, only to find that the ultimate goal still eluded us. The calculations which led to our conjectures are quite extensive and I am willing to share them with any interested reader on request.

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All Canadian resources and manipulatives for Grades K-12.

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# ▲ DIRECTORS' DIALOGUE



**PETER SAARIMAKI**
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**TORONTO BOARD OF EDUCATION**

*These notes are from the last meeting of the 1996/97 OAME Board of Directors held February 8, 1997.*

## Introduction

The OAME is currently experiencing a gradual decline in membership and continues to explore news ways of reaching out to teachers of mathematics and providing services teachers would otherwise not have access to. In the past years OAME, with OMCA, has produced several resources to bridge the gap between Ministry expectations (general) and classroom practice (specific), especially as they relate to assessment.

In addition, we are moving to catch up with technology, with OAME's website at the University of Guelph and several chapter websites (see below). You are urged to visit the site, and to encourage your colleagues, and to avail yourselves of the information and connections provided there. As well there is the OAME conference on the Electronic Village (ENOREO), which has lurched and groped along with many interesting discussions over time.

### Guest

Ivan Johnson, the NCTM rep for Western Canada, attended the meeting and presented information regarding NCTM Regional and Annual meetings into the 21st Century. OAME Futures committee is looking into hosting a meeting, with the possibility we look to a fall date so as not to conflict with the traditional OAME Annual conference each spring.

## Executive Reports

Included in many administrivia items, necessary to the smooth running of the organization (and with all the important items, we were still finished by 4:00 pm: thanks to Barry, Dave and Bonnie for keeping us on track!!), were several topics the reader should find of interest.

An eagerly awaited tradition for the chapter reps was the ceremonial "Handing Out of the Rebate Cheques"!!. From the annual fees members pay to OAME, a percentage (24) is returned to the designated chapter for local events and assistance. (If no chapter is designated, OAME retains the funds for general provincial use.) So make sure you avail yourself of your contribution to your local chapter through its activities.

With the provincial reduction and restructuring of school boards, several OAME chapters may have to look at their names, coverage and/or executive makeup. This may be the opportunity for many of chapters to look at their names, geographic range and possible redefinitions.

## Chapters

Most chapters this spring prepared and supported one or more teams for the Ontario Math Olympics. (Did you know that CHAMP held preliminary contests for students grades 7–11, involving almost 7800 students from over 30 elementary and almost 40 secondary schools. Way to go!! I guess that's why they are known as CHAMPions!!) More info will be available from the OMO committee.

In addition many chapters were planning to host miniconferences and/or Annual General Meetings with a wide range of workshops and interesting speakers, including: Rena Upitis (Dean of the Faculty of Education at Queens, with degrees in math and music); Todd Romiens, the Math Man; and Peter Ryback (on Astronomy, Space Camp and Mathematics).

Many chapters have newsletters by which their members keep in touch are reach out to other educators in their area. They also will make note of other chapter events since many teachers have access to several chapters. Check out: ISOMA NEWS, CHAMP NEWS, $Y^4$MA's The Exponent, and the COMA Newsletter.

Several chapters also have conferences planned for well in the future, including:

- Sept 23, 1997 $Y^4$MA is $E^4$: Engaging, Enriching, Exciting and Empowering (this will be their 10th anniversary and so there will be lots of special "stuff")
- Sept 26/27, 1997 NOMA/AMNO's fall conference will be held in Sudbury. In addition, NOMA, with the Nipissing Math Council is running the 1998 Annual OAME conference in North Bay.

- May 20 23, 1998 the GATEWAY conference in North Bay: plan for an exciting train trip to receive a major dose of Northern Hospitality: for those who made it to Sudbury in 1985, you will know what that can mean: if you missed Sudbury, don't miss North Bay in '98.

**Chapter websites for you ... and your students ... to visit:**

(I won't give away their origin, so you will have to zip to the site to see who's there!)

**www.math.carleton.ca/coma**

**www.mathstat.uoguelph.ca/oame**

**www.golden.net/~gvma**

**www.quinte.net/isoma/**

[If I have missed any, please let me know at: Peter.Saarimaki@eqao.com]

You might also want to visit these too:

**www.edu.gov.on.ca** Ministry of Education: press releases, news, background papers

**www.oct.on.ca** Ontario College of Teachers

**www.eqao.com** Testing in Ontario at grades 3, 6, 9 plus SAIP, TIMSS, OACTIP

## Committees

I've said it before, and I'll say it again: an elephants faithful ... sorry, that's another story! .... but it does involve math! Committees are what makes OAME! Read on.

**AWARDS:** Elaine Harvey was awarded a life membership at the meeting (after being discretely sent out of the room for the announcement and vote) and was genuinely surprised and appreciative to receive this prestigious recognition.

I remember sitting beside Bob Robinson when he was awarded this award. It's the kind of presentation that leaves the recipient speechless, at least for the moment.

The Life Award is not necessarily awarded every year, so those so recognized are very deserving. Congratulations are extended to Elaine along with a very big Thank You for the years of service she has contributed to OAME and the teachers of mathematics in Ontario.

Another tradition of the last directors meeting is the thanks given to retiring members and the best wishes given to members running for another term. Those retiring members included Vince DeLisi, Lynda Colgan, Gareth Haddock and Elaine Harvey. Their time and efforts are greatly appreciated, and we look forward to the volunteer efforts of our newly elected members.

The Publications committee reported that the Junior LAIM document was garnering rave reviews by the teachers using the activities and having brisk sales.

This committee is also monitoring the set up of the OAME website. When you visit the site, please let your colleagues know and offer your suggestions for additions and other hotlinks.

The various divisional committees continue to plan and produce supports for teachers at every level of education. The Primary/Junior people are part of the planning for the Leadership conference next fall. Traditionally, the Leadership Conference inspires aspiring leaders and provides school boards with additional sources of support. This is especially needed as education in Ontario undergoes continuing decline in its curriculum support personnel.

The Primary/Junior committee is also looking into authentic learning activities, especially with respect to answering the questions, "Where's the Math??" They are also in the planning stages for a Primary version of LAIM, and will be looking for classroom tested activities and sites to field test materials. Interested?? Contact your chapter rep (see the back page) or a member of the executive to get in line.

The Transition Years committee is exploring Assessment Units as a part of the challenge process. The Specialization Years committee passed out copies of its paper to the Ministry as part of OAME's response to the Secondary Reform initiatives. Special Projects is also an integral force in the Reform enterprise. You should be keeping in touch with your chapter rep or federation for up-to-date information on the directions of the Ministry. You should also follow the meetings at the Fields Institute through our president.

The OAME is well represented on the Ministry Expert Panel as they discuss the directions of the Secondary Reform.

The Futures committee has drafted Guidelines now for the Ontario Math Olympics to assist each year's volunteer group. There was also discussion regarding the hosting of a regional NCTM conference in the future. This also was Mickey Sandblom's last meeting. Mickey, as a past President of OAME, has continued to serve beyond his, first, retirement. He is now retiring again (a new rep should be announced by now).

Leadership '97 is in good hands with Judy Crompton organizing a program to Explore the Future. The main focus will be for teachers of Grades 1–6 plus secondary levels.

## The Future of OAME

Judy Crompton, President elect at the time, led the directors in a soul searching exercise focusing on how best to serve our members by stating that "mathematics education is in transition!" and then posing the question "What does this mean for OAME?" Directors then considered what OAME should be doing for its members and how we should position ourselves over the next few years.

Some of the suggestions included:

- be current with new directions and developments in mathematics education
- be current with all MET policies and directions in terms of their effect on mathematics education
- be pro-active in encouraging MET policy that will be consistent with our vision of ME
- be aware of the opinions of our membership on key issues
- reflect the various views of our membership in our actions
- keep our membership informed regarding our policies, actions and results

How are we doing? What suggestions do you have to add? Let Judy know. Au revoir and have a rejuvenating summer break.


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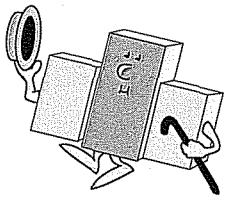
Susan Stuart

Margaret Warren

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Frances Schatz

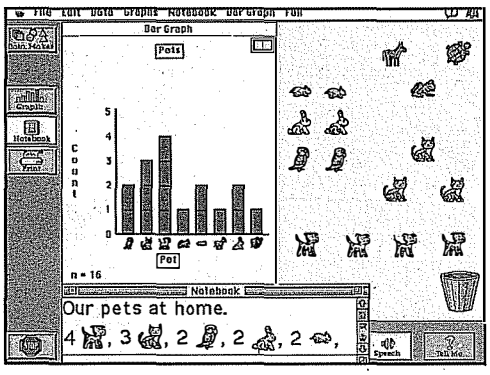
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# Graphers

## Graphers exposes young learners to data graphing!




**Graphers** helps young students learn about graphs and empowers them to use graphs to solve problems. **Graphers** is a tool for graphing data that can be counted or sorted. Students create pictorial data chosen from over 30 data sets and arrange the objects on the screen. They easily manipulate data and create their graphs using simple point, click, and drag motions with a mouse.

**Graphers** can represent students' data in a table or with six types of graphs:

- pictographs
- bar graphs
- circle graphs
- loops
- grid plots
- line graphs

The **Warm Up** section enables you or your students to quickly make, manipulate, and graph simple data. Among the data sets are:

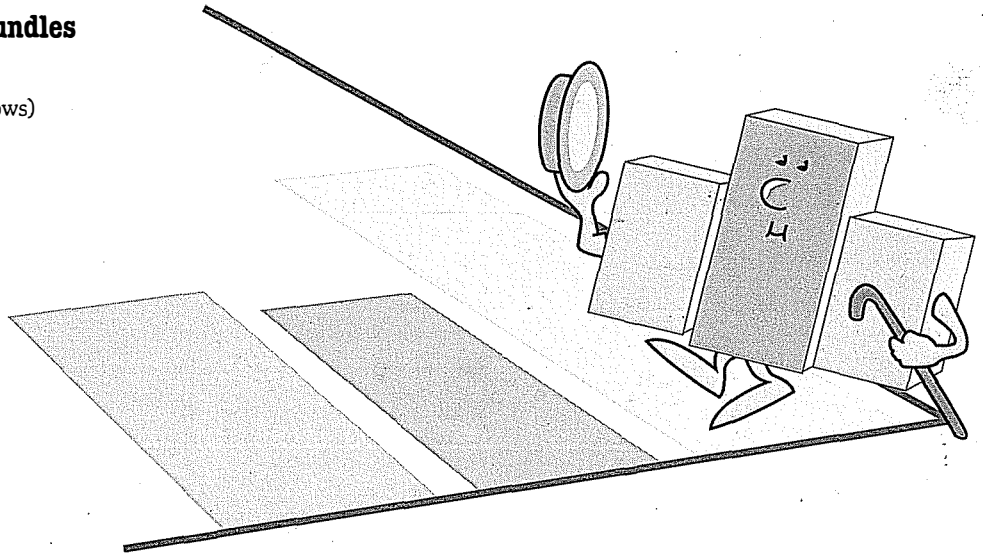
- colours
- pets
- sports
- fruit
- ice cream
- numbers

The **Work Out** section offers many more types of data and graphs. The additional sets of data include data with more than one attribute, called Sorting Data, and several sets of Time Data.

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