Submission of Articles

The *Ontario Mathematics Gazette (OMG)* is looking for articles, news and "good ideas" useful to mathematics teachers or mathematics teacher education. We are seeking submissions from mathematics teachers K-12 and tertiary, that describe teaching mathematics content in an innovative way.

The criteria for manuscript submission to the *OMG* includes the following:

- The ideas/activities are of interest to the readership.
- The article/ideas are not widely known or practised and are innovative.
- The mathematics content is appropriate for the readership.
- The mathematics content is accurate.
- The article is well written and grammatically correct.
- The article/ideas are free of sexual, ethnic, racial and other bias.
- The article has not been published previously, nor is out for review by other publications.

Articles must be original. Your article must be word processed, preferably using *MS WORD*, double spaced on one side with wide margins, not exceeding 10 numbered pages of text and prepared according to the *Publication Manual of the American Psychological Association Fifth Edition*. Diagrams and figures should be drawn by computer if possible or neatly drawn in black ink in camera-ready form. Embedded images should also be submitted separately in jpeg or tif format. Proof of the photographers' permission is required. For photos of students, parent or guardian permission is required. Please number pages.

You must submit **4 hard copies** of your article to the Editor **Marilyn Hurrell** whose address appears on page 2. Your name may not appear anywhere in your article, including web sites. Your name and full mailing address and e-mail address must be included on a separate sheet. Your article will be sent out for blind review. If it is accepted as is, or with major/minor revisions or rejected, you will be notified. Once an article is accepted for publication, computer files of text, original graphics and photos will be required. E-mail submissions are welcome but must be followed by hard copies.

The Editor reserves the right to edit manuscripts before they are published. Once an article is published it becomes the property of OAME.

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The *Ontario Mathematics Gazette*—ISSN 0030–3211—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

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Advertisements for the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** at the above address. Courier is recommended to avoid possible delays. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be camera-ready and colour advertisements should have no bleeds.

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Advertising Rates are available by telephoning, e-mailing, or writing to the Advertising Manager.

▲ Editor's Message

MARILYN HURRELL, EDITOR



Although I, and everyone associated with the production of the *OMG*, strive for perfection, I did miss an error in the December 2007 (Vol. 46 # 2) issue. This has been dealt with and I am pleased to print the corrected version of "Financial Literacy and the Rule of 72 (Some Intriguing Applications for Grades 10-12)" in this issue of the *OMG.*

Thanks and apologies to the author John Kitney, who drew the errors to my attention.

(The errors were in the logarithmic equations on page 35 of the version printed in December.)

We in NWOAME are busily working on our first conference since Nov. 2005. It is coming together nicely and, we hope, will be a huge success. This is not only a shameless plug for the event but also an encouragement to those members of other Chapters to get involved in an event for your Chapter! I have met some very good and enthusiastic young teachers who are volunteering their tine and energy on behalf of NWOAME.

Remember to check out some activities for Pi Day on the Internet (see below) and involve your students in a fun mathematical activity on March 14. I thought the Pi mobile on the web site "Math With Mr. Herte" was a good one.

Please VOTE on-line for your next OAME Board members!

Marilyn

From *News Bulletin* [National Council of Teachers of Mathematics], March 2007, Volume 43, Issue 7, p. 1. See **http://www.nctm.org/news/articles/2007_03nb_piday.htm**

March 14 – A Day to Celebrate Math and Science.

It is said that March comes in like a lion and goes out like a lamb. And perhaps it does if the lion has a wild mane like Einstein and the lamb has a tail that seems to go on forever like pi. On March 14, in the middle of the month when the lion and lamb meet, we celebrate Einstein and the intriguing number pi. Why March 14?

Cont'd on page 30

▲ President's Message

JACQUELINE HILL, PRESIDENT



Happy Hippy Hoppy Spring One and All;

I think this can go on record as one of the most interesting winters in terms of weather that Ontario has experienced in a long time.

As my year as President comes to a close I would just like to take inventory. In my first communications to the board of directors of OAME I revealed my message of "In the classroom and beyond". I have spent my year trying to keep this in mind. I believe that everything has to start with teachers in the classroom and then grow. From this vantage point I also kept in mind the vision of past presidents Dan Charbonneau (The Environment) and Kathy Kubota Zarivni (Complexity Theory). I wove these two themes as well as Differentiated Instruction into the board meetings. In my President's Messages I gave a "before/during and assessing DI" as well as a "key phrases to keep in mind when you think about DI " piece.

As you think about the picture of mathematics and how Differentiated Instruction fits in, you need to keep at the forefront of your mind strategies that are going to give the students choice in their learning. One way to go about giving choice to the students is to use portfolio assessments with them. It doesn't matter if students are in grades 1-12 you can still give them choice that relates to their learning style. An example of a generic math portfolio could be;

Choose two of the following

1) create a model for your understanding of the concept
2) write a short story demonstrating your understanding of the concept
3) research and critique two math websites
4) create a "Dear Ann Landers" column of a newspaper that describes 3 everyday uses for the concepts in this term
5) create a three fold brochure to "sell" the math concepts in this unit

Choose three of the following

6) write a poem or Rap song that will help you remember the content in this unit
7) create a crossword puzzle that has the definitions from this unit in it
8) create a word search for the definitions in this unit
9) create a slide show (or power point show) to convey your understanding of the unit content
10) make a dictionary of terms from this unit
11) create a radio program to help listeners understand the content in this unit
12) Create a card game (e.g., Go fish, or concentration) dealing with a topic covered

Submit both of the following

13) Your best five journal responses
14) Completed Survival guide (with all exam review questions)

I ended my message last time with what I thought was a nice math problem. The problem was stated as "Using the last four digits of a phone number, how many times will the first three digits add up to the final digit?" (e.g., 2316 is 2 + 3 + 1 = 6) That is, the first three digits add up to the last digit (the 6). Explain your answer using two different methods.

When I think of the process expectations it is obvious that problem solving has to be done in order to solve the problem. You will need to:

- select tools and computational strategies to go about solving the problem.
- use reasoning and proving when you think you have all of the possible outcomes.
- reflect on your solution to ensure it is reasonable.

It is all about connecting when you consider having to organize information. Have students organized in different contexts that they could call upon here. To answer using two different methods may present the representation of a variety of mathematical ideas. And to answer the question, you would be communicating your mathematics either orally, visually or in writing. Many methods may be employed. One such method could be referred to as The Sum of the Digits.

Let's look at the last digit being a 0. So to get a sum of 0, the digits must be 0000 (all zeros). Thus one outcome.

Let's look at the last digit as a 1. You could have 1001 or 0101 or 0011; thus 3 outcomes.

Let's look at the last digit as a 2. You could have 0022 or 0202 or 0022 or 1102 or 1012 or 0112; thus 6 outcomes.

Similarly if the last digit is a 3, you could have 0033 or

0303 or 0033 or 1113 or 0123 or 0213 or 1023 or 2013 or 1203 or 2103. Thus there are 10 outcomes.

In total there are 220 ways to have a four digit phone number such that the first three digits sum to the fourth digit.

Finally… in my first President's Message I referred to those inspirational people in my life who helped me along my "mathematical path". I would like to close by again counting my blessings and thanking them for their support and encouragement along the way.

Yours in Mathematical fun,

Jacqueline Hill



VOTE
MAR 1 - APR 10

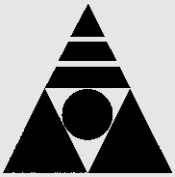
ALL MEMBERS TAKE NOTE!!
OAME ELECTIONS ARE ONLINE

Annual elections for positions on the Executive and Board of Directors will be online from March 1- April 10. To vote, you must activate your online account on the website at **www.oame.on.ca** if you have not already done so. Click on "Members Only" and follow the directions to activate your account.

You will find your membership number and expiry date on the envelope of your *Gazette*.

From March 1 to April 10 you will find "OAME Elections" in the links on the left side, after you have logged in using your username and password created when you activated your account. Just follow the instructions. Short biographies for all candidates are included.

If you need assistance, contact the Executive Directors



▲ LETTER TO THE EDITOR

Dear Editor:

Please invite teachers you teach or work with to participate in the Canada-wide Math Performance Festival. The Festival is sponsored by the Fields Institute, the University of Western Ontario, and the Canadian Mathematical Society. See http://mathfest.ca

We are also running a Math Performance Contest for grades 4-6 students and teachers (and preservice teachers). The adjudication committee includes:

1. Susan Aglukark (3-time Juno Award winner)
2. Tracy Bone (2007 CAMA Best Female Artist)
3. Douglas Coupland (author of bestselling *jPod*)
4. Bob Hallett (of *Great Big Sea*), and
5. Jay Ingram (of *Daily Planet*).

Medals (and other prizes to be announced) will be given to the top 3 student and top 3 teacher performances selected by each celebrity, as well as the top 10 student and top 10 teacher performances overall. The deadline for submissions is June 30th, 2008.

We are in the process of sending Math Performance Festival posters to all elementary schools across Canada, in English, French & Inuktitut. We will also incorporate all three of these languages in the Festival website.If you are teaching preservice teachers please invite them to create and submit performances, individually or in groups (as an alternative assignment for your course??). Please also encourage them to involve their own students.

There is a wide variety of resources at **http://mathfest.ca**, including:

6. Performing Mathematics: A Guide for Teachers & Students (pdf)
7. Teachers chat about preparing for the Festival (videos)
8. How to Write and Perform a Math Poem
9. How to Interview a Math Concept
10. How to Perform a Math Problem
11. How to Organize a Math Performance Festival at your School …… and a wide variety of math performances

Thank you so much,
George Gadanidis
Associate Professor, Faculty of Education,
University of Western Ontario
http://publish.edu.uwo.ca/george.gadanidis ▲

▲ Technology Corner: Collecting Data with Sensors

MARY BOURASSA



Mary Bourassa teaches mathematics at Lisgar Collegiate Institute in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She was also past-president of COMA, gives workshops around the world and is a National Instructor for Texas Instruments.

The mathematics we teach is much more accessible to students if it is real-life mathematics. One way to achieve this is to collect data, model it and make predictions based on our model. There is much data that can be collected without technology but sensors allow us to collect many different types of data. The most commonly used sensor is the motion detector, also known as a CBR, CBR2 or Go!Motion sensor. These tools have completely changed how we teach students about graphing and have had a huge impact on their understanding of graphs. They can be used for such activities as matching distance-time graphs, collecting ball bounce data and collecting sinusoidal data by having students walk in circles.



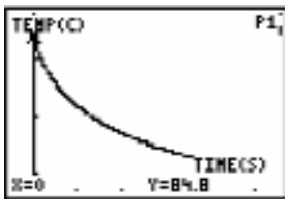
Vernier makes over 50 different sensors (the full list is available at **http://www.vernier.com/probes/**). They can be used with LoggerPro software, TI-InterActive! software, Fathom software as well as EasyData and DataMate apps for the TI-84 family of graphing calculators. Some Go! products (Go!Motion and Go!Temp) connect directly to a computer, while Go!Link can be used as the intermediary for 43 compatible Vernier sensors, used mostly for science experiments. In order to use other sensors with graphing calculators you need a CBL2 as the intermediary. We'll take a look at two activities using sensors: modeling exponential data and sinusoidal data. Both will be explained using graphing calculators but they are easily modified for use with computer software.

Activity 1

Newton's Law of Cooling is an easy activity to complete when studying exponential functions. Collecting cooling data can be time-consuming but there is a way of collecting good data, quickly. Attach the temperature sensor to channel 1 of the CBL2 and the CBL2 to your TI-84 and run the DataMate app. It should recognize the temperature sensor and display the current room temperature. Take note of this value.

Collecting the Data:

- Get a cup of very hot water.
- Place the temperature sensor in the water. The temperature will be displayed and you should see it rise. Once the temperature reaches its maximum press 2:START to begin collecting data.
- Remove the temperature probe from the water so that it will cool in the air. This will effectively simulate the action of a cooling cup of liquid while allowing you to collect the data in a relatively short period of time.
- You will see the data being plotted as it is collected for three minutes (you can change the length of time over which data is collected if 3 minutes is not sufficient). Your data should show temperature values that decrease rapidly at first then level off as the time values increase.



- If your data is not satisfactory, repeat the experiment.

Once you are satisfied with your data, quit DataMate. The time data will be stored in List1 of your calculator and the temperature data in List2.

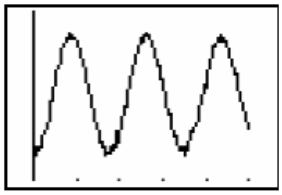
- Students working in groups should now transfer the data to their group mates who will need to set up their own scatter plots in order to analyze the data. They should determine the type of model to use then find

an equation of the form $y = ab^x + c$ to fit the data. The calculator's exponential regression will not produce a good fit as its equation is of the form $y = ab^x$ thus students are forced to model the equation themselves using the knowledge they have acquired about the parameters of the equation. You can then pursue follow-up questions tying this to Newton's Law of Cooling. You can also ask them to determine whether they should add milk to a hot drink when they make it or after a few minutes in order to keep the drink as hot as possible.

Activity 2:

This activity, which produces sinusoidal data, can be performed with a tuning fork or by blowing in the top of a bottle. As students like to make noise, the latter is the option I tend to choose! As with the previous activity, start by taking the sensor, in this case a Vernier microphone, and plug it into channel 1 of the CBL2 then start the DataMate APP.

Collecting the Data:



- One person should hold the microphone while the other creates the sound. When ready press 2:START – it will beep then start sampling.
- If you are not happy with the results press ENTER and start again.
- Once you have a good sinusoidal graph press ENTER then choose 5:TOOLS then 1:STORE LATEST RUN.
- Choose option 6:QUIT and you should get a screen indicating that time is stored in list1 and frequency in list2.
- Press ENTER.
- You can now disconnect the calculator and CBL2.

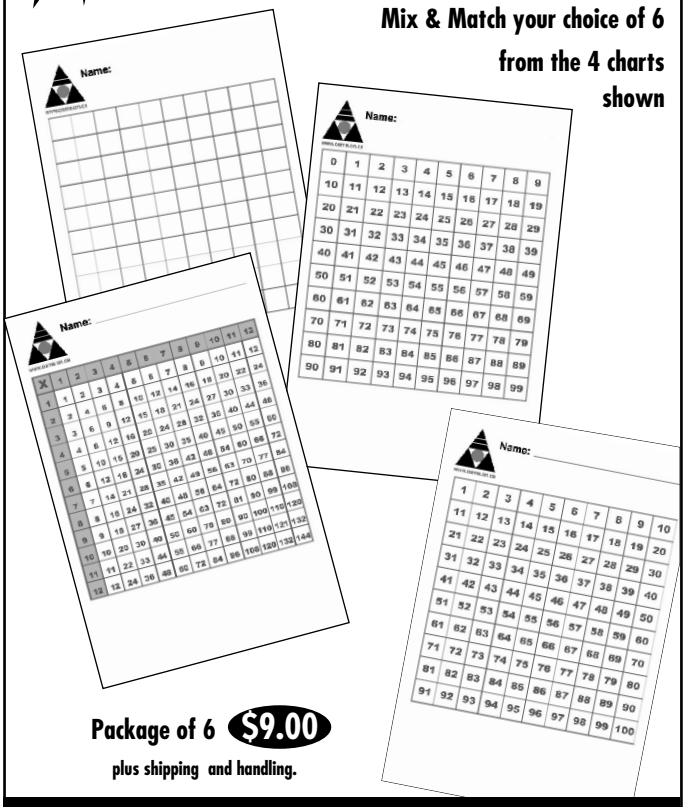
Students will now need to transfer the data to the other members of the group who should then set up a scatter plot. They will need to model the data using a cosine equation of the form $y = a \cos b(x - c) + d$. You can choose how much scaffolding your students need to get that equation. I ask for a cosine equation instead of a sine equation as there is no cosine regression available on the TI-84. Once students have a model, they can work out the frequency of the sound they made. If using a tuning fork, they can compare their calculated value with the one imprinted on the tuning fork. They can then experiment with the loudness and pitch of the sound and see if they can correctly determine how changes in the sound will affect the graph and equation.

These are just two examples of how sensors can connect the mathematics we teach to the real world. The data collection in both activities is quick leaving plenty of time for modeling and answering questions. Students will solidify their understanding of concepts when they apply them to a concrete problem and they will enjoy doing so. These activities can also be performed as demonstrations and the data set obtained for each can then be given to all students to model. If you would like classroom-ready versions of these activities, please feel free to e-mail me. ▲

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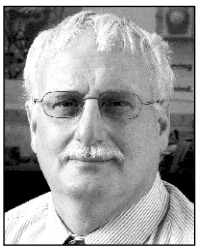


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▲ OAME/NCTM Report March 2008



JOHN KERSLEY

In my last few columns I have mentioned several NCTM web sites. Most people, however, start with **www.nctm.org** and navigate from there. Another approach is to start with **my.nctm.org/eBusiness**. From here members can sign in and find a list of all publications, math materials and apparel offered by NCTM.

New Publications From NCTM

- ***Mathematics Teaching Today: Professional Standards for Teaching Mathematics*** This publication provides a vision for effective mathematics teaching and describes various aspects of mathematics teaching including teaching practices, professional supervision, interacting with peers and professional growth.
- ***Cartoon Corner*** Grab students' attention and imagination with cartoons collected and adapted from *Mathematics Teaching in the Middle School*
- ***Algebra in the Early Grades*** A comprehensive look at issues in algebra in the early grades.
- ***Navigating through Numbers and Operations in Grades 3 – 5*** The latest addition to the popular navigation series.
- ***Second Handbook of Research on Mathematics Teaching and Learning*** (two volume set) This set allows educators to see where mathematics education research has been and where it should be going in order to provide new conceptualizations of research problems and to suggest possible research areas.

A glimpse in the past and a look in the future. NCTM is about to add several of its out-of-print articles and book chapters to its collection of online resources. The web site is **www.nctm.org/publications/forms.aspx?ekfrm=12520**.It is interesting/surprising to find out how mathematics was taught years ago. Much has drastically changed, but a lot is still much the same. Future publications include: 'The High School Curriculum Project' which looks at the grades 9 – 12 curriculum from a different perspective from NCTM's *Curriculum Focal Points*; and a new three volume series, tentatively titled, *Essential Understandings* for mathematics coaches and mentors.

Web Sites of Interest to Mathematics as Suggested by NCTM

- **www.casioeducation.com/onlinetraining** free online training for Casio's latest graphing calculator, PX-9860G Slim
- **www.onlinemathcourses.org** eight graduate level mathematics courses with content geared for teachers
- **www.igenstudio.com** from innovation generation. A fun, educational, interactive site designed for children to explore how technology and science affect their lives
- **www.funbrain.com** learning activities for parents and teachers including flashcards and games
- **www.notsohumblepi.com** mathematics cartoon greeting cards for a low cost with profits supporting juvenile diabetes
- **www.mathbits.com** offers wide variety of mathematics material for students and teachers including lesson plans, explanations and resources for algebra, geometry and trigonometry

NCTM Professional Development Opportunities

Events are available at **www.nctm.org/profdev** This site also includes resources, (books, online and articles) for the focus of the year 2007 – 2008: 'Becoming Certain about Uncertainty: Data Analysis and Probability'.

A complete list of workshops on line can be found at **www.nctm.org/workshops.aspx**

NCTM Conferences www.nctm.org/meetings

NCTM Annual Meeting and Exposition Salt Lake City, Utah "Becoming Certain About Uncertainty" April 9 – 12, 2008 Washington, D.C. April 22 – 25, 2009

NCTM Regional Conferences and Exposition for 2008 Oklahoma City, Oklahoma October 2, 3, 2008 Cleveland, Ohio October 16, 17, 2008, Reno, Nevada November 6, 7 2008

Affiliate Conferences

A list of affiliate conferences can be found at **www.nctm.org/affiliates/conferences.htm** ▲

▲ Photo Math - The Fine Art of Viewing the World Through a Mathematical Lens

RON LANCASTER



Ron Lancaster is a Mathematics Lecturer in the Department of Curriculum, Teaching and Learning, the Ontario Institute for Studies in Education (OISE/UT) at the University of Toronto, 252 Bloor St West, Rm 11-162, Toronto Ontario M5S 1V6

Teachers and students are encouraged to put on a pair of mathematical glasses and become a math spotter, someone who sees mathematics everywhere and then possibly poses mathematical questions based on the images taken. Ron Lancaster, the current Editor for Photo Math, welcomes photos with questions for possible publication in future columns. Material can be sent via e-mail or by regular mail to either of the two addresses given above.



Photograph 1 Photographer: Ron Lancaster
Location: San Francisco, California

(a) Is the surface area of each digit the same? What could you do to know for certain?

NOTE: Micah Lexier, a famous Canadian artist, created a piece of art work based on the theme of digits having the same surface area. The name of the work is *All Numbers Are Created Equal*. It can be seen at the National Gallery of Canada. Information can be found at **http://www.umanitoba.ca/schools/art/galleryoneoneone/lexier3.html** and **http://query.nytimes.com/ gst/fullpage.html?res=9501E7D8133AF933A05752C1A9659C8B63**

(b) Find three consecutive primes that add up to 731.

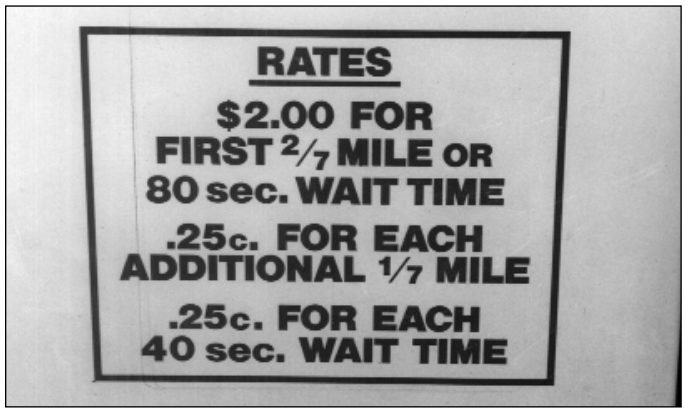
(c) Find all numbers less than 731 that can be written as a sum of three consecutive primes.

(d) Are all numbers that can be written as a sum of three consecutive primes, themselves prime numbers? How do you know?

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Photographs 2, 3 and 4. Photographer: Ron Lancaster. Location: Orlando, Florida

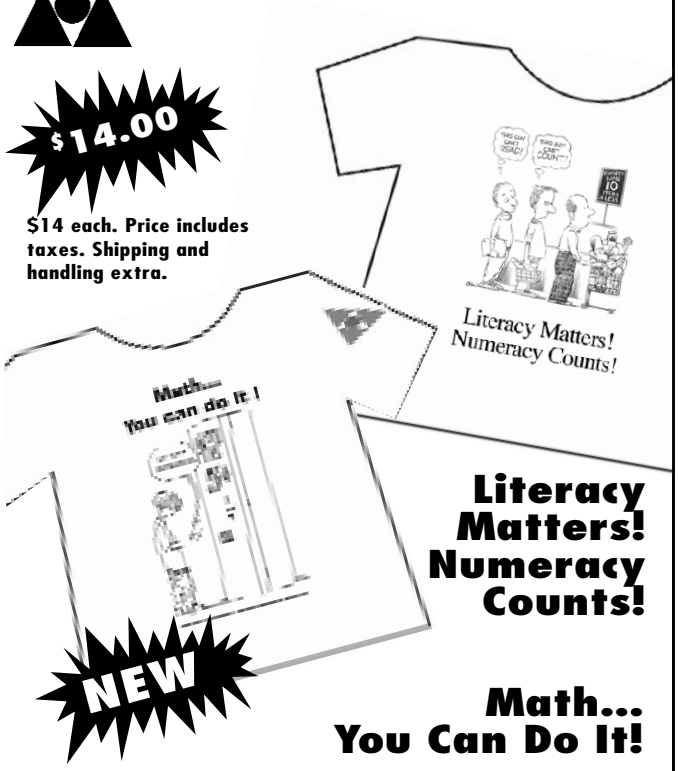
There are no regulations on cab fares at the airport in Orlando, Florida. Cabs must post their fares on the side of the vehicle and customers can pick any cab from the line up. The three photographs show some of the many different rates that are used by a number of cab companies and individual owners. A question that begs to be asked is: "are all of these rates basically the same?"

(a) Suppose that you traveled 56 miles in a cab and that due to traffic congestion and stop lights, you were sitting still for 4 minutes. Find the fare for each of these three cabs. Are the fares all the same? If not, is the one that is the lowest in this case, always the lowest? How do you know?

(b) Is it a good idea to regulate the fares charged by cabs? Do most cities have regulations?▲

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OAME T-SHIRTS



$14 each. Price includes taxes. Shipping and handling extra.

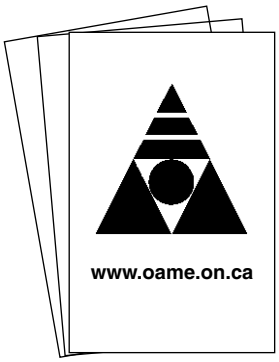
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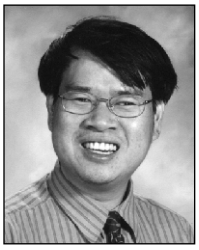
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▲ Get Interactive!

Louis Lim, with Richard Costen, Jackie Dawson-North, Bob Hosken, Diana Malile, Lesley Ropp, and Tanya Tucker



Louis Lim is department head of mathematics at Richmond Hill High School, York Region DSB, and Additional Qualifications instructor with Queen's University and York University. Louis' involvement with the OAME includes former editor of the Ontario Mathematics Gazette, presenter at chapter and annual conferences; and elected board of director. In spring 2005, he was recipient of the OAME's "Exceptional and Creative Teaching in Secondary Mathematics" Award.

This edition of the *"Get Interactive"* follows the December 2007 issue in which I featured web sites from my summer 2007 Honour Specialist Mathematics class from Queen's University. One of their assignments was to identify and describe 3 web sites that make connections to mathematics. I selected 1 web site from each student to share in this column with the broader mathematics education community. Once again, links to the environment are made. I appreciate their contributions.

1.) Mathematics and Woodworking

Contributed by Richard Costen, Toronto District School Board

Woodworking projects require mathematics to measure wood for cutting, measure angles for cutting wood, and to test if a corner is exactly 90 degrees. From a hardware store, **http://www.rockler.com/articles/display_article.cfm?story_id=91** describes the importance of mathematics to measure sides and mitre or bevel angles and how to use it in projects such as attaching a diagonal support brace and building with regular polygonal-shaped structures. **http://www.rockler.com/articles/display_article.cfm?story_id=97** has mathematical tables used in woodworking and a summary of the formulas used. Included are online calculators for finding angles and sides of right-angled triangles and regular polygons, approaching the same concepts from a more practical "angle". These links are good connections for the MFM2P trigonometry unit. A third link, useful for the MEL4E measurement unit, **http://www.rockler.com/articles/display_article.cfm?story_id=102**, shows how to buy lumber based on different pricing methods.

2.) Mathematics and Art

Contributed by Jackie Dawson-North, Durham District School Board

Art is a wonderful way to express thoughts and feelings. The web site, **http://www.princetonol.com/groups/iad/lessons/middle/geomath.htm**, has links to lesson plans that incorporate different art components in mathematics such as sacred geometry, where patterns, designs, and structures are created in nature, and mandalas, the circle of eternity. Its valuable information could be integrated into many mathematical concepts such as describing real-life patterns and circle geometry, all of which can engage student interest and enthusiasm. My personal favourite is the use of patterns and tessellations. Students will definitely enjoy creating brilliant art masterpieces during mathematics class.

3.) Math and Sports

Contributed by Bob Hosken, Upper Grand District School Board

The ESPN Sports Figures web site, **http://sportsfigures.espn.com/sportsfigures/**, "provides an athlete's perspective into the subjects of math and physics."

It has numerous lesson plans which connect mathematics and physics to sports. Lessons include "The Hoop Variable", which promotes the importance of graphing to see trends, as well as "Cartesian Baseball", which can be used as an introduction to the Cartesian coordinate system and vectors. There are also video clips that the teacher can use to help introduce the topic.

4.) Mathematics and Music

Contributed by Diana Malile, Toronto District School Board

Students may find connections between music and mathematics helpful. The following web site, **http://nrich.maths.org/public/leg.php?group_id=8&code=-365#results**, provides some great problems that are applicable to elementary and secondary students, covering topics such as transformations, fractions, and ratios. The web sites **http://plus.maths.org/issue40/features/wardhaugh/index.html** and **http://plus.maths.org/issue35/features/rosenthal/index.html** convey the importance of mathematical techniques such as Euclid's algorithm and logarithms to musicians by

connecting ratios and musical intervals. The Euclidean algorithm and logarithms can be applied to musical questions such as finding the number of tones in an octave or the number of major thirds in a fifth or in an octave. A third web site, **http://www.bced.gov.bc.ca/careers/aa/lessons/aom15.htm**, connects mathematics to a range of notes played on a piano or other instrument.

5.) Mathematics and Weather

Contributed by Lesley Ropp, Waterloo Region District School Board

Project SkyMath, **http://www.eo.ucar.edu/skymath**, is a pilot project designed and funded by the University Corporation for Atmospheric Research and the National Science Foundation, that uses current environmental and real-time weather data suitable for middle school mathematics. The project, free for download, is a self-contained module with sixteen temperature related activities that teach concepts in data analysis, graphing, number patterns, statistics and measurement. *Project SkyMath* is intended to be used as a replacement unit in Grades 5-8 classrooms and emphasizes the use of technology through working with thermometers, spreadsheet software, Internet data, and e-mail.

6.) Mathematics & Environmental Awareness

Contributed by Tanya Tucker, Peterborough Victoria Northumberland Clarington Catholic District School Board

The web site, **http://www.pbs.org/teachers/mathline/concepts/earthday.shtm** provides teachers with activities which can be used in the mathematics classroom from grades 4 through 9. The investigations promote environmental awareness and are great activities for Earth Day. Included is a link to an answer key and to more lesson plans which connect mathematics to various topics mathematics and nature, farming and gardening, weather, elections and voting, architecture, math and the movies and many others.

Another web site **http://earthmath.kennesaw.edu/** provides a variety of classroom projects involving real data that deals with environmental issues. Some of these projects include investigating fuel economy, predicting stream flow, CO_2 emissions from passenger cars, water consumption and a few others. This web site would be great for secondary school teachers. Many of the projects included involve the use of technology. ▲

▲ What's the Problem?: A Cheesey Problem

SHAWN GODIN



Shawn is the head of Mathematics, Business and Technology at Cairine Wilson S.S. in Orleans, ON.

Welcome back problem solvers. Last issue I left you with the following problem.

A mouse eats his way through a 3 x 3 x 3 cube of cheese by tunneling through all of the 27 1 x 1 x 1 sub-cubes. If he starts at one of the corner sub-cubes and always moves onto an uneaten adjacent sub-cube, can he finish at the center of the cube? (Assume that he can tunnel through walls but not edges or corners.)

This problem appeared as H286 from the April 2001 issue of *Crux Mathematicorum with Mathematical Mayhem* and the solution appeared in the February 2002 issue. You can check out more problems at the *Crux Mathematicorum with Mathematical Mayhem* web site **http://journals.cms.math.ca/CRUX/**

How do we attack such a problem? One way might be just try it out and see if you can do it. You could build a model with linking cubes and remove the blocks as you tunnel. I will model the cheese with a series of tables. I will number the sub-cubes as I tunnel my way through.

Attempt #1

Top Layer			Middle Layer			Bottom Layer		
1	6	7	18	17	16	19	20	21
2	5	8	13	14	15	24	23	22
3	4	9	12	11	10	25	26	27

Attempt #2

Top Layer			Middle Layer			Bottom Layer		
1	2	3	24	23	14	21	22	13
6	5	4	25	16	15	20	17	12
7	8	9	26	27	10	19	18	11

Attempt #3

Top Layer			Middle Layer			Bottom Layer		
1	18	17	2	3	16	5	4	15
8	19	20	7	22	21	6	27	14
9	10	11	24	23	12	25	26	13

In each of the three attempts above I tunnel through the sub-cubes following the rules. In some cases I hit the center early so that I can still get through all the cubes. I haven't been able to finish on the center cube but that doesn't mean it can't be done. To see if there is any pattern to my attempts, I will put them all together in one set of tables.

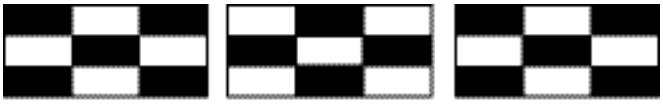
Top Layer		
1,1,1	6,2,18	7,3,17
2,6,8	5,5,19	8,4,20
3,7,9	4,8,10	9,9,11

Middle Layer		
18,24,2	17,23,3	16,14,16
13,25,7	14,16,22	15,15,21
12,26,24	11,27,23	10,10,12

Bottom Layer		
19,21,5	20,22,4	21,13,15
24,20,6	23,17,27	22,12,14
25,19,25	26,18,26	27,11,13

Oh my, that is interesting! If you look at the numbers in each cell they are either all even, or all odd … how odd. If we colour each cell in out table light or dark, so that as you move from cell to cell you change colours you get something like below.

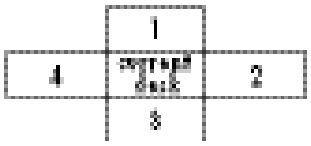
Top Layer Middle Layer Bottom Layer



The first cube and the last cube (number 27) are both odd, so if you start on a corner (shaded) you can only end on one of the shaded cubes. Since the center cube is not shaded it is impossible to end there. It is interesting to note that that there are only 13 non-shaded cells, so if you started on one of them and followed the rules you would not be able to pass through all of the cubes!

This problem deals with parity, each of the cells are assigned to one of two groups (like whole numbers are assigned to two groups even or odd). You may want to play with the following two problems.

1. A domino covers 2 squares on a checkerboard so that the board can be completely covered with 32 non-overlapping dominoes. If squares from opposite corners are removed from the board can the board be completely covered with 31 non-overlapping dominoes?
2. In a classroom there are 25 desks arranged in 5 rows and 5 columns. If 25 students fill these seats can each student move to a new desk that is adjacent to his current desk? That is he may move to either of the 4 desks pictured below.



Now for Your Homework

The letters A, B, C, D, E, F, G and H represent distinct digits. Determine their values given the two products below are true. (Note the first digit of a number must be non-zero)

A B C D	B F D G
___ x E	____x G
D C B A	G D F B

Until next time, happy problem solving! ▲

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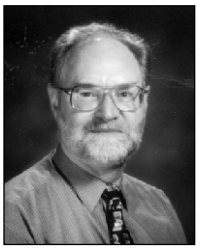
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▲ CAS Corner: Factoring Trinomials on a CAS Device

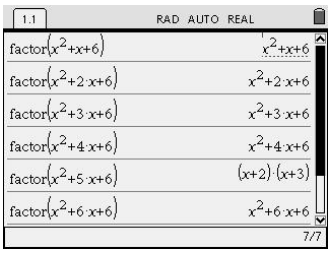
FRED FERNEYHOUGH

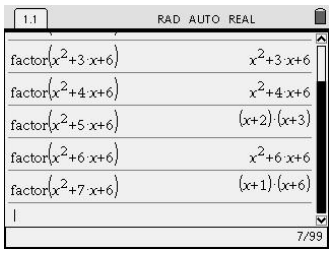


Fred Ferneyhough is former head of mathematics at Harold M. Brathwaite SS in Brampton. He retired from there in February 2006. Fred is an avid user of and advocate for the use of technology in the mathematics classroom. He is also an author and tech consultant for several books and a National Instructor for Texas Instruments.

One of the problems when teaching students to factor trinomials (over the integers) is that many students think that every trinomial can be factored, when in fact, this can only be done with a very small percentage of trinomials. Using a CAS device to explore factoring makes this venture relatively easy and reinforces the concepts developed when expansion was taught.

For years, most of us have taught factoring using a very traditional approach, creating examples that always have nice factors and each example is independent of the ones before it and those that followed it. Using a CAS and algebra tiles can change all of that.



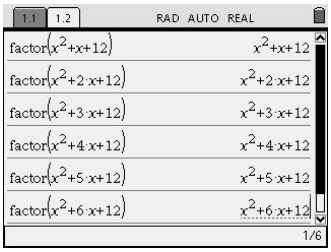


Consider the first two screens. In these screens, the quadratic term has been fixed at x^2 and the constant at 6. The coefficient of the linear term increases in each screen. In most of the lines, the same trinomial is returned as the result, indicating that the trinomial cannot be factored. Yet, two of the attempts were successful, which begs the question "Why?" This is an ideal situation in which to use algebra tiles to investigate.

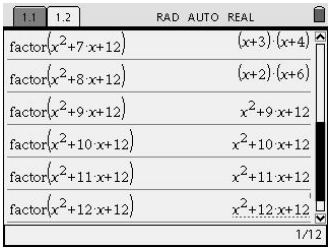
In each case, we would like our students to examine the values given in the trinomial and compare them to the constants in the factors. Hopefully, our students will remember the pattern from expanding – that the two constants in the factors must add to produce the coefficient of the middle term of the trinomial and multiply to produce the constant term of the trinomial – wouldn't it be nice if they actually expressed that statement in those words!

It would be ideal if they realized that the two successful cases occurred for the only two pairs of values which have a product of six and that they realized that they could quit after the last attempt shown.

As mentioned before, using a CAS allows students to generate a large number of examples quickly, so another example is presented for them to attempt. This is the first screen in this set. Since the constant is 12, we might hope that students could predict that the factors that work would be the pairs {2, 6}, {3,4} and {1,12}.



If they do recognize this fact, then hopefully, at least one student will notice that there should be at least one more line with a trinomial that factors.



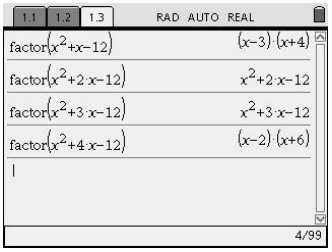
At this point, we could put students into pairs or groups of two or three and assign each group a constant to work with. They will try to factor trinomials of the form $x^2 + k*x + c$, where c is the constant assigned to them and k represents a natural number that varies from 1 to c. The idea here would be to have only the successful attempts done and find all of them for the constant assigned. We could have the successful results either displayed on the device or on the board. And yes, it's quite likely that you are going to have to explain what a natural number is – sigh.

An extension of this idea would be to compile the results in the table on the next page. Each group would contribute their results. If it is, write the factors in the space provided and highlight that cell in the table.

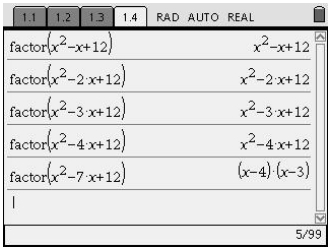
	c = 1	c = 2	c = 3	c = 4	c = 5	c = 6	c = 7	c = 8	c = 9
b = 1	x^2+x+1	x^2+x+2	x^2+x+3	x^2+x+4	x^2+x+5	x^2+x+6	x^2+x+7	x^2+x+8	x^2+x+9
b= 2	x^2+2x+1	x^2+2x+2	x^2+2x+3	x^2+2x+4	x^2+2x+5	x^2+2x+6	x^2+2x+7	x^2+2x+8	x^2+2x+9
b = 3	x^2+3x+1	x^2+3x+2	x^2+3x+3	x^2+3x+4	x^2+3x+5	x^2+3x+6	x^2+3x+7	x^2+3x+8	x^2+3x+9
b = 4	x^2+4x+1	x^2+4x+2	x^2+4x+3	x^2+4x+4	x^2+4x+5	x^2+4x+6	x^2+4x+7	x^2+4x+8	x^2+4x+9
b = 5	x^2+5x+1	x^2+5x+2	x^2+5x+3	x^2+5x+4	x^2+5x+5	x^2+5x+6	x^2+5x+7	x^2+5x+8	x^2+5x+9
b = 6	x^2+6x+1	x^2+6x+2	x^2+6x+3	x^2+6x+4	x^2+6x+5	x^2+6x+6	x^2+6x+7	x^2+6x+8	x^2+6x+9
b = 7	x^2+7x+1	x^2+7x+2	x^2+7x+3	x^2+7x+4	x^2+7x+5	x^2+7x+6	x^2+7x+7	x^2+7x+8	x^2+7x+9
b = 8	x^2+8x+1	x^2+8x+2	x^2+8x+3	x^2+8x+4	x^2+8x+5	x^2+8x+6	x^2+8x+7	x^2+8x+8	x^2+8x+9
b = 9	x^2+9x+1	x^2+9x+2	x^2+9x+3	x^2+9x+4	x^2+9x+5	x^2+9x+6	x^2+9x+7	x^2+9x+8	x^2+9x+9

The question is "what fraction or percentage of trinomials are factorable". Certainly one table won't answer this question, but it gives students an insight into the fact that a very small percentage are factorable. In addition, it is another opportunity to reinforce the pattern of "two numbers that sum to b and multiply to give c".

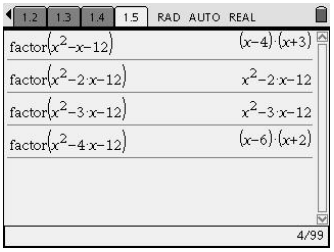
There are three other variations that can be investigated. The first variation is shown to the right. In this approach, the constant is changed to a negative while the coefficient remains positive and increases. To work in this context, the students need to consider integer values for the constant term in each factor. The first four in a series are shown and two were successful. Are there others in this series which would be successful?



A second variation would be to change the value of the coefficient of the linear term to a negative value but keep the constant term positive. The screen to the right shows the first four attempts in a series, as well as one found later that was successful. As above, students could be challenged to find other successful trinomials and ensure that they have found all possible results.



Finally, students could investigate possible outcomes when both of these values are negative. The approach of investigating patterns assists students with what might normally be an abstract exercise that is attainable by students who intuitively understand such algebraic manipulations or those who rely upon memorization. Retention of skills has been found to be better in a larger number of students when such an approach has been used. ▲



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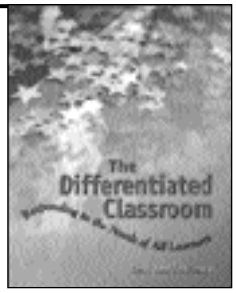
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▲ Book Review: The Path of Differentiated Instruction

REVIEWED BY SUE MELVILLE

Sue Melville teaches mathematics and chemistry in Sudbury, Ontario. She represents the Northern Ontario Mathematics Association (NOMA) chapter on the OAME board. Sue is currently chair of the OAME Equity Committee.

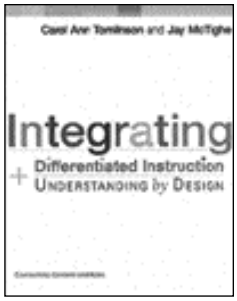
The theme of the 2008 OAME provincial conference is "The Path is Made by Walking". One of the many paths you will explore at the conference is that of differentiated instruction. In this article, we provide a brief annotated bibliography of some of the print resources on differentiated instruction that are available for teachers of mathematics.



A quick Internet search on **www.google.ca** lists 417 print resources on differentiated instruction (D.I.). If you're just becoming interested in this topic, or are not sure what differentiated instruction is, the following books provide a good overview of the topic, goals, and strategies of D.I.



The Differentiated Classroom: Responding to the Needs of All Learners was published by Carol Ann Tomlinson in 1999. Tomlinson is a prolific writer on this topic as well as a sought-after speaker and consultant. In this book, Tomlinson defines differentiated instruction, outlines its principles, and describes examples of differentiated lessons. This book is a great introduction to the topic. Published by the Association for Supervision and Curriculum Development. ISBN: 0-87120-342-1.



Tomlinson's *How to Differentiate Instruction in Mixed-Ability Classrooms, 2nd Ed. (2001)* is a good follow-up to that listed above. It goes into more depth than the 1999 book, devoting one chapter each to the steps required to differentiate lessons by readiness, interest, student learning profile, content, process, and product. I like this particular book because of its emphasis on very practical elements that will help any classroom teacher. Association for Supervision and Curriculum Development. ISBN: 0-87120-512-2.

Differentiated Instructional Strategies: One Size Doesn't Fit All, written by Gayle Gregory and Carolyn Chapman (2006) is another good introduction to the topic of D.I. Gregory is a Burlington, Ontario-based writer, speaker, and consultant. She too, has written a series of books on this topic. In this book, Gregory emphasizes the reasons for differentiating instruction and techniques for creating an inclusive environment based on the characteristics of each learner in the class. Corwin Press. ISBN: 1412936403.

In addition to these general books on differentiated instruction, many devote much if not all of their text to the specifics of mathematics instruction. The list of resources geared specifically to teaching mathematics in the elementary panel is not extensive, but here are a few examples:

Differentiating Math Instruction: Strategies That Work for K-8 Classrooms, by William N. Bender (2005) includes sections entitled "Ideas from Teachers," "Top Ten Tactics," and "Web Site Reviews." Bender includes research on brain-compatible teaching as well as strategies for getting students to visualize, vocalize, and move during math lessons. Bender is also the author of Differentiating Instruction for Students with Learning Disabilities (2002). Corwin Press. ISBN (Strategies for K-8): 0761931481. ISBN (Students with Learning Disabilities): 0761976205.

In *Differentiated Instruction: A Guide for Elementary School Teachers*, by Amy Benjamin (2003), the author dispels eleven myths of differentiated instruction before suggesting where to start for the beginning practitioner of D.I. Benjamin devotes one chapter to "Teaching for Abstract Thinking" and includes some advice for teaching mathematics using creative dramatics and visualization. This book will be of interest to teachers who have progressed past an introduction to D.I. Eye on Publication, Inc. ISBN: 1930556551.

Differentiated Instruction for the Middle School Math Teacher: Activities and Strategies for an Inclusive Classroom, by Joan D'Amico and Kate Gallaway (2008) may be of interest to intermediate teachers of mathematics. It includes lessons, suggestions for making the home-school connection to mathematics, and descriptions of how to adapt learning centres for different learners. John Wiley & Sons, Inc. ISBN: 078798468X.

In my experience, most of the publications written about differentiated instruction are geared to teachers in the elementary panel, although not specifically addressing mathematics instruction. As a classroom teacher currently teaching Grade 10 Applied math, I am always looking for ideas about how I can truly differentiate my lessons so that my strong students remain challenged while the struggling students feel included, and those in between continue to achieve. I have found some success using some of the following resources:

Differentiation in Practice: A Resource Guide for Differentiating Curriculum by Carol Ann Tomlinson and Cindy A. Strickland (2005) contains a variety of sample differentiated units of study from seven areas of study at the secondary level. It is a handy reference during unit planning and suggests a variety of strategies for meeting the needs of secondary learners. Association for Supervision and Curriculum Development. ISBN: 1-4166-0050-7.

Integrating Differentiated Instruction & Understanding by Design by Carol Ann Tomlinson and Jay McTighe (2006) brings together two models of curriculum instruction. The book begins with a rationale for incorporating the two models into one before moving on to devote six chapters to various aspects of learning in diverse classrooms. McTighe is the author of Understanding by Design (1998), written with co-author Grant Wiggins. Association for Supervision and Curriculum Development. ISBN: 1-41660284-4.

Amy Benjamin's *Differentiated Instruction: A Guide for Middle and High School Teachers (2002)* may be of interest. It uses the same format as Benjamin's book for elementary teachers and includes case studies for many subjects, including mathematics. One of my ongoing concerns about differentiating mathematics lessons at the high school level is that I can't always picture what these lessons will look like. Benjamin's book looks promising because of her emphasis on using it interactively and collaboratively with colleagues. Eye on Publication, Inc. ISBN: 193055639X.

An important piece of advice that sustains me as I travel down the path of differentiated instruction is to celebrate the successes and learn from those lessons that did not turn out as desired. The importance of knowing one's own students' levels of readiness, ability, and interest cannot be overstated. Developing a lexicon of strategies for differentiating by content, process, and product takes time. Enjoy the journey! ▲



▲ Why You Should Learn Geometry

WALTER WHITELEY
PROFESSOR OF MATHEMATICS AND STATISTICS, GRADUATE PROGRAMS IN MATHEMATICS, IN EDUCATION, AND IN COMPUTER SCIENCE, YORK UNIVERSITY, TORONTO, ONTARIO



Walter Whiteley teaches in the Mathematics and Statistics Department at York University, and supervises graduate students in Mathematics, Computer Science, Mathematics Education, and Interdisciplinary Studies. He also coordinates a part-time MA program for Teachers of Mathematics at York. His research is in Applied Geometry (working with engineers, computer scientists, biochemists and mathematicians) as well as research in Visual and Spatial Reasoning with mathematics educators. He taught in the Quebec CEGEP system for 20 years (courses between Grade 11 high school and university), before coming to York University. Much of his current teaching is geometry courses for future teachers, and for in-service teachers in the MA program for Teachers.

This article was written as a reply to an article titled "Why you should learn algebra" which appeared in the Jan 5 2007 Los Angeles Times. See: From the Los Angeles Times, Friday, January 5, 2007. See ***http://www.latimes.com/news/printedition/asection/la-oe-eggen5jan05, 1,2836776.story***

This article by Walter Whiteley also appeared in the St Louis Confluence, Volume 4 #1 2007.

The following was my response to a January 5, 2007 *Los Angles Times* article "Why you should learn algebra" by David Eggenschwiler, an English professor emeritus at USC. There is much in the article that I agree with, including the value of mathematics and science, in developing the 'habits of mind' that creative people apply in effective problem solving in many fields. What attracted my attention was the identification between 'algebra' and 'mathematics'.

As a geometer and educator, I am struck, once again, by the confusion represented in the piece, between mathematical thinking, as a rich range of activities including the associated ways to learn effective reasoning for mathematics, physics, etc., and the narrower slice associated with algebra.

Perhaps the author and the readers would find the story of Michael Faraday helpful for context. Faraday apparently showed both dyslexia and dyscalculia. He did not reason with formulae (algebra). Nevertheless, he could do engineering, and built the first electric motor, using visual reasoning recorded in his note-books. (See, for example, the studies of David Goodings.) This alternative approach, too, is part of the range of what mathematics and science include. James Clerk Maxwell (the physicist and geometer) at one point said that some of his key early work was translating Faraday's pictural reasoning into formulae. Both versions are what I would call 'mathematical reasoning'. Good science can be expressed in multiple ways and we are more effective working with such multiple representations.

It is a sad carryover of the curricular excesses of the last four decades in mathematics (including excesses at the University level), that people would speak as if algebra is central and, implicitly, geometry and associated visual reasoning are marginal. As long as we exclude people who excel at visual reasoning, and struggle with algebraic reasoning, we exclude people of enormous potential to contribute to science, engineering and mathematics. For some reflections on these choices (nuanced by interviews with a number of current mathematicians), I suggest some chapters in very recent book: *The King of Infinite Space* by Siobhan Roberts, subtitled: *Donald Coxeter: The Man Who Saved Geometry.*

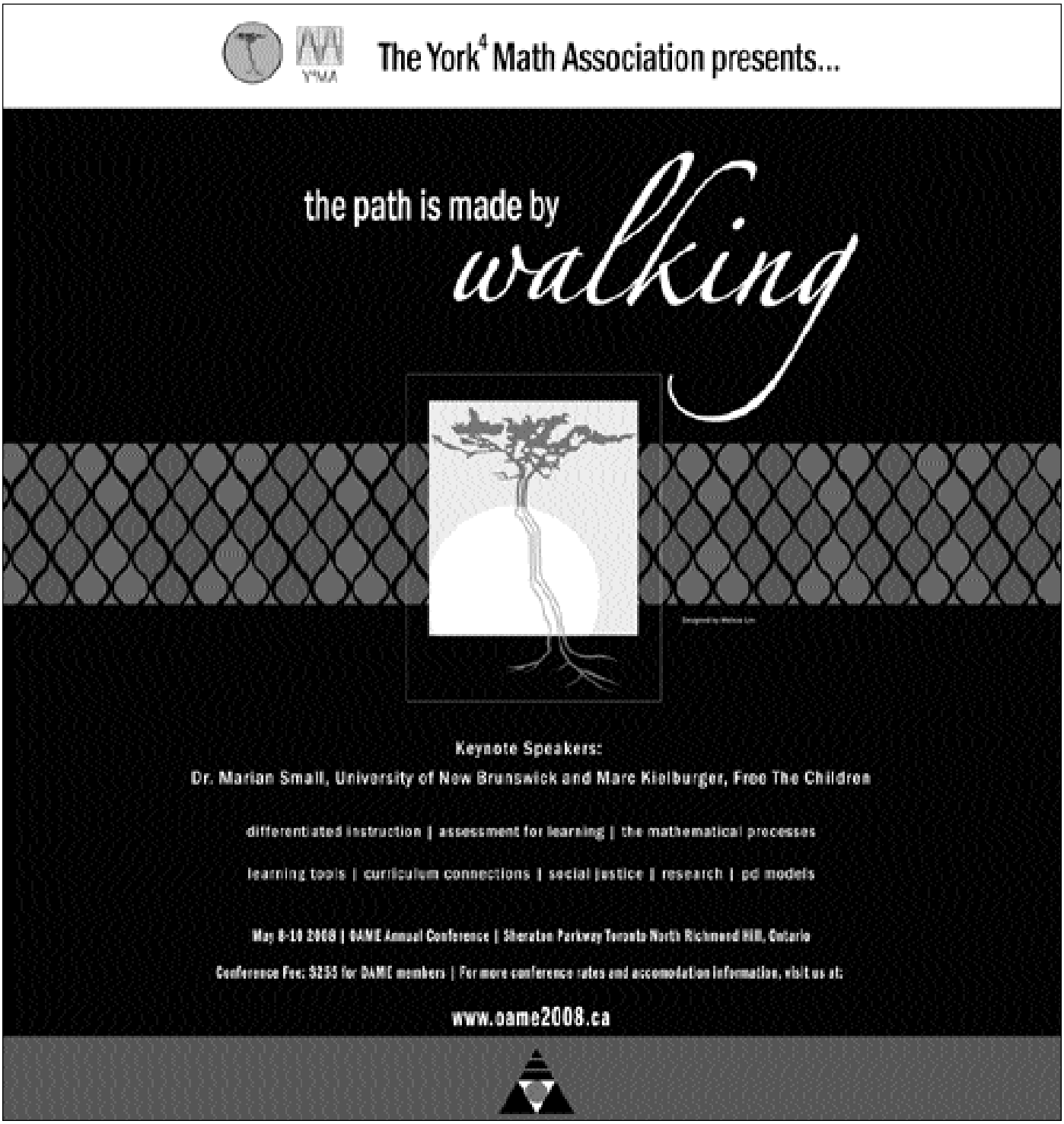
In the web of connections within human cognition, correctly evoked in the piece, algebra, geometry, probability, and the connections of these (and other) areas all play a role in developing mathematical reasoning. One should take care not to narrow the entrances to mathematics to the single door associated with algebra, nor to restrict the wanderings of interested students within the wide landscape of mathematics. The capacity of this richer landscape to entice, motivate, and train students, including students interested in the trades and applications, should be used to the full.

In fact, one area that supports the rise of geometry these days, is the deep, unavoidable need of so many applied areas to use geometry in the solution of their problems. My own work in discrete applied geometry has included collaborations with structural, mechanical and electrical engineers, as well as computer scientists,

biochemists and biophysicists. Algebra is not the way this communication starts, or ends, through it has a parallel role in suggesting and supporting connections and insights.

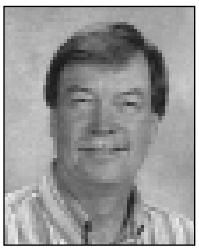
Here are a few sites that offer further links:

- Geometry, Space and Technology: Challenges for teachers and students: **http://wiki.math.yorku.ca/index.php/CMESG**
- Wiki on Spatial Reasoning **http://wiki.math.yorku.ca/index.php/Spatial_Reasoning_Links**
- Learning to See Like a Mathematician: **http://www.math.yorku.ca/~whiteley/** ▲



▲ Financial Literacy and the Rule of 72 (Some Intriguing Applications, Grades 10-12)

JOHN KITNEY



John Kitney recently retired from Bayridge Secondary School in Kingston Ontario and was Head of Mathematics. John is committed to working with teachers to integrate financial literacy in the classroom.

This article is reprinted with corrections, from the December 2007 issue of the OMG, Vol 46 #2.

The integration of financial literacy with the elementary and secondary curricula is a matter that deserves greater attention. The lives of our students will be increasingly affected by the monetary framework on which our society is built. Their successes, which include financial independence, will be determined to a considerable extent by their degree of fiscal understanding. My personal interest in this subject was piqued 6 or 7 years ago with the opportunity to participate in a summer program involving business as well as fellow mathematics teachers. The focus of the in-service was a resource called "Taking Stock In Your Future". At that time the sponsoring parties included the Investor Education Fund (Ontario Securities Commission), OISE/UT and the Toronto Stock Exchange. In this article I propose to share some applications that relate to this free resource which provides meaningful starting points for an understanding of compound interest and building personal wealth.

The most instructive part of my experience with the "Taking Stock …" program has been the opportunity to work with similarly motivated business teachers and share our two perspectives on topics in financial literacy. I found myself eager to understand the various ratios and indicators used to assess investment types and relative performance, along with their curious names. I also found that, under the financial literacy umbrella, there was a great deal of mathematics that was easily accessible to students as early as the junior and intermediate levels. To this end the Investor Education Fund continues to support the development of resources for use in the classroom as well as background and reference material for teachers. The website: **www.investorED.ca** is an excellent place to start. "Teacher's Corner" is a link to updates on the Fund's latest classroom initiatives and a range of supplementary activities. Workshops are available, at no cost, to introduce teachers to the Investor Education mandate with any of the three main resources:

1) Taking Stock In Your Future: Intermediate
2) Taking Stock In Your Future: Senior
3) Fair Play

One of many ideas that connects the business and mathematics curricula has the intriguing title of "Rule of 72". As with many other concepts that we present to our students from different perspectives, here is one of those ideas that I had not encountered as a math teacher, in such simplicity. At the same time, I found that many business teachers had either forgotten or rarely taken the chance to fully investigate the mathematical implications or limitations of this little gem. The accessibility of the graphing calculator (TI-83 series or equivalent) in our high schools and the emphasis on a functions approach to mathematics in the Ontario secondary curriculum provide an ideal framework for a discussion of this relationship.

The Rule of 72

A single payment compound interest investment will double its initial value if the product of the interest rate and the number of conversion periods is approximately 72 (using the interest rate as the percentage for each conversion period). That is to say, $1000 invested at 8% compounded annually will become about $2000 in 9 years. If this interest rate is changed to 12% then the "doubling time" is about 6 years. This immediately begs the question of whether or not we can determine the exact time required to precisely double the principal of a compound interest investment. In the familiar formula, $A = P(1 + i)^n$, this means we want to calculate the value of the exponent, n, when $(1 + i)^n$ is equal to 2. Even without an understanding of logarithms our grade 10 students can solve for the desired exponent by using the intersection of two functions. The doubling time at 12% p.a. compounded annually is the value of x that satisfies both $y = 1.12^x$ and $y = 2$ at the same time. A graphing calculator allows an easy determination of the point of intersection by entering the functions under [Y =] then selecting "5:intersect" in the menu from [2nd] CALC (found on the [TRACE] key). Repeating this process for a range of interest rates we can closely examine the Rule of 72. The chart below shows six such "intersections". It was set up using domain [–1, 20] and range [0, 5] and

was inspired by Exploratory Activity B (Activity 1.6) in the Taking Stock In Your Future: Senior[1]. The first calculator display (Fig. 1) shows six exponential functions along with the line y = 2. The second display shows the table of values for $y = 1.04^x$ (Fig 2).

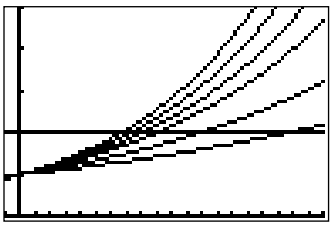


Fig 1

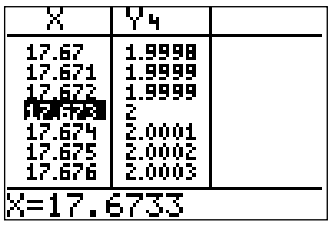


Fig 2

i %	*x*	Doubling time (n)	i x n
4	17.6730	17.67	70.7
5	14.2067	14.21	71.1
6	11.8956	11.90	71.4
7	10.2448	10.24	71.7
8	9.0065	9.01	72.1
9	8.0432	8.04	72.4
10	7.2725	7.27	72.7
11	6.6419	6.64	73.0
12	6.1163	6.12	73.2

Chart 1 Is i x n always equal to 72?

We can conclude, from the chart, that the Rule of 72 works "best" when the rate of interest is close to 8%. Of course, there is a certain amount of variation in the product i x n (using interest rates from 4% to 12%) but none-the-less it is a "good" rule.

At this point the grade 11 student might be interested in examining the amount of variation in the final amount (A) of an investment if the Rule of 72 was strictly applied. The following chart uses the compound interest formula, $A = P(1 + i)^n$, \$1000 for the principal, P, various rates of interest, i, and values of n such that i x n = 72.

Chart 2 Examples of "doubling" when i x n = 72

i %	n	(1+ i)^n *	\$ A
4	18	$(1.04)^{18}$	2 025.82
5	14.4	$(1.08)^{14.4}$	2 018.95
6	12	$(1.06)^{12}$	2 012.20
7	10.29	$(1.07)^{10.29}$	2 006.13
8	9	$(1.08)^{9}$	1 999.00
9	8	$(1.09)^{8}$	1 992.56
10	7.2	$(1.10)^{7.2}$	1 986.22
11	6.55	$(1.11)^{6.55}$	1 980.91
12	6	$(1.12)^{6}$	1 973.82

[1] Kelly et al, Taking Stock In Your Future: Senior, Investor Education Fund, 2002, (page 38).

*Students who are interested in the programming option can create a short algorithm on the TI-83 calculator that will input the interest rate, i, and provide the value A(\$).

Once again the Rule of 72 has optimum values between 7 and 8% and there is also some variation over the range of percentages that we have chosen to examine (a variation of 1.3% at our extremes of 4 and 12 percent).

A more sophisticated treatment of this topic will fit very nicely with the recently revised curriculum for grade 12 Advanced Functions (MHF 4U) as it provides an excellent application of logarithms. This involves a comparison of two functions, each one related to a chart of data that we have investigated above. The functions define doubling time (n) in each of the two ways that we have discussed. The first can be seen to be in the family of rectangular hyperbolas where n is a function of i and n x i = 72. In other words, $n = \frac{72}{i}$. The second function will produce the exact values of doubling time, n, from a ratio of logarithms, also dependent on i. We are interested in a comparison of the shape of these functions, where exactly they might intersect, and how quickly they deviate from each other over the domain of input, i.

Students in the new MHF course will learn the following manipulations in their unit on logarithms.

If $a = b^y$ then $y = \log_b a$, but we can also observe that

$$\log a = \log b^y$$

$$\log a = y \log b$$

$$\text{so } y = \frac{\log a}{\log b}$$

and this is considerably more "calculator ready" than using $a = b^y$.

We can now develop the log function that relates to the doubling time of our investment at various interest rates. We want to examine when $2 = (1 + i)^n$ or $n = \log 2 \div \log (1 + i)$. Recall that i is the percentage rate as a decimal. With the syntax of the graphing calculator we will use the [Y=] feature to input the following:

$$Y1 = \frac{\log(2)}{\log\left(1 + \frac{x}{100}\right)}$$

as well as the function that we are comparing it to,

$$Y2 = \frac{72}{x}$$ (see footnote[2])

Using a domain ("window") of x values [0, 20] and a range [0, 40] the graphs are only barely distinguishable

[2] x represents the interest rate as the percentage value, for both functions: 4 for 4%, 5 for 5% etc.

(Fig. 3 & 4). On the other hand, we know that they are not identical because of our previous investigations (chart 1). Using the [ZOOM] menu (2: Zoom In) two or three times with the cursor (+) located near the value x = 8 (for 8%) on the curve(s), we can quickly begin to see the variation between the functions.

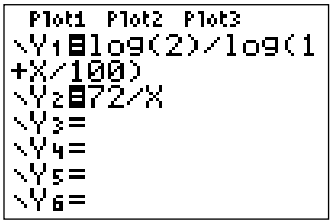


Figure 3

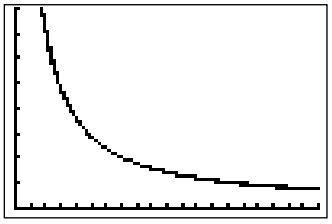


Figure 4

Another idea would be to graph the "difference function" which is very conveniently generated using Y3 = Y1 – Y2. This requires the use of [VARS] followed by the [Y-VARS] key, 1:Function, etc. Using domain [0, 20] and range [-1, 1] to get started, we can see where the difference between the Y1 and Y2 functions is minimal (values for Y3 near zero). The x-value of the point of intersection of the two original functions is the "zero" of the difference function, Y3 (Fig. 5 & 6). This value, x = 7.8468715, can be determined with the [2nd] CALC feature (using the [TRACE] key) and selecting "2: zero" from the subsequent menu. The point of interest on the parent functions (Y1 and Y2) is approximately (7.85, 9.12).

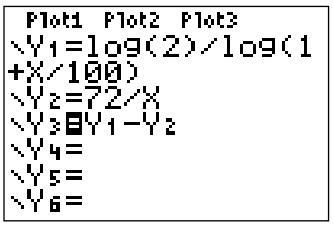


Figure 5

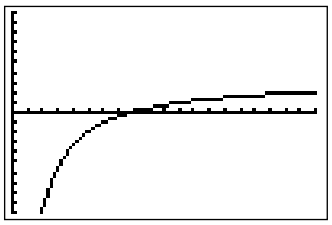


Figure 6

So we have now determined that the Rule of 72 works beautifully at 7.85% and surprisingly well in the neighbourhood, we can see yet again that it is a pretty good approximation over a reasonable domain of commonly used rates.

If we return to our grade 11 investigation (chart 2) where the investment was $1000, we can check the rate of 7.8469% p.a. with conversion periods, n = 72 ÷ 7.8469 = 9.1756. This represents an annual compounding of interest for just over 9 years.

$$A = 1000(1.078469)^{9.1756}$$

$$= 2\,000.0001$$

while, of course, 7.8469 x 9.1756 = 72.0000.

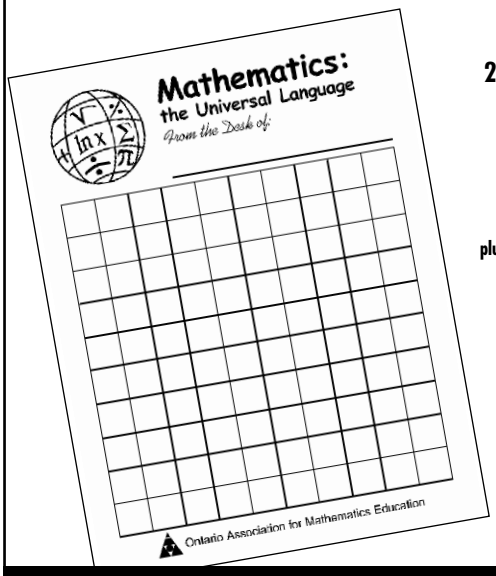
The mathematics of investment affords many opportunities for applying high school math skills of patterning, formula manipulation and graphical analysis. In this discussion we have examined one instance where a simple memory aid from the business sphere (the Rule of 72) provides an opportunity to apply the skills of mathematical modelling. The use of graphing technology provides easy access to the visual displays of the respective functions and makes some theoretical concepts very real.

The Taking Stock In Your Future guide is a very helpful starting point for investigations of this type. It provides a range of activities that allow discussion and analysis at various entry levels. In "Part 1: Building Personal Wealth as a Component of the Curriculum" teachers can find a careful review of concepts as well as an introduction to the language of investment for those who are less familiar.

These resources (along with workshops to interested groups) are available free of charge from the Investor Education Fund **www.investorED.ca**. ▲

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▲ Growing Professional Learning Communities In Superior Greenstone

WAYNE MELVILLE, ASSISTANT PROFESSOR
LAKEHEAD UNIVERSITY
COLLEEN KAPPEL, SUPERINTENDENT OF EDUCATION
SUPERIOR-GREENSTONE DISTRICT SCHOOL BOARD
HOLLY STORTINI, VICE PRINCIPAL
SUPERIOR-GREENSTONE DISTRICT SCHOOL BOARD



Wayne's research interests are in science as inquiry and teacher professional learning. In addition, he teaches biology and general science curriculum and instruction courses. Before migrating to Canada in 2005, Wayne taught secondary science and mathematics for seventeen years in Australia.



Colleen has a keen interest in Professional Learning Teams as a primary model for professional development and improving student achievement. Colleen has over eighteen years experience in education as a teacher, curriculum coordinator, Special Education Officer and principal.



Holly has been teaching for 27 years. Her areas of interest include mathematics, computers and primary education. Numeracy has been her board responsibility for the last 5 years and she has given many workshops in Math within SGDSB and local areas. More recently she has been involved with ministry training in numeracy.

1. Introduction

The Superior-Greenstone District School Board is located in Northwestern Ontario and contains ten elementary and five secondary schools serving ten fairly isolated rural communities and 2300 students. These communities are currently facing economic challenges due to the ongoing difficulties being encountered by the forestry industry. A serious consequence of these challenges has been a recent, steep decrease in the local school-age population. One school has lost 38% of its students in the past year, as families have moved in response to retrenchments.

Despite the challenges that the area is facing, the Board has been implementing, since August 2006, a program in which volunteer math teachers in grades 7 and 8 have been provided with opportunities to establish professional learning communities (PLC's). Through this article, we would like to share what we have learnt from this experience and the subsequent improvements in student math outcomes. In particular, we will focus on the administrative changes that have been implemented in order to initiate and develop PLC's.

In developing the notion of PLC's, the Superior-Greenstone District School Board has been mindful of two recent themes in teacher professional learning research. The first is that, while teacher professional learning is central to improvements in student outcomes, there is limited evidence to correlate teacher professional learning to improvements in student performance (Kubitskey, Fishman & Marx, 2004; Wilson & Berne, 1999). Second, research suggests that there are three important principles in teacher professional learning:

- Professional learning should involve 'communities of learners that are redefining teaching practice.'
- Professional learning 'ought not be bound and delivered but rather activated' (original emphasis).
- Teachers need to struggle with 'how to build trust and community while aiming for a professional discourse that includes and does not avoid critique' (Wilson & Berne, 1999, pp.194-195).

These findings have guided our thinking about the development of PLC's.

2. Background: Making Professional Learning More Accessible for Teachers

Within the Board's elementary schools, many teachers teach multi-age classrooms and are the only teacher of a specific grade(s) in the school. While not uncommon in rural areas, this situation can also limit the scope and depth of school-based professional conversations. Another issue affecting access to professional learning was the Board's large geographical area. In the past, teachers were required to travel great distances to attend Board-sponsored workshops and

training sessions. One strategy that the Board implemented in 2005 to address this challenge, was to group the schools into three 'families'. This allowed teachers and administrators within each 'family' to localize professional learning opportunities. By localizing these opportunities, the financial, personal and time costs are now reduced, thus increasing the accessibility of professional learning.

Implications

Ask the question: *Are professional learning opportunities accessible to teachers in this district? If not, think of ways to improve accessibility.*

3. Planning and Building Administrative Support

In 2006 a group of administrators attended a conference on PLC's. Believing that the PLC model provided opportunities for teacher professional learning and improving student outcomes, it was decided to develop the school 'family' structure into a series of PLC's. The first stage of implementing this decision revolved around building the administrative support needed to support the PLC's. This task was entrusted to the Board Superintendent, who made four key decisions. The first was to provide ongoing training for administrators in the concept of implementing and sustaining PLC's, which must be viewed as long-term undertakings. Second, release time of 0.5 days per month for teachers was to be provided, along with a recognition that teachers would also invest after school time in the program. Third, the Board's Numeracy Lead was appointed to implement the PLC's. The Numeracy Lead is a district level person responsible for teacher professional development in numeracy. The position is part of the Ontario Ministry of Education's emphasis on literacy and numeracy. In selecting math as the focus of the PLC's, the Board also acknowledged the fact that money that had already been spent on resources and training in the area of mathematics, but there was little evidence of student improvement. Finally, the reasoning behind the implementation of the PLC model was discussed throughout the Board area, in order to give stakeholders a rationale for the changes. This understanding is crucial in sustaining a PLC for the long-term.

Implications

If you are considering implementing a PLC, you must recognize the need for long-term commitment and support, both tacit and explicit, from both Boards and schools. PLC's are not a short-term proposition.

If you are an administrator, demonstrate confidence in the ability of both your teachers and those entrusted to implement and sustain the PLC.

4. Implementation

The Board understands that the greatest improvements in student outcomes come through the ongoing professional learning of its teachers. In establishing the PLC's, the Board also recognizes that professional learning is not a linear process, and that progress will vary for individual teachers, both over time, and between the schools. In September 2006, volunteer grade 7 and 8 teachers in two 'families' of schools agreed to participate in the program. The third 'family' of schools was not involved in the project. The Numeracy Lead coordinated the implementation of the program with the teachers. By appointing a specialist in mathematics to work with the PLC's, and providing teacher release time for the task, the Board provided both the supportive leadership and administrative conditions necessary for PLC's to develop. Having provided these foundational supports, the Board then needed to trust in the professionalism of its teachers, for at this point, professional accountability was crucial to the success of the program.

During the regularly scheduled weekly meetings, teachers progressively focused their conversations around teaching and learning in an area that was considered problematic: problem solving and mental mathematics. These conversations have progressively come to focus on the teachers' procedural and conceptual knowledge; student learning, and teaching. Selected excerpts from The Teaching Gap (Stigler & Hiebert, 1999) have been important in promoting these conversations.

Implications

Be careful in selecting the person to initiate the development of a PLC. They must be an educational leader in their subject who has the respect of both teachers and administrators.

To ensure the long-term sustainability of the PLC, the responsibility for leadership must eventually reside with the teachers. A crucial aspect of PLC's is the development of teacher leadership.

Having established the administrative framework for a PLC, allow teachers to develop strategies

and materials that deal with the specific challenges that exist in their schools.

5. The Community in Practice

The teachers have reported that the project has improved their teaching, rather than being the threat to their teaching that they had originally envisaged. These improvements, shaped in the monthly conversations, encompassed the content, pedagogy and assessment of problem solving and mental math. The fundamental shift in teacher thinking in these areas centred on a desire to teach for understanding of the material, rather than just 'teaching the text book'. The teaching strategies that were developed centred on developing students' conceptual knowledge of problem solving and mental math strategies, in a manner similar to that described for Japanese students in *The Teaching Gap*. When presented with mathematical problems, the classes discussed various strategies for producing a mathematically acceptable answer, or answers. Students were also given opportunities every week to investigate problems in order to produce an answer which could then be presented to the class for verification. As the year progressed, the complexity of the problems presented by the teacher, or which the students developed for themselves, moved from single step to multi-step problems. A specific comment that the teachers made was that they no longer accepted textbook questions and answers uncritically, and were more prepared to use resources that were more appropriate, or challenging, for their students.

In terms of formative assessment, it was noted that many students experienced difficulty in mental math computations, hindering their problem solving work. To address this specific issue, the teachers developed a diagnostic tool for mental math. Knowledge of multiplication tables was also noted as a barrier to student achievement, prompting two actions. The first was to provide assistance to those students who struggled in this area. The second was to communicate the concern to teachers in earlier grades, with the intention that teachers in those grades could assist students in their classes. In terms of summative assessment, common assessments were developed that allowed comparisons to be made between classes and teaching emphases.

Implications

Be respectful of teachers' concerns about the implementation of the PLC. Many teachers are wary of short term educational fads.

Accept that ambiguity is part of working through complex issues of teaching and learning. Don't impose a quick fix 'solution' to fit another agenda (including your own!).

6. Student Outcomes

The student-centred PRIME Mathematics, published by Thomson Nelson, is designed to facilitate an 'improvement in elementary school mathematics instruction and student performance' (see http://www.prime.nelson.com/PRIME/). Both PLC and non-PLC schools introduced the PRIME Mathematics materials at the same time. However, the test results in the Number and Operation strand would seem to indicate that the materials, of themselves, are not as important to improving student outcomes as the changes that the PLC teachers implemented in their classrooms. In analysing these data, it should be remembered that the sample size is relatively small, with less than 100 students in any one population. Even with that qualification, the data from November 2006 and May 2007 do appear to support the notion that functioning PLC's enhance student outcomes:

Table 1: Comparison of* PRIME Mathematics *mean scores and percentage change (November 2006-May 2007).

	Professional Learning Community		Non-Professional Learning Community	
	Mean Scores			
	Number	Operation	Number	Operation
November 2006	3.56	3.40	3.89	3.51
May 2007	4.11	3.63	3.99	3.49
% Change	15.45	6.76	2.57	-0.57

On both the Number and Operations phases, the PLC results were better than the non-PLC results, even though the non-PLC schools showed higher mean scores in November 2006. The improvement, it would appear, is related to the teaching strategies adopted in the PLC's. These strategies centred on developing students' conceptual knowledge of problem solving and mental math. In particular, it would seem that the emphasis on developing students' conceptual knowledge in the PLC's has allowed students to move more easily from the *PRIME Mathematics* 'beginner' phase to the abstract 'flexible' phase. This movement is not evident in the results for the non-PLC students mean scores. The plateauing of the non-PLC mean scores reinforces the view that traditional North American problem solving pedagogies do not facilitate the conceptual

understanding required for abstract thought (Stigler & Hiebert, 1999).

Implications

If mathematics teachers only develop a surface understanding of new materials, then there is a very real risk that they will not attend 'to the underlying mechanism of developing students' mathematical reasoning through problem solving' (Lewis, Perry, Murata, 2006, p. 5).

When reforming teaching, classroom teachers must seek to develop an in-depth understanding of the meanings associated with any changes to content or pedagogy.

7. Final Notes

PLC's are dynamic, and that dynamism must be respected. The apparent success of the Superior-Greenstone PLC's to date is partly due to the initial efforts that the Board put into developing an understanding of the concept among administrators. This understanding is important if they are to support the long-term work of teachers in the difficult task of improving teaching and learning. Understanding that the process takes time, resources and commitment is imperative, as it avoids the danger that administrators focus on the 'what' of change, rather than the 'how'. Equally important, teachers must be respected and supported as they seek to improve their practice. Given our experience, it is essential that other education authorities that are looking at establishing PLC's, consider the planning work that must be undertaken before attempting to implement any changes in their jurisdiction.

8. Acknowledgement

This ongoing program was initially funded with a MISA (Managing Information for Student Achievement) grant from the Ontario Education Ministry (see http://www.noelonline.ca/index.php?pid=12), through the Northern Ontario Education Leaders.

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OAME Chapter Conferences and Events

February 21 - 23 2008	Leadership Holiday Inn Toronto Airport	OAME
Feb 28, 29	Regional Conference "Focus on Math:	
March 1 2008	Success in Our Classrooms" Travelodge Airlane Hotel Thunder Bay ON	NWOAME
April/May	Spring Mini Conference	ISOMA
April/May	Spring Mini Conference	WOMA
Early May 2008	Spring mini conference Brock University	GOLDEN
May 30, 31 2008	OMO: Trent University	PRMA
Oct 23, 2008	Fall Conference U of Waterloo: 3:30 pm	GVMA

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▲ Fantasy Fractions: When a Grade 5 Class Creates Computer Mathematics Games

Chantal Buteau with Sarah Camilleri, Kamel Fodil Michèle-Elise Lacroix and Joyce Mgombelo

Chantal Buteau and Joyce Mgombelo are professors at Brock University in the departments respectively of Mathematics and Preservice (education). Sarah Camilleri is a Brock mathematics graduate working in the private industry. Kamel Fodil, former mathematics high-school teacher, is Principal of francophone elementary École Nouvel Horizon school. Michèle-Élise Lacroix works now at Bureau satellite de la Pédagogie culturelle in Welland..

It's Thursday morning, another beautiful sunny day at École Nouvel Horizon in Welland, and the Grade 5 class is debating computer games. But instead of arguing about Super Mario Brothers, the students are discussing the design of their own computer mathematics games about fractions. A small group in the right corner of the classroom is talking about its Blue Apple Tree game: should the fraction 1/3 be in the difficulty level 1, 2 or 3? Some already imagine themselves as Jean wearing his green smiley face t-shirt, and running to challenge the wizard on the bridge, wondering if they will succeed at creating an ascending melody with the water glasses. In a few weeks, the grade 5 students will play mathematics on the computer with their games, the ones they designed. Can you imagine it?

Fantasy Fractions/Fractions Fantastiques Learning Object[1] is the result of a collaborative class project between a Grade 5 class from francophone elementary school École Nouvel Horizon (Welland, Ontario) and Brock University. The principal of the elementary school sought partners for collaborations aiming to enrich the science and mathematics experience of his students. A strong wish to directly involve the students in the collaboration had been expressed. The project yielded a Honour's thesis[2] project for a mathematics student at Brock University that developed, in French and in English, a fun, interactive, and highly engaging computer Learning Object based on the Grade 5 classes' ideas. In what follows, we briefly report on this collaborative project that took place in 2006-2007.

Who Was Involved?

The project directly involved all École Nouvel Horizon grade 5 students, as well as the school principal, Kamel Fodil, and the grade 5 teacher, Michèle-Elise Lacroix. It also involved Brock University Mathematics Honour's student, Sarah Camilleri, who carried out her Honour's thesis project around this collaborative Learning Object, co-supervised by Chantal Buteau and Joyce Mgombelo, respectively professors in the Departments of Mathematics and Pre-service at Brock University. At the event celebrating the completion of the project, eight Brock pre-service students, the Chairs of both Departments, and the grade 6 teacher at École Nouvel Horizon also participated.



The designer team for Fantasy Fractions Learning Object at the Celebration Day at Brock University on 26 October 2007.

How Did Such a Collaborative Project Run?

A first meeting took place in June 2005 with the school principal and both Brock professors to discuss ideas and needs. The initial idea to create a project around a Learning Object first arose at this meeting. Sarah agreed to take part in the collaborative project as the main focus of her Honour's thesis project. The group decided that the Learning Object should be translated into French due to the francophone character of the

[1]*MICA Student Project Website, URL:* ***http://www.brocku.ca/mathematics/studentprojects/***

[2]*Camilleri, S. (2007b).* Fantasy Fractions Learning Object: A Collaborative Grade 5 Class Project. *Unpublished Honour's Thesis, Brock University.*

school.

Sarah's main responsibilities were to research the features of good Learning Objects; to give a brief presentation to the school teacher and principal in September 2006; to identify the mathematics topic together with the teacher; to organize the mathematics and graphics material from the grade 5 students; to modify the students material to achieve a programmable state; to implement the Learning Object; and finally, to present *Fantasy Fractions* to the grade 5 class in April 2007. Sarah also organized the computer graphing and French translation of the Object. Her Honour's thesis reports in detail about the project, and includes a theoretical discussion about Learning Objects and a brief analysis of *Fantasy Fractions*. Chantal's and Joyce's roles were mainly to supervise the progress of the project.

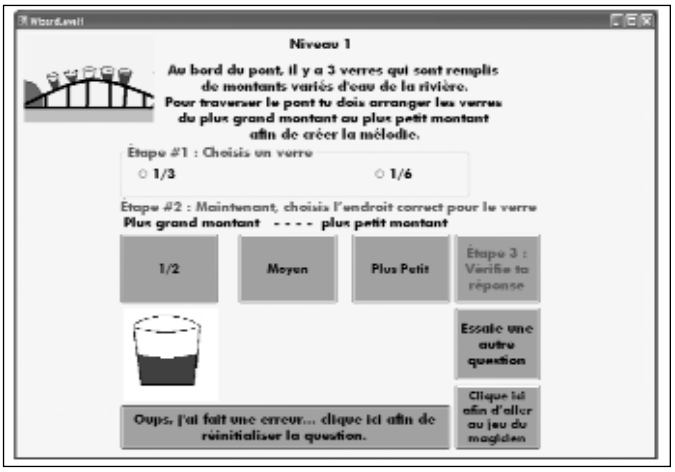
Michèle-Elise's primary responsibilities were to select the mathematics topic (fractions) with Sarah, to describe the project to her class, to coordinate the class work for developing the mathematics games, lessons, Learning Object theme, story line and characters, to collect the students' material, to translate the material into English and pass it on Sarah, and to keep an overview on the Learning Object design progress.

The grade 5 students played a central role in the project. Under the supervision of their teacher and school principal, they designed interactive lessons and games in groups of two or three, and jointly created the Learning Object story line and characters. They prepared descriptions and drawings of their games, lessons and story line. Details such as the blue apples and the smiley face on Jean's t-shirt were all integrated in the Learning object in order to render the students' design as faithfully as possible. *Fantasy Fractions* is really the result of their creativity!

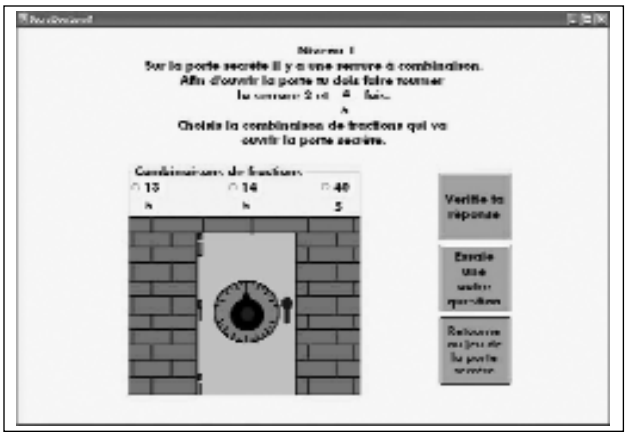
What is the *Fantasy Fractions/Fractions Fantastiques* Learning Object?

Fantasy Fractions is a Learning Object that guides the user through three lessons on fractions with corresponding games: equivalent fractions, mixed and improper fractions, and comparing fractions. It is intended to be a fun way for users to learn fractions for the first time or brush up on old skills. A storyline is incorporated into the Learning Object for users to follow as they learn. In the narrative, the user is a student who falls asleep while studying for a math test and wakes to find a treasure map beside him/her. The user must then follow the treasure map to get to the treasure by completing the lessons.

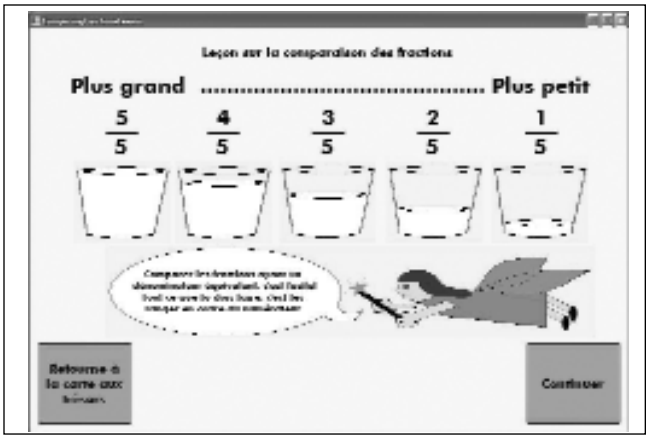
This Learning Object was designed to be attractive and challenging to any grade 5-6 school user. It was created to include not only the grade 5 class ideas but also the features of a good Learning Object. The games and the lessons are all highly interactive. Users have the chance to go through all of the three different lessons while following examples and trying some questions on their own. The games are also very interactive and can be fun for users to play with. Each game has three levels of difficulty to ensure that the Learning Object is challenging and reusable. Users also have full navigation control in the Learning Object, as they can always access the treasure map and decide where they would like to go next. Multiple animations, graphics, and bright, vibrant colors were added to make the Learning Object more attractive to the user. Two different characters, Anna and Jean, were implemented to increase identification and interest. Users choose which character they want to be at the beginning of the game and that same character is seen throughout the whole program to make the user feel involved in the storyline.



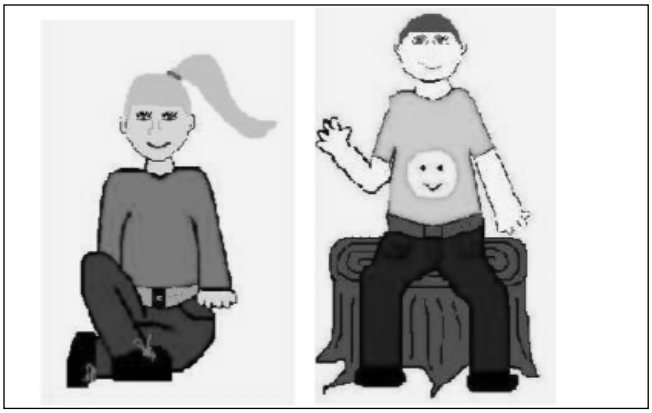
Wizard Game (level 1): To cross the bridge, the user needs to create the ascending melody with the water glasses. First Select a fraction (Step 1), then click on the right orange box to place the water glass (Step 2), and check your answer by clicking on the top-right orange box (Step 3). The bottom box reinitializes the water glasses, if desired. The middle right box randomly generates 3 new fractions to order. Level 2 has 3 fractions but does not show the glasses as one orders the fractions. Level 5 has 5 fractions to order without showing the glasses, but gives the user hints in case of mistakes. The bottom right box brings the user back to the treasure map.



Secret Door Game (level 1): To open the secret door in order to access the key, the user has to input the right combination, e.g. 2 4/5 turns is 14/5. The top right box checks the answer; if it is right, the lock turns and the door opens. If it is wrong, it encourages the user to try again. The level 2 game deals with more difficult fractions. Level 3 has 5 doors to open simultaneously.



The fairy shows the user how to compare fractions with same denominator.



Anna and Jean, the two main characters in Fantasy Fractions Learning Object which were created by the grade 5 class of École Nouvel Horizon.

What do the Grade 5 Students and Teacher Say?

The students enjoyed the creative process and were extremely proud of their accomplishments. They were delighted to hear that they would contribute to the creation of an enjoyable educational game for other students, a game they could call their own. The students were highly motivated and followed the deadlines carefully. Brainstorming took place every morning in order to gather ideas and to find amusing ways of incorporating mathematical concepts into fun games. Each student had a particular role in the visual designs of the game as well. The teacher, Michèle-Élise, saw the project as an innovative way of teaching the grade 5 mathematical concepts. The students had to master the concepts in order to create a comprehensive game for others. She felt the collaboration with Sarah and Kamel (school principal) was very valuable. The project promoted cooperation, creativity, active participation and leadership among the students.

What does the School Principal Say?

For the school principal, Kamil, it was an opportunity to show the students that math can be fun. From day one, our grade 5 students were extremely motivated and engaged in developing this tool that will be used by students from other schools. The project provided the opportunity to work in collaboration with Michèle-Élise (teacher) and Sarah (university student). He felt that it was wonderful to see the students working in groups, sharing ideas and most importantly respecting the project deadline.

What does the University Student Say?

Sarah felt that the project was a unique experience that will stay with her forever. It was amazing for her to see what unique ideas the grade 5 class could come up with and give her to work with. The students of the grade 5 class generated many innovative ideas for characters, games, and lessons. The students came up with most of the basic concepts of the Learning Object which made some things easier and some more difficult. Their ideas were a huge help when trying to design the graphics, storyline and treasure map. Although Sarah used the students' ideas for the games, they sometimes had to be reworked so that the games could be programmable. This Learning Object also took a lot more time to develop since there were so many people involved with so many different ideas. This made the process more difficult and more time consuming, but worth it in the end.

The best part of this project was presenting it to the grade 5 class and being able to witness their reactions. The children were amazed to see their ideas and drawings all incorporated into the Learning Object. It was an extraordinary experience to collaborate with the class, teacher and principal. Sarah says that the project taught her about team work, co-operation, and time management and she would recommend a project similar to this to anyone who loves programming.

What does the Education Professor Say?

Joyce said that, for her, the project took place in the context of Sierpinska's[3] plenary presentation at the Canadian Mathematics Education Study Group (CMESG) conference, which urged mathematics educators to focus their research on what is common to all mathematics stakeholders while arguing that any good mathematics education research should be structured around mathematics activities or tasks. Indeed, this project was a collaboration with mathematician, principal, teacher, honours student, and more importantly grade 5 students, and around the design and implementation of *Fantasy Fractions*.

The grade 5 students' ideas for the learning object provides insight on how students learn mathematics through games, story/narrative, fiction (blue apples!), etc. and it indicates a need for more research in learning objects and mathematics learning.

What does the Mathematics Professor Say and what is Mathematics Integrated with Computers and Applications (MICA) Brock University Program?

Chantal notes that the collaboration with elementary school Ecole Nouvel Horizon built on a long tradition in the Brock University Department of Mathematics, established by Eric Muller (Professor Emeritus), of involvement in the school mathematics community. The *Fantasy Fractions* project engaged a Brock Honour's student to expand on her MICA project experience, a first of this kind.

Mathematics Integrated with Computers and Applications (MICA)[4,5,6] is the core undergraduate mathematics program at Brock University, with 4 streams: pure mathematics, applied and computational mathematics, mathematics education, and statistics. Two of the guiding principles are :

(a) to develop mathematical concepts hand in hand with computers and applications and (b) to strengthen the concurrent education program. In addition to revised traditional courses (e.g. Calculus and Algebra courses), new innovative computer project-oriented courses MICA I-II-III were introduced in which students complete individual original final projects. In particular future teachers can design and implement a Learning Object (see the web site[1] for viewing examples of first -third year student projects). Based on our observations, MICA students get a rich experience of learning mathematics and mathematics didactics through the design and implementation of Learning Objects[7]. What? The Grade 5 Students at University?!?

The fall breeze is blowing and the trees are changing colors all around, as the students from École Nouvel Horizon hop off the bright yellow school bus. Where are they heading? Into Brock University! The now grade 6 students are taking a tour of the University where they make a stop at a large lecture room.They imagine themselves all grown up and attending a University class. But that is not why they are here. Today, they will receive a certificate attesting to their roles as the designers of the *Fantasy Fractions* Learning Object. And they do not know it yet, but Anna, Jean, the wizard, fairy, gardener and two apple trees walked out the virtual world of *Fantasy Fractions* and are in the other room preparing mathematics games for them. When the students head back home, walking proudly with a Brock University bag on their shoulder, they may wonder: were

[3]*Sierpinska, A. (forthcoming). Research in Mathematics Education: Through a Keyhole. In Elaine Simmt and Brent Davis (Eds.).* Proceedings of the Annual Meeting of Canadian Mathematics Education Study Group, 2003*, Acadia University.*

[4]*Buteau, C. (2007). Technology in University Mathematics Instruction? At Brock, Yes! In* Ontario Mathematics Gazette, Vol 45#2

[5]*Ben-El-Mechaiekh, H., Buteau, C., and Ralph, B. (2007).* MICA: A Novel Direction in Undergraduate Mathematics Teaching. In Canadian Mathematics Society Notes 39*(6).*

[6]*Brock Teaching (2001), URL:* ***http://www.brocku.ca/ctl/ pdf/Brock_Teaching_1.pdf***

[7]*Buteau, C., & Muller, E. (2006). Evolving technologies integrated into undergraduate mathematics education. In L. H. Son, N. Sinclair, J. B. Lagrange, & C. Hoyles (Eds.), proceedings of the* ICMI Study 17 Conference: Digital technologies in mathematics education—Rethinking the terrain, *Hanoi University of Technology, Hanoi, Vietnam.*

those fraction characters creations of our minds or was that all real?

Appendix - Welcome Song to Grade 5 Class at their Brock Celebration Day

The song is to the tune *Hakuna Matata* from *The Lion King* and was created by Brock pre-service students (2007-08) Jill Couture and Julie Wilms.

(Brock University
What a wonderful place!
Brock University
Welcome to our space!
It means lots of learning
For all of our school days
It's the place to be, just wait and see!
Brock University)

Why, when Jean and Anna came out to play
When Jean and Anna came out to play.
Very nice.
Thanks!
They found a math adventure dealing with fractions
With mysterious blue apples, such an attraction!
So Jean and Anna decided to take the quest
To earn keys to open the treasure chest

Now we have more of an adventure for you
Now let us introduce you to the crew,
[Introduce ourselves using character names]

(repeat refrain) ▲

Cont'd from page 2 **- March 14 – A Day to Celebrate Math and Science.**

The date is Einstein's birthday and it represents the common three-digit approximation for pi-3.14.

Pi Day is a great day to have some fun with mathematics and science. Why not plan a pi recitation contest followed by a pi eating contest? Make your pies square, instead of round, since "pie are squared" is the surface area bounded by a circle. For more Pi Day ideas, visit the "Math with Mr. Herte" web site, at **http://www.mathwithmrherte.com/pi_day.htm**, or the Pi Day Resources page at **http://www.mobot.org/education/megsl/pi.html** *. ▲

▲ The Role of Language in Problem Solving

PAUL GIGANTI, JR.

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Many students and adults who love mathematics appreciate the subject for its precise and consistent nature. It can be a tremendous comfort knowing that 2 + 2 equals 4, has for thousands of years, and will be true for all eternity. Comfort comes in knowing that once you have mastered addition, you can always say, "I can do this—because I have done it before."

However, most adults and students often feel quite differently about mathematics when it comes to problem solving. While there are many different kinds of problems, many problems in mathematics often involve language as well as numbers. This can confound many people since good problems are seldom as simple as computation. For instance, a problem might read, "When is 2 + 2 NOT equal to 4?"

Why does mathematics become so different when we wrap words and situations, or context, around numbers? Actually, the words are the difference; they bring language into mathematics, and along with language comes complexity. Since words and sentences can have different meanings, the mathematics that used to be a simple, straightforward calculation is now subject to interpretation. How we interpret the words wrapped around the numbers in problem solving allows us to arrive at different conclusions about the same problem.

Take this word problem, for example:

If you have two-dozen chickens, and purchase another two-dozen chickens, how many chickens do you have?

Depending on how you interpret the problem, the answer could be four-dozen chickens, 48 chickens, twice as many, or 24 pairs of chickens. All of these answers are correct, given different interpretations of the problem.

Why should we have students go to the trouble of learning problem solving in mathematics when it is so much simpler to just ask them to calculate 2 + 2 = ? Problem solving and the language of good problems perform a very important function in mathematics; they allow us to bring a touch of reality into the mathematics classroom and help students prepare for the

mathematics they will do in real life.

Real life rarely hands you a simple calculation but puts mathematics in the context of some real situation. Outside the classroom we are continually faced with mathematical situations (problems) that are ambiguous, fuzzy, lacking information (or having too much information), and subject to interpretation. Teachers must help their students to become good problem solvers. Here are some possible ways.

- Find interesting and challenging problems for students at their level.
- Help students learn to clarify problems before they begin solving them.
- Suggest questions students can ask when they get stuck on a problem.
- Teach students different problem-solving strategies to use in their work.
- Create a classroom environment that allows for mistakes as well as successes.
- Ask students to reflect on their answer when they think they have solved a problem.
- Encourage students to find and discuss different solutions and strategies.
- Give students good problems to solve frequently, daily if possible.

Here are two sample problems:

A farmer has a corral in the shape of a square that is 16 metres on each side. What is the area of the corral? What is the perimetre of the corral?

A farmer wants to build the largest rectangular-shaped corral possible for his cows. He only has 48 metres of fencing, but his property borders a river. How big can he build his corral if he uses the river as one side of the corral?

We can use these two related but very different problems to emphasize positive ways to teach problem solving to students of all ages.

Find interesting and challenging problems for students at their level.

There are many problems that teachers can use with students, but you have to look a bit harder to find good problems. Finding the best problems for students to solve makes a great difference. First and foremost, a good problem should catch the interest of students; they are much more likely to persevere in solving a problem if it is interesting to them. Motivation can be a big factor in problem solving. Second, good problems should squeeze as much mathematics out of the situation as possible. Since all problems are more than simple calculations, a good problem makes students work harder mathematically and gives them opportunities to demonstrate more mathematical knowledge and abilities.

Which of the two sample problems do you think students would find more interesting? While it is true that the first problem is an easier problem, the second problem is more challenging and provides an interesting context. Which problem will tell the teacher (and the student) more about their understanding of mathematics concepts and skills? The second problem is a more complex situation that uses the same mathematics concepts as the first, but requires that students demonstrate a deeper understanding of the same concepts. The best problems not only have interesting contexts, they also involve more than one mathematical concept and sometimes ambiguity.

Help students learn to clarify word problems.

There are three ways a teacher can assist students in clarifying a problem before they start to solve it. The first step should always be to read a word problem together. This first reading is important; it is amazing how many students begin a problem before reading it all the way through. For younger children this usually means reading the problem aloud. For older students tell them to read the problem to themselves and then give them some quiet time to do so. For younger students, reading the problem aloud removes reading skills as a possible impediment. For older students, it is simply good for students to learn that the first step is to carefully read the problem, usually more than once. However, this is not enough to ensure that the problem is clear to students.

The second step is to force students to restate a problem in their own words, either vocally or in writing. Once they have read the problem, they should be able to do this without looking at the problem. If their restatement of the problem is incorrect or missing key information, ask them to go back and read the problem again, and then restate the problem in their own words again. Only after they are able to restate the problem correctly should they begin to solve it. This use of spoken or written language creates a solid connection with the word problem and the student's brain, an important beginning.

The third step is to teach students to ask clarifying questions. Since problem solving in a classroom is supposed to be practice for problem solving in real life, and problems in real life are seldom presented in crystal-

clear terms, students need to learn to ask questions that clarify the task ahead. Teachers should pose good problems that require students to ask clarifying questions, then help students practice asking good questions before they set pencil to paper.

In the first sample problem, the possible questions are: What is area? What is perimeter? What is a metre? These questions demonstrate the students' lack of knowledge of the mathematical terms at the core of both problems. However, if the students already know the answers to these first questions, then for the second problem they might ask: Is the corral a rectangle? Are we looking for the largest area or perimetre for the corral? Do we need to use decimal or fractional lengths for the sides of the corral?

Teaching students to ask clarifying questions is not as simple as asking them if they have any questions. Since students aren't used to asking good questions, you have to push them a bit. A simple classroom rule that seems to work well in prying good questions out of students is to make students ask at least three clarifying questions about a problem before any student is allowed to begin working on that problem.

Suggest questions students can ask when they get stuck on a problem.

There are generic questions that students should ask themselves that work wonderfully to break up a problem-solving logjam. Teachers should train students to automatically ask themselves these questions before, during, and after solving a problem: What am I being asked to find? What information do I have? What information, if any, is missing or not needed? What strategy might be best to start with in this problem?

In the heat of manipulating numbers, concepts, and/or shapes while working on a problem, we sometimes lose sight of the objective—what we were looking for in the first place. These generic questions work with any problem to focus a student on the problem-solving task at hand.

Teach students different problem-solving strategies.

A wise person once said, "You know you have a problem when you do not know what to do." We can turn that into a definition: Problem solving is knowing what to do when you do not know what to do. Problem-solving strategies are simply the things you try when you do not instantly know what to do. When a student confronts a good problem, there is always a moment, no matter how brief, when he or she does not know exactly what to do. But what the student does next is important and that is where problem-solving strategies are valuable.

Strategies for solving problems are like tools in a tool box: the more tools you have in your personal tool box and the better you know how to use those tools, the better you are prepared to do something. Teaching students different problem-solving strategies, how to use them, and for which types of problems they work best, helps students build their own problem-solving tool kit that they will carry with them into adult life. The box below gives a short list of great problem-solving strategies.

- Think of a similar problem you have solved before.
- Do a simpler problem that is an easy version of the problem.
- If a problem has several answers, look for a pattern within the answers.
- Make a table or an organized list of possible answers, and then eliminate wrong answers systematically.
- Make a diagram or picture of the situation in the problem.
- Write an equation or formula.
- Use guess and check.
- Work backwards.
- Act it out or model the problem.
- Sleep on it!
- Do research on the problem.

For instance, in the second sample problem, students might begin by using the strategy, draw a diagram. Once they have a sketch of the corral and the river, they might use another strategy, guess and check, by trying a few combinations of length and width to see which gives them a larger corral. Once they are on the right track, they might use the strategy, make a systematic list, so they can be sure they did not skip over the largest possible answer while working.

Unfortunately, students are not born knowing how to use these strategies; they must be taught. One of the best ways to teach an individual problem-solving strategy is to choose a problem that lends itself to that strategy, then demonstrate to all the students how the strategy can be used to solve that problem. The teacher can do this before a problem-solving session or after

students have solved (or tried to solve) a problem.

Some strategies are harder to demonstrate than others, and not all strategies work well with all problems. Demonstrate the simpler strategies first and always look for problems that clearly can be solved using a particular strategy. You will also need to teach your students that a good problem often requires more than one problem-solving strategy.

Create a classroom environment that allows for mistakes as well as successes.

When you teach problem solving, your students will make many mistakes since problem solving can be very frustrating. Creating a safe learning environment where students can be wrong on their way to being right is critical. Students need to understand that problem solving is hard work and often frustrating. Unlike simple calculations that, once mastered, are simply repeated, problems present students with new and unfamiliar situations, situations where they do not automatically know what to do since they may not have dealt with that situation before. In these situations, people often make mistakes before they succeed.

To be able to learn and benefit by one's mistakes, a student must first learn that everyone makes mistakes, but that it is important to keep trying. The best problem-solving strategy for teaching this life-lesson is guess and check. In fact, we used to call this method trial and error! This is by far the best strategy to use when you do not have a clue how to begin. Anyone can guess an answer, plug it in, and see if it is correct. But guess and check teaches something very powerful: you can learn from your mistakes. A mistake can tell you important information that leads to a better second guess, which can lead to an even better third guess, and eventually to the correct answer. It is certainly not the most elegant problem-solving strategy, but it does teach that it is OK to make mistakes as long as you benefit from them. For instance, in the Corral sample problem, one guess alone would be a waste of time unless you were extremely lucky. How will you know when you have found the largest corral unless you make a second guess and a third guess? While it will always be critical for students to get the right answer, a few mistakes along the way should not hurt, and might really help.

Ask students to reflect on their answer when they think they have solved the problem.

It is tempting for students to think that, once they have an answer, they are finished—but this is not always the case. How often have you seen students write down an answer that does not make sense at all and quickly move on to the next problem? Language can come to the rescue here as well. A few good questions asked in reflection after a problem is solved can sometimes prevent a ridiculously wrong answer. Your students might try asking themselves these questions: Does my answer seem reasonable? Did I find what I was asked to find? Did I copy the numbers from the problem correctly? Do my calculations look correct? Do I feel confident that my answer is correct and complete?

Reflection is often skipped over by students—yet is every bit as important as any other part of the problem-solving process. It also gives students time to pause and feel good about their work before going on to the next problem because there will always be another problem to solve.

Encourage students to find and discuss different solutions and strategies.

Even if students go through all the steps above in solving a problem, two things will always happen—some students will find very different ways to solve the same problem, and some students will get the answer wrong. If you encourage your students to discuss their findings and strategies after finishing a problem, even more learning can take place. This can be beneficial in either a whole group setting or in smaller groups. Students who did the problem incorrectly will have an opportunity not just to hear the right answer but also to see how other students solved it and may find out where they made their mistakes. Until students have the opportunity to hear and see how others have solved the same problem, many students cannot imagine that there is another way to solve it. This open sharing of solutions and strategies can greatly assist students in learning to use those strategies and new techniques in solving future problems.

Once again, this takes time. Allow for an extra five to ten minutes of class time after students complete a given problem for this very important sharing session. This, in effect, is teaching time—it is the students who are doing the teaching.

Give students good problems to solve frequently.

The importance of practice in learning mathematics has been known for ages. However, its importance extends beyond learning basic mathematics facts or computation; practice is also true in learning problem solving. There is no substitute for experience, and the

only way to get experience in problem solving is to solve many problems. Learning how to use different problem-solving strategies, how to ask questions that clarify a problem, learning from mistakes, and even knowing when you have a correct answer are all skills learned by grappling with one problem after another. Mathematics programs in other countries, such as Japan, focus each lesson on a good problem as a beginning. The problem provides an opportunity to teach a new concept in context.

Unfortunately, this suggestion and all of the other suggestions for teaching problem solving require a serious time commitment in a classroom, a minimum of 10 minutes a day. Many teachers have adopted a strategy called "Problem of the Day" where the students' mathematics period begins with a problem written on the board. Once the problem is completed, including as many of the process and language dynamics mentioned previously, teacher and students move on to the new mathematics topic of that day. While this takes time away from teaching new concepts—unless, of course, you embed the new concepts into the problem solving—over time it generates capable, fearless problem solvers.

Listed below are some good problems you can start with, but you will soon be looking for more. Luckily, there are hundreds of great books packed with good problems for your students. A good rule of thumb is: students should solve one good problem every day. This allows students at all levels to hone their skills by gradually working harder problems and applying new mathematics skills and concepts as they grapple with and solve new problems.

A Few Good Problems for Upper Grades

1. Thirty-two people enter a ping-pong tournament. When a person loses a game, he/she is eliminated from the contest. How many games must be played to figure out who is the best ping-pong player?
2. A hollow box is made entirely of one-inch cubes glued together to form all of its sides, its top, and bottom. If the box measures 6" x 5" x 4", how many one-inch cubes are needed to build it?
3. At a fire, a fireman stood on the middle step of a ladder shooting water on the fire. As the smoke cleared, he went up three steps. A sudden burst of flames forced him to go down five steps. A few minutes later he climbed up seven steps and worked there until the fire was out. Then he climbed the seven remaining steps and entered the building. How many steps does the ladder have?
4. At a party of 12 friends, the activities begin with each person shaking every other person's hand exactly once. How many handshakes take place?
5. Osgood Smart glued together 125 small cubes to make one big solid cube. Then he painted all six sides of the big cube bright red. Later when he broke it down into small cubes again, he found that some cubes had three sides painted, some cubes had two sides painted, some cubes had one side painted, and some cubes had no paint on them at all. How many of each kind of cube did he have?
6. How many completely different shapes (including sizes) can you cut out of a 3" x 3" grid of squares if you are only allowed to cut on the lines?
7. A 200-pound man and his two daughters, who each weigh 100 pounds, are standing on the bank of a river teeming with Piranha. They want to get to the other side, but their canoe can hold no more than 200 pounds. How can they get safely across?
8. A giant hero sandwich that is 30 feet long has been created. It is to be divided into four parts so that each section is one foot longer than the previous section. What will be the lengths of the four pieces?
9. If you have chocolate, strawberry, and vanilla ice cream, how many different double-dip cones can you make? Remember, some people like to eat their strawberry before they eat their chocolate ice cream, and some like it just the other way around.
10. A glazier is fitting colored glass into a window that has three sections across and three sections up and down. In each section she can fit a red pane or a yellow pane. How many different windows are possible using these colors? ▲

▲ Call for Manuscripts

The *Ontario Mathematics Gazette* is inviting manuscripts for all grade levels.

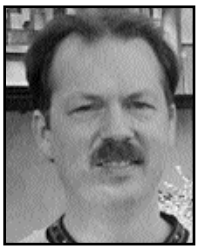
Instructions for submission of manuscripts are found on page 1 of any *OMG*.

Contact the Editor

Marilyn Hurrell
for further details.

▲ Tangrams Are Not Toys

CORNELIS DE GROOT



Dr. Cornelis (Kees) de Groot is an assistant professor of elementary education at SUNY New Paltz in New York State. His interests are related to learning transitions in mathematics and performance assessment. Dr. de Groot is director of the NYS K-8 Math Mentor Network and is the editor for Secondary Math Exemplars.

This article first appeared in the OMG*, Vol 40 #3 March 2002.*

It is possible to assess much of peoples' thinking while observing their struggle to put a puzzle together. Ask anyone to put the seven Tangram pieces back together to form a square, and you will find her or him using trial and error strategies even that person admits to having done this before; adults and children alike. Although people have solved the Tangram puzzle, it is perplexing that each time they encounter the puzzle, it appears just as problematic as before. Each semester, I give this assignment to my pre-service elementary teachers and this ritual consistently repeats itself. Why does this phenomenon happen?

In this article, I describe paper-folding techniques that assist learners in discovering important underlying structures in the Tangram puzzle. The main ideas that I put forward are well suited for upper elementary and middle school teachers and maybe of interest to others as well. I will make the point that knowing these underlying structures enable learners to reconstruct the square from the seven discrete pieces, using mathematical reasoning and strategies. It is more effective to comprehend a structure than it is to remember the actual puzzle.

Manipulatives are occasionally characterized as toys, and sometimes starting as early as second or third grade, students are told that manipulatives, such as cubes or counting with fingers, are for little children. This negative characterization of manipulatives implicates that toys are not appropriate learning tools and that play or manipulation wastes time in the learning of mathematics. I think that this stance is problematic if we believe that the development of learning mathematics is conceptually facilitated by the development of insight into mathematical structures and mathematical relationships. It has been well documented that concrete experiences are needed for many children to accomplish the transition to more abstract mathematical knowledge (for example, Campbell & Johnson, 1995; Cobb, Wood, Yackel, & McNeal, 1993; and Thompson & Lambdin, 1994).

Relationships Among the Pieces

To begin the discovery of the underlying structures of the Tangram, I recommend that learners first develop a good sense of how the seven pieces (two small triangles, one medium triangle, two large triangles, a square, and a parallelogram) are related to each other. There are many activities in books or on the Internet that can assist with helping students form such relationships. You can find a selection of resources at the end of this article. It is particularly important that students realize that all the pieces can be constructed from the small triangle; that is, the medium triangle, the square, and the parallelogram are all composed of two small triangles and the large triangle is made from four small triangles. In Figure 1, you can see that the seven pieces together are equivalent to 16 small triangles. This allows the learner to discern the part of the Tangram square that is represented by each piece. For example, the large triangles are each four sixteenths or one fourth of the Tangram square. The medium triangle, the small square, and the parallelogram are each two sixteenths or one eighth of the Tangram square, which leads to the making of a Tangram using these relationships and spatial and logical reasoning. This construction can be demonstrated using a square piece of paper.

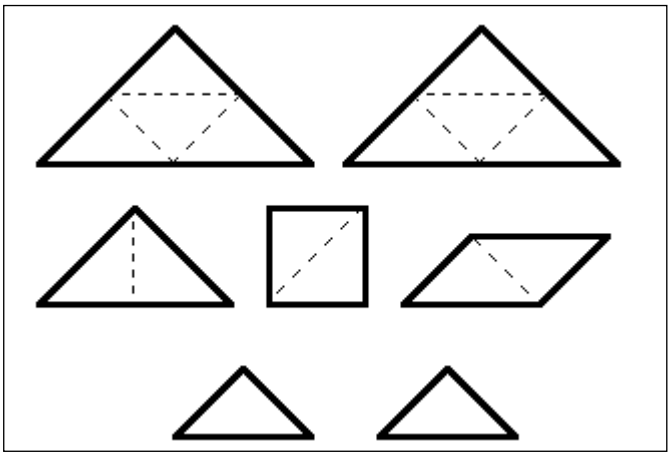


Figure 1. Seven Tangram Pieces are equivalent to 16 small triangles.

Unfolding the Tangram Structure

In this folding method, we will fold along or perpendicular to the diagonals of the square. Others have used a rectangular grid to reveal the structure of Tangrams (see, for example, the University of Exeter Centre for Innovation in Mathematics Teaching web site), but this does not lend itself well to working with the triangular shapes and the parallelogram. As a teacher, you can choose to pre-cut the Tangram squares, but I suggest that you use the diagonal folding methods, so often used in folding paper airplanes. This method might assist the students in viewing triangular pieces that fold on top of each other as congruent (see Figure 2).

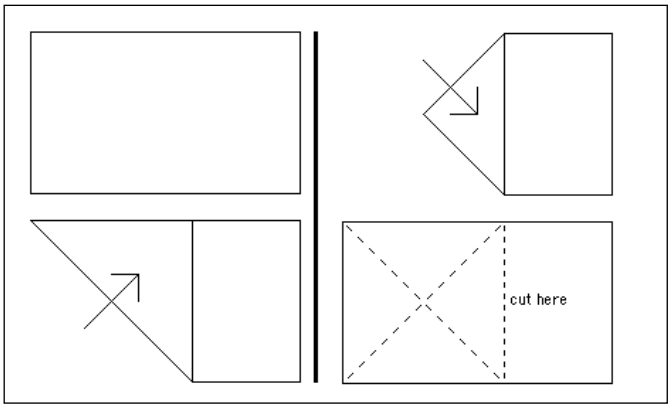


Figure 2. Making a square by diagonal folding.

Upon completion of the construction of the Tangram square, you will be able to identify four triangular regions, each representing one fourth of the entire square (see Figure 3a). The two large triangles must be positioned in two of these regions. This is a good moment to investigate how many ways the two large triangles can be positioned in the Tangram square. It appears that there are only two possibilities: adjacent to or opposite of each other (see Figure 3b). Determining the number of ways you can place given shapes is a very important step, since it will lead to an efficient strategy in re-construction of the puzzle later on.

Next, fold each vertex of the square to its centre and unfold it (see Figure 3c). In this way, a smaller square has been formed, connecting all the midpoints of the original square. It is important to stop here for a moment and examine the size of this inner square. Students may see that the inner square is half the size of the Tangram square. They might also conclude that the two large triangles fit inside this square (see Figure 3d). While this is true, it must be discussed that this strategy is not fruitful, since it will be impossible to place the square and the parallelogram in this construction. What can be discovered next? The medium triangle piece can be put in the space on each of the four corners and subsequently be flipped over to the inner square. This construction will show that the Tangram square is made up of eight medium triangles and that the inner square is made of four medium triangles. This in itself would

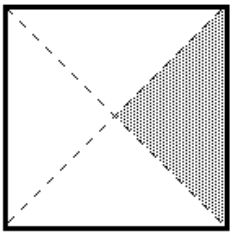


Figure 3a. One fourth of the figure is shaded.

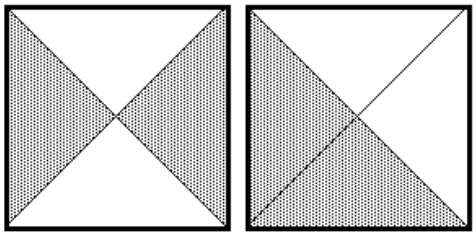


Figure 3b. Ways to place two large triangles.

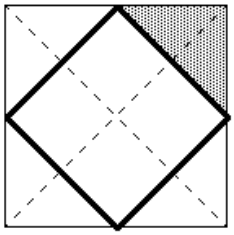


Figure 3c. One eighth of the figure is shaded.

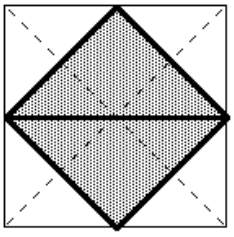


Figure 3d. Placing large triangles here prevents the placement of the square and parallelogram.

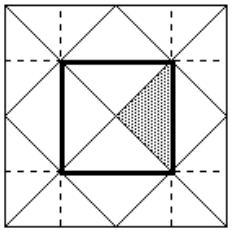


Figure 3e. One sixteenth of the figure is shaded.

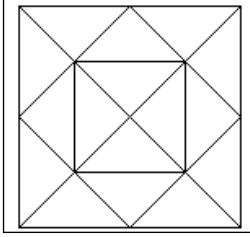


Figure 3f. The fundamental structure for the Tangram.

constitute a visual proof of the previously stated relationship.

Continue folding the vertices back to the centre and again fold the vertices to the centre of the smaller square. I found it was more convenient to fold when I first turned the paper over for this purpose. (You may recognize this construction from your childhood as a fortune telling game. Children often call this a “cootie catcher.” It can be folded open and closed with one’s thumb and index finger from both hands inserted.) Again, a smaller square is formed. This time, do not immediately unfold it. What is the relationship between this small square, the previous medium square, and the original large square? Unfold the paper completely. Several structures are visible (see Figure 3e). First, the three squares that were previously discussed are recognizable. Second, do you see the 16 small triangles? To make this more apparent, have students trace the fold lines that are solid in the figure with a marker, and Figure 3f will emerge from this.

We now have the fundamental structure underlying the Tangram. We have already determined where the two large triangles can go and where the medium triangles could fit. First, let us investigate where the square and the parallelogram might fit. The small triangle is the least of our concern, because it can fit in 16 places. You can see that there are only four spots for the square (see Figure 4a). Note that due to the symmetry it does not matter which of these four spots is chosen. You can see that the parallelogram must go along the sides of the Tangram square (see Figure 4b). There are two possibilities on each side, making a total of eight placements for the parallelogram.

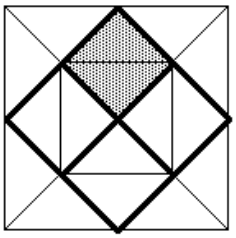


Figure 4a. Placing the square.

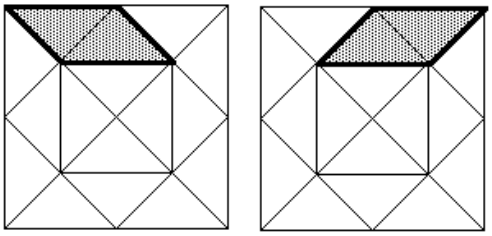


Figure 4b. Placing the parallel-ogram.

Putting It Together

We are now ready to reason backwards to a solution of this puzzle. We will start with the pieces that have the least number of possible placements and work our way up to those with the most: the two large triangles, then the square, then the parallelogram or medium triangle, and finally the two small triangles.

If we start with the two large triangles in opposing positions, we have two places left for the square (see Figure 5a). Placing the square in one possibility forces the placement of the parallelogram in the space opposite the square (see Figure 5b). There are only two possibilities for this. However, either choice presents a problem; it will be impossible to place the medium triangle anywhere (see Figure 5c). We have now excluded the possibility of the two large triangles in opposition. Therefore, they must be adjacent!

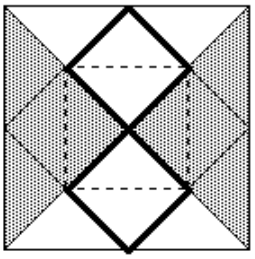


Figure 5a. The two big triangles are placed opposite. Two choices for the square.

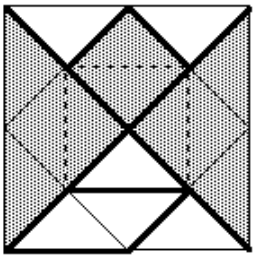


Figure 5b. The parallelogram must go in the space opposite the small square.

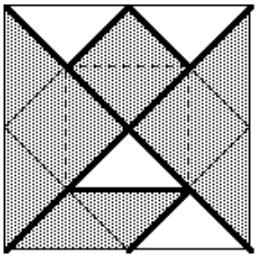


Figure 5c. Placing the medium triangle is impossible.

Start again by placing the two large triangles adjacent to each other (see Figure 6a). There are two possibilities for placing the square. Because of the symmetry, it doesn’t matter where it is placed (see Figure 6b). In turn, this leaves only two possibilities for the parallelogram on the remaining side. One possible placement of the parallelogram will make it impossible to

place the medium triangle, but the other allows the placement of all the remaining pieces (see Figures 6c and 6d). Now the puzzle is solved once and for all.

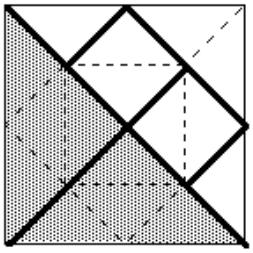


Figure 6a. Large triangles are adjacent and the small square can go in two places.

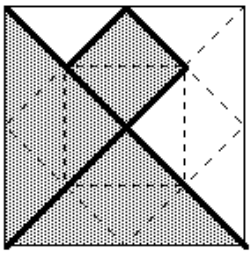


Figure 6b. By symmetry, the square can be arbitrarily placed in one of the two options.

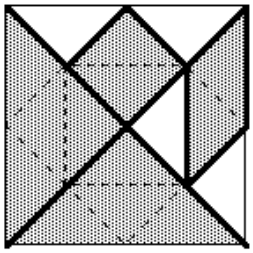


Figure 6c. Only two places for the parallelogram. This option prohibits the placement of the medium triangle.

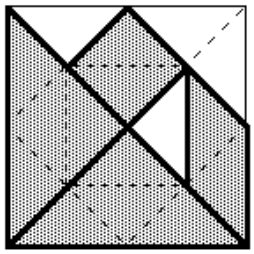


Figure 6d. The puzzle is solved.

The Payoff

We have reached a point where we can reconstruct the Tangram puzzle at any time using the underlying structure, the relationships among the pieces, and logical reasoning. This addresses a large number of important objectives all at once, making it very worthwhile to spend two or three lessons investigating Tangrams. The big payoff is that these strategies model particular mathematical ways of thinking and problem solving, allowing for a transition toward recognizing and describing the underlying structure, which in turn becomes embedded as the mental image for the Tangram. Using such a structure helps students to be successful at solving figures made with the Tangram pieces, as well as deciding whether or not it is possible to construct certain geometric shapes with all seven pieces or a subset of these. I have included a list of additional references for those who are interested in Tangrams and other mathematical puzzles or games. Enjoy!

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University of Exeter Centre for Innovation in Mathematics Teaching web site: **http://www.ex.ac.uk/cimt**.

Math Forum at Swarthmore College web site, with an excellent list of links and resources: **http://forum.swarthmore.edu/trscavo/tangrams/contents.html.** ▲

▲ Finding Trigonometric Ratios as Side Lengths



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and David R. Duncan
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Teachers are always looking for ways in which the six trigonometric functions can be isolated and examined. Although these functions are typically expressed as ratios in a right triangle, they can also be depicted as segment lengths. We will discuss how this can be done.

Consider the triangle of Figure 1: Construct this triangle so that m (<BAC) is larger than 45°.

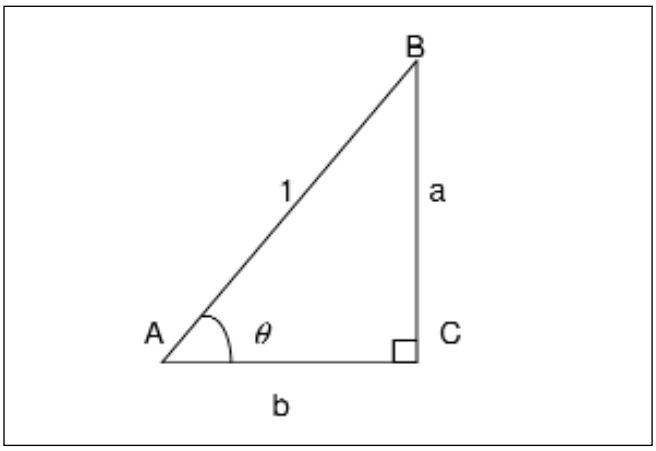


Figure 1

By the usual definitions of right triangle trigonometry:

$$\sin\theta = \frac{a}{1} = a$$

$$\cos\theta = \frac{b}{1} = b$$

We next expand Figure 1 into Figure 2 by locating a line segment ($\overline{DE}$) of length 1 with D on $\overleftrightarrow{AB}$ and E on $\overleftrightarrow{AC}$ so that $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.

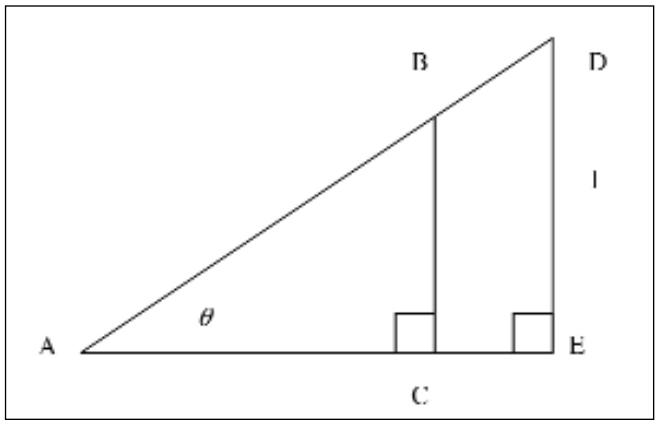


Figure 2

In this larger ΔADE, we note that:

$$\sin\theta = \frac{1}{AD} \rightarrow AD = \frac{1}{\sin\theta} = \csc\theta$$

$$\tan\theta = \frac{1}{AE} \rightarrow AE = \frac{1}{\tan\theta} = \cot\theta$$

Two more trigonometric functions of angle θ are thus expressed as segment lengths.

Finally, we expand Figure 2 into Figure 3 by locating a point G on $\overleftrightarrow{AC}$ so that AG = 1. We then locate F on $\overleftrightarrow{AD}$ so that $\overleftrightarrow{FG} \perp \overleftrightarrow{AG}$.

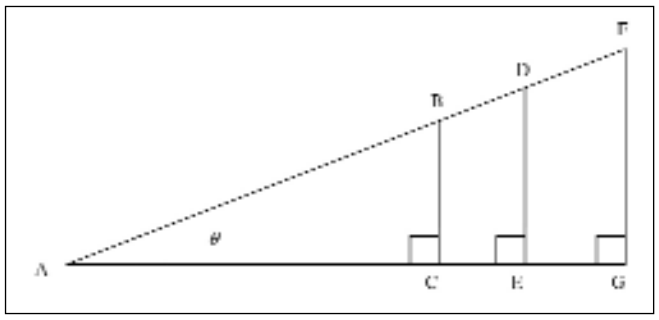


Figure 3

Then $\tan\theta = \frac{FG}{1} = FG$

$$\cos\theta = \frac{AG}{AF} = \frac{1}{AF} \rightarrow AF = \frac{1}{\cos\theta} = \sec\theta.$$

Consequently all six trigonometric functions of angle θ are expressed as lengths in Figure 3.

Teachers and students are invited to investigate other trigonometric relationships that might emerge on these figures. ▲

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