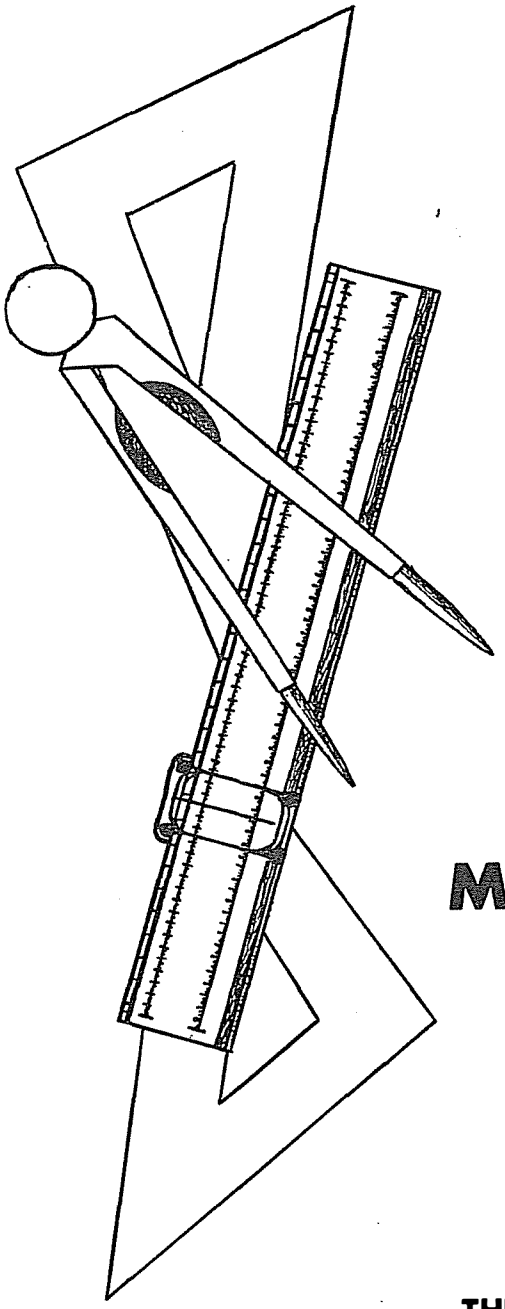


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ONTARIO MATHEMATICS GAZETTE

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THE ONTARIO MATHEMATICS COMMISSION
THE ONTARIO ASSOCIATION OF
TEACHERS OF MATHEMATICS

EDITORIAL

The recent budget of the Ontario Legislature is an indication of the current explosion in education. While expansion and quantity are the chief reasons for the budgetary amounts, the problems are compounded by the urgent need for continuing developments in curriculum reform and improved teaching methods, with associated implications in relation to teacher-training and updating.

In order to best utilize our resources it is essential to have good communications between the various organs of education (e.g. the School Systems, the Department, the Universities, the Curriculum Institute, and the recently formed Ontario Institute for Studies in Education). There is evidence that communication between these groups is improving compared with the past, but in some instances a limiting factor is a general lack of knowledge by members of one group of the purpose and functions of other educational organizations. Such knowledge is essential, especially in view of the almost terrifying rate at which some of these organizations are expanding or changing.

In this somewhat reflective issue of the Gazette, we present the first of a series of articles by which we hope the various institutions of education may be better understood. Our view is that criticism should not be censored, but that it should be informed.

Mr. Frank Kinlin, a member of the Policy and Development Council on education, recently appointed by the Minister, has written an article pertaining to the

Department of Education, and in particular its recent history in connection with mathematics curriculum reform. He has qualifications for this task which are exceptional. He has had experience as teacher and inspector in both the elementary and secondary school systems, and was one of the early mathematics Inspectors for the province. He participated in the now historical but momentous Lakefield meeting on mathematics reform (1959), attended by representatives of the Schools, the Universities, and the Department. Later in the Curriculum Branch his courage and vigour were not unimportant reasons why we are not five additional years behind in experience at evaluating and improving our schools.

It is to be hoped that cooperation of the type initiated by Mr. Kinlin will grow and deepen in the years to come.

J.E.H

EIGHT YEARS IN MATHEMATICS WITH THE DEPARTMENT OF EDUCATION

by

J. F. Kinlin,

Member, Policy and Development Council

I consider that I have been most fortunate to have been involved in mathematics with the Department for the period August 1, 1957 to April 30, 1965. In a world giving promise of a still greater acceleration of changes, one would be foolhardy to claim that the particular time represented will go down in history as the period of greater change in mathematics in Ontario. However, for the short term, and in the restricted point of view, there will probably be no argument that the changes, as far as the Department is concerned, have been more significant and more rapidly developing than in any previous eight years.

Peculiarly, the same eight years encompass a period of great change within the Department itself, and it may be interesting to dwell on the possibility of inter-relationships or even to carry out a matching of the changes in an attempt to determine which was cause and which was effect.

The first significant Departmental change in the period was the adoption in late 1956 of the policy of appointing staff inspectors for academic subjects. It is true that the policy applied to all subjects, but the first appointments for 1957 included two for mathe-

matics (Mr. A.W. Bishop and me) and one for each of Moderns and English -- and mathematics has always involved a greater number of persons than any other academic subject. There is no doubt that mathematics in Ontario profited from this concentration of attention. However, the interesting point for conjecture is whether the mid-fifties stirrings in mathematics pointed up the need of staff inspectors in that subject and led to the adoption of this new Departmental policy.

Another new Departmental policy in which Mathematics had a pioneering role is that of integration of elementary and secondary education. Although not indicated at the time as being for the purpose of integration and not necessarily productive of any achievements toward that objective, Mathematics has an early place in the records of the development in that my appointment in 1957 was the first case when an elementary inspector became an inspector of academic subjects in secondary schools. Again, in the Curriculum Division where integration was a stated objective as early as 1960, Mathematics scored early in integration when, beginning in 1962 one person became responsible for and operated completely in the full range of Kindergarten to Grade 13. Perhaps it was by chance or perhaps it was by its very nature that Mathematics took its place in the vanguard of this movement toward the worthy objective of integration of elementary and secondary education.

When the educational history of Ontario is written, that part dealing with the early sixties will certainly feature a development of active cooperation and close

collaboration between the Department and educational associations established by organizations outside the Department. The example given will likely be the Ontario Curriculum Institute or perhaps the newly-formed Ontario Institute of Studies in Education; bodies with aims which, to the uninitiated, might suggest competition with the Department, and yet supported by the Department by grants and having Departmental personnel on the governing executives. Yet it was the Ontario Mathematics Commission which broke new ground leading to this type of cooperation. It was Dr. A.J. Coleman who, in early 1960, in negotiation with the then Chief Director of Education, cleared the way for full-fledged membership by Departmental personnel on the Commission and for yearly Departmental grants to the Commission, and thereby set the stage for this new relationship between the Department and non-Departmental bodies. Only those close to the situation at that time can appreciate the difficulties associated with the development of this cooperation between the Department and the Commission. Such a development caused surprise outside the Department and consternation within. No such thing had ever been attempted before and acceptance (or tolerance) of the situation by certain senior officials required considerable tact and diplomacy. Timing of approaches, forms of presentations, even strategy in keeping certain persons apart at crucial times - all required careful planning. It was providential in those delicate times that Frank Asbury was on the Commission's executive. Perhaps in my position in the Department and at the

same time on the Commission's executive, I am in the best position to appreciate the total contribution made to this achievement by his good judgment and his policy of frank and continual consultation.

Everyone now accepts the value of field pre-trials of curriculum materials. I submit that it was the work of the Ontario Mathematics Commission, in co-operation with the Department, which set the pattern in Ontario for field pre-trials of curriculum materials. Furthermore, it provided a workable and effective scheme for a feed-back from the trials to the Department and the Commission, and a utilization of feed-back in the developments of further curriculum materials by non-Departmental agencies, and in the preparation of new course outlines by Departmental Committees. The actual extent of these pre-trials is indicated by the following list of number of schools participating:

1959-60: 19 schools in Grade 9

1960-61: 37 schools in Grade 9

1961-62: 35 schools in Grade 9;
17 schools in Grade 10

1962-63: 23 schools in Grade 10;
12 schools in Grade 11

1963-64: 10 schools in Grade 7;
25 schools in Grade 10
18 schools in Grade 11
7 schools in Grade 12

1964-65: 6 schools in Grade 12; several schools on specified Problems material in Grade 13

1965-66: Several schools on specified Problems material in Grade 13

The procedures of the committees preparing mathematics courses also instituted a significant new practice in Department Committee work, which is now showing signs of acceptance as Departmental policy. Previously, the deliberations of curriculum committees were treated as strictly confidential and the thinking and plans of the committees remained secret from the teachers until the receipt of the printed course outline. The new practice was instituted in 1961 when Mr. Bishop and I interviewed approximately 1,000 mathematics teachers, singly and in groups, and reported to them the proposals of the RP 12 committee in order to elicit reactions and opinions for referral back to the working committee. This practice was continued on a broader front with the later I 12 and S 12 committees in that the members were encouraged to discuss committee proposals with their confreres. This practice justifies itself not only because the confreres' reactions were helpful in committee deliberations, but also because teachers had some prior knowledge of possible content and were able to up-date their knowledge on the topics long before the course outlines were released.

Traditionally, Departmental courses of study were developed to last, and in some cases, course have lasted for at least a generation. It is now generally accepted that such an attitude slows down innovations dictated by a rapidly-changing world and even delays necessary and obvious adjustments. Mathematics courses showed the way in this modern thinking by a built-in implication of short duration. It is true that the necessity of early revision of the new Grade 7 course and of the new Grade

11 course, because of changing courses prior to these levels, resulted from the accident of a too-late start which dictated concurrent rather than completely consequent revisions. However, despite the lack of idealism in the motivation, the short-term character of these courses has shown the way to a new outlook in Departmental course development.

There is some evidence that the increasing rapidity of curriculum decisions in mathematics is changing the outside image of the Department and the inside image of the Curriculum Division. At the risk of appearing prosaic, I am recording some of these decisions in mathematics over recent years, not only to indicate number but also to allow the accelerating rate to presage the future.

1. June 2, 1958 - the decision (momentous at that time) to accept Geometry in Grade 11, and Algebra in Grade 12 as standard sequence (Remember the years of controversy on this question!)
2. November 1961 - The decision to revise the courses of Grades 11, 12 and 13, to provide a two-paper Grade 13 (after study of the question by a 10-member committee)
3. March 1962 - approval of the new Curriculum RP 12 for mathematics (particularly significant because it allowed new approach concepts at Grade 9 on an optional basis)
4. January 1963 - Setting up of two Curriculum Committees for revision of 11 - 12 - 13 and of 7 - 8 - 9 - 10 respectively
5. September 1963 - Approval of Curriculum S 12 A (Grade 11) for compulsory use beginning in September 1964 (Distributed November 1963)

6. October 1963 - Approval of Curriculum I12 A (Grade 7) for optional use beginning in September 1964 (Distributed February 1964)
7. April 1964 - Approval of Curriculum I12 B (Grade 8) for optional use in Grade 8 after successful completion of Curriculum I12 A (Distributed July 1964)
8. June 1964 - Approval of Curriculum S12 B (Grade 12) for compulsory use in Grade 12 beginning in September 1965 (Distributed September 1964)
9. Spring of 1965 - Approvals of Curriculum I12 C (Grade 9) and S12 C (Grade 13)

The records indicate some accomplishments, but the task is far from complete. Course outlines are needed for sections still untouched. Revision of new courses will be needed even if the school situation remained static, and it should not, and one would be unpardonably naive to think that new course outlines mean new insights for pupils. The successful use of new ideas requires a continual process of teacher up-dating. There is no doubt that old traditions of procedure will need to be broken. Perhaps the pattern set by the Department in regard to the problem in Grades 7 mathematics points the way. Only those close to the situation will be able to appreciate how drastic some of the actions really were for the Department. First of all, a long-hallowed tradition was broken when a certificate course was offered in the summer of 1963 for teachers of Grade 7 and 8 before there was an officially-approved course of study! Secondly, in 1963 and 1964, the traditional Departmental dividing lines separating Teacher Education, Professional Development, and Curriculum Branches were broken as never before in a pooling of total resources directed toward assistance to teachers in courses

organized by inspectors and by Federation groups! Finally, in April 1964, all of the above-mentioned Departmental Branches combined forces in an epochal short course for elementary school inspectors, the first time in the memory of the oldest inspectors that the Department had offered such a program to assit them in their work of improving instruction in the classrooms! It is probable that Ontario experienced two years of the most dedicated concentration of teacher and supervisory effort that history reveals in this jurisdiction. The fact that over 99 per cent of the Grade 7 pupils of the province are studying from course outlines that are still optional reflects a truly remarkable situation to the credit of all who participated and are participating. Yet we must remember that even with 100 per cent pupil participation in Grade 7, the task is far from complete in that one grade -- and that is only one grade!

Mathematics has changed! Most now accept that mathematics needed some changing. Everyone -- well at least everyone outside the Department -- agrees that the Department should change even if hopes for a rapid change are meagre. Yet, the Department has changed to some degree in each instance recorded above. Did mathematics serve as the cause? Or just the vehicle? Or are the events merely unrelated sequences? Perhaps logic will not give the answer. But at least coincident with the soarings of mathematical change, the Department has broken some old traditions related to structure, procedures, and attitudes and has attained some unexpected heights of achievement.

Perhaps Mathematics <u>does</u> have another hard-earned star in its crown!

MATHEMATICS: THE FOUR YEAR PROGRAMME

by W.G. Anglin
Head, Department of Mathematics
Q.E.C.V.I. Kingston

During the month of July, 1965, a study committee formed by the Ontario Mathematics Commission and financed by the Ontario Curriculum Institute met to consider some of the problems which will be faced by students of the four year mathematics programme during the next few years.

New curricula for the five year programmes have been developed, and are in process of being put into operation. The same is true for Grades 7 and 8. The work of this study committee marked the first review of four year mathematics courses in the light of the new material being presented to students in the elementary schools and in the five year programmes of the secondary schools.

The committee was made up of fourteen members: five of them concerned with the Business and Commerce courses; four with the Arts and Science courses; five with the Science, Technology and Trades courses; and all keenly interested in the state of mathematics in our secondary schools.

In trying to make decisions about what could and/or should be taught in the four year programmes the committee worked on the basis of predictions obtained from several members of the Department of Education. The general feeling of Department officials is that the calibre of student in the four year courses will be improving as enrollment in the two, four and five year programmes is

redistributed. It is predicted that the percentage of the total school population in the five year programme will decrease while the percentages in the four and two year programmes will increase until about fifty percent of the student population is in the four year programme. This should tend to shift the poorer, and many of the average, five year students into the four year programme and the lowest of the four year programme into the two year programme. The overall effect of this should be to improve the calibre of four year students. The committee considered therefore that eventually the four year streams will contain the 'average' students while the five year and two year streams will have the 'above average' and 'below average' respectively.

In the past there has been some reluctance on the part of mathematics teachers to teach the 'new approach' mathematics to four year students. However, the demands of our future society will require our present students to be well exposed to the basic concepts of mathematics so that they can be retrained as the need arises. Predictions are that workers of the future will not remain at one job throughout their working lives but will probably have to be retrained two or even three times as demands change. The aim of the 'new approach' is to present the work to the student in a way that he is able to see a structure which can unite apparently isolated skills. This ought to foster more understanding of mathematics and what it can do and remove the boring, repetitive, mechanical methods used in the traditional teaching of mathematics.

In considering the needs of these students the committee kept in mind that if it is hoped to make these acceptable to both industry and institutions providing further training, they must be given worthwhile courses, sound in fundamentals on which they can build. This is especially clear in the case of the Business and Commerce Branch where, in the past, the course offered has been strictly an arithmetic course. This was the branch which the committee felt needed the most radical change in order to bring its courses into line with current thinking about the mathematics needs of Ontario students. On the other hand, much of the material in trigonometry and mechanics which is now being taught to senior Science, Technology and Trades students was felt to be very satisfactory and important enough to remain in any new curriculum.

The Department of Education's curriculum HS 1 lists mathematics as a compulsory subject only in grade 9. Although, in practice, very few schools provide any alternative courses, especially in S.T. and T., the committee felt that the need for more mathematics training should be clearly indicated by making the subject compulsory to the end of grade 10. For most of the shop courses it would continue to be a requirement right through to the end of grade 12 but in the Business and Commerce Branch it might be made optional in these grades with, as an alternative, a business machines and arithmetic course. In this way, a solid four year mathematics programme could be introduced which would provide a wide background for four year students heading for educational institutions after completing grade 12.

The committee, after much discussion, agreed that the basic course given in each of the grades of each of the branches should be the same in content. This diverges widely from the present set-up but it was felt that there is enough material in mathematics which is basic to all areas of life, to justify giving all the students the same core courses. It was felt that the different slant needed in the S.T. and T. Branch and the B. and C. Branch could be provided by the examples associated with each topic. For example, the topic of ratio and proportion is equally important to all three branches. The applications in the B. and C. Branch could be to simple and compound interest, taxation, commission, foreign exchange rates, etc.; while in S.T. and T. programmes the applications might be to gear trains, levers, pressure-temperature-volume problems, trigonometry, etc. It was felt, too, that each branch would benefit from some of the examples used in the other branches. The three courses should not be disjoint as they are at present.

The committee also gave some consideration to the present methods of evaluating the student's progress in the four year courses. The common practice now is to push the student on to the next grade as long as he has a reasonable (perhaps 60%) percentage in June even though he may have one or two failures. This leads him into over-difficult work in the subjects which he failed. It would be much better for the student himself, and for all the others involved, if he were permitted to repeat those subjects he had failed and proceed to the next level of difficulty in the ones which he had passed. It was even

suggested by several members of the committee that serious consideration be given to the possibility of a completely ungraded system. With present staffs and facilities this would be out of the question, but there is the possibility that the results of such a system might be so much better than our results under the present scheme to justify the upheaval necessary to make such a change.

A study group based on London, is now working to produce experimental blocks of materials for use in the core course at Grade 10 level. This follow-up of the work of the Summer Committee should be regarded with much interest by all concerned with mathematics in the secondary schools of Ontario.

SOME THOUGHTS ON THE NEW GRADE 11 COURSE

by A. W. Harris

Althouse College of Education

The introduction, in September 1964, of Curriculum S12A into Grade 11 of the Five Year Program, as the official course of study for Mathematics may have been one of the most important changes of subject content ever made in Ontario. It certainly has been one of the most discussed. The purpose of this paper is an interim evaluation of the experience with the new course in Western Ontario schools. I shall try to compare progress in the same grade this year with that of last year, and also comment on the situation in Grade 12 this year.

S12A formed the first step in a major revision of the Mathematics courses of the Intermediate and Senior Divisions. It was followed by S12B for Grade 12, in September 1965, and this year, all across the province, seminars and courses are being held in preparation for the introduction of S12C into Grade 13 next September.

The story of the events which led up to the revision is well known to most of you. However, a brief review may be helpful, especially if you are new to our system. Most would agree that the Ontario Mathematics Commission "started the ball rolling". O.M.C. proposals for a change were tested in a number of Ontario Secondary Schools from 1959 on. In 1962 and 1963 respectively, the Department of Education, which had sanctioned the earlier trials, approved the use of the experimental textbooks in

Grades 9 and 10, as an alternative to the traditional course of study.

Then at the end of 1962 the Department decided to appoint Curriculum Revision Committees for the Senior and Intermediate Divisions. The Senior Division Committee met for the first time on February 8, 1963. The Grade 11 course was completed by summer of that year; the committee finished its work in June 1965, with the recommendation of the New Grade 13 course S12C to the Minister of Education.

During the winter of 1963-64, preparations for the next year moved slowly as most schools waited for a textbook. Although three publishers had Grade 11 textbooks in the hands of the teachers by summer 1964, there was considerable difficulty in September due to slow deliveries to the schools.

The release of S12A and the publication of the textbooks unleased a flurry of seminars, short courses, and meetings. During the summer of 1964 the O.S.S.T.F. and the Department held courses which were well attended. The College of Education which had been doing "new approach" work for several years, devoted considerable time to the new Grade 11 in its programmes. However, in spite of all these efforts, there were still many Grade 11 teachers who for one reason or another knew little or nothing about the new course before September 1964.

One of the transitional problems was that most of the Grade 11 classes had received no previous training in the terminology of sets and the so-called "number structure" approach. Although Unit 1 was written as a basic introduction to the new concepts, the inexperience of the teachers and lack of background of the pupils led to a wide variety

of experiences with Unit 1, some of them bad.

My report is based on personal observations and on interviews with a dozen Western Ontario department heads held recently. I was pleased to hear the other day that the Department of Education plans to do a thorough survey of experience with S12A toward the end of this school year.

In all but one of the schools I checked, the Grade 11 classes of 1964-65 did not bring any knowledge of the new concepts from Grade 10. In spite of this handicap all but three schools covered the course; omissions were usually from the Analytic Geometry. In one school where pupils had been on the new work from Grade 9 on, the class found time to do the optional topics. However, nearly all of the Department heads reported that they had been rushed, having been far behind at the end of Unit 1, and catching up later on in the year.

Most of the schools reported normal failure rate. A couple had difficulty with Christmas results which were cleared up later in the year. In a few schools individual teachers had examination difficulties. Two schools reported abnormally high failure rates, which caused trouble at promotion time.

Sample examination papers sent out by the Department were bound to be helpful. They were mainly used as a guide and as a source of individual questions.

Half the schools "switched" textbooks this year and reported improvement.

Although it is too early to be certain, all schools feel that Grade 11 is going better this year. Comments were, for example, "....much easier", on schedule",

"doing better", "a better text, pupils better prepared". There seem to be two factors involved: first, of course, the teachers have the benefit of a year's experience; secondly, in most cases this year's Grade 11 pupils either had the new work in Grade 9 and 10, or at least had an introduction to sets and number systems last May, as was suggested by the Department.

This year's Grade 12 classes should give a clue to our success with Grade 11. Nearly all of the department heads said they are concerned about the pupil's lack of retention of the Grade 11 work. One said retention of highlights is good; another feels that he may be expecting too much in form and substantiation. Several mentioned failure to recall formulas of Analytic Geometry, a topic which was hurried over during the spring term. One noted unusual difficulty with the application of a concept out of context: e.g. integers, when used as exponents.

The majority of the heads I talked to liked the new course, in spite of the difficulties. There were, however, criticisms. One or two mentioned dissatisfaction with the Deductive Geometry. A couple felt that too many different ideas had been introduced - a jumble, said one - resulting in a lack of time to learn anything properly. One person said he thought the introduction of S12A should have been delayed one year. The eventual problem of over-lap between the experimental Grade 10 textbooks and the new Grade 11 course was also mentioned.

Pupil reaction to the new course varied from enthusiasm to passive resistance. In this respect S12A fared about as well as its predecessor. One of the heads asked his Grade 11 "repeaters" for their preference; they liked thenew course better. One case of negative pupil reaction reported to me occurred where pupils with no previous preparation were mixed with pupils who had been on the new work in Grade 9 and 10.

My own general impressions of the 1964-65 school year tend to confirm the opinion of the department heads I interviewed. The depth of treatment of Unit 1 was difficult to gauge. The textbooks contributed to this by their prolixity. The teachers, being unfamiliar with the course of study and the textbook, and often the vocabulary, made slow progress. As a result, they were "scrambling" for most of the year. However, at the end most felt that the course had gone better than they had anticipated.

Mr. W.T. Laing, Western Ontario Area Superintendent allowed me to investigate the statistics on promotion results in the Division. I chose at random a sample of 20 schools (out of approximately 100) and compared the distribution of marks for Grade 11 Geometry and Five Year Mathematics in June 1964 and June 1965 respectively. With regard to percentages of passes, 8 schools increased the percentage of passes in 1965 by at least 3%, 7 schools decreased the percentage of passes by at least 3%, and 5 differed by less than 3% from 1964 to 1965. Distributions of marks over the scale showed no significant change in pattern from one year to the next. I realize

that this means very little; however, in that group of schools, promotions in June did not present any special problem in June 1965.

Not all of the advantages claimed for the new course have been realized as yet. The subject is not unified by the inclusion of a variety of units of work labelled Geometry, Trigonometry, etc. For one thing, the sought for unification of language is not apparent, except perhaps from textbook to textbook. The authors tended to follow the prescribed course rather closely.

The advantage of the number structure approach and the use of set terminology will be more apparent when these ideas are introduced earlier. They are in the new Grade 7 and 8 courses. On the other hand, the bringing down of the Trigonometry and Analytic Geometry to Grade 11 was well received.

The great increase in the amount of graphical work, which is a big improvement, presents special problems in classroom management, which not all have solved yet. We need to experiment with new devices and methods for showing graph work.

In summary, the following points occur to me.

(a) The first year went better than expected. Many difficulties were of a transitional nature, being due to the lack of experience of the pupils and teachers.

(b) Things are going better this year.

(c) A fair evaluation of the course can only be made after the pupils come up from Grades 9 and 10 on the new programme.

(d) At that time there will be considerable overlap from Grade 10 to 11; this will mean another re-

vision of Grade 11.

(e) Grade 12 pupils are getting along fairly well this year, but show some effects of the transitional year.

The last few years have been interesting but hectic ones for Mathematics teachers. The new courses have required us to study more Mathematics and re-appraise our aims and objectives. We do not all agree on what should be taught. The situation is confused by the changes taking place in the Intermediate Division. However, the pupils have probably benefited from our new-found enthusiasm, even if it is a by-product of "the great debate".

ONTARIO ASSOCIATION OF TEACHERS OF MATHEMATICS

Dear Colleague:

Are you still debating whether to attend the Convention or not? If so, May I suggest that you read carefully the programme that has been organized for you. Never before has the O.A.T.M. presented in a single convention programme such an impressive array of speakers, who will speak on a great variety of topics. If you are planning to attend, complete and return the advance registration form of the O.A.T.M. (and the O.E.A. membership application form) to:

Mr. Ross Leigh

as soon as possible. A fair number of teachers have already sent these in. We expect a record high in membership enrolment.

Once again we will be displaying students' projects at the Mathematics Fair.

For the wives of teachers we have arranged a special programme. One item will be a fashion show by Simpsons. It will be held on APRIL 12TH at the Inn, at 2:00 p.m. This will be followed by a talk on Interior Decorating by an expert from T. Eaton Co., Ltd. There will also be a shopping excursion to Yorkdale Plaza on April 13TH. So invite your wife to the convention. Perhaps you could go the whole hog and treat her to a few days of luxury living at the Inn. I'm sure she richly deserves this holiday and will certainly appreciate it at this time of the year. In

fact, the change will be good for both of you. There are many advantages in staying at the Inn, not the least of which is the convenience of their family service.

I hope we will see you there at Easter.

Sincerely,

J.W. Fencott,
Publicity Representative,
O.A.T.M.

75TH ANNUAL MEETING

Inn-on-the-Park

Toronto

April 11, 12, 13, 1966.

April 11	7:30 p.m.	REGISTRATION (Reception Hall-New Wing)
	8:30 p.m.	RECEPTION (for guests and members) M.C. - Dr. K.D. Fryer, University of Waterloo.
April 12	9:15-9:30 a.m.	REGISTRATION (Reception Hall-New Wing)
	9:30-10:30 a.m.	DR. MARY P. DOLCIANI Hunter College, New York, N.Y. "Development of Intuitive Geometry" (Elementary and Secondary Schools)
	11:00-12:00 a.m.	SEMINARS (See list below)
	12:00-1:30 p.m.	LUNCH
	1:30-2:30 p.m.	DR. W.W. SAWYER University of Toronto "Some Topics in High School Mathematics"
	3:00-4:00 p.m.	G.A. KAYE Assistant Superintendent, Curriculum Division, Department of Education, "Your Evolving Curriculum"
	7:00 p.m.	BANQUET THE HONOURABLE WM. G. DAVIS Minister of Education Province of Ontario.
April 13	8:45-9:15 a.m.	BUSINESS MEETING
	9:15-9:45 a.m.	PRESENTATION OF AWARDS - MATHEMATICS FAIR
	9:45-10:45 a.m.	SEMINARS (See list below)
	11:00-12:00 a.m.	DR. T.E. HULL University of Toronto "Random Number Generators"

11:00-12:00 a.m.	DR. W.W. SAWYER University of Toronto "Mathematical Reforms on Both Sides of the Atlantic" (Elementary and Secondary Schools)
12:00-1:30 p.m.	I.B.M. Luncheon. The first 250 delegates registering for the convention will be the guests of I.B.M. at this luncheon.
1:30-2:30 p.m.	DR. MARY P. DOLCIANI Hunter College, New York, N.Y. "Vector Notions that Occur in all Mathematics Courses"
3:00 p.m.	Tour of the IBM plant.

SEMINARS

Topics

The Place of Geometry in the Elementary Schools
- A.R. Hanwell, Inspector of Public Schools, Ottawa, Ontario

Grade 13 Mathematics - Calculus
- Dr. G.F. Duff, Department of Mathematics University of Toronto

Programmed Instruction in Mathematics
- Ross W. Leigh, Owen Sound Collegiate & Vocational Institute

Basing High School Mathematics on Vectors
- A.W. Harris, Althouse College of Education, London, Ontario

Testing Students in Mathematics and Evaluating Results
- Dr. H.H. Russell, Ontario Institute for Studies in Education

Sequences and Series
- Dr. Kenneth Fryer, Department of Mathematics, University of Waterloo

In-Service Training of Teachers
- D.G. Fox, Bell High School, Ottawa, Ontario

Logic
- Dr. D.A. Clarke, Department of Mathematics, University of Toronto

Trends in Kindergarten-Grade VI Mathematics Courses
- J.K. Crossley, Assistant Superintendent, Curriculum Division, Ontario Department of Education

NOTE - The seminars will be repeated to enable teachers to attend more than one session.

A NOTE ON AN ANALYTICAL GEOMETRY QUESTION

by

G.V. Helwig

Central Technical School, Toronto

Question 4(a) of the Geometry Paper in the Senior Matriculation Examination in June 1965 reads as follows:

> "Find the value of k for which the ellipse represented by $k^2x^2 + 9y^2 = 13$ intersects orthogonally each member of a set of parabolas represented by the equations $y^2 = 4px$. (Two curves intersect orthogonally if the tangents at the point of intersection are perpendicular)."

The kind of solution most likely to be proposed by students able to solve the above problem is as follows:

Let $P\ (x_1, y_1)$ be a point of intersection of the ellipse and parabola.

The tangent to the ellipse at P is, $k^2 x x_1 + 9yy_1 = 13$, with slope $\dfrac{-k^2 x_1}{9 y_1}$

and the tangent to the parabola is, $y y_1 = 2p(x-x_1)$, with slope $\dfrac{2p}{y_1}$.

If the conics cut orthogonally

$$\frac{-k^2 x_1}{9 y_1}$$

$$\therefore k^2 = \frac{9y_1^2}{2px_1}$$

but $y_1^2 = 4xp_1$ since $P(x_1,y_1)$ lies on the parabola

$$\therefore k^2 = \frac{9.4px_1}{2px_1} = 18.$$

$$k = \pm \sqrt{18} = \pm 3\sqrt{2}$$

in the above solution the condition that $P(x_1,y_1)$ lies on the parabola is quoted and used. However, the condition that the same point lies on the ellipse, which is assumed, did not appear to be imposed. This led to a treatment of the problem, which required $P(x_1,y_1)$ to be only on the parabola. Not only did this give the correct solution, but it also yeilded some interesting properties of sets of parabolas and ellipses.

Let $P(x_1,y_1)$ be any point on a parabola of the set, not lying on the ellipse.
The equation

$$k^2 xx_1 + 9yy_1 = 13$$

is similar in form to the equation of the tangent at a point on the ellipse, but as $P(x_1,y_1)$ does not lie on the ellipse, it is the equation of the polar of P with respect to the ellipse. *

The slope of the tangent to the parabola at $P(x_1,y_1)$ is $\frac{2p}{y}$, (as above) and the slope of the polar to the ellipse is $-\frac{k^2x_1}{9y_1}$, it followed that if the tangent and the polar are perpendicular $k^2 = 18$,
as in the solution of the problem given above.

P may be considered to move along the parabola until

it coincides with a point of intersection of the parabola and the ellipse at P' (x', y') , say. As P moves, the polar of P with respect to the ellipse approaches and finally coincides with or becomes the tangent to the ellipse at P'. As the tangent to the parabola and the polar to the ellipse are always perpendicular (with $k^2 = 18$),it follows that the parabola and ellipse cut orthogonally.

A different solution to the problem has been obtained.

It should be noted that exactly the same solution to the problem would be obtained for any ellipse of the set

$$k^2x^2 + 9y^2 = q, \text{ where } q \text{ is a parameter.}$$

The equation of the set of ellipses

$$18x^2 + 9y^2 = q$$

may be written

$$2x^2 + y^2 \quad \frac{q}{9}$$

so that the equation of the set may be written in a more convenient form

where the new parameter b is the length of the semi-minor axis. Thus, more general properties of the set of parabolas

$$y^2 \quad 4px$$

and the set of ellipses

$$2x^2 + y^2 = b^2$$

become evident as follows:

The tangent at any point on a parabola of the set is perpendicular to the polar of the point with respect to any ellipse of the set of ellipses;

the tangent at any point on an ellipse of the set is perpendicular to the polar of the point with

respect to any parabola of the set;

and, the parabolas and ellipses all cut each other orthogonally.

To refer back to the solution given at the beginning of this paper there is, in fact, no valid objection to the solution. The expression "the condition that the same point lies on the ellipse, which is assumed, did not appear to be imposed "was used advisedly. The condition that this point lies on the ellipse is, in fact, imposed when the equation of the tangent is taken to be

$$k^2xx_1 + 9yy_1 = 13$$

In order to derive this equation it has to be assumed that the point (x_1,y_1) lies on the ellipse so that

$$kx_1^2 + 9y_1^2 = 13.$$

Foot Note *

The properties of the polar of a conic are given in most standard College textbooks on Analytical Geometry. See, for instance, Analytical Conics by Sommerville (Bell and Sons). The equation of the polar has the same form as the equation of the tangent to a conic. When the pole lies on the conic, the polar becomes the tangent at that point. One property of the pole and polar is that if any line is drawn through the pole P to cut the polar at Q and the conic at U and V, then PQ is divided internally and externally in the same ratio by U and V, so that PQ.UV is a harmonic range.

I wish to acknowledge gratefully the helpful suggestions and criticisms of Professor Hogarth and his associates at Queen's University which resulted in this paper.

THE MANY FACETS OF MATHEMATICS*

by

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Mathematics is often described as a logical subject. I thought I would begin to-day by explaining just exactly what this means. The key to the explanation is, of course, the word 'logical' - it is necessary to understand what logic is in order to understand what mathematics is.

Logic may be defined as the rational reconstruction and analysis of the results of creative thought. That is, logic provides a mechanism for criticizing and analysing an argument, for assessing it, finally, as correct or incorrect. An argument, in turn, is a detailed sequence of steps, each appealing to some logical rule or principle, by which the truth of a statement - the conclusion - is shown to follow inexorably from the truth of other statements, the hypotheses. An argument is then valid, or

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* A summary of a talk prepared for the Canadian Mathematics Seminar, Ottawa, Ontario, April 22, 1965.

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correct, when the rules of logic have been correctly applied and the whole construction is logically sound and valid. It has been, for years, the purpose of the schools in teaching Euclid's geometry, for example, to teach the principles of logic, or sound reasoning, through the construction and analysis of arguments in a comparatively simple context.

There are a number of observations that may be made in connection with this definition. In the first place, a logical argument - or, in mathematical language - a theorem , is valid whether or not the hypotheses are, in fact, true. It is simply a matter of whether or not the rules of logic are correctly applied. This brings up the whole question of mathematical truth. A mathematical subject is a sequence of theorems, each depending, apart from the rules of logic, upon preceding theorems. If we think of following this sequence backwards, to the beginning, we come to the conclusion that there must be, somewhere a number of statements upon whose truth the truth of all the theorems depends. We cannot have an argument without hypotheses and we have got to get some hypotheses from somewhere to start the whole thing off. In mathematics these initial statements, whose truth is not questioned, are called axioms. Axioms are true in the sense that the rules of chess or checkers or bridge or ping-pong are true - because we say they are. Indeed, in this pure sense, all of mathematics is a game, played out according to purely arbitrary rules. Mathematics has been described as applied symbolic logic, and the description has some truth in it. Every mathematical theorem is, effectively, a conditional statement to the effect that, if certain axioms are true then, necessarily, the theorem is likewise true.

Another comment about our definition concerns the word 'results' in the expression 'results of creative thought'. Logic is not thought -- creative or otherwise. The thinking has already been done when logic enters the picture. The thought process in mathematics takes various

forms - there is the conjecture, which is formulated as a result of insight, intuition or sheer guesswork. Then there is the devising and development of a valid proof - or disproof - of the conjecture. In this connection, trial-and-error, and analogy, are useful. Then, only when the conclusion has been formulated and the argument constructed, does logic play its critical role in evaluating the result as a valid consequence of the hypotheses. There is, indeed, more to it than this. The formulation of interesting problems, whose solution requires the introduction and analysis of new mathematical concepts, is clearly not a logical process. Thus, in geometry, we make axioms about points and lines, but introduce, in terms of the axiomatic objects, the concepts of triangle, rectangle, circle, congruence, similarity, angle, locus about which many deep and interesting facts can be established. One cannot emphasize too highly this creative aspect of mathematics. Creativity is as characteristic of mathematics as it is of music, literature, art or any other intellectual activity. A first-rate mathematician is a creative artist, above all else, who works with abstract concepts, always seeking deep insight combined with simplicity and elegance. The so-called new mathematics being introduced into the schools to-day represents, basically, an effort to make mathematical concepts known and understood by a wider audience.

Mathematics, in the sense that I have defined it, did not develop historically as an abstract, unworldly, intellectual pursuit. Indeed, it has always had its roots in very worldly, matter-of-fact situations. It is almost certain that Euclid and his contemporaries thought of their geometry and of the real world in which they

lived and moved about as the same thing. It was only when other systems of geometry were discovered that the idea that geometrical arguments and constructions might not be relevant to real life was seriously considered. For then there arose the embarrassing question of which system was, in fact, the correct one. Similarly, in analysis, intuition, based on physical and geometrical analogies, led to the rapid development of calculus and related subjects - developments which at the time were ruthlessly criticized as irrational and nonsensical. There followed a period of critical reappraisal in which the whole matter was solidly based upon the properties of real numbers, which in turn were completely characterized by axioms due to Peano, Dedekind and others. Recently, the abstract or axiomatic approach has gained in popularity, and in acceptance as the only ultimate source of mathematical truth.

This realization, then, that mathematics consists essentially of abstract systems, devoid of any real-life content, the product entirely of the human creative imagination, has come only after a long, rich history.

Let me conclude by commenting briefly upon some of the other facets of this rich and fascinating subject.

Mathematics finds many applications to physics, chemistry, economics, business and, indeed, to real life in general. We are reminded of the words of Lord Kelvin: "When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you

have scarcely - - - advanced to the stage of science." What Kelvin seems to be saying, in the context of to-day's talk, is that a mathematical model is a very real and important - perhaps the only - source of genuine insight into real-life situations. This seems strange to those of us accustomed to looking upon abstract mathematics as rather more difficult to understand than real-life phenomena. The history of mathematics shows the very real connection between mathematical and physical concepts - or - demonstrates our ability to abstract mathematics from real-life. All of what is called applied mathematics may be described as the creation and development of mathematics for the express purpose of exploiting the insight which it gives us into real-life situations. Pure mathematicians and applied mathematics are distinguished only by their motivation - pure mathematicians pursue mathematical research because there is much of intrinsic interest to be done, challenging problems to be solved, and because of the pleasure that follows a new insight or discovery- applied mathematicians have as their motivation the development of understanding and insight with respect to some real-life situation. John von Neumann has remarked upon the danger that the pure mathematician will be led to follow the path of least resistance, to pursue the easy problems and to ignore the difficult ones until his subject becomes shallow, uninteresting and over-specialized. The applied mathematician, on the otherhand, must tackle whatever problems are relevant to his ultimate goal, and nature has not been known to make things easy for him. I for one believe that applied mathematics still offers the most fruitful source of new mathematics.

Finally, there is the impact which modern computing devices have made upon mathematics. To understand this, we may consider two general methods for proving theorems - the constructive and the non-constructive. A constructive proof gives a step-by-step procedure, or an algorithm, for proving the result, or solving the problem in hand. Thus a constructive proof of the fundamental theorem of algebra would consist of a well-defined process for finding a root of an algebraic equation. Such a process is known only in the case of equations of fourth or lower degree. The algorithm may be complicated, and involve many thousands or even millions of arithmetic operations, yet it may nonetheless be considered constructive. A non-constructive proof asserts the existence of a solution, but does not show one how to find it. One common type of non-constructive proof is the indirect proof - you say "suppose the equation doesn't have a root" and then proceed to derive some contradiction, concluding from this that the equation must have a root. But you have no idea how to find it. Other forms of non-constructive proof are based upon axioms of existence - that there exist numbers, or functions with certain properties, or that one may choose sequences or subsets with certain properties. No indication is given of how the choice is to be made in a particular case. Indirect proofs are often used in Mathematics, with most powerful and useful results. Charles Babbage, the real inventor of automatic computers, spoke of the legislative and executive branches of mathematics, meaning by executive that facet of mathematics concerned with the practical solution of particular problems with the aid of constructive algorithms. The twent-

ieth century emphasis on abstract, axiomatic mathematics has greatly advanced the legislative branch. But, just at the time when constructive mathematics seemed doomed to oblivion, the development of automatic computers gave tremendour power to carry out the most elaborate algorithms, and attention has been focussed on constructive, algorithmic mathematics as a sort of oounterpose to the abstract, axiomatic and essentially existential modern development. It is rather restrictive to insist upon constructive solutions to all the problems of mathematics, yet the challenge is great and much progress has been made.

SOLUTIONS, GRADE XIII PROBLEMS, 1965.

1. Find the general solutions of the equation

$$2 \cos \frac{x}{3} - \sin \frac{x}{2} = 2$$

$$2\left(1 - 2 \sin^2 \frac{x}{6}\right) - \left(3 \sin \frac{x}{6} - 4 \sin^3 \frac{x}{6}\right) = 2$$

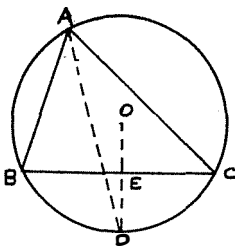
$$4 \sin^3 \frac{x}{6} - 4 \sin^2 \frac{x}{6} - 3 \sin \frac{x}{6} = 0$$

$$\sin \frac{x}{6} \left(2 \sin \frac{x}{6} + 1\right) \left(2 \sin \frac{x}{6} - 3\right) = 0$$

$\sin \frac{x}{6} = 0 \quad x = 6n\pi$, n I

$\sin \frac{x}{6} = -\frac{1}{2} \quad x = (12n-1)\pi$ or $(12n+7)\pi$, n I.

2. The bisector of an interior angle of a scaline triangle meets the right bisector of the opposite side at the point P. Show that P lies outside the triangle.



Draw the circumscribed circle of triangle ABC. The perpendicular bisector of chord BC bisects the arc BDC. The bisector of <BAC bisects the arc BDC. Thus the bisectors meet at D on the circumscribed circle which is outside the triangle.

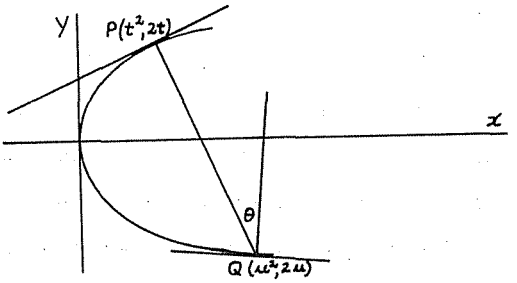
3. (a) Show that the sum of two positive varibles whose product is constant decreases when their difference decreases.

From the identity $(x+y)^2=(x-y)^2+4xy$ where xy is constant it is evident that when $(x-y)^2$

decreases then $(x+y)^2$ decreases. Since x and y are positive then $|x-y|$ decreases when $x+y$ decreases.

(b) When the speed of a motor car is v miles an hour the car uses gasoline at the rate of $(\frac{1}{2} + \frac{x^2}{1800})$ gallons an hour. Find the value of v for which the mileage per gallon of gasoline is greatest. The mileage per gallon is greatest when the number of gallons per mile is the least, i.e. when $\frac{1}{2v} + \frac{v}{1800}$ is least. Since $\frac{1}{2v} \cdot \frac{v}{1800}$ is constant the least value occurs when $\frac{1}{2v} = \frac{v}{1800}$ or $v = 30$.

4. PQ is a chord of the parabola $y^2=4x$ and also is a normal to the parabola at P. If θ is the angle between the normals at P and Q, express $\tan\theta$ as a function of the slope m of the tangent at P.



Tangent to $y^2=4x$ at $P(t^2,2t)$ has equation

$y \cdot 2t = 2(x+t^{-2})$

i.e. $ty = x + t^2$

The slope of the tangent is $\frac{1}{t}$.

The slope of the normal at P is $-t$.

Similarly the slope of the normal at $Q(u^2,2u)$ is $-u$. The slope of PQ is $\frac{2t - 2u}{t^2 - u^2} = \frac{2}{t+u}$

then $\frac{2}{t+u} = -t$ and $u = \frac{-2}{t} - t = -2m - \frac{1}{m}$

$$\tan\theta = \frac{-t-u}{1+tu} = \frac{-\frac{1}{m} - 2m - \frac{1}{m}}{1 + \frac{1}{m}(-2m-\frac{1}{m})}$$

Upon simplification $\tan\theta = 2m$.

5. Let P be any point on a hyperbola with centre 0 and foci F_1 and F_2. Show that $PO^2 = PF_1 . PF_2$ if and only if the hyperbola is rectangular.

Let equation of the hyperbola be $\frac{x^2}{2} - \frac{y^2}{2} = 1$.

Show $PO^2 = PF_1 . PF_2$ if and only if $a^2 = b^2$

$$x^2 + y^2 = (ex-a)(ex+a)$$

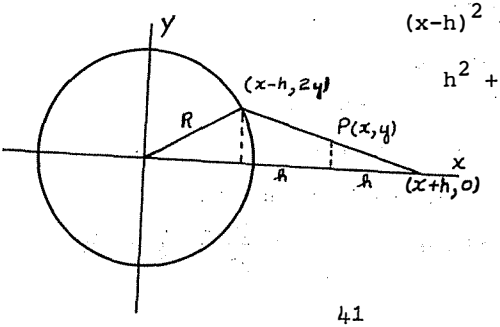
Since $y^2 = \frac{b^2}{2}(x^2-a^2)$ or $(e^2-1)(x^2-a^2)$

then $x^2 + (e^2-1)(x^2-a^2) = e^2x^2 - a^2$.

hence $e^2 = 2$.

Since the steps are reversible $PO^2 = PF_1 . PF_2$ if and only if $a^2 = b^2$.

6. A wheel of radius R, with its centre fixed at the origin, rotates in the xy plane. A rod of length 2R has one end pivoted at the rim of the wheel and the other end is constrained to lie in the positive x-axis. Find the equation of the locus traced out by the midpoint of the rod.



$$(x-h)^2 + 4y^2 = R^2$$

$$h^2 + y^2 = R^2$$

eliminating h using $h = \pm\sqrt{R^2 - y^2}$ we get

$(x \pm \sqrt{R^2 - y^2})^2 + 4y^2 = R^2$ or $x^2 + 3y^2 = \pm 2x\sqrt{R^2 - y^2}$.

Desired locus has equation $(\frac{x^2 + 3y^2}{2x})^2 + y^2 = R^2$.

7. A rational number A_m (m is a positive integer) is defined by $A_m = \frac{(2m+1)!}{m!(m+1)!}$

(a) Prove A_m is an integer less than 4^m

$\frac{(2m+1)!}{m!(m+1)!}$ is the coefficient of x^m in the binomial expansion of $(1+x)^{2m+1}$ and as such is an integer. The sum of the coefficients in the expansion is 2^{2m+1}. Since the coefficient of x^m is equal to the coefficient of x^{m+1} then the coefficient of x^m is less than $\frac{1}{2}(2)^{2m+1}$ or 4^m.

(b) Any prime number between $m+1$ and $2m+1$ divides A_m.

Any prime number greater than $m+1$ and equal to or less than $2m+1$ is a factor of the right side of

$$m!\,(m+1)!\,A_m = (2m+1)!$$

and hence is a factor of the left side. Since it is not a factor of $m!$ or $(m+1)!$ it must be a factor of A_m.

8. Let the function f of two variables α and β be defined by

$$f(\alpha,\beta) = \frac{\alpha^2\beta^2}{\alpha-\beta}\left(\frac{1}{\alpha^7} - \frac{1}{\beta^7}\right).$$

If $\alpha = \frac{1}{2}(\sqrt{5}-1)$ and $\beta = \frac{1}{2}(\sqrt{5}+1)$ prove that $f(\ ,\ )$ is an integer and compute this integer.

$\alpha+\beta = -1$ and $\alpha\beta = -1$

$$f(\alpha,\beta) = -\frac{1}{\alpha^5\beta^5}\left(\frac{\alpha^7-\beta^7}{\alpha-\beta}\right)$$

$$= \frac{-1}{-1}(\alpha^6+\alpha^5\beta+\alpha^4\beta^2+\alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5+\beta^6)$$

$$= (\alpha^6+\beta^6)+\alpha\beta(\alpha^4+\beta^4)+\alpha^2\beta^2(\alpha^2+\beta^2)+\alpha^3\beta^3$$

$$= (\alpha^6+\beta^6)-(\alpha^4+\beta^4)+(\alpha^2+\beta^2)-1$$

$$\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta = 3.$$

$$\alpha^4+\beta^4 = (\alpha^2+\beta^2)^2-2\alpha^2\beta^2 = 7$$

$$\alpha^6+\beta^6 = (\alpha^2+\beta^2)(\alpha^4-\alpha^2\beta^2+\beta^4)$$

$$= 3(7-1)$$

$$= 18$$

$$f(\alpha,\beta) = 18-7+3-1$$

$$= 13 \quad \text{and} \quad f(\alpha,\beta) \quad \text{is an integer.}$$

9. Let a sequence of numbers f_n be defined by the rules $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}(n>1)$.

(a) Prove that $f_n^2 - f_{n-1}f_{n+1} = (-1)^{n-1}$

$$f_3 = f_1 + f_2 = 1+1 = 2$$

$$f_2^2 - f_1f_3 = 1-2 = -1$$

$f_n^2 - f_{n-1}f_{n+1} = (-1)^{n-1}$ is true for $n = 2$.

$$f_{n+1}^2 - f_nf_{n+2} = f_{n+1}^2 - f_n(f_{n+1}+f_n)$$

$$= f_{n+1}(f_{n+1}-f_n) - f_n^2$$

$$= f_{n+1}f_{n-1} - f_n^2$$

$$= -(f_n^2 - f_{n+1}f_{n-1})$$

If $f_n^2 - f_{n+1}f_{n-1} = (-1)^{n-1}$ then

$$f_{n+1}^2 - f_nf_{n+2} = (-1)^n.$$

(b) Defining $g_n = \frac{f_{n+1}}{f_n}$ prove that the sequence

g_{2n+1} is increasing and that the sequence g_{2n} is decreasing.

$$g_{2n+1} - g_{2n-1} = \frac{f_{2n+2}}{f_{2n+1}} - \frac{f_{2n+1}}{f_{2n}} + \frac{f_{2n+1}}{f_{2n}} - \frac{f_{2n}}{f_{2n-1}}$$

$$= \frac{f_{2n+2}f_{2n} - f_{2n+1}^2}{f_{2n+1}f_{2n}} + \frac{f_{2n+1}f_{2n-1} - f_{2n}^2}{f_{2n}f_{2n-1}}$$

$$= \frac{-(-1)^{2n}}{f_{2n+1}f_{2n}} + \frac{-(-1)^{2n-1}}{f_{2n}f_{2n-1}}$$

$$= \frac{1}{f_{2n}}\left[\frac{1}{f_{2n+1}} - \frac{1}{f_{2n-1}}\right] > 0$$

g_{2n+1} is increasing.

Similarly it can be shown g_{2n} is decreasing.

10. Tan A and Tan B are the roots of the equation $x^2 + cx + D = 0$. Show that

$\sin^2(A+B) + C \sin(A+B)\cos(A+B) + D(\cos^2(A+B) = D.$

$\tan A + \tan B = -C$ and $\tan A \tan B = D.$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B} = \frac{-C}{1-D} \quad D \neq 1$$

$$\sin^2(A+B) + C \sin(A+B)\cos(A+B) + D\cos^2(A+B)$$

$$= \frac{\sin^2(A+B) + C \sin(A+B)\cos(A+B) + D\cos^2(A+B)}{\cos^2(A+B) + \sin^2(A+B)}$$

$$= \frac{\tan^2(A+B) + C\tan(A+B) + D}{1 + \tan^2(A+B)} \quad (\cos(A+B) \neq 0)$$

$$= \frac{\left(\frac{-C}{1-D}\right)^2 + C\left(\frac{-C}{1-D}\right) + D}{1 + \left(\frac{-C}{1-D}\right)^2}$$

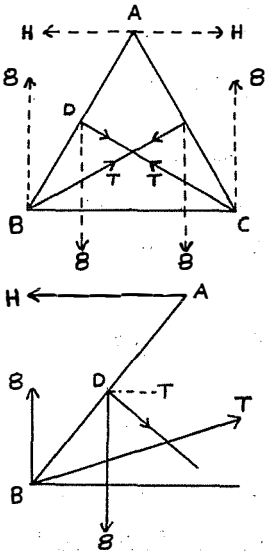
$= D.$

If $D = 1$ then $(A+B) = \frac{\pi}{2}$ and $\cos(A+B) = 0$.

and the identity is still true.

11. Two equal uniform beams AB and AC, each of weight 8 pounds are connected by a frictionless hinge at A, and are placed in a vertical plane.

Their ends B and C rest on a smooth horizontal plane, each beam making an angle of 60° with the horizontal. They are kept from slipping by strings connecting B and C with the mid points of AC and AB. Find the tension in one of the strings and the force at the hinge on one of the beams.



Because of the symmetry of the figure it is sufficient to consider only half of the diagram.

Let the side AB be 2 units in length. By resolving

$$H = 2T \cos 30^{\circ} = T\sqrt{3}.$$

Take moments about B.

$$H.\sqrt{3} = T + 8(\tfrac{1}{2})$$

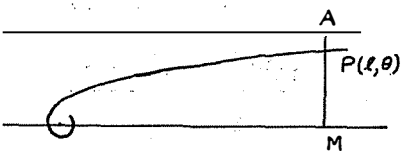
$$3T = T + 4$$

$$T = 2$$

$$H = 2\sqrt{3}.$$

12. Consider the curve whose equation in polar coordinates is $\ell\theta = a$ wherein a is a constant.

(a) Find limit $\ell \sin\theta$ as $\theta \to 0$



$$\lim_{\theta \to o} \ell \sin\theta = \lim_{\theta \to o} \ell\theta \frac{\sin\theta}{\theta} = \lim_{\theta \to 0} a \frac{\sin\theta}{\theta} = a$$

(b) Show that the curve $\theta = a$ has an asymptote and find its equations.

Consider the line whose equation is $\sin\theta = a$. This is a line parallel to the initial line and at a distance a from that line.

$$PA = a - \ell\sin\theta$$

$$\lim_{\ell\to\infty} PA = \lim_{\theta\to 0} PA = \lim_{\theta\to 0} (a - \sin\theta)$$

$$= a - a = 0.$$

Hence the line parallel to the initial line and distance a units from it is an asymptote.

Its equation is $\ell\sin\theta = a$.

13. (a) Show analytically that if z_1 and z_2 are two complex numbers whose sum is 1 then

$$z_1 + z_2 \geq 1$$

What is the geometric interpretation of this inequality?

Let the complex numbers be $x_1 + iy_1$ and $x_2 - iy_1$ then $x_1 + x_2 = 1$.

Since $z_1 = \sqrt{x_1^2 + y_1^2}$ then

$$z_1 \geq x_1 \geq x_1$$

and

$$z_2 \geq x_2 \geq x_2.$$

The geometric interpretation is that the sum of two sides of a triangle is greater than the third. If the three vertices become collinear, then the strict inequality becomes greater than or equal to.

$$a_1 + z_2 \geq x_1 + x_2 = 1.$$

(b) Find all the roots of the equation

$z^4 + z^3 + z^2 + z + 1 = 0$ and show their positions on an Argant diagram.

$$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$$

The roots of $z^5 - 1 = 0$ are

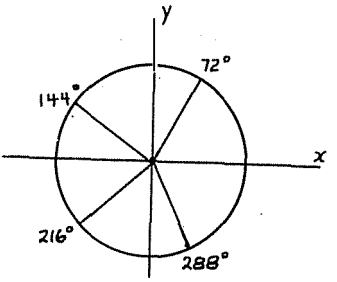
$\cos \frac{2\pi}{5} + i \sin$

$\cos \frac{4\pi}{5} + i \sin$

$\cos \frac{6\pi}{5} + i \sin$

$\cos \frac{8\pi}{5} + i \sin$

and 1



The first four are the roots of the required equation.

14. (a) Two dice are thrown simultaneously. Find the probability of throwing (i) a "four" (ii) a "seven" (iii) a total which is neither "four" nor "seven".

(i) There are three ways to get a total of "four". Since number of equally likely outcomes is 6×6 or 36 the probability is $\frac{3}{36}$ or $\frac{1}{12}$.

(ii) There are six ways to get a total of "seven". The probability is $\frac{6}{36}$ or $\frac{1}{6}$.

(iii) The probability of neither a "four" nor "seven" is $1 - \frac{1}{12} - \frac{1}{6} = \frac{3}{4}$.

(b) Two dice are thrown a number of times until a "four" or a "seven" is obtained. Find the probability of throwing a "four" before a "seven".

There are 9 ways of getting a "four" or a "seven".

There are 3 ways of getting a "four".

The probability of a "four" before a "seven" is $\frac{3}{9}$ or $\frac{1}{3}$.

15. Using a truth table prove that

$[(A \rightarrow B) \wedge (C \rightarrow B)] \leftrightarrow [(A \vee C) \rightarrow B]$

A	B	C	$A \rightarrow B$	$C \rightarrow B$	$(A \rightarrow B) \wedge (C \rightarrow B)$	$A \vee C$	$(A \vee C) \rightarrow B$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	F	T	F
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	F	F	T	F
F	F	F	T	T	T	F	T

A comparison of the entries in the last and third last columns establishes the proposition.

THE LIMIT CONCEPT*

(PART II)

A.H. Lightstone - Queen's University

3. On Computing the Limit of a Sequence

In the preceding section we introduced the notion of the limit of a sequence, and demonstrated that if (a_n) is convergent we could actually construct the real number, $\lim (a_n)$, associated with (a_n) and called the limit of (a_n). In this section we shall develop an efficient method of computing the limit of a sequence, which we shall have occasion to use throughout this article. The idea is to reduce the problem of computing the limit of a given sequence, to that of computing the limit of possibly several, simpler sequences. This goal is achieved by the four following theorems:

THEOREM 3.1: $(a_n + b_n)$ is convergent and $\lim (a_n + b_n) = \lim (a_n) + \lim (b_n)$, provided (a_n) and (b_n) are convergent.

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THEOREM 3.2: $(a_n \cdot b_n)$ is convergent and $\lim (a_n \cdot b_n) = [\lim (a_n)].[\lim (b_n)]$, provided (a_n) and (b_n) are convergent.

THEOREM 3.3: $(a_n - b_n)$ is convergent and $\lim (a_n - b_n) = \lim (a_n) - \lim (b_n)$, provided (a_n) and (b_n) are convergent.

THEOREM 3.4: $(\frac{a_n}{b_n})$ is convergent and $\lim (\frac{a_n}{b_n}) = \frac{\lim (a_n)}{\lim (b_n)}$,

provided (a_n) is convergent, (b_n) si convergent, $\lim (b_n) \neq 0$ and $b_n \neq 0$ for each n.

We may regard the sequences $(a_n + b_n)$, $(a_n.b_n)$, $(a_n - b_n)$ and $(\frac{a_n}{b_n})$ as being the sum, product, difference and quotient, respectively, of the sequences (a_n) and (b_n). Then Theorem 3.1 asserts that the limit of the sum of two convergent sequences, is the sum of their limits; Theorem 3.2 asserts that the limit of the product of two convergent sequences, is the product of their limits; Theorem 3.3 asserts that that the limit of the difference of two convergent sequences, is the quotient of their limits -- provided the specified conditions are met.

To explain just how these theorems help us find the limit of a given sequence, observe that if we can express the given sequence as the sum, say, of two sequences whose limits we know, then Theorem 3.1 tells us that the sum of the two limits is the limit of the given sequence. For example, the sequence $(5 + \frac{1}{10^n})$ is the sum of the sequences (5) and $(\frac{1}{10^n})$, whose limits are 5 and 0 respectively. Therefore, by Theorem 3.1,

$$\lim(5 + \frac{1}{10^n}) = \lim (5) + \lim (\frac{1}{10^n}) = 5 + 0 = 5.$$

Similarly, the sequence $(\frac{5}{10^n})$ can be expressed as the product of the sequences (5) and $(\frac{1}{10^n})$; therefore, by Theorem 3.2

$$\lim (\frac{5}{10^n}) = [\lim (5)].[\lim (\frac{1}{10^n})] = [5].[0] = 0.$$

As another illustration, consider the sequence $(\frac{2 - \frac{1}{n}}{3 + \frac{2}{n}})$ which is the quotient of the sequences $(2-\frac{1}{n})$

and $(3 + \frac{2}{n})$. It is easy to see that both these sequences are convergent. For, by Theorem 3.3

$$\lim (2 - \tfrac{1}{n}) = \lim (2) - \lim (\tfrac{1}{n}) = 2 - 0 = 2;$$

and by Theorem 3.1,

$$\lim (3 + \tfrac{2}{n}) = \lim (3) + \lim (\tfrac{2}{n})$$

$$= 3 + [\lim(2)].[\lim(\tfrac{1}{n})]=3+[2].[0]=3.$$

Therefore, by Theorem 3.4, $\lim(\dfrac{2 - \frac{1}{n}}{3 + \frac{2}{n}}) = \dfrac{\lim(2 - \frac{1}{n})}{\lim(3 + \frac{2}{m})} = \frac{2}{3}$.

Having demonstrated the value of the four fundamental theorems, we are now ready to prove these theorems. Note how readily we prove these theorems by appealing directly to the definition of the limit concept.

Proof of Theorem 3.1: Suppose $\lim (a_n) = A$ and $\lim (b_n) = B$. We shall prove that given p we can find q such that $|a_n + b_n - (A + B)| < \frac{1}{10^p}$ whenever $n>q$.

Let p be any natural number. Since $\lim (a_n)=A$, there is a natural number r such that $|a_n-A| < \frac{1}{10^{p+1}}$ whenever $n>r$; since $\lim (b_n)=B$, there is a natural number s such that $|b_n-B| < \frac{1}{10^{p+1}}$ whenever $n>s$. Let q be the larger of r and s; then $|a_n-A| < \frac{1}{10^{p+1}}$ and $|b_n-B| < \frac{1}{10^{p+1}}$ whenever $n>q$. Therefore,

$$|a_n + b_n - (A + B)| = |(a_n - A) + (b_n - B)|$$

$$\leq |a_n - A| + |b_n - B|$$

$$< \frac{1}{10^{p+1}} + \frac{1}{10^{p+1}}$$

$$< \frac{1}{10^p} \quad \text{provided } n>q.$$

This establishes that given p we can find q such that $|a_n + b_n - (A + B)| < \frac{1}{10^p}$ whenever $n>q$; thus, the sequence $(a_n + b_n)$ is convergent and $\lim (a_n + b_n)=A+B$.

Proof of Theorem 3.2: Suppose $\lim (a_n)=A$ and $\lim (b_n)=B$. We shall prove that given p we can find q such that $|a_nb_n-AB|<\frac{1}{10^p}$ whenever $n>q$. First, let k be any natural number such that $|A|<10^k$ and $|B|<10^k$ Now let p be any natural number. Just as in the proof of Theorem 3.1, there is a natural number q such that

$$|a_n - A| < \frac{1}{10^{p+k+1}} \quad \text{and} \quad |b_n - B| < \frac{1}{10^{p+k+1}}$$

whenever $n>q$. In particular, this means that $|a_n| < |A| + \frac{1}{10^{p+k+1}}$. Therefore

$$|a_n.b_n-A.B| = |a_n.(b_n-B)+B.(a_n-A)| \leq |a_n|.\ |b_n-B| + |B|.\ |a_n-A|$$

$$< |a_n| \cdot \frac{1}{10^{p+k+1}} + |B| \cdot \frac{1}{10^{p+k+1}}$$

$$< (|A| + \frac{1}{10^{p+k+1}}) \cdot \frac{1}{10^{p+k+1}} + |B| \cdot \frac{1}{10^{p+k+1}}$$

$$< \frac{1}{10^{p+1}} + \frac{1}{10^{2p+2k+2}} + \frac{1}{10^{p+1}}$$

$$< \frac{1}{10^p} \quad \text{provided} \quad n>q.$$

This establishes the required result; thus the sequence $(a_n.b_n)$ is convergent and $\lim (a_n.b_n)=A.B$.

Proof of Theorem 3.3: Suppose that $\lim (a_n) = A$ and $\lim (b_n) = B$. Now, $(a_n - b_n)=(a_n + -b_n)$; furthermore, $\lim (-b_n) = [\lim(-1)].[\lim (b_n)] = - B$. Therefore, by Theorem 3.1, $\lim (a_n - b_n) = \lim (a_n) + \lim (-b_n) = A+-B = A - B$. This establishes Theorem 3.3.

Proof of Theorem 3.4: Suppose $\lim (a_n) = A$, $\lim (b_n) = B$,

$B \neq 0$ and $b_n \neq 0$ for each n. We shall prove that the sequence $(\frac{a_n}{b_n})$ is convergent and $\lim (\frac{a_n}{b_n}) = \frac{A}{B}$. First, we prove that the sequence $(\frac{1}{b_n})$ is convergent and $\lim(\frac{1}{b_n}) = \frac{1}{B}$. Let k be a natural number such that $|B| > 10^{-k}$; then there is a natural number t such that $|b_n| > 10^{-k-1}$ whenever $n > t$. Now let p be any natural number. Since (b_n) is convergent, there is a natural number r such that $|b_n - B| < \frac{1}{10^{p+2k+1}}$ whenever $n > r$. Let q be the larger of t and r; then

$$|\frac{1}{b_n} - \frac{1}{B}| = \frac{|B - b_n|}{|b_n| . |B|} < \frac{10^{-p-2k-1}}{10^{-k-1} . 10^{-k}} = \frac{1}{10^p}$$

whenever $n > q$.

We have shown that given p, we can find q such that $|\frac{1}{b_n} - \frac{1}{B}| < \frac{1}{10^p}$ whenever $n > q$. It follows from Theorem 2.3 that $(\frac{1}{b_n})$ converges and $\lim (\frac{1}{b_n}) = \frac{1}{B}$. By Theorem 3.2,

$$\lim (a_n \cdot \frac{1}{b_n}) = [\lim(a_n)] . [\lim(\frac{1}{b_n})] = A . \frac{1}{B} = \frac{A}{B}.$$

This completes the proof of Theorem 3.4.

As we have pointed out and illustrated, the four fundamental theorems discussed here, enable us to compute limits efficiently. To demonstrate the power of these theorems, we prove

THEOREM 3.5: (a_n^k) is convergent and $\lim (a_n^k) = [\lim(a_n)]^k$, provided (a_n) is convergent -- whenever k is a natural number.

Proof. We shall demonstrate the theorem by mathematical induction. Clearly, the result is true if $k = 1$. Assume

true for k = t; ie.e we assume (a_n^t) is convergent and $\lim (a_n^t) = [\lim (a_n)]^t$. We shall show that the result is true for k = t + 1. By Theorem 3.2, the sequence $(a_n^t . a_n)$ is convergent, and $\lim (a_n^t . a_n) = [\lim (a_n^t)]$. $[\lim (a_n)]$; thus, the sequence (a_n^{t+1}) is convergent and $\lim (a_n^{t+1}) = [\lim (a_n)]^{t+1}$. This establishes Theorem 3.5.

Excerises

Use the technique of this section to compute the following:

		Answer
1.	$\lim (\frac{1}{n} + \frac{1}{n^2})$	0
2.	$\lim (\frac{n + 1}{n})$	1
3.	$\lim (4 + \frac{4}{n} + \frac{1}{n^2})$ $\lim (2 + \frac{1}{n})$	2
4.	$\lim (5/10^n + a_n)$ where a_n is 2 to n decimal places	$2^1/2$
5.	$\lim (2/a_n)$ where a_n is 2 to n decimal places	$2^1/2$

6. Show that $\lim (\frac{a^4_n - A^4}{a_n - A}) = 4A^3$, assuming that $\lim (a_n) = A$ and $a_n \neq A$ for each n.

7. If $\lim (b_n) = B$ and $\lim (a_n - b_n) = L$, show that $\lim (a_n) = B + L$.

4 More Theorems about "lim"*

In this section we shall establish several theorems about the limit of a sequence which are useful in calculus. The proofs of the following theorems are simplified, and in fact become much more meaningful, if we observe that the statement "lim (a_n) = A" means that given a permitted error, say 10^{-p}, there is a term of the sequence (a_n) such that all subsequent terms of the sequence differ from A by less than 10^{-p}.

Theorem 4.1: If (a_n) and (b_n) are convergent sequences such that $a_n \leq b_n$ for each n, then $\lim (a_n) \leq \lim (b_n)$.

Proof. Let $\lim (a_n) = A$ and $\lim (b_n) = B$. Suppose $B < A$, contrary to the theorem. Then there is a natural number k such that $10^{-k} < A - B$. But there is a term of (a_n) such that all subsequent terms of (a_n) differ from A by less than 10^{-k-1}, and there is a term of (b_n) such that all subsequent terms of (b_n) differ from B by less than 10^{-k-1}. In particular, there is a natural number t such that a_t differs from A by less than 10^{-k-1}, and b_t differs from B by less than 10^{-k-1}. Since $10^{-k} < A - B$, we must conclude that $b_t < a_t$, contrary to the assumption of the theorem. This contradiction establishes Theorem 4.1.

Corollary 4.1: If (a_n) is convergent and $a_n \leq b$ for each n, then $\lim (a_n) \leq b$.

Corollary 4.2: If (b_n) is convergent and $a \leq b_n$ for each n, then $a \leq \lim (b_n)$.

Theorem 4.2: If the sequence (a_n) is such that $a_n \leq b$ and $a_n \leq a_{n+1}$ for each n, then (a_n) is convergent and $\lim (a_n)$ = l.u.b. $\{a_n \mid n \in N\}$.

Proof. Since b is an upper bound of $\{a_n | n \in N\}$, it follows from the Completeness Property of the real number system that $\{a_n | n \in N\}$ has a least upper bound, say L. We shall show that $\lim (a_n)=L$. Let p be any natural number; then $L - 10^{-p}$ is not an upper bound of $\{a_n | n \in N\}$. This means there is a member of $\{a_n | n \in N\}$, say a_q, such that $L - 10^{-p} < a_q$. Since $a_t \geq a_q$ whenever $t>q$, it follows that a_t differs from L by less than 10^{-p} whenever $t>q$. In other words, given a natural number p, there is a term of the sequence such that all following terms differ from L by less than 10^{-p}. Hence, $\lim (a_n)=L$. This establishes Theorem 4.2.

A sequence (a_n) is said to be an increasing sequence if $a_n \leq a_{n+1}$ whenever n is a natural number. A sequence (a_n) is said to be bounded above if there is a number, say b, such that $b \geq a_n$ for each n. Using these terms, Theorem 4.2 asserts that each increasing sequence which is bounded above is convergent.

To illustrate this theorem, consider the sequence (a_n) where $a_n = r + r^2 + r^3 + \ldots + r^n$ and $0<r<1$. Since this is an increasing sequence, we need only show that the set of terms of this sequence has an upper bound, in order to demonstrate that the sequence converges. But, for each n, $r.a_n = r^2 + r^3 + r^4 + \ldots + r^{n+1}$ hence, $a_n - r.a_n = r - r^{n+1}$; therefore

$$a_n = \frac{r - r^{n+1}}{1 - r} = \frac{r}{1 - r} - \frac{r^{n+1}}{1 - r} < \frac{r}{1-r}.$$

Thus, $a_n < \frac{r}{1-r}$ for each n. This establishes that (a_n) is increasing and bounded above; therefore, (a_n) is convergent. Furthermore, it is easy to see that

l.u.b. $\{r + r^2 + r^3 + \ldots + r^n \mid n \quad N\} = \frac{r}{1 - r}$. Hence, by Theorem 4.2, $\lim (r + r^2 + r^3 + \ldots + r^n) = \frac{r}{1 - r}$ provided $0<r<1$.

We shall now establish one more result concerned with the sequence $(\frac{a_n}{b_n})$ discussed in Theorem 3.4 of the preceding section. In that theorem it is assumed that (b_n) is convergent and $\lim (b_n) \neq 0$. However, sequences $(\frac{a_n}{b_n})$ such that $\lim (b_n) = 0$ are of special interest. Obviously, Theorem 3.4 does not help here. Actually, we can't say much about the convergence of $(\frac{a_n}{b_n})$; however, we can demonstrate the important result that $\lim (a_n) = 0$ is a <u>necessary</u> condition for the convergence of $(\frac{a_n}{b_n})$, given that $\lim (b_n) = 0$.

<u>THEOREM 4.3:</u> The sequence $(\frac{a_n}{b_n})$ is not convergent if $\lim (b_n) = 0$ and $\lim (a_n) \neq 0$ -- assuming that $b_n \neq 0$ for each n.

<u>Proof.</u> First, let us be clear about the meaning of the statement "$\lim(a_n) \neq 0$". We mean that 0 is <u>not</u> the limit of (a_n) ; that is, either $\lim (a_n)$ is a number other than 0, or else (a_n) is not convergent. In short, we mean that the statement "given any natural number p, we can find a natural number q such that $|a_n|<10^{-p}$ whenever $n<q$" is false. Therefore, $\lim (a_n) \neq 0$ iff there exists a natural number t such that

(1) given q we can find n such that $n>q$ and $|a_n| \geq 10^{-t}$.

Now, suppose, contrary to the theorem, that $(\frac{a_n}{b_n})$ is convergent. Then there is a natural number k such that $|\lim (\frac{a_n}{b_n})| <10^k$. It follows that $|\frac{a_n}{b_n}|$ does not exceed 10^k if n is large enough; more preciselyk there exists a natural number q_1

such that

(2) $\left|\frac{a_n}{b_n}\right| \leq 10^k$ whenever $n > q_1$.

Since $\lim (b_n) = 0$, there exists a natural number q_2 such that

(3) $|b_n| < 10^{-k-t}$ whenever $n > q_2$.

Let q be the larger of q_1 and q_2. Then, from (1), there exists a natural number, say s, such that $s > q$ and $10^{-t} \leq |a_s|$. Furthermore, from (3) $|b_s| < 10^{-k-t}$. Therefore, $10^{-t}, |b_s| < 10^{-k-t} |a_s|$; i.e. $10^k < |a_s/b_s|$. But, from (2), $|a_s/b_s| \leq 10^k$. Contradiction! This establishes Theorem 4.3.

The following is merely a rewording of Theorem 4.3.

COROLLARY 4.3: $\lim (a_n) = 0$ if the sequence (a_n/b_n) converges and $\lim (b_n) = 0$.

We shall make use of the following result subsequently.

THEOREM 4.4: The sequence (c_n) converges and $\lim (c_n) = L$ provided that:

(i) $a_n \leq c_n \leq b_n$ whenever $n \in N$

(ii) $\lim (a_n) = \lim (b_n) = L$.

Proof. The proof is left as an exercise.

We now consider some useful results.

DEFINITION 4.1: A sequence (b_n) is said to be a subsequence of (a_n) iff (b_n) can be obtained from (a_n) by deleting specified terms of (a_n).

For example, $(1,3,5, \ldots, 2n - 1, \ldots)$ is the subsequence of (n) obtained by deleting alternate terms of (n). Also, (a_{n+5}) is the subsequence of (a_n) obtained by deleting the first five terms of (a_n).

THEOREM 4.5: If (b_n) is a subsequence of (a_n) and

(a_n) converges, then $\lim (b_n) = \lim (a_n)$.

Proof. The details are left as an exercise.

THEOREM 4.6: Let (b_n) and (c_n) be subsequences of (a_n), and suppose that each term of (a_n) is a term of (b_n) or of (c_n); then $\lim (a_n) = L$ if $\lim (b_n) = \lim (c_n) = L$.

Proof. The details are left as an exercise.

DEFINITION 4.2: A sequence (a_n) is said to be bounded if there is a real number K such that $K \geq |a_n|$ whenever $n \in N$.

For example, the sequences $([-1]^n/n)$ and $(1/n^2)$ are bounded, whereas (n) is not bounded.

THEOREM 4.7: Each bounded sequence possesses a convergent subsequence.

Proof. The details are left as an exercise.

THEOREM 4.8: If $\lim (a_n) = A$ and $a_n \geq 0$ for each n, then $\lim (\sqrt{a_n}) = \sqrt{A}$.

Proof. In view of Corollary 4.2, $A \geq 0$; hence, there are two cases.

1. Assume $A=0$. Given p, we want to find q such that $\sqrt{a_t} < 10^{-p}$ whenever $t>q$. By assumption, corresponding to the natural number $2p$, there is a natural number q such that $a_t < 10^{-2p}$ whenever $t>q$; therefore, $\sqrt{a_t} < 10^{-p}$ whenever $t>q$.

2. Assume $A>0$. Given p_1, we want to find q such that $|\sqrt{a_t} - \sqrt{A}| < 10^{-p_1}$ whenever $t>q$. We begin by choosing a natural number p such that $p>p_1$ and $10^{-p} < \sqrt{A}$. We shall find a natural number q such that $|\sqrt{a_t} - \sqrt{A}| < 10^{-p}$ whenever $t>q$; clearly, this will achieve our purpose. Now, $\sqrt{a_t}$ differs from $\sqrt{A}$ by less than 10^{-p} if

(1) $\sqrt{A} - 10^{-p} < \sqrt{a_t} < \sqrt{A} + 10^{-p}$

But if c and d are any positive numbers, then $c<d$ iff $c^2<d^2$; therefore, (1) is true iff

(2) $(\sqrt{A} - 10^{-p})^2 < a_t < (\sqrt{A} + 10^{-p})^2$

Subtracting A from each inequality, we see that (2) is true iff

(3) $(\sqrt{A} - 10^{-p})^2 - A < a_t - A < (\sqrt{A} + 10^{-p})^2 - A$

We now make use of the assumption of our theorem. Choose p_2 so that $10^{-p_2} < |(\sqrt{A} - 10^{-p})^2 - A|$ and $10^{-p_2} < (\sqrt{A} + 10^{-p})^2 - A$. Since $\lim (a_n) = A$, there is a natural number q such that $|a_t - A| < 10^{-p_2}$ whenever $t>q$. We have found our q! Clearly, (3) is true provided $t>q$; thus, reading back through our chain of inequalities, $|\sqrt{a_t} - \sqrt{A}| < 10^{-p} < 10^{-p}1$ provided $t>q$. This establishes our theorem.

It is easy to generalize Theorem 4.8.

THEOREM 4.9: If $\lim (a_n) = A$ and $\sqrt[k]{a_n}$ exists for each n, then $\lim (\sqrt[k]{a_n}) = \sqrt[k]{A}$.

Proof. A proof can be constructed from the proof of Theorem 4.8; the details are left as an exercise.

Exercises

1. Assume that the sequence (a_n) is such that $a_n \geq b$ and $a_n \geq a_{n+1}$ whenever n N. Prove that (a_n) is convergent and $\lim (a_n) = $ g.l.b. $\{a_n | n \quad N\}$.
2. Prove Theorem 4.4.
3. Prove Theorem 4.5.
4. Construct a sequence (a_n) and a subsequence of (a_n), such that the subsequence is convergent though (a_n) is not.

5. Prove Theorem 4.6.

6. A sequence (c_n) is said to be monotonic iff either $c_n \leq c_{n+1}$ for each n, or $c_n \geq c_{n+1}$ for each n. Prove that each sequence possesses a monotonic subsequence.

   Hint: Consider the sequence $(a_i, a_j, \ldots)$ where a_i is the largest term of (a_n), the given sequence, a_j is the largest term of (a_{n+i}), and so on. What happens if (a_n) has no largest term?

7. Prove Theorem 4.7.

8. Assume that (b_n) is convergent and $a_n \geq b_n$ for each n; is (a_n) necessarily convergent?

9. Show that the statement: "Given a natural number p we can find a natural number q such that $|a_n| < 10^{-p}$ whenever $n>q$" is false iff the statement: "There exists a natural number p such that for any choice of q we can find a natural number n with the property that $n>q$ and $10^{-p} \leq |a_n|$" is true.

10. Observing that $1 + 1/1! + 1/2! + \ldots + 1/n! \quad 2 + 1/2 + 1/2^2 + \ldots + 1/2^n < 3$ whenever $n \quad N$, prove that the sequence $(1 + 1/1! + 1/2! + \ldots + 1/n!)$ is convergent.

11. * Prove the Cauchy Criterion: (a_n) converges iff given any natural number p there is a natural number q such that $|a_n - a_g| < 10^{-p}$ whenever $n>q$.

12. Prove Theorem 4.9.

5. An Important Application

It was stated in the introductory page of Part I of this article that many apparently different concepts can be expressed in terms of sequences. We shall show in this section how the geometric concept "the line tangent to a curve at a point on the curve" can be characterized using sequences.

Let f be a function whose graph is the curve shown in the diagram; suppose (a, b) f so that the point (a,b) is on the curve. We wish to determine the line tangent to the curve at (a,b). First of all, we must be sure that we know just what is meant by the line tangent to a curve at a point on the curve. The intuitive idea is to join the given point to a nearby point on the curve, so obtaining a line; then to obtain another line by joining the given point to a still closer point; repeating without end, we obtain a sequence of lines which somehow point out one line, the limiting line -- and this line is the tangent line. Notice that here there is an undefined idea, the limiting line of a sequence of lines; moreover, there is the vague idea of a "nearby" point; so the intuitive idea presented here will not do as a mathematical definition of this concept. However, we can quite easily clear up the blurry concepts presented

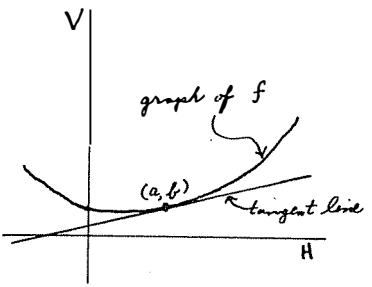


Figure 5.1

above, and so reformulate the intuitive idea as a precise mathematical definition. First, let us consider the family of all lines through the point (a,b). Different members of this family have different slopes, so a member of the family is known once its slope is known. Since the tangent line is a member of this family, we see that what we are after is the slope of the tangent line!. In other words, we must produce a formula which tells us, in terms of the given function f and the given point (a,b), the slopes of the line tangent to the curve at (a,b). This we accomplish as follows: Let (a_n) be any sequence such that $\lim (a_n) = a$, $a_n \neq a$ and a_n is a member of the domain of f for each n. Think of (a_n) as providing us with first-co-ordinates of points on the curve.

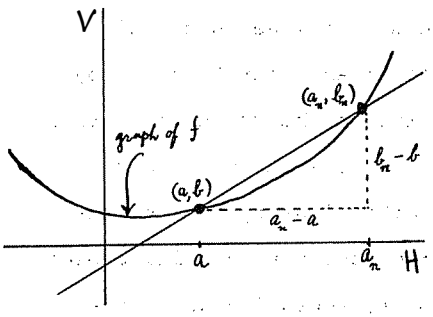


Figure 5.2

Thus, if a_n is the first coordinate of a point on the curve, the second co-ordinate, which we denote by "b_n", is $f(a_n)$. Now consider the line through the points (a,b) and (a_n,b_n). The slope of this line is

$$\frac{b_n - b}{a_n - a} \quad \text{which is} \quad \frac{f(a_n) - f(a)}{a_n - a}$$

With each term of the sequence (a_n) we can associate the slope of the line joining (a,b) to the point on the curve whose first coordinate is that term of the sequence.

In this way, we obtain another sequence, namely, the sequence of slopes $\left(\frac{f(a_n) - f(a)}{a_n - a} \right)$.

If this sequence has a limit, say m, and if this limit is independent of the particular sequence chosen, in this case (a_n), then we say that m is the slope of the line tangent to the curve at (a,b). Furthermore, in case

$$\left(\frac{f(a_n) - f(a)}{a_n - a} \right) = \left(\frac{1}{c_n} \right)$$

where $\lim (c_n) = 0$, we shall say that the tangent to the curve at (a,b) is parallel to V. Otherwise, there is no line tangent to the curve at (a,b).

To illustrate the exceptional case of the definition, let us find the line tangent to the graph of $\{(a,b) \mid b=a^{1/2}\}$ at $(0,0)$. Let (a_n) be any sequence such that $\lim (a_n) = 0$, $a_n \neq 0$ and $0<a_n$ for each n. We consider the sequence of slopes: $\left(\frac{a_n^{1/2}-0}{a_n - 0} \right) = \left(\frac{1}{a_n^{1/2}} \right)$. Since $\lim (a_n) = 0$, it follows that $\lim (a_n^{1/2}) = 0$; hence the tangent line is parallel to V and passes through $(0,0)$, and so is V.

To illustrate the general definition, let us determine the line tangent to the graph of x^2 at $(3,9)$. Applying our definition, we consider any sequence (a_n) such that $\lim (a_n) = 3$ and $a_n \neq 3$ for each n. Forming the sequence of slopes, we obtain the sequence $\left(\frac{a_n^2 - 9}{a_n - 3} \right)$; that is $(a_n + 3)$. Let the slope of the tangent line be m; then $m = \lim (a_n + 3) = \lim (a_n) + \lim (3) = 3 + 3 = 6$. Clearly, the value of m is independent of the sequence (a_n) chosen.

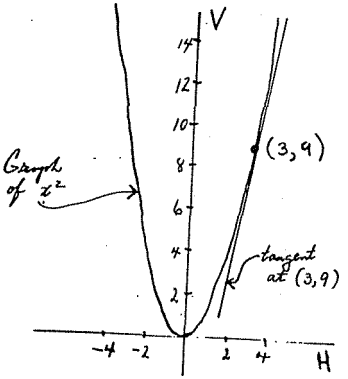


Figure 5.3

Therefore, the line tangent to the graph of x^2 at (3,9) has slope 6, and so is the graph of $6x - 9$.

It is just as easy to find the slope of the line tangent to the graph of x^2 at (a,a^2). Let (a_n) be any sequence such that $\lim (a_n) = a$ and $a_n \neq a$ for each n.

Denoting the slope of the tangent line by m,

$$m = \lim \left(\frac{a_n^2 - a^2}{a_n - a}\right) = \lim (a_n + a) = \lim (a_n) + \lim (a)$$

$$= a + a = 2a.$$

Again, this result is independent of the particular sequence (a_n) chosen. Therefore, the slope of the line tangent to the graph of x^2 at (a,a^2) is $2a$.

As another example, let us find the slope of the line tangent to the unit circle with centre at the origin, at the point (a,b) on the circle, where $b \neq 0$ (then $|a| \neq 1$). Note that the upper half of the circle is the graph of the function $\{(a,b) | b = \sqrt{1-a^2}\}$, while the lower half of the circle is the graph of the function $\{a,b) | b = -\sqrt{1-a^2}\}$. Let (a,b) be a point on the upper semi-circle where $b \neq 0$; we shall compute the slope of the tangent to the upper semi-circle at (a,b). Let (a_n) be any sequence such that $\lim (a_n) = a, a_n \neq a$ and $-1 \leq a_n \leq 1$ for each n. Constructing the sequence

of slopes, we obtain the sequence

$$\left(\frac{\sqrt{1-a_n^2} - \sqrt{1-a^2}}{a_n - a}\right).$$ But

$$\frac{\sqrt{1-a_n^2} - \sqrt{1-a^2}}{a_n - a} = \frac{a^2 - a_n^2}{(a_n-a)(\sqrt{1-a_n^2} + \sqrt{1-a^2})}$$

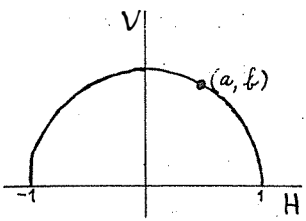
$$= - \frac{a + a_n}{\sqrt{1-a_n^2} + \sqrt{1-a^2}}.$$ Hence,

$$\lim\left(\frac{\sqrt{1-a_n^2} - \sqrt{1-a^2}}{a_n - a}\right) = - \lim\left(\frac{a + a_n}{\sqrt{1-a_n^2} + \sqrt{1-a^2}}\right)$$

$$= - \frac{\lim (a + a_n)}{\lim(\sqrt{1-a_n^2} + \sqrt{1-a^2})} = - \frac{\lim (a) + \lim (a_n)}{\lim(\sqrt{1-a_n^2}) + \lim(\sqrt{1-a^2})}$$

upper semi-circle

$$= - \frac{-2a}{2\sqrt{1-a^2}} \quad \frac{-a}{\sqrt{1-a^2}} = - \frac{a}{b}.$$



Note that $\lim(1-a_n^2) = \lim (1) - \lim(a_n^2)$ $= 1-a^2$; therefore by Theorem 4.8, $\lim (\sqrt{1-a_n^2} - = \sqrt{1 - a^2}$.

Figure 5.4

This establishes that the slope of the line tangent to the upper semi-circle at (a,b) is $- \frac{a}{b}$ provided $b \neq 0$. In the same way, we can show that the slope of the line tangent to the lower semi-circle at (a,b) is $- \frac{a}{b}$. Together, these results tell us that the slope of the line tangent to the circle at (a,b) is $- \frac{a}{b}$, provided

$b \neq 0$.

Next we determine the tangent to the upper semi-circle with radius 1 and center at the origin, at the point $(1,0)$. Let (a_n) be any sequence such that $\lim (a_n)=1$ and $-1 \leq a_n<1$ for each n. Constructing the sequence of slopes we obtain the sequence $(\sqrt{1-a_n^2}/(a_n-1))$.

Now $$\frac{\sqrt{1-a_n^2}}{a_n-1} = \frac{\sqrt{(1-a_n)(1+a_n)}}{1-a_n} = -\frac{\sqrt{1-a_n}\sqrt{1+a_n}}{1-a_n}$$

$$= -\frac{\sqrt{1+a_n}}{\sqrt{1-a_n}} = c_n; \text{ But } \lim (1/c_n)=\lim(-\frac{\sqrt{1-a_n}}{\sqrt{1+a_n}}$$

$$= \frac{-\lim(\sqrt{1-a_n})}{\lim(\sqrt{1+a_n})} = 0 \quad \text{since } \lim(\sqrt{1-a_n}) = 0 \text{ and}$$

$\lim(\sqrt{1+a_n}) = \sqrt{2}$. Therefore, our tangent is parallel to V.

Finally, let us determine the points of intersection of the circle and its tangent at (a_1,b_1) where $b_1 \neq 0$. We must find all ordered pairs in both $\{(a,b) \mid a^2 + b^2 = 1\}$ and $\{(a,b) \mid b = \frac{-a_1}{b_1}(a-a_1) + b_1\}$, where $a_1^2+b_1^2=1$. Obviously, (a_1,b_1) is in both sets. Suppose (a,b) is in both set; then

$$a^2 + b^2 = 1 \quad \text{and} \quad b = \frac{-a_1}{b_1}(a-a_1)+b_1 = \frac{-a_1a+(a_1^2+b_1^2)}{b_1}$$

$$= \frac{1-a_1a}{b_1}. \quad \text{Therefore } a^2 + (\frac{1-a_1a}{b_1})^2 = 1, \text{ and so}$$

$$a^2b_1^2 + 1 + a_1^2a^2 - 2a_1a = b_1^2 \text{ ; hence}$$

$$a^2(a_1^2 + b_1^2) - 2a_1a + (1 - b_1^2) = 0, \text{ thus}$$

$a^2 = 2a_1 + a_1^2 = 0$, $(a - a_1)^2 = 0$, therefore $a = a_1$.

It follows that $b = b_1$; hence, (a_1, b_1) is the only ordered pair in both sets. Thus, this tangent to the circle meets the circle at one point only.

BOOKS AND REVIEWS

The response to our request for your help in this new department has been less than spectacular. This is your magazine, not ours. Your reviews editor cannot continue to impose on his colleagues for contributions. Besides, we want to spread the word around about books which you have found interesting and helpful. To repeat from the previous issue:

We seek your cooperation. Anyone who wishes to submit book titles with or without reviews (preferably with) is urged to do so. Please send them to:

D. A. Clarke, Reviews Editor,
Ontario Mathematics Gazette,
c/o Department of Mathematics,
University of Toronto,
Toronto, Ontario.

Two secondary-school teachers have submitted reviews for this issue at our request. To them and the other reviewers, we extend your thanks.

How many of you know about the Schaum's Outline Series, Schaum Publishing Company, 257 Park Avenue South, New York, N.Y., 10010? This is a series of the usual "outline" type that has a number of mathematical topics in its list. They are of considerably higher quality than is usual in this type of book. Their titles are for College and University subjects, but since this refers to the United States, many of them are applicable, at least in part, to our revised mathematics curricula.

Each one is packed with examples and worked exercises; and is written so that each chapter is, as nearly as possible, independent of the rest. The subject matter ranges from elementary to moderately advanced, sometimes within the same book.

Titles include: Trigonometry, Analytic Geometry, First Year College Mathematics, College Algebra, Calculus, Set Theory and Related Topics, Modern Algebra, Matrices, Statistics, and others.

Last year the Ontario Curriculum Institute published, in pamphlet form the reports of two committees of the Ontario Mathematics Commission whose work had been underwritten by the Institute. These pamphlets are: Mathematics, the four year programme, and Mathematics, report of the committee considering the mathematics programme (K to 6). It has been brought to our attention that the distribution of these pamphlets has not been as wide as is desirable. These pamphlets should be read by every teacher. If you have not seen these pamphlets, check to see if copies are lying around your school somewhere. Otherwise, copies may be obtained from:

The Ontario Curriculum Institute,
Suite 303,
344 Bloor Street West,
TORONTO 4.

The Price is 75 cents each.

DAC

Meserve, Pettofrezzo, Meserve, Principles of Advanced Mathematics, L. W. Singer Company, 1964.

Chapters in this book are entitled: Sets of Elements -- Permutations, Combinations and Probability Measurements of Geometric Figures -- Circular Systems of Numbers -- Linear and Quadratic Expressions -- Polynomials Special Functions (Power Exponentials, Absolute Value, Probability, density, other density functions) -- Sequences, Series, and Limits -- Vectors -- The Algebra of Matrices.

The Teacher's edition has 758 pages, including 166 pages of a Teacher's Manual and numerous annotations in red print throughout the book.

According to the preface, Principles of Advanced Mathematics is designed to prepare students for a course in analytic geometry and calculus as their next course in

Mathematics. Throughout the book there is an emphasis upon the underlying mathematical structure and the use of proven pedagogical approaches to the subject matter. Careful attention has been given to readability. New terms are defined; accuracy and precision are considered; topics are related to the students experience in algebra and geometry.

With reference to Curriculum S. 12C, Mathematics B, Units 1, 2, 3, Sets; Permutations; Mathematical Induction and Binomial Theorem; Probability, are covered fairly well in Chapter 1 and 2 with the exception that the fundamental counting principles are not all covered.

Chapter II covers unit 4, Vectors, in the manner suggested in S 12C, in that the introduction to vectors is made without the use of a coordinate system. Later in the chapter coordinates are introduced. All the material in Unit 4 is covered quite adequately, except that no reference is made to vectors as ordered pairs of triples. This chapter also includes most of the work in Unit 5, Equations of Lines and Planes, S 12C. There is an excellent summary at the end of the chapter in the work that has been introduced.

The material in the first part of Chapter 7 and in Chapter 12 provides good background material for Unit 6, Systems of Linear Equations, and Unit 7, Matrices and Linear Transformations, S 12C. The reader will have to be selective to get the required content of Units 6 and 7. Even then, not all sections of these two units are covered. The last part of chapter 6 covers unit 8, S 12C, Complex numbers and Polar coordinates, with the exception that solutions of quadratic equations and absolute value are not covered. Unit 9, S 12C, Groups, is not adequately covered in this book.

In Mathematics A, S 12C Unit 1, Logic is not covered in this book. Units 2 and 5, Functions and Transformations in the Plane, are not dealt with from the mapping approach, although a great deal of these sections are discussed in Chapters 7 and 12, but not from a mapping point of view.

Unit 3, Second Degree Equations in the Plane,

is covered to a great degree in chapter 7, with the exception that the focus -- directrix definition and development of the equation of ellipse and hyperbola are not given. One section of this chapter, deals with obtaining the equations of the conics by suitably performing a translation or rotation of the axes or the graph with equations

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, A^2 + B^2 + C^2 \neq 0.$$

Chapters 4 and 5 deal with the work in Unit 4, Trigonometry.

Units 6, S 12C, Slopes and Simple Derivatives, is partially covered in this book. It includes a section on limits, finite and infinite sequences, and limits of such. Alsi, a section on area under a curve, by the summation process, is included. Chapter 10 concludes with a section on Limits of Functions and rates of change. There is no further work on calculus in this book.

In conclusion, I feel that this book would be a valuable reference book for the new Grade XIII Mathematics Program.

L. A. Towe

Glicksman and Ruderman, Fundamentals for Advanced Mathematics, Holt Rinehart and Winston, 1964.

Although this book was written for the "College capable" student in the United States, it could also serve a useful purpose in Ontario, as a supplementary textbook for the A and B mathematics courses of grade 13, since it covers most of the topics in Curriculum S 12C.

The first six chapters - The Nature of Mathematics, Logic, Methods of Proofs, Sets, Algebra of Sets, and Relations and Functions - give a meticulous treatment of the foundations of mathematics. They alone might well constitute the basis of a good inservice training program for prospective teachers of grade 13. Since some of the basic definitions and notations differ from

those in most other textbooks, the reader would be well advised to study thoroughly these early chapters before proceeding further in the book. The remaining eighteen chapters make consistent use of these earlier fundamentals, and may be studied in a variety of sequences, as suggested by the tree diagram which is found opposite page 1 in the text. These chapters include: Mathematical Induction, Permutations, Combinations, Probability, Limits, Introduction to Calculus, Trigonometry, Analytic Geometry, Vectors in 2 and 3 Dimensions, Complex Numbers, and Coordinate Systems.

The authors have taken a precise, and in some cases original, approach to the material. This is immediately evident in Chapter 1, where the terms proposition, statement, sform (statement formula) and sentence are each assigned a unique meaning, different from usual definitions. Thus the inconsistencies of "closed sentence" and "open sentence" are cleverly circumvented. In chapter 2, we see a distinction made between a conditional and an implication. A conditional (denoted $p \rightarrow q$, and read "if p then q") may be either true or false (not both) for all permissible substitutes, whereas an implication (denoted $p \Rightarrow q$, and read "p implies q") is defined as a true conditional for all permissible substitutes. This precision is evident in the exercises; for example, "Prove that $(p \wedge q) \Rightarrow (p \rightarrow q)$". However these few examples can hardly do justice to the detailed systematic development of the fundamentals found in the first 185 pages.

The style of the book is mathematically sound. The authors have clearly stated the undefined terms, and have developed the topics by means of definitions, axioms, and theorems, interspaced with meaningful examples. The exercises are abundant, interesting, and varied in style and purpose.

As the title suggests, the authors have attempted to write a book for the capable high school student, which establishes with firmness and accuracy the essential fundamentals for more advanced mathematical studies - a book which will ease the transition from high school to university. In the opinion of this reviewer, they have succeeded.

G. A. Scroggie

Aleksandrov, A.D.; Kolmogorov, A.N.; Lavrent'ev, M.A. (Editors) Mathematics: Its Content, Methods, and Meaning The M.I.T. Press Cambridge Massachusetts 1963 Vol. I (358 p.p.), Vol. II (376 p.p.), Vol. III (351 p.p.) $30. a set.

This three volume set of books contains 20 chapters collected into six parts. Each chapter gives andaccount of andimportant branch of mathematics, written by a Russian expert in the field. The set has been translated from the 1956 Russian version by the American Mathematical Society.

The level of exposition varied from chapter to chapter. Some chapters would be intelligible to a good high school student while others would require a certain background in the subject for a full appreciation.

The following is a list of the chapters: I: A GENERAL VIEW OF MATHEMATICS II: ANALYSIS III: ANALYTIC GEOMETRY IV: ALGEBRA: THEORY OF ALGEBRAIC EQUATIONS V: ORDINARY DIFFERENTIAL EQUATIONS VI: PARTIAL DIFFERENTIAL EQUATIONS VII: CURVES AND SURFACES VIII: THE CALCULUS OF VARIATIONS IX: FUNCTIONS OF COMPLEX VARIABLE X: PRIME NUMBERS XI: THE THEORY OF PROBABILITY XII: APPROXIMATIONS OF FUNCTIONS XIII: APPROXIMATION METHODS AND COMPUTING TECHNIQUES XIV: ELECTRONIC COMPUTING MACHINES XV: THEORY OF FUNCTIONS OF A REAL VARIABLE XVI: LINEAR ALGEBRA XVII: NON-EUCLIDEAN GEOMETRY XVIII: TOPOLOGY XIX: FUNCTIONAL ANALYSIS XX: GROUPS AND OTHER ALGEBRAIC SYSTEMS.

The set resembles in many ways the familiar book "What is Mathematics?" by Courant and Robbins. However, the discussions in the Russian book are considerably more detailed, and a larger number of topics is covered. Thus the Analysis chapter, and the chapter on Curves and Surfaces could be used by a beginner to pick up the rudiments of the subject. Many of the chapters are somewhat unconventional for an introductory account. For example the chapter on Analytic Geometry has an extensive discussion of affine, orthogonal and projective transformations.

The motivation for many of the chapters is the consideration of some physical problem. Thus the chapter just mentioned has a discussion of the Lorentz transformation of special relativity, and before discussing the solution of partial differential equations, the author actually derives some.

A few chapters (notably XI on Probability and XIX on Functional Analysis) are too short or too compressed to give anyone, except an expert, more than the vaguest ideas about these subjects. A useful feature of the book is the way history is interwoven with mathematics. For example, Chapter IV has a very interesting account of the attempts to solve algebraic equations leading up to the work of Lagrange, Abel and Galois. Group theory and Galois theory are then explained and current problems in this field are stated.

A somewhat irritating aspect to the book is the over emphasis on Russian discoveries even in cases where non-Russian work on the subject is at least equally important. An example of this is the exclusive use of the name of the mathematician Ostrogrodskii to refer to Stokes' theorem (as it is called in western countries).

Some chapters are out of date (Chapter XIV on Computing Machines) because of advances in technology in the past ten years; other chapters (Chapter VIII on the Calculus of Variations) were out of date when they were written because the author has used an old-fashioned approach.

The translation is reasonably good and quite readable, although some phrases sound more Russian than English. In the odd place the transliteration of Russian names is not consistent. Thus A.M. Lyapunov and A.M. Ljapunov are both given separate listings in the index and different page references, but they are actually the same person.

Despite the above criticisms the three volumes should be in every high school library, for they provide a unique useful and readable reference for many branches of mathematics.

R. A. Ross

P.J. Davis, The Mathematics of Matrices, Blaisdell, New York, 1965.

This is an entertaining and delightful book intended as a high school or first year university textbook. The author introduces matrices as arrays, and brings out the connection with linear transformations in chapter 4. The development is carried as far as the definition of characteristic values. The subject is treated concretely rather than abstractly with stress on two and three dimensions and a casual approach to rigour. As explained in the preface, the author approaches matrices with the eye of the analyst and the experience of the applied mathematician and thus regards matrix theory as mathematical grammar rather than fiction or belles-lettres. The style is informal, the exposition clear, and the pace leisurely. There is a wealth of treasure in the 350 pages of this book. It should be read by anyone who teaches matrices.

G. C. Bush and P. Obreanu, Basic Concepts of Mathematics, Holt Rinehart and Winston, New York, 1965

This textbook is based upon first year lectures at Queen's University and is designed to lay a suitable foundation for later mathematics courses. Many topics are introduced but not developed in much detail. An idea of the subjects discussed can be gleaned from the chapter headings which are as follows: Introduction--The Language of Mathematics, Part One - The Theory of Sets (Algebra of Sets, Equivalence Relations, Functions, Order Relations), Part Two - Number Systems and Algebraic Systems (The Integers, Rational Numbers and Real Numbers, Complex Numbers, Groups, Rings and Fields), Part Three - Analytic Geometry (Vector Algebra, Lines and Planes), Part Four - Calculus (Sequences and Series, Limits and Continuity, Derivative and Integral).

The emphasis throughout is theoretical rather than manipulative. The style is clear but austere and no special effort is made to motivate the reader. Although much terminology is introduced, there is also genuine mathematical content. There are many well chosen problems and a serious reader should work a large fraction of them.

Although this reviewer is inclined to doubt that the contents of this book form a suitable diet for first year undergraduates, he recommends the book strongly to a high-school teacher well trained in mathematics as it was taught twenty or thirty years ago who wishes to learn the contemporary idiom.

W. T. Sharp

Now that Probability has become an integral part of the Grade 13 curriculum, many teachers will be looking for reliable reference material from which to draw information and enrichment material for Probability. The following list of books may prove useful in this regard.

1. Emile Borel, Elements of the Theory of Probability (translated by J. E. Freund), Prentice-Hall, 1965.
2. William Feller, An Introduction to Probability Theory and its Applications, John Wiley second Edition, 1964.
3. Samuel Goldberg, Probability on Introduction, Prentice-Hall, 1960.
4. Fredrick Mosteller, R.E.K. Rourke, G. B. Thomas, Probability and Statistics, Addison Wesley, 1961.

Of these books items 3 and 4 will be directly useful in classwork because they are meant for students at approximately the Grade 13 level. Item 4 was the text for Continental Classroom, the early morning educational programme put on by one of the main television networks in the U.S. Its exposition is clear and direct and it contains many worked examples.

Item 3 covers in great detail the material in the new Grade 13 curriculum from roughly the same standpoint and includes many useful combinatorial problems. I recommend it highly.

Items 1 and 2 will not be directly useful in the classroom for they are meant for a more mature audience than the average Grade 13 class. Item 2 is meant for non-mathematicians and does not employ sophisticated notions. However, the book covers a surprising amount of material considering that it uses only elementary methods. Secondary school teachers

will find that reading only a few chapters of this book will provide them with knowledge in depth which should be an invaluable aid in teaching Grade 13 Probability.

Item 1 is of historical as well as pedogogical interest, for it is a translation of the second edition of a book which first appeared in 1909 when modern Probability Theory was in its infancy. The first edition has had a profound effect upon the directions which modern Probability took. This translation of the second edition is accessible to anyone with a slight knowledge of the calculus and presents many fascinating problems which relate elementary Probability Theory to other disciplines.

B. Brainerd.

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