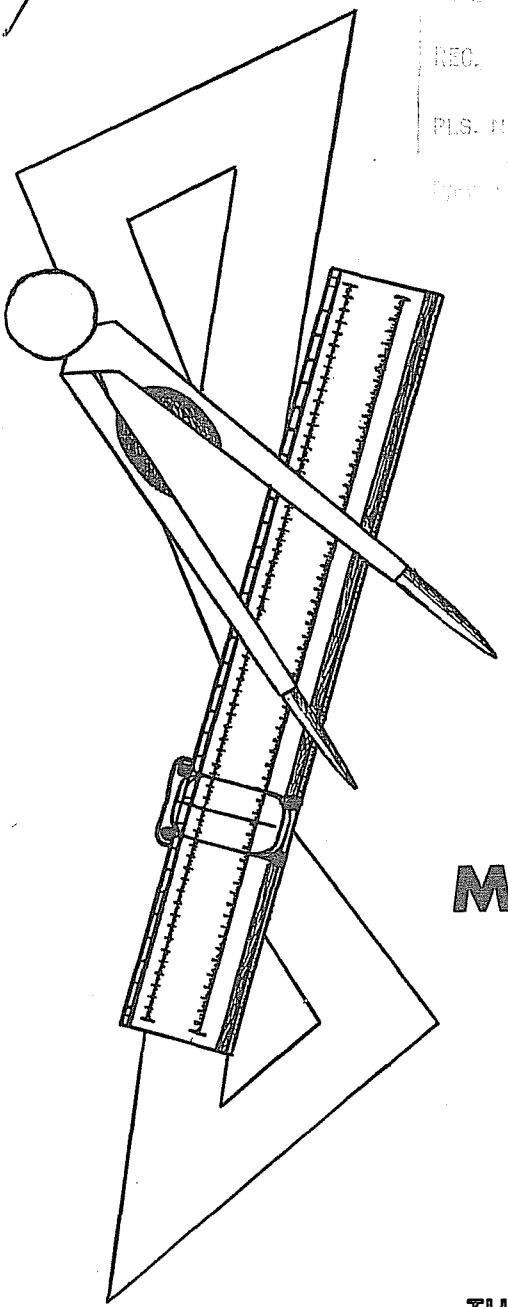
REC. DEC 16 1965
PLS. NOTE



VOLUME 4
NUMBER 1
DECEMBER 1965

ONTARIO MATHEMATICS GAZETTE

sponsored by
THE SAMUEL BEATTY FUND
THE ONTARIO MATHEMATICS COMMISSION
THE ONTARIO ASSOCIATION OF
TEACHERS OF MATHEMATICS

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EDITORIAL

The summer of 1967, a period when Canadians generally will find themselves in the midst of centennial festivities, will also be for Ontario a milestone in the forward progress of reform in mathematics education. At that time, the secondary schools will have graduated the first class to be exposed to the recently announced new mathematics programme for Grade 13. At the same time virtually all of the new elementary school graduates will have completed the updated curriculum for Grades 7 and 8. Furthermore, it will then be reasonable to expect to find a new beginning-at the beginning, in Kindergarten.

Thus at the present time we find ourselves in the middle of the first wave of curriculum development and reform. It is perhaps too early to attempt any assessment of this first effort, but a few facts seem to be encouraging.

Differences of opinion have been expressed in many places (including the Gazette!) as to the suitability and desirability of the new material. Some contend the changes are modest; a few consider them extreme. There is little unanimity as to the rate at which we should attempt to converge on the "ideal" mathematics programme, nor indeed exactly what that programme should be. The exciting thing is that the debate does exist - the old lethargy is gone. Many more people are involved. Mathematics is becoming a vibrantly living subject.

The mechanism of curriculum reform, which includes the department, the teacher, and the professional mathematician is constantly being enlarged. And more people are writing more and better text-books.

To the surprise of many people, the most enthusiastic response to mathematics curriculum reform has been in the elementary schools. Three quarters of the Grade 7 classrooms in Ontario voluntarily accepted the new programme in 1964-5. This year the new programme is in virtually all the schools, still on a non-compulsory basis. This is substantial evidence that increasing numbers of teachers are becoming prepared to

accept their proper role as continuing students.

The student role of teachers at all levels is undoubtedly the most important key to progress in an age of science that is typified to a degree by a sign which appeared ten years ago in a Canadian Government Research Laboratory, and which read, "If it works, it is obsolete".

But to be a student involves time, even for the teacher. It is vitally essential that teaching loads be reduced to permit continuing study. This is particularly urgent now for those teachers who are involved in presenting new courses, or are about to do so.

About this issue

One of the more immediate tasks facing mathematics teachers in Ontario is the implementation of the new Grade 13 programme. To assist in this endeavour many universities are offering special courses which include suitable background material.

In this issue of the Gazette we include three feature articles which are intended to be directly helpful to the teacher in connection with the new Grade 13 curriculum.

Dr. Clarke, our review editor, has prepared short reviews of some of the reference books which are suitable (at least in part) as background material.

One of the most important new topics is the elementary calculus, for which the most fundamental concept is that of limit. Included in this issue is the first of two parts of an article by Dr. Lightstone on "The Concept of a Limit". His material is not aimed at the geometrical approach intended in the curriculum, but rather exemplifies the more mature understanding which the teacher should possess.

Finally, Dr. Coleman, one of the enthusiastic pioneers of mathematics reform in Ontario, presents his views of the perspective of the new course, as well as a description of its chief concepts.

J.E.H.

GRADE 13 MATHEMATICS - A GOOD BEGINNING

A. J. Coleman - Queen's University

Background

The changes which are currently being introduced in the Mathematics curriculum in the school systems of most of the technologically advanced countries of the world are remarkable not because they are radical and abrupt. They are neither. To any objective historian of education their extraordinary aspect must surely be the long delay in the general recognition of the archaic nature of the traditional curriculum and the gingerly trepidation with which change is finally being embarked upon.

The contrast between the Mathematics Curriculum for the Ontario High School of 1966-67 and that of 1936 is insignificant compared with the radical developments in Physics, Chemistry and French, for example. Yet these changes in Math are being undertaken with great fear and trembling on the part of officials in the Department of Education, Inspectors, Principals and teachers. Parents seem more anxious about these minor changes in Maths than they ever were about major revolutions in other subjects. The students seem the least worried and many (perhaps most) seem to prefer the "new math" to the "old math".

To carry out faithfully the spirit of the new Ontario curriculum will require a major effort on the part of teachers and text-book authors. We must struggle out of the old familiar ruts in which we have been comfortably embedded for decades. Even so, the new syllabus represents only a beginning of necessary and inevitable change. In my view, a good beginning, but only a beginning.

In many ways the school mathematics curricula in Denmark, Belgium and parts of Switzerland have already moved much farther in the direction of contemporary relevance than we can hope Ontario's curriculum will be even in 1970 after the effect of the current revision at the Grade Nine level has worked its way up. We are undoubtedly considerably behind the better

schools in New York and California. Even some of the English Public Schools - often regarded as the last stronghold of conservatism, - have embarked on mathematics programmes which are excessively daring compared with our new Ontario curriculum.

Like all curricula everywhere, our new Mathematics programme in Grades 11, 12 and 13 is the result of compromise. It has been designed by a Committee appointed by the Department of Education almost three years ago. The Committee had the great advantage of immediate access to the intense and creative thinking about the syllabus which has been going on within the Ontario Mathematics Commission since 1960. It is largely because of the work of the O.M.C. that teachers have been awakened to the need for change.

The curriculum changes so far announced are pertinent chiefly to the five-year stream, that is 40%-60% of the High School population including those students who will ultimately fill positions as engineers, social scientists, physicists or mathematicians in which the highest possible mathematical and statistical acumen will be necessary if they are to enable Canada to maintain her place among the technologically developed countries. If the well-being of society were the only criterion determining the new curriculum it would be mathematically very much more demanding than it actually is. The excessive influence (fortunate or unfortunate, I do not debate here) of the Grade 13 examinations in deciding the professional fate of students was constantly in the minds of the Senior Division Committee. The more radical proposals for change were not given the serious consideration which some of them deserved, because of the realistic assessment of the magnitude of change which could be attempted in one stage without introducing too great disruption in the teaching process.

From my above remarks the reader will easily infer that I do not believe that Ontario has yet entered Mathematical Utopia. Nonetheless, I look on the new curriculum with great satisfaction. We are much farther ahead in Ontario than I dreamed could be possible, five years ago. For our substantial

progress we must thank the Ontario Mathematics Commission, the excellent leadership of Mr. Frank Kinlin and his associates in the Department of Education, the new atmosphere of sympathetic give-and-take between teachers and university professors and, perhaps chiefly, the increasing sense of professional responsibility manifest by the members and leaders of the Ontario Federation of Teachers.

Curriculum Content

Since most of the old mathematics curriculum in Ontario was sound, the main bulk of the new Grade 11, 12 and 13 programme is familiar material. However, the emphasis has definitely swung from concentration on successive isolated topics to an attempt to present mathematics as one integral and meaningful discipline; from techniques to ideas.

In the remainder of this article, I attempt to illustrate this last sentence by commenting on what I regard as the main new ideas in the Grade 13 programme. Space does not permit me to deal with all that I consider important. We shall consider:

1) the general notion of a function conveyed through the mapping approach;
2) the idea of limit and derivative;
3) probability, as a set function;
4) vectors and linear spaces;
5) groups.

(1) The function concept is implicit in ordinary language and at many parts in the mathematics programme. For example, absolute value enables one to immediately define a function, $x \to |x|$, from the real numbers to the non-negative real numbers. There has been some debate in Ontario about the "best" or "proper" definition of function. To my mind, some of the debaters have been excessively dogmatic. Any well-informed mathematician, knows that the concept of function is of central importance to mathematics today, that its meaning has evolved over the last 150 years and that only recently - about sixty years ago - has the meaning of the term been made precise. Even today there is not universal agreement about

the terminology connected with the idea.

There are at least two widespread and equivalent ways of defining function: as a set of ordered pairs, or as a mapping. For both it is necessary to specify (explicitly or implicitly) two sets (1) A, the domain of definition, or the domain of the function, and (2) B, a set in which the values of the function are contained. It will probably be some years before mathematicians agree on a name for this latter set. According to the observations of myself and my colleagues at Queen's the term co-domain is apparently gaining popularity. A function is known, if in addition to being aware of the domain and co-domain of the function, one knows, without ambiguity, which element of B is associated to each element of A . The subset of the co-domain, containing all the values actually occurring in the image of A under the function, is usually called the range of the function. The range may or may not coincide with the co-domain.

For each element, x , of the domain of the function, f , there is a unique element of the range which corresponds to x . We can write $f: x \to f(x)$ [read: "The function f which maps x on f at x "] or simply $x \to f(x)$ [read: "The function x to f at x "] . Thus the notion of function is essentially equivalent to that of unambiguous correspondence. The elements of the domain and range of the function can be almost anything. The main development in Grade 13 consists in discussing functions for which the domain and co-domain are not necessarily the good old real numbers on which the author and the reader were raised!

(i) A transformation in the plane is a function which maps points onto points; (ii) a binary operation on any set A is a function from $A \times A \to A$; (iii) a permutation of a set of objects is a one-to-one function. The teacher should stand on his head, if necessary, to help students grasp the comprehensiveness of this simple notion of function.

(2) Limit and derivative are the basic ideas of calculus. The notion of limit distinguishes topology from algebra.

Indeed, a topological space consists of the bare idea of a set together with enough paraphenalia to enable one to give a meaning to the idea "close". When is one point close to another? The points $A = 0.999$ and $B = 1.0000$ on the real line might be thought to be close for some purposes. However, there are an infinity of rational points between them. This suggests that A and B are far apart!

Units 6, 7 and 8 of Part A of the curriculum are very modest in their scope, aiming merely to give a first, intuitive approach to differential calculus, sufficient it is hoped to be of some use in the elementary parts of mechanics.

In discussing derivatives, the teacher should carefully avoid the old-fashioned mystical and mystifying talk about "infinitesimals". In a knock-down-and-drag-out debate in the 18th Century, Bishop Berkely finally said to Newton's successor "I do not insist on any particular answer, but kindly begin a sentence with the phrase "An infinitesimal is..." and convey some meaning to me". To which the great professor replied that Berkely was an ass and should busy himself with Confirmations and avoid mathematical arguments and that whereas he himself did not understand what an infinitesimal was he had a friend who claimed to do so!

The key fact to emphasize (in the argument leading to the definition of the derivative) is that Δx is never zero. It may be small but it remains finite. Thus the problem of dividing by zero never arises and never should arise!

(3) The new course proposes that in Grade 13 we should prepare the minds of students for a satisfactory definition of probability. For this it is necessary, in connection with any probabilistic situation, to enumerate all possible elementary outcomes and attach a probability to each (I am thinking only of finite sample spaces as does the syllabus!). The particular event which you are hoping for (e.g. receiving a large inheritance from a maiden aunt) might be implied by several of the elementary outcomes. Thus an event is defined as a sub-set of S , the set of elementary outcomes, that is,

a subset of the so-called sample space. A probability, measured as a real number between 0 and 1 , is attached to subsets of S . Thus the basic concept in probability is a real valued set function. Sound frightening? What are volume and area?

(4) A strong case could easily be made for the contention that, practically the most useful idea in the whole school programme is that of a linear or vector space. When the die is finally cast and a physicist, engineer, economist, biologist or psychologist actually calculates the answer to a problem the chances are very great that he will solve a system of linear equations. Perhaps a system of 3 equations, or if he is designing the wing of a large aeroplane, possibly 3000 equations! The electronic computer's favourite occupation is solving linear equations.

If you are asked to define a Frenchman, you do not begin by the definition of an individual Frenchman, but rather with the class of Frenchmen. Then, with a smile, you can explain that a Frenchman is a particular instance of Frenchmen!

So it is with vectors. You can not define a vector. You can define the idea of a linear or vector space. Then a vector is one object in the set you have characterized.

Though vectors are associated in our minds with arrows, forces and velocities - the famous directed distance idea - the notion has much greater generality and the new curriculum expects the student to begin exploring beyond physical examples even though in this initial introduction of the topic into the mathematics syllabus any attempt to the utmost generality is carefully eschewed.

(5) The last Unit of the B part of the Grade 13 is devoted to the subject of Groups. It would be a great mistake on the teacher's part to leave this idea to the last hectic weeks of the year. The notion and its name should appear in the earlier Grades. The Translation Group is already implicit in the experience of the pre-school child when he shoves his blocks around on the floor. The multiplicative group of

fractions appears in Grade 3; the additive group of integers in Grade 8; translation and rotation groups appear early in Grade 13 in Part A. The right moment for the notion of Group to become explicit depends on the ability of the student and the interests of the teacher.

The explicit treatment of Groups in Unit 9, Part B of the new syllabus provides an opportunity for the teacher to expatiate on the real meaning of mathematics and mathematical structure, to emphasize its abstract character, and to point out that the idea of group has been present, like a golden thread, giving coherence to the whole mathematics programme from Kindergarten to Grade 13. Thus he can open a door through which the student may glimpse something of the rich and fascinating mathematical vistas which still lie before him.

75th ANNUAL

O.A.T.M.

EASTER CONVENTION

April 12 & 13, 1966

at

The Inn on the Park, Toronto

Speaker: Dr. Mary Dolciani, Prof. W. W. Sawyer

Banquet Speaker: The Honourable W. G. Davis,
Minister of Education

BOOKS AND REVIEWS

With this issue of the Gazette, we begin a new permanent department. We call it "Books and Reviews" rather than "Book Reviews" because it is our hope to bring as many books to your attention as possible. Not all of these books will be reviewed in the usual sense; some will simply be listed with a brief description of their contents and relevance.

We seek your cooperation. Anyone who wishes to submit book titles with or without reviews (preferably with) is urged to do so. Please send them to:

D. A. Clarke, Reviews Editor,
Ontario Mathematics Gazette,
c/o Department of Mathematics,
University of Toronto,
Toronto, Ontario.

In keeping with the spirit of the rest of this issue, "Books and Reviews" is devoted this time to the new Grade XIII curriculum. Books likely to be of assistance to teachers in their preparation for the new curriculum are introduced. This is only the beginning; we hope to continue in subsequent issues (but gradually we shall devote more space to other areas). Again, your participation is solicited.

The comments and reviews in this issue are by two of our colleagues, and they follow a few remarks of our own (with some assistance from Lewis Carroll).

"...The name of the function is called 'Polysyllabic'."

"Oh, that's the name of the function, is it?" Alice said, trying to feel interested.

"No, you don't understand", the Knight said, looking a little vexed. "That's what the name is called. The name really is 'Tangent'."

"Then I ought to have said 'That's what the function is called'?" Alice corrected herself.

"No, you oughtn't: that's quite another thing! The function is called 'Trigonometric': but that's only what it's called, you know!"

"Well, what is the function then?" said Alice, who was by this time completely bewildered.

"I was coming to that", the Knight said. "The function really is $\{(x,y) \in R \times R \mid y = \tan x\}$: and the notation's my own invention."

Are you as bewildered as Alice? If so, you're not alone. A great deal has been written about functions and relations lately, and much of it is wasted effort.

If you want to see all the answers stated fully and correctly by an expert (probably the expert), read the "Introduction" (the first 68 pages!) of

A. Church, An Introduction to Mathematical Logic, Vol. I (Vol. II has never been published), Princeton U. Press, Princeton, N.J. 1956.

This is by no means easy reading, but anyone willing to invest the time and effort required, will be rewarded with a comprehension of this subject which will put him on top of it. It is recommended that the voluminous footnotes be ignored on a first reading.

(The absence of the "set-builder" notation and ordered pairs can be explained by the fact that the Knight failed to communicate his ideas to Professor Church.)

"Beware the Jabberwock, my son!"

DAC

Ford and Ford, Calculus, McGraw-Hill, Toronto, 1963.

This very nice freshman Calculus text has a novel approach to limits and continuity (Chapters 2,3) which can be found nowhere else (to the reviewer's knowledge). The approach neatly bridges the gap between intuition and so-called rigour. It

greatly enhances comprehension and makes the proofs seem very much simpler than usual (although they are essentially the same old ones). As a matter of fact, the proof of Theorem 2.1 (continuity of $mx + b$, x^2 and $1/x$) can be made even simpler than the authors make it.

DAC

Melvin Hausner, A Vector Space Approach to Geometry, Prentice-Hall, Englewood Cliffs, 1965.

This book, an outgrowth of a one-year in-service course for teachers, explores on an elementary level the relationship of geometry and linear algebra. Vector spaces, matrices and linear transformations are discussed informally without full axiomatic detail and applied to a wide variety of geometrical topics. Examples of geometric topics discussed are: Ceva's theorem, area of polygons, the oriented n-simplex and its boundary, the Euclidean transformations of 3-space, and the "Erlangen Program". Little knowledge of geometry is assumed (Ontario grade 12 would surely suffice) but some parts of the book require considerable mathematical sophistication. Reference to function spaces and probability theory is made several times. A few sections require calculus but these can be omitted without major loss. The whole book is unconventional in nature and reflects the personal viewpoint of the author. At times one may not agree with his approach (thus affine geometry and vectors are motivated by a discussion of the physical notion of center of mass, an approach which does not seem helpful to this reviewer, a physicist) but it is indeed refreshing to read an elementary book that reflects the views of an individual rather than a committee.

The text is clearly written and very readable, although some readers will find that it does not always adhere to their concept of good English. The author is at times authoritarian in tone: thus on page 43 the reader is exhorted to memorize a

formula that seems highly insignificant to the reviewer. Determinants are treated axiomatically with a minimum of computation. The exercises are unusually stimulating but do not emphasize manipulative skills. In conclusion, this seems to me a very good book, one to be highly recommended to anyone who will make the effort to read it seriously, but one not well suited for reference use. It contains a substantial body of genuine mathematics, presented in an interesting and clear (although informal) manner.

Jacob T. Schwartz, Introduction to Matrices and Vectors, McGraw-Hill, New York, 1961.

This little book should be known to every teacher of grade 13 mathematics. The approach is concrete and computational: thus matrices are arrays rather than representations of linear transformations. The text is readable and the pace slow, but in 160 pages substantial results are achieved. The viewpoint is algebraic and little geometry is included. Any good grade 13 student should be able to read the book by himself. The author, a distinguished analyst, provides a counter example to the common conjecture that top research mathematicians cannot write good books on the high school level. It is to be hoped that this book will become available in a paperback edition and hence accessible to a wider audience.

W. T. Sharp

P. M. Cohn, Linear Equations, Routledge and Kegan Paul, 75 pages (paperback).

Covers vector spaces, matrices, systems of linear equations, determinants. Most of the first four chapters of this little book are on the new Grade 13 curriculum.

A feature found in too few other linear algebra books is its use of geometric language and pictures to motivate and illustrate the algebraic manipulations.

P. M. Cohn, Solid Geometry, Routledge & Kegan Paul, 75 pages (paperback).

Covers vectors and coordinate systems, lines and planes, coordinate transformations, spheres, central quadrics. At the beginning of chapter four, the author wishes to find the points of intersection of a sphere and a plane. While he has been using purely vector methods for lines and planes these are insufficient here and so he must do the problem in terms of components with respect to a basis. This illustrates the limitation to lines and planes of the use of vectors in analytic geometry and shows that we really cannot get so far from our old high school analytic geometry after all. Most of the first three chapters are on the new Grade 13 curriculum. Both this book and its companion Linear Equations are worthwhile reading for teachers preparing to teach the new course.

T. R. Munkres, Elementary Linear Algebra, Addison Wesley.

Contains essentially the same material as Linear Algebra by P. M. Cohn; but the terminology and methods are more modern or at least are closer to those outlined in the new grade 13 curriculum. It is an abstract presentation with little use of geometric ideas for illustration.

A. Tarski, Introduction to Logic and to the Methodology of Deductive Sciences, Oxford Galaxy Books, (250 pages) (paperback).

Excellent reading for any teachers who wish to acquire a real understanding of such things as variables, constants, statements, open sentences, equality, deductive method and axiomatic systems. There are numerous pedagogically important explanations, examples and exercises. Contains a list of

suggested readings for those wishing to pursue further some topic he has only touched on. One drawback: the book was written in 1941 and some of the terminology and symbolism differs from current usage.

F. Dinkines, Introduction to Mathematical Logic, Appleton - Century Crofts (paperback).

This book contains one of the best treatments of conditional and implication that I have seen. It is divided into two sections: the Statement Calculus, which treats statements and connectives; and the Predicate Calculus, which treats quantifiers and shows how they interact with connectives. It contains much material on laws of inference and is notable for the many interesting English sentences and arguments which are translated back and forth into symbols.

The book is one of three paperbacks by the same author, the other two being on set theory and abstract mathematical systems. They are also available bound together in hardcover.

R. R. Christian, Introduction to Logic and Sets, Ginn and Company, 70 pages (paperback).

This little booklet has the novel feature of not including the conditional among the sentential connectives, but only introduces it after the correspondence between sets and properties has been established. Then implication is defined as corresponding to the relation of inclusion between sets. In this way all the difficulties of the conditional wherein the antecedent and consequent may be unrelated statements are put off until the last part of the book. The book is packed full of many ideas which are not found in other presentations, but is very well organized and coherent.

R. L. Wilder, Introduction to the Foundations of Mathematics- 2nd Edition. Wiley.

This book is invaluable for all those wishing to understand

the axiomatic method and infinite sets. The first part of the book, covering the theory of these two ideas goes into a detailed analysis of the axiomatic method, infinite sets, cardinal numbers, well ordered sets and ordinal numbers.

The second part of the book describes the development of the various viewpoints on foundations and talks about the controversies which have raged, and are still raging today, among mathematical philosophers.

T. Dantzig, Number, The Language of Science, Doubleday Anchor

To quote Albert Einstein, -

"This is beyond doubt the most interesting book on the evolution of mathematics which has ever fallen into my hands. If people know how to treasure the truly good, this book will attain a lasting place in the literature of the world. The evolution of mathematical thought from the earliest times to the latest constructions is presented here with admirable consistency and originality and in a wonderfully lively style."

W. W. Sawyer, What is Calculus About?, Random House (120 pages) (paperback).

An excellent book on how the Calculus could be taught to Grade 9 or 10 students. One can do no better to describe Sawyer's ideas than to quote from chapter one of the book.

"In the past, people often thought of calculus as an extremely difficult subject. Then, particularly in England, teachers began to realize that many things could be done by calculus in a way that was much simpler and more interesting than anything in algebra. In English high schools a student may have two or even three years of calculus. But then some mathematicians say that this is not good; that calculus is really more complicated than it appears, and that it should only be taught by a very well qualified mathematician. Where

does the truth lie in all these conflicting views?

A comparison may be helpful. An old lady lives in a quiet village, and every Sunday she drives herself to church. You ask her if it is easy to drive a car. "Oh, yes," she says, "I have no mechanical aptitude, and I find it quite simple." She might find it less simple if she had to drive in the middle of New York, or take a heavy truck across the Rockies. But there is no denying the fact; she can drive a car. And, if she ever did have to drive in heavy traffic, her experience of handling a car would be of some use to her. She would not be so helpless as someone who had never driven at all.

The situation in calculus is somewhat similar. Elementary calculus is like elementary car driving, not difficult to learn and it enables you to do many things you could never manage otherwise. But if you wish to push calculus as far as it will go, you will run into things that are more complicated.

How should calculus be taught then? Should we bother the beginner with warnings that only become important in more advanced work? If we do so, the beginner will be confused because he will not see any need for these warnings. If we do not, we shall be denounced by mathematicians for deceiving the young.

I believe the correct approach is to do one thing at a time. When you take a student into a quiet road to drive a car for the first time, he has plenty to do in learning which is the brake and which the accelerator, how to steer, and how to park. You do not discuss with him how to deal with heavy traffic which is not there, nor what he would do if it were winter and the road were covered with ice. But you might very well warn him that such conditions exist, so that he does not overestimate what he knows.

If you try to tell him the whole truth, he probably

cannot take it in all at once. An even more important objection is - we do not know the whole truth. Our student is young. Perhaps he will live to drive a car in the first Martian expedition. And who knows what difference in driving technique will be needed on Mars?"

O. Toeplitz, The Calculus: A Genetic Approach, Phoenix Science Series (190 pages) (paperback).

To quote Alfred L. Putnam in the preface to the American edition:

"This is not a textbook in the calculus, nor is it a history of the calculus. Over a period of many years Toeplitz sought in his teaching to evolve what he described as a "genetic approach" to the subject. He aimed for nothing less than a presentation of the calculus that would do justice to the growth and development, in the course of time, of its central ideas. The choice of topics is that of a mathematician concerned for what have proved to be crucial concepts and techniques of the calculus. The organization and exposition are determined by the historical evolution of these themes from their beginnings with the Greeks down to the present. Toeplitz was convinced that only through this genetic approach could students come to a full understanding of the significance of the concepts and techniques which constitute the calculus. To be sure, he was addressing himself to a German audience with the mathematical training provided by the Gymnasium, but what he has offered in this volume is equally relevant for serious students and teachers of the calculus in America."

Dolciani et al, Modern Introductory Analysis, Teachers Edition, Houghton-Mifflin.

This book will interest anyone who would like to see how the topics of the new grade 13 curriculum would be presented

in a high school text. For some topics it goes further than our new curriculum, for some not as far.

The first chapter on logic and sets covers approximately as much as our new curriculum and does it very well. It puts the ideas in the forefront and leaves the symbolism to the end of the chapter using it only as a shorthand method of illustrating the connectives and proving the rules of inference.

The second chapter on the real number system as an ordered field is poor. Instead of defining equality in terms of the replacement property, it gives only the three properties of symmetry, reflexivity and transitivity which alone do not characterize equality. Then it gives the equality of sums and products of equals as an axiom. Actually this and the three other properties of equality follow trivially from the replacement property. The result of this is a number of theorems throughout the rest of the book which would also follow trivially if equality were defined properly. Besides being inefficient, this treatment is very bad from a motivational point of view since the student is required to learn proofs of theorems which appear obvious and actually are.

The third chapter on mathematical induction, sequences, and series goes much further than our curriculum and introduces limit theorems to deal with convergence of sequences and series. In my opinion it makes a mistake when it uses the formula for the sum of a geometric series (which has been derived in an earlier book of the series by the usual method of cancelling terms after multiplying by the ratio and subtracting) without proving it again by mathematical induction. Unless it is emphasized that mathematical induction is an indispensible method of proof for statements of the form "for all n, $P(n)$" the student may wonder why we need mathematical induction at all, especially since all the examples proved in the book could be done by manipulation of an indefinite number of terms similar to the method of proof of the geometric series described above.

The fourth chapter on vectors is good, defining immediately

a vector as an ordered pair and avoiding the difficulties which may arise in changing over from physical vectors to ordered pairs. The answer to question number 13 on page 124 is wrong and could hardly be just a misprint.

The chapter on complex numbers takes some of their properties from those of vectors where they have been proven since both are represented by ordered pairs. The method of introduction of multiplication of complex numbers is weak.

The chapter on calculus goes deeper than our curriculum, but not as far. It goes into continuity and limit theorems, but has very few applications and no integration.

These are only a few comments on some of the chapters of the book. There are other chapters on: Trigonometry, Matrices, Analytic Geometry, Logarithms, Exponentials, Probability, Systems of Equations, Functions. Taken as a whole, the book is more difficult than our new Grade 13 curriculum.

If you buy it, get the teachers edition which contains a booklet of advice and explanations bound to the text; also, the pages of the text are over-printed with comments and references to this booklet.

D. P. Wakfer

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THE LIMIT CONCEPT*

(PART I)

H. A. Lightstone - Queen's University

1. Sequences

It is well known that many different concepts of mathematics, such as the length of a curve, area, instantaneous velocity, an infinite sum, all involve a common idea -- the concept of a sequence of real numbers. Here, we shall explore this basic concept and develop some of its properties.

The intuitive content of the notion of a sequence of real numbers, is that of a list of real numbers with a beginning -- but no end. One thinks of an animated cartoon in which the characters are real numbers who present themselves one after another announcing: "I am 5.17", "I am π", "I am $2^{\frac{1}{2}}$", "I am -3", and so on. Here, the notion of order is vital. The real numbers appear in a certain order; this order is not accidental, but is deliberate. We can use the natural numbers to illustrate this idea: let us write down the natural numbers in their obvious order (i.e. using the relation less than). So, we obtain a list with a beginning but no end. If only we could transform, perhaps by some magician's trick, each natural number on this list, into a real number. Ah! Let μ be a particular mapping of N, the set of all natural numbers into R, the set of all real numbers. Then all we need do, is replace each natural number n on the list by $\mu(n)$ -- the real number associated with n by the mapping μ. This transforms the list of natural numbers into a list of real numbers, as desired.

One final, simplifying step. Since the mapping μ contains the essence of the notion of a sequence of real numbers, let's say it is a sequence of real numbers. This means, in particular, that a sequence of real numbers is a set; this is

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in accordance with the fundamental program of expressing mathematical objects in terms of sets or ordered n-tuples. We have arrived at the

DEFINITION 1.1: Any mapping of the natural numbers into the real numbers is called a sequence of real numbers.

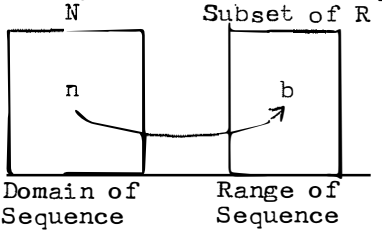
Thus, a sequence of real numbers is any set of ordered pairs in which first members are natural numbers, second members are real numbers, and each natural number is a first member of precisely one ordered pair. For example,

$\left\{(n,b) \mid n \in N \text{ and } b = \frac{1}{10^n}\right\}$ is a sequence of real numbers,

and

$\left\{(n,b) \mid n \in N \text{ and } b = 2^{\frac{1}{2}}\right\}$ is a sequence of real numbers.

Note that a sequence is nothing more than a special kind of function -- namely, a function whose domain is N, the set of natural numbers. Thus, a sequence associates a real number with each natural number. The following diagram is intended to convey this view of a sequence:



The real number b is associated with the natural number n, by this particular sequence.

It is easy to generalize this concept. Let S be any non-empty set, and let μ be any mapping of the natural numbers into S. Then μ is said to be a sequence of members of S. For example, let $S = \{1,2\}$; then $\{(n,b) \mid n \in N$, and $b \in S$, and $b = 2$ iff n is even$\}$ is a sequence of members of S.

Here, we shall be concerned only with sequences of real numbers. For this reason, let us agree to call a sequence of real numbers, a sequence -- for short.

Though our definition of a sequence is precise and exact, it does not emphasize the important fact that a sequence

involves the notion of order. Admittedly, this notion is implicit in the definition, since the natural numbers are ordered, and so an order is imposed -- in a natural way -- on the real numbers appearing in the sequence. Our job, now, is to bring out, explicitly, this notion of order. We do this as follows. Given a sequence, let's write down the ordered set whose first term is the real number paired with 1 , whose second term is the real number paired with 2 , whose third term is the real number paired with 3 , and, in general, whose kth term is the real number paired with k . Of course, this ordered set completely characterizes the sequence from which it is constructed. Therefore, we agree to denote the sequence by this same ordered set. That is, this ordered set, we agree, is a name of the given sequence. For example, the sequence $\{(n,b) | n \in N$ and $b \in R$ and $b = 1 + \frac{1}{10^n}\}$ is denoted by the ordered set

$$(1.1,\ 1.01,\ 1.001,\ldots,\ 1 + \frac{1}{10^n},\ldots) \quad .$$

One more important convention. In order to discuss sequences abstractly, we shall want to talk about a sequence without specifying a particular sequence, i.e. we shall want to write down the name of a sequence without stating which sequence we have named. Since a sequence may involve infinitely many numbers, what we need is a method of talking about infinitely many numbers without actually specifying the numbers. Let us adjoin "1" to the letter "a" , obtaining the expression "a_1" . The symbol "1" is said to be a subscript. We can suppose that the expression "a_1" is the name of a real number. Again, adjoining the symbol "2" as a subscript to the letter "a" , we obtain the expression "a_2" , which we may suppose is the name of a real number. In general, adjoining the symbol "k" as a subscript to the letter "a" , we obtain the expression "a_k" , which we may suppose is the name of a real number. Incidentally, "a_1" is read "a sub one" , "a_2" is read "a sub two" , "a_k" is read "a sub k" . Now that we have a system

of naming real numbers, consider the sequence $\{(n,b) | n \in N$ and $b \in R$ and $b = a_n\}$. This sequence is denoted by the ordered set $(a_1, a_2, a_3, \ldots, a_n, \ldots)$. We shall call the members of this ordered set, the terms of the sequence; and we shall call a_n the nth term of the sequence, for each n. One further agreement: we shall sometimes omit the nth term of the sequence, writing down only the first few terms -- provided it is clear which sequence is being denoted. For example, we shall denote the sequence

$$\{(n,b) | n \in N \text{ and } b \in R \text{ and } b = n^2\}$$

by writing

$$"(1,4,9,\ldots,n^2,\ldots)" \text{ or } "(1,4,9,\ldots)" ;$$

and we shall denote the sequence

$$\{(n,b) | n \in N \text{ and } b \in R \text{ and } b = 5\}$$

by writing

$$"(5,5,5,\ldots,5,\ldots)" \text{ or } "(5,5,5,\ldots)" .$$

One final convention regarding notation. In an abstract discussion it is convenient to abbreviate the name $"(a_1,a_2,a_3,\ldots,a_n,\ldots)"$ by writing down only the nth term of the sequence concerned, so obtaining $"(a_n)"$. More precisely, we denote a sequence by writing down between parentheses, the prescription by means of which the nth term is calculated. For example, the sequence $\{(n,b) | n \in N$ and $b \in R$ and $b = n^2\}$ is denoted by $"(n^2)"$. Much simpler and highly effective! In the following sections we shall make use of the above conventions for naming sequences.

2. The Limit of a Sequence

We have seen that one way of constructing a sequence is by rounding off a given real number to one decimal place, then to two decimal places, then to three decimal places, and so on. The resulting sequence can be called a sequence of approximations to the given number. Here we have the

fundamental idea of calculus! In a moment we shall say exactly what we mean by a sequence of approximations to a given number. First, note that the term approximation is meaningful only if a permitted error is stated. For example, if the permitted error is 10, then 7 is an approximation to 2, since 7 differs from 2 by less than 10, the permitted error. If the permitted error is $.1$, then 7 is not an approximation to 2, since 7 differs from 2 by more than $.1$. The point is that any approximation must be accompanied by a statement of the permitted error. Now we can state our definition.

DEFINITION 2.1: Let L be a real number and let (a_n) be a sequence. We shall say that (a_n) is a sequence of approximations to L iff given any permitted error we can find a term of (a_n) such that each of the following terms is an approximation to L.

Thus, (a_n) is a sequence of approximations to L iff corresponding to any given permitted error, there is a term of the sequence such that each of the following terms differs from L by less than the permitted error.

Example 1: Show that $(\frac{1}{10^n})$ is a sequence of approximations to zero.

Solution: Consider a permitted error, say k. By counting the number of zeros immediately to the right of the decimal point, we readily find a natural number q such that $\frac{1}{10^q} < k$. Then each term of $(\frac{1}{10^n})$ following the q^{th} term differs from zero by less than k, the permitted error. Hence by Definition 2.1, $(\frac{1}{10^n})$ is a sequence of approximations to zero.

THEOREM 2.1: (a_n) is a sequence of approximations to L iff given any natural number p there is a term of (a_n) such that each of the following terms differs from L by less than $\frac{1}{10^p}$

Proof: Clearly, if (a_n) is a sequence of approximations to L, then (a_n) has the stated property. Suppose, now, that given any natural number p there is a term of (a_n) such

that each of the following terms differs from L by less than $\frac{1}{10^p}$. We shall show that (a_n) is a sequence of approximations to L. Choose a permitted error, say k where $k > 0$. Next, choose a natural number p such that $\frac{1}{10^p} < k$ (note: this means $10^p > \frac{1}{k}$). By assumption there is a term of (a_n) such that each of the following terms differs from L by less than $\frac{1}{10^p}$; therefore, each of the following terms differs from L by less than k. This proves that (a_n) is a sequence of approximations to L.

Example 2: Show that $(\frac{1}{n})$ is a sequence of approximations to zero.

Solution: Let p be any natural number; consider the natural number q, where $q = 1 + 10^p$; then $q > 10^p$, thus $\frac{1}{q} < \frac{1}{10^p}$. But each term of $(\frac{1}{n})$ that follows the q^{th} term differs from zero by less than $\frac{1}{10^p}$. This proves that $(\frac{1}{n})$ is a sequence of approximations to zero.

Notice that Theorem 2.1 is expressed in words for the most part. We now translate the verbal expressions contained in this theorem, into mathematical terms.

THEOREM 2.2: (a_n) is a sequence of approximations to L iff given any natural number p there is a natural number q such that:

$$\left| a_n - L \right| < \frac{1}{10^p} \quad \text{whenever} \quad n > q .$$

Proof: Apply Theorem 2.1.

In case (a_n) is a sequence of approximations to L, we shall say that (a_n) is a convergent sequence, that (a_n) converges to L, and that L is the limit of the sequence (a_n). Finally, let us agree to denote L by writing "$\lim (a_n)$". In other words, the mapping lim is defined as follows:

DEFINITION 2.2: $\lim (a_n) = L$ iff (a_n) is a sequence of approximations to L.

Applying Theorem 2.2, we readily establish

THEOREM 2.3: $\lim (a_n) = L$ iff given any natural number p there is a natural number q such that $\left|a_n - L\right| < \frac{1}{10^p}$ whenever $n > q$.

Example 3: Show that $\lim (5,5,5,\ldots,5,\ldots) = 5$.

Solution: Let p be any natural number, and consider the first term of (5). Clearly, each of the following terms differs from 5 by less than $\frac{1}{10^p}$, since $\left|5 - 5\right| < \frac{1}{10^p}$. Therefore, $\lim (5) = 5$.

We have just introduced the term "convergent sequence"; let us be quite clear about the definition.

DEFINITION 2.3: The sequence (a_n) is said to be convergent iff there is a real number, say L, such that $\lim (a_n) = L$.

For example, the sequence $\{(n,b) \mid n \in N$ and $b = 2$ if $n < 5$ and $b = 7$ if $n > 4\}$ is convergent since $\lim (2,2,2,2,7,7,7,\ldots,7,\ldots) = 7$. On the other hand, the sequence $(2,7,2,7,2,7,\ldots,2,7,\ldots)$ is not convergent.

Suppose that (a_n) is a convergent sequence; then any real number L which has the property that given p we can find q such that $\left|a_n - L\right| < \frac{1}{10^p}$ whenever $n > q$, is called the limit of the sequence (a_n).

We have spoken of the limit of a sequence, rather than a limit of a sequence. The following theorem justifies this practice.

THEOREM 2.4 (The Uniqueness Theorem): Each convergent sequence has exactly one limit.

Proof: We shall indicate the idea of the proof and leave the details as an exercise. Suppose that (a_n) is a convergent sequence such that both L and M are limits of (a_n). Assuming that $L \neq M$, there is a natural number t such that L and M differ by more than $\frac{1}{10^t}$. Taking $\frac{1}{10^{t+1}}$ as the

permitted error, we easily reach a contradiction.

So far in the discussion, the two concepts of a convergent sequence and the limit of a convergent sequence, have been completely intertwined. To establish that a given sequence is convergent requires that we first know the limit of the sequence -- see Definition 2.3. Of course, we might try an inspired guess, and apply Definition 2.2 to prove that we are right; but this procedure is really "trial and error". What we need is a method of computing the limit of a given convergent sequence from the terms of the sequence; that is, faced with a convergent sequence, we want a procedure which will automatically produce the limit of the sequence. Fortunately, such a procedure exists.

A CONSTRUCTION FOR THE LIMIT OF A SEQUENCE: Suppose that the sequence (a_n) is convergent. We construct from (a_n) another sequence which we denote by "$R(a_n)$" . First, we round off each term of (a_n) to one decimal place; it may happen that the resulting numbers -- with the possible exception of the first few -- are the same, say d_1 . If so, then d_1 is the first term of $R(a_n)$. Next, we round off each term of (a_n) to two decimal places; it may happen that the resulting numbers -- with the possible exception of the first few -- are the same, say d_2 . If so, then d_2 is the second term of $R(a_n)$. In general, we round off each term of (a_n) to k decimal places; it may happen that the resulting numbers -- with the possible exception of the first few -- are the same, say d_k . If so, then d_k is the k^{th} term of $R(a_n)$. It is a matter of fact that the numbers $d_1, d_2, d_3, \ldots, d_k, \ldots$ are determined by this procedure, with at most one exception. There may be one stage in the preceding construction, say the p^{th} , at which the p^{th} term of the sequence $R(a_n)$ is not determined. If so, then we say that the p^{th} term of $R(a_n)$ is d_{p+1} rounded off to p decimal places.

Having constructed the sequence $R(a_n)$ from the given convergent sequence (a_n) , we run through a second construction process which will yield a sequence called "$DR(a_n)$" . You may recall that in constructing the least upper bound of

a set of real numbers, one uses the operation of "chopping off" digits. We now apply this same operation to the sequence $R(a_n)$. Writing down the terms of $R(a_n)$, chopped off after the first decimal place, we observe that all the numbers so obtained are the same, say e_1 , with the possible exception of the first few. Then e_1 is the first term of the sequence $DR(a_n)$. Writing down the terms of $R(a_n)$, chopped off after the second decimal place, we observe that all the numbers so obtained are the same, say e_2 with the possible exception of the first few. Then e_2 is the second term of the sequence $DR(a_n)$. In general, writing down the terms of $R(a_n)$, chopped off after the k^{th} decimal place, we observe that all the numbers so obtained are the same, say e_k , with the possible exception of the first few. Then e_k is the k^{th} term of the sequence $DR(a_n)$. In this manner, the sequence $DR(a_n)$ is constructed from the sequence $R(a_n)$. But the sequence $DR(a_n)$ defines a real number, say L , in the following sense. The first term of $DR(a_n)$ is the integral part and the first decimal place of L ; the second term of $DR(a_n)$ is the integral part and the first two decimal places of L ; and in general, the k^{th} term of $DR(a_n)$ gives us the integral part and the first k decimal places of L . This number, L , that we have constructed from the given convergent sequence (a_n) , is the limit of (a_n) . Given time for a really penetrating study of the real number system and the limit concept, we could prove our assertion that $\lim (a_n) = L$. Since this is out of the question here, we content ourselves with several illustrations of the construction.

Example 4: Compute $\lim (a_n)$ where

$$(a_n) = (5.176,\ 5.1761,\ 5.17601,\ 5.176001,\ 5.1760001, \ldots) .$$

Solution: $R(a_n) = (5.2,\ 5.18,\ 5.176,\ 5.176,\ 5.176, \ldots)$; hence,

$$DR(a_n) = (5.1,\ 5.17,\ 5.176,\ 5.176,\ 5.176, \ldots) .$$

Therefore, $\lim (a_n) = 5.176$.

Example 5: Compute $\lim (a_n)$ where

$(a_n) = (5.3749, 5.37501, 5.37499, 5.375001, 5.374999, 5.3750001, \ldots)$.

Solution: $R(a_n) = (5.4, 5.38, 5.375, 5.375, 5.375, \ldots)$; hence,

$DR(a_n) = (5.3, 5.37, 5.375, 5.375, 5.375, \ldots)$.

Therefore, $\lim (a_n) = 5.375$.

Example 6: Compute $\lim (a_n)$ where a_n is π rounded off to n decimal places.

Solution:

$(a_n) = (3.1, 3.14, 3.142, 3.1416, 3.14159, 3.141593, 3.1415927, \ldots)$;

hence

$R(a_n) = (3.1, 3.14, 3.142, 3.1416, 3.14159, 3.141593, 3.1415927, \ldots)$;

thus

$DR(a_n) = (3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, 3.1415926, \ldots)$,

Therefore, $\lim (a_n) = \pi$.

Example 7: Compute $\lim (a_n)$ where

$(a_n) = (-17.00049, -17.000501, -17.0004999, -17.00050001,$

$-17.000499999, \ldots)$.

Solution:

$R(a_n) = (-17, -17, -17.001, -17.0005, -17.0005, -17.0005, \ldots)$;

hence,

$DR(a_n) = (-17, -17, -17, -17.0005, -17.0005, -17.0005, \ldots)$.

Therefore, $\lim (a_n) = -17.0005$.

We state without proof an important aspect of our construction.

THEOREM 4.2.5: (a_n) is convergent iff $DR(a_n)$ defines a real number in the sense of the text.

THE 75th ANNUAL O.A.T.M. EASTER CONVENTION--APRIL, 1966.

Preparations for the Seventy-Fifth Annual Easter Convention of the Ontario Association of Teachers of Mathematics are well under way and things are beginning to fall into place. For the first time in its long history the Association will be holding its convention at the Inn-on-the-Park in Don Mills, a suburb of Toronto, to mark this auspicious occasion.

The dates of the convention are April 12th (Tuesday) and April 13th (Wednesday), 1966, though registration will commence on April 11th (Monday) at 7 p.m. It will be followed by a reception in the lounge of the hotel, when those present will have an opportunity to meet the guest speakers and socialize with friends. Light refreshments will be served at this function which is being organized by Dr. K. Fryer, University of Waterloo, and his committee.

Advance registration forms will be mailed to all teachers of mathematics in Ontario in the very near future. The committee will appreciate it if you could send them the required information as early as possible.

The two main speakers at the Easter convention will be Dr. Mary Dolciani, a graduate of Hunter Collete, N.Y., and Dr. W. W. Sawyer, a well-known mathematician, who is a member of the Department of Mathematics, University of Toronto.

Dr. Mary P. Dolciani is an Associate Professor of Mathematics at Hunter College, New York. In her many years of mathematics teaching she has held many positions of responsibility. She has been a member of the School Mathematics Study Group as well as a director and teacher in numerous National Science Foundation Institutes. She is at present a visiting secondary school lecturer for the Mathematical Association of America and is closely associated with the New York State Department Institutes for Mathematics Teachers. Dr. Dolciani is a well-known author of many books. In my opinion, she is a charming person, an excellent teacher of mathematics and is an interesting speaker. She is fully aware of the problems

confronting High School teachers at this critical juncture in curriculum development.

Dr. W. W. Sawyer distinguished himself while still at school by winning scholarships to Highgate Grammar School and St. John's College, Cambridge. At Cambridge, he specialized in the mathematics of quantum theory and relativity. With his colleagues he investigated the applications of mathematics to industry. From 1948 to 1950 he was Head of the Mathematics Department of University College, Gold Coast. He was lecturer in Mathematics at Canterbury College, Christchurch, New Zealand, from 1951 to 1956. From 1956 to 1958 he was Associate Professor at the University of Illinois. He has written a number of books on mathematics, which have been widely acclaimed. He edits the Mathematics Students Journal. Since September Dr. Sawyer has been on the staff of the University of Toronto, Department of Mathematics.

Once again, at this year's convention we will be holding many small-group seminars on a variety of interesting topics, details of which will be published later.

It is also our intention to organize the Mathematics Fair and to offer cash and/or book prizes for the best students projects in the six sections.

I am sure the members will be pleased to know that the accommodation at the Inn-on-the-Park is excellent. The room rates are competitive. The meals served at the Inn are not only excellent but are also reasonably priced. This luxurious modern hotel is located in one of the best suburbs of Metro Toronto and offers many special features. There is a parking lot large enough to accommodate one thousand cars. Your executive realizing that many wives would want to accompany their husbands to the convention has organized an interesting programme for them during their stay at the Hotel.

The banquet speaker for the occasion will be the Minister of Education.

J. W. Fencott, Publicity Representative.

WHITHER SCHOOL MATHEMATICS?

by Douglas H. Crawford
Queen's University

Introduction

Today, school mathematics is in a state of flux. In the U.S.A., many experimental programmes liberally financed by the Federal Government and other agencies are being tried out, Similar developments are taking place, though more gradually, in Canada and elsewhere. What is it all about, and what does it mean for the future of school mathematics in our country?

Mathematics as the study of abstract systems.

Let us look together at an apparently simple problem, and see what it can teach us about the nature of mathematics.

What is the sum of two and two?

Most of you will say "four", and write your solution as

$$2 + 2 = 4 .$$

Now suppose that I showed you my solution, and it looked like this

$$2 + 2 = 10 .$$

No doubt some of you would protest at this, yet what I have written can be defended. I would agree that two and two are four, but simply point out that I had decided to base my system of writing numbers on a group of four, and not ten as you are accustomed to. In my symbol language then, 10 means 1 group of four and no units!

Now, a mathematician friend of mine might confuse you still further by saying that in some circumstances, the sum of two and two is zero i.e. $2 + 2 = 0$.

If asked to explain, he would have to admit that he was thinking of a special kind of addition called modular addition modulo four. In such a system, we think of each number as equivalent to the remainder it leaves when divided by the

particular modulo, in this case four. Although the sum of two and two is four, four leaves no remainder when divided by four, and so

$$2 + 2 = 4 = 0 \text{ (modulo 4)} .$$

In still other circumstances, it is possible to have

$$2 + 2 = 2 .$$

In this case, we are using logical addition.

What can we learn from these examples? First and foremost, I think, we are made to realize that when we do mathematics, we are making assumptions. So when I write

$$2 + 2 = 4$$

I am using the decimal system of writing numbers, and I am assuming that you know this.

Secondly, by making a study of mathematical sentences like

$$2 + 2 = 4 , \quad 2 + 2 = 0 \quad \text{and} \quad 2 + 2 = 2$$

we learn that we can carry out different, yet similar operations on the same numbers, and obtain strikingly different results.

Let us consider two further examples, this time from the realm of algebra.

What are the solutions if any, of the equation

$$x + 1 = 0$$

where x stands for an unknown number?

If I asked a child of eight say, the chances are that he or she would say there was no such number x . Yet most of you would answer without hesitation that there is exactly one number x which fits these conditions and that it is -1 .

Suppose now the equation had been

$$x^2 + 1 = 0 .$$

Now, let us agree that we all know that the square of -5 is 25 , and that in general, no negative number can have a

square less than 0 . Are there any solutions to the equation?

The answer is, "yes, there are two, provided that we introduce a new kind of number called an imaginary number into our thinking"!

What can we learn from this second group of examples? Firstly, that the limits of mathematics recede as we learn more about the subject, second that, in asking whether solutions exist, the answer depends on the set of numbers we have in mind.

$x + 1 = 0$ has no solution, if we are thinking of the set of counting numbers 1,2,3, and so on, but it has if we are thinking of the set $0, \pm 1, \pm 2,$ and so on.

To a mathematician, the key words in the above discussion are SETS, NUMBERS, OPERATIONS.

In arithmetic and algebra we consider certain sets of numbers, and carry out certain operations upon them. Consideration of these operations shows us that they possess certain properties. Thus, addition in the set of counting numbers is commutative. Adding x to y gives the same result as adding y to x .

As mathematics has expanded in the last century or more, increasing attention has been focused on these and related ideas. Mathematicians have built up new logical systems with unspecified objects called elements (replacing numbers) and operations with well-defined properties based on those of the four arithmetical operations, yet differing according to the assumptions that the mathematician makes.

At the same time, the study of geometry has shown that what we can prove depends on what we assume, so that in most cases the systems we study may have no immediately obvious relation to the real world in the way they are set up. They are abstract systems created out of assumptions, elements, operations and logical reasoning.

The real problem for school mathematics is - how much of this study of structure, as opposed to computation, should there be, and at what levels should it become explicit? The hope is, that the ability to compute will arise as a by-product

of the understanding of structure.

To give some guide to where these new ideas and emphases are occurring in revised programmes in Canada, I could say roughly that

(i) different notational or numeration systems for numbers (2 + 2 = 10 etc.) are appearing by Grade 7;

(ii) explicit study of sets of numbers and the properties of addition, subtraction, etc. are also appearing from Grade 7 up;

(iii) algebraic solution of equations arises from Grade 9 up;

(iv) new kinds of numbers and operations which lead to the idea of mathematical system tend not to appear till the last year of high school.

But remember, the seeds of all these ideas are being planted in even the kindergarten, through the use of concrete materials, used to give a structural emphasis.

Learning and Teaching Mathematics

Regardless of the content of the mathematics curriculum, there arises the problem of how best to teach it. This aspect of school mathematics is also the subject of much inquiry. Traditionally, mathematics has been taught mainly by the expository or lecture method, and to a large extent, the pupil has been shown rules, and told to follow them, even when he didn't understand them. Of course, this is a gross over-simplification of the situation, but there is a fair measure of truth in this assertion.

In any case, there has been a strong reaction against this view. Mathematicians contend that students should discover mathematics, for only in this way, will they learn to do mathematics. Psychologists have conducted fascinating experiments which show conclusively that learning by rote is less effective, in terms of both immediate and long term recall. In addition, there is impressive evidence that students taught by rote methods develop poor attitudes to mathematics whereas those taught by discovery methods develop good attitudes.

The case for a discovery approach is strong then, yet there remain many limitations to its use, both practical and theoretical, which means that we must continue to search for improved learning and teaching.

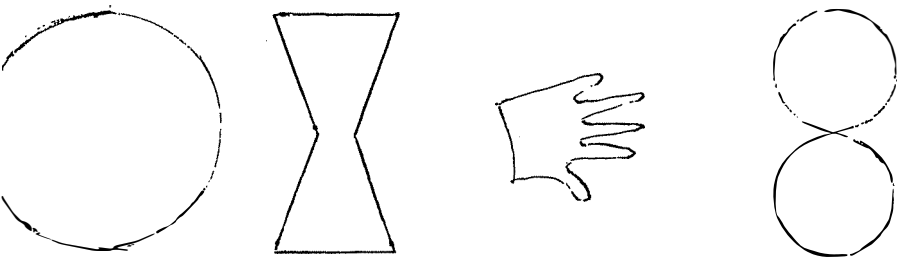
What about Geometry?

In closing, let me try to give a glimpse of where I believe geometry fits into these new trends. First, although this is still in the future, I think I can predict fairly confidently that geometry will receive much more attention in the elementary grades. As a balancing mechanism to the trend to structure and abstraction in arithmetic, and from other points of view, particularly the need for a sound knowledge of figures and solids, and their basic properties, it is important for children to handle, construct, and measure geometric objects. Much can be learned using elastic bands and a lattice or peg board on which polygons of different kinds can be made and studied. Models of cubes, cylinders, etc. also generate great interest, and are the raw materials out of which bridges, tunnels, and airplanes are constructed.

Little of this is "new" mathematics - rather it has been neglected mathematics. Yet, it does lead to the study of analytical geometry and vectors and through these to a more modern wedding of algebra and geometry in the study of vector spaces and linear algebra.

At high school level, a more logical and deductive study of Euclid's Geometry, relieved of some of the flaws and inconsistencies persisting until now, still affords the student his first detailed look at a sizeable sequence of mathematical theorems based on certain stated assumptions, and proved by rigorous deductive reasoning.

Finally, and perhaps most exciting of all, there is the prospect of at least a taste of the new branch of geometry known as topology. In this geometry, we concentrate on very general properties of geometric figures, namely those associated with the idea of continuity.



1 2 3 4

Topologically only one of the four curves above is different from the rest. Which is it? To find out, imagine that each has been drawn in chalk on a rubber sheet. By pulling and pushing your sheet could you distort curve 1 to become curve 2 ? to become curve 3 , curve 4 ? The answer is that curves 1, 2, and 3 are topologically equivalent while curve 4 belongs to a different "genus". One way of seeing this is to notice that in each of the first three curves, a given point A can be in one of only three regions - namely, inside the curve, on it, or outside it. What about curve 4 ?

It is still not certain where this kind of geometry fits into school mathematics, but that it is important mathematics, there is little doubt. It is comforting to know at any rate that, topologically, coffee cups and doughnuts go together. Perhaps after trying to struggle through this article, you would rather use them in a non-mathematical way!*

Conclusion

In this short article, emphasis has been placed on what the new trends in school mathematics are. It does not follow that traditional mathematics leading to the study of calculus and other established branches of mathematics is to be neglected. What is needed is a balance between the "inward-looking" mathematics of structure and the "outward looking" mathematics of application.

*If you want to know more, buy and read "Mathematics" (Life Science Library).

For many children, a more concrete and applied approach to mathematics may be preferable, and it is tempting to argue that "this abstract structural stuff" should only be for the best pupils. We simply do not know the best curriculum for each child. But we must adapt our curricula to the needs of society, too. The answer lies in more research and experimentation.

Our world is changing rapidly. We have automatic machines of all kinds to do our routine work and calculation. If we are to be masters of these machines, then we must strive to learn ideas and not only facts; relationships and not only formulas. Yet we must ensure that we make changes sensibly, testing as we go, and, in the process, fit the education to the child.

ONTARIO MATHEMATICS COMMISSION REPORT

The members of the Ontario Mathematics Commission have been very active during this past year trying to fulfil the objectives set out in its constitution. In particular the one objective that appears to have had the most concentration is "Undertaking curriculum research aimed at keeping the province abreast of the best contemporary practice, and circulating such information to interested bodies".

The two most active committees are the "Advisory Committee on Elementary Education" and the "Advisory Committee on Secondary Education". The former of these two committees submitted a brief on teacher training to the Minister's Committee on Teacher Training. During the year the Elementary Committee has given much thought and discussion to the mathematics designed for Kindergarten to Grade 6 level. With the help of the Ontario Curriculum Institute a summer study group was formed to consider the sequence of topics taught in elementary schools throughout Canada, United States and several European Countries. A report on this study will be published by the Ontario Curriculum Institute and should be available before the end of 1965. This report should have an influence on future courses in mathematics designed for the elementary school.

The Secondary Education Committee has also had a busy year. Recommendations were made a year ago to the Department of Education to have the following topics included in the Problems paper for 1965, Complex Numbers, Probability, Logic and Polar Coordinates. Copies of brochures dealing with these four topics were distributed to all schools without charge and additional copies were sold at a nominal fee. The committee has again made recommendations to the Department of Education to have Calculus, Polar Coordinates and Probability included in the Problems paper for 1966. These recommendations have also been accepted. Copies of brochures written for these three topics may also be obtained by writing to the Ontario Mathematics Commission office. A subcommittee has prepared a brief relative to the General Level

grade 13 Mathematics Course that is suggested for the future. A study committee set up by the Secondary Education Committee also received a grant from the Ontario Curriculum Institute to make a study of the mathematical needs of the four-year non-university bound student. Their report will also be available before the end of 1965 and should be read by every secondary school mathematics teacher.

The topics of mathematics and their uses required to follow the technological advances in the world are changing continuously. It is essential to have an organization like the Ontario Mathematics Commission to follow these changes and keep the teachers informed. Since its formation, some of its aims and objectives, as well as its operation, have changed. With this in mind an enlarged constitution committee has been set up to bring the constitution up-to-date. Be sure that the representative from your organization or a substitute for him is present at the next general meeting to make the revised constitution effective. The next meeting will be held on February 19th, 1966.

J. Norman C. Sharp,
O.M.C. President.

GEOMETRY: WHAT APPROACH?

H. A. Elliott - Royal Military College

It is six years since Professor Dieudonné used the slogan "Euclid must go" in his address to the Royaumont conference on mathematics sponsored by the Organization for European Economic Cooperation.* This was frequently interpreted to mean that high school geometry must go but Dieudonné in the same address stated "My quarrel is therefore not with the purpose but with the methods (his italics) of teaching geometry, and my chief claim is that it would be much better to base that teaching not on artificial notions and results which have no significance in most applications, but on the basic notions which will command and illuminate every question in which geometry intervenes."** The argument about high school geometry is then not whether geometry, and the many familiar properties, should be learnt, but how geometry should be taught. In fact the discussion started many years before 1959 but the present situation in mathematics has accentuated the need for a decision on the method to be used in contemporary teaching.

The discussion on the place of Euclid in the schools was already raging in full force at the beginning of the century. In 1909 the English Board of Education issued a regulation on Teaching Geometry and Graphic Algebra in Schools. This proposed that great emphasis should be placed on intuitive geometry and only when the pupils were old enough and clever enough (about 14 years old in Grammar School) should they have to learn geometry in strict logic. Similar recommendations were made in other European countries. In Germany proposals went further with the suggestions of the Erlangen

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* This has now been succeeded in the NATO Organization by the Organization for Economic and Cultural Development.

** New Thinking in School Mathematics, Organization for European Economic Coooperation 1960, Queen's Printer, Ottawa.

program. This program is the forerunner of the present dynamic (motion or bewegungs) geometry.

In the 1923 Report of the Mathematics Association of Great Britain, The Teaching of Geometry in Schools, great stress was laid on a gradual and progressive development of geometry throug all the school grades. It is interesting to note that this report stressed that "there ought to be no formal separation between plane and solid geometry" and that "symmetry should have a larger place in teaching than has been usual". Neither of these recommendations has been adequately implemented here in Canada even yet, and possibly the main reason is the conservatism and inertia of the teachers of mathematics.

The Second Report on the Teaching of Geometry in Schools by the Mathematical Association in 1938 re-emphasized most of the conclusions of the earlier report and based this re-emphasis on the fifteen years of experience since the earlier report. In particular it stressed the need for starting with an experimental and intuitive approach up to the age of fourteen. It followed this with a second stage when it was recommended that the pupil should learn to prove a few closely associated theorems based on a selection of more fundamental theorems which are to be assumed. In the third stage the isolated groups of deductively proved results are systematised into one coherent pattern. Only after this is "the organisation of the primitive propositions" (the congruency, parallelism and similarity basic theorems) and basic axioms considered suitable. Even this is accompanied by the statement, "experience has shown that (this last stage) is suitable only for top divisions and not for the majority of boys". As the total set of boys (and girls) considered are those in academic grammar schools, about the top 15%, this last statement is a very severe restriction.

Another comment made at the same time on The Teaching of Mathematics in Secondary Schools* was "although algebra,

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* Mathematical Gazette Vol. XXV, No. 258, 1940, G. Bell and Sons Ltd., London.

trigonometry, the use of graphs, the calculus all depend on the rules of Greek Geometry, scarcely more than a dozen from the two hundred propositions of Euclid's elements are essential to help us in understanding how to use them. The remainder are complicated ways of doing things which can be done much more simply when we know later branches of mathematics... the person who wants to understand the place of mathematics in modern civilisation is merely distracted and disheartened by them."

The controversy reached the press in Canada about three years ago when Professor Coleman of Queen's was quoted as predicting "that geometry as a special subject will eventually be eliminated from the curriculum (as) in England and Denmark it already had been eliminated."* Again this was wrongly interpreted by many to mean that geometry must go. Professor Coxeter of Toronto and one of the world's foremost geometers, put the case for geometry in a letter** "let us teach our children some of the wonderful discoveries that have been made in geometry (as well as in algebra and analysis) since 1800, stressing the fact that geometry describes an important aspect of the world in which we all live; but let us hear no more about "eliminating" a subject that has been loved for its own sake, and applied in many directions, for at least 2,500 years." Again the emphasis is on the enrichment of geometry beyond the traditional content. Professor Porter of Toronto speaking at the same meeting as Professor Coleman pointed out that "engineering is drawing and would suffer greatly if geometry were eliminated".* Professor Coxeter adds "he might just as well have spoken also for the architects, physicists, chemists, and crystallographers, all of whom make essential use of geometry."** What Professor Porter did not point out was that much of the geometry of interest to the engineer, for example Desargue's theorem, is not, in fact, taught in

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* Globe and Mail, 7 March, 1962, Toronto, Ont.

** Globe and Mail, 10 March, 1962, Toronto, Ont.

high schools. Also three dimensional geometry is almost an unknown subject to the academic high school student of mathematics, though as engineering drawing some is taught to technical students. Obviously again the need expressed is for more geometry, not less, and for geometry taught in such a way that the needs of both contemporary mathematics and contemporary science and technology are met.

What then are the possible choices for the teaching of geometry? Basically the following possibilities exist:

(i) Euclidean geometry may be preserved in its traditional state on a semi-axiomatic basis,

(ii) Euclidean geometry may be modified and its basic axioms revised to be more rigorous,

(iii) geometry may be reduced to the discovery of properties solely by measurement and experiment and then these generalized intuitively,

(iv) geometry may be down graded to a purely descriptive study with some results "proved" in analytical methods,

(v) the concepts of vectors and transformations may be developed intuitively as a starting point for "proofs" by the use of such concepts.

Other possibilities which would attempt a completely rigorous approach are not listed here as they are considered to be completely unsuitable and impracticable for school use. Let us examine the five possibilities listed above.

The first possibility has familiarity in its favour if only two dimensional properties are taught. It is an old friend and provided that excessive emphasis on quoting theorems by number and book is avoided it has pedagogical value. It preserves the proofs of the congruency results by superposition, the undesirability of which is not a matter of universal agreement among teachers of mathematics. For example, Honsberger* claims that "the banishment of superposition robs congruence

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* Ontario Mathematics Gazette Vol 3, No. 4, p. 27, February 1965.

of its only sensible grade ten meaning by blanking out 'equal distances'." Also Berman* states; "This method (superposition) is actually very desirable since in more advanced work they will consider Euclidean geometry as the study of geometric figures which remain fixed when they undergo rigid motion." As will be seen later a vector and transformation approach essentially reinstates superposition in modern dress and somewhat more formally.

However three dimensional geometry is very difficult taught from the traditional Euclidean approach and it would seem unlikely that its study could be introduced in the schools except as purely descriptive treatment with some mensuration. This is a grave disadvantage to the technical and general student alike. Also the methods of proof and most of the concepts involved in the traditional courses do not lay an adequate foundation on which future mathematicians and physical, biological or social scientists can build the more modern mathematics needed in their later studies. Too many of the two dimensional metric properties on which we concentrate do not generalise or transfer to other surfaces or dimensions.

The second possibility is essentially that pursued by the School Mathematics Study Group texts and many other experimental programmes in North America. The geometry sections of the new mathematics curricula introduced in Ontario in Grades 7, 8 and 9 and Grades 11 and 12 follow the same trend with some small modifications. Congruency proofs by superposition are replaced by the so-called congruency axioms, which are more correctly assumptions as they are not independent axioms. Attempts are made to cover some of the gaps which the traditional Euclidean treatments leave in the topics of "betweeness" and order. The language of sets is used, sometimes in such a way as to obscure the simplicity of the concept. This sometimes unnecessary use of the language of sets is regarded adversely by some mathematicians.

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* Ten Refresher Lectures in Modern Mathematics, Stanton and Fryer, University of Waterloo, 1960.

Geometers themselves have raised the additional argument that the S.M.S.G. approach sometimes replaces purely geometrical concepts by mensuration concepts in terms of a unit of measurement. The example most frequently quoted is that by which the algebraic assumption that $b \times h$ is equal to the area of a rectangle is used to prove the theorem that two triangles on the same base and with their other vertices on a line parallel to that base are equal in area. This theorem can be proved without any such assumption about the numerical measurement of an area.

However a more vital criticism is that the S.M.S.G. type of approach does not ease the problems of three dimensional geometry and its only contribution to the background of the future university student is a partial presentation of an axiomatic approach. In fairness it must be stated that the S.M.S.G. texts themselves are more thorough than most of their imitators in their treatment of the basic axioms, and they do more three dimensional work. Apparently this more rigorous treatment has proved to be too difficult and abstract for most teachers and students. This is in line with the British experience mentioned earlier.

Perhaps the third possibility might be described as the coward's way out. However, it can meet most of the immediate needs of the average citizen who needs only practical geometry. It also has the advantage that it is possible by measurement to discover many three dimensional results as well. Even Desargue's theorem in both two and three dimensions could be illustrated with models though whether a pupil would really believe that it was always true without exception is difficult to decide! But such an approach does nothing to forward the mathematical education of the pupil in the sense that he learns no new mathematical concepts, methods or principles. In an age which will depend more and more on the mathematical and scientific literacy of its citizens this is a serious shortcoming. Even if the man-in-the-street cannot do the mathematics for himself, he should be able to read it -- and one hopes appreciate it --

as much as he does history, geography or literature.

The fourth possibility suffers from the disadvantage that it is limited to two dimensions in earlier years because of difficulty with a three dimensional coordinate system. Even two dimensional work must be delayed until a coordinate system has been developed in algebra. It also suffers from the disadvantage that geometry loses its separate identity and becomes part of algebra. The "proofs" of many results are rather difficult algebraically in coordinate geometry and it is difficult to base them on truly rigorous foundations. This is especially true for those proofs involving angle properties which require a considerable knowledge of trigonometry also.

The fifth and final possibility uses the basic ideas of motion and directed lengths, and children appear to have little difficulty with these concepts if they are introduced first in concrete situations. Symmetry, reflection, translation and rotation are the fundamentals of this treatment. One important advantage of this approach is that the extension to three dimensions is much simpler than in the case of Euclidean methods. Another advantage to the university bound student is that the language and ideas do form a sound intuitive and experimental base for more advanced work in linear algebra, especially vectors and matrices.

H. M. Cundy* mathematics master at Sherbourne School states a case for this approach as follows: "This...approach leads naturally to the idea of a vector - the embodiment of a translation, gives a number of examples of groups and lends itself easily to the development of the ideas of half-line (ray) and half plane required for linear programming. Similarity transformations (including congruency transformations) link it up with mensuration and arithmetic."

In fact this fifth approach is really the logical conclusion to the studies of the Mathematical Association mentioned earlier.

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* Mathematical Gazette Vol XLVII, No. 362, p. 333, 1963. G. Bell and Sons Ltd., London.

The early approach is of necessity experimental. Translation and rotation are literally the translation and rotation of models and drawings--in fact as Dieudonné remarks "...much research and experimentation has lately been going on in educational circles (especially in Belgium) concerning the methods by which this teaching of geometry as a part of physics, so to speak, can be conducted. I think this development should be highly encouraged, provided that it puts the emphasis not on such artificial things as triangles but on basic notions such as symmetries, translations, composition of transformations, etc.". Most teachers would disagree with the idea that triangles are artificial, but there is no reason why both triangles and the other factors cannot be emphasised! Dieudonné also emphasises firmly that in this first stage (pre 14 years of age) "no attempt (should be made) at axiomatisation". In all this experimental work it must be remembered that three dimensional studies are considered to be as important as two dimensional work. This parity is quite possible to achieve even in constructions, if the straight edge and compass restriction of Euclid is deleted, and no engineering draughtsman would retain it anyway!

In the second stage it is probable that trigonometry is not yet available to the student and so the more formal study is confined to the addition and subtraction of vectors and multiplication by scalars. This restriction automatically selects some of the theorems and properties of Euclidean geometry and leaves others to a later stage. Congruency could be based on two assumptions,

(i) two triangles are congruent if a sequence of simple translations can superimpose one on the other,

(ii) two triangles with a common vertex are congruent if a sequence of simple rotations about that vertex can superimpose one on the other.

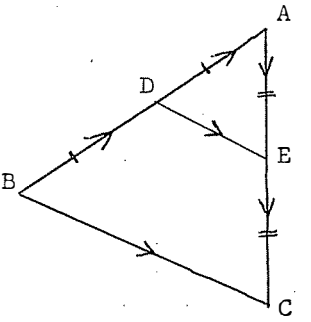
A <u>simple translation</u> is a translation in a given direction which leaves the distances between any pairs of corresponding points in space invariant (constant).

A <u>simple rotation</u> is a rotation about a given point which

leaves the angles between any pairs of corresponding rays invariant (constant).

In fact we might replace the word triangles by geometrical figures (plane or solid) and give a general definition of congruency. Other suitable assumptions can be introduced to permit a wide coverage of Euclidean topics in both two and three dimensions. Also, coordinate systems will have been introduced into algebra by this stage and the two topics can be linked together. The ease with which this can be done will of course depend on how much intuitive work has been done experimentally in geometry and with graphs of sets of points on squared paper in algebra in earlier grades. As an example of a vectorial treatment consider the following deduction:

Question: In ΔABC, D is the mid point of AB and E of AC, prove that DE is parallel to and equal to half of BC.



Proof:

$\vec{DE} = \vec{DA} + \vec{AE}$ (addition rule)

$\vec{DA} = \frac{1}{2}\vec{BA}$ (given)

$\vec{AE} = \frac{1}{2}\vec{AC}$ (given)

$\therefore \vec{DE} = \frac{1}{2}\vec{BA} + \frac{1}{2}\vec{AC}$

$= \frac{1}{2}(\vec{BA} + \vec{AC})$

$= \frac{1}{2}\vec{BC}$ (addition rule)

$\therefore$ DE is parallel to BC and equal to half of BC.

At the next stage multiplication of vectors and possibly a more axiomatic approach may be used.

The second stage might well take place at the grade 9 or 10 level, while the third stage would be at grade 11 or 12.

Although the above suggestions for the second and third stages are discussed in terms of vectors and Euclidean geometry, the experimental first stage also leads easily into projective geometry and may use transformations other than simple translations and rotations. With today's projectors, other visual aids and transparent plastics, the projective relationships

between different figures is easily demonstrated. For example it can be shown that ellipses and circles are projectively related and the pupil can discover what geometrical properties for circles are true also for ellipses. Circles can also be transformed into ellipses by stretching transformations, and invariant properties can be discovered in this case also.

In conclusion it is felt that a search for a more rational approach, a simpler approach and a more modern mathematical approach to geometry in the schools leads to one solution: an approach utilising the ideas of translation, rotation, reflection and symmetry together with the algebra of vectors. A short bibliography on these topics is appended below and the works listed include many references to further sources.

BIBLIOGRAPHY

1. Report of the Commission on Mathematics, Appendices 10, 17, 19 and 20, College Entrance Examination Board, New York, 1959.
2. Enrichment Mathematics for the Grades, Chapter 23, Twenty-seventh Year Book, National Council of Teachers, Washington, 1963.
3. An Introduction to Vectors, Ontario Mathematics Gazette, Ontario Mathematics Commission, Toronto, Vol 3, No. 2, 1964.

REPORT RE: THE SCARBOROUGH "SATURDAY PROBLEMS CLASSES" (JANUARY TO MAY, 1965)

The purpose of these classes was threefold:

(1) To challenge the mathematically gifted student;

(2) To provide adequate preparation for students attempting the Departmental Problems Paper;

(3) To provide "In-Service" training for the less experienced teachers in Grade 13 Mathematics beyond the regular course of study.

The classes were held at Winston Churchill C.I. on Saturday mornings from 9:30 to 11:30 a.m., commencing January 9 and concluding May 8, 1965.

Each session took either of two forms:

(1) A two-hour lecture by a university professor, or

(2) A one-hour lecture by a Scarborough teacher and a one-hour "tutorial" in which 5 or 6 small groups attacked problems under the guidance of a Scarborough teacher; (one or more teachers per group).

The attendance of both students and teachers was gratifying. Approximately 80 students attended the opening lecture on "Diophantine Equations", delivered by Professor R.G. Stanton, University of Waterloo. At least 50 students attended regularly throughout.

The other visiting lecturers, and their topics, were as follows:

Prof. D. A. Clarke, University of Toronto, "Logic";

Prof. D. C. Russell, York University, "Topology";

Prof. Paul Wakfer, Scarborough College, "Complex Numbers";

Prof. J. Kalbfleisch, University of Waterloo, "Probability";

Prof. F. A. Sherk, University of Toronto, "Limits".

The students responded well to this series of special classes, and comments of the visiting lecturers were favourable. A comparative study of 1964 and 1965 examination results also indicates that these classes were worthwhile.

In June, 1964, Nineteen Scarborough students sat for the Problems Paper. The marks ranged from 10 to 52, and the average was 26.0; whereas in June, 1965, thirty-five students wrote the paper, the range went from 2 to 70, and the average 21.8.

The following points would seem to be directly related to the Problems classes:

(1) Almost twice as many Scarborough students wrote Problems in 1965 as in 1964. (This was not true throughout the province; 1072 students wrote in 1965; 961 in 1964. The difference is comparatively small.)

(2) The high mark of 70 (1965) is outstanding. (provincial figures show that only 12 students out of 1072 - slightly more than 1% - obtained a mark exceeding 67.)

(3) The lower average in 1965 obtained by the Scarborough students is due to the fact that 8 students received less than 10 marks. It is reasonable to conclude that these students would not have attempted the paper in previous years. If these are excluded, the average mark for 1965 increases to 27.0. (It is noteworthy that the Provincial average for 1965 was slightly lower than that of 1964.)

(4) Only one Scarborough student obtained a mark in excess of 49 in 1964, whereas three did so in 1965. (There is no "pass mark" in the Problems paper. This year only the top 10% of all students throughout the province obtained a mark in excess of 40.)

Furthermore, it was noted that the percentage of "Firsts" in Mathematics papers, obtained by Scarborough students, was significantly higher in 1965 than in 1964.

Every effort will be made to launch a similar but more effective programme of special classes or seminars in 1965 - 1966. A committee has already been appointed by the Mathematics Department Heads in Scarborough to work out the details of this project.

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WESTERN ONTARIO REGIONAL MEETING

OF ONTARIO ASSOCIATION OF TEACHERS OF MATHEMATICS

Place - New Althouse College of Education, London

(corner of Western Road and Sarnia Gravel, just west of U.W.O. campus. Enter from Western Road - Plenty of parking in rear of College).

Time - 10 am. - 3 pm. Saturday, January 22, 1966.

PROGRAMME

10.00 to 10.30 - Registration and Coffee

10.30 to 11.30 - Vectors in Three Dimensional Geometry - Dr. D. A. Clarke, University of Toronto.

Dr. Clarke was a member of the committee which prepared the new Grade XIII course, and will present valuable background material for this Grade XIII topic.

11.30 to 12.30 - Discussion seminars on the implications of this topic for the Grade XIII next year.

12.30 to 2.00 - Dinner at Althouse College and informal inspection of the new building.

2.00 to 3.00 - Seminar discussion groups on the following:

a - 4 year programmes in Mathematics

b. - 5 year Grade 9 and 10 courses

c - 5 year Grade 11 and 12 courses

All Mathematics teachers and those interested in teaching Mathematics are cordially invited to attend.

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