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EDITORIAL

This fall I have been visiting a few schools, and have been watching with interest the way in which various teachers have been approaching the new Grade 11 Curriculum. Ideally, I had hoped that many teachers would be consulting all three of the available new texts in order to have as much background as possible. Unfortunately, this is not the case. Why it is not the case is the subject of this editorial.

Secondary-school teachers of mathematics should be professional people with a firm grasp of their subject who are keeping up-to-date. A good specialist should consult and compare all available texts; he should have available other reference material. He will wish to read The Mathematics Teacher and the Ontario Mathematics Gazette. He will read and use articles from one or both of the American Mathematical Monthly or the British Mathematics Gazette. He will read professional journals such as the OSSTF Bulletin. And he will often be giving a special Problems class with enrichment material for advanced students.

But where do our mathematics teachers get time for all this? The answer is, of course, that they do not.

The OSSTF has done a fine job in raising teacher salaries. But now is the time for it to put emphasis on the teacher as a professional scholar — a person who must keep up with his subject just as alertly as does a medical doctor. To do this, teachers must have more free time for study. Based on a forty-period week, I believe that immediate and strong attempts should be made to attain the goal of no more than twenty teaching periods for department heads, no more than thirty for department members. This would be a beginning at meeting the crucial problem of providing teachers with the time indispensable for keeping themselves up-to-date as scholar-teachers.

At least one principal of my acquaintance will be strongly opposed to this idea. He feels it is his duty to keep his "best teachers" teaching every period of the day so that they can process as many students as possible. This is a tempting theory, but I believe it is illusory. A well may be pumped dry. And how long will his best teachers really be up-to-date if they are so loaded down with work and assignment marking that they rarely, if ever, manage to squeeze in the time to read a different text, a more ample reference, or a journal article?

[continued on page 14]

Mathematics and Physics Section, O.A.T.M.P.

WHAT IS APPLIED MATHEMATICS?

[Mr. Lloyd D. Auckland has forwarded us this article, reported by Mrs. Leppard, on "What is Applied Mathematics" by Dr. T. E. Hull, Computing Centre, University of Toronto. This was given at the noon luncheon of the O.A.T.M.P. and was originally published in the May-June issue of the Canadian School Journal.]

To understand what is meant by applied Mathematics, it is useful to begin with some indication of the importance of models in the solving of scientific problems. The engineer uses physical models (such as wind tunnels, and flight simulators). The physicist and chemist use imagined models of atoms and molecules. All scientists will use, at least some of the time, abstract or mathematical models in the course of solving problems. His techniques may involve algebra, or calculus, or probability theory.

We use a model in place of the real thing because the model is less expensive and easier to work with. Unfortunately, the model may not be perfect and so we have to be very careful about the possible effect of any approximations we have introduced with our model.

The applied mathematician comes into the picture as a person who is an expert on these models which are mathematical in nature. The pure mathematician may be interested in the same models, but he is interested in them for their own sake, and not because of their possible applications. He often is interested in mathematical systems which seem to be quite unrelated to any possible applications. For him the manipulation of a mathematical system is a game to be played with symbols according to certain rules. He is, of course, free to change the rules if he wishes, and to invent new games, whereas the applied mathematician is restricted to those games, or models, which have some relation to reality. The difference between applied and pure mathematicians is mainly a difference in point of view, or motivation.

The effect of computers on all these ideas is to enormously extend the scope of the models which we are now able to work with.

The computer carries out the operations we prescribe. We no longer do so ourselves; we programme the computer to carry out the steps in the process. We spend our time studying the various processes which are possible, with a view to determining which are the most effective and most efficient.

We can now handle problems involving vastly greater quantities of information and requiring vastly more complex manipulations.

Of course, all this is having an important effect on problems in traditional areas of science such as engineering and physics. But it is also effecting newer areas, and quite a few of these might well lead to material for a high school curriculum.

In summary, we have emphasized the importance of using models in many problem-solving situations, especially in science. An applied Mathematician is concerned with models which are mathematical in nature. The computer enables him to handle extremely large and complex models. Many new applied mathematical models are now being developed and used, and quite a few examples of these could be used in high school courses. The result would be to provide motivation for much of the mathematics that is being presented, and to increase both the usefulness and the understanding of that mathematics.

We have been asked to print, for reference, the names of the following elementary journals. Subscriptions can be obtained by writing to the addresses given:

American Mathematical Monthly, Prof. H. M. Gehman, Mathematica Association of America, SUNY at Buffalo, Buffalo, N.Y., U.S.A.

Price: $8.00 (American) per annum.

Canadian Mathematical Bulletin, Secretariat of the Canadian Mathematical Congress, 985 Sherbrooke Street, West, Montreal.

Price: $5.00 per annum.

Mathematical Gazette (Britain), G. Bell and Sons Ltd., York House, Portugal St., London, W.C. 2, England.

Price: £ 1/1/- per annum.

All of these contain useful material for Problems classes in larger schools; teachers will find many short articles which they can adapt.

Try a year's subscription for the School Library.

[Editor's Note: The following is the first of a series of articles on vectors and their applications. This is a continuation of the articles on proposed topics or enrichment material for Grade 13 written by the authors for the Bulletin of the OSSTF.

AN INTRODUCTION TO VECTORS

Gerald Berman and K. D. Fryer

Introduction.

Vectors are usually introduced at the elementary level using some physical interpretation. The student is asked to think of 'forces', 'velocities', and such objects having 'direction and magnitude'. Vectors are used almost immediately to discuss physical applications in statics or mechanics. The student begins to identify vectors with forces and other such physical quantities, and never realizes the simplicity and power of vectors as a mathematical tool.

In this article, vectors are introduced in terms of directed line segments and applied to elementary analytic geometry. In succeeding articles, topics in trigonometry, geometry, and algebra will be presented as illustrations of further applications of vectors.

Directed line segments and vectors.

Consider the two directed line segments

(i) from O(0,0) to A(1,1), and

(ii) from B(3,5) to C(4,6).

These line segments have the same length and make the same angle, 45°, with the positive X-axis. Indeed, each member of the infinite family of line segments joining P(x,y) to Q(x+1,y+1) for every x, y, has length 1 and makes an angle of 45° with the positive X-axis. Conversely, the length 1 and the direction 45° determine the entire family of line segments. Notice that the above family is also determined by the ordered pair of real numbers [1,1] (the differences of corresponding co-ordinates of P and Q); the particular directed segments (i) and (ii) are representatives of the family.

In general, any ordered pair of number [a,b] determines or represents, or is an infinite family of line segments: the line segments P(x,y) to Q(x+a,y+b), for every x,y; we call such an ordered pair of real numbers a vector (actually

a planar or two-dimensional vector). Thus the pair of real numbers [1,1] is a vector corresponding to the line segments in the above example. The numbers a and b are the components of the vector, and the directed line segment PQ is a representative of the vector. When working geometrically with the vector, any representative may be used.

Notice that the simplest representative of [a,b] is the directed line segment OA where O is the origin and A is the point (a,b); [a,b] is sometimes referred to as the position vector of A (relative to the origin).

Notation.

Symbols such as $\underline{u}$ and $\underline{v}$ will be used to designate vectors; in text books bold faced letters are used. Thus we might write $\underline{u} = [a,b]$, $\underline{v} = [c,d]$, etc. A representative of the vector, the directed line segment joining P and Q, wil be denoted by PQ. The vector [0,0] is denoted by $\underline{0}$.

Length and direction.

As described above, the vector $\underline{u} = [a,b]$ has associated with it the "physical properties" of magnitude and direction.

The length of $\underline{u}$ is the length of any one of its representatives, and hence is equal to the distance between the orig and (a,b). Denoting the length of $\underline{u}$ by $|\underline{u}|$, we have

$$|\underline{u}| = \sqrt{a^2 + b^2}.$$

Looking at this in another way, if P and Q denote any two points, then $|\overrightarrow{PQ}|$ is the distance between them.

The direction of the vector $\underline{u}$ is defined to be the angl θ that any representative makes with the X-axis. Taking the representative OA, where A = (a,b), it is immediate that

$$\tan \theta = b/a.$$

For example, the vector [3,4] has length $\sqrt{3^2 + 4^2} = 5$, and makes an angle θ with the X-axis, where $\tan \theta = 4/3$.

It is easy to construct a vector of length one having the same direction as $\underline{u}$. In fact, the vector [a/c,b/c], where $c = |\underline{u}|$, has the same direction as $\underline{u}$ but length 1, as is easily checked. The components a/c and b/c of this unit vector (vector of length 1) are sometimes called the direction cosines of $\underline{u}$, since they are the cosines of the angles θ and $90^\circ - \theta$ that any representative of the vector makes with the X and Y axes respectively, that is,

$$\cos\theta = a/c, \quad \sin\theta = b/c.$$

In trigonometry, these equations are sometimes taken as the definitions of $\cos\theta$ and $\sin\theta$.

Thus, the vector $\underline{u} = [3,4]$ has length 5 and direction cosines $3/5$, $4/5$; the vector $[-3,-4]$ also has length 5, but direction cosines $-3/5$, $-4/5$. Notice that if θ is the direction angle, $\tan\theta = 4/3$ in both cases. This is the property that characterizes parallel vectors. Notice that the vectors $[3,4]$ and $[-3,-4]$ differ only in the 'sense' or order of naming the end points of the representative; that is, if $[3,4] = \overrightarrow{PQ}$, then $[-3,-4] = \overrightarrow{QP}$.

Arithmetic Operations on vectors.

As we have seen, vectors are the same only if they have the same components. Thus the two vectors $\underline{u} = [a,b]$ and $\underline{v} = [c,d]$ are equal if and only if $a = c$ and $b = d$.

Two arithmetic operations are defined for vectors in a very natural way, multiplication of a vector by a scalar number, and addition of vectors. In symbols

$$k\underline{u} = [ka,kb];$$

$$\underline{u} + \underline{v} = [a+c,b+d].$$

Each of these operations is now considered geometrically.

Product of a vector and a scalar.

Since the point (ka,kb) lies on the line joining 0 to (a,b), it follows that $\underline{u}$ and $\underline{v} = k\underline{u}$ have the same direction; if $k > 0$ they have the same sense; they have the opposite sense if $k < 0$. The length of $\underline{v}$ is k times the length of $\underline{u}$.

Notice that if $k = 1/|\underline{u}|$, then $\underline{v}$ has unit length and the same direction as $\underline{u}$. Thus, as defined above, the components of $\underline{v}$ are the direction cosines of $\underline{u}$.

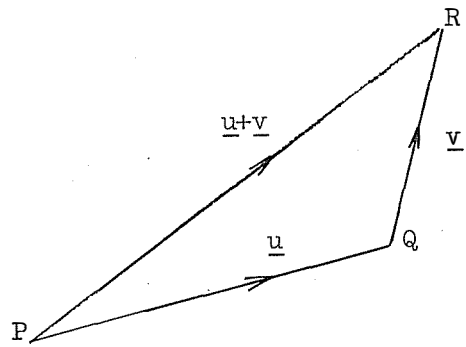
Further, if $k = -1$, we have

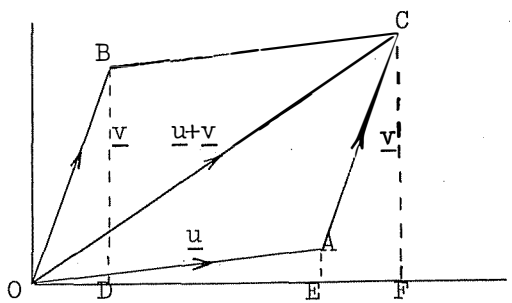
$$(-1)\underline{u} = [-a,-b] = -\underline{u};$$

in terms of representative directed segments, if $\underline{u} = \overrightarrow{PQ}$, then $-\underline{u} = \overrightarrow{QP}$, that is, if the components of $\overrightarrow{PQ}$ are $[a,b]$, then the components of $\overrightarrow{QP}$ are $[-a,-b]$.

Addition of vectors.

The sum of vectors can also be interpreted geometrically. If we take the line segment from the origin to the point A(a,b) as the representative of $\underline{u}$, and the line segment from the origin to B(c,d) as the representative of $\underline{v}$, then it is easy to check that the diagonal OC of the parallelogram with sides OA and OB will be a representative of $\underline{w} = \underline{u} + \underline{v}$.





$$OF = OE + OD = a + c$$

$$FC = EA + DB = b + d$$

Notice that $\overrightarrow{AC}$ is also a representative of $\underline{v}$, so that the vector equation $\underline{w} = \underline{u} + \underline{v}$ is represented geometrically by the statement

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

concerning the triangle OBC, with sides $\underline{u} = \overrightarrow{OA}$, and $v = \overrightarrow{OB}$. More generally, if P,Q, and R denote any three points, then

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}.$$

This implies that if $\underline{u} = \overrightarrow{PQ}$ and $\underline{v} = \overrightarrow{QR}$, then $w = \overrightarrow{PR}$, and the sum of any two vectors can be constructed geometricall in this way.

An Example.

Suppose $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$,

and we wish to find the components of $\overrightarrow{P_1P_2}$. From the above, we may write

$$\overrightarrow{P_1P_2} = \overrightarrow{P_1O} + \overrightarrow{OP_2} = \overrightarrow{-OP_1} + \overrightarrow{OP_2}$$

$$= -[x_1,y_1] + [x_2,y_2] = [-x_1,-y_1] + [x_2,y_2]$$

$$= [x_2 - x_1,\ y_2 - y_1] .$$

Of course, this can be abbreviated to

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = [x_2,y_2] - [x_1,y_1] = [x_2 - x_1, y_2 - y_1].$$

Notice that

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} ,$$

using the formula for the length of a vector. This is the usual distance formula in analytic geometry.

Equations of a line.

Let P denote any point on the line P_1P_2; then

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P}.$$

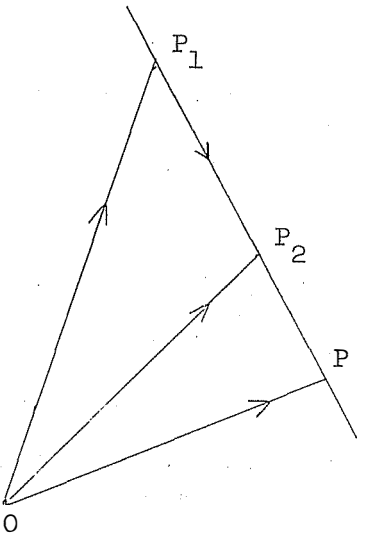
Now $\overrightarrow{P_1P}$ is along $\overrightarrow{P_1P_2}$; hence, by the previous discussion, there is a scalar k such that

$$\overrightarrow{P_1P} = k\,\overrightarrow{P_1P_2};$$

thus we have

(1) $$\overrightarrow{OP} = \overrightarrow{OP_1} + k\,\overrightarrow{P_1P_2}$$

For every value of k, the point P is on the line P_1P_2, and every point on the line can be represented in this way for an appropriate choice of k. Thus (1) is the vector equation of the line P_1P_2.



Noting that

$\overrightarrow{OP} = [x,y]$, $\overrightarrow{OP_1} = [x_1,y_1]$, and $\overrightarrow{P_1P_2} = [x_2-x_1, y_2-y_1]$

from the above example, and using the algebra of vectors, we must have

$$[x,y] = [x_1 + k(x_2-x_1),\ y_1 + k(y_2-y_1)] .$$

Since vectors are equal if and only if they have the same components, then

$$x = x_1 + k(x_2-x_1), \quad y = y_1 + k(y_2-y_1) .$$

These are the parametric equations of P_1P_2. By eliminating k, the usual two-point form of the equation of a line is obtained.

The various other standard forms of the line are easily obtained from a vector argument. The reader will find it instructive to obtain the intercept form and the normal form of the equation of the straight line by a vector argument.

Division of a line segment.

Suppose we wish to find the point P_0 which divides the line segment P_1P_2 in the ratio m/n. Then

$$\overrightarrow{P_1P_0} = (m/m+n)\ \overrightarrow{P_1P_2};$$

hence $k = m/(m+n)$ in equation (1), that is,

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OP_1} + (m/m+n)\overrightarrow{P_1P_2} \\ &= \overrightarrow{OP_1} + (m/m+n)(\overrightarrow{OP_2} - \overrightarrow{OP_1}) \\ &= \frac{n}{m+n}\ \overrightarrow{OP_1} + \frac{m}{m+n}\ \overrightarrow{OP_2} \quad .\end{aligned}$$

If we now substitute the components for the vectors and perform the necessary algebra as in the previous example, we obtain the usual formulae

$$x_0 = (n\ x_1 + m\ x_2)/(m+n), \quad y_0 = (n\ y_1 + m\ y_2)/(m+n) .$$

In particular, the midpoint M is obtained by setting $m = n = 1$; thus

$$\overrightarrow{OM} = \tfrac{1}{2}\ (\overrightarrow{OP_1} + \overrightarrow{OP_2}) .$$

Applications.

The properties of vectors may be applied directly to problems in Euclidean Geometry. It is important to note that it is not necessary to use components to prove results, unless of course the components themselves are necessary in the solution. We illustrate this principle by proving two well known propositions as examples.

(i) The line segment joining the midpoints of two sides of a triangle is parallel to the third side and is one half as long.

Let P and Q denote the midpoints of the sides AB and AC of the triangle ABC. Let the origin O be any point in the plane. It is sufficient to show that

$$\overrightarrow{PQ} = \tfrac{1}{2}\overrightarrow{BC} .$$

By the midpoint formula,

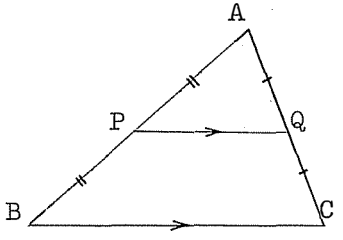
$$\overrightarrow{OP} = \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}), \quad \overrightarrow{OQ} = \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC}) .$$

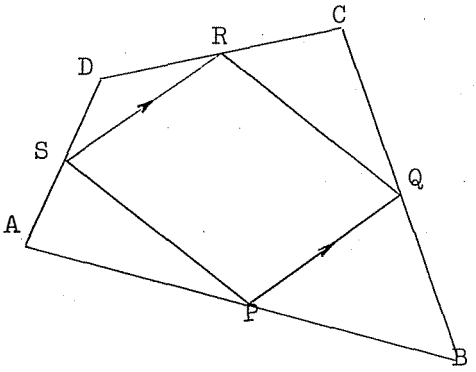
From the definition of addition,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} .$$

Consequently,

$$\overrightarrow{PQ} = \tfrac{1}{2}(\overrightarrow{OC} - \overrightarrow{OB}) = \tfrac{1}{2}\overrightarrow{BC}.$$





(ii) The midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

If P, Q, R, and S are the midpoints of the sides AB, BC, CD, and DA respectively, it is sufficient to prove that $\overrightarrow{PQ} + \overrightarrow{RS} = \underline{0}$. This is equivalent to showing that two opposite sides are equal and parallel.

Using the fact that P, Q, R, and S are midpoints, we have

$$\begin{aligned}\overrightarrow{PQ} + \overrightarrow{RS} &= (\overrightarrow{OQ} - \overrightarrow{OP}) + (\overrightarrow{OS} - \overrightarrow{OR}) \\ &= \tfrac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}) - \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) + \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OD}) \\ &\qquad - \tfrac{1}{2}(\overrightarrow{OC} + \overrightarrow{OD}) \\ &= \underline{0} .\end{aligned}$$

This is the required result.

Curves.

It was shown that all the points on the line P_1P_2 are represented by the vector equation (1). The points on any curve C may also be expressed in vector form once its parametric equations have been obtained. For suppose that the parametric equations of C are $x = f(t)$, $y = g(t)$; then the position vector of any point P on the curve is given by

(2) $$\overrightarrow{OP} = [f(t), g(t)] .$$

For example, parametric equations for the circle $x^2 + y^2 = r^2$ are

$$x = r \cos t, \quad y = r \sin t;$$

the corresponding vector equation is

$$\overrightarrow{OP} = [r \cos t, r \sin t] .$$

Similarly, the parabola $y = px^2$ has vector equation

$$\overrightarrow{OP} = [t, pt^2],$$

and the standard ellipse and hyperbola and have vector equation

$$\overrightarrow{OP} = [a \cos t, b \sin t].$$

and $\overrightarrow{OP} = [a \sec t, b \tan t]$

respectively.

By expressing a curve in the form (2), the powerful vector methods may be applied. This is the standard procedure in Differential Geometry where the properties of curves are studied with the aid of Calculus.

Editorial - continued from page 3

Closely related to the need for more time for study is the need to design our schools differently. We have classrooms and we have a staff room. But where can a teacher find a spot for an hour's intensive study or concentration? Giving him time is not enough — he must also be given space. Just as university professors have offices where they may carry on their own study and research, so secondary-school teachers need offices. Let us hope that soon schools will be built such that every teacher has either a private office or shares an office with at most one other teacher.

The scholar-teacher is an ideal which I am sure we all admit is desirable. Let us actively promote this ideal, especially through the OSSTF. Let us strive to obtain more time for teachers to engage in scholarship, and let us also urge provision of proper quarters for them in which to carry on their study. This is a matter of professional pride, professional duty, and urgent professional need. It is easy to pay lip-service to education, but boards must be made to realize the need for adequate time and space allowances.

With sincere hopes for realization of these two steps forward in the professional status of the teacher,

Yours until the next issue,

Ralph G. Stanton

EDITOR-IN-CHIEF.

A CONFUSION OF QUADRANGLES

[A sequel to "A Profusion of Squares"]

Norman Miller, Queen's University

In "A Profusion of Squares" (this Gazette, Vol. 3, No.1), which in this paper will be referred to as P.S., the problem was considered of constructing a square with sides passing, one through each of four pre-assigned points. This problem has a number of ramifications. We consider here the related problem in which the square is replaced by a quadrangle of given shape: to describe a quadrangle similar to a given quadrangle and with its sides passing, one through each of four specified points. Since the problem for squares has, in general, six solutions for the same set of four points, it might be conjectured that the number of solutions for the similar-quandrangle problem is also six. A bit of experimentation with the construction discloses a surprising result, which prompts the word confusion in the title of this paper.

Our point of departure is the construction, described in P.S., of a square by means of circles. Fig. 1 will recall

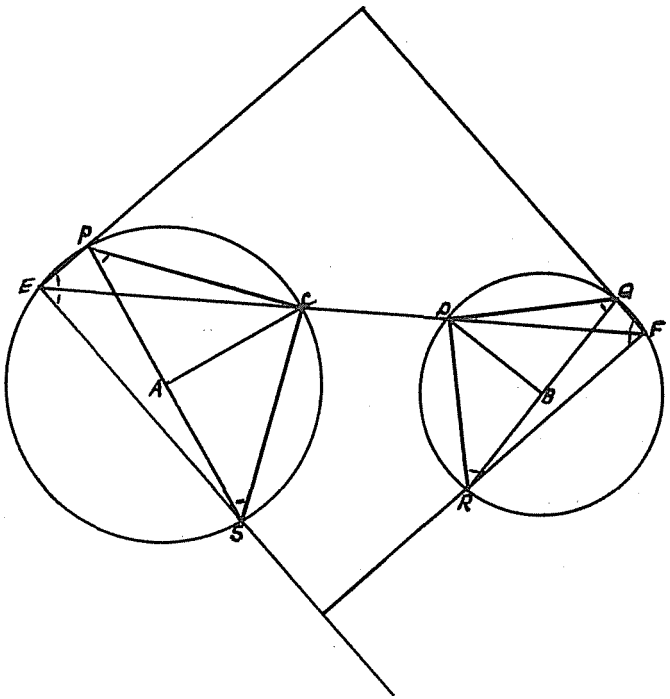


Figure 1

this construction. Now take a quadrangle (see Fig. 2) which will serve as a pattern for the similar quadrangles. In Fig. 2 consider, for definiteness, the diagonal that

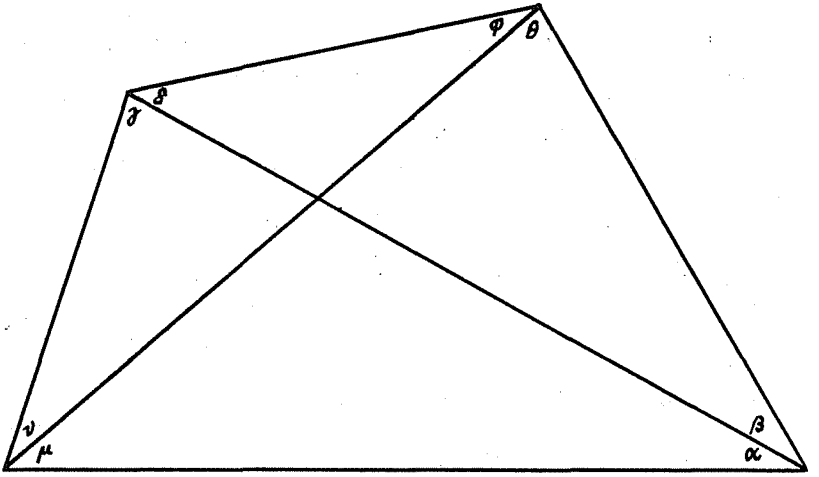


Figure 2

makes with the sides the angles α, β, γ, δ. Any other quadrangle in which a diagonal makes with the four sides the angles α, β, γ, δ, in the same relative order as in Fig. 2, is similar to this quadrangle.

Each of the angles marked at the end of the diagonal EF of the square constructed in Fig. 1 is half a right angle. For the construction of a quadrangle similar to that of Fig. 2, these angles should be α, β, γ, δ, in a proper order. Hence, instead of construction circles on PS and QR as diameters, we shall begin by constructing a first circle with a chord PS that divides the circle into two segments of which one contains an angle $\alpha + \beta$, and a second circle with a chord QR such that one of the segments bounded by QR contains an angle $\gamma + \delta$.

In P.S. the number of constructions of squares with the given incidence properties was investigated. These result from two sets of possibilities:

(1) the two circles can be drawn through PQ and RS, or PR and SQ, or PS and RQ;

(2) for any pair of circles each of the radii perpendicular to its diameter can be drawn in either of two opposite directions. This results in $3 \times 2 \times 2$ or 12 constructions These twelve constructions yield identical squares in pairs so that, in general, the number of distinct squares is six.

Consider now the number of constructions for the similar quadrangles. Figure 3 shows two circles through PS and QR. The smaller of the two angles PAS equals $2\alpha + 2\beta$ and the larger of the two angles QBR equals $2\gamma + 2\delta$. The angle PAS can be divided into parts 2α and 2β in two ways according as the angle 2α has AP or AS as one arm. For each of these ways there is just one way of dividing angle QBR into parts 2γ and 2δ so that α and γ will be on the same side of a diagonal, as similarity of the quadrangles requires. Next, the circle through P,S can be drawn so that the angle PAS $(= 2\alpha + 2\beta)$ is on the opposite side of the chord PS; likewise for the circle through Q,R. Considering the possibilities so far mentioned, we can have $2 \times 2 \times 2 = 8$ constructions. This number is again doubled because of the fact that the angles α and β can be used with the circle through Q,R, and γ and δ with the circle through P,S. Our count is now 16, but the end is not yet. Taking account of the fact that three pairs of circles can be drawn, as indicated by PQ, RS; PR, QS; PS, QR, we bring the count of possible constructions to $16 \times 3 = 48$. This number is again doubled when we make use of the second diagonal of our pattern quadrangle whose angles with the sides are μ, υ, θ, Φ. This brings the final count of possible constructions to 96. As suggested by the problem of the squares, these 96 constructions fall into 48 pairs in each of which the quadrangles, though not the constructions that produce them, are identical. In fact, the quadrangles obtained by using the angles μ, υ, θ, Φ, at the ends of one diagonal, merely repeat those obtained from α, β, γ, δ, at the ends of the other. Figures 3 and 4 show two constructions that produce the same quadrangle EGFH. If these, or two other corresponding constructions, were made on two loose sheets, the identity of the quadrangles could be shown by superposing one sheet on the other and holding them up to the light.

A word of caution may be offered to any Platonic devotees of ruler and compasses. Do not make your quadrangles a parallelogram unless you wish to experiment with a problem that leads to fewer than 48 similar quadrangles. (See the exercise at the end of this paper.) Take the four fixed

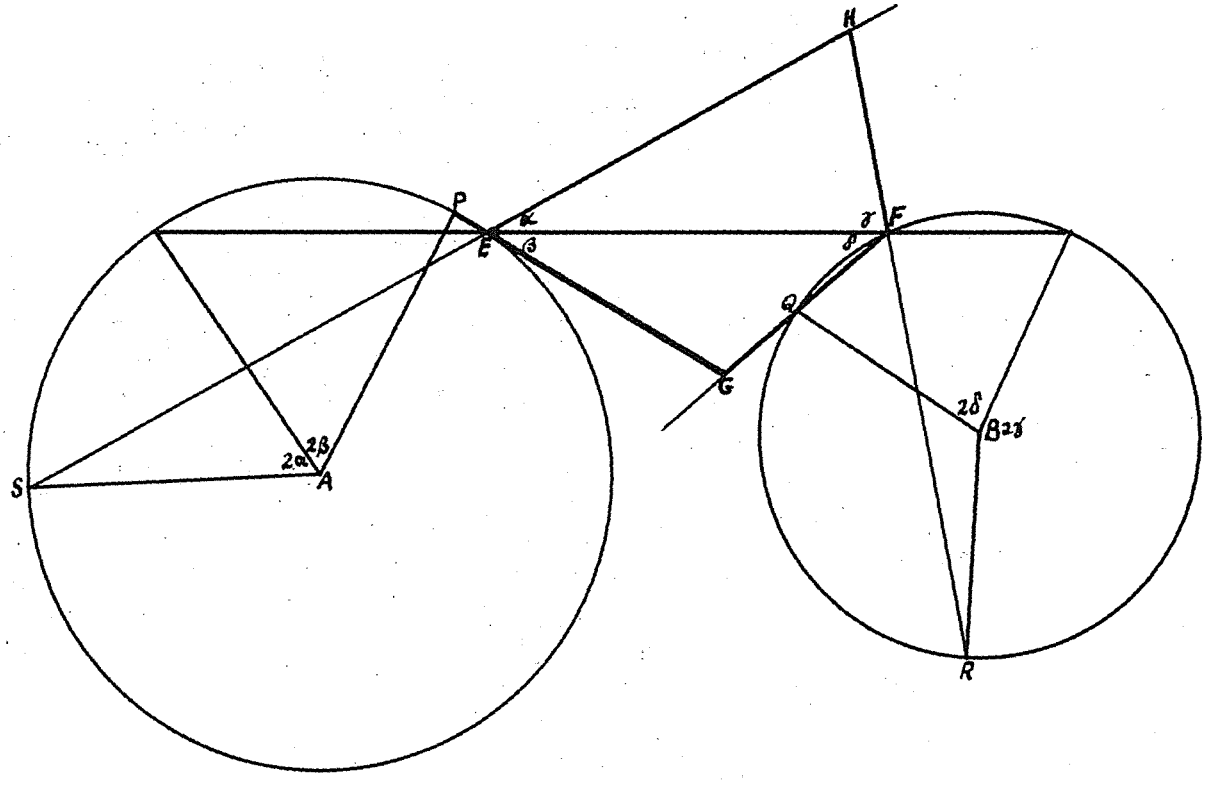


Figure 3

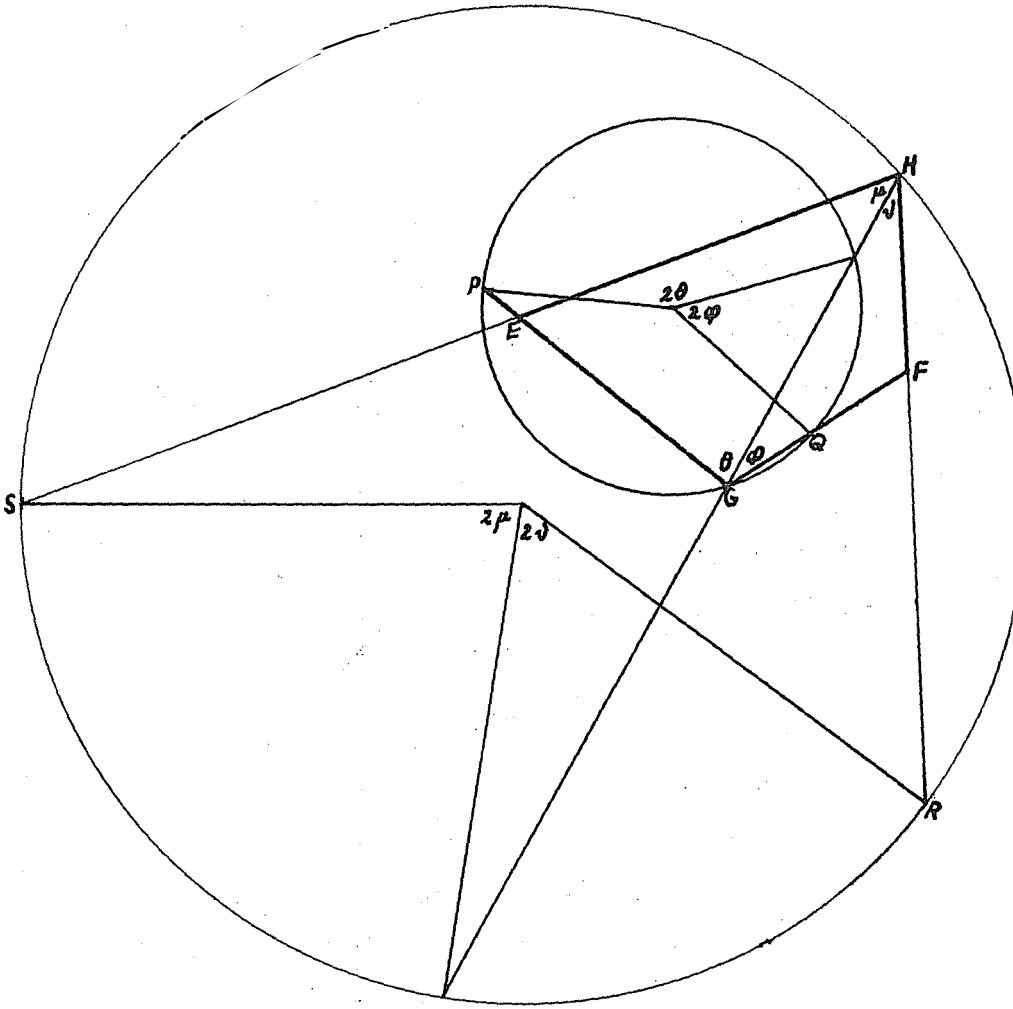


Figure 4

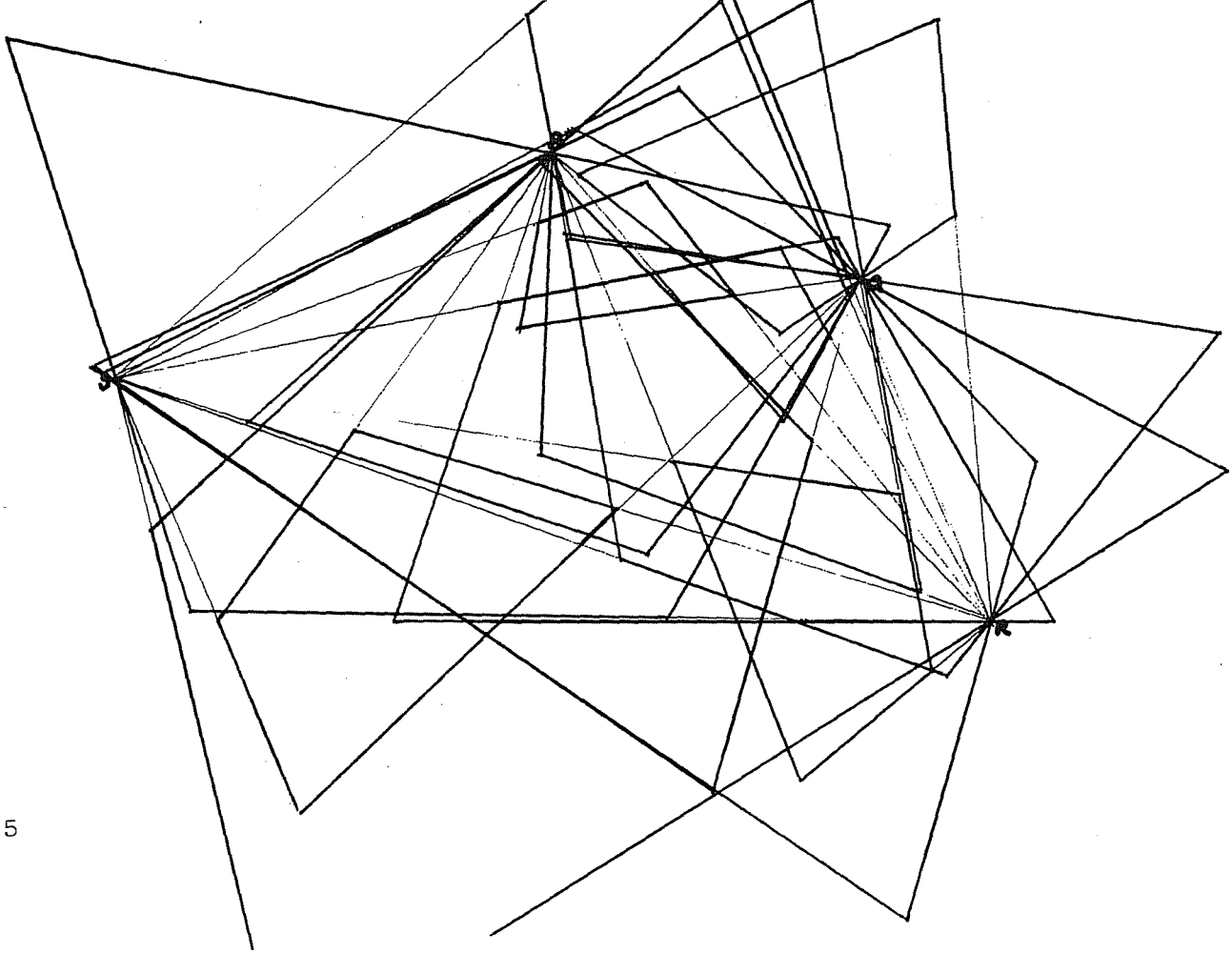


Figure 5

points in an irregular pattern and all contained within a circle of about six inches diameter. Do not make any angles of your pattern quadrangle too acute. With these precautions, you should be able to get each of the figures on a standard sheet, $8\frac{1}{2}$" by 11". When you have decided on the four fixed points, locate them from one sheet to another by means of a needle.

You may find it surprising to see similar quadrangles sprout, as it were, from all points of the compass. Figure 5 shows 14 of the 48 quadrangles that are similar to that of Fig. 2 and whose sides are incident with the points P,Q,R,S. The constructions that give rise to these quadrangles are omitted from the figure. You may find some amusement in locating the 14 quadrangles. Why only 14? The number is arbitrary but a limit had to be imposed before confusion became worse confounded.

Does the problem of the quadrangles admit an extension to other polygons? Take first triangles. A pupil in grade X will have little difficulty in proving that, given three fixed points, an infinite set of triangles similar to a given triangle exists in which the sides of each triangle pass, respectively, through the three fixed points. Now take a pentagon ABCDE (Fig. 6).

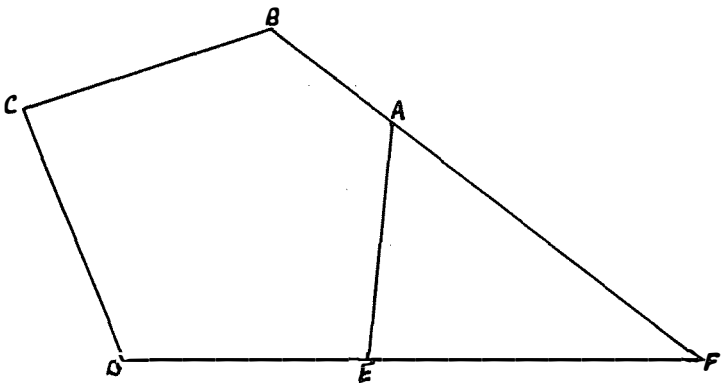


Figure 6

Let DE and BA be produced to meet at F. Now a finite number (48) of quadrilaterals similar to BCDF can be constructed whose sides are incident, respectively, with four preassigned points. Once such a quadrilateral is drawn, a unique side, corresponding to AE, that will complete a pentagon similar to ABCDE, can be drawn. However, the chance that this fifth side will pass through a fifth preassigned point is zero.

The preceding argument shows that, in a plane, the problem of constructing a polygon of n sides similar to a given polygon, having its sides incident each with one of n preassigned points, has, in general, a finite non-empty set of solutions only when $n = 4$. A triangle does not impose enough restrictions to reduce the solutions to a finite number; an n-gon with $n > 4$ imposes more restrictions than can be satisfied except in special cases. The key number that unlocks this problem is four.

Exercise:

Show that, for a non-specialized set of four points, the number of solutions of the quadrangle problem for a non-square rhombus is 12 and for a non-square rectangle, 24.

PROBLEM FOR TRANSLATION

$(x)[x \in \{y|y \text{ is a rose}\} \supset x \in \{z|z \text{ is red}\}]$

$(x)[x \in \{y|y \text{ is a violet}\} \supset x \in \{z|z \text{ is blue}\}]$

$(\exists x):[x \in \{y|\ \text{I love y's}] \cdot [(P)(x \in P = \text{you} \in P)]$

anon.

JUNIOR MATHEMATICS CONTEST

D. Attridge, Orangeville

As the summer holidays have paid their last visit for another year, thoughts are once again turned toward future activities in our schools. The District 10 Junior Mathematics Contest committee met during the summer to discuss future plans and it hopes that the decisions will prove favourable to all concerned.

Last year's contest was expanded to include not only District 10 schools but also schools from the rest of Ontario. The province was divided up into the O.S.S.T.F. districts and a shield was provided by the University of Waterloo for each district having two-thirds of its schools participating. From 30 districts 3837 students wrote the contest and the results were most gratifying to the committee. The competitio was keen and the results were excellent.

Unfortunately the contest was limited last year to 15 students per school, thereby excluding a large number of excel lent students from the larger schools. It is hoped that this year the contest will be open to any number of students from grades 9, 10, and 11 from any school in order to stimulate a more widespread interest in mathematics. The top 3 contestant from any school will still make up the team from that school, but many more students will now have the opportunity to participate in the contest.

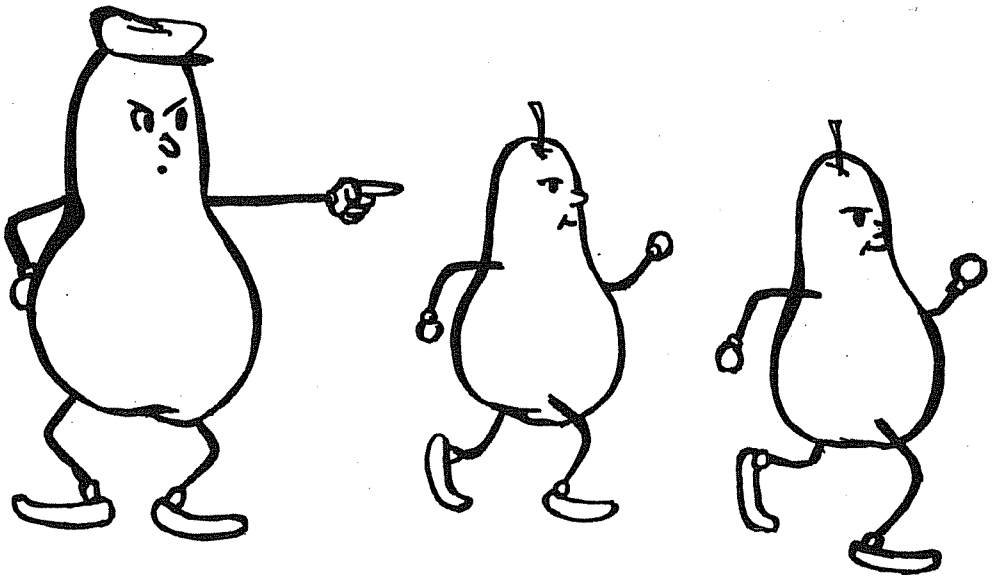
Reprints of the past two years' contests will be availabl at a later date. These will assist students in preparing for the future contests. An entry fee of $2.00 per school will allow 20 students to participate. Additional students may enter at a rate of 10 cents per contestant. These fees will help defray costs in printing and handling which are incurred by the committee.

Some generous grants have been received to promote this contest for which the committee is particularly grateful. It also wishes to express thanks to the many district chairmen who voluntarily gave of their valuable time to assist in getting the information and materials out to the schools in their districts. Also thanks is given to the numerous school supervisors who promoted and administered the contest in their own schools. It is through their efforts that the contest is a success and that interest in mathematics is generated.

The committee hopes to provide a week's seminar during the summer for the top 30 to 40 students in the province in

order to allow these students to dig further into the field of mathematics at a different level and thereby create in themselves and their schools a greater interest in mathematics.

Further news about 1965's contest will be contained in the next Gazette. It is hoped that the new policies of the committee will allow all of our mathematics "students" to participate and that the lack of restrictions will create a much more competitive and interesting contest. Further information may be obtained by writing to Ed Anderson, Wingham; Don Attridge, Orangeville; Ron Dunkley, St. Mary's; or Bill Nediger, Clinton.



BOOK REVIEW

Notes on Plane Coordinate Geometry - B. Abrahamson (London, Butterworths, 1963).

This book, by a member of the Toronto mathematics department, is a stimulating discussion of many aspects of coordinate geometry. While it does not cover the same ground as the Analytic Geometry of Grade 13, superior teachers and students will find many thought-provoking ideas. It seems to this reviewer that a reader who has already encountered Analytic Geometry is likely to gain more from the conceptual discussions in the book.

There are six chapters entitled: Introduction, Motions, The Straight Line, Internal and External, Foundations of Trigonometry, Polygons. In Chapter 2, the author firmly bases all subsequent work on a definition of a Motion: A motion is a transformation M which transforms points P into other points P' in such a way that for any two points P and Q, PQ = P'Q'; then he discusses translations, rotations, and vectors. Chapter 3 introduces triangles and the concept of congruence, proves the congruence theorems, and introduces work with scalar and vector products, incidence of lines and points, the various forms of the line equation. Chapter 4 discusses internal and external division of a line segment, the two sides of a line, the interior and exterior of a triangle. In Chapter 5, one finds the definition of a directed angle: A directed angle $\angle PAQ$ is the configuration consisting of two half-lines AP and AQ in that order, meeting in a common vertex A; the half-lines AP and AQ are called the arms of the angle. From this the author develops the usual basic trigonometric relations, making considerable use of components of vectors. A noteworthy remark: "Instead of regarding the trigonometric functions sine, cosine, and tangent as functions of angles, we can consider them as functions of real numbers; this is their most convenient form for the purposes of analysis". Finally, in Chapter 6, area is introduced and it is shown that a simple oriented polygon has an interior and an exterior.

In recapitulation, a good book for its stress on concepts. There are few exercises. A school with a problems class should have this book on its reference shelf.

N. N. H.

CORRESPONDENCE

The Editor,
Ontario Mathematics Gazette,

Dear Sir:

While there is no doubt that you have been most successful in including in the Gazette many interesting and important ideas pertinent to the teaching of High School Mathematics, I must object to what appears to me as your misguided zeal in exploiting your position to ride a favourable hobby-horse about the definition of function. While the definition, which presumably you approve, expounded on P. 30 of the December issue is one that is widely accepted by mathematicians, and which I, myself, employ on occasion, it is quite misleading for you to maintain that it has any real advantage over the definition of a function as a set of ordered pairs satisfying the single valuedness condition. Your so-called philosophical arguments are unsound and your examples irrelevant. From the latter one would conclude that you consider a function to be a meat-grinder or a marriage ceremony. While in both instances some function (in the everyday use of the word function) is performed, neither analogy conveys the essential point about a function as clearly as does the definition as a set of ordered pairs.

Prof. M. E. Munroe, who has written one of the standard works on Lebesgue integration, begins his recent book entitled "Modern Multi-Dimensional Calculus" (Addison-Wesley Publishing Company, 1963) on P.1 as follows: "A function is defined as a set of ordered pairs of numbers no two of which have the same first entry, The set of all first entries in the ordered pairs is called the domain of the function. The set of all second entries is called its range." This definition corresponds with the definition of logarithms and trigonometrical functions. Your predeliction for defining functions with the help of formulas has only a very minor application in mathematics as a whole, and is certainly not very helpful to the High School student in connection with the above functions.

Nor is their any validity in the implication of the article on P.30 and in Professor Staal's article that defining functions as sets of ordered pairs is a recent fad indulged in by incompetent mathematicians. Few books have had a greater effect on recent advanced mathematics than those of Professor Lefschetz on Algebraic Topology published in 1930 and 1942 by the American Mathematical Society. As you are aware, the A.M.S. is the most distinguished association of research mathematicians in the world. On P.2 of the 1942 edition Lefschetz defines a

function by beginning with a set of ordered pairs. It is true that he is a bit vague as to whether he is thinking of the function, as the set of ordered pairs, or as the correspondence spelled out by means of the ordered pairs, but this is as it should be since both definitions are logically equivalent and no one has mastered the concept who is not thoroughly familiar with both approaches.

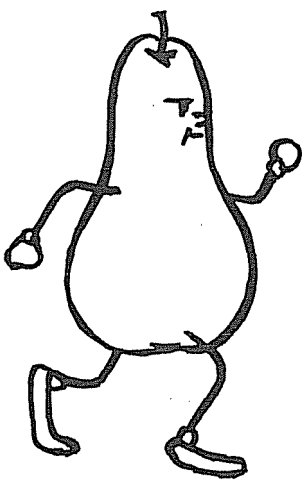
Your author, who is too timid or too modest to sign his name, asks a question which is in brackets near the bottom of page 32. The fact of the matter is that the shoe is on the other foot. The correspondence from the independent to the dependent variable presupposes the notion of ordered pair as a logically undefined idea. Only after assuming that does your author succeed in defining a function. Thus, not only is the definition of a function as a set of ordered pairs more concrete and closer to the actual experience of students in using tables and physicists in recording the results of experiments, but also it is logically prior to the one proposed by your author.

Yours sincerely,

(signed) John Coleman

A. J. Coleman,
Head,
Department of Mathematics,
Queen's University.

[Editor's note: While we do not agree with a number of the statements made by Professor Coleman, we welcome his contribution. The more we can stimulate comment and discussion, the greater will be the success of the Gazette in increasing interest in mathematics in Ontario.]



A DIALOGUE ON THE APPLICATIONS OF MATHEMATICS*

Archimedes: Your Majesty! At this late hour! This is a surprise! To what do I owe the honour of a visit from King Hieron in my modest home?

Hieron: My dear friend Archimedes, we had a dinner in my palace this evening to celebrate the great triumph of our small city Syracuse over the mighty Romans. I invited you, but your place remained empty. Why didn't you come, you above all, to whom we owe today's victory? Your huge concave brazen mirrors set afire ten out of the twenty big ships of the Romans; they speeded like fiery torches out of the harbour in the southwest gale, but all went down before reaching the open sea. I could not go to sleep without thanking you for delivering our city from the enemy.

Archimedes: They may come back, and we are still surrounded on the mainland.

Hieron: We shall speak about this later. Let me first hand over to you a present, which is the best I can give.

Archimedes: A wonderful masterpiece indeed!

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* This dialogue was read at the University of Toronto on March 30, 1964. The main aim of the dialogue is to discuss certain basic problems concerning mathematics and its applications in a form which — it is hoped — brings these questions nearer to the non-expert. The historical setting as well as the dialogue form serve both these purposes. In view of this the author considered it admissible to attribute to Archimedes — besides results and views which he certainly had (see The Works of Archimedes with the Method of Archimedes, Edited by T.L. Heath, Dover, New York) — also some results and views for which there is no direct evidence, but which he may have had. Thus for instance some ideas belonging to cybernetics and also the invention of a sieve device (which was in fact constructed only in our day by D. H. Lehmer) are attributed to Archimedes. While these bold conjectures can certainly not be proved, they can not easily be disproved either. (After all, in solving for instance certain problems, which, according to present day terminology, consist in evaluating definite integrals. Archimedes was ahead of his age by at least 1400 years.) As regards the historical background, the main departure from known facts is that King Hieron died about three years before the dialogue could have taken place, that is 212 B.C.; the author hopes to be excused for making use of "poetic license" in disregarding this fact.

Hieron: This tray is of pure gold; you may test it with your method, you will find no trace of silver in it.

Archimedes: The reliefs show the adventures of Odysseus. In the middle I see the unsuspecting Trojans pulling the giant wooden horse into their castle—I always wondered whether the Trojans used some sort of compound pulley to accomplish this? Of course the horse stood on wheels, but the road to the castle must have been rather steep.

Hieron: My dear Archimedes, by Zeus, forget for a moment your pulleys. You know how astonished I was when you alone launched the heavy ship I wanted to send to King Ptolemy, by turning the handle of your triple pulley. But have a look at the other scenes on the tray.

Archimedes: I recognize the Cyclops and Circe as she changes the companions of Odysseus into pigs, and here the concert of the sirens, to which Odysseus listens while he is chained to the mast of this ship: if you look at his face you can almost hear the enticing song. And there is Odysseus in the nether world, meeting the shadow of Achilles, and here he is frightening the charming Nausikaa and her girls, and finally of course the scene where Odysseus disguised as an old beggar spans his bow and squares with the suitors—a marvelous piece of art really. I thank you my gracious king: this is really a king's gift.

Hieron: It is the best piece of my treasure-house but you deserve it. I have chosen it not only for its beauty and value, but for a third reason. What you did today for Syracuse can be compared only with the trick of Odysseus. Both are triumphs of a sharp mind over brute force.

Archimedes: You make an old man blush. But let me remind you again that the war is not over yet. Would you like to hear the advice of an old man?

Hieron: I even bid you as your king to tell me frankly your opinion.

Archimedes: This is the moment when you should make peace with the Romans; since the war started we have never been in such a favourable bargaining position. If Marcellus does not send his envoy to you before midnight, you should send yours to him before dawn, and make peace before the sun sets again. Marcellus is eager to withdraw his troops which besiege the city, because he needs them against Hannibal. Moreover, if he can reach an agreement tomorrow, he can report to Rome a victory, if only a diplomatic one and not just the sad news that half of his fleet is lost. After the report about today's battle reaches Rome, the Romans will be so furious that they will not be satisfied with anything less than total victory.

Hieron: Your analysis is correct. As a matter of fact, I got a message from Marcellus this evening in which he offered peace and withdrawal of his troops under certain conditions. If you knew these conditions you would be less keen on making a deal with the Romans.

Archimedes: What does Marcellus want?

Hieron: Well, he wants ten new ships, for the ten ships that we sank today, further that all our forts be demolished except one in which a garrison of Roman soldiers would be stationed; of course he wants a lot of gold and silver, and that we declare war on Carthage, but then he wants my son Gelon, my daughter Helena and yourself as hostages. He promises however that no harm will be done to the city and its inhabitants as long as we adhere to the treaty.

Archimedes: Perhaps he does not insist on everything. Of course he will insist that you should hand me over to him.

Hieron: You speak coolly about this. By all the gods of Olympus, as long as I am alive I will not give my children into the hands of the enemy, nor will I give you to them! I do not mind the gold and the ships: he can have them. But what I dislike most in his conditions is that if we fulfil them we shall be completely at his mercy. What guarantees are there that he will keep the treaty: he does not give me hostages.

Archimedes: Take care not to question that he will stick to his words; Romans are sensitive about their honour, at least during negotiations. But perhaps you can avoid giving him your children.

Hieron: And what about you. Would you be ready to make this sacrifice for your city?

Archimedes: Is this a question or a request?

Hieron: Of course a question only. Do you want to know what I replied to Marcellus?

Archimedes: You answered already?

Hieron: Yes. I accepted all his conditions except giving you as a hostage but I agreed to give my children as hostages only under the condition that he sends me two of his children as hostages too. As regards you, I told him that your age does not permit your living in camp. But knowing that he wants you really not as a hostage but he wants your wisdom, I promised that you would write for him a full description of all your inventions which are of military importance.

Archimedes: I will never write anything about my inventions concerning warfare.

Hieron: Why not? If there will be peace, we do not need them anymore. Could you explain why you refuse to write about your inventions?

Archimedes: If you have the patience to listen to my reasons, I shall do so.

Hieron: I am ready to listen. I want to remain awake and wait for Marcellus's answer.

Archimedes: Then we have a lot of time, because Marcellus will need some time to formulate his answer. It will sound like a whip.

Hieron: Do you think he will discontinue the negotiations?

Archimedes: Surely. You contested his honour. He will never forgive this, and there will be no agreement.

Hieron: You may be right.

Archimedes: I always admired your artful diplomacy and your psychological insight into your opponents' hearts. But this time you neglected this art.

Hieron: I have to admit this. Perhaps I was too drunk with wine and with victory. But what is done is done. Still, I want to hear your reasons.

Archimedes: Though the question became an academic one, nevertheless I shall explain my point of view. You compared my machines with the wooden horse of Troy. Well, your comparison is really very close, but in a quite different sense. Odysseus used the wooden horse to smuggle himself and some Greek soldiers into Troy. I used my machines to smuggle an idea into the public mind of the Greek world: the idea that mathematics — not only its elements, but also its most subtle parts — can be applied with success to practical purposes. I have to confess that I hesitated quite a long time before doing this, because I hate war and murder. But the war was here anyway, and this was the only way to make myself understood. I have tried other ways but in vain. May I remind you that some years ago when I invented a pump to pump out the water from your mines so that the people who work there should not wade to their hips in water — you were not interested. The supervisor of your mines told me that he did not care about the legs of the slaves getting wet — they were not made of sugar, these were his words. And do you remember when I proposed to make a machine to irrigate your fields? I was told that slave work was cheaper. And when I proposed to use the force of steam to drive the mills of King Ptolemy what was his answer? He said that the mills which served his ancestors would serve him as well. Shall I remind you of other examples? There were at least a dozen others. All my

endeavours to show the world what mathematics can do for them in peace were in vain. But when the war came, suddenly you remembered my pulleys, cogwheels and levers. In peace-time everybody regarded my inventions as toys, unworthy of a serious grown-up citizen, still less of a philosopher. Even you, who always supported me, and helped me to realize my ideas, did not take them quite seriously; you have shown them to your guests, to entertain them but that was all. Then the war came and the Roman ships closed the harbour; I ventured a casual remark, that by throwing stones on them by a catapult we could drive them away; you jumped at the idea. I could not take back what I had said, and had to go ahead. Once I started on this road, I had to continue. But my feelings about it were mixed from the start. I was of course happy that my inventions were not ridiculed any more and at last I had a chance to show the world what mathematics in action really is. But this was not the sort of action by which I wanted to prove the practical value of mathematical ideas. I saw men killed by my machines and this made me feel guilty. I made a solemn oath to Athene that I would never tell the secret of my war machines to anybody either by word or in writing. I tried to soothe my conscience by saying to myself that the news about Archimedes defeating the Romans by mathematics would reach all corners of the Greek speaking world, and this would be remembered even when the war was over and the secrets of my war-machines were buried with me.

Hieron: It is true my dear Archimedes I am getting letters from kings who are my friends asking about your inventions.

Archimedes: And what do you answer them?

Hieron: I tell them that these questions are classified as long as the war goes on.

Archimedes: I hope that you understand now that I have other reasons for not publicizing my secrets. I succeeded in keeping the secrets even from those who are carrying out my plans. Each man knows about some detail only. I am glad you never asked me questions because I would have refused to answer.

Hieron: But now I shall ask you some questions. Don't be afraid, I shall not ask for your secrets, only about the underlying general principles.

Archimedes: I think I can answer such questions without breaking my oath.

Hieron: Before asking you such questions, I want to ask you something else. Why was it so important to you that your ideas about the usefulness of mathematics be accepted?

Archimedes: Perhaps I was a fool but I thought that I could change the course of history. I was worried about the future of our Greek world. I thought that if we applied mathematics on a large scale--after all mathematics is a Greek invention, and I think the best achievement of the Greek spirit—we might save our Greek way of life. Now I realize that it is too late. The Romans will conquer not only Syracuse but all other Greek cities too; our time is over.

Hieron: I think that even in this case, our Greek culture will not be lost: the Romans will take it over. Look how they try to imitate us already. They copy our statues, translate our literature--and you see that Marcellus is already interested in your applied mathematics.

Archimedes: The Romans will never really understand it. They are too practical-minded, and they are not interested in abstract ideas.

Hieron: They are certainly interested in its practical uses.

Archimedes: These things can not be separated. One has to be a dreamer of dreams to apply mathematics with real success.

Hieron: This sounds rather paradoxical. I thought that to apply mathematics one should first of all have a good practical sense. This leads me to my first question. What really is the secret of the new science which you invented—let us call it applied mathematics, and what is the main difference between your applied mathematics and that sort of mathematics—let us call it pure mathematics—which is taught in the schools?

Archimedes: I am sorry that I have to disappoint you. There does not exist any other kind of mathematics besides that which your teachers taught you, and not without success as I recall. Applied mathematics as an art, which is different and separated from mathematics as such does not exist. My secret is so well hidden, because it is not a secret at all: its very obviousness is its best disguise. It is hidden like a golden coin thrown into the dust of the street.

Hieron: Do you mean that your marvelous machines are based on that sort of mathematics that every educated man knows?

Archimedes: You are getting nearer the truth.

Hieron: Could you give me an example?

Archimedes: Well, let us take as an example the mirrors which did such a good job today. What I did was simply to remember a well known property of the parabola: take any point P of a parabola, connect it with the focus, and draw through P a line which is parallel to the axis: these two

lines form equal angles with the tangent to the parabola in the point P. You can find this theorem in the books of my distinguished colleagues of Alexandria.

Hieron: It is hard to believe that you have destroyed half the fleet of Marcellus by this theorem, which is one out of hundreds of similar geometrical propositions. I remember it vaguely, though I forget its proof.

Archimedes: Probably when you heard one of its ingenious proofs you understood it, and perhaps even admired its beauty and elegance, but that was all. Some mathematicians went one step further, they explored some of its purely geometrical consequences, or invented new proofs, but there they stopped. I simply went one simple step further, I looked at its non-mathematical consequences also.

Hieron: I thought you invented some new law of optics.

Archimedes: Optics is, after all, nothing but another branch of geometry. What I used from optics, the law of reflection of a ray has also been known a long time.

Hieron: Do you mean that in order to apply mathematics one does not need to get new mathematical results, only to fit together a practical situation and its mathematical counterpart: some well known mathematical proposition?

Archimedes: No, it is not quite so simple. It often happens that the theorem which you need does not exist, and you have to find and prove it yourself. But even if this is not needed, to find the mathematical counterpart—as you call it (I prefer to call it a mathematical model), of a practical situation is not the same as matching gloves. First of all, one can construct many mathematical models for the same practical situation, and one has to choose the most appropriate one, that which fits the practical situation as closely as your practical aims require (it can never fit completely) and at the same time is not too complicated, but is still mathematically feasible. These are of course conflicting requirements and a delicate balancing of these two is usually necessary. You have to approximate closely the real situation in every respect which is important for your purposes, but to neglect everything which is of no importance for your actual aims. A model need not to be similar to the modeled reality in every respect, only in those respects which really count. On the other hand, the same mathematical model can be used to fit quite different practical situations. For instance I used the properties of parabolas also in construction catapults, because the track of a stone thrown by a catapult can be approximated to some extent by a parabola. I also used parabolas in computing how deep a ship will dip into the sea under the weight of its load. Of course, the cross section of a ship has not exactly

the shape of a parabola, but a more realistic model would not have been mathematically manageable. The results were nevertheless in fairly good agreement with the facts. Expecially I was able to find out under what conditions a ship would be able to stand upright when tilted by the wind and the waves because its center of gravity tends to be in the deepest possible position. In trying to describe such a complicated situation even a very rough model may be useful because it gives at least qualitatively correct results, and these may be of even greater practical importance than quantitative results. My experience has taught me that even a crude mathematical model can help us to understand a practical situation better because in trying to set up a mathematical model we are forced to think over logically all possibilities, define all notions unambiguously, and to distinguish between important and secondary factors. Even if a mathematical model leads to results which are not in accordance with the facts, this may be useful because the failure of one model can help us to find a better one.

Hieron: It seems to me that applying mathematics is similar to warfare: sometimes a defeat is more valuable than a victory, because it helps us to realize the inadequacy of our arms or of our strategy.

Archimedes: Now you have really grasped the essential point.

Hieron: Tell me something more about your mirrors.

Archimedes: I have told you the basic idea already. Of course after I hit on the idea of using the mentioned property of parabolas, I had to solve the question of how to grind and polish a metal mirror into the form of a concave paraboloid of revolution; but I would prefer not to speak about this. Of course I also had to select an appropriate alloy.

Hieron: Without intruding into your secrets, it is clear that besides the properties of the parabola you had to know a lot about metals and the art of dealing with them too. This shows, it seems to me, that a knowledge of mathematics is not sufficient if somebody wants to apply it. Isn't a man who wants to apply mathematics in a similar position to a man who wants to ride on two horses at the same time?

Archimedes: I would change your simile slightly: he who wants to apply mathematics is like a man who wants to harness two horses to his carriage. This is not so difficult to do, some knowledge of horses as well as of the carriage is of course needed, but any of your coachmen knows how to do this.

Hieron: Now I am quite confused: every time I think that applied mathematics is mysterious, you show me that it is really quite simple: but when I am convinced that the whole thing is really simple, you point out that it is much more complicated than I imagined.

Archimedes: Its principles are obvious, but the details are sometimes quite involved.

Hieron: I do not understand yet what you mean by a mathematical model. Tell me more about this.

Archimedes: Do you remember the sphere which I constructed some years ago to imitate the motions of the sun, the moon and the five planets, by which it was possible to show how the eclipses of the sun and the moon take place?

Hieron: Of course, it is one of the things in my palace which I show to all my visitors; everybody thinks that it is marvelous. Is this a mathematical model of the universe?

Archimedes: No, I would call it a physical model. Mathematical models are invisible, they exist only in our mind, and can be expressed by formulae. A mathematical model of the universe is that which is common to the real universe and to my physical model. In the physical model to each planet for instance there corresponds a tiny ball about the size of an orange. In my mathematical model of the universe the planets are represented simply by points.

Hieron: I think that I begin to understand what you mean by a mathematical model. But let us return to the simile about horses. The art of harnessing horses to a carriage and driving them is quite different from breeding and raising horses. Isn't the art of applying mathematics quite distinct from making mathematics: finding and proving theorems?

Archimedes: You are of course right about horses, though the man who raised a horse usually knows most about it and can drive it better than anybody else. As regards mathematics, as I pointed out earlier, in order to be able to apply it successfully one has to have a deep understanding of it, and if somebody wants to apply mathematics in an original way he has to be a creative mathematician. Conversely, the occupation with applications can aid in pure mathematical research.

Hieron: How is this possible? Could you give an example?

Archimedes: Perhaps you remember that some time ago I was very interested in a question of mechanics, namely finding the center of gravity of a body. The results which I obtained about centers of gravity helped me to build machines, and also they helped me to prove new geometrical propositions. I have developed a peculiar method which consists of investigating geometrical problems by means of mechanical considerations concerning centers of gravity. This method is of a heuristic character; that means that it does not furnish exact proofs. Many theorems first became clear to me by using this method of reasoning. Of course as the method does not furnish actual demonstrations, I afterwards proved the theorems which I conjectured by means of my mechanical

method by the traditional methods of geometry. But it is much easier to supply the proof if one has previously acquired by the method of mechanical analogies some knowledge of the question, and thus knows what he has to prove, than without such previous knowledge.

Hieron: Tell me one theorem which you have found in this strange way.

Archimedes: The area of any segment of a parabola is four thirds of the area of the triangle which has the same base and height. After finding the result with my method, I found a proof along traditional lines too.

Hieron: If you found this theorem by mechanics, why did you bother about the geometrical proof?

Archimedes: When I first discovered my method, the results which I got with its help were not all correct; later, by analysing the cases in which the method misled me, I developed it so far that now it never misleads me. But still I can not prove that every result which I get in this way is really true; maybe somebody will prove this some day, but until that time comes I do not have complete confidence in the method.

Hieron: Are strict proofs in applied mathematics really necessary at all? After all, as you said the mathematical model is only an approximation to reality. If you use approximately correct formulae your results will be still approximately close to reality and they can never be absolutely correct anyway.

Archimedes: You are mistaken my king. Just because the mathematical model is only an approximation and there is always a certain discrepancy with the facts, one has to take care not to increase this discrepancy further by a careless use of mathematics, one has to be as accurate as possible. By the way in regard to approximations, it is a common misunderstanding that using approximations means a departure from mathematical precision. Approximations have a precise theory, and results about approximations — for instance inequalities — have to be proved as rigorously as identities. Perhaps you remember the approximations which I gave for the area of a circle with unit radius, I proved them with the rigour which is usual in geometry.

Hieron: What other results did you find by your mechanical method?

Archimedes: This method led me to discover also that the volume of a sphere is two thirds of the volume of a circumscribed cylinder.

Hieron: I heard you say that if you die you want this theorem to be shown on your gravestone. Does it mean that you consider this your most outstanding result?

Archimedes: I think the method itself is more important than any of the particular results which I got with its help. Do you remember when I told you once, speaking about levers: "Give me a place to stand on and I can move the earth". Of course there doesn't exist such a fixed point in the world. However if one is doing mathematics, one has such a fixed point to lean on, namely the axioms, and logic is indispensable. Applying mathematics to problems of the real world means to move the earth from the fixed point of mathematics.

Hieron: You always speak about applying mathematics, but all the examples you give are applications of geometry. In regard to geometry, I see now how it can be applied. For instance the functioning of a machine depends on the form and size of its components, the track of a stone thrown by your catapult is a curve, approximately a parabola as you said. But what about other branches of mathematics, say number theory? I can hardly imagine how it can have any practical importance. Of course I am not speaking about the elements of arithmetic which is clearly used in every sort of computatation; I mean things like divisibility, prime numbers, least common multiple and other similar topics.

Archimedes: Well if you connect two cogwheels with a different number of teeth, the least common multiple comes in inevitably. Does this simple example convince you? Recently I got a letter from my friend Eratosthenes of Cyrene in which he wrote about a simple but ingenious method — he calls it the sieve method — which he invented to find prime numbers. Pondering about his method, I made a sketch of a machine which realizes his idea. This machine works with a set of cogwheels of different sizes, if you turn this machine a number of times say n, and look into a hole, then if you can look through it the number n is a prime, but if the view is closed then it is composite.

Hieron: This is really amazing. If the war is over you must build this machine. My guests will love it.

Archimedes: If I am alive I shall certainly do it. This will show that machines can solve mathematical problems. Perhaps this will help mathematicians to realize that even from their own point of view they may gain something from studying the relation of mathematics and machines.

Hieron: Speaking about gain, I recall a story about Euclid; one of his students studying geometry asked Euclid; "What shall I gain by learning these things?" Whereupon Euclid called his slave and said: "Give him threepence, since he wants to gain from what he learns". It seems to me that

this story shows that Euclid thought it unnecessary for a mathematician to bother about the practical use of his results.

Archimedes: I have of course heard the anecdote, but you will be surprised if I tell you that I sympathise completely with Euclid, in his place I would have said something similar.

Hieron: Now I am confused again. Up to now you spoke with enthusiasm about the applications of mathematics, and now you agree with the purists who think that the only reward to which a scientist should look forward in doing mathematics is the pleasure of knowledge.

Archimedes: I think you and most people misunderstand the story about Euclid. It does not mean that he was not intereste in the practical consequences of mathematical results; and that he considered them unworthy of a philosopher. This is pure nonsense; he has written as you certainly know a book called Phaenomena, about astronomy, and a book on Optics and he is probably the author of the book Catoptrica too, which I used in constructing my mirrors; he was also interested in mechanics. As I understand the story, Euclid wanted to emphasize the remarkable fact that mathematics rewards only those who are interested in it not only because of its rewards but also because of itself. Mathematics is like your daughter Helena who is suspicious every time a suitor appears that he is not really in love with her, but is interested in her only because he wants to be the son-in-law of the king. She wants a husband who loves her for her own beauty, her wit and charm and not for the wealth and power which he can get by marrying her. Similarly mathematics reveals its secrets only to those who approach it with pure love, for its own beauty. Those who do this are of course also rewarded with results of practical importance. But if someone asks at each step "What can I profit by this?" he will not get far in mathematics You remember when I told you that the Romans will never be really successful in applying mathematics, now you see why; they are too practical-minded.

Hieron: I think we should learn from the Romans, then we would have better success in fighting against them.

Archimedes: I do not agree with this. If we try to win by giving up the ideas we stand for and by imitating our opponents we are lost before the battle begins. Even if we could win the war in this way, it would not be worth while, such a victory is worse than a defeat.

Hieron: Let us not speak about the war but return to mathematics. Tell me, how do you construct your mathematical models?

Archimedes: It is difficult to explain this in general terms. Perhaps a simile will help. A mathematical model of a real

situation is something like its shadow on the screen of the mind.

Hieron: It seems to me that your philosophy is the exact opposite of Plato's. He says that the real things are the shadows of ideas, while if I correctly understood the meaning of your words, you are saying that ideas are the shadows of reality.

Archimedes: The two points of view are not so far from each other as it seems. Plato was puzzled by the correspondence between mathematical ideas and reality, and he thought that the main task of philosophy was to explain this correspondence. Up to this point I agree with him completely. I do not agree with his explanation, but at least he saw the problem clearly and tried to work out one of the logically possible answers. But I think we have to leave philosophy and return to the facts of life, because I hear somebody knocking at the door. I am going to open the door.

Hieron: Let me do it, it must be my envoy with Marcellus's answer.....Here is the message.

Archimedes: What is his answer?

Hieron: Read it yourself.

Archimedes: Let me see. "Marcellus sends his greetings to King Hieron and announces that he will conquer Syracuse before the new moon, then King Hieron will realize that a Roman sticks to his words."

Hieron: Now what do you think about this?

Archimedes: His Greek is really not bad. As regards the contents, it is what I expected.

Hieron: Truly, your prediction was as correct as if you had found it by your method.

Archimedes: Well at least we know now what to expect.

Hieron: I must go now, I need some sleep, tomorrow we have to prepare ourselves for a new attack. Thank you for this interesting talk.

Archimedes: I enjoyed it very much. I don't often have the opportunity to talk about mathematics nowadays. Thank you again for the wonderful tray.

Hieron: I am glad you like it. Good night my friend, I think you need rest too.

Archimedes: Good night my king. I will not sleep yet. I want to finish a letter to my friend Dositheus of Pelusium about my latest results. Now that the Roman fleet is gone there will certainly be ships leaving the harbour tomorrow, but perhaps after tomorrow the Romans will set up the blockade again. I want to use this occasion; it may be the last one.

SOME DIOPHANTINE PROBLEMS - III

R. G. Stanton

1. Introduction.

In this article we shall consider a particular problem, namely, the solution of

$$x^2 - 3y^2 = 1.$$

Equations of this type are usually treated by using continued fractions; we shall show how the solution can be exhibited using elementary methods only. The procedure is of considerable interest involving geometric illustration, problems with inequalities, and Fermat's "method of descent".

2. A Family of Solutions.

Clearly (p_1, q_1) is a solution, by inspection, where $p_1 = 2$, $q_1 = 1$. We show how to use this solution to give an infinitude of solutions.

$$p_1^2 - 3q_1^2 = 1,$$
$$(p_1 + q_1\sqrt{3})(p_1 - q_1\sqrt{3}) = 1,$$
$$(p_1 + q_1\sqrt{3})^n(p_1 - q_1\sqrt{3})^n = 1^n = 1.$$

Now suppose $p_1 + q_1\sqrt{3}$ is raised to the nth power by the binomial theorem; we end up, after collecting terms, with

$$(p_1 + q_1\sqrt{3})^n = p_n + q_n\sqrt{3},$$

where p_n comes from those terms where $\sqrt{3}$ occurs to an even power, $q_n\sqrt{3}$ comes from those terms where $\sqrt{3}$ occurs to an odd power.

Similarly,

$$(p_1 - q_1\sqrt{3})^n = p_n - q_n\sqrt{3}.$$

Thus

$$(p_n + q_n\sqrt{3})(p_n - q_n\sqrt{3}) = 1,$$
$$p_n^2 - 3q_n^2 = 1.$$

We thus have the

Theorem: From the particular solution (p_1,q_1), we can derive an infinite family of solutions (p_n,q_n) where

$$p_n + q_n\sqrt{3} = (p_1 + q_1\sqrt{3})^n.$$

We can derive a useful recursion relation for computing (p_n,q_n) in

Corollary 1. $$p_n + q_n\sqrt{3} = (p_1 + q_1\sqrt{3})^{n-1}(p_1 + q_1\sqrt{3})$$
$$= (p_{n-1} + q_{n-1}\sqrt{3})(2 + \sqrt{3}).$$

Multiplying out and comparing coefficients, we at once obtain

$$p_n = 2p_{n-1} + 3q_{n-1}, \quad q_n = p_{n-1} + 2q_{n-1}.$$

Using these relations, we at once obtain

$(p_1,q_1) = (2,1)$ $\qquad$ $(p_4,q_4) = (97,56)$

$(p_2,q_2) = (7,4)$ $\qquad$ $(p_5,q_5) = (362,209)$

$(p_3,q_3) = (26,15)$ $\qquad$ etc.

Corollary 2. All members (p_n,q_n) of the family are distinct solutions.

This follows at once from the fact that

$$p_n = 2p_{n-1} + 3q_{n-1} > p_{n-1}$$

for all n. So no two of the first elements p_n of the solution pairs are identical.

3. Geometric Interpretation of the Solutions.

The solutions $P_1(2,1)$, $P_2(7,4)$, $P_3(26,15)$, $P_4(97,56)$, $P_5(362,209)$, etc., can be plotted in the Cartesian plane, and all lie on the hyperbola

$$x^2 - 3y^2 = 1.$$

See Figure 1.

However, we can also look at these points as representing single numbers, just as we represent complex numbers on a plane. Thus P_1 can be identified with the single

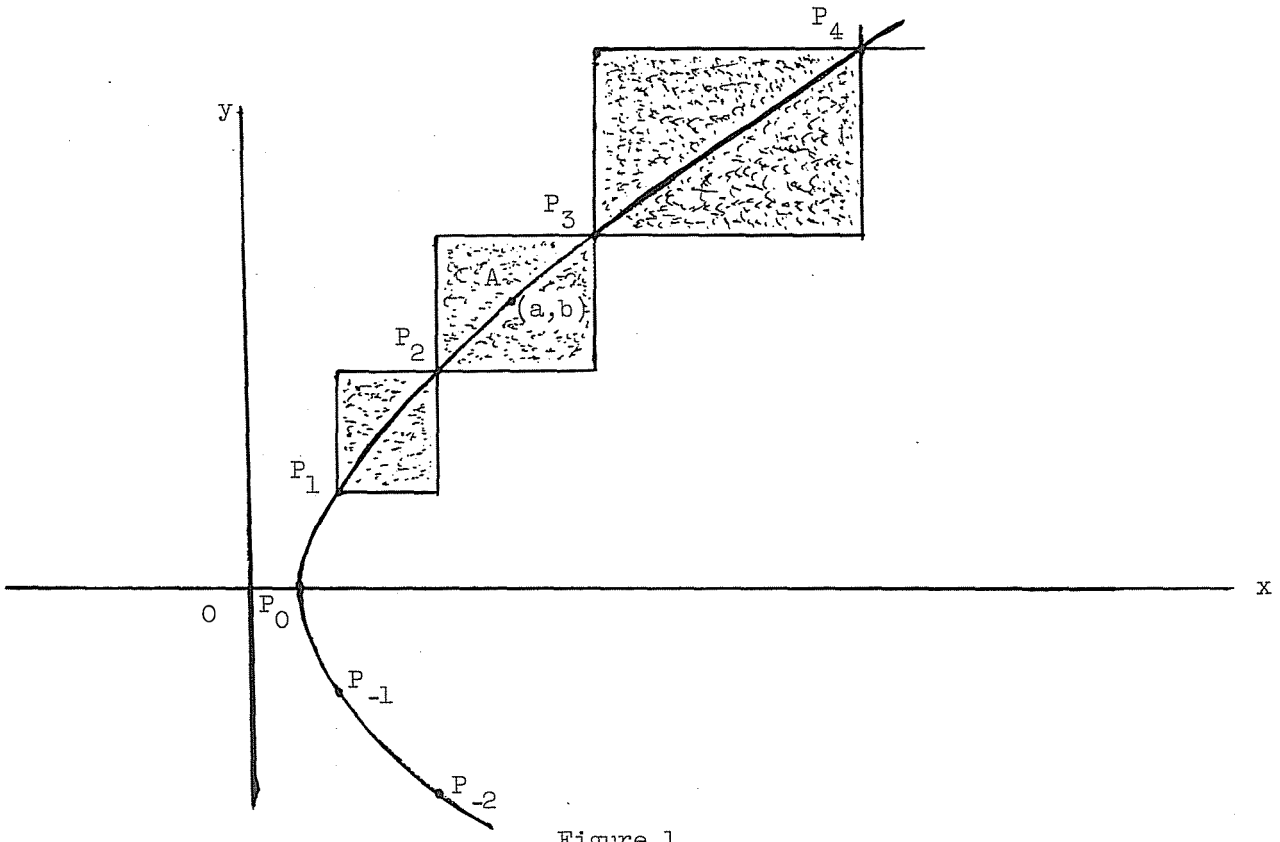


Figure 1

irrational number $2 + \sqrt{3}$. P_2 is identified with $P_1 \times (2 + \sqrt{3}) = 7 + 4\sqrt{3}$, P_3 is identified with $P_2 \times (2+\sqrt{3}) = 26 + 15\sqrt{3}$, etc. In short, each member of the point sequence $P_1, P_2, P_3, \ldots$ is obtained by multiplying the preceding member by $2 + \sqrt{3}$.

Similarly, any member can be obtained from the member succeeding by dividing by $2 + \sqrt{3}$. Thus

$$P_3 = \frac{P_4}{2 + \sqrt{3}} = P_4 \times (2 - \sqrt{3})$$

$$= 26 + 15\sqrt{3} \quad .$$

This principle carries on to give other solutions on this branch of the hyperbola; thus

$$P_0 = \frac{P_1}{2 + \sqrt{3}} = 1 + 0\sqrt{3} \quad .$$

$$P_{-1} = \frac{P_0}{2 + \sqrt{3}} = 2 + (-1)\sqrt{3} \ ,$$

$$P_{-2} = \frac{P_{-1}}{2 + \sqrt{3}} = 7 + (-4)\sqrt{3}, \quad \text{etc.}$$

In short, the relation

$$P_{i+1} = P_i \times (2 + \sqrt{3})$$

holds for all i.

In the sequel, we restrict outselves to solutions (u,v) with $u > 0$, $v > 0$.

4. Other Possible Solutions.

Our next endeavour is to show that, save for changes of sign and inclusion of the trivial solutions $(1,0)$ and $(-1,0)$, the family (p_i,q_i) gives all solutions of this equation. We first (See Figure 1) draw a sequence of rectangles with opposite vertices P_i, P_{i+1}, and with sides parallel to the co-ordinate axes. It is clear geometrically that any arbitrary solution (a,b) lies in this set of rectangles $(a > 0, b > 0)$.

To prove this, we note that a must lie between two successive values of p, say

$$p_i < a < p_{i+1}.$$

Then
$$p_i^2 - 3q_i^2 = 1,$$
$$a^2 - 3b^2 = 1,$$
$$p_{i+1}^2 - 3q_{i+1}^2 = 1,$$

since we have three solutions. It immediately follows that

$$p_i^2 - a^2 = 3(q_i^2 - b^2),$$
$$p_{i+1}^2 - a^2 = 3(q_{i+1}^2 - b^2).$$

Hence we find

$$q_i < b < q_{i+1}.$$

This proves that (a,b) lies within the rectangle with vertices P_i and P_{i+1}.

5. A Sequence of New Solutions.

We saw that all members of the sequence $P_1, P_2, P_3, \ldots$ can be obtained by successive multiplications by $2 + \sqrt{3}$. This suggests that the solution $A(a,b)$ might also generate a sequence of solutions; we now show that this is the case. We shall obtain from $A(a,b)$ a solution $C(c,d)$ lying to the left of A.

Define the point C by

$$C = \frac{A}{2 + \sqrt{3}} = \frac{a + b\sqrt{3}}{2 + \sqrt{3}}$$

$$= (a + b\sqrt{3})(2 - \sqrt{3}).$$

Then $c = 2a - 3b,$

$d = 2b - a.$

We compute

$$c^2 - 3d^2 = (2a - 3b)^2 - 3(2b - a)^2$$

$$= a^2 - 3b^2 = 1.$$

This proves that $C(c,d)$ is a solution, in short, the

Theorem: Any solution (a,b) generates another solution $(2a - 3b, 2b - a)$.

Now we ask where point C lies. We shall have to use

$$a = \sqrt{1 + 3b^2}, \quad b = \sqrt{\frac{a^2 - 1}{3}}.$$

Then
$$\begin{aligned} c = 2a - 3b &= 2\sqrt{1 + 3b^2} - 3b \\ &> 2\sqrt{3b^2} - 3b \\ &= (2\sqrt{3} - 3)b \\ &> 0. \end{aligned}$$

Also, using the fact that $a > 2$,

$$\begin{aligned} c = 2a - 3b &= 2a - 3\sqrt{\frac{a^2 - 1}{3}} \\ &< 2a - 3\sqrt{\frac{a^2 - a^2/4}{3}} \\ &= 2a - 3a/2 \\ &= \frac{1}{2}a. \end{aligned}$$

Thus the abscissa c is positive and C lies to the left of A.

A similar computation works for d; thus

$$\begin{aligned} d = 2b - a &= 2b - \sqrt{1 + 3b^2} \\ &< 2b - \sqrt{3b^2} \\ &= (2 - \sqrt{3})b \\ &< b. \end{aligned}$$

On the other hand,

$$\begin{aligned} d = 2b - a &= 2\sqrt{\frac{a^2 - 1}{3}} - a \\ &> 2\sqrt{\frac{a^2 - a^2/4}{3}} - a \\ &= 0. \end{aligned}$$

Thus d is positive and less than b, that is, C lies

below A.

In short, we have proved that C lies on the segment of the hyperbola between A and P_1.

6. The Method of Descent.

The "method of descent" is a variant of the indirect method of proof. We suppose, if possible, that there exists a solution A(a,b) with $a > 0$, $b > 0$, which is not one of the points P_i. Then the construction described in Section 5 produces a solution C(2a-3b,2b-a) which lies to the left of A. Then the same procedure applied to C produces a solution C_1 which lies to the left of C. Continue this process.

Now A has co-ordinates (a,b), P_1 has co-ordinates (2,1), and the sequence of points C, C_1, C_2, ... lie on the segment P_1A, successive members of the sequence being closer to P_1. However, there are only a finite number of abscissae between the abscissae of P_1 and A (namely, a - 3 abscissae). So after a finite number k of steps one must either reach a point P_α or one must reach a point with abscissa 3.

If one reached P_α, then

$$P_\alpha \times (2 + \sqrt{3})^k = A,$$

and this is impossible since A is not a member of the sequence P_1, P_2, P_3, ... Hence we find that there must be a solution with $x = 3$. But then $y = \sqrt{8/3}$, which is not an integer. Thus we reach a contradiction.

Since the assumption of the existence of a solution A(a,b) not belonging to the sequence P_1, P_2, P_3, ... leads to a contradiction, we must conclude that there is no such "extra" solution A(a,b).

This method, whereby one assumes a solution A, and

derives from it an entire "descending" sequence of solutions A, C, C_1, ... until one descends to a "solution" $(3,\sqrt{8/3})$ which is impossible, is called the method of descent.

LETTER TO THE EDITOR:

Dear Dr. Stanton:

I would like to make a request from you regarding the Ontario Mathematics Gazette. It arises from the fact that the librarian in a Northern Ontario high school asked if there should be a copy of the Gazette in the school library. The answer given was an emphatic yes.

Would it be possible for you to find a small space in a future issue of the Gazette in which you could suggest to the Head of the Mathematics Department of a school that he place a copy of the Gazette in the school library for the use of the other teachers and the students. Thank you. If this could be done it would save writing a letter making this request to the schools.

I have had letters from teachers and a student expressing their appreciation of the Ontario Mathematics Gazette.

Yours sincerely,

Henry J. Campbell,
Secretary-Treasurer,
Ontario Mathematics Commission.

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