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FEBRUARY 1964

ONTARIO MATHEMATICS GAZETTE

sponsored by
THE SAMUEL BEATTY FUND
THE ONTARIO MATHEMATICS COMMISSION
THE ONTARIO ASSOCIATION OF
TEACHERS OF MATHEMATICS AND PHYSICS

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ONTARIO MATHEMATICS GAZETTE

E D I T O R I A L S T A F F

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E D I T O R I A L

This editorial is concerned about SHORTFALL in governmental approaches to the problems of higher education in Ontario.

At the present moment, there are about 35,000 university students in Ontario. Various estimates have been made of the number who will be in University in the year 1970. There seems to be no doubt that 100,000 is a conservative estimate.

Where will these students come from? That question is easily answered; they are now in secondary and elementary schools in the Province of Ontario. They are not hypothetical bodies; they are in class now and they can be counted. Indeed, the number is likely to go up if the percentage of students who continue on to University increases in the future.

Where will these students be educated? One would hope that they could be educated in Ontario Universities. And it is pleasant to report that the Universities of Ontario are doing their best to meet this "wave of students" or "baby bump" as it is called. It is not so pleasant to report that the Government of Ontario does not seem to be keenly aware of the problem, and is not doing its share to meet the problem.

And this brings us to the problem of SHORTFALL, an ingenious term (coined, I believe, by President Claude Bissell of the University of Toronto) which aptly describes governmental approach to the problems of University education.

The government is not facing the problem squarely.

Recently, the Ontario Council of University Faculty Associations presented a brief to the Premier of Ontario. This brief is published in full in the current issue of the Bulletin of the Canadian Association of University Teachers, and it is with difficulty that I restrain myself from making long quotations from it. In perfectly polite language, it sketches the problem, but the end result is an indictment of governmental University policies.

For the past number of years, a highly respected political figure has been premier of Ontario. While his administration made such progress, he did not seem keenly alive to the approaching University problem, and not enough progress was made in confronting it. Now that he has retired, he is in charge of the University Affairs Committee, the committee which advises the government on University Affairs. That is, he is in charge of one of the areas where his government did not shine.

Here are two statements from the brief.

1. "With all due respect, we observe that the members of the Advisory Committee cannot be expected to have an intimate knowledge

of University matters, since none of them has had experience as a university teacher or administrator, and they are all busy men with many other responsibilities, public and private."

Or as a newspaper said, they are part-time amateurs.

2. "The successful achievements of the Department of Education are due in no small measure to the fact that most of the senior civil servants have had teaching experience in the school system of Ontario - they know how it works, and they have continuing contacts with it. We are aware of no one in the Department of Education who has University teaching experience of this sort, or who has continuing contacts with the universities."

In short, the Government does not have adequate facilities for learning about University problems and assessing these problems.

The Government of Ontario has not faced the problem of the coming bulge in University education in the past. What is it doing now?

It does not appear to be paying much attention to the brief submitted by the Ontario Council of Faculty Associations.

In the year 1963, the Universities of Ontario requested 96 million dollars to meet the urgent problems of expansion occasioned by the increase. The government gave 70 million.

SHORTFALL FOR 1963 WAS 26 MILLION.

In the year 1964, the Universities of Ontario requested 140 million dollars to meet the expansion problems. Things were more urgent; time was a year later; the governmental SHORTFALL of 1963 had to be made up.

The government gave 101 million.

SHORTFALL FOR 1964 WAS 39 MILLION - UP 50 PERCENT.

University expansion and planning take time. One can not receive a grant in September and have something done with it by October. The present SHORTFALL has already seriously compromised the ability of the Universities to meet their problems five years from now.

A cynic might claim that, since the annual SHORTFALL is running at about thirty percent, the government will be satisfied if 70 percent of future university applicants can find places. I can not believe that such an unpolitical view is meant. Nor can I believe that the government wishes an education "only 70% as good as should be provided". I believe that the sole problem is lack of governmental information.

[continued on page 47]

ONTARIO ASSOCIATION

OF

TEACHERS OF MATHEMATICS AND PHYSICS

EASTER CONVENTION

PROGRAMME

TUESDAY, March 31, 1964

12:45 p.m. Registration and Examination of Book Displays

1:15 p.m. Minutes; Financial Statement; President's Remarks

1:30 p.m. Dr. Myron F. Rosskopf, Columbia University -
"Where Must Euclid Go".

2:30 p.m. Mr. J. F. Kinlin, Department of Education -
"Report from the Intermediate Curriculum Revision Committee".

3:15 p.m. Report on N.C.T.M.

3:45 p.m. Seminars

I Grades 7 and 8

Topic: Language and the Use of Sets in Elementary School Mathematics

Speakers: Mr. D. Mumford, Ontario College of Education
and
Mr. W. W. Bates, Mathematics Consultant, Board of Education, Toronto.

Chairman: Mr. W. W. Bates, Toronto.

II Grades 9 to 11

Topic: The Introduction of Rational Numbers by the Use of Fryer's Blocks

Speaker: Dr. K. D. Fryer, University of Waterloo.

Chairman: Mr. G. F. Osborne, Napanee.

III Grade 13

Topic: Sequences, Series and Limits

Speaker: Dr. R. G. Stanton, University of Waterloo.

Chairman: Mr. R. L. McNeil, Beamsville.

IV Vocational Mathematics

Topic: Technical Mathematics

Speakers: Panel headed by Mr. David Campbell, H.B.Beal Secondary School, London.

Chairman: Mr. W. Nediger, Clinton.

4:45 p.m. Adjournment

Wednesday, April 1, 1964

9:15 a.m. Dr. D. Clarke, University of Toronto -
"The Place of Logic in Secondary School Mathematics".

10:00 a.m. Dr. M. Rosskopf: "What is a Mathematical System".

11:00 a.m. Mr. R. Mitchell, Department of Education -
"Report of the Senior Curriculum Revision Committee".

11:45 a.m. Introduction of New Executive
New Business
Adjournment for Luncheon.

12:30 p.m. Luncheon at Bloor Street United Church.

SOME DIOPHANTINE PROBLEMS - II

R. G. Stanton

1. Introduction.

We shall, in this article, discuss linear Diophantine equations, in particular, the case of one equation in two unknowns and the case of two equations in three unknowns. We shall need to be able to use the Euclidean Algorithm for finding the g.c.d. of two integers [See, for example, Algebra and Solid Geometry (Stanton, Davis, Fryer), University of Waterloo Press]. We briefly recall this algorithm.

Let two integers, say 5627 and 13294, be given. Successive divisions at once give (using the remainder at any stage as the new divisor):

$$\begin{aligned} 13294 &= 5627(2) + 2040 \\ 5627 &= 2040(2) + 1547 \\ 2040 &= 1547(1) + 493 \\ 1547 &= 493(3) + 68 \\ 493 &= 68(7) + 17 \\ 68 &= 17(4) + 0 \end{aligned}$$

Since, at any stage, only three terms are involved in these relations, we find (using gcd as the abbreviation for greatest common divisor):

$$\begin{aligned} \text{gcd of } 13294 \text{ and } 5627 &= \text{gcd of } 5627 \text{ and } 2040 \\ &= \text{gcd of } 2040 \text{ and } 1547 \\ &= \text{gcd of } 1547 \text{ and } 493 \\ &= \text{gcd of } 493 \text{ and } 68 \\ &= \text{gcd of } 68 \text{ and } 17 \\ &= 17 \end{aligned}$$

In short, this process (the Euclidean Algorithm) has produced the gcd of the two original integers.

Now let us reverse the procedure and express 17 as a linear combination of the two original integers. We work back through the Euclidean Algorithm to obtain:

$$\begin{aligned} 17 &= 493 - 68(7) \\ &= 493 - 7[1547 - 3(493)] \\ &= 22(493) - 7(1547) \\ &= 22[2040 - 1547) - 7(1547) \\ &= 22(2040) - 29(1547) \\ &= 22(2040) - 29[5627 - 2(2040)] \\ &= 80(2040) - 29(5627) \\ &= 80[13294 - 2(5627)] - 29(5627). \end{aligned}$$

Our conclusion is that

$$17 = 80(13294) + (-189)5627.$$

If we were to carry this procedure through in the general case of literal coefficients, we would obtain the

Theorem. If d is the gcd of two integers a and b, then there exist integers u and v such that $d = au + bv$.

This theorem will allow us to discuss linear Diophantine equations.

2. One Equation in Two Unknowns.

Suppose one is given an equation

$$ax + by = c.$$

Two cases arise.

Suppose d is the greatest common divisor of a and b. Then we may write

$$a = da_1, \quad b = db_1,$$

and have

$$d(a_1x + b_1y) = c.$$

If d is not a divisor of c, there are no solutions. If d is a divisor of c, then $c = dc_1$ and we may divide by d to obtain an equation of the form

$$a_1x + b_1y = c_1$$

where a_1 and b_1 are reltaively prime. For example,

$$16x + 29y = 23.$$

The method of solution is to write

$$1 = \text{g.c.d. of } 16 \text{ and } 29.$$

$$\begin{aligned} \text{Thus } 29 &= 1(16) + 13 \\ 16 &= 1(13) + 3 \\ 13 &= 4(3) + 1 \end{aligned}$$

and $1 = 13 - 4(3)$
$= 13 - 4[16 - 13] = 5(13) - 4(16)$
$= 5[29 - 16] - 4(16) = 5(29) - 9(16)$.

Multiply by 23 to give

$$23 = 16(-207) + 29(115).$$

Thus $x = -207$, $y = 115$, is a <u>particular solution</u> of the given equation. In general, the Euclidean Algorithm provides a particular solution for any equation

$$a_1x + b_1y = c_1$$

where a_1 and b_1 are relatively prime. We now prove the

<u>Theorem.</u> The general solution of

$$a_1x + b_1y = c_1$$

is given by

$$x = x_1 - b_1t, \quad y = y_1 + a_1t ,$$

where (x_1,y_1) is a particular solution and t is an arbitrary (integral) parameter.

<u>Proof</u>. Let (x,y) be a general solution. Then

$$a_1x + b_1y = c_1$$

and

$$a_1x_1 + b_1y_1 = c_1$$

since (x_1,y_1) is a particular solution. Subtracting,

$$a_1(x - x_1) = -b_1(y - y_1).$$

Now a_1 and b_1 are relatively prime. Hence a_1 divides $y - y_1$, say,

$$y - y_1 = a_1t$$

Substitute, and we find $x - x_1 = -b_1t$. This shows that

$$x = x_1 - b_1t, \quad y = y_1 + a_1t,$$

where t is an integer.

Try this solution, and we find

$$\begin{aligned} a_1x + b_1y &= a_1(x_1 - b_1t) + b_1(y_1 + a_1t) \\ &= a_1x_1 + b_1y_1 \\ &= c_1. \end{aligned}$$

In short, this expression is a solution for any integral value of t.

Thus the general solution to the example

$$16x + 29y = 23$$

is

$$x = -207 - 29t, \quad y = 115 + 16t$$

where t is any integer.

3. Two Equations in Three Unknowns.

Let the equations be

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2.$$

It is necessary that the g.c.d. of a_1, b_1, c_1, must divide d_1; also the g.c.d. of a_2, b_2, c_2, must divide d_2.

For example,

$$27x + 18y + 45z = -63,$$
$$10x + 45y - 35z = -15.$$

Here the two necessary conditions are satisfied; so there may be solutions. We find by division that

$$3x + 2y + 5z = -7,$$
$$2x + 9y - 7z = -3,$$

where both g.c.d.'s are now unity.

Four cases may arise, only the last of which is particularly interesting.

Case 1. $a_1 : b_1 : c_1 : d_1 = a_2 : b_2 : c_2 : d_2$.

If these two quadruplet ratios are equal, there is really only one equation. For example,

$$3x + 4y + 7z + 5 = 0,$$
$$6x + 8y + 14z + 10 = 0.$$

We let $z = t$ and then get one equation in two unknowns,

$$3x + 4y = -5 - 7t.$$

Solve this by the methods of the last section to give

$$x_1 = 5 + 7t, \quad y_1 = -5 - 7t.$$

The general solution then is

$$x = 5 + 7t - 4u, \quad y = -5 - 7t + 3u, \quad z = t.$$

This gives a doubly-infinite family of solutions.

Case 2. $a_1 : b_1 : c_1 = a_2 : b_2 : c_2$ only,

that is, $a_2 = ka_1$, $b_2 = kb_1$, $c_2 = kc_1$,

but $d_2 \neq kd_1$.

In this case there is no solution. For example,

$$3x + 4y + 7z + 5 = 0$$
$$9x + 12y + 21z + 8 = 0$$

Combining these equations leads to the impossible result:
$3(5) - 8 = 0$.

Case 3. $a_1 : b_1 = a_2 : b_2$ only.

Thus $a_2 = ka_1$, $b_2 = kb_1$, $c_2 \neq kc_1$

(we presume that the unknowns are so ordered that the equal ratio coefficients are written first). Then, for example, we might have

$$3x + 2y - 7z + 16 = 0,$$
$$6x + 4y + 9z - 14 = 0.$$

Solving for z, we find a unique value, $z = 2$. Substitute $z = 2$ and we find

$$3x + 2y = -2,$$

an equation with particular solution $x_1 = -2$, $y_1 = 2$. The general solution then is

$$x = -2 - 2t, \quad y = 2 + 3t, \quad z = 2,$$

a one-parameter family of solutions having z uniquely determined.

Case 4. This is the interesting case; no coefficient ratios are equal. For example,

$$3x + 2y + 5z = -7,$$
$$2x + 9y - 7z = -3.$$

By elimination of z, we at once obtain

$$31x + 59y = -64.$$

Since the g.c.d. of 31 and 59 is 1, we obtain

$$1 = 31(1216) + 59(-640).$$

So we may take a particular solution

$$x_1 = 1216, \quad y_1 = -640.$$

The general solution then is

$$x = 1216 - 59r, \quad y = -640 + 31r.$$

Substitute in the original equations and we find that

$$z = -475 + 23r.$$

4. Conclusion.

Systems of equations involving more unknowns may be handled similarly. The reader might like to try the method outlined on the system

$$13x + 11y - 6z = 18,$$
$$12x - 5y + 7z = 26.$$

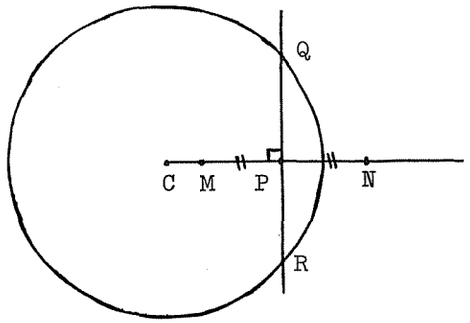
GUILE FOR THE GULLIBLE

Ross Honsberger - Northview Heights

The following fallacies are to be found, among others, in Professor E. A. Maxwell's delightful little book "Fallacies in Mathematics", published in 1963 by the Cambridge University Press (95 pages, 95 cents). Their resolutions are deliberately omitted here. If you are unable to figure them out, it is suggested that you get the book and see for yourself.

I. - To Prove That Any Given Point Inside A Circle Really Is On The Circumference

Let C be the centre of a circle and M a given point inside. Draw the line CM and mark on it the point N such that $CN = \frac{r^2}{CM}$, r the radius of the circle. Since $CM < r$, the value of CN will exceed r, placing N outside the circle.



Now draw the right bisector of MN to meet MN at P and the circle at Q and R. Then

$CM.CN = (CP-MP)(CP+PN) = (CP-MP)(CP+MP)$, since $PN = MP$.

i.e. $CM.CN = CP^2-MP^2 = (CQ^2-QP^2) - (MQ^2-QP^2)$, by Pythagoras.

i.e. $CM.CN = CQ^2-MQ^2 = r^2 - MQ^2$, since $CQ^2 = r^2$.

But, by the definition of CN, $CM.CN = r^2$. Consequently, we have $r^2 = r^2 - MQ^2$, giving $MQ^2 = 0$ and $MQ = 0$.

Therefore M is at Q on the circumference!

II - To Prove That Zero Is Greater Than Zero

Let $X = 1 + 1/3 + 1/5 + 1/7 + \ldots$ and

$Y = 1/2 + 1/4 + 1/6 + 1/8 + \ldots$.

Multiplying Y by 2, we get $2Y = 1 + 1/2 + 1/3 + 1/4 + 1/5 + \ldots$.
That is, $2Y = X + Y$. As a result, $Y = X$ and $X - Y = 0$.

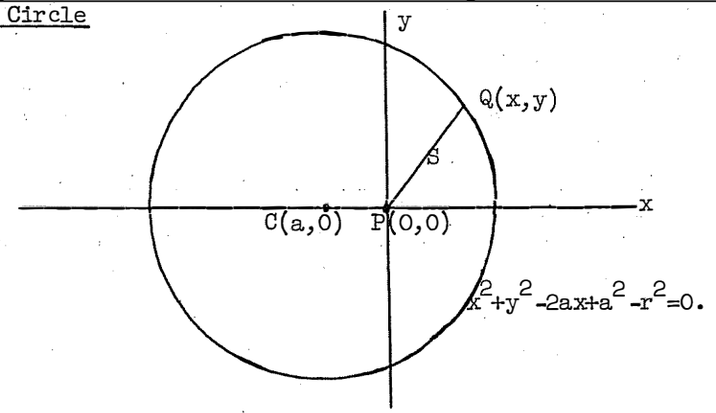
But upon subtracting Y from X we get

$$(1 - 1/2) + (1/3 - 1/4) + (1/5 - 1/6) + \ldots ,$$

which is a series consisting entirely of positive terms. Thus $X - Y > 0$.

Combining results we have $0 > 0$.

III - To Prove That There Is No Point On A Circle Which Is Nearest To (Nor A Point Which Is Farthest From) A Given Point Inside The Circle



Let the circle have centre C, radius r, and let P be any point inside the circle. Take coordinate axes so that CP is the x-axis and P the origin. Then the equation of the circle is, letting C be (a,0),

$$(x - a)^2 + y^2 = r^2, \quad \text{or} \quad x^2 + y^2 - 2ax + a^2 - r^2 = 0.$$

For any point Q(x,y) on the circle, the distance $s = QP$ is simply the square root of $(x^2 + y^2)$, since P is the origin. Then $s^2 = x^2 + y^2$. But Q(x,y) lies on the circle, hence the co-ordinates of Q satisfy the equation of the circle, giving

$$s^2 = x^2 + y^2 = 2ax - a^2 + r^2, \quad \text{where } a \text{ and } r \text{ are fixed.}$$

Then, differentiating, $2s\frac{ds}{dx} = 2a$, giving $\frac{ds}{dx} = \frac{a}{s}$.

Now clearly "a" is not zero, and "s" never becomes zero nor infinite. Consequently, ds/dx never takes the value zero. Thus $s = QP$ never attains a maximum or minimum!

IV - To Prove $2 = 1$

The expansion for the binomial theorem is

$$(x+y)^n = x^n + nx^{n-1}y + [n(n-1)/2]x^{n-2}y^2 + \ldots + nxy^{n-1} + y^n.$$

Now for $n = 0$, we get $(x+y)^0 = x^0 + 0 + 0 + \ldots + 0 + y^0$, since all the terms in the expansion except the first and last have a factor n. This gives

$$1 = 1 + 1, \quad \text{or} \quad 2 = 1.$$

V - To Prove $\pi = 0$

We begin with the formula $e^{iy} = \cos y + i\sin y$ from the field of complex numbers. For $y = 2\pi$, we get

$$e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1 + 0 = 1.$$

The $e^{ix} = e^{ix}.e^{2\pi i}$ (since $e^{2\pi i} = 1$) $= e^{i(x+2\pi)}$.

Raising each side to the i^{th} degree, we get

$e^{-x} = e^{-(x + 2\pi)}$, since $i^2 = -1$.

Now multiplying each side by $e^{x+2\pi}$, which is not zero for any x, we get $e^{2\pi} = e^0$, giving $2\pi = 0$ and $\pi = 0$.

VI - <u>To Prove $2 = 1$</u>

Clearly $$\int_1^2 f(x)dx = \int_0^2 f(x)dx - \int_0^1 f(x)dx.$$

For a moment let's consider just the first integral on the right, $\int_0^2 f(x)dx$. If we substitute $2z$ for x, we get $\int_0^2 f(x)dx$ $= 2\int_0^1 f(2z)dz$. ($dx = 2dz$, and the range of integration would change to (0 to 1) because as x goes from 0 to 2, z goes only from 0 to 1.). Now it doesn't matter what letter is used to name the variable;

$$2\int_0^1 f(2z)dz = 2\int_0^1 f(2x)dx.$$

Thus $$\int_1^2 f(x)dx = 2\int_0^1 f(2x)dx - \int_0^1 f(x)dx.$$

For functions which have the property that $f(2x) = \frac{1}{2} f(x)$, this becomes

$$\int_1^2 f(x)dx = 2\int_0^1 \tfrac{1}{2} f(x)dx - \int_0^1 f(x)dx = 0.$$

Now the function $f(x) = \frac{1}{x}$ possesses this property that $f(2x) = \frac{1}{2}f(x)$. Consequently,

$\int_1^2 \frac{dx}{x} = 0$. That is, $\log 2 - \log 1 = 0$, giving

$\log 2 = \log 1$ and $2 = 1$.

Dr. Maxwell concludes his book with a few "howlers" which he believes to be the products of innocent students. The five which follow may give you a laugh.

I - In solving $(x + 3)(2 - x) = 4$, the procedure went

$x + 3 = 4$ giving $x = 1$, or

$2 - x = 4$ giving $x = -2$, which is correct.

II - In determining the largest angle of the triangle with sides 4, 7, 9, the solution went -

$\sin A = 9/7$

$= 1.2857.$

Now $1 = \sin 90^\circ$

and $.2857 = \sin 16^\circ 36'$

Therefore $1.2857 = \sin 106^\circ 36'$ giving

$A = 106^\circ 36'$, which is correct.

III - In proving the converse of the theorem of Apollonius, that is, if W is a point in the base YZ of $\triangle XYZ$ such that $XY^2 + XZ^2 = 2XW^2 + YW^2 + WZ^2$, then W is the midpoint of YZ, the proof proceeded:

let K be the midpoint of YZ. By Apollonius' theorem,

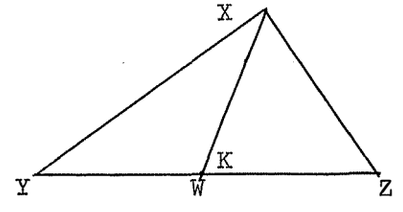
$XY^2 + XZ^2 = 2XK^2 + YK^2 + ZK^2$ $(YK = ZK)$.

Then, using the given,

$2XW^2 + YW^2 + ZW^2 = 2XK^2 + YK^2 + ZK^2$

Factoring gives $W(2X^2 + Y^2 + Z^2) = K(2X^2 + Y^2 + Z^2)$.

Dividing out, we get $W = K$. Thus W is the midpoint of YZ.



IV - A simplification problem went -

$$\frac{2^3 \cdot 4^8 \cdot 16^{-n}}{16^2 \cdot 4^{-2n}} = \frac{22^{11-n}}{20^{2-2n}} = \frac{22^{11-n}}{20^{-n}} = 22^{11-n} - 20^n = 2^{11},$$

which is correct.

V - In solving $(5 - 3x)(7 - 2x) = (11 - 6x)(3 - x)$ we have

$5 - 3x + 7 - 2x = 11 - 6x + 3 - x$

$12 - 5x = 14 - 7x$

$2x = 2$

$x = 1$, which is correct.

THE DIRECTION COSINE FORM OF THE LINE EQUATION

[Of all the forms of the line equation, the symmetrical direction cosine form, which generalizes to 3-dimensional geometry, is probably the most important. The following note will provide some exercises for superior Grade XIII students.]

Suppose that a line ℓ_1 is given in the Cartesian plane; draw a line ℓ_2 through the origin parallel to ℓ_1. Then ℓ_2 will make an angle α with the positive x axis and an angle β with the positive y axis; the angle α may be acute (Figure 1) or obtuse (Figure 2), but the angle β is always acute.

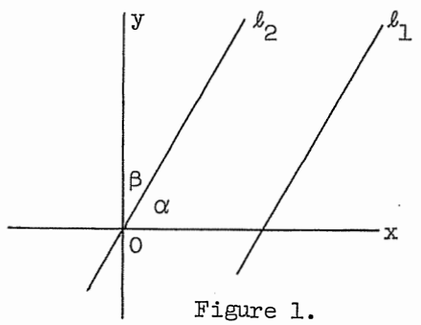


Figure 1.

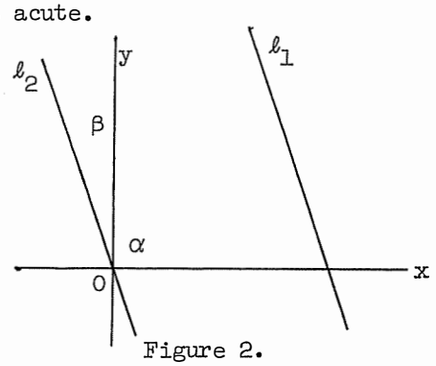


Figure 2.

The angles α and β are called the direction angles of the line and the quantities $\cos\alpha$ and $\cos\beta$ are called the direction cosines of the line.

Problem. Find the equation of the line through (x_1,y_1) with direction angles α and β.

Solution. In Figure 3, we are given ℓ_1, through $P_1(x_1,y_1)$.

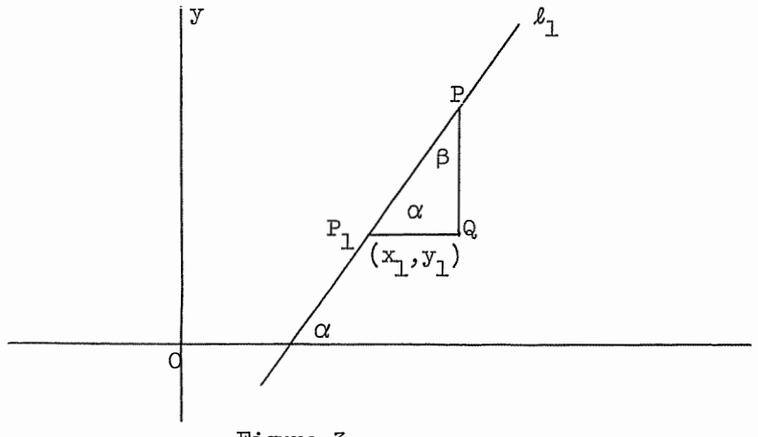


Figure 3.

Let $P(x,y)$ be a general point on ℓ_1. Draw the run P_1Q and the rise QP. Then it is clear that

$$\sphericalangle PP_1Q = \alpha,$$

$$\sphericalangle P_1PQ = \beta.$$

Consequently, if we assign the letter d to the distance P_1P, we find

$$P_1Q = x - x_1 = d \cos \alpha,$$
$$QP = y - y_1 = d \cos \beta.$$

We thus obtain

$$\boxed{\begin{aligned} x &= x_1 + d \cos \alpha \\ y &= y_1 + d \cos \beta \end{aligned}}$$

as the equations of the line ℓ_1.

Note that x_1, y_1, $\cos \alpha$, $\cos \beta$, are fixed, but d is a parameter which varies; as d changes, we get every point on the line.

Example 1. Find the direction cosine form of the equation of the line through $(2,3)$ with direction angles $\alpha = 60^\circ$, $\beta = 30^\circ$.

Solution. The line is

$$x = 2 + d \cos 60^\circ = 2 + \frac{1}{2} d,$$

$$y = 2 + d \cos 30^\circ = 2 + \frac{\sqrt{3}}{2} d.$$

Notice that we can plot points on the line by assigning values to d; if $d = 4$, we get the point $(4, 2 + 2\sqrt{3})$ which is 4 units from the initial point $(2,3)$; if $d = -2$, we get the point $(1, 2 - \sqrt{3})$ which is 2 units from $(2,3)$ in the other direction along the line.

Example 2. Find the direction cosines of the line $3x + 4y - 13 = 0$.

Solution. The slope of the line is $-3/4$. But the slope is equal to $\tan \alpha$. From a diagram, Figure 4, it is clear that

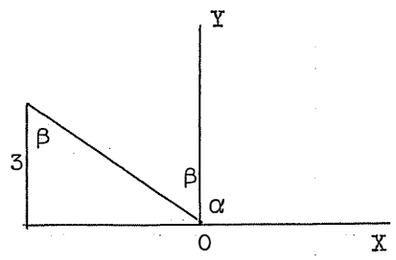


Figure 4.

$\cos\alpha = -4/5$. But (Figure 4) $\beta = \alpha - 90^\circ$, and so $\cos\beta$ may be read from the diagram as $3/5$.

Example 3. Prove $\cos^2\alpha + \cos^2\beta = 1$.

Solution. There are 2 cases.

(1) Consider Figure 1 (α acute).

Then $\cos^2\alpha + \cos^2\beta = \cos^2\alpha + \cos^2(90^\circ - \alpha)$

$= \cos^2\alpha + \sin^2\alpha$

$= 1.$

(2) Consider Figure 2 (α obtuse).

Then $\cos^2\alpha + \cos^2\beta = \cos^2\alpha + [\cos(\alpha - 90^\circ)]^2$

$= \cos^2\alpha + [\cos(90^\circ - \alpha)]^2$

$= \cos^2\alpha + \sin^2\alpha$

$= 1.$

Example 4. Use the direction cosine form of the line equation to prove that if a secant from P_1 to a circle meets the circle at Q and R, then $P_1Q.P_1R$ is constant.

Solution. Take axes through the centre of the circle so that the circle has equation $x^2 + y^2 = r^2$.

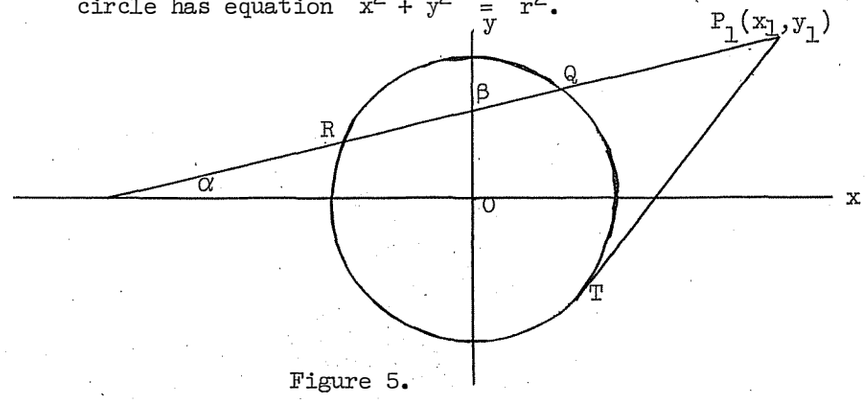


Figure 5.

Let P_1QR be a secant with direction angles α and β. Then this secant has equations

$$x = x_1 + d\cos\alpha,$$
$$y = y_1 + d\cos\beta.$$

Solve the line with the circle; the result is

$$x_1^2 + 2dx_1\cos\alpha + d^2\cos^2\alpha + y_1^2 + 2dy_1\cos\beta + d^2\cos^2\beta - r^2 = 0,$$

which simplifies to

$$d^2 + 2d(x_1\cos\alpha + y_1\cos\beta) + x_1^2 + y_1^2 - r^2 = 0.$$

This is a quadratic in d, and the two roots will be the distances P_1Q and P_1R from the point P_1 to the two intersections Q and

R of the line and the circle. So the product of the roots, namely,

$$P_1Q.P_1R = x_1^2 + y_1^2 - r^2,$$

which is constant.

As a corollary, we note that this constant is equal to the square on the tangent P_1T (letting Q and R approach one another).

Example 5. Find the distance from $P_1(x_1,y_1)$ to $Ax + By + C = 0$ along a line with direction cosines given by $\cos \alpha = \lambda$, $\cos \beta = \mu$.

Solution. Consider Figure 6.

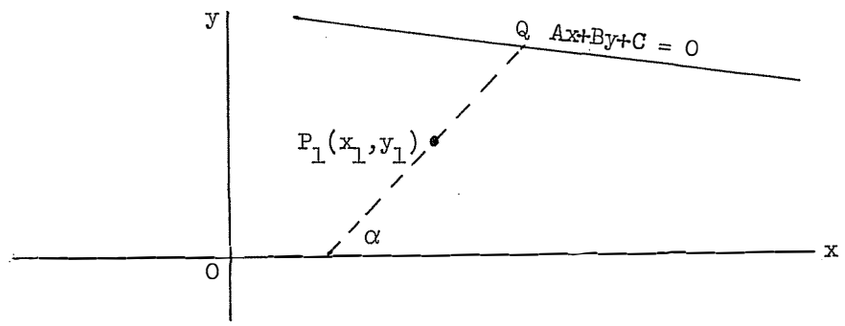


Figure 6.

The line P_1Q has equations

$$x = x_1 + d \cos \alpha,$$
$$y = y_1 + d \cos \beta.$$

Solve with $Ax + By + C = 0$ to give

$$A(x_1 + d \cos \alpha) + B(y_1 + d \cos \beta) + C = 0.$$

Then

$$d = -\frac{Ax_1 + By_1 + C}{A \cos \alpha + B \cos \beta}.$$

We have thus obtained the value of the parameter d corresponding to the point Q of intersection of the two lines. But the parameter d represents the distance along P_1Q measured from P_1. Hence

$$P_1Q = -\frac{Ax_1 + By_1 + C}{A \cos \alpha + B \cos \beta}.$$

Corollary. Suppose P_1Q is the perpendicular distance. Then $\tan \alpha$ is equal to the slope of P_1Q, namely, B/A. We find

$$\cos \alpha = \frac{1}{\sec \alpha} = \frac{1}{\pm\sqrt{1 + B^2/A^2}} = \frac{\pm A}{\sqrt{A^2 + B^2}}$$

$$\cos \beta = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \frac{A^2}{A^2 + B^2}} = \frac{\pm B}{\sqrt{A^2 + B^2}}.$$

Note that $\frac{\cos\beta}{\cos\alpha} = \frac{B}{A}$.

Substituting, the perpendicular distance is

$$\pm \frac{Ax_1 + By_1 + C}{\frac{A^2}{\sqrt{A^2+B^2}} + \frac{B^2}{\sqrt{A^2+B^2}}} = \pm \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} .$$

We should note that the distance may come out positively or negatively depending upon which side of the line P_1 lies; from one side, P_1Q is positive; from the other side P_1Q is negative. Which side is which is completely arbitrary (for example, it is reversed if we write a line as $3x - 4y - 8 = 0$ rather than $4y - 3x + 8 = 0$).

Exercises.

1. Prove in both cases, α acute and α obtuse, that $\tan\alpha = \cos\beta/\cos\alpha$.
2. Find the direction cosines of the following lines:
   (a) with slope $-20/29$;
   (b) with x. intercept 5, y intercept 6;
   (c) with equation $5x - 12y + 6 = 0$;
   (d) perpendicular to $3x + 4y - 6 = 0$;
   (e) through $(2,1)$ and $(-3,4)$;
   (f) with angle of inclination $\pi/3$;
   (g) perpendicular to $y + 2x = 0$.
3. Find the direction cosine form of the equation of the line through $(-2,-1)$ and
   (a) parallel to $2y - x + 6 = 0$;
   (b) perpendicular to $x + 3y + 1 = 0$;
   (c) through $(0,-5)$;
   (d) parallel to the y axis;
   (e) parallel to the x axis;
   (f) making equal intercepts on the two coordinate axes.
4. Find the distance from
   (a) $(-3,2)$ to $3x - 4y + 8 = 0$ along a line with direction angles 120° and 30°;
   (b) $(-1,-5)$ to $x + y - 2 = 0$ along a line with direction cosines $\frac{-5}{13}$ and $\frac{12}{13}$;
   (c) $(0,1)$ to $6x + 5y + 1 = 0$ along a line perpendicular to $6x + 5y + 1 = 0$;

(d) $(3,-1)$ to $5x - 4y + 6 = 0$ along a line of inclination $\alpha = \mathrm{Tan}^{-1} -4/5$.

5. Find the distance from $(-2,-8)$ to $x + 3y + 6 = 0$ along
   (a) a line parallel to $2x - y + 5 = 0$;
   (b) along a line of inclination 45°.

6. Write the following lines in a direction cosine form [you will have to pick a point (x_1,y_1) on the line].
   (a) $3x - y + 8 = 0$;
   (b) $6x + 5y - 2 = 0$;
   (c) $x - 9 = 0$;
   (d) $y = 3x$;
   (e) $y = 6$.

7. Eliminate the parameter and write the following lines in the form $Ax + By + C = 0$:
   (a) $x = 2 - \frac{1}{2} d$, $y = 4 + \frac{\sqrt{3}}{2} d$;
   (b) $x = -1 + \frac{3}{5} d$, $y = \frac{4}{5} d$;
   (c) $x = 6 - \frac{5}{13} d$, $y = 1 + \frac{12}{13} d$.

8. Find the points on the line
   $$x = 2 - \tfrac{3}{5} d, \quad y = -3 + \tfrac{4}{5} d,$$
   corresponding to the values $d = \pm 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 10, \pm \frac{1}{2}$. Plot these on a large graph, and draw the line.

9. Plot the lines given in parametric form as
   (a) $x = 3 + 2t$, $y = 6 - 5t$;
   (b) $x = 4 - t$, $y = 5 - t$;
   (c) $x = 2 + 6t$, $y = 4$.
   In each case, write the line in the form $Ax + By + C = 0$ after plotting it.

10. Write the following parametric equations in the direction cosine form, and state in each case the relation between the given parameter t and the distance parameter d:
   (a) $x = 1 - 3t$, $y = 2 + t$;
   (b) $x = 4 + 6t$, $y = 3 - 8t$;
   (c) $x = -2t$, $y = 2 - 5t$;
   (d) $x = 5 - 3t$, $y = 1 + 7t$.

11. Use the direction cosine form of the line equation to find
   (a) the locus of a point $P_1(x_1,y_1)$ where P_1 moves according to the law that it lies on the midpoint of a moving

chord (of fixed inclination α) of the circle $x^2 + y^2 = r^2$;

(b) the locus of $P_1(x_1,y_1)$ moving so that it is always the midpoint of a chord of fixed inclination α for the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

(c) the locus of $P_1(x_1,y_1)$ moving so that it is always the midpoint of a chord of fixed inclination α for the curve $y^2 = 4px$.

12. Use the direction cosine form of the line equation to investigate whether the law

$$P_1Q \cdot P_1R = \text{constant}$$

established for a secant to a circle [Example 4] also holds for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

13. Use the direction cosine form of the line equation to establish that, provided $h^2 - ab \neq 0$, the general second-degree curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is characterized by possessing a "centre" $P_1(x_1,y_1)$ with the property that <u>every</u> chord through P_1 is bisected by P_1 (write the equation of a chord of inclination α and show that the bisection condition leads to two equations for x_1 and y_1; solve these).

14. Use the direction cosine form of the line equation to find:

(a) the locus of the midpoint $P_1(x_1,y_1)$ of a moving chord of length 4 units whose ends always lie on the axes;

(b) the locus of the midpoint $P_1(x_1,y_1)$ of a moving chord of length 6 units whose ends always lie on the lines $y - x = 0$ and $x = 3$.

N O T I C E

The Gazette has received a communication from Mr. Henry J. Campbell, Secretary-Treasurer of the Ontario Mathematics Commission (1260 Bay Street, Toronto 5, Ontario) in which he informs us that they have a fair stock of the following back copies of the Ontario Mathematics Gazette: Vol. 1, Nos. 2 and 3; Vol. 2, Nos. 1 and 2. Mr. Campbell is of the opinion that some of the younger teachers might appreciate having a copy of these numbers. Copies can be obtained by writing to Mr. Campbell.

A PROFUSION OF SQUARES

Norman Miller

An old "chestnut" in geometry, that appears in some high school text-books, is to describe a square whose sides shall pass, one through each of four assigned points in a plane. A side shall here mean an infinite straight line containing two adjacent vertices of the square. My attention was recalled to this problem recently by a solution of it given by Fred M. Campling, a pupil in Grade XI. His solution, which is reproduced in Section 1 below, was the inspiration for this paper.

1. Solution by synthetic geometry.

Let P, Q, R, S be the four given points. They can be grouped in pairs in any one of three ways; we shall take PS and QR. On PS and QR as diameters, describe circles and call their centres A and B respectively. Draw radii AC and BD at right angles to these respective diameters. Join CD and let this line cut the circles again in E and F. Then EF is a diagonal of a required square (Fig. 1).

[See Figure 1]

Once the diagonal is known, the square is readily constructed. The proof that the figure is a square can be left to the reader with the hint that each of the angles marked in Fig. 1 is half a right angle.

2. The multiplicity of solutions.

Since each of the right bisectors AC and BD can be drawn in either of two opposite directions, four combinations of these radii can be taken. These give four different constructions and provide four squares satisfying the required incidence conditions. The points P and S lie on adjacent sides of each of these squares as also do Q and R. It has been already noted that the points P, Q, R, S can be paired in three ways (P can be taken with Q or R or S). Each of these three choices will, by the foregoing construction, provide four squares. This suggests that the number of squares determined by the four points in the manner demanded may be twelve. One should, if he is interested, make these twelve constructions on twelve different sheets. When you have marked the four points on one sheet, place a blank sheet under it and locate the same four points on the blank sheet by means of a needle. When the twelve figures were drawn, it was found that they fell into six pairs in each of which, though the construction lines were very different, the squares were identical. This was made apparent by superposing one sheet on another and holding them up to the light. The result of this experiment suggests but does not prove that there are, at least in general, six and only six different squares. To replace conjecture by proof, analytic geometry is clearly called for.

3. Analysis of the problem.

Let P, Q, R, S be four given points through which the sides of a square, in circular order, are to pass. Set up axes as shown in Fig. 2 so that the given points are (0,0), (a,0), (b,c), and (d,e).

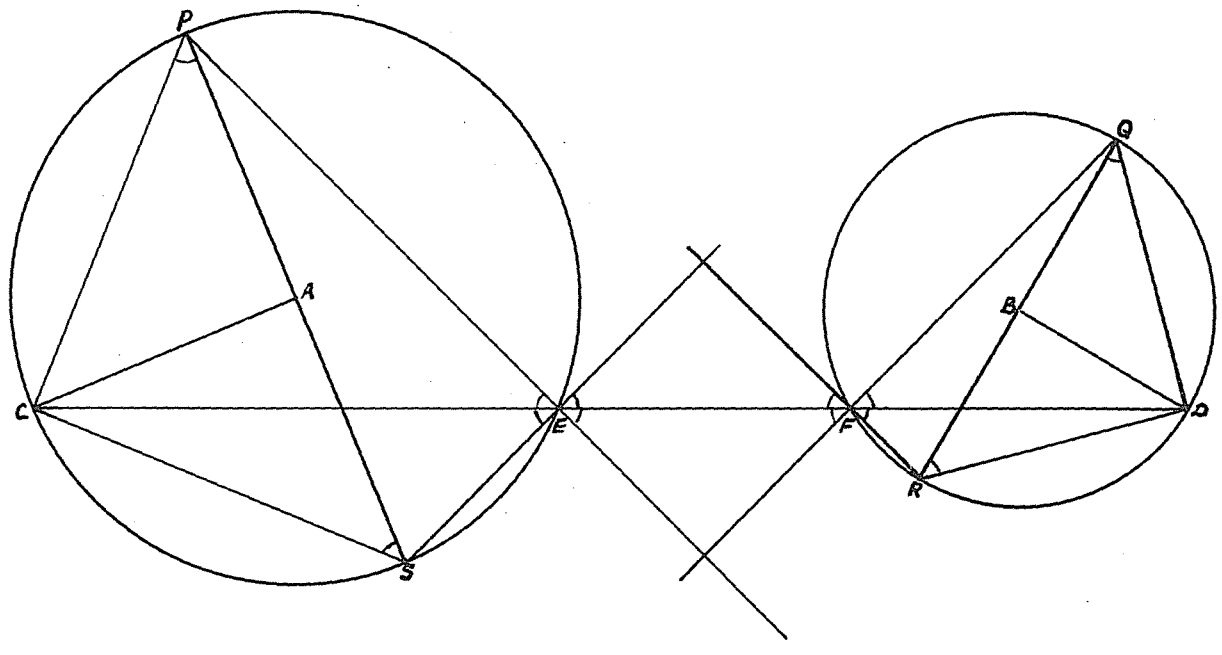


Figure 1.

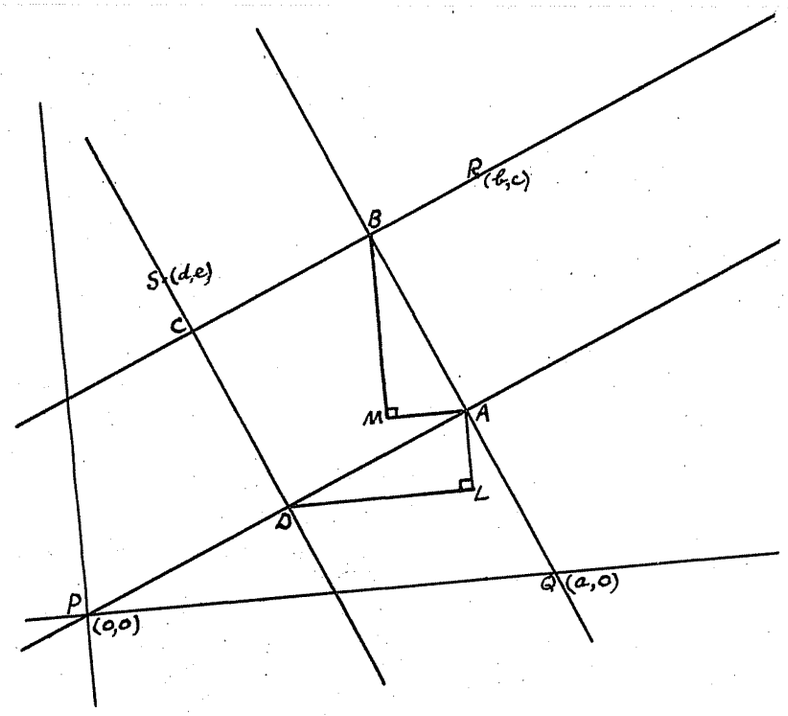


Figure 2.

Five constants, a, b, c, d, e, are involved. Let m denote the slope of the sides that are to pass through P and R and $-1/m$ the slope of the sides through Q and S. A rectangle can then be constructed with any value of m. The problem is to express m in terms of a, b, c, d, e, in order that this rectangle be a square.

By using the equations of the four sides of the rectangle ABCD (Fig. 2) we find the following co-ordinates of A, B, and D:

$$A: \left(\frac{a}{m^2+1}, \frac{ma}{m^2+1}\right); \quad B: \left(\frac{a-mc+bm^2}{m^2+1}, \frac{m(a-b)+c}{m^2+1}\right); \quad D: \left(\frac{me+d}{m^2+1}, \frac{m^2e+md}{m^2+1}\right).$$

The condition that the rectangle ABCD be a square is that the similar triangles AMB and ALD be congruent, or that DL = MB. It is necessary and sufficient that

$$x_A - x_D = y_B - y_A$$

or that

$$\frac{a - me - d}{m^2+1} = \frac{ma - mb + c - ma}{m^2+1}$$

whence $$m(b - e) = c - a + d \qquad (1)$$

If $b - e \neq 0$, this equation determines a unique value for m. Hence, in this general case, just one square exists whose sides pass through P, Q, R, S, in a given circular order. Since there are

six possible circular orders of the four letters, there are, in general, six squares that satisfy the required incidence conditions.

If $b - e = 0$ and $c - a + d \neq 0$, then an exception arises in the algebra but not in the geometry; the square studied above still exists, its sides being parallel, respectively, to the coordinate axes.

4. Infinite families of squares satisfying the same incidence conditions.

It remains to consider the case

$$b - e = 0 \quad \text{and} \quad c - a + d = 0 \qquad (2)$$

in which (1) is satisfied by any value of m. Hence an infinite family of squares exists, whose sides satisfy the given incidence conditions with any four points whose coordinates satisfy (2). These conditions can be translated into simple geometrical terms. They imply that $PM = NS$ and $NQ = MR$ (Fig. 3) from which it

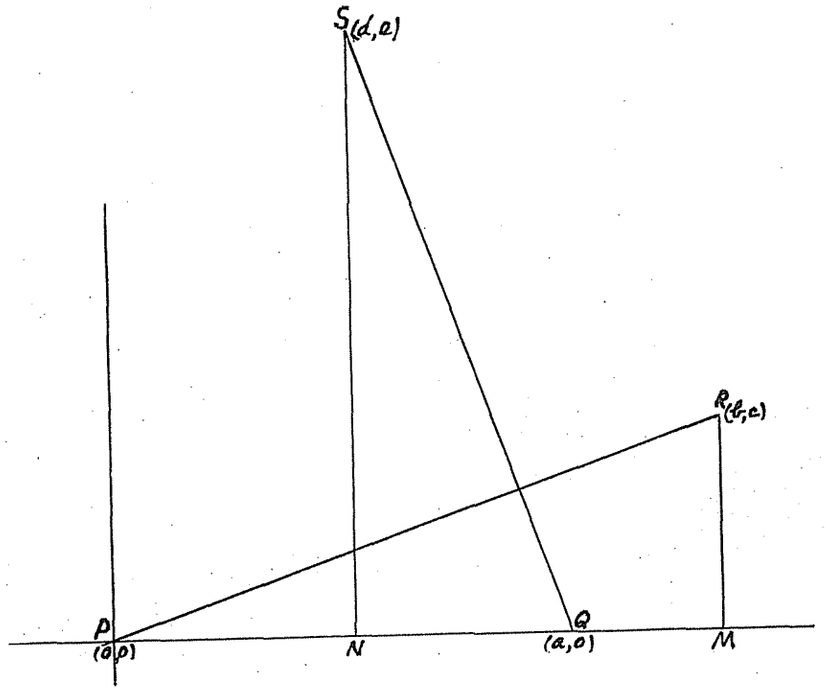


Figure 3.

follows that PR = SQ. Also the product of the slopes of PR and SQ is

$$\frac{c}{b} \cdot \frac{e}{d-a} = \frac{c}{b} \cdot \frac{b}{-c} = -1.$$

Hence SQ and PR are perpendicular. That the line segments PR and SQ are both equal and perpendicular is the geometrical meaning of equations (2).

Now let PR and SQ be two segments anywhere in the plane with the two properties of equal length and perpendicular direction. Let a pair of parallel lines in any chosen direction be drawn through P and R and a second pair of parallel lines perpendicular to the first pair through S and Q. It is intuitively clear and is easily proved independently that these two pairs of parallel lines will enclose a square. Figure 4 shows two such equal and perpendicular segments PR and SQ with five squares of the infinite family whose sides satisfy the required incidence conditions. Of this family one square degenerates to a point, the intersection of PR and QS.

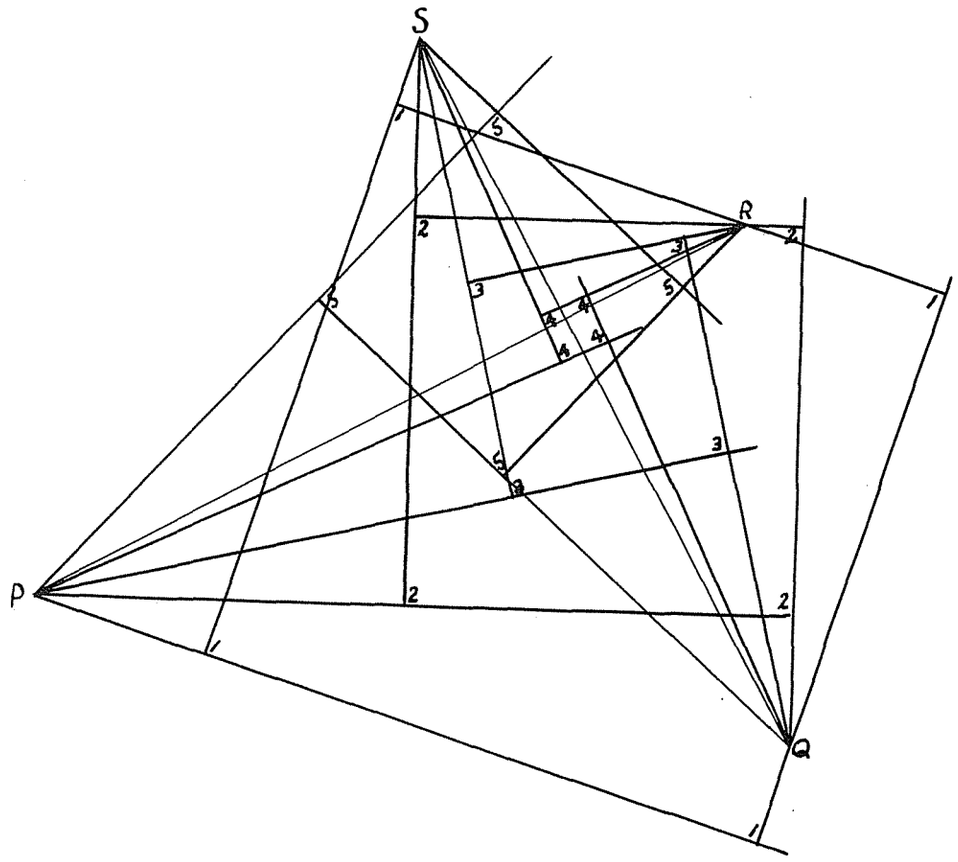


Figure 4.

Are there in our schools any disciples of Walt Disney who might make an animated cartoon showing the square changing its size and position as the lines through the four points rotate?

We follow the exceptional case a step further. Let three points P, Q, R be arbitrarily fixed. Through Q let a line be drawn perpendicular to PR. On this line cut off two segments QS' and QS" each equal to PR. Then the two sets of points {P,Q,R,S'} and {P,Q,R,S"} have the property that, for each, an infinite set of squares exists whose sides have the required incidence properties with the four points and in which also P and R are on opposite sides of each square. If it is required that P and Q be on opposite sides or that Q and R be on opposite sides, there will exist similarly two infinite families of squares for each requirement. For three given points, this discussion has disclosed the existence of six infinite families of squares in which every square has three sides incident with the three given points and in which the fourth sides of the squares of any one family pass through a fourth fixed point.

5. An alternative construction.

The foregoing discussion leads to an alternative construction for the six squares that have the required incidence properties with four non-specialized points*. If the points are labelled P, Q, R, S, choose two of these to lie on opposite sides. This can be done in three ways. Consider first the three points P, Q, R and take P and R to be an opposite pair. Then find fourth points S' and S" by the construction of the preceding paragraph. Then, taking S'S or S"S as a side of a square, we can complete a square whose sides pass, respectively, through P, Q, R, S. Thus we get two squares with P and R on opposite sides. Likewise two other squares have P and Q on opposite dies and two others have P and S on opposite sides.

It is an instructive exercise, using this method, to draw the six resulting squares. This can be done on six sheets by locating the four points from one sheet to the others by means of a needle as suggested in 2 above. If the squares are all drawn on one figure, care should be taken not to have the points too close together on account of the many construction lines; it is also advisable to use different colours of ink or crayon for the different squares.

A student or teacher who likes geometry will find it interesting to draw figures for some particular cases. The following exercises are suggested.

Exercises.

1. Make the circle construction given in Section 1 for the exceptional case of Section 4 to see how the multiplicity of squares arises.
2. Construct the six squares when the right bisector of PS is also the right bisector of QR.
3. Make a circle construction for the case in which the right bi-

* See, for example, N.F.Dupuis, Elementary Synthetic Geometry,New York, Macmillan, 1906, page 82.

sector of PS coincides with the right bisector of QR and the distance between PS and QR is half the sum of their lengths. Show how the infinite set of squares arises.

4. Take P,Q,R,S in a straight line. In this case three of the squares are reflections of the other three in this straight line.

INVERSE TRIGONOMETRIC FUNCTIONS

If a curve is reflected about the line $y = x$, the roles of x and y are interchanged. Thus the curve $y = x^2$ becomes $x = y^2$ if it is reflected about the line $y = x$. Similarly, if $y = \sin x$ is reflected about this straight line, a new curve $x = \sin y$ or $y = \sin^{-1} x$ is the result [see the article on symmetry in Vol. 2, No. 1 of the Gazette].

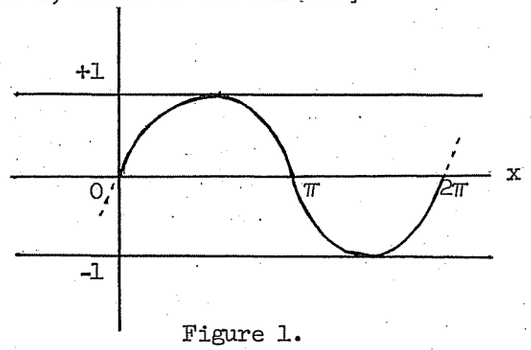


Figure 1.

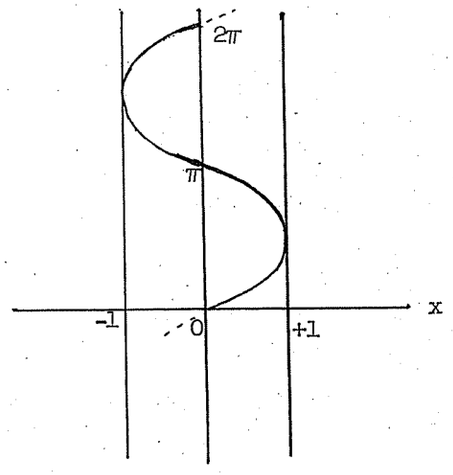


Figure 2.

Figure 1 represents one period of the curve $y = \sin x$, while Fig. 2 is the graph of this curve after a reflection about the line $y = x$; in other words, it is part of the graph of $y = \sin^{-1}x$. Since the sine curve extends indefinitely in the horizontal direction to both right and left of the y-axis, the curve $y = \sin^{-1}x$ extends vertically indefinitely both below and above the x-axis.

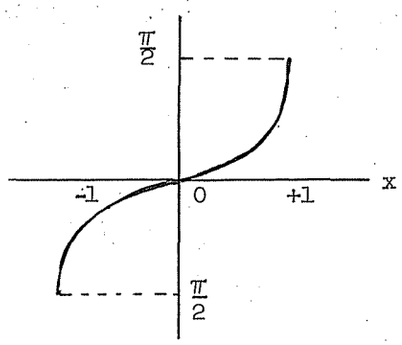


Fig. 4.

We say that the equation $y = \sin x$ defines y as a <u>function</u> of x, since a vertical line parallel to the y-axis cuts the sine curve in Fig. 1 in only one point. However, a vertical line would cut the curve in Fig. 2 in an infinity of points. So we say that the equation $y = \sin^{-1}x$ is a <u>relation</u> between x and y [earlier, it was customary to say that the equation defined y as a multiple-valued function of x]. Then Fig. 2 is the graph of this relation, and the graph is given in greater detail in Figure 3.

The curve would continue up and down the y-axis indefinitely. However, in our diagram, if we were to restrict the values of y so that $-\pi/2 \leq y \leq \pi/2$, the graph would then represent the graph of a function which is usually called $y = \mathrm{Sin}^{-1}x$. In the equation $y = \sin^{-1}x$, x must be in the interval $-1 \leq x \leq 1$. For any value of x in this interval, the corresponding value of y could be found.

Figure 4 represents the graph of the <u>function</u> $y = \mathrm{Sin}^{-1}x$, while Fig. 3 represents the graph of the <u>relation</u> $y = \sin^{-1}x$. If we were asked to find the value of y such that $y = \mathrm{Sin}^{-1}1/2$, we would use the graph of the function given in Fig. 4, and the unique answer would be $\pi/6$ radians. Similarly, $\mathrm{Sin}^{-1}-1/2 = -\pi/6$ radians; $\mathrm{Sin}^{-1}-1 = -\pi/2$ radians; $\mathrm{Sin}^{-1} - \sqrt{3}/2 = -\pi/3$ radians.

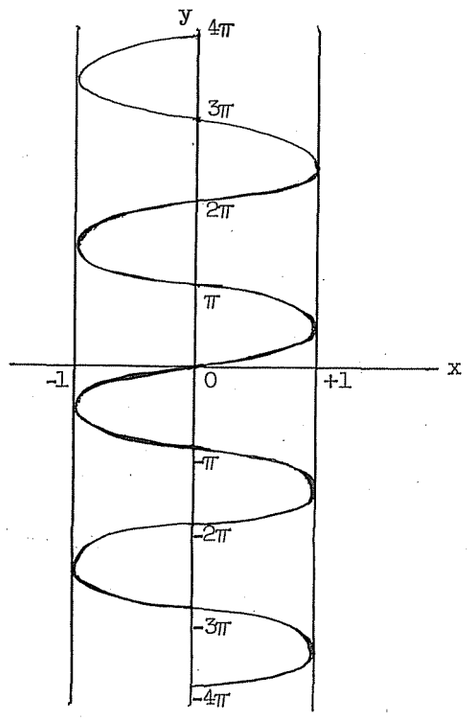


Figure 3.

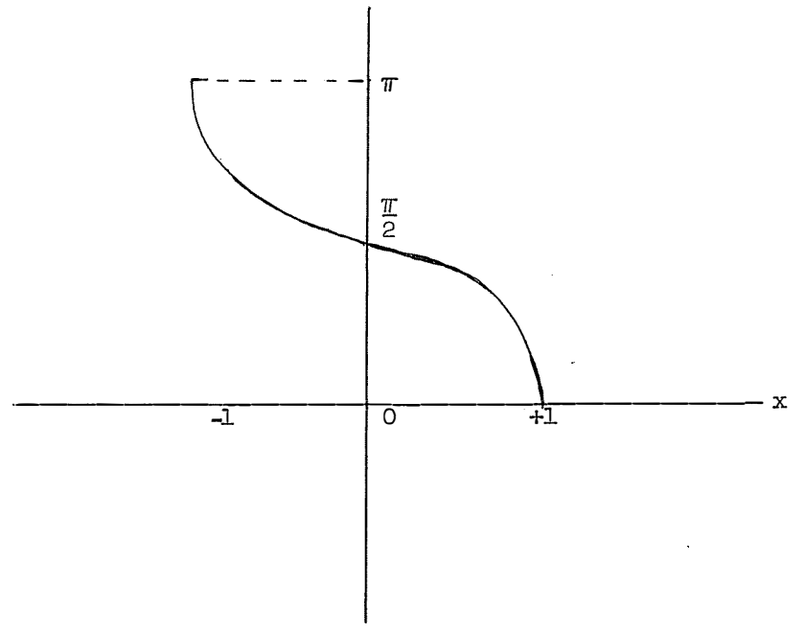


Figure 5.

Figure 5 represents the graph of the function $y = \text{Cos}^{-1}x$. Since the cosine function has positive value in both the first and fourth quadrants, if x is negative we take our values from the second quadrant. Thus, if x is some positive number between

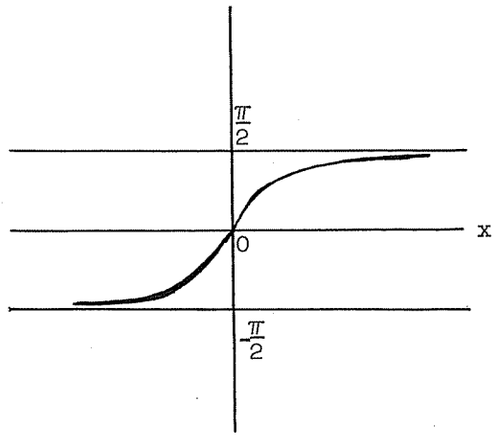


Figure 6.

0 and 1 inclusive and if $y = \text{Cos}^{-1}x$, then y will have a value between 0 and $\pi/2$ radians. On the other hand, if x is negative, y will be between $\pi/2$ and π. Thus, $\text{Cos}^{-1}-1/2 = 2\pi/3$ radians; $\text{Cos}^{-1}-1 = \pi$ radians.

The graph of the function $y = \text{Tan}^{-1}x$ is given in Fig. 6. This graph is similar to the graph of the function $y = \text{Sin}^{-1}x$ in that positive values of x give an answer for y in the range $0 \leq y \leq \pi/2$, while negative values of x will result in y being assigned a value between 0 and $-\pi/2$ radians: for example, $\text{Tan}^{-1}\sqrt{3} = \pi/3$ radians; $\text{Tan}^{-1}-1 = -\pi/4$ radians.

EXERCISE

1. Draw the graphs of the following relations:
   (a) $y = \sin^{-1}x$ (b) $y = \cos^{-1}x$ (c) $y = \tan^{-1}x$

2. Draw the graphs of the following functions:
   (a) $y = \text{Sin}^{-1}x$ (b) $y = \text{Cos}^{-1}x$ (c) $y = \text{Tan}^{-1}x$

3. Using the graphs of Exercise 2, write down the value of:
   (a) $\text{Sin}^{-1}\sqrt{3}/2$ (b) $\text{Tan}^{-1}-1/\sqrt{3}$ (c) $\text{Cos}^{-1}1$
   (d) $\text{Sin}^{-1}-1$ (e) $\text{Tan}^{-1}1$ (f) $\text{Cos}^{-1}-\sqrt{3}/2$
   (g) $\text{Cos}^{-1}1/\sqrt{2}$ (h) $\text{Cos}^{-1}-1/\sqrt{2}$ (i) $\text{Sin}^{-1}-1/\sqrt{2}$
   (j) $\text{Tan}^{-1}0$ (k) $\text{Cos}^{-1}0$ (ℓ) $\text{Tan}^{-1}-\sqrt{3}$

Some Answers:

(h) $3\pi/4$ radians (i) $-\pi/4$ radians (ℓ) $-\pi/3$ radians.

LETTER TO THE EDITOR:

In a fit of something, I looked through last February's Gazette, and thought of a piece that would fill the bottom of two pages! Being as it is a question and answer! Thus:

Q: The Swiss navy is composed of squadrons of ten ships each. What, other than zero, can be substituted for x, the place-holder (I don't know the terminology, but the Math will shine through) for number of squadrons, to give the right number of ships, which is, of course, zero?!

(Answer on p. 46]

THE GENERAL SECOND-DEGREE EQUATION

1. Introduction.

In this article, the aim is to give a general definition of a conic section and show that every conic section has an equation of the second degree. We then establish conversely that every equation of the second degree represents a conic section (real or imaginary). Finally, we use invariants to establish a few properties of the general second-degree equation.

2. Definition of a Conic.

We define a conic section, or simply a conic, as follows. Let a fixed point F, a fixed line ℓ, and a constant number $e > 0$ be given. We call F the focus, ℓ the directrix, e the eccentricity. Then a conic is defined to be a curve consisting of all points P such that

$$\frac{\text{distance } PF}{\text{distance from } P \text{ to } \ell} = e.$$

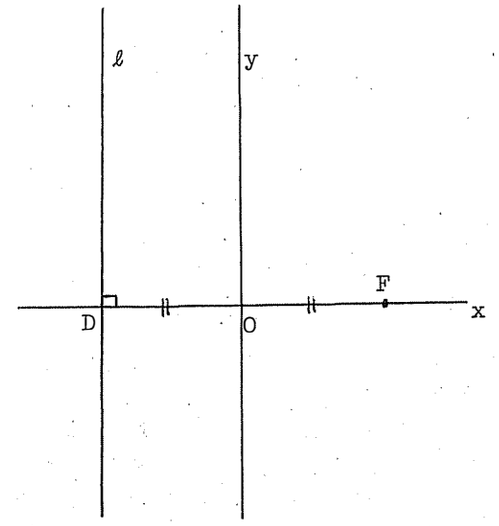
Later we shall extend the definition to permit the possibility $e = 0$.

We treat several cases.

Case 1 : $e = 1$.

In this case, we draw a line through F perpendicular to ℓ at D. There will be only one point O on this line equidistant from F and from ℓ; choose O as origin and FD as the x axis, as shown in the diagram. Then if we call the constant length FO by the name a, it is easy to use the usual development (cf. any Grade 13 Geometry text) to find that the locus of P has equation

$$y^2 = 4ax.$$



This curve, for which $e = 1$, is called a parabola. Note that it includes the case $a = 0$ (the focus F lies on the directrix ℓ); in this case, we say we have a degenerate parabola $y^2 = 0$ (two coincident straight lines).

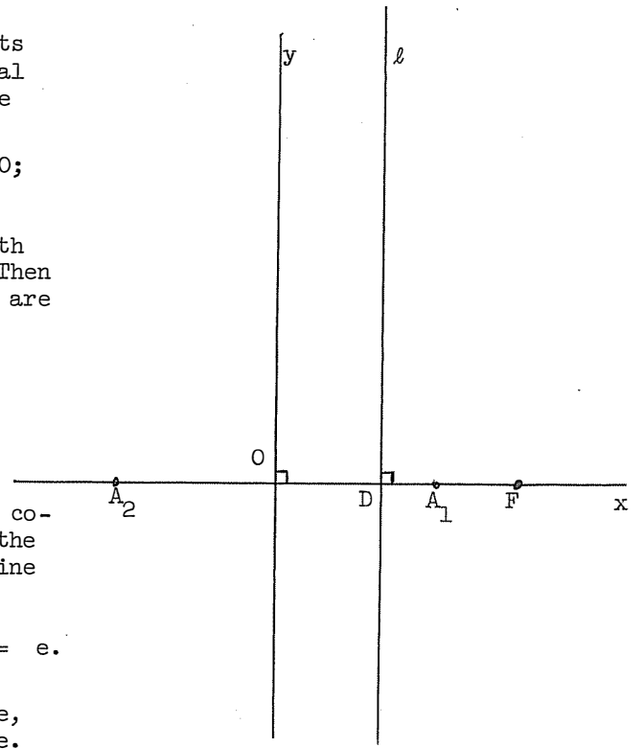
Case 2 : $e > 1$.

Again, draw a line through F perpendicular to ℓ at D. There will be two points A_1 and A_2 on this line such that

$$\frac{FA_1}{A_1D} = \frac{FA_2}{A_2D} = e.$$

These will be the points of internal and external division of FD in the ration e/1.

Now bisect A_1A_2 at O; select O as origin and FD as the x axis. We call the constant length OA_1 by the name a. Then $A_1(a,0)$ and $A_2(-a,0)$ are points on the curve.



Furthermore,

$$\frac{FA_1}{A_1D} = \frac{FA_2}{A_2D} = e.$$

Let the focus F have coordinates (u,0) and the directrix ℓ be the line x = v. Then

$$\frac{u - a}{a - v} = \frac{u + a}{a + v} = e.$$

Thus $u + ev = a + ae,$
$u - ev = -a + ae.$

Solving these equations, we find:

$$u = ae, \quad v = a/e.$$

In short, the focus is (ae,0), the directrix is $x = a/e$.

Now let P(x,y) be a general point on the required curve. Then

distance from P to $\ell = x - a/e,$

distance PF $= [(x - ae)^2 + y^2]^{\frac{1}{2}}.$

Thus

$$[(x - ae)^2 + y^2]^{\frac{1}{2}} = e(x - a/e) = ex - a.$$

It follows that

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2).$$

[Note: This equation permits a to be zero, that is, the focus F lies on the directrix ℓ. The equation then is

$$x^2(e^2 - 1) - y^2 = 0,$$

which represents two intersecting straight lines. We say that these 2 intersecting lines represent a <u>degenerate hyperbola.</u>]

When $a \neq 0$, we proceed as follows. Since $e^2 - 1 > 0$, it is customary to set

$$a^2(e^2 - 1) = b^2.$$

Then $b^2x^2 - a^2y^2 = a^2b^2$.

This curve, for which $e > 1$, is called an hyperbola. Because it is symmetrical about the x and y axes, we see that there is another focus at $(-ae,0)$ and a corresponding directrix at $x = -a/e$.

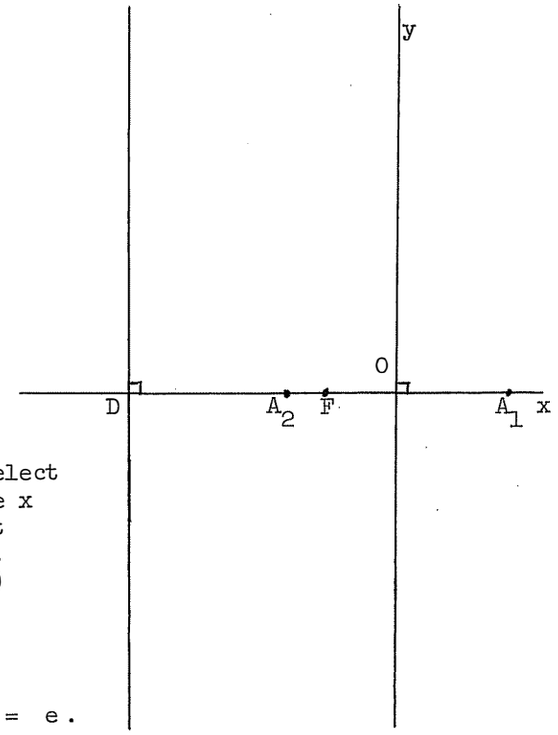
Case 3 : $0 < e < 1$.

As in Case 2, draw a line through F perpendicular to ℓ at D. There will be two points A_1 and A_2 on this line such that

$$\frac{FA_2}{A_2D} = \frac{FA_1}{A_1D} = e.$$

These will be the points of external and internal division of FD in the ratio $e/1$.

Now bisect A_1A_2 at O; select O as origin and FD as the x axis. We call the constant length OA_1 by the name a. Then $A_1(a,0)$ and $A_2(-a,0)$ are points on the curve.



Furthermore

$$\frac{FA_1}{A_1D} = \frac{FA_2}{A_2D} = e.$$

Let the focus F have co-ordinates $(-u,0)$ and the directrix ℓ be the line $x = -v$. Then

$$\frac{u + a}{v + a} = \frac{a - u}{v - a} = e.$$

Thus

$$u - ev = ae - a,$$

$$u + ev = ae + a.$$

Solving these equations, we find

$$u = ae, \quad v = a/e.$$

In short, the focus is $(-ae,0)$, the directrix is $x = -a/e$, in the figure we have used.

Now let $P(x,y)$ be a general point on the required curve.

Then

$$\text{distance from } P \text{ to } \ell = x + a/e,$$
$$\text{distance } PF = [(x + ae)^2 + y^2]^{\frac{1}{2}}.$$

Thus

$$[(x + ae)^2 + y^2]^{\frac{1}{2}} = e[x + a/e] = ex + a.$$

It follows that

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2).$$

[Note: This equation permits a to be zero, that is, the focus F lies on the directrix ℓ. The equation then represents the single point $(0,0)$. We refer to this point as a degenerate ellipse.]

When $a \neq 0$, we proceed as follows.

Since $1 - e^2 > 0$, it is customary to set

$$a^2(1 - e^2) = b^2.$$

Then $$b^2x^2 + a^2y^2 = a^2b^2.$$

This curve, for which $e < 1$, is called an ellipse. Because it is symmetrical about the x and y axes, we see that there is another focus at $(ae,0)$ and a corresponding directrix at $x = a/e$.

Case 4 : $e = 0$.

It is customary to conclude by extending the definition of a conic to include the limiting case $e = 0$. If we look at Case 3 and permit e to approach zero, regarding the intercepts A_1 and A_2 as fixed, we see that this limiting case corresponds to $ae^2 \longrightarrow 0$, $a/e \longrightarrow \infty$. Thus the focus F coincides with the origin 0, and the directrix ℓ recedes indefinitely far from 0.

Analytically, this corresponds to the case

$$b^2 = a^2(1 - e^2) = a^2.$$

Thus $$x^2 + y^2 = a^2.$$

This curve is called a circle; we regard it as the limiting case of an ellipse as $e \longrightarrow 0$.

3. The General Second-Degree Equation.

In Section 2, we showed that the focus-directrix-eccentricity definition of a conic leads to one of four standard equations, when axes are suitably chosen, namely,

$$y^2 = 4ax,$$

$$b^2x^2 - a^2y^2 = a^2b^2,$$

$$b^2x^2 + a^2y^2 = a^2b^2,$$

$$x^2 + y^2 = a^2.$$

In this section, we consider the converse problem. Suppose we are given a general second-degree equation of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

We seek to show that if such an equation represents a curve containing any real points at all, then that curve is one of the conic sections.

The main tools needed in establishing this result are translation and rotation of axes. Both of these topics are treated in the standard analytic geometry texts; so we shall merely quote the necessary results.

I. If the co-ordinate axes are translated to a new parallel position with new origin at the point (h,k), then the new co-ordinates (X,Y) of a point are related to the old co-ordinates (x,y) of the point by the equations

$$x = X + h, \quad y = Y + k.$$

II. If the co-ordinate axes are rotates through an angle θ to a new position, the origin remaining fixed, then the new co-ordinates (x',y') of a point are related to the old co-ordinates (x,y) of the point by the equations

$$x = x' \cos \theta - y' \sin \theta,$$

$$y = x' \sin \theta + y' \cos \theta.$$

4. Removal of the xy Term.

We begin by proving the

Theorem. If we select θ so that

$$\tan 2\theta = \frac{2h}{a - b},$$

then rotation of the axes through an angle θ will place them in a position so that, referred to the new axes, the curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

will have an equation of the form

$$a'x'^2 + b'y'^2 + 2g'x' + 2f'y' + c' = 0.$$

Proof. Let us rotate axes through an angle θ so that the new

axes give us an $x' - y'$ co-ordinate system. Then

$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta.$$

The curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

will have an equation, referred to the new co-ordinate system, which is

$$a(x' \cos\theta - y' \sin\theta)^2 + 2h(x' \cos\theta - y' \sin\theta)(x' \sin\theta + y' \cos\theta)$$
$$+ b(x' \sin\theta + y' \cos\theta)^2 + 2g(x' \cos\theta - y' \sin\theta)$$
$$+ 2h(x' \sin\theta + y' \cos\theta) + c = 0.$$

Collecting terms we find:

$$(a \cos^2\theta + 2h \sin\theta \cos\theta + b \sin^2\theta)x'^2$$
$$+ 2(-a \sin\theta \cos\theta + h \cos^2\theta - h \sin^2\theta + b \sin\theta \cos\theta)x'y'$$
$$+ (a \sin^2\theta - 2h \sin\theta \cos\theta + b \cos^2\theta)y'^2$$
$$+ 2(g \cos\theta + h \sin\theta)x' + 2(-g \sin\theta + h \cos\theta)y' + c = 0.$$

In order that the $x'y'$ term vanish in this equation, θ must be chosen so that

$$h(\cos^2\theta - \sin^2\theta) + (b - a) \sin\theta \cos\theta = 0,$$
$$h \cos 2\theta = (a - b) \sin\theta \cos\theta,$$
$$\frac{2h}{a - b} = \tan 2\theta.$$

Since the tangent function can assume all values from $-\infty$ to $+\infty$, we see that, no matter what value is taken by $2h/(a - b)$, we can find 2θ so that

$$\tan 2\theta = \frac{2h}{a - b}.$$

With this choice of θ, the equation of the given curve has the form (referred to the $x'y'$ co-ordinate system)

$$a'x'^2 + b'y'^2 + 2g'x' + 2f'y' + c' = 0,$$

where $a' = a \cos^2\theta + 2h \sin\theta \cos\theta + b \sin^2\theta,$
$b' = a \sin^2\theta - 2h \sin\theta \cos\theta + b \cos^2\theta,$
$g' = g \cos\theta + h \sin\theta,$

$$f' = -g \sin\theta + h \cos\theta,$$
$$c' = c.$$

This completes the theorem.

5. Square Completion to Identify Conics.

We saw in Section 4 that it was possible to choose the co-ordinate system so that the cross-product term was missing in the general second-degree equation. We now assume this has been done and that the equation has the form

$$a'x'^2 + b'y'^2 + 2g'x' + 2f'y' + c' = 0.$$

[Actually, we might well for sheer convenience now drop the dashes; we shall merely keep them to stress the fact that the co-ordinate system used is not the same as the original one.] There are several cases.

Case 1. $a' \neq 0$, $b' \neq 0$.

We write:

$$a'(x'^2 + 2\frac{g'}{a'}x' + \frac{g'^2}{a'^2}) + b'(y'^2 + 2\frac{f'}{b'}y' + \frac{f'^2}{b'^2})$$
$$+ c' - \frac{g'^2}{a'} - \frac{f'^2}{b'} = 0.$$

Let us now translate the $x'y'$ axes to a new position so that we have new co-ordinates (X,Y) related to the old by

$$x' + \frac{g'}{a'} = X, \quad y' + \frac{f'}{b'} = Y.$$

[This is achieved by translating to a new origin through $(\frac{-g'}{a'}, \frac{-f'}{b'})$.]

The new equation of the curve, referred to the XY co-ordinate system, is:

$$a'X^2 + b'Y^2 + c' - \frac{g'^2}{a'} - \frac{f'^2}{b'} = 0.$$

This has the form

$$AX^2 + BY^2 - C = 0,$$

where $A = a'$, $B = b'$, $C = -c' + \frac{g'^2}{a'} + \frac{f'^2}{b'}$.

Then

$$AX^2 + BY^2 = C.$$

If $C \neq 0$, we write

$$\frac{X^2}{C/A} + \frac{Y^2}{C/B} = 1.$$

If $\frac{C}{A}$ and $\frac{C}{B}$ are positive, this curve is an ellipse.

If $\frac{C}{A}$ and $\frac{C}{B}$ are both negative, there are no real points at all which satisfy the equation. (Sometimes, one says that the locus, in this case is an "imaginary conic".)

If $\frac{C}{A}$ and $\frac{C}{B}$ differ in sign, this curve is an hyperbola.

On the other hand, if $C = 0$, then the equation takes the form

$$AX^2 + BY^2 = 0.$$

If A and B have the same sign, this locus reduces to the single point $(0,0)$ (degenerate ellipse).

If A and B have different signs, then we can write

$$(\sqrt{A}\,X + \sqrt{-B}\,Y)\,(\sqrt{A}\,X - \sqrt{-B}\,Y) = 0,$$

and recognize the locus as two intersecting straight lines (degenerate hyperbola).

Case 2. a' or b' is zero.

There is no loss in generality in assuming $a' = 0$. The equation then takes the form

$$b'(y' + 2\frac{f'}{b'}\,y' + \frac{f'^2}{b'^2}) + 2g'x' + c' - \frac{g'^2}{a'} - \frac{f'^2}{b'} = 0.$$

Let us now translate the $x'y'$ axes to a new position through the point $(\frac{-c'}{2g'} + \frac{g'}{2a'} + \frac{f'^2}{2g'b'}, \frac{f'}{b'})$. Then the new co-ordinates (X,Y) are related to the old by

$$x' + \frac{c'}{2g'} - \frac{g'}{2a'} - \frac{f'^2}{2g'b'} = X, \quad y' + \frac{f'}{b'} = Y.$$

The new equation of the curve, referred to the XY co-ordinate system, is:

$$b'Y^2 + 2g'X = 0,$$

$$Y^2 = -\frac{2g'}{b'}\,X.$$

This has the standard form for the equation of a parabola ($g' = 0$ corresponds to the degenerate case).

Case 3. $a' = b' = 0$.

Since we see (Section 4) that

$$a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta = 0,$$
$$a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta = 0,$$

it follows by addition and subtraction that

$$a + b = 0,$$

$$a\cos 2\theta + 2h\sin 2\theta - b\cos 2\theta = 0.$$

Then $a\cos 2\theta + h\sin 2\theta = 0$.

But θ satisfied the equation

$$h\cos 2\theta - a\sin 2\theta = 0.$$

Squaring and adding:

$$a^2 + h^2 = 0.$$

Thus $a = h = b = 0$; in short, Case 3 can not occur since we assume we were originally given a genuine equation of the second degree.

Our conclusion to this section can then be summed up as a

Theorem. After rotating axes through a suitable angle θ, the general second-degree equation takes the form

$$a'x'^2 + b'y'^2 + 2g'x' + 2f'y' + c' = 0.$$

It is not possible to have $a' = b' = 0$. If exactly one of a' or b' is zero, the curve is a parabola (including the possibility of the degenerate case of two coincident lines). If neither a' nor b' is zero, then the equation may not be satisfied by any real points; however, if there are real points on the locus, then it represents an ellipse (including the possibility of the degenerate case of a single point) or an hyperbola (including the possibility of the degenerate case of two intersecting straight lines).

6. Finale.

This article will be followed by another which will try to do two things. (1) It will give illustrations which show how, in actual practice, the manipulations of Sections 4 and 5 are not as formidable as they appear when done literally. (2) It will give some simple discussion of invariants, and will develop immediate criteria which will tell whether or not a given conic is an ellipse, parabola, or hyperbola; whether or

not a given conic is proper or degenerate.

Finally, in conclusion, a pedagogic note. Should the development of this article be a student's first approach to conics?

Our answer is a resounding NO. After a student has studied the individual conics, he is in a position to appreciate the unified approach of Section 2 and to see the reason for discussing the general equation of degree 2 as done in Sections 4 and 5. But he should know something about conics before plunging into a general discussion of the nature of the present article. This work on the general conic should be a follow-up to the sort of work done at present in Grade 13; it is an enrichment of known work, not an introduction.

P O E T R Y S E C T I O N

George Bellamy - Georgetown

GONE FOR A BURTON

To be or not to be: that is the sentence
That's open to be filled -
Filled with the members of solution sets,
Or values got by simple transposition.
To solve, to satisfy, to find, these
Are too simple for our disjoint time,
And we must pause to think what else may range
Through and beyond our limited domain:
Whether the truth set that we seek
Is bounded nether heaven or nether hell,
Ranges the real infinite, or lies
In vain imaginaries, or numbers too complex
For Grade IX comprehension.
To solve, to satisfy, would nullify demand
For esoteric texts with curlicues,
Bring rigour to those members of sub-sets
Of teachers and professors who engage
In the production of profound null sense
For hire and salary. This were a concept
Too deep for our delight, and in't
We'd drown our sorrows with a holomorph.

REPORT ON MATHEMATICAL INSTRUCTION IN CANADIAN UNIVERSITIES

2 June 1963 — Universite Laval, Quebec

Present: Professors H. I. Schiff (McGill),
Nominated by the Chemical Institute of Canada;
E. R. Pounder (McGill), G.M. Volkoff (B.C.),
Nominated by the Canadian Association of Physicists;
A. Gauthier (Montreal), R. L. Jeffery (Acadia),
G. de B. Robinson (Toronto), M. Wyman (Alberta),
Nominated by the Canadian Mathematical Congress.

1. Background. At the meeting of the International Mathematical Union in Sweden last summer the Report of the International Commission on Mathematical Instruction made it clear that certain difficulties are arising throughout the world between departments of mathematics, physics and chemistry. The story of what is being done elsewhere to meet this situation stimulated some of us to organize the present meeting in the hope that we might be able to suggest remedies applicable in our Canadian Universities.

Prior to our meeting the accompanying article by Weinstock was circulated to members of the group and the following comments were received:

(a) Rigour is over-stressed in courses in mathematics attended by students of physics and chemistry,

(b) Mathematicians are unwilling to meet the needs of scientists for more advanced courses.

No disagreement with these comments or with the position adopted by Weinstock was expressed by any members of the group. It was agreed that more co-operation is needed. How to achieve this co-operation was the subject of lengthy discussion. In the experience of those present however, there appeared to be one chief source of difficulty.

2. The choice of his major subject by an honours student may be too long delayed. In order to summarize the discussion on this point it should be emphasized that students come to Canadian Universities at two distinct levels. In some institutions they take a first course in the Calculus in their first year (e.g. Alberta, Toronto), in others in their second year (e.g. B.C., McGill, Montreal). For clarity, let us call the year in question the common introductory year. All of us agreed that such a common year should exist for honour students and we shall discuss its mathematical content in Sections 3 and 4 below, but we were also agreed that to offer mathematical courses to students of mathematics, physics and chemistry, together beyond this common year may force the lecturer to compromise the rigour required by the professional mathematician for the benefit of those seeking a working knowledge of the subject. Whatever compromise is reached, the result may be unsatisfactory to all concerned. Thus we would recommend that students state their major interest at the end of

their common introductory year and that subsequent courses in mathematics be arranged to meet the needs of each group.

3. Mathematical courses suitable to the common introductory year. We would suggest that two courses are appropriate, one in calculus and one in algebra and geometry (see Section 4), though it may be convenient to give the course in algebra and geometry in the following year.

With regard to these two courses, we would suggest that the physicists and chemists should think carefully what specific techniques they actually require so that the mathematicians may plan the courses. It should be realized that to develop a subject properly certain time intervals are necessary. Thus these courses should be integrated with the appropriate courses in physics and chemistry so that each lecturer may know when the significant ideas with which he is concerned will be treated by his colleagues. This requires continuous co-operation throughout the year.

It was suggested that joint problems might be prepared at crucial points, but it was pointed out that if the mathematician reaches certain topics at known times and his colleagues know and use the same language in referring to mathematical concepts, it may be no disadvantage to have applications made by physicists and chemists indèpendently. Some repetition is desirable at this early stage.

It was emphasized that much of the difficulty in this common introductory year arises through the inability of young lecturers to recognize the importance of the intuitive approach. Rigour should follow familiarity not precede it, even for students planning to become professional mathematicians.

4. Further discussion of this common introductory year emphasized the fact that modern physicists and chemists are utilizing vectors, matrices and groups almost as soon as the calculus, so that two introductory courses are really involved. Both these courses should be given by the mathematics department but in close co-operation with the departments of physics and chemistry. Mathematicians should remember that the origin of most mathematical ideas is physical and geometrical, and intuition springs from this common ground. On the other hand, physicists and chemists should recognize the continually increasing abstraction necessary to describe adequately natural phenomena. Students must have some practice in abstraction if they are to do more than maipulate formulae; this is why the trained mathematician is essential to the progress of science. Abstraction however, should not be confused with rigour; it is the method used to isolate certain aspects of reality common to many contexts. The physicists and chemists are seeking the same goal in this respect and need the language of mathematics to pass from the particular to the general.

5. Discussion of the common introductory courses in mathematics raised the question of the High School curriculum. The group was unanimously of the opinion that a conservative approach benefits

the greatest number of students. Certainly, changes should be made, but to introduce ideas which cannot be applied by the students or a rigorous form of language useful only to a few is frustrating and useless. To a large extent these comments apply to the common introductory courses at the University as well. The notation of the calculus can cause confusion if not properly developed and explained, but it has proved a marvellous tool for 250 years and its value should be exploited from every point of view. These introductory courses should provide the student with a well developed technique and an understanding of its use, as well as a knowledge of the pitfalls to be expected and avoided.

6. Courses beyond the common level will vary from university to university, since they should depend to a large extent on local needs and emphases. In those universities where theoretical physics or applied mathematics is recognized in the Calendar, common ground may exist even for two more years. In such cases, it is desirable for the student to study mathematical models as well as mathematical methods in close conjunction. In some universities such courses are offered in the physics department and in others in the mathematics department. Personalities and historical patterns of development enter here and we can only stress the need for understanding and co-operation at these higher levels also.

In this connection we would recommend that a sufficient number of mathematical options at the higher level be available for students of physics and chemistry. Since it is undesirable that courses on special functions, orthogonal expansions, etc. be sandwiched into courses on theoretical physics, we would suggest that a half year course devoted to the appropriate mathematics might precede a course in its applications for such students, and a more rigorous course in analysis for students of mathematics.

7. Mathematics in biology, economics, psychology, etc. Finally, we would make a plea for the teaching of mathematics in biology, economics, psychology etc. to be done by persons trained in the appropriate mathematical field. Abstraction and model building are becoming increasingly important in these disciplines also but Canadian universities have been only too ready to minimize their significance. In particular, the advent of the electronic computer has made it possible to handle statistics in a manner previously out of the question. The need for co-operation here is just as pressing, if universities are to prepare their students for graduate work elsewhere.

H. I. Schiff
E. R. Pounder
G. M. Volkoff
G. de B. Robinson (Chairman)
A. Gauthier
R. L. Jeffery
M. Wyman

BOOK REVIEW

TOPICS IN MODERN MATHEMATICS, edited by R. G. Stanton and K. D. Fryer, Prentice-Hall Inc., Englewood Cliffs, 1964.

The following quotation is taken from the dust jacket of the book.

"Whether the reader's mathematics background is limited or extensive, whether he wants to survey modern mathematics generally or explore one topic particularly, he will find this is the single one-volume source to which he can turn to find a self-contained discussion of each of the following topics:

Groups and Fields · Set Theory · Boolean Algebra · Logic and Computing · Vector Spaces and Matrices · Numerical Analysis Functions of a Single Variable · Fundamental Concepts of Calculus Probability Theory and Statistics · Some Types of Geometry

The authors represented, in addition to the editors, are:

G. Berman, J. W. Graham, D. G. Wertheim, and D. A. Sprott.

Just as their articles span a broad range of topics from the relatively simple to the relatively complex, so their levels of presentation vary correspondingly. Their styles, however, are uniformly informal and provocative; their language and methods a careful blend of modern and traditional usage; and their emphasis on concepts rather than manipulation.

Carefully chosen illustrative examples, diagrams, and problem materials, are generously provided for each chapter. And lastly, to stimulate thought and discussion, there is an epilogue devoted to some controversial aspects of the modern mathematics programme."

Ontario readers will recognize portions of the present volume which originally appeared in mimeographed form under the title "Ten refresher lectures in modern mathematics". However, the present volume has been considerably revised and amplified; many new topics have been added, and the improvement is general.

Perhaps a few specific comments should be made. The book as a whole would be an excellent addition to any school mathematics library and would provide stimulus to the better students in the upper grades. The chapters on "Groups and fields" and on "Boolean algebra" are quite difficult, and will not provide too much usable material. The chapters on "Probability and statistics" and on "Some types of geometry" contain quite a bit of thought-provoking material as well as some more advanced work that will be above most students. The other six chapters should be well within the capacities of good grade 13 students, almost in their entirety. Students who are interested in functions will find particularly valuable and unusual illustrations in the two chapters on "Functions of a single variable" and "Fundamental concepts of calculus"; the illustrated examples in these two chapters are especially instructive and provide real insight.

The chapter on "Logic and computing" is particularly lucid, and many of the illustrations could be used as early as grade 11. The work on "Set theory" and on "Vectors and matrices" provides excellent material on topics that are beginning to receive more and more stress in revised curricula and in proposals for further revision.

The three page epilogue shows a tempered enthusiasm for modern mathematics, but raises some pertinent questions about modern mathematics in the secondary school curriculum. These questions are important for us, and should make us all think very seriously about some of the thornier problems.

The main strength of the book is the fact that the various chapters are written as separate essays and can stand on their own feet. I would think that any student or teacher could find some of the "topics" which would please his palate, and I do recommend the volume. One word of criticism: there are a number of exercises included in the book, but I do not think the number is adequate. The book would be improved if the exercise material were considerably increased.

N. N. H.

Answer to Question on page 31.

Simply Mathematical induction:

1 squadron $\longrightarrow$ 10 ships
2 squadrons $\longrightarrow$ 20 ships
3 squadrons $\longrightarrow$ 30 ships
.
.
.
N squadrons $\longrightarrow$ NO ships!

R. S. D. Thomas,
University of Toronto.

Editorial - continued from page 4

But it is time that the government received more information. It is time that this political SHORTFALL ceased.

In conclusion, one bright note. One University in the province seems to be receiving adequate operating grants. According to the Financial Post of October 12, student population at York was 500 students. The operating grant at York was 1.3 million, an average of 2600 dollars per student. At Queen's University, for example, student population was 3792. The operating grant at Queen's was 3.45 million, an average of 910 dollars per student. The other universities were in the same range as Queen's.

Everyone must rejoice that York is given an adequate operating grant. This is an excellent thing. Similarly, everyone must hope that other Universities may in the future receive the same treatment, and that they too will receive adequate operating grants.

And now, farewell for another time. This is the first time I have written an editorial about the problems of education at the University level. But the problem is serious, and I thought that all my secondary school friends would want to know more details about it. After all, where are the students who, at present indications, will not be getting as good a university training as they should (if they get it at all)?

THEY ARE IN YOUR CLASSES AT THIS VERY MOMENT.

Ralph G. Stanton

EDITOR-IN-CHIEF

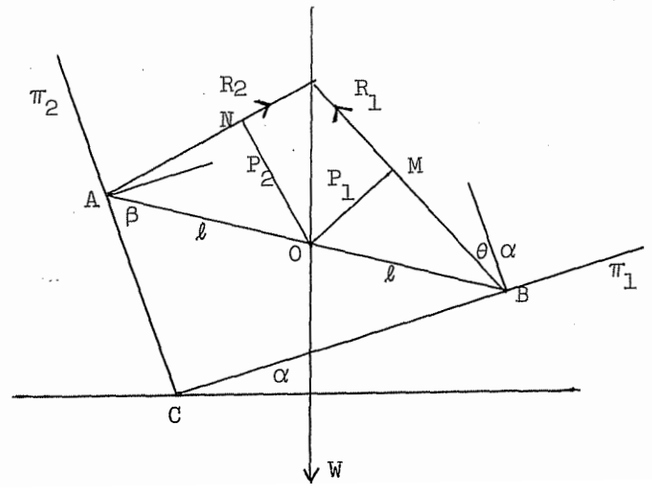
CORRESPONDENCE

Dear Dr. Stanton,

May I suggest the following alternative for the last question on the 1964 "Problems" paper.

R_1 and R_2 represent the number of units in the resultant reactions of π_1 and π_2 on the ends of the rod AB and are concurrent with line of action of "weight of rod".

$\angle CAB = \beta$
$AB = 2\ell$
$OM = p_1$
$ON = p_2$



(1) Resolve R_1 and R_2 <u>horizontally</u>

$$R_1 \sin(\theta + \alpha) = R_2 \cos(90 + \theta - \beta)$$

$$\therefore \frac{R_1}{R_2} = \frac{\sin(\beta - \theta)}{\sin(\theta + \alpha)}$$

(2) Take moments about O

$$R_1 \times p_1 = R_2 \times p_2$$

$$R_1 \times \ell \cos(\theta + \alpha) = R_2 \times \ell \sin(90 + \theta - \beta)$$

$$\frac{R_1}{R_2} = \frac{\cos(\beta - \theta)}{\cos(\theta + \alpha)}$$

$$\therefore \frac{\sin(\beta - \theta)}{\sin(\theta + \alpha)} = \frac{\cos(\beta - \theta)}{\cos(\theta + \alpha)}$$

i.e. $\sin(\beta - \theta - \theta - \alpha) = 0$

And <u>one</u> solution of this equation is $\beta = 2\theta + \alpha$. This is obviously the angle which the rod makes with π_2.

Sincerely,
Jean Leppard.
