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ONTARIO MATHEMATICS GAZETTE

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THE ONTARIO ASSOCIATION OF
TEACHERS OF MATHEMATICS AND PHYSICS

EDITORIAL

Happy New Year for 1965. I am going to start out the year by making a hopeful plea for better procedures in improving Mathematics curricula in Ontario. I am sure we are all in agreement with such an aim; let me point out where I think we could do better.

(1) "Approving" of Textbooks. At the present time, there is a distressing "sameness" about high school texts. I am convinced that this occurs because of the requirement that textbooks be "approved" by the Curriculum Branch of the Department of Education. This leads publishers to strive for mediocre acceptability. I am sure that some authors would like to write more "traditional" texts; others would like to write more "modernistic" texts. And, in striving for the least common denominator, each group is dissatisfied with the mould into which it is forced.

I believe there is a solution. Let the Curriculum Branch give up "approval"; after all, let us face the fact that there are no people in the Curriculum Branch who are competent to pass judgment on textbook authors, most of whom are far more knowledgeable than any officials of the Curriculum Branch. Instead, let us have an independent board made up of University Professors who are prepared to approve any textbook provided it is mathematically correct. Then any group of authors could produce what they feel to be the ideal text; free competition would do the rest. The board need only concern itself with mathematical correctness.

(2) Curriculum Committees. Like Canadian Cabinets, Departmental Curriculum Committees seem formed to "represent" varied interests - inspectors, teachers, O.C.E., Universities, teachers' colleges. But is this a good method? The Curriculum Committee currently considering Grade 13 revisions has about 20 members, only 3 of whom are University Professors. The new Grade 13 advanced programmes will be a preparation for University Mathematics; yet decisions concerning topics on these programmes are being made by a committee with inadequate University representation. Indeed, some proposed topics are completely beyond the ken of some committee members. This is an unhealthy situation. Surely the best qualified people to discuss content should be more adequately represented on such committees.

(3) Conflicts of Interest. Recently there was considerable scandal in the Province of Quebec arising from the

fact that various textbook writers were on the board approving their own textbooks for school use. I am afraid that, unless greater care is exercised here in Ontario, we may find our Province encountering similar problems. Surely it is an anomalous situation when the Ontario Mathematics Commission, which is well-stocked with producers of new texts, keeps pressing for changes which will necessitate the writing of further textbooks; when the Department of Education sets up Curriculum Committees, well-stocked with producers of new texts, to recommend changed curricula; when Departmental inspectors are in charge of evaluating teachers who are using books written by these same inspectors. This sort of procedure is careless, if not improper, and it is especially heartening to note that the new Ontario Curriculum Institute has proposed a regulation, for its own members, which will eliminate this problem.

Conclusion. In conclusion, let me express a hope for elimination of Departmental "approval" of texts in favour of a qualified evaluation of correctness; a hope for greater University participation in curricular development; a hope for separation of curricular development from new textbook interests. And now, until the next issue, all good wishes.

Ralph G. Stanton

Ralph G. Stanton,
Editor, Ontario Mathematics Gazette.

On a University Education (Continued from page 30)

useless system of ideas. By being tied to the practical, the human reason exposes only the mundane and the commonplace - by being free, the intellect charts the unexpected, the divine. The university is the last preserve of freedom of the spirit - it is the responsibility of all of us to see that this sanctuary does not vanish.

I have not told you what you will find at university, but rather what I believe you should expect to get from universities. I wish you good luck.

ONTARIO ASSOCIATION OF TEACHERS OF MATHEMATICS AND PHYSICS

ONTARIO EDUCATIONAL ASSOCIATION

EASTER CONVENTION 1965

The Executive and Councillors of the Ontario Association of Teachers of Mathematics and Physics have been busy planning the Easter Convention and are hoping that 1965 will be the best meeting yet. This meeting has been planned for the Teachers and has several points of special interest arranged, with talks on curriculum, films for viewing, seminars for real discussion, and many exhibits in the first Mathematics Fair. In brief, the programme is this:

Tuesday, April 20

9.00 - registration, films, displays.
10.00 - Mr. Garth Kaye: Senior Curriculum report.
10.30 - Dr. Brainerd from the University of Toronto.
11.30 - Mr. Arnold Harris: Intermediate Curriculum report.
12.15 - Luncheon with an address by Dr. Fehr from Columbia University.
3.00 - Seminars and films.
4.00 - Adjournment

Wednesday, April 21

8.45 - displays, exhibits.
9.15 - Address: Dr. Fehr.
10.15 - Presentation of Awards for exhibits at the Mathematics Fair.
10.30 - Business meeting.
11.00 - Seminars and films.
12.00 - Adjournment.

Several innovations are being tried at this meeting. Films will be available for viewing and will be shown at three different times during the convention. This is to enable the teachers to see what is available in films pertaining to mathematics, and to ascertain what may be useful in the classroom.

Seminars, that truly will be seminars will be repeated on two occasions. Thus teachers will be able to attend more than one of the interesting sessions that has been provided for both the elementary and secondary group. Under the chairmanship of Mr. J. MacLean and Mr. J. Lesage, ten topics have been prepared by outstanding elementary, secondary and university personnel. Attendance at each of these groups will

be limited to approximately twenty people so that true seminar discussion may take place. The ten seminar topics are:

Geometry and Measurement
The Integers
Interpretation of Data
"In Depth" Teaching in Grade XIII Geometry, Algebra and Trigonometry
Suggestions for effective Teaching in Topics of the the Grade XI - 5-year Course
Use of New Approach Mathematics for 4-year Students.
Some Technical Mathematics Topic
Functions - as They are in the Grade XII - 5-year course
A Practical Approach to Calculus
Liaison between Elementary and Secondary Schools in Mathematic

I am sure there are several topics among this group that you would like to discuss and to hear discussed.

Curriculum has been and will continue to be a topic filled with thoughts, questions, and ideas. Two of our members from the Department of Education, Mr. Garth Kaye and Mr. Arnold Harris will report on the progress of the curriculum planning in the Intermediate Division (Grades 7, 8, 9 and 10) and in the Senior Division (Grades 11, 12 and 13). Our main speaker will also bring us information about curricula from a wider area.

Our guest speaker, Dr. Howard Fehr, who spoke to the Easter Convention several years ago, is a person of wide teaching and writing experience. Beginning his career as a high-school teacher of Mathematics in his native Pennsylvania and in New Jersey, he moved on to college teacher of Mathematics at the State Teachers' College at Montclair, New Jersey, attaining the rank of Professor in 1946. In 1946 he started teaching at Teachers' College, Columbia University, becoming Professor of Mathematics in 1948, and Head of the Department of the Teaching of Mathematics in 1949. He belongs to many mathematical organizations including the National Council of Teachers of Mathematics of which he was president from 1956 to 1958. He is a member of numerous foreign mathematical associations and of the International Commission on Mathematics Instruction. He is the author of many books on various topics of mathematics and was the editor of The Twenty First Year Book of the National Council of Teachers of Mathematics on

the topic The Learning of Mathematics. He has written over 200 articles for various leading magazines and periodicals. With all this background he is coming to us to give two lectures "Some Significant Innovations in European Mathematics" and "A Genuinely Modern Programme in Mathematical Education".

Our other guest speaker will be Dr. Brainerd from the University of Toronto. At the request of several of our members he will show what mathematics is necessary to do a research problem and how it is applied. He will give the necessary mathematical background and you will be able to see the problem worked to a conclusion.

One of the highlights of the convention will be the display of the best items from the Mathematics Fair exhibits. Under the chairmanship of Mr. J. W. Fencott, a vast amount of work has been done to ensure the success of this new project. All the teachers and most of the students in the province should have received notice of this endeavour, and many have signified already of their intentions of entering. The entries will be pre-judged at Queen's University, at the Ontario College of Education in Toronto, at the University of Toronto and at the University of Waterloo, with the final judging to take place in Toronto. These exhibits will be displayed to show the initiative and ingenuity of the students and teachers of mathematics in this province. It should give you the teachers an idea of what can be done by individuals in your own schools.

In the business meeting the revision of the constitution of this organization, which was originally set down in 1891, should bring us further up to date.

Further information will be given as to the exact location of all of these activities but it is expected that all the facilities of O.C.E. will be available to us. The stage has been set for an interesting informative and challenging time at this year's convention. All that is required to make it a success is for you, the teachers of Ontario, to support the O.A.T.M. and P. and to be at these meetings. Further details will be sent to all the teachers of mathematics in the province, in a letter giving you the opportunity of joining the Ontario Educational Association, the National Council of Teachers of Mathematics, and of ensuring yourself a place at this year's convention.

Lloyd D. Auckland,
Publicity Representative.

CORRESPONDENCE

The Editor-in-Chief,
The Ontario Mathematics Gazette,
University of Waterloo,
Waterloo, Ontario.

Dear Sir,

Professor Coleman's enthusiasm for ordered pairs as the true essence of the idea of function is inspiring to behold.

But I am afraid that Professor Coleman's belief that the definition of a function as a set of ordered pairs is close to the experience of physicists in recording the results of experiments, is a mistaken belief. Indeed, a physicist would be in bad trouble if he were not able to distinguish between the ordered pairs of numbers that record his measurements, and the function which expresses the underlying pattern or natural law. A great deal of the history of physics could be written in terms of the effort to find the function when those ordered pairs are given.

And, of course, because of experimental errors, the function that expresses the natural law need not fit any of the recorded pairs, as anyone will realize who ever took some measurements in a laboratory.

Thus Professor Coleman's favored definition fails to take into account a distinction which is crucial to any adequate understanding of physics: the distinction between law and empirical data. That, in addition, it is exceedingly awkward, will be clear to anyone who ever tried to translate the following simple statement into terms of ordered pairs: "The volume of a given mass of water is a decreasing function of the temperature between $0^{o}C$ and $4^{o}C$, and an increasing function above $4^{o}C$."

The whole matter is a rather striking illustration of the fact that there may be one or two important considerations involved in the teaching of mathematics beyond those that legitimately shape the mathematical publications of a "most distinguished association of research mathematicians".

Sincerely yours,

Alexander Wittenberg,
Professor of Mathematics
York University.

DIOPHANTINE PROBLEMS IV

R. G. Stanton

1. Introduction.

It is time to draw this series to a close, and it is only appropriate to conclude with the most famous of all Diophantine problems: Fermat's Last Theorem.

On a page in his copy of the works of Diophantus, where that master is discussing Pythagorean Triplets, Fermat points out that the equation

$$x^n + y^n = z^n ,$$

where x, y, z, and n are positive integers, has no solution for $n > 2$. He adds that he has discovered "a truly marvellous proof of this theorem which unfortunately the margin is too small to contain".

The moral is obvious: use wide margins! Succeeding generations of mathematicians, especially with the help of computers, have verified the truth of Fermat's statement for $n \leq 4000$. Probably Fermat did not have a general proof for this result. However, he has left a proof of the impossibility for the case $n = 4$.

2. Fermat's Last Theorem for n = 4: Preliminaries.

Actually, Fermat proved that there is no solution in positive integers for the Diophantine equation

$$x^4 + y^4 = z^2. \tag{1}$$

This is clearly a stronger result than the impossibility of

$$x^4 + y^4 = z^4.$$

Start now with Equation (1), and let (x_1, y_1, z_1) be a solution with z_1 minimal. If any prime p divides x_1 and y_1, we can write

$$x_1 = px_2, \; y_1 = py_2.$$

Then $$p^4 \{x_2^4 + y_2^4\} = z_1^2.$$

Thus p^2 must divide z_1, say $z_1 = p^2 z_2$. It follows that

$$x_2^4 + y_2^4 = z_2^2,$$

and this is an equation of the form (1) with smaller values of the intergers involved ($z_2 < z_1$). Since z_1 was minimal, this is impossible.

We can thus assume that, in Equation (1), the integers x and y have no common factor.

A similar argument shows that x and z have no common factor; y and z have no common factor.

3. Use of the Pythagorean Results.

Now write Equation (1) in the form

$$(x^2)^2 + (y^2)^2 = z^2.$$

This is a primitive Pythagorean equation, so we must have the general solution

$$x^2 = m^2 - n^2,$$

$$y^2 = 2mn,$$

$$z = m^2 + n^2.$$

Here m and n are relatively prime. Also one is odd and the other even in order to keep x and z relatively prime.

If m were even and n were odd, we could write

$$m = 2a, \quad n = 2b + 1.$$

Then $$x^2 = 4a^2 - (4b^2 + 4b + 1)$$

$$= 4(a^2 - b^2 - b) - 1.$$

But every perfect odd square has the general form $4c + 1$; so this is impossible and thus m must be odd, n even. Set $n = 2b$. Then

$$x^2 = m^2 - (2b)^2,$$

$$y^2 = 4\,bm,$$

$$z = m^2 + (2b)^2.$$

Now b and m are relatively prime; yet their product is $y^2/4$, a perfect square. Thus b and m are both perfect squares in their own right, say

$$b = c^2,\ m = d^2.$$

This allows us to write

$$x^2 = (d^2)^2 - (2c^2)^2,$$

$$z = (d^2)^2 + (2c^2)^2.$$

Write the first of these equations as

$$x^2 + (2c^2)^2 = (d^2)^2.$$

This is a Pythagorean equation. Furthermore, it is primitive in form, since $2c$ and d are relatively prime. Hence we must have (again using the general result for the solution of the Pythagorean equation):

$$x = r^2 - s^2,$$

$$2c^2 = 2rs,$$

$$d^2 = r^2 + s^2.$$

Here r and s are relatively prime, one being odd and the other even. As before, the fact that

$$c^2 = rs$$

shows that r and s are perfect squares, say

$$r = t^2,\ s = u^2.$$

Thus $$d^2 = (t^2)^2 + (u^2)^2;$$

in short, $$t^4 + u^4 = d^2.$$

But this shows that (t, u, d) is a solution of Equation (1).

4. The Method of Descent.

Consider Equation (1) as

$$x_1^4 + y_1^4 = z_1^2,$$

where z_1 is minimal. In Section 3, we showed that this supposition leads to another solution

$$t, u, d$$

in which

$$z_1 = d^4 + 4c^4,$$

That is, $d < z_1$.

But z_1 was supposedly a solution having least possible z. Thus we have a contradiction; in brief, no solution (x_1, y_1, z_1) can exist, since its assumed existence leads to a contradiction.

O.M.C. NEWS

This is a brief report from the O.M.C. giving the substance of the general meeting held on Saturday, March 7, 1964, in Room 241 of the Larkin Academic Building of Trinity College, Toronto.

The meeting opened with remarks by President Elliott who reported that there had been a great demand for the brochure "What is Number?". He reported that the Negotiations Committee of the O.M.C. has had two meetings with representatives of the Ontario Curriculum Institute and has received verbal assurance that the Institute was willing to underwrite the Commissions' budget for the present year to the estimated financial expenditure. A second meeting had been inconclusive. Dr. Elliott pointed out that the O.C.I. resolution which states that no person shall be employed by the Institute who is under contract with a publisher to write a text book might be restrictive if applied to members of the O.M.C.

After the report of the Secretary-Treasurer, the report of the Executive followed. The Executive has been able to secure the services of Mr. Henry J. Campbell to succeed Mr. Frank C. Asbury as Secretary-Treasurer. It has authorized the granting of $150.00 to the group which started the Junior Mathematics Contest in District 10. Some criticism was expressed of the editorial policy of the Mathematics Gazette. A favourable reply has been received from the Department of Education concering experimentation in Grade XIII with the problems paper. A motion was carried to the effect "that a prize of the value of $25.00 be given in honour of Mr. Asbury to the winner of the Junior Mathematics Contest and that the matter of further prizes be considered at another time, the prize to be continued for the life of the commission and/or the life of the Contest".

Mr Totton reported on the work of the Scholarship Committee and its questionnaire concerning School Mathematics Clubs.

Mr. J. Ryan reported for the Elementary Education Committee. The Committee has studied the Cuisenaire method of teaching arithmetic, and has sent out to the Inspectors of the Elementary schools asking for a report on any experimental work being done in their inspectorates. The Committee felt that a campaign should be started to make mathematics a specialized subject in the elementary schools.

Dr. Hogarth reported for the Secondary Education Committee. The Committee had looked into the 8-7 versus 5-10 division of periods in Grade XIII and reported on a possible pair of courses which might fit into an 8-7 period division. The meeting then adjourned for lunch.

In the afternoon session, it was moved "that the Department of Education be asked to grant permission for the trial under the usual conditions of new Grade XII teaching material in certain schools during 1964-65". A lengthy discussion then followed concerning possible experimentation in Grade XIII and content of Grade XIII. There was considerable difference of opinion as to the amount of stress which should be placed on new terminology. There seemed to be considerable feeling that the proposed Grade XIII course was more difficult than present courses and was too long.

Mr. Sharp presented the report of the Research and Publications Committee, indicating that Dr. Howard Russell

is continuing with his project of evaluating the Grade IX work done through the new approach and through the old. Consideration is being given to discover some means of solving the problem of overlapping in mailing of the Ontario Mathematics Gazette. A grant of $500.00 had been received from The Samuel Beatty Fund to help finance the Gazette.

Mr. Sharp pointed out that his committee was a clearing house for brochures; he felt that, to a great extent, it had been erroneously thought that these brochures were to be taken into the classroom and used in a Grade XIII class. Mr. Sharp pointed out that the purpose of brochures was to give the teacher some background on material which may be introduced into the course of study. It was reported that the Gazette had been the recipient of the Greer award.

The meeting concluded with an important discussion of the Ontario Curriculum Institute. A discussion was held concerning the restrictive clause of the O.C.I. whereby persons in the employ of the O.C.I. may not enter into a contract with a publisher to write a text book. It was also pointed out that the O.C.I. was willing to underwrite the expenses of the Commission for the year 1964. One member of the Commission pointed out that although a definite proposal had been received from the O.C.I., it seemed that there had been a failure to get from them encouragement equal to the encouragement received at the last annual meeting of the Commission. He felt that the objectives of the two bodies were different and that the questions of research and the giving out of research are different. He questioned whether the Commission could carry on with the Gazette, brochures, the work of the Scholarship Committee, Mathematics Clubs, etc., under the O.C.I. He felt that we needed a firm commitment or a clear statement from the O.C.I. as to just what it is and what it will do.

Father Egsgard and Mr. Sharp summarized their conclusions from the last meeting of the Programme Committee of the O.C.I. They felt that the O.C.I. would like the Commission to do its work in the realm of mathematics but it is not prepared to finance the Commission's other works. Anyone who works for the O.C.I. must do so within the frame of reference of its "Textbooks - Authors" declaration. It seems to be a matter of the O.M.C. and the O.C.I. going their separate ways or of the O.M.C. being absorbed in an entirely different form. The Institute representatives said that they would only be willing to take the Committees

[Continued on page 43]

JOINT VARIATION

Lucian R. King

It might be of interest to readers to include a proof of the confusing Grade 13 Theorem. If z varies as x when y is fixed, and if z varies as y when x is fixed, then z varies as the product xy when both x and y may vary.

Proof. Since z varies as x (y fixed), we may write

$$z = k_1(y)\, x,$$

where $k_1 \neq 0$. Note that the "constant of variation" is constant with respect to x. However, it can be a function of y; that is for any particular y, $k_1(y)$ is constant, but it may change for different y's.

Similarly z varies as y (x fixed); so

$$z = k_2(x)\, y,\ k_2 \neq 0,$$

where the "constant of variation" is constant with respect to y, but may involve x.

We now have $k_1(y)\, x = k_2(x)\, y$, and obtain

$$\frac{x}{k_2(x)} = \frac{y}{k_1(y)}.$$

This identity must hold for all values of x and y; however, the LHS is a function of x and the RHS is a function of y. If x or y appeared in these functions, they could not be identically the same. Hence they are constant, and

$$\frac{x}{k_2(x)} = \frac{y}{k_1(y)} = \frac{1}{c},$$

where c is a numerical constant, $c \neq 0$.

Thus we get $k_2(x) = cx$, whence

$$z = c\, x\, y, \text{ as required.}$$

THE PLACE OF LOGIC IN SECONDARY SCHOOL MATHEMATICS

D. A. Clarke, University of Toronto

1. Before we begin to discuss this question, we should first have some idea of what logic is, and is not.

a) Logic is really impossible to define, in the same way that mathematics is impossible to define. The best we can do is to give an operational definition by answering such questions as: What do logicians study? What questions do they ask? etc. The following is a partial list.

(i) Logicians study language, with particular reference to such notions as meanings, definition and definability truth and falsity (validity).

(ii) Logicians study the nature of proof, asking such questions as: What is proof? What are the connections between proof and validity?

(iii) Hence, logicians become concerned with explicit rules for proof and validity. This leads to the development of formal systems of logic.

(iv) Because of (iii), and for other reasons, logicians are interested in the general theory of algorithms, i.e., effective, mechanical procedures of computability, solvability, and decidability.

(v) Logicians study axiomatic systems and models thereof. They ask, for example, "What are suitable definitions for, and rules for determining consistency (Do the axioms yield only those theorems we want?), completeness (Do the axioms yield theorems we do not want?), independence (Do some of the axioms yield the rest as theorems?), and categoricity (Do the axioms have one model or many?).

(vi) Finally, logicians study a number of special problems and in general delve into the meaning and nature of all the fundamental notions of mathematics.

b) Logic is not Aristotle's syllogistic, nor is it the study of truth tables. This is meant in the same sense that mathematics is not the study of the addition and multiplication tables. Furthermore, logic is neither philosophy,

nor a branch of philosophy (although it may be a useful tool therein).

c) In summary, the role of logic with respect to mathematics is to clarify and justify the basic concepts and fundamental principles of mathematics: number, figure, set, function, proof, definition, construction, algorithm, axiom, validity, inference, deduction, axiomatic theory, model, etc.

2. One more question needs to be considered before we can get to our main topic. What is the relation of logic to mathematics? Is logic a part of mathematics or vice versa, or something else? I shall attempt to answer this only from the point of view of mathematics and avoid a larger philosophical question.

The following points can be made.

(i) Logic is a useful and unavoidable tool in the study of mathematics. This is so if the logic is treated explicitly or implicitly.

(ii) Logic therefore becomes a legitimate study of mathematicians, so as to understand their subject better and improve its usefulness.

(iii) Techniques from other branches of mathematics (especially algebra and topology) have proved to be very useful in research in logic.

(iv) The study of, and research into, logic has created new and important branches of mathematics, e.g., the general theory of algorithms, model theory.

(v) Many recent results and techniques in logic have contributed greatly to other branches of mathematics.

So we conclude that logic is a part of mathematics and vice versa. The two are inter-connected in a complex way, and it is really impossible to give an exact answer.

3. Now let us deal with our main topic by posing three questions.

a) Why study logic? As we have seen, this is unavoidable even if the study is only implicit. And I suggest that

the implicit approach is possibly the right approach in the secondary schools. Consider an analogy. It is difficult (or meaningless) in a history class to discuss the early history of the United States without some reference to Hamiltonian and Jeffersonian philosophies. It does not necessarily follow that either philosophy or political science should be courses in the secondary-school curriculum. It does follow that the secondary-school history teacher should have sufficient knowledge and understanding of these matters to treat them in a way which is adequate under the circumstances, i.e., to give meaning to the historical discussion at hand and not go beyond the previous training and mental capacity and maturity of the student.

The language of logic (and set theory) has considerable value as a linguistic tool. The power of a good notation is well known. Indeed, the history of mathematics could be, in part, a study of the development of good notations. Logic is one major part of this development. The right notation does a lot of the routine, mechanical thinking.

b) How should we study logic? I shall come back to this in my summary, and I have said something about it alread In my own opinion, the secondary-school teacher should treat the necessary logical notions and introduce useful logical notations bit by bit while covering other topics. These notions and notations should be introduced when three conditions are satisfied. (i) they are useful, (ii) the student is ready to understand them, and (iii) the teacher is prepare to discuss them.

One could give as good an argument for a formal course in numerical analysis as one could for logic, but I have not heard anyone suggest it. Perhaps because any significant amount of numerical analysis seems to be too difficult. But any significant amount of logic is too difficult also. Remember, logic properly understood does not amount to learning truth tables. Of course, just as in the case of logic, a certain amount of numerical analysis is already implicit in the curriculum.

c) What should be studied? This cannot be explicitly and categorically answered if we adopt my point of view. I list instead a few examples of the type of topic I feel secondary-school teachers should be prepared to discuss adequately as the need arises.

(i) The nature of mathematical statements: propositions, predicates (and the logical connectives), theorems, definitions.

(ii) The nature of the other basic notions of mathematics: e.g. set, function, relation, axiom and axiomatic systems, number and number systems, algebraic structures, etc.

(iii) The proper manipulation of formal expressions of all kinds with an understanding of what they mean and why.

(iv) The nature of proof, and what constitutes a proof. (What are the basic principles of proof? How does one conjecture and formulate a theorem --- along with the necessary preliminary definitions, look for a proof or disproof, and check an alleged proof to see if it is correct?)

In conjunction with this list, let me add a list of some of the topics and ideas which I have found that students understand poorly.

(i) The need for, and nature of, precise definitions and formulations of theorems.

(ii) The meaning of implication, equivalence, contrapositive, modus ponens, etc.

(iii) The meaning, use, and manipulation of quantifiers. ("for all x ", "there exists an x ").

(iv) Proof by cases.

(v) Reductio ad absurdum.

(vi) The use of counter-examples.

(vii) Mathematical induction.

(viii) Functional notation, and the use and meaning of free and bound variables. (the difference between f, $f(x)$ and $f(3)$; the difference between the x in $f(x)$ and

$$\frac{df(x)}{dx}\text{; etc.).}$$

4. In summary, it is my opinion that a formal course in logic has no place in the secondary-school curriculum.

But it may well belong in teacher-training curricula. What is required is an adequate understanding on the part of the teacher, so that any of the above topics which come under the name of logic in the broadest sense can be adequatel dealt with as and when required; and only to that extent which is consistent with the training and maturity of the student. Above all, let us avoid a dry, routine, rote, memory course of logical formulae and truth tables.

CORRESPONDENCE

Dear Dr. Stanton:

Please permit me to mention a couple of errors in the reference to the British Mathematical Gazette in the October issue of The Ontario Mathematics Gazette.

G. Bell and Sons, Ltd., may be the publishers; of this I am not sure as recent issues have been arriving postmarked Belfast. However, the address for subscriptions is

Hon. Treasurer, The Mathematical Association,
Gordon House, 29 Gordon Square,
LONDON, W. C. 1.

Also, the subscription rate was raised from 21 shillings to 30 shillings per annum as of January, 1964.

...

An article in a recent issue of the British Gazette included some suggestions for the teaching of the addition of positive and negative numbers, which might be of interest to Ontario Grade-nine teachers.

Another teachers' journal in English is "The Australian Mathematics Teacher." While rather insignificant, it has contained some recent articles of interest on proposed changes in the mathematics courses in Australia, similar to those taking place in Ontario. The rate is 15 shillings (Australian) for three issues per year, and the address is The Treasurer, Mathematical Association, Teachers' College, University Grounds, Newtown, N.S.W., Australia.

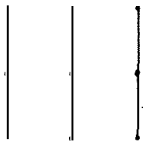
Yours sincerely,

(signed) G. S. Prunner

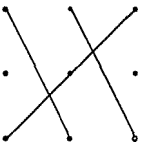
A SEARCH FOR PATTERNS IN A SYSTEM

Myron F. Rosskopf,
Teachers College, Columbia University

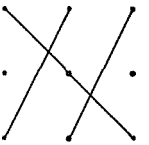
Suppose you have (1) three elastic strips between pegs on a pegboard and have three removable pegs. Several patterns can be made (2) without intertwining the elastic strips by (3) moving the ends of the strips. Let us consider the three patterns illustrated below. We shall name these patterns I, J, and K.



I



J

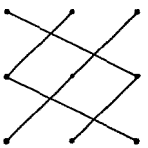


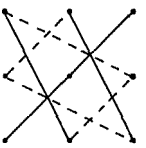
K

We wish to consider the set of patterns {I, J, K}. Now, in order to have a system to consider, we introduce an operation, which we shall denote by " * " . The operation " * " will mean one pattern followed by another and remove the middle row of pegs. For example, shown below in the first drawing is K followed by K. The second drawing shows the result of removing the middle row of pegs. What is the resulting pattern equal to? Compare it with J. It "looks" a bit different, but the elastic strips connect the pegs that form pattern J. We write K * K = J, meaning that K * K and J are names for the same pattern.

K

K



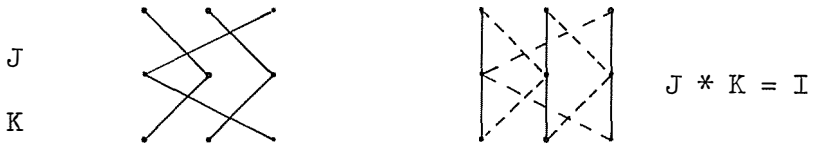


K * K = J

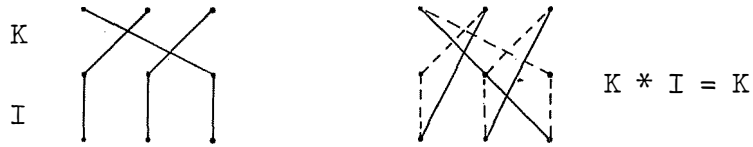
This is very interesting. The operation applied to two particular members of the set resulted in a member of the set. Is this always true? Let us fill in the table below. One entry we know; there are eight others to find.

*	I	J	K
I	I	J	K
J	J	K	I
K	K	I	J

Let us do a few. First, let us see what J * K is. The diagrams below are self-explanatory. Clearly J * K = I. Again the result is a member of the set.



This time let us see what K * I looks like.



Again the result is a member of the set. It turns out that this is always so. Any member of the set followed by another or the same member of the set is a member of the set. The set is closed under the operation " * ".

Exploring closure yielded more than we had expected. Inspection of the table indicates that I is the Identity element for the operation " * ". For I * I = I; I * J = J * I = J; I * K = K * I = K. Another unexpected result is that "*" is commutative. Clearly the identity element, I, is both a left and right identity. Also J * K = K * J, the only other entries in the table to check.

We can use the table to check associativity as well. The items to check are:

$(I*I)*I = I*(I*I)$ $(J*J)*J = J*(J*J)$ $(K*K)*K = K*(K*K)$

$I*I = I*I$ $K*J = J*K$ $J*K = K*J$

$I = I$ $I = I$ $I = I$

$(J*I)*J = J*(I*J)$ $(K*I)*K = K*(I*K)$ $(K*I)*I = K*(I*I)$

$J*J = J*J$ $K*K = K*K$ $K*I = K*I$

$K = K$ $J = J$ $K = K$

$(J*I)*I = J*(I*I)$ $(K*J)*I = K*(J*I)$ $(K*I)*J = K*(I*J)$

$J*I = J*I$ $I*I = K*J$ $K*J = K*J$

$J = J$ $I = I$ $I = I$

Clearly, " * " is associative.

The next item to check occurs whenever one has a system to test for properties. One tries to find for each element a member of the system such that under the operation " * " the two yield the identity element. Again the "operation table" is of great help. The equations to solve are:

$I * x = I$; $K * x = I$; $J * x = I$.

The solution x is called the inverse element. We list these, having obtained them from the table.

Element of Set	Inverse Element
I	I
J	K
K	J

Are the inverse elements members of the set? Yes, indeed. So our system under " * " possesses all the "good" properties. We list them:

a. closure

b. associativity

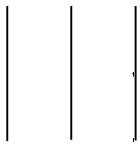
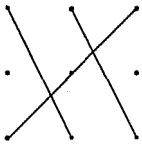
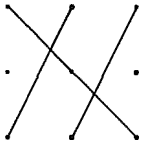
c. identity element

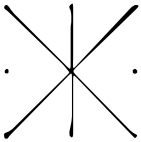
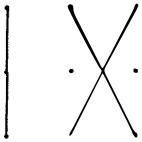
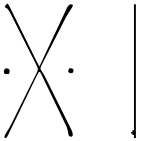
d. inverse elements

e. commutativity

Our little elastic strip system is a commutative group.

Earlier we said there were many patterns that could be formed with the elastic strips, a pegboard, and no intertwining, but we would consider just three. Now, let us see what happens in a "richer" set under the operation " * ". This time we shall consider six patterns, shown and named below.

I  J  K 

L  M  N 

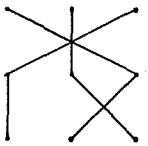
The first task is to fill in the entries in the table.

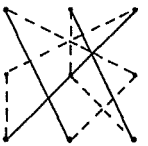
*	I	J	K	L	M	N
I	I	J	K	L_2	M_2	N_2
J	J	K	I	N_3	L_3	M_3
K	K	I	J	M_3	N_3	L_3
L	L_2	M_4	N_5	I_5	J_1	K_1
M	M_2	N_5	L_5	K_5	I_5	J_1
N	N_2	L_5	M_5	J_5	K_5	I_5

Some of the entries we already know from our experience with the system consisting of the set I, J, K and the

operation " * ". We fill these in without any trouble. Of course, now we are working with the system consisting of the set I, J, K, L, M, N, and the operation " * ". We shall calculate some of the entries just to be sure we can visualize what is going on.

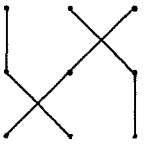
L
M

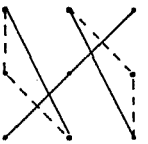




L * M = J

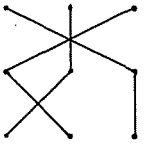
M
N





N * N = J

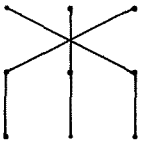
L
N

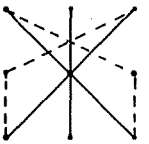




L * N = K

L
I

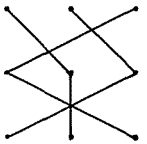


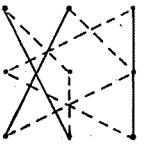


L * I = L

It looks as if I again is both a right and left identity element for the set under " * ". But let us continue to fill in the table, for we should like to have our system possess the property of closure.

J
L

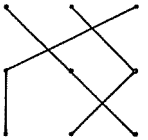


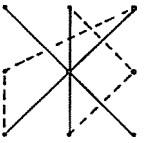


J * L = N

J

M

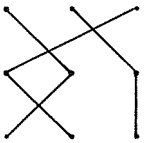


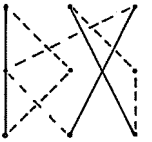


J * M = L

J

N

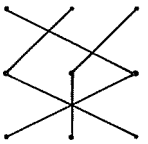


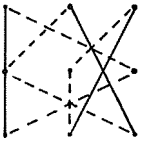


J * N = M

K

L

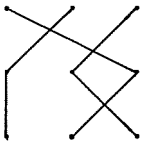




K * L = M

K

M

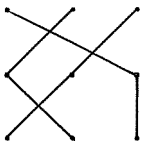


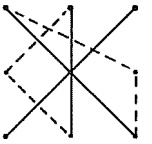


K * M = N

K

N



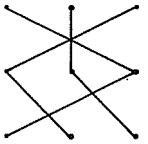


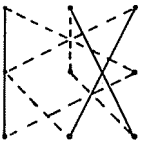
K * N = L

We cannot assume commutativity; as we must calculate L * J as well as J * L.

L

J



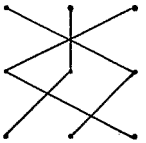


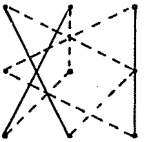
L * J = M

It was a good thing we were careful! For, L * J $\neq$ J * L. Immediately we conclude that the system is not commutative under " * ". Remember, just one exception,

one counter-example, is needed. Now, the best that we can hope for is that the system is a group. We shall have to give up commutativity. We continue with the table.

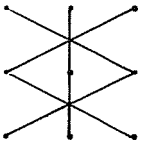
L
K

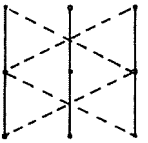




L * K = N

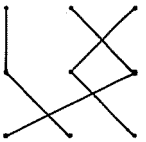
L
L

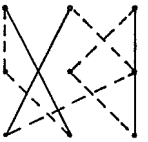




L * L = I

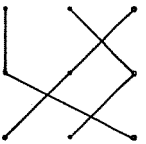
M
J

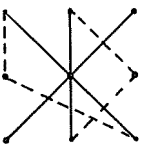




M * J = N

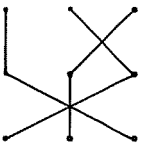
M
K

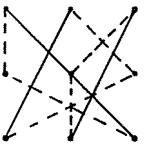




M * K = L

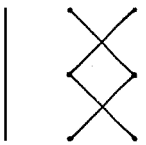
M
L

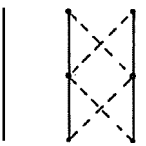




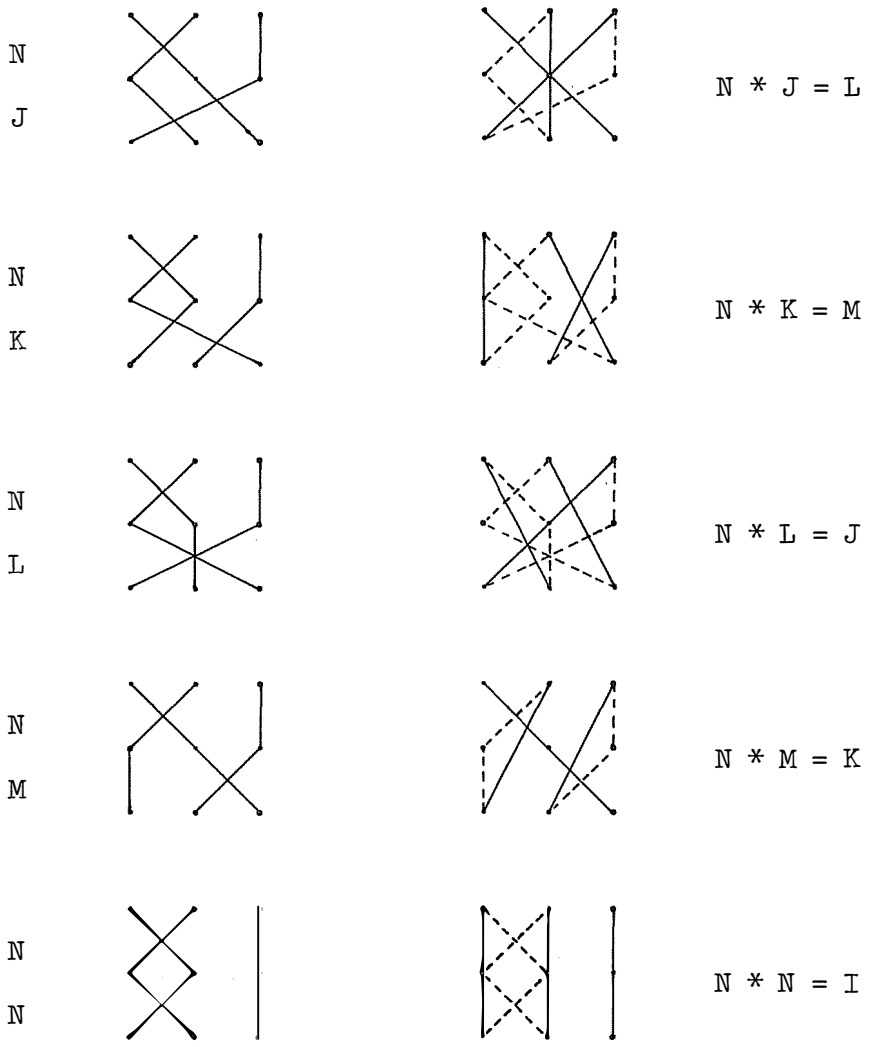
M * L = K

M
M





M * M = I



So the system is closed under the operation " * ".

Next, using the table, let us check associativity. The task would take too long for the time available. We shall satisfy ourselves that the system is indeed associative by checking just a few examples.

$(L*K)*N = L*(K*N)$ $\qquad$ $(M*N)*J = M*(N*J)$ $\qquad$ $(N*L)*K = N*(L*K)$

$N * N = L * L$ $\qquad$ $J * J = M * L$ $\qquad$ $J * K = N * N$

$I = I$ $\qquad$ $K = K$ $\qquad$ $I = I$

$$(M*M)*L = M*(M*L)$$

$$I * L = M * K$$

$$L = L$$

The next property to check is whether each element of the set has an inverse that also is an element of the set. Again the table is helpful. We consider each element and ask ourselves, "What element following this on yields I under " * " ?

Element	Inverse Element
I	I
J	K
K	J
L	L
M	M
N	N

So the system of six elements and the operation " * " is a group but not a commutative group. Our first set, somewhat small, turned out to be a commutative group. But when we added more elastic strip patterns via our pegboard, we lost the property of commutativity. Somehow, intuitively, one expects to have a "richer" system to have not only the properties of a "poorer" system but additional ones as well. But we lost a property. However, there are these questions that could be asked. Notice that our set of six elements contains the set of three as a subset. This subset forms a group. Then the set of three elements under " * " is a subgroup of the set of six elements under " * ". Does the set contain any other subgroups? Does the set of three elements contain any subgroups? If there are certain questions about the richer group that must be answered in the negative, still there are other questions to be asked and to be investigated that are trivial of answer or impossible to ask for the three element group. Always there are further problems to be studied. Success in solving one problem in mathematics leads one to answering questions that lead to investigations of other problems. Experiences and successes in such experiences furnish training for asking the sort of questions mathematicians ask.

ON UNIVERSITY EDUCATION

[A statement made by Professor George Cross of the University of Waterloo at a panel discussion on University Education at the annual high-school Mathematics Seminar, April 2, 1964.]

As I understand these annual high-school seminars, the winners of the mathematics contest are invited to a university campus to give them a glimpse of university life - this panel is only one of several items on your programme to accomplish this end. The fact that you have done so exceedingly well on what is essentially a mathematical contest does not mean that you are a mathematician, that you will become a mathematician, or, indeed, that you should become a mathematician. In view of this, although I work in the field of mathematics myself, I feel that I should make some remarks about university life and education in general.

It seems to me that there is much public ignorance or misunderstanding of the nature and function of a university. At a time when universities were securely and independently contained within their own walls - and there was a time not too long ago when this was so - popular knowledge of universities was not important since the public had no influence on university life. In the present generation, when universities are being so heavily financed by public funds, public knowledge of university life is necessary, public ignorance to be deplored.

There has been a tendency recently to try to make universities more practical, more vocational - to look on universities as schools which a student could and should attend in order to be trained to earn a living - schools which everyone attends automatically after successfully completing Grade XIII or its equivalent. To illustrate this point I can recount the story told by a Dean at the University of British Columbia who had had an interview with the mother of a freshman student who had failed his first year exams very badly. The Dean suggested that perhaps the student's abilities could be applied better in other directions. The mother's reply was: "Oh, he must go to university - he's not smart enough to earn a living without a university degree".

I would like to think that a university is a place

where students like yourselves can receive an education, not a card that entitles them to a job; I would like to think that you and most of the other students who enrol at universities over the next few years do so in order to become educated, not just to qualify for a job. To my way of thinking, a person has become educated when he has learned to read understandingly, to speak intelligently, and to write intelligibly on a wide variety of subjects. So defined, the set or class of educated people contains too few members. I think it should be the aim of everyone to make the set larger.

You will discover, if you have not already done so, that if you enrol in an honours course in mathematics at most universities in Ontario (and I am sure that a similar statement is true for Physics, Engineering, and other disciplines) that you will be able to satisfy the requirements by successfully completing 4 or five general elementary courses and a large block of courses in mathematics. I do not think you should consider that your responsibility ends when you have satisfied the formal requirements for a degree, because I question whether the completion of such a programme alone could be said to have created an educated person as I have defined him. What I suggest is that if you want to call yourself a student in good standing (and everyone in this room is a student as I now use the word), you must study not only the works of Euclid, Descartes, Einstein, and Zygmund (i.e., the mathematical writers) but also the writings of Plato, Scotus, Erigena, Russell, Sartre, Camus, and Dostojevsky (i.e., the philosophers, the novelists, the humanists, etc.). This appears to me the great challenge facing students entering university today - in the face of more and more knowledge and more and more specialization, the challenge to finish university not as a specialist alone but as an educated person with a speciality.

So far, I have touched on just one facet of university life - the learning of known facts. There is another facet - the searching for the unknown. Most professors spend part of their time teaching so that students may learn, and part of their time doing research in their respective fields of interest. There is always the demand by some people that what is done in research have some practical applications. But, again, I would like to think that the university is a place where the human mind, the spirit, the intellect, the soul, call it what you will, is unfettered and free to explore the most esoteric and apparently

[Continued on page 3]

BOOK REVIEW

Limits - The Concept and Its Role in Mathematics - Norman Miller (New York, Blaisdell Publishing Company, 1964)

An idea of the scope of this book is given by the headings of the seven chapters which are: Introduction; Real Numbers, Sets, Variables, and Functions; Limits of sequences; Limits of Functions; The role of limits in infinite series; An important limit - the derivative; Another important limit - the definite integral. There are also four appendices on: The set of rational numbers; The limit of a bounded monotone sequence; The limits of the Darboux sums; Limits of functions of more than one variable.

In the preface it is explained that this exposition of the role of limits in Mathematics is intended for the use, mainly, of teachers and students in High Schools. And there are a number of things in the book which will provide considerable interest for such an audience. Some of the points which I particularly liked are: the intuitive notion of limits as introduced in Chapter I; the examples on limits of sequences and limits of functions in Chapters 3 and 4; the excellent discussion of series given in Chapter 5; the work on asymptotes in Chapter 6; the discussion of upper and lower sums in the work on integration in Chapter 7.

Teachers will find considerable variety in the topics discussed and considerable difference in the level of difficulty of different parts of the book. For example, there are some excellent and extremely useful pages connected with the geometric series, the harmonic series, and the so-called "p" series, in which the n'th term is the reciprocal p'th power of n. On the other hand, some of the material is rather difficult; for example, many readers will find the section using the ratio test for convergence of a series to be relatively heavy going; and the work on Darboux sums and limits of functions of more than one variable is likewise rather complicated.

Summing up, I think that the reader will find this book a treasure-house of interest and even the more difficult parts will richly reward the effort expended.

N. N. H.

THE EXPERIMENTAL CLASS IN "LOGIC" HELD IN LONDON CENTRAL COLLEGIATE, JULY 6 TO AUGUST 7, 1964

H. J. Campbell, Ontario Mathematics Commission

This experiment originated with the Secondary Education Committee of the Ontario Mathematics Commission. At the suggestion of Inspector A. W. Harris, the London Board of Education agreed to offer the course at their summer school. It was given for two hours a day for five weeks. The expense of the course was shared by the London Board of Education and the Ontario Mathematics Commission.

The purpose of the course was to evaluate student reaction to material which might be used in future mathematics courses. It was hoped that it would provide a sound enrichment for students taking Grade 13 mathematics and later university mathematics. The instructor was Mr. David Boyle of London. The textbook used was the first four chapters of "Introduction to Finite Mathematics" by Kemeney, Snell, and Thompson. Nineteen students registered in the class: Grade 13 - 1, Grade 12 - 14, Grade 11 - 4. Of the 19 students, 12 took a second summer school subject, 10 selected subjects for enrichment, and 2 registered in make-up courses. Three members attempted a full-time job; two eventually dropped out, and the third did not make a passing grade. The results for 17 students were 90 - 100%-4, 80 - 89%-8, 70 - 79%-5, 60 - 69%-1; below 60%-1. The highest mark was 98, the average mark was 80, and the median 82.

From comments of the pupils to certain questions asked, most agreed that it was interesting, challenging, and worthwhile. They were glad they had taken it. Mr. Boyle felt the course had been an outstanding success.

Mr. Boyle gave a comprehensive report of the summer school class to the Secondary Education Committe of the Ontario Mathematics Commission. Since the first chapter of the "Introduction to Finite Mathematics" is the reference for Logic in the problems classes, the Secondary Education Committee suggested to Mr. Boyle that he prepare a brief report on the teaching of Logic which would be helpful to the teachers. The report follows.

SOME SUGGESTIONS FOR THE PRESENTATION OF LOGIC

David Boyle

In the summer of 1964, a course in logic based upon the text "Introduction to Finite Mathematics", was given to a selected group of senior high-school students. The following suggestions for the presentation of the subject material are based upon the experience gained through the teaching of this course.

Venn diagrams, introduced at the beginning of the course, can then be used to illustrate the various logic connectives.

The text makes some of the truth tables appear very complicated. Truth tables with fewer entries would help the student to follow the basic steps that lead to a solution.

Page 13 Figure 15

p q r	[p $\rightarrow$ (q $\vee$ r]	$\wedge$	$\sim$[p $\leftrightarrow$ $\sim$r]
T T T	T T T T T	T	T T F F T
T T F	T T T T F	F	F T T T F
Step #	1 3 1 2 1	5	4 1 3 2 1

A simplified version would be:

p q r	[p $\rightarrow$ (q $\vee$ r]	$\wedge$	$\sim$[p $\leftrightarrow$ $\sim$r]
T T T	T T	T	T F
T T F	T T	F	F T
Step #	2 1	3	2 1

In both instances the step numbers indicate the order of table completion.

Since necessary and sufficient conditions may be difficult concepts for a high-school student, the section should be expanded considerably. Tree diagrams, on the other hand, could be condensed with no loss in pupil comprehension.

Additional material should be provided for the translation of some English phrases into their corresponding symbols. For example:

If p, then q	$p \longrightarrow q$
p if, and only if, q	$p \longleftrightarrow q$
p is a sufficient condition for q	$p \longrightarrow q$
p is a necessary condition for q	$p \longleftarrow q$
p only if q	$p \longrightarrow q$

In Section 10, Valid Arguments, the impression is given that the problems must be solved by means of truth tables. This is a very time-consuming procedure that can lead to errors. Consequently, some time can be profitably spent by the class in discovering that equivalent compound statements have identical truth tables. For example:

$$p \vee \sim q \equiv \sim (\sim p \wedge q)$$

$$\sim p \vee \sim q \equiv \sim (p \wedge q)$$

$$p \wedge \sim q \equiv \sim (\sim p \vee q)$$

Thus, one rule may be formulated:

To change from conjunction to disjunction (or vice versa),

1) change the sign of each statement;

2) change the symbol;

3) place a negation sign outside the complete expression.

It may also be shown that:

$p \longrightarrow q \equiv \sim p \vee q$

$p \longleftarrow q \equiv p \vee \sim q$

$\sim p \longrightarrow q \equiv p \vee q$

Thus, a second rule is obtained:

To change from the conditional to disjunction,

1) change the sign of the "if" part;

2) change the symbol.

With the aid of these two rules, most statements that occur in any given question can easily be changed from the conjunction or disjunction forms to their equivalent conditional form. These may then be arranged in order of implication.

Example: Test the validity of the argument

$\sim(q \wedge r)$

$q \vee \sim p$

$s \longrightarrow r$

$\underline{s \vee t}$

therefore $p \longrightarrow t$

Solution: Expressed in equivalent conditional forms,

$\sim(q \wedge r) \equiv \sim q \vee \sim r \equiv q \longrightarrow \sim r$

$q \vee \sim p \equiv p \longrightarrow q$

$s \longrightarrow r \equiv \sim r \longrightarrow \sim s$

$s \vee t \equiv \sim s \longrightarrow t.$

Arrange in order of implication,

$p \longrightarrow q,\ q \longrightarrow \sim r,\ \sim r \longrightarrow \sim s,\ \sim s \longrightarrow t;$

Therefore $p \rightarrow t$ is valid.

The last section, The Application of Logic to Switching Circuits, provides interesting practical applications which can be easily demonstrated using the following equipment:

6-volt D.C. source

6-volt bulb and bulb holder

four-pole double-throw switches

wiring

Example: Section 12, Page 53, Question 8

Solution: Let the four switches be represented by P, Q, R, and S. (To simplify the wiring the switches used should be multipoled double-throw.)

Let P indicate the switch in position one.

Let P' indicate the switch in position two.

The lights turn on when an even number of switches are closed. The lights will be on when

a) all the switches are in position one

P Q R S

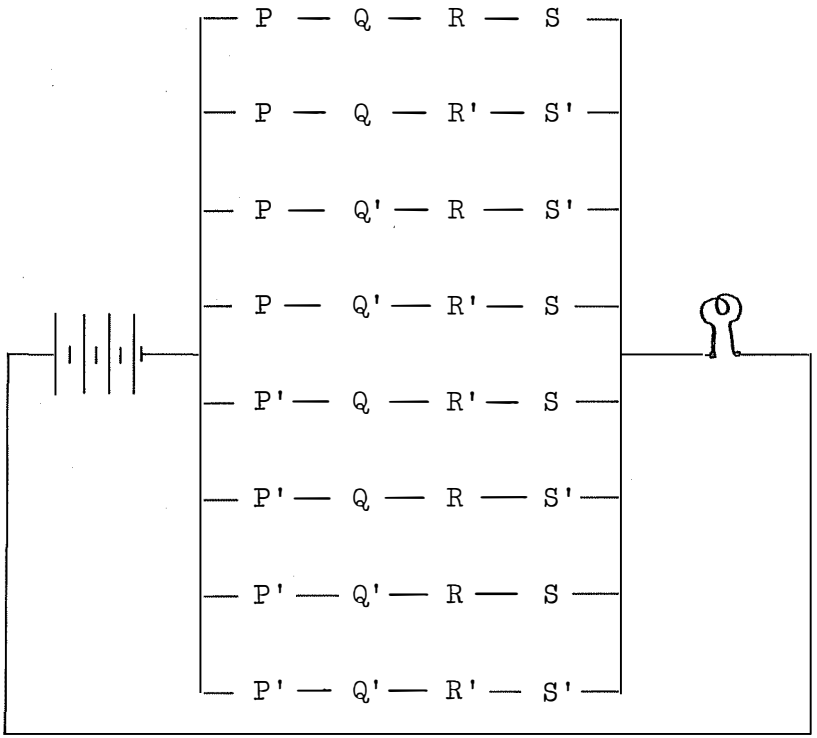
b) two of the switches are in position one

P Q R' S'
P Q' R S'
P Q' R' S
P' Q R' S
P' Q R S'
P' Q' R S

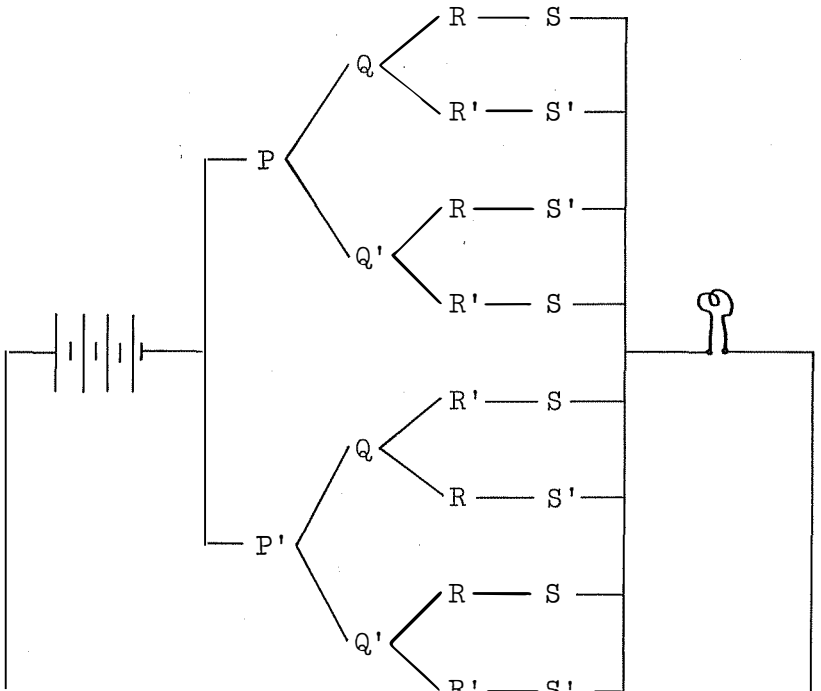
c) none of the switches is in position one

P' Q' R' S'

Theoretical Diagram



Simplified Theoretical Diagram



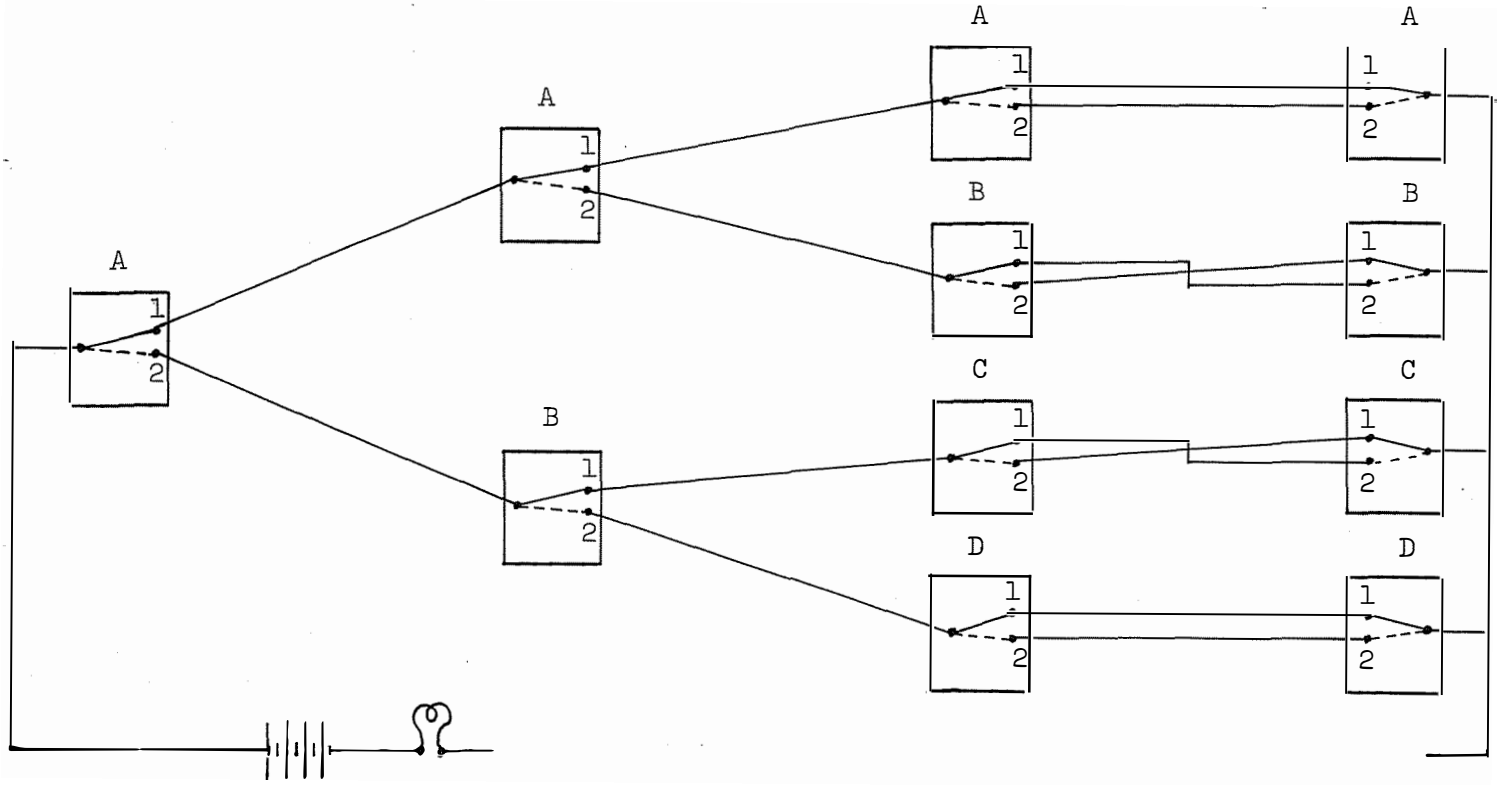
Practical Diagram

Switch P
(1P D-T)

Switch Q
(2P D-T)

Switch R
(4P D-T)

Switch S
(4P D-T)



MATHEMATICS GAZETTE RECEIVES THE GREER AWARD

L. D. Auckland (St. Thomas)

At the 1964 Convention of the Ontario Educational Association, the Mathematics Gazette was presented with the Greer Award for a distinctive, necessary, and useful contribution to education in Ontario. The cash award was presented to Mr. W. R. Cunnington, president of the Ontario Association of Teachers of Mathematics and Physics, which is one of the sponsors of the Mathematics Gazette. The presentation was made by Robert H. Wallace a former O.E.A. president, in honor of the late Dr. V. K. Greer, a leader in the field of education in Ontario for many years.

Dr. Greer was born at Winchester, Dundas County, in 1885. He began his professional career as a teacher in a rural school near Hamilton and shortly after became Principal of Tweed Continuation School. Later he attended Queen's University and obtained his Master's Degree in Mathematics. This led to his appointment as teacher of Mathematics in St. Thomas Collegiate Institute. From this point, Dr. Greer's regular work was in the field of elementary education. He became successively Inspector of Public Schools in his home County of Dundas; Master at Stratford Normal School; Inspector of Public Schools and later Inspector Administrator of Public Schools in London, and Chief Inspector of Public and Separate Schools for Ontario in 1925. In 1944 he was promoted to the office of Superintendent of Elementary Education for the Province.

Among his other positions, he was chairman of the Teachers' and Inspectors' Superannuation Commission; Financial Advisor to the Minister of Education; and Assessor to the Royal Commission on Education, until his death in November 1945.

The Greer Award has been made almost annually since 1947 by a Committee of the Ontario Education Association. The Mathematics Gazette is pleased to be a recipient of this award for its work in the field of education in Ontario, from the fund established as a memorial to this former educational leader.

THE HERON FORMULA FOR THE AREA OF A TRIANGLE

R. G. Stanton

In a recent Gazette, we published a development of the Heron formula without using half-angle formulae; developments appear in the various new Grade 11 texts. The purpose of this note is to sketch a proof which I recently encountered in a 1722 edition of the works of Nicolas de Malezieu (1650-1727) a mathematician and poet. While the proof is probably not the most parsimonious, the idea is interesting and allows use of some geometric theorems which one would not expect to crop up.

Let us consider a general scalene triangle with $a < b < c$ as shown in Figure 1.

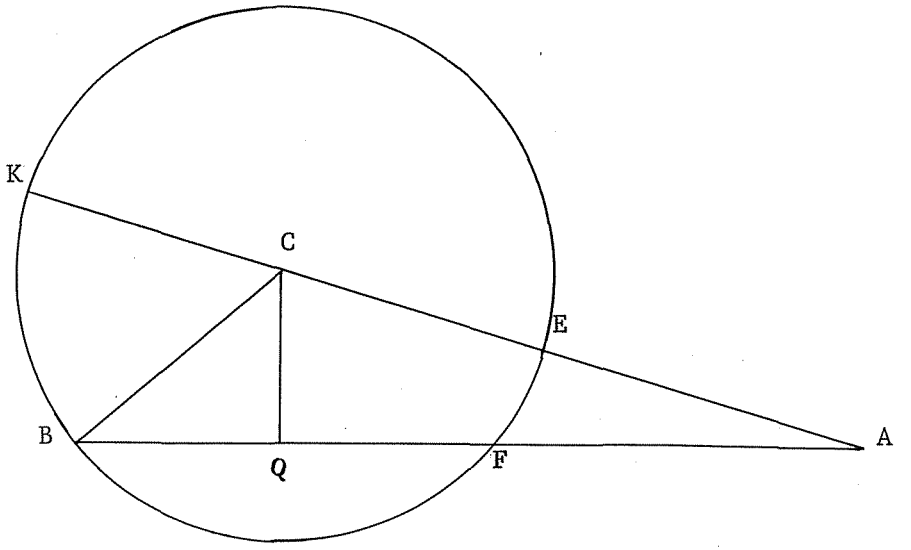


Figure 1

With centre C and radius a describe a circle cutting CA at E and AB at F; produce AC to meet the circle at K, and drop the perpendicular CQ.

We know, from the properties of secants, that

$$AE.AK = AF.AB \ .$$

Now $AE.AK = (b-a)(b+a) = b^2-a^2$.

If we use x to denote the base BQ, we have

$$AF.AB = (c-2x)c \ .$$

Thus $(c-2x)c = b^2 - a^2$,

whence $$x = \frac{c^2-b^2+a^2}{2c} \ .$$

Knowing x, it is easy to compute the altitude from

$$\begin{aligned} CQ^2 &= a^2 - x^2 \\ &= (a-x)(a+x) \\ &= \left[\frac{b^2 - (a-c)^2}{2c}\right]\left[\frac{(a+c)^2 - b^2}{2c}\right] \\ &= \frac{(b-a+c)(b+a-c)(a+c-b)(a+c+b)}{4c^2} \end{aligned}$$

Introducing the usual abbreviation

$$a + b + c = 2s \ ,$$

we then have

$$\begin{aligned} CQ^2 &= \frac{(2s-2a)(2s-2c)(2s-2b)(2s)}{4c^2} \\ &= 4s(s-a)(s-b)(s-c)/c^2 \ . \end{aligned}$$

Since $\triangle ABC = \frac{1}{2}\, AB.CQ = \frac{1}{2}\, c(CQ)$,

we end up with the usual result:

$$\triangle ABC = \sqrt{s(s-a)(s-b)(A-C)} \ .$$

[Note: Of course the development for the cases a=b or a=b=c is trivial without this or any other special method.]

GRADE XIII PROBLEMS, 1964

ultural experiment, there are b fields,
plots of land. In these plots v
re planted subject to the four condi-

e variety be planted in each plot;
ts in the same field contain the same
grain;
ty be planted in a total of r plots;

rieties occur together in the same
tly λ times.
and that $r(k - 1) = \lambda(v - 1)$.

r of varieties planted, including
but the total number of plots of
ore $bk = rv$.
icular variety, say grain A , and
ies in the field where A is planted.
lds, each time with k-1 other varieties.
other varieties is then $r(k - 1)$.
other varieties, and each is planted
herefore $r(k - 1) = \lambda(v - 1)$.

g to give a "graphical" solution as
l be represented by a blue dot and
by a red dot. Join a blue dot to
henever the variety of grain repre-
is planted in one of the k plots

that readers
We use Figure
found by
endicular to

C

.BC

(1)

CE = CD, we find

(2)

s - c. (3)

eral and hence
B respectively,
hus

of the field represented by the blue dot. The lines may be counted in two ways: there are b blue dots each joined to k red dots, i.e., kb lines; similarly, there are v red dots each joined to r blue dots (since each variety is planted r times), i.e., vr lines. Thus $vr = bk$.

To derive the second equation, use green dots to represent each of the $v(v - 1)/2$ pairs of varieties. Join a green dot to a blue dot by a line whenever the pair of varieties is planted in the corresponding field. Notice that each pair of varieties occurs in exactly λ fields, and each field contains $k(k - 1)/2$ pairs of plots. Counting these lines just as in the first case leads to the equation

$$\frac{v(v - 1)}{2}\,\lambda = \frac{k(k - 1)}{2}\,b .$$

Substituting $bk = rv$ gives $r(k - 1) = \lambda(v - 1)$, as required.

2. Develop a formula for the sum of the fourth powers of the first n natural numbers, that is, for

$$1^4 + 2^4 + 3^4 + \ldots + n^4 .$$

Solution.

Set $x = 1, 2, \ldots, n$ in the identity

$$(2.1) \qquad (x + 1)^5 - x^5 = 5x^4 + 10x^3 + 10x^2 + 5x + 1 .$$

Add the resulting identities to give

$$(2.2) \qquad (n + 1)^5 - 1 = 5\,S_4 + 10S_3 + 10S_2 + 5S_1 + n$$

where $S_i = 1^i + 2^i + \ldots + n^i \quad (i = 1, 2, 3, 4)$.

Solving for S_4 leads to the formula

$$S_4 = \frac{1}{5}[(n+1)^5 - 10S_3 - 10S_2 - 5S_1 - n - 1] ,$$

where $S_3 = n^2(n+1)^2/4$, $S_2 = n(n+1)(2n+1)/6$,

$S_1 = n(n+1)/2$.

If desired, S_4 can be expressed in terms of n by substitution. The result is

$$n(n+1)(2n+1)(3n^2 + 3n - 1)/30 .$$

3. By considering the identity

$$(3 + 2\sqrt{2})^n = a_n + b_n\sqrt{2}$$

show that there is an infinite number of solutions in pairs of positive integers (a_n, b_n) for the equation

$$x^2 - 2y^2 = 1 .$$

Exhibit two such solutions.

Solution.

Setting $n = 1, 2$ in the identity

(3.1) $$(3 + 2\sqrt{2})^n = a_n + b_n\sqrt{2}$$

yields $a_1 = 3$, $b_1 = 2$, $a_2 = 17$, $b_2 = 12$. It is easily checked that the points $(3,2)$ and $(17,12)$ satisfy the equation

(3.2) $$x^2 - 2y^2 = 1.$$

Induction may be used to complete the proof that (a_n, b_n) satisfies (3.2) for every n. Assume (a_r, b_r) satisfies (3.1) and (3.2). Then

$$a_r^2 - 2b_r^2 = 1$$

and

$$a_{r+1} + b_{r+1}\sqrt{2} = (3 + 2\sqrt{2})^{r+1}$$

$$= (a_r + b_r\sqrt{2})(3 + 2\sqrt{2})$$

$$= 3a_r + 4b_r + (3b_r + 2a_2)\sqrt{2} .$$

Hence $a_{r+1} = 3a_r + 4b_r$, $b_{r+1} = 3b_r + 2a_r$.

Thus $a_{r+1}^2 - 2b_{r+1}^2 = (3a_r + 4b_r)^2 - 2(3b_r + 2a_r)^2$

$$= a_r^2 - 2b_r^2$$

$$= 1$$

showing that (a_{r+1}, b_{r+1}) satisfies (3.2). Since $a_{r+1} > a_r$, all the solutions obtained are distinct.

4. Consider the sequence of real numbers defined by

$$x_1 = 2, \quad x_{n+1} = \sqrt{5 - x_n + 2x_n^2} \quad \text{for } n \geq 1 .$$

(a) Prove

(i) $x_n > 3$ for all positive integers $n > 1$;

(ii) $x_{n+1} > x_n$ for all positive integers n;

(iii) $x_{n+1} > 1.2x_n$ for all positive integers n.

(b) Investigate the existence of a real number

$$a = \lim_{n \to \infty} x_n,$$

and find a if it exists.

Solution.

(a) These three inequalities can be proved by induction. They are easily checked for the lowest value of n specified. The proofs may be completed as follows, on the assumption that the inequalities hold for $n = r$.

(i) $x_{r+1} = \sqrt{5 + x_r(2x_r - 1)}$

$> \sqrt{5 + 3(5)}$

> 3

(ii) $x_{r+1} = \sqrt{5 + x_r(2x_r - 1)}$

$> \sqrt{0 + x_r . x_r}$ $\quad [2x_r - 1 > x_r$ since $x_r > 3$ by (i)].

$= x_r$

(iii) $x_{r+1}^2 - \frac{36}{25} x_r^2 = (5 - x_r + 2x_r^2) - \frac{36}{25} x_r^2$

$= 5 - x_r + \frac{14}{25} x_r^2$

$= 5 + x_r (\frac{14}{25} x_r - 1)$

$> 5 + x_r (\frac{14}{25} . 3 - 1)$

> 0 $\quad$ [using (i) again].

(b) Using (iii), it follows by induction that

$$x_n > (\frac{6}{5})^{n-1}(2).$$

Thus x_n is unbounded as $n \to \infty$; so $\lim_{n \to \infty} x_n$ does not exist.

An alternative proof can be obtained by assuming the existence of a real number α such that $\alpha = \lim_{n \to \infty} x_n$.

Then α must satisfy

$$\alpha = \sqrt{5 - \alpha + 2\alpha^2} \text{ , that is, } \alpha^2 - \alpha + 5 = 0.$$

However, this quadratic has no real roots.