8-PAGE INSERT ELEMENTARY MATH

Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

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March 2013

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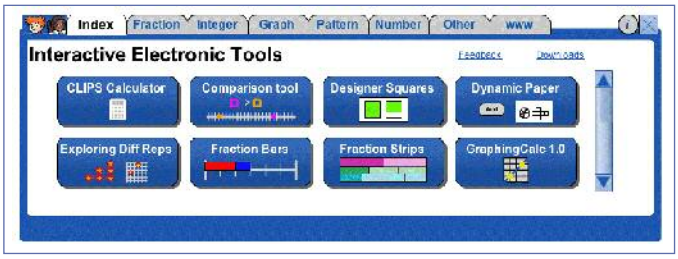
See In the Middle – The Museum of Mathematics Special Opening on 12-12-12

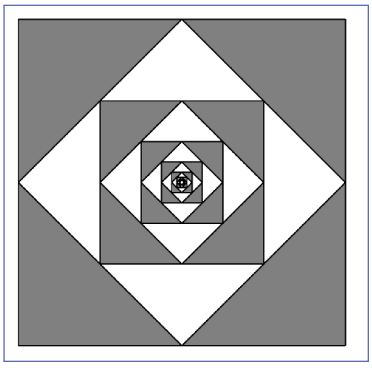


See Celebrate The International Year of Statistics With Census At School!



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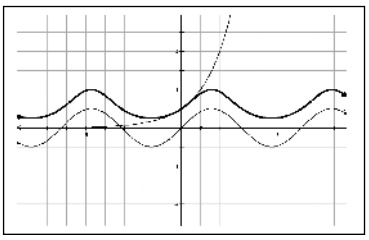




See What's the Problem? Open It Up

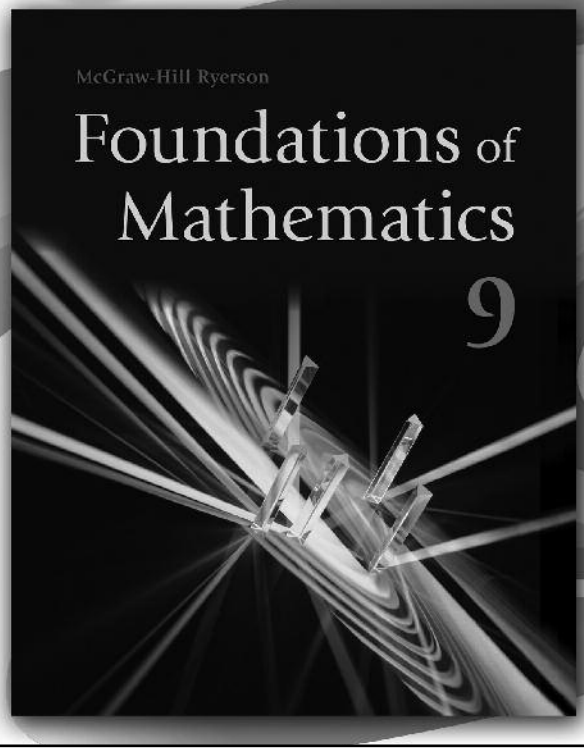
See Stratgies for Teaching Composition of Functions

$f(x) = 2^x$ $g(x) = \sin(x)$ $f(g(x)) = 2^{\sin(x)}$



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Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Immaculate K. Namukasa, along with **separate files for each of the text, graphics, and/or photographs**. Please e-mail all files to Immaculate K. Namukasa at **immaculate.namukasa @oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and e-mail address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME.

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Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be camera ready and colour advertisements should have no bleeds.

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▲ Editor's Message

IMMACULATE KIZITO NAMUKASA



Immaculate Kizito Namukasa is an associate professor at the University of Western Ontario. Her current research interests in mathematics education include mathematical thinking and activity, mathematics teacher education, non-routine problem solving and learning environments, complexity research framework, and critical mathematics education. Namukasa is also interested in globalization and internationalization of schools and schooling in general.

Teaching with technology is not only mandated by the curriculum, but it is also expected by twenty-first-century learners. It engages and motivates learners. This first 2013 issue of the OAME *Ontario Mathematics Gazette* is comprised of two special features sharing a web-based resource for teaching mathematics, in particular, the Critical Learning Instructional Paths Supports, CLIPS. One of these special features shares experiences of two teachers who used CLIPS. The other, the second part to the piece that appeared in the December issue of the *Gazette*, illustrates how to use new tools now available within CLIPS. The article, *Does anyone have another way?* by Kajander, Boland, and Casasola, offers ways of supporting students' alternate methods of solving mathematics problems. Eliciting multiple solutions and non-standard algorithms is central to encompassing new visions of teaching mathematics. In the article entitled *Strategies for teaching composition of functions*, Sibbald shows how characteristics of functions being composed in turn influence the characteristics of the composition function. This focus pushes the boundary beyond the standard mechanics of procedures. Yan and Gibbs share emerging places where learning and teaching resources formerly available through the educational outreach program at the Statistics Canada website may be accessed. Throughout the remainder of this issue of the *Gazette*, regular contributors: (a) share resources and activities for teaching (e.g., the NCTM feature by Romiens; the *What Is the Problem?* feature on non-routine geometrical mathematics problems by Godin; and the *In the Middle* feature, in which Rosenbloom

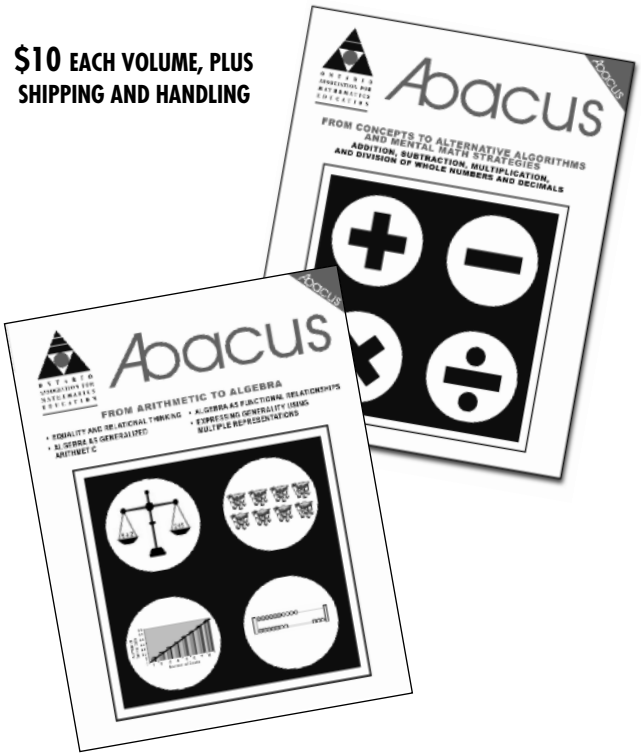
shares her experience of a new Math Museum); (b) expound on big ideas in mathematics (e.g., *M4BT* feature on distinctions between continuous and discrete data); (c) offer professional learning ideas for teachers (e.g., the *President's Message* on the role of positive attitudes in learning); and (d) report on activities in the mathematics education community (e.g., Fields and NCTM events of interest to readers). As can be seen, activities in the mathematics and mathematics education community are diverse and continue to expand in directions that were previously unforeseen.

If you are looking for more ways of showing your students that mathematics is useful for understanding and addressing pressing global issues, listen up! On March 5, 2013 the UNESCO launch of the Mathematics of Planet Earth 2013 (MPE2013) will take place in Paris. Endorsed by UNESCO, MPE2013 is a worldwide initiative that aims to bring awareness to planetary issues and to the role that mathematics can play in their solution. The MPE2013 initiative has Canadian origins: Professor Christiane Rousseau from the University of Montreal, brought it to life by assembling, first, Canadian, and then world, mathematicians and other scientists, to create international research efforts focused on planetary challenges. The Canadian launch of MPE2013 took place in December at the 2012 Canadian Mathematical Society (CMS) Winter meeting. To become more knowledgeable about this initiative, one can access the MPE2013 website, **http://mpe2013.org/**, where educational materials (including teaching resources, posters, podcasts, and web links) are shared. It is there that I found an intriguing podcast titled, *Sounding the alarm*, by Walter Craig, from McMaster University, that outlines the mathematics involved in detecting and alerting those at risk in the event of a tsunami. The educational materials, most of which are lesson activities organized by grades, can be retrieved via the educational tab on the website. For instance, curriculum materials are available at **http://mpe2013.org/curriculum-material/**. This drop-down menu also provides links to several NCTM lessons on two themes of the MPE2013 initiative: *A Planet to Discover* and *A Planet Organized by Humans*. The other two themes are: *A Planet Supporting Life* and *A Planet at Risk*. MPE2013 themes cleverly showcase how mathematics can be learnt in conjunction with other school subjects such as social studies and biological sciences. Selected MPE2013 themes could be adopted when incorporating ecological or social justice issues in teaching mathematics. Evidently, the mathematics we teach in elementary and secondary schools is useful in diverse domains of work and life. ▲

ABACUS: A COMPILATION

Volume 46 – This resource for teachers includes the concepts of addition, subtraction, multiplication, and division of whole numbers and decimals, alternative algorithms, mental math strategies, manipulatives, and links to literature. It will be useful for exploring these aspects of teaching and learning personally or with a team of teachers. Includes one two-coloured bead string.

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Volume 47 – This resource for teachers includes the concepts of equality and relational thinking, algebra as generalized arithmetic, algebra as functional relationships and expressing generality using multiple representations. With links to manipulatives and literature, it will be useful for exploring these concepts personally or with a team of teachers.

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▲ President's Message – Attitude Is Everything!

CATHY HALL, PRESIDENT



"Mathematicians stand on each other's shoulders."

I was recently lucky enough to visit a classroom where math enthusiasm and excitement literally jumped off the walls… and I'm not referring only to the amazing, interactive Math Word Wall that was being generated and curated by the students themselves. That in itself was impressive, since most Word Walls, if they even exist, are the bailiwick of the teacher. However, the teacher in this case had encouraged his students to make the Math Wall their own, to unlock new math vocabulary and concepts in a very real and authentic way for themselves and their peers. As inconsequential as this may seem on the surface, I learned that it was one of the main contributing factors in a turnaround in his students' attitude and success in math. As I talked with the students and observed interactions within their groups, I was struck by how comfortable they seemed to feel as they discussed new concepts and explored possible problem-solving avenues and strategies.

I neglected to mention that this teacher had some inspirational "wall words" of his own, posted prominently for all to see and, hopefully, internalize. My gaze immediately landed on a brightly coloured banner lettered in a funky script. It read, "ATTITUDE IS EVERYTHING!" I'm a sucker for motivational messages—and this one definitely struck a chord with me. I firmly agree with Judy Willis's premise in *Learning to Love Math* (2010) that "The first step to success in math is a positive attitude… When you help your students build a positive attitude toward math, they become engaged in the material and motivated to excel in mathematics because they value it." She then goes on to show how this can result in a "supportive math community," where students feel comfortable getting involved, taking risks, setting goals, and fully investing themselves in their learning.

My favourite inspirational catch phrase, both in my own classroom and in facilitating PD workshops, has always been "MAKE IT HAPPEN!" I like to think of it as a softer, friendlier version of the slogan made famous by Nike: "Just do it!" However, the underlying message is the same—that readiness to learn and the effort to carry through must come from within the individual.

In an ideal world, this positive attitude would permeate the teaching and learning experience in each of our classrooms. However, the reality is that too often the daily stressors and inherent math anxiety, when it comes to math itself, get in the way. I was struck recently by the following passage in an autobiography I was reading. Although it referred to a music classroom, I could easily visualize a similar scenario being played out in a math class, or any other curriculum discipline for that matter.

"In Mr. Wainwright's music room, I discovered that I was petrified of standing up in front of the class and singing. It wasn't shyness so much as a fear of being singled out and made to look ridiculous… It was for this reason that I discovered the Fake Sick Trick." *

It's hard to believe that the anxious, performance-phobe being referred to here was Rod Stewart, the music icon who went on have an amazing music career, selling more than 200 million records! He has also been twice inducted into the Rock and Roll Hall of Fame, is a Grammy Living Legend, and has played the world's biggest concert. Still and all, it took him many years to adopt a more positive attitude toward his "strange, raspy voice" and build up his confidence after the negative experience of those early days in music class.

All of this set me to thinking about how our students feel as they enter our math classrooms. What type of atmosphere greets them as they come through the door? I remembered having come across a student's perspective on this very question and hunted down the passage in the book *Classroom Instruction That Works*. I'd like to share a bit of it with you:

> Marisa and her friend, Alex, two middle school students, are on their way to their second period class. Alex complains that Mr. Sommerville's class isn't very interesting, and he doesn't learn much in it.
>
> **Marisa:** I'm looking forward to my class with Ms. Hastings. She makes everyone feel like they can learn math. She always says "Hi" to us when we come into class, and she expects us to get right

** I won't give you all the nasty details here, but I will divulge that the trick involved mashed potatoes and other assorted remnants from the school lunch program!*

down to work. She has a problem on the board for us to solve on our own, or we check in with our base group.

Alex: What's a base group?

Marisa: It's a group that stays together for a semester. We do some team building activities and other fun stuff to get to know one another. That makes it easier to speak up in class.

Alex: Mr. Sommerville is usually busy at his desk when we come in, and we just talk until a few minutes after the bell. Then he asks us to be quiet, and he starts his lecture. Mostly, we're just bored. Some kids even put their heads down on their desks.

Marisa: We'd never do that in Ms. Hasting's class! She'd think we were sick or something. She expects us to do our best every day, and she tells us stories of people who worked and succeeded. She likes us to share stories like that about ourselves or other people we know. We even keep track of how much effort we're putting into our work and how much we're learning. Sometimes I get discouraged, but Mr. Hastings always says something to help me see what I *do* know and what I can do to get better.

Alex: I wish Mr. Sommerville was more like Ms. Hastings. Maybe then I'd care more about learning and get better grades.

These students have said it much better than I can. They've painted a vivid picture of two very different classroom approaches and performance expectations… and their very real effects on students. One teacher created a rich classroom atmosphere, where creative problem solving and risk taking are encouraged. A key element in this type of environment is the understanding that individual student achievement levels are not predetermined by their past performance. When students are given this vote of confidence and the chance to prove themselves, the results can be quite dramatic. Not only do they begin to invest themselves more fully in their learning, but they also start to actually believe that they *can* be successful in math! And, once that momentum is set in place, the stage is well and truly set for learning on a grand scale!

As I told you earlier, one of my favourite mantras with teachers and students alike is "Make it happen!" You might think that this is a tough call for the math-phobes out there, but I don't agree. The tools are all in place; we just have to use them. One of the most powerful tools is the curriculum itself. All the PD in the world will not, and cannot, make a difference when it comes to effective curriculum delivery and student success if teachers are not familiar with the curriculum expectations themselves. Once again, attitude is everything. If you view your math curriculum as a never-ending checklist of expectations that you "can't possibly cover" within the school year, then you're probably feeling just like those students who are so overwhelmed—or bored—that they can be found with their heads on their desks. Instead, try to find the bits and pieces within that curriculum that excite you… and then take a minute to analyze why that is. Chances are that the same real-world connections or just plain weird math that you discover will be the key to unlocking a more positive attitude among your students. Then give yourself permission to run with that excitement and let your students do likewise. I never cease to marvel at the wild and wonderful learning that can result from the excitement generated by powerful, and often unexpected, paths in math. Some of the detours that I've seen work amazingly well are storytelling about famous mathematicians (e.g., Gauss, Fibonacci, Pascal), hands-on construction projects showing the usefulness of the Pythagorean Theorem, read-alouds of books linked to math concepts (e.g., *Ten Black Dots, Chasing Vermeer, The Number Devil*), linkages to the arts (e.g., M.C. Escher, the *Golden Ratio*), puzzles (e.g., *Tower of Hanoi*) and game playing (e.g., the game of Sprouts). No matter which direction you choose to take, there can be amazing "Aha!" moments for your students along the way as they make connections to their prior math knowledge.

Of course, we all know that similar "Aha!" moments can occur in our regular lessons as well. However, they do not magically materialize without employing a few helpful tips and strategies to help things along. One of the biggest mistakes we make as teachers is keeping our lesson goals and intended outcomes a secret from students. Instead, how about using explicit cues to alert students to what they are about to encounter in the lesson? It should go without saying that we need to take the time to identify and discuss what our students already know about the topic or problem at hand. However, with time pressures and tight course outlines, we sometimes skip this critical step in the process… and we shouldn't. If we take the attitude that we don't have time to help students access their prior knowledge through the use of inferential questions or advance

organizers, then we need to reset our thinking. Students' prior knowledge must be brought to the forefront in order to set the stage and allow them to engage with new information being presented or investigated.

A word of caution, however. As recent research shows, teachers have been found to ask an inordinate number of questions during class, sometimes as many as 120 questions per hour! Unfortunately, 70 to 95 of those questions are lower-order, rote-memory questions. Even more alarming is the finding that the wait time between asking the question and eliciting a response is generally less than two seconds! Consequently, students experiencing math anxiety can be seen to exhibit a number of blocking/coping strategies when confronted with this rapid-fire questioning, e.g., averting their gaze, adopting a pondering countenance while maintaining complete silence, waiting patiently until the teacher moves on to another student or actually gives in and answers the question, or even escaping to the washroom to avoid being put on the spot. Remember that "Fake Sick Trick" that Rod Stewart felt he had to employ? Well, he's not alone! I have a sneaking suspicion that if we asked our students how many of them have felt the need to use one or more of these coping strategies on a regular basis, we would get a much-needed insight into how our attitudes and teaching strategies are affecting many of our students. If, like me, you firmly believe that our intentions are the best and that we truly care about providing a nurturing learning environment for our students, don't be afraid to take a step back on a regular basis and view your math class through a different lens. Despite the untold stresses and pressures inherent in our profession, you can "make it happen" by ensuring that your classroom is one where "attitude is everything"!

References

Dean, C.B., Hubbell, E.R., Pitler, H., & Stone, Bj. (2012). *Classroom instruction that works: Research-based strategies for increasing student achievement.* (2nd ed.). Alexandria, VA: Association for Supervision and Curriculum Development.

Doctorow, R. (2002). *Mathematics education: A summary of research, theories, and practice.* Toronto: Thomson Nelson.

Willis, J. (2010). *Learning to love math.* Alexandria, VA: Association for Supervision and Curriculum Development.

▲

▲ Pennies are History

WILLIAM LUNDY



Bill is a retired public school educator and retired reserve force officer. As a teacher, he served 40 years with the Hastings and Prince Edward District School Board (HPEDSB). As a reserve force officer, Bill began as an officer with Cadets, then moved to the primary reserve, where he had a number of duties that culminated in a tour of duty as Commanding Officer of HMCS ONTARIO from 2000 to 2004.

Bill was invited to become a member of Advanced Education's Education Consultant team in September 2011. In this capacity he has worked with K-12 classrooms from Brampton to Cornwall, and delivered workshops, seminars, and formal SMART training courses. He is a certified SMART trainer for Notebook Software versions 10 and 11, for SMART Notebook Math Tools, and is a SMART Certified Lesson Developer. He currently serves as a director for the Ontario Association for Mathematics Education (OAME) and was past co-chair of its annual conference held in Kingston in 2012.

If you are looking for a resource to help your students understand more about the demise of the penny as a medium of cash exchange, and how to help them grasp the skills of rounding off to the nearest five cents, check out the following file at the SMART™ Exchange (**www.exchange.smarttech.com**). "Pennies Are History" is a SMART Notebook file that links to a well-made video explaining the "Why" about the penny's demise; it then uses a number-line model to show several examples of rounding up and down; finally, there are some practice pages.

When you go to the SMART Exchange, simply type "Pennies Are History" into the search box, and you should find that this file is the first result. Look for the "No Penny" image on the title page. The full URL to the file is **http://exchange.smarttech.com/details.html?id=3b8102fc-3834-4bcb-8443-322cb42402d7**

Here's a screen capture of the title page:



▲ Technology Corner – the New Tools of Teaching

MARY BOURASSA



Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

Where do you find new ideas? I have mentioned that Twitter is my number one source for inspiration. I heard about most of what you will read in this column through Twitter. It is a great way of finding new ways to teach old topics and is also a great place to share what you are doing with others. This method of collaborating from afar allows us all to grow as teachers and springboard off of each other's ideas. Following the #mathchat hashtag will also give you many new ideas and links to great blogs and activities.

1. Mathalicious

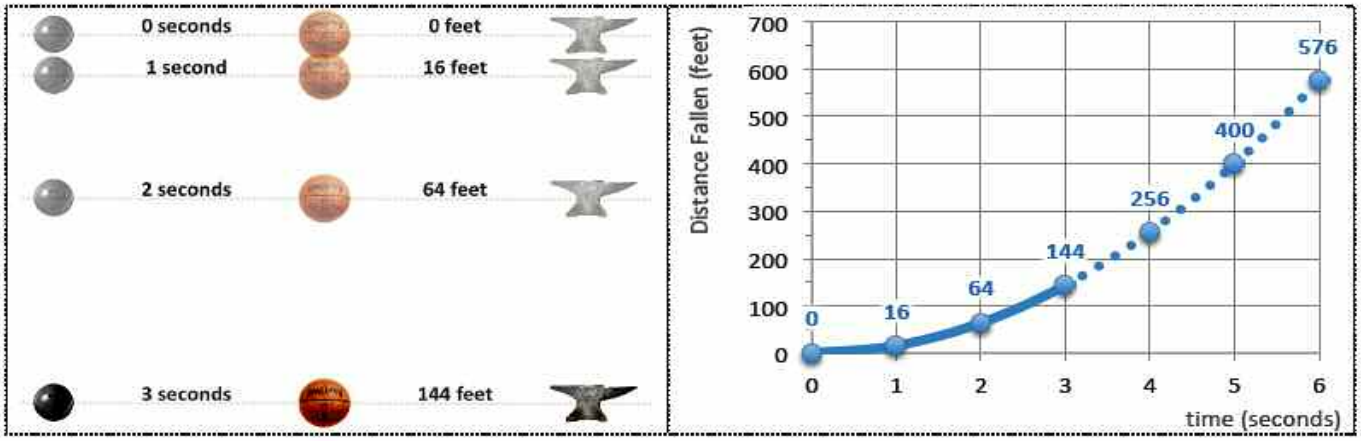
Mathalicious (**www.mathalicious.com**) is a website for math teachers by math teachers. The lessons found on this site deal with real-world topics that will engage your students. They also intend to inform your students to make them better consumers, more informed citizens and allow them to make healthier choices. Students' critical-thinking and problem-solving skills are developed through open-ended questions requiring justification and connections between algebraic, numerical, and graphical solutions.

Lessons are sorted by grade level, elementary through high school, and by topic. Each lesson comes with a fantastic lesson guide (containing solutions) and a student handout. The student handout is broken into three acts, and act four allows for extensions.

Below is a sample of Lesson Guide for Act One of a lesson:

Act One **20-25 min.**

In Act One, students watch an animation of three objects falling and graph the data. They use this to predict the distances after 4, 5 and 6 seconds, and use their reasoning to determine the height of Inspector Javert's bridge.



From the animation, students should recognize that with each additional second, the objects fall further and further, i.e. the distance is not proportional to the time. **This is not intuitive**, and you may want to spend a few minutes having students discuss what this graph really means before having them predict the distances at 4, 5 and 6 seconds. Questions might include:

- *How far did the object fall in its first second? In its second?*
- *When you double the time (e.g. from 1 to 2 seconds), does the distance also double?*
- *Why doesn't the distance increase by the same amount with each additional second?*
- *If the distance* did *increase by the same amount, what would the graph look like? (Ans: a straight line)*

A Mathalicious subscription is available on a "pay what you can" system, designed for teachers paying out of pocket. Departmental subscriptions are also available. Well worth it!

2. Three-Act Catalog

http://mr-stadel.blogspot.ca/p/3-act-catalog_17.html

Andrew Stadel has a catalog of three-act math tasks tagged by topic. Many are great for middle school, but there are also some for Grades 9 and 10. I have used the Styrofoam Cups activity (before it was posted here) and the Basketball Shots activities with my Grade 10 Applied class. You can also find links, among others, to Dan Meyer's blog, 101 Questions (a site dedicated to pictures and videos that may perplex you enough to ask a question), Estimation 180 (Andrew's site aimed at developing number sense through estimation), and Math Mistakes.

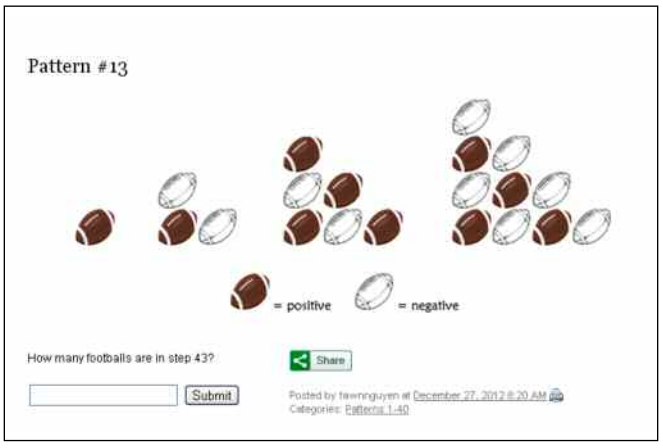
3. Visual Patterns

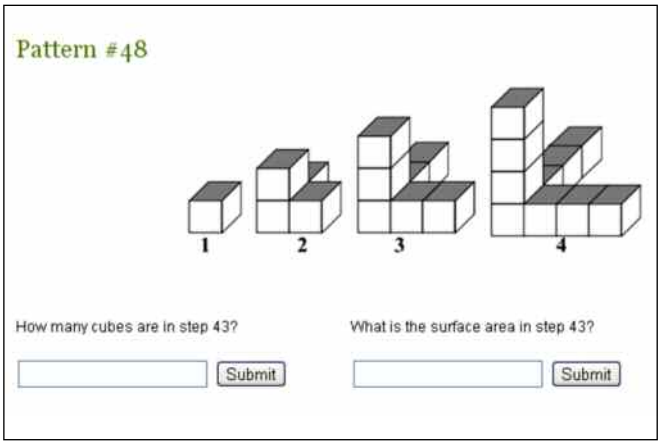
http://visualpatterns.org

This site, created by Fawn Nguyen, is dedicated to showing patterns of all sorts. There are blocks and numerous shapes and even an input–output machine. You are prompted to enter the number of items in step 43. If you enter the correct value, the equation representing the pattern will pop up! (Kudos go to Chris Robinson @absvalteaching for that feature.) If you search http://fawnnguyen.com, you can also find a document with the equations to all the patterns.

Teachers and students can also submit their own patterns. Not only does this site give you numerous patterns to use with your students, but it may also spark you to create new patterns.

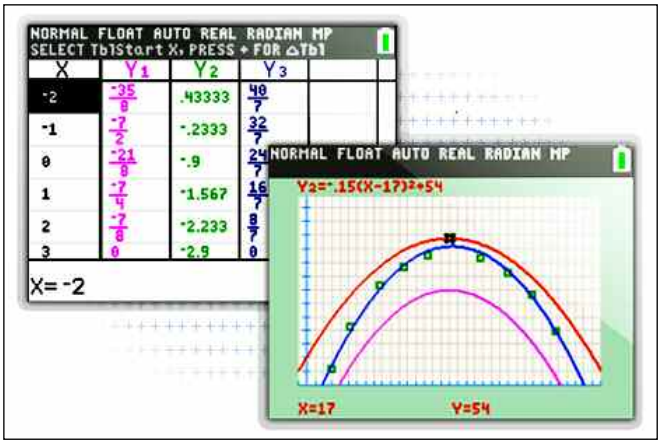
Here is a sample of what can be found:





4. Colour TI-84

The TI-84 Plus C Silver Edition is the latest in the line of TI-83/84 graphing calculators. It features a high-resolution, full-colour backlit display. This allows for greater connection between equations, data, and the corresponding graphs through colour coding. There are also grid lines available, making graphs easier to read. Pictures can be imported and inserted as the graph background, with equations being graphed on top, allowing for more real-world connections. These calculators now come with rechargeable batteries (via USB cable or charging station). It will be available this Spring. TI continues to provide activities on their website (**http://education.ti.com**)—there are over 2500 that use the TI-84 family of graphing calculators!



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▲ OAME/NCTM Report March 2013

TODD ROMIENS



Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a Past President of OAME and a life member, as well as a previous editor of the Gazette*.*

Did you know that "56 percent of students would rather eat broccoli than do math homework." This was the result of a survey taken and reported in NCTM's *SmartBrief* in December 2012. There was very little information about the survey, except for this result, so I wouldn't put much credence in generalizing. I just thought it was an interesting question on which to collect data.

NCTM is producing a new series of books called Annual Perspectives in Mathematics Education (APME). One volume will be produced each year. The first APME volume, called *Using Research to Improve Instruction*, will be published in 2014 and will focus on bringing research to practice and practice to research in mathematics education. Teachers are invited to submit chapters that discuss research projects, including classroom-based action research projects and collaborations that present findings and information that are relevant to both researchers and practitioners.

NCTM has created another neat resource called *Dynamic Paper*. This was created by teachers, for teachers. It allows you to create nets, graphs, spinners, pattern blocks, tessellations, and much more, then export the image for use in Word or PowerPoint or download an 8.5" x 11" PDF, ready for student use.

This applet allows you to create the following:

Nets – two-dimensional outlines of three-dimensional shapes, including regular polyhedra, prisms, pyramids, cylinders, and cones

Graph Paper – coordinate graphs, polar coordinates, logarithmic graph paper

Number Lines – including positive and negative coordinates

Number Grids – hundreds boards and the like

Tessellations – tiling patterns involving triangles, quadrilaterals, and hexagons

Shapes – pattern blocks, attribute blocks, and coloured tiles

Spinners – up to 16 sectors with adjustable sizes

Select the tab for the type of image you want, enter the values for the parameters, and click "Add" to insert the image into the workspace. On the workspace, you can adjust the size of the image. To change the other characteristics, you will need to change the values of the parameters on the dashboard.

The "Reset" button will restore the default values for all parameters.

The "+" and "–" keys can be used to create additional pages, and the arrow keys can be used to move back and forth between them. Images that you create will be inserted on the current page; if you delete the page, images are deleted also.

The eraser button will delete all images on the current page. The trash icon will delete all pages and erase all images, and a new blank document can be started.

Under the NCTM illuminations, you can find a nice little program for younger students to work with equivalent fractions. Students can create equivalent fractions by dividing squares or circles, and then matching each fraction to its place on a number line. Google "NCTM illuminations equivalent fractions."

Along the top, choose a square or a circle. Use the sliders to increase or decrease the number of columns and rows or sectors.

Once the shape is divided as you want it, click on the sections to colour them. Make the green and blue fractions equivalent to the red one. Then click on the "check" buttons to see if your fractions match. (Note that you must use different denominators for your fractions.)

Use the "new fraction" button to generate a new red fraction. You can also Build Your Own red fraction to start.

You can also do some problem solving by building fractions with prime number denominators and trying to find appropriate equivalent fractions. Ask, "Is it possible to create a fraction such that no equivalent fractions can be created with this applet?"

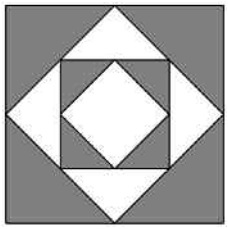
Also, remember the NCTM annual conference is in Denver, Colorado from April 17–20, 2013. If you are able to attend, it's always a great learning experience. ▲

▲ What's the Problem? Open It Up

SHAWN GODIN



Shawn is the head of Mathematics, Technology, and Family Studies at Cairine Wilson Secondary School in Orleans, Ontario. He is a past and present member of the OAME Board of Directors and is currently Editor-in-Chief of Crux Mathematicorum with Mathematical Mayhem.



Welcome back, problem solvers. Last time, I left you with the following problem:

Tell me everything you can about the diagram to the right.

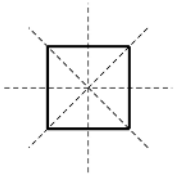
At this point, I usually tell you the source of the problem. In this case, I won't because… I don't remember! I know that the problem is modified in some form, and I drove myself crazy searching for the original "inspiration." I couldn't find it, but that isn't a problem because the diagram should look familiar to most of you. The difference is I just opened the problem up.

For the last couple of semesters, I have been using this problem very early in the Grade 9 course. My intention is to engage students' prior learning and to get them thinking about some of the mathematics that we will be discussing shortly. Each time I have used this, students have provided me with interesting answers. I find it also helps to train the students to open their minds and think. Even if the question were asking something specific, like "What fraction of the figure is shaded?" (which I think was the original question), it is worthwhile to ask yourself: "What mathematics do we know that could be applied to this situation?," "What else could be asked/can be determined?," and (my favourite question) "What if…?" So let's consider some of the things my students and I have thought about this problem. This won't be a complete list, but it will give you an idea of the richness of topics and sophistication a nice starting point yield.

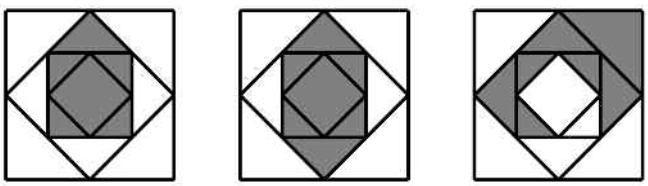
We'll start with the most obvious idea: shape. What types of shapes can we see in the figure and how are they related?

Obviously, we have four squares of different sizes and orientations. At each "stage," the squares are rotated by 45° and diminished in size. As such, if we advance two stages, the square is back to the same orientation. At this point, we could talk about the square having rotational symmetry, so that if a square is rotated by 90°, 180°, 270°, or 360°, its orientation is unchanged. Similarly, we could talk about the fact that we can reflect a square in any of its four axes of symmetry (see diagram below) and it remains unchanged. This could lead to a talk about group theory, specifically dihedral groups, depending on the class.

The next shapes that we encounter are right isosceles triangles. This could lead to a discussion of the various specific names we have for triangles based on the size of their biggest angle (obtuse, right, and acute) and on the number of equal sides they have (equilateral, isosceles, and scalene). We note that there are twelve triangles, arranged in three groups of four. This could lead to a discussion on congruence and similarity. The squares that we first encountered are all similar. For the triangles, we have three sets of four congruent triangles, yet all the triangles are similar. It is interesting to note that **all** squares are similar and, correspondingly, **all** right isosceles triangles are similar. Yet all right triangles are **not** similar. An interesting activity would be to find some other shapes such that all figures of this type are similar, as well as noting which type of shapes are not always similar.

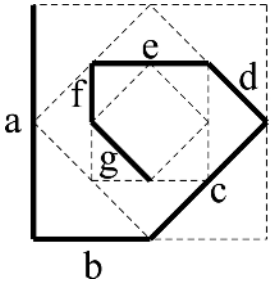


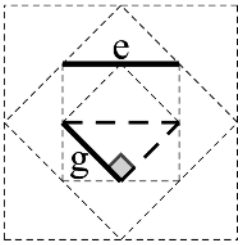
What other shapes can we find? Using only segments from the original diagram, you can find all types of different shapes. Three examples are pictured below. The first two are convex (pentagon and hexagon), while the third is a concave decagon. This could lead to a discussion about properties of convex and concave polygons, as well as a general discussion on polygon names.



What else can we determine about the original figure? Notice that there are line segments of different lengths which we label in the diagram below. By the

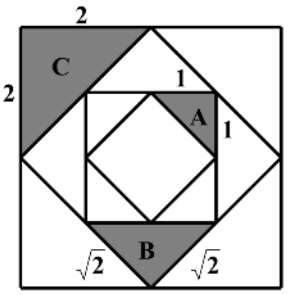
construction of the figure, we know $b = \frac{1}{2}a$, $d = \frac{1}{2}c$ and $e = \frac{1}{2}f$. Using the Pythagorean theorem yields $g^2 = f^2 + f^2$, hence $g = \sqrt{2}f$. Similarly, $e = \sqrt{2}d$, and $c = \sqrt{2}b$. If we draw some diagonals into the original figure, we would see more relationships of this type. The second diagram below shows that $e = \sqrt{2}g$; therefore, $d = g$.

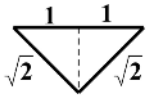




Thus, if we let the smallest length, f, be 1 unit, then the lengths of the segments in the figure, from smallest to largest, are $f = 1$, $d = g = \sqrt{2}$, $b = e = 2$, $c = 2\sqrt{2}$, $a = 4$.

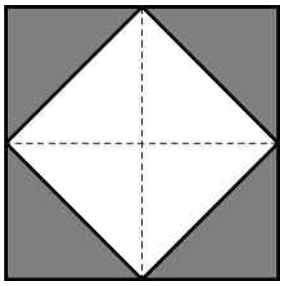
Now we are in a position to calculate some areas. If we look at our triangles, as in the diagram below, we see that $A_A = \frac{1}{2}(1)(1) = \frac{1}{2}$ square units. Similarly $A_B = 1$ square unit and $A_C = 2$ square units. We see the interesting fact that as we move from smaller to larger triangles, the linear dimensions are $\sqrt{2}$ times as large, yet the area is $\left(\sqrt{2}\right)^2 = 2$ times as large. The left-hand diagram below shows this relationship between the areas of figures A and B by drawing the median to the hypotenuse of figure B.





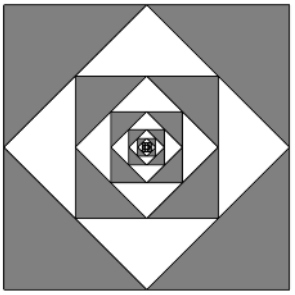
This highlights a property of similar figures. If figure B is similar to figure A, with a scale factor s, then the area of figure B is s^2 times the area of figure A.

If we look at two consecutive similar squares, we can compute the areas and come to the same conclusion. If we draw in some extra lines, as in the diagram below, we can tell without computation that the large square has twice the area of the smaller square.

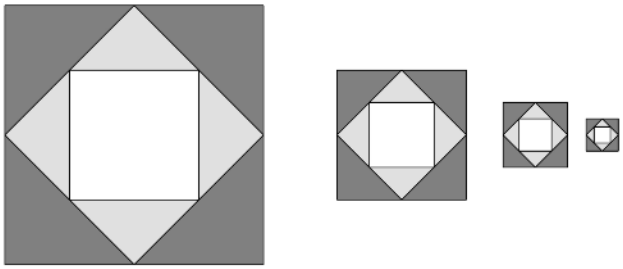


From all these computations of area, we could determine that $\frac{5}{8}$ of the original figure is shaded. It would be a worthwhile exercise to find different ways of showing this without doing direct calculations.

I will finish with one "what if?" question. What if the process outlined were continued indefinitely? If we keep dividing things up, we get the figure below.



What kind of mathematics can we find here? If we change our unit of measure so that the longest side is 1 unit, then the large square has area 1 square unit. A quick calculation shows that the total area of the four largest triangles is $\frac{1}{2}$ square units. The next largest shaded triangles have a total area of $\frac{1}{8}$ square units. If we list the total areas of the shaded triangles of a specific size, we get $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}$, an infinite geometric sequence with initial term $a = \frac{1}{2}$ and common ratio $r = \frac{1}{4}$. Thus, the total shaded area is the sum of an infinite geometric sequence $S_\infty = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}$. We can actually see that this is true by separating the square into a number of square "rings" like the ones shown below.



At each stage, the shaded part of the ring (dark grey) is twice as large as the unshaded part (light grey). Hence, at each level, $\frac{2}{3}$ of the ring is shaded. So, since the original square is made up of a number of rings that are each $\frac{2}{3}$ shaded, the whole square must be $\frac{2}{3}$ shaded.

So we have taken a simple picture that you could ask any number of questions about, but it can lead to: classifying shapes, similarity, congruence, the Pythagorean theorem, area, geometric sequences, group theory, and more. Giving your students the task of using their knowledge to see what they can find will help them make connections, as well as letting you know areas of strength and weakness individually and within the group.

Now for your homework:

1. Keep looking at this problem for yourself. Is there any other mathematics that you can find there that we haven't talked about? Are there any nice "what if…?" questions? I'd love to hear anything interesting you have found.
2. A student has two open-topped cylindrical containers. The larger container has a height of 20 cm, a radius of 6 cm, and contains water to a depth of 17 cm. The smaller container has a height of 18 cm, a radius of 5 cm, and is empty. The student slowly lowers the smaller container into the larger container. As the smaller container is lowered, the water first overflows out of the larger container and then eventually pours into the smaller container. Determine the depth of the water in the smaller container when the smaller container is resting on the bottom of the larger container.

Until next time, happy problem solving!

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▲ Hey, It's Elementary: "The Great Finger Math Debate"

LYNDA COLGAN



Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.

"EVERY STUDENT CAN LEARN, JUST NOT ON THE SAME DAY, OR THE SAME WAY."
~ George Evans

It is given that every elementary teacher has read, heard, and more importantly, lived Evans' profound observation about children's learning. Intuitively, experientially, and empirically, we concur that some children in our classrooms learn best kinaesthetically, while others thrive through auditory or text-based teaching approaches. Some children are isolates—preferring opportunities to problem-solve individually, while many need the support and comfort of peer collaboration and team-learning activities. We know that on occasion, a visual representation one moment magically turns on the proverbial light bulb, even though it was met with a child's blank stare just days beforehand. More often than not, a standard algorithm that was akin to ancient hieroglyphics in November can be executed flawlessly in April. Learning is complex, deeply personal, and absolutely insensitive to arbitrary timelines and grade designations.

So, why is it that some myths prevail to the detriment of children and mathematics learning? I am speaking specifically about taking advantage of those sophisticated *digit*-al computers you and I commonly call fingers.

Across the country, there is an interesting phenomenon—abacus and *chisenbop* (finger math) schools are popping up in major centres and in smaller

communities. The after-school programs, which are modelled after ones in Japan and India, use low-tech tools—such as fingers—to teach children to understand place value and perform complex calculations without the help of pen or paper.

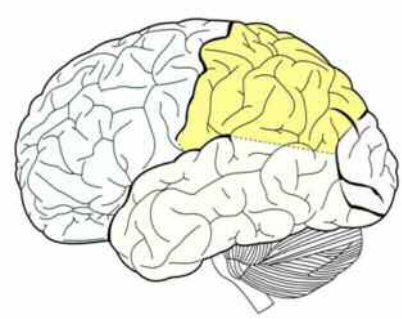
Yet many of us were taught to associate counting on our fingers with either cheating or failing to know our number facts. I vividly remember Miss Richardson (a pseudonym), my Grade 3 teacher at Bruce Public School in Toronto, asking all members of the class to lay their fingers flat on the top of the desk while she asked us one by one, row by row to answer number-fact questions or compute answers to mental math exercises. Sadly, my own lasting image of her is one of a hawkish-like creature peering over the top of cat eye-framed glasses, scanning the room for the slightest finger movements, poised and primed to swoop down on her prey at the minutest movement of muscle.

Could I meet Miss Richardson today, this is what I would say to her: "When a child uses her fingers to do arithmetic, we should not assume that her understanding is immature—she is, in fact, activating the math part of the brain."

Brian Butterworth, a Professor at University College, London, England, is currently working with colleagues on the neuropsychology and the genetics of mathematical abilities. His unique research offers a completely original picture of how our brains do math.

As Butterworth explains, one of the reasons the brain was a mystery for so long is that it is protectively encased by the skull. Under ordinary circumstances, one does not have easy access to see the brain. With contributions from science, engineering, and mathematics came the advent of imaging tools for seeing inside the skull—CAT scans and MRIs—meaning that it is now possible to generate three-dimensional images of the brain that not only allow static views, but also movies that show the dynamic behaviour of the brain.

By producing movies of brain activity from subjects "doing math," Butterworth has generated data that he believes suggests that the left parietal lobe of the brain is where math happens. We all know that some of us are good at math and some of us are not; however, Professor Butterworth contends that the reason a person falters at math is usually not because of the wrong gene or "engine part" in the left parietal lobe, but because he or she has not fully developed the mathematics sense with which we were born.



Butterworth uses an example to illustrate his point. Imaging tells us that the left parietal lobe is also where fingers are registered in our brain—a fact that Butterworth believes not only can explain the reason we naturally count on our fingers, but cries out for why classroom instruction should use our built-in digital computers to enhance computational fluency. Butterworth points out that just as Einstein's brain has been found to have unusually densely packed cells in the left parietal lobe, so musicians have been shown to have a bigger and more elaborately connected motor cortex—the region of the brain that controls hand and finger movements—than non-musicians. This is not a sign that these people (i.e., Einstein or musicians) are predisposed toward math or music, but, argues Butterworth, simply illustrates the fact that that finger practice—with numbers or notes—makes perfect.

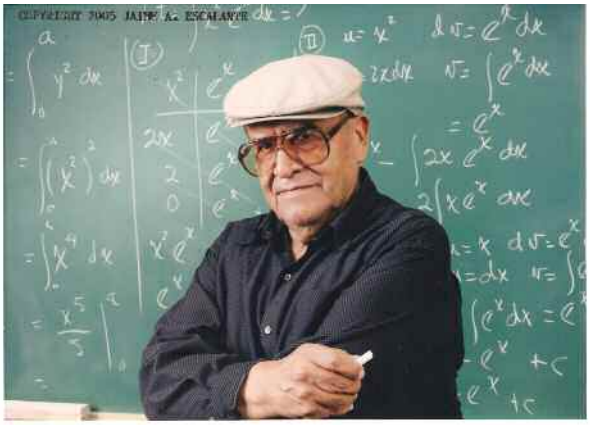


One of the strategies that Butterworth suggests to keep the left parietal lobe dense is to try to find new ways to do old things.

Do you remember learning the multiplication tables? For most people, these are learned, not as abstract patterns of number, but through language, almost like a poem, and it is not unusual for people to recite the table in order to find the answer to a mathematical problem.

There are, however, more brain-based ways to learn how to multiply. Do you remember the movie *Stand and*

Deliver? In this 1988 classic, a true story based on the life of dedicated mathematics teacher, Jaime Escalante, a group of dropout-prone students are inspired by Mr. Escalante in a *Pygmalion*-like turnaround. Except this unique teacher doesn't teach grammar, posture, and manners à la Henry Higgins in *My Fair Lady*, but rather, challenges his students to learn calculus to build up their self-esteem. This cadre of tough guys and street kids does so well that, sadly, they are accused of cheating on a nation-wide Advanced Calculus Placement examination.



In an early scene from the film, mathematics teacher Jaime Escalante confronts Chuco, a defiant gang member, during his class at Garfield High School in East Los Angeles. The encounter goes like this:

Escalante: Ohh. You know the times tables?

Chuco: I know the ones… twos… three.

[On "three," Chuco makes a very impolite finger gesture to Escalante.]

Escalante: Finger Man. I heard about you. Are you The Finger Man?

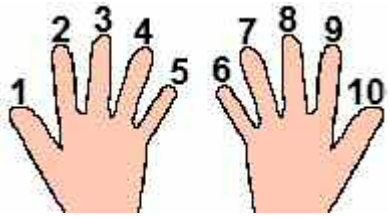
Well, just so you know, I'm the Finger Man, too. Do you know what I can do? [Escalante is holding his hands up to the class, moving his fingers in time with each number fact.] I know how to multiply by nine! Nine times three. What you got? Twenty-seven. Six times nine.

One, two, three, four, five, six. What you got? Fifty-four.

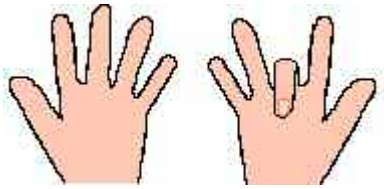
You wanna hard one? How about eight times nine? One, two, three, four, five, six, seven, eight. What do you got? Seventy-two.

To capture Chuco's attention, Escalante was using a well-known finger-multiplication strategy: multiplying by nines by using his fingers as a visual display.

Hold both your hands up with palms facing you. Number the fingers from left to right as one to ten.



Now hold down the finger of the number you want to multiply by nine. In this example, we are going to multiply by eight, so the eighth finger is held down.



The fingers to the left as the tens and the fingers to the right are units. In this example, there are seven fingers to the left and two to the right, which makes 72.

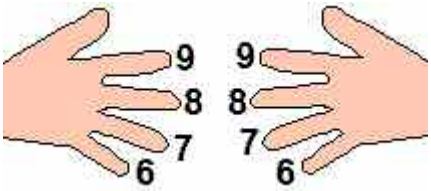


Here is how you would multiply 4 x 9, using your fingers.



You can also use finger math to multiply by six, seven, eight, or nine. This particular method does not apply to multiplying by one to five and is limited to computing products from 6 x 6 to 9 x 9. It is said that this method was widely used in medieval Europe.

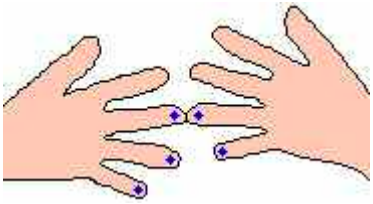
Hold your hands with palms facing you and fingers toward each other. Number the fingers on each hand from bottom to top as six to nine. (The thumb could be ten, but it's easy to multiply by ten, so don't bother!)



Now touch the two fingers of the numbers that you want to multiply. You may need to twist your hand slightly to do this. This example is multiplying eight by seven, so the finger number 8 touches the finger number 7 on the other hand.



Now for the calculation. First, you add the fingers below and including the touching ones. These are the tens. Here there are five of them, three on one hand and two on the other, so that makes 5 x 10 = 50.



Then you multiply the fingers on the left hand above the touching ones with the fingers on the right hand above the touching ones. These are the units. Here there are two fingers on the left and three fingers on the right, so that makes 2 x 3 = 6. Add this to the 50 you've already got, and that makes 56.



In the example above, we separated the tens and units. What happens if you try to multiply 6 x 6? You should find that the tens figure is 2 x 10 = 20, and the units figure is 6 + 6 = 12. Added together, this makes 36, which is correct.

Before these finger-based multiplication methods are dismissed as "tricks" having no mathematical value, I present the following explanations about why such digit-y wizardry works.

The secret to the 9 x 9 finger-math system lies in the fact that for a number to be a multiple of 9, the sum of its digits equals 9. For example, 18: 8 + 1 = 9; or 36: 3 + 6 = 9.

The second part to the secret is that when we multiply a number by 9, the first digit in the product is always 1 less than the number by which we are multiplying. For example, 8 x 9 = 72 and 8 – 1 = 7 ; or 3 x 9 = 27, and 3 – 1 = 2.

When we fold down our finger to multiply by 9, we are basically subtracting 1 from the number by which we're multiplying, and so the fingers to the left of the folded finger give the first number of your product. Then, whatever number it takes to add to 9 is your second number. That's the number of fingers to the right of the folded finger.

The digital system for computing the 6 times table and up was used extensively through medieval Europe and is still used in some rural communities in France.

This method was first described by Leonardo da Vinci in the 1500s in his application of knowledge about *complementary arithmetic*.

If you are multiplying 8 x 7, you would hold your hands this way.



8 x 7: Touch the 8 finger on hand one (your middle finger) with the seven finger on the other hand (index finger). Count the fingers touching and those below to get five. Each of these is a group of ten. Take the remaining fingers on each hand above and multiply (2 x 3 = 6). Next, add this product of 6 to the 50 you found earlier, to get 56.

Compare this to Leonardo's method.

First, write the numbers.

8
7

Next, write the 10-complement for each by subtracting the number from 10.

8	2
7	3

Next, subtract the 10-complement of one number from the other number: 8 – 3 or 7 – 2. This results in 5, the digit for the 10s place. Multiplying the complements gives 6, the value in the ones column.

8	2

7	3
5	6

When you use your fingers, you are acting out this method.

There is little resistance to using beans, beads, blocks, or bingo chips to teach basic calculation and reinforce computational skills and strategies, yet whether or not reliance on finger-based representations is beneficial or detrimental remains the subject of an ongoing debate between researchers in neurocognition and mathematics education.

From the neurocognitive perspective, finger counting provides multi-sensory input, which conveys both cardinal and ordinal aspects of numbers. Recent data indicate that children with good "finger sense," have enhanced mathematical skills. Conversely, research in mathematics education recommends first using finger counting, then concrete structured representations and, finally, mental representations of numbers to perform numerical operations.

I opened this article with the quotation from Evans that underscores the uniqueness of every child's learning trajectory. I close by saying that it is unlikely that a single system can offer everything to everyone, fingers included. And while I welcome debate on the topic, I am prepared to take my stand: I believe that fingers count.

▲

▲ Call for Manuscripts

The *Ontario Mathematics Gazette* is inviting manuscripts for all grade levels.

Instructions for submission of manuscripts are found on page 1 of any *OMG*.

Contact the Editor for further details.

▲ Celebrate The International Year of Statistics With *Census At School*!

ALISON L. GIBBS, UNIVERSITY OF TORONTO



Alison Gibbs is a faculty member in the Department of Statistical Sciences at the University of Toronto. She is interested in the promotion and teaching of statistical thinking at all levels. At the moment, she is developing an introductory statistics Massive Open Online Course (MOOC), a free, open-access, online course designed for large-scale global participation. She currently leads the Census at School Canada project team for the Statistical Society of Canada.

As you may have read in the September issue of the *Gazette*, Statistics Canada discontinued its Education Outreach program in June 2012, including their operation of *Census at School Canada*. We are grateful to the folks at Statistics Canada for their development and stewardship of the *Census at School* program in Canada. In the summer of 2012, the Statistical Society of Canada officially took over its operation.

The Statistical Society of Canada (SSC) is a community of statisticians from academia, government, and industry, run by volunteers. The society's mission is to encourage the development and use of statistics and probability, including the development of public awareness of the value of statistical thinking and the importance of statistics and statisticians in Canadian society and promoting the highest possible standards for statistical education and practice in Canada. See **www.ssc.ca** to learn more about the SSC. Enthusiasm about statistics education and outreach to schools has been growing recently in the SSC. Tangible evidence of this is the formation of a new section in Statistics Education through which we hope to grow our connections with Canadian teachers and students. It is also reflected in our commitment to maintain and continue to build *Census at School Canada*.

Census at School is an international classroom project designed to engage students aged 8 to 18 in statistical reasoning, using data collected about themselves and about participating students from around the world. The *Census at School* website contains detailed instructions on how to get your students involved and lesson plans and ideas for students of all ages. Your students can examine the data they collect about themselves and their classmates and see how they compare to other students from across Canada and other participating countries. You can find the Canadian component of the project at **www.censusatschool.ca**.

What's New at *Census at School Canada*?

New look, same great project.

If you've participated in *Census at School Canada* in previous years, you'll notice a new look. Otherwise, the project runs the same as previously. If you haven't yet used *Census at School* with your classes, you'll find excellent training resources that were developed at Statistics Canada to lead you through.

We're celebrating the International Year of Statistics!

The International Year of Statistics is a worldwide event, supported by more than 1400 organizations, to recognize and celebrate the importance of statistics in virtually every aspect of everyone's life, including the food they eat, the games they play, the technology they use, and the information they receive. For more about the International Year of Statistics and to keep up-to-date on planned activities for schools, see **www.statistics2013.org**.

The 2011–2012 Canadian summary data have been posted.

How does your class compare to the latest cohort of Canadian students? What's been happening in Canada over time?

One of the measurements students collect in *Census at School* is their reaction times on a simple task. At the time of writing this column, I was entering the 2011–2012 summary data on the website. One thing that struck me immediately was an apparent trend with age in the reaction time data. See Table 1 for some of the summary data.

Average reaction time in seconds				
Age (years)	All boys	Right-handed	Left-handed	Ambidextrous
8	0.55	0.54	0.75	0.51
9	0.51	0.51	0.46	0.51
10	0.49	0.49	0.48	0.51
11	0.46	0.46	0.48	0.46
12	0.43	0.43	0.44	0.42
13	0.41	0.40	0.42	0.42
14	0.42	0.42	0.43	0.40
15	0.40	0.41	0.41	0.36
16	0.40	0.40	0.37	0.38
17	0.39	0.38	0.39	0.42
18	0.39	0.38	0.43	0.47
19	0.39	0.39	0.39	Not available

Table 1: *Average reaction time, by age and dominant hand, for boys participating in* Census at School Canada *in 2011–2012. From* ***www.censusatschool.ca/data-results/2011-2012/average-reaction-time****.*

Figure 1 shows how the average reaction time for boys changes with age. Reaction times appear to decrease with age, with the rate of decrease levelling off in the teenage years (Figure 1(A)). (What function might be a useful model to describe the relationship for all the boys?)

The relationship for right-handed boys (Figure 1(B)) is very similar to that for all boys. Of course, this is not surprising, since the right-handed boys comprise more than 80 percent of all boys. (See **www.censusatschool.ca/data-results/2011-2012/handed**.) But what happened to the left-handed and ambidextrous boys (Figures 1(C) and 1(D))? Does the age–reaction time relationship differ, depending on handedness?

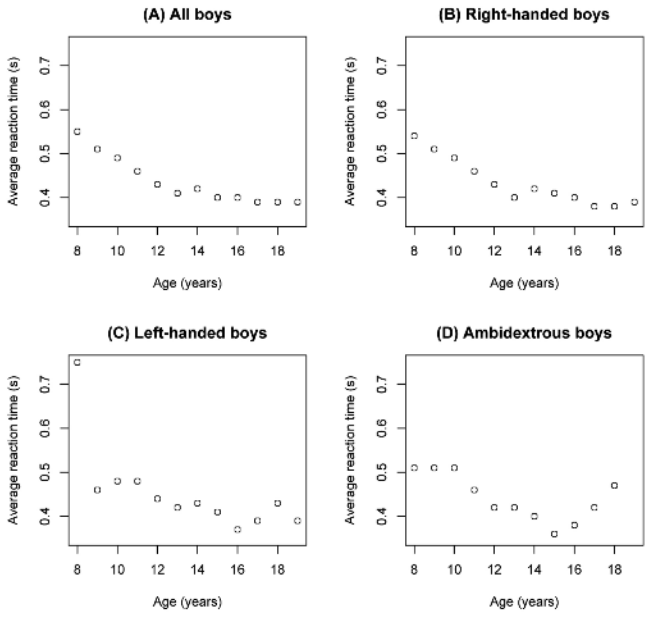


Figure 1: *Plots of average reaction times (in seconds) versus age (in years) for (A) all boys, (B) right-handed boys, (C) left-handed boys, and (D) ambidextrous boys participating in* Census at School Canada *in 2011–2012.*

A more plausible explanation for the apparent differing relationships in the left-handed and ambidextrous boys is sample size. In Canada in 2011–2012, only 52 8-year-old boys participated in *Census at School*. (See **www.censusatschool.ca/data-results/2012-2012/participation-by-age**.) Of all participating boys in elementary schools, 10.4 percent were left-handed and 8.9 percent were ambidextrous. (See **www.censusatschool.ca/data-results/2011-2012/handed**.) For secondary students, 10.7 percent and 7.1 percent of boys were left-handed and ambidextrous, respectively. Moreover, times until an event happens, such as reaction times, typically have right-skewed distributions. For an illustration of the distribution of reaction times, see Figure 2 for histograms of reaction times for random samples of 50 elementary and 50 secondary students. With a right-skewed distribution, the average of a small sample can be unusually large, such as in the case of the 8-year-old left-handed boys. For these boys, a large value from the right tail of the reaction- time distribution was observed, artificially inflating the average. Something similar happened for the 18-year-old ambidextrous boys, resulting in a larger than expected average reaction time. However, the average of a small sample from a right-skewed distribution can also be unusually small, as happened for the 8-year-old ambidextrous boys. For this group of participating boys, by chance, we missed getting any observations in the right tail of the distribution, resulting in a smaller average.

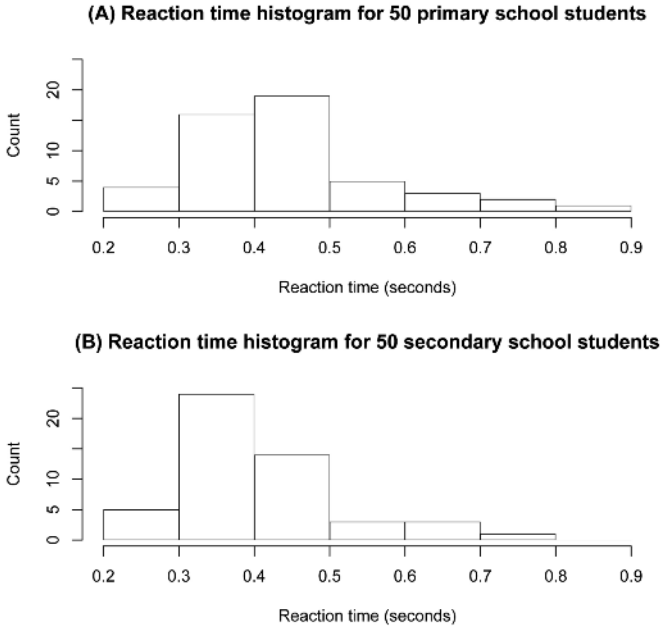


Figure 2: *Histograms of reaction times (in seconds) for random samples of 50 of (A) elementary students and (B) secondary students participating in* Census at School Canada *in 2011–2012. The data were obtained from the* Census at School *random data sampler at* **rds.censusatschool.org.uk**.

When learning about the mean and other measures of centre, students typically investigate the influence of outlying values on the mean. Here we have an illustration of the importance of that concept. When working with means, the possibility for outlying values, as we expect might be observed with data from right-skewed distributions, can distort relationships.

For some related ideas for using *Census at School* data with your class, see the following lesson plans with teacher notes at **www.censusatschool.ca/learning-activities**:

- *How weird is our class?* is designed for students in Grades 4 to 8 to compare class data with the rest of Canada, including guided discussion about why they might differ.
- *Relationships between variables* is designed to guide secondary students in investigating relationships between quantitative variables and the role of controlling for other variables.

What interesting relationships can you find in the 2011–2012 data? What trends can you find over the years? Let us know at **censusatschool@ssc.ca**. ▲

▲ In the Middle – The Museum of Mathematics Special Opening on 12-12-12

CAROLINE ROSENBLOOM



Caroline Rosenbloom is currently on secondment as an Elementary Coordinator in the J/I Midtown Technology Option and Mathematics Instructor in the pre-service program at OISE/UT. She has enjoyed working with intermediate students and teachers as a middle school teacher, Mathematics Instructional Leader in the Toronto District School Board, and Education Officer at EQAO. Caroline is co-author of Investigating Mathematics Using Polydron.

Art museums, science museums, natural history museums, air and space museums… there are many different museums to explore! A year ago, I had read about the construction of a museum dedicated to math. Mathematics will finally become a face in the public eye!

The **Museum of Mathematics (MoMath)**, is located in Manhattan, New York at 11 East 26th Street, right near Madison Square Park. It had a special private opening on December 12, 2012 and was open to the public on Dec. 15, 2012. It is now open seven days a week from 10:00 a.m. to 5:00 p.m.





Walking through the doors was magical. You see math everywhere; with a "pi" door handle, geometric shapes on the walls and floor, and a friendly greeter with her MoMath geometric badge. My family couldn't wait to get started.









On the first floor, you can take a smooth ride on a bumpy floor in a **square-wheeled tricycle**. Other interesting exhibits included a **Hyper Hyperboloid** and a cool **Coaster Roller**.

The basement floor is also filled with fun activities. As you enter, you see a large, interactive, computerized **Voronoi diagram** on the floor and a **Harmony of Spheres**, a touch-sensitive musical instrument. There is also a rolling activity investigating **geometric attributes**. The investigator puts two halves of a 3D shape together and then rolls the resulting shape in hopes it will trace out a target path. Kids and adults persisted and tried a variety of shapes. The best part of the visit was the language and reasoning taking place as a result of the museum exhibits.

The giant SOMA cube was probably the exhibit where we spent the most time. It was one of our favourites. Kids who like Block-by-Block by Thinkfun will love this life-size version of the puzzle. People were baffled and persisted when trying to solve this famous problem.

I would have liked to see many more of the exhibits, but many were under repair. Hopefully by the summer holidays, the complete museum will be up and running.

I'm thrilled that there is a museum focused on mathematics, helping people see math in a new light, making it tangible and interesting. I hope the museum is expanded so that it provides more hands-on exhibits and age-group diversification. Experiencing interesting patterns, puzzles, and connections to the world helps people see the value and beauty of mathematics.

So take a drive down to New York and see it for yourself!

Reference

National Museum of Mathematics. (n.d.). Retrieved from **http://momath.org/** ▲

▲ Vote March 4 – April 28

ALL MEMBERS TAKE NOTE!! OAME ELECTIONS ARE ONLINE

Annual elections for positions on the Executive and Board of Directors will be online from March 4–April 28. To vote, you must activate your online account on the website at **www.oame.on.ca** if you have not already done so. Click on "Members Only" and follow the directions to activate your account. You will find your membership number and expiry date on the envelope of your *Gazette*.

From March 4 to April 28, you will find "OAME Elections" in the links on the left side, after you have logged in using your username and password, which you created when you activated your account. Just follow the instructions. Short biographies for all candidates are included. If you need assistance, contact the Executive Directors

▲ Fields Institute – MathEd Forum Notes

STEWART CRAVEN



Stewart Craven presently teaches courses for the Continuing Education Department and the Transition Years Program at York University. He is an active participant in the Fields Mathematics Education Forum.

The meeting dedicated to the memory of Margaret Sinclair, held at York University in October, was a wonderful blend of presentations, hands-on mathematics (e.g., partially filling pyramids with water in order to examine the attributes of two-dimensional surfaces), and casual conversations among participants including a number of Sinclair family members and friends. This meeting reminded us that Margaret Sinclair set in motion a number of research initiatives that will continue through the ongoing work of many of her colleagues. It can be said that Margaret Sinclair's legacy is certainly intact. I would like to acknowledge the work put in to organize the meeting and the hospitality extended by the hosts at York University.

The theme of the November meeting was centred on the use of the Arts as a context through which researchers can investigate mathematics education and students can come to better understand mathematics. **Eva Knoll** (Mount Saint Vincent University) explored the topic, "*Arts-Based Research in (School) Mathematics.*" **Bill Ralph** (Brock University) followed with a talk entitled "*Seeing Is Conceiving,*" which he described as a tour of visual media in education, art, and research. **George Gadanidis'** (Western University) topic was "*Mathematics Through an Arts Lens*." He works with teachers in elementary schools to help them incorporate the Arts (e.g., song, poetry, and visual arts) into their teaching. The final talk for the day was "*Geometric Abstraction as Art and Language—The Work of Zanis Waldheims,*" by **Yves Jeanson** and **Raymond Guy** (Collège Boréal). The abstract read as follows: "Latvian-born Zanis Waldheims (1909–1993) immigrated to Canada in the early 1950s. He developed a structured aesthetic approach to represent concepts that describe the human being in its integrality. He produced a body of over 600 works of geometric art." As is always the case, the discussions about all of these talks was lively and informative.

The MathEd Forum Research Day was held at the Fields Institute on Saturday, January 26 from 10:00 a.m. to 2:00 p.m. Submissions and abstracts have been solicited and the agenda will be posted on the website at **www.fields.utoronto.ca/programs/mathed/**.

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▲ MB4T... Mathematics By And 4 Teachers – How Close Can We Get? Discrete and Continuous Data

ANN KAJANDER



Ann Kajander is an experienced classroom teacher currently teaching mathematics and mathematics education at Lakehead University. Her research interests relate to teachers' enhanced learning of mathematical concepts. She and her classroom-teacher colleague, Tom Boland, are currently writing a new book for teachers called Math Without Memorization: Big Ideas in Modelling *and* Reasoning for Elementary Teachers and Other Learners.

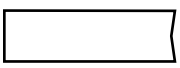
In the previous column, the development of the circle area formula with elementary students was explored. In that discussion, some "higher-level" mathematical ideas, such as the notions of limits and infinity, were used in an intuitive way. The need to draw upon relatively more sophisticated mathematical notions does sometimes occur in mathematics learning. The early use of the terms "discrete" and "continuous" in the Ontario curriculum is an example; in fact, the topics of discrete and continuous data are referred to as early as Grade 2 and Grade 5 respectively, in the Ontario Curriculum.

In Early Years mathematics, number sense begins with whole-number values. Counters may be used to illustrate particular values, such as the number 3:



The next possible larger value, using these counters, is fairly clearly "4." The data represented by these counters is called *discrete*. There is no possible counter of the illustrated type to represent any value between 3 and 4.

On the other hand, it is possible to have a length of tape that is 3 cm long:

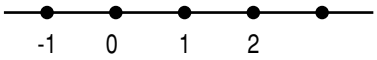


In this case, it is not necessary that the next largest length of tape is 4 cm. In fact, depending on how we cut the tape, it is at least theoretically possible to have any tape length *in between* 3 cm and 4 cm. The tape might be an illustration of *continuous* data.

The counters and tape examples illustrate ideas that even young children can intuitively understand. Some number representations have only specific allowable values, with "gaps" or "spaces" in between these values; others include "all" (infinitely close) values *in between* any two given values. The counters and tape examples provide an analogy to illustrate the fundamental distinction between *discrete* sets of numbers and *continuous* ones.

Number lines, which are often used in school mathematics in particular, can illustrate the relative sizes of values. Number lines are typically used informally at the elementary level, and often it is not specified which type (or set) of numbers or data is intended on a particular number line. An "open" number line uses only labelled values as needed. For example, we might get a rough idea of the relative sizes of 1.7 and $\frac{5}{8}$ by showing them on a number line and then comparing them with whole-number values that we know, such as 0, 1, and 2, which are often called "benchmarks."

Number lines representing whole numbers or integers, without any fractions or other numbers in between, really just imply separate locations or "points" on the line because there are no numbers between points. So while we typically illustrate an integer number line with points on a line,



we really are implying a set of points like this (with no values between them):



Nevertheless, the "line" is usually drawn to connect the values, and generally referred to as, in the case of integers for example, the integer "number line."

In contrast to the integer line, if we want a number line which *also* includes all the decimals and fractions—including "strange" numbers such as the one we call "pi"—then we can draw a line *in between* the integer points. Mathematicians call this number line the "Real" number line, and the group or set of numbers which fill in this line the "Real" numbers. Without defining it in careful mathematical language, the "Real" numbers include

values as close to any given value as we want, and always contain a value between any two given values. For example, a simple way to find a number between, say 5.12345 and 5.12346, is by using a simple average:

$$\frac{5.12345 + 5.12346}{2} = 5.123455$$

Since we can always do such an operation to find a value in between any two real numbers (and we can keep doing that), we find there must be an *infinite* number of values on the Real number line; mathematicians use the word *dense* to describe it. A "dense" line implies we can get "infinitely close" to any given value on the line. The Real number line represents *continuous* data—no spaces between values—while whole numbers and integers are *discrete*—there are spaces or "gaps" between them on a number line.

Students might wonder if data that involves decimals or fractions can still be discrete, even though the allowable values might be less than one unit apart. (Can you think of an example of such data?) Further uncertainty might be added if a measuring device were involved. Let's think about a context.

Students in one class are being timed on a run, using a digital timer that measures in minutes and seconds. Students in a second class are being timed on the same run, but using a timer that has minutes, seconds, and tenths of a second.

Using this scenario, think about which of the following represent discrete or continuous data:

- the actual time of the students' runs in the first class
- the recorded data of the students' times from the first class
- the actual time of the students' runs in the second class
- the recorded data of the students' times from the second class

This example illustrates a situation in which the actual situation (the run times), and the data collected on the timers which estimates these times, are not the same type of data. Theoretically, the time it takes to run a set distance can vary among any continuously measured lengths of time between students' shortest and fastest times; so within this reasonable span, the domain of the data is a continuous set of values. But in both classes, the measurement device used to collect the data was estimating the data using a discrete timer. In the first case, the smallest unit was a second; in the second case, the data was represented as finely as in tenths of a second. However, there were still "gaps" between the possible times using the devices, including the one measured in tenths, so theoretically even the second more accurate measuring device still provided only a discrete estimate of the data.

While elementary students do not need fully rigorous definitions of continuity to navigate these ideas, discussions such as the one provided here can shed some light on the intuitive distinctions among types of data. Many real-world contexts are available to help ground and make sense of such discussions.

As we progress along in mathematics, there are still other types of numbers to be encountered. For example, the square roots of negative numbers, such as $\sqrt{(-1)}$, are called imaginary numbers. While students do not generally need numbers such as these in elementary school, it might be helpful to have an answer if a student asks, "What if you try to take a square root of a negative?" Mathematicians centuries ago were shocked to find such numbers existed, and even tried to cover up the existence of some of the stranger mathematical numbers in order to keep mathematics "pure"! Today, mathematicians are more comfortable with the idea that some numbers are hard to model with concrete objects. The mystery and intrigue of mathematics continues to beckon those who are willing to live with such surprises!

▲



▲ Stratgies for Teaching Composition of Functions

TIM SIBBALD



Tim Sibbald is an assistant professor at the Schulich School of Education at Nipissing University and a vice-president of OAME.

One of the overall goals of the Advanced Functions (MHF4U) course, as stated in *The Ontario Curriculum, Grades 11 and 12: Mathematics (Revised)* (2007), is to "determine functions that result from… the composition of two functions, describe some properties of the resulting functions,…" (Expectation D.2., p. 95). Often the emphasis is placed on student understanding of the mechanics of the process, particularly with attention to different representations of functions (i.e., as sets of coordinates and as algebraic relationships). Attention is often given to understanding the technical outcome in terms of the algebra of the original functions. While these are important aspects of composition, another important feature is the overall character of composition in terms of the overall characteristics of the contributing functions.

Within the Advanced Functions course, several key types of functions are used: polynomials, rational functions, trigonometric functions (including reciprocal functions), exponential and logarithmic functions. The functions were chosen for study in the Grade 12 curriculum because of the variety of unique characteristics they portray (see Table 1). The objective here is not to go through every composition combination, but rather to use select examples to show how characteristics of the functions being composed influence the characteristics of the composition function. This facilitates higher-order thinking about the meaning of composition in terms of function properties, rather than just technical details. It is hoped that, through the discussion of a set of examples, teachers may feel more at ease about encouraging student exploration of the many other possible combinations.

Throughout this article, graphs of compositions are used. In the graphs, the underlying functions are shown as thin lines, while the composition is shown as a thick line. Any graphs including trigonometric functions have been made using radians for the independent variable.

Table 1
Key Characteristics of Functions

Function Type	Characteristics
Polynomial	• Two equivalent forms: standard form and factored form. • End effects same or different, based on even/odd value of degree. • Number of roots bounded by the degree. • Positive end effect (in X direction) is positive or negative (in Y direction) according to the sign of the leading coefficient. • Domain is all real numbers. • Range is all real numbers for odd degree, but bounded for even degree.
Rational	• May have vertical asymptotes. • May have horizontal asymptotes. • May have oblique asymptotes. (Note that higher-order end effects are not included in the Advanced Functions course.) • Domain may have holes. • Range may be all real values or constrained to specific intervals.
Exponential	• Exhibits growth or decay. • Domain is all real numbers. • Range is bounded.
Logarithmic	• Exhibits growth or decay. • Function is the inverse of an exponential. • Domain is bounded. • Range is all real numbers.
Trigonometric	• Exhibit periodic behaviour. • May have vertical asymptotes. • Domain may have an infinite set of equally spaced holes. • Range may be an interval (for example, a sine function), all real numbers (tangent function), or intervals (cosecant has range of all real values, except between -1 and 1).

Exponential of a Rational

As a first example, consider the following composition:

$$f(x) = 1.1^x \quad g(x) = \frac{x-2}{x} \qquad h(x) = f\big(g(x)\big) = 1.1^{\left(\frac{x-2}{x}\right)}$$

A few choices have been made in the construction of this example, and it is instructive to point this out to students. The base in the exponential function was

chosen so that it does not grow too fast; a slow change in behaviour facilitates the use of Geometer's Sketchpad® (or other graphing tools). A second choice was to have a simple rational function with the vertical asymptote at $x = 0$. It could be placed elsewhere, but would not be as instructive. If students are put in the position of having to choose their own functions, some will voice concern over how to make the choices, and it is a key opportunity to encourage experimentation.

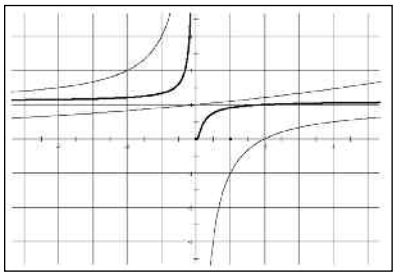


Figure 1. *Exponential of a rational.*

What does the graph look like? Examine just the algebra for a moment. Consider it from the eyes of a student who has followed the algebraic steps and arrived at the algebraic answer. Clearly the composition has an asymptote when $x = 0$. Many students have trouble focusing on the horizontal asymptote and the bounded range of $f(x)$. The graph appears in Figure 1. Now that the graph has been unveiled, the key task is interpretation of the details and verifying that the technology has generated an accurate result. First, we note the vertical asymptote when $x = 0$. The horizontal asymptote of the rational function is $y = 1$ for positive and negative large x, and this gives the composition a horizontal asymptote $y = 1.1$. The other feature that is apparent in the composition function is the abrupt end of the right *branch* which, in calculus terms, the limit as x approaches zero through decreasing positive x values. Students in Advanced Functions do not yet have the calculus background to articulate the limiting behaviour; however, developing the reasoning skills is beneficial. Suggest to students that they think about what happens when $x = 1/2$ or $1/10$, which allows them to see numerically that the range restriction of the exponential is leading to a range restriction in the composition. In symbols, one can say $f(x) > 0$ implies that $f(g(x)) > 0$, which means that (0, 0) is not a point in the composition. Alternatively, $g(0)$ is undefined (division by zero), so $f(g(0))$ is also undefined.

Exponential with Trigonometric Functions

It is important initially to constrain the overall choice of functions. This is particularly true with trigonometry, and it is instructive to constrain this to the sine and tangent functions, which allow periodic zeros along with range restrictions and asymptotes, respectively. Once students have experience, then differentiated instruction using a larger range of choices of trigonometric functions is feasible, but this is after introductory teaching.

Consider the following example:

$$f(x) = 2^x \quad g(x) = \sin(x) \quad f(g(x)) = 2^{\sin(x)}$$

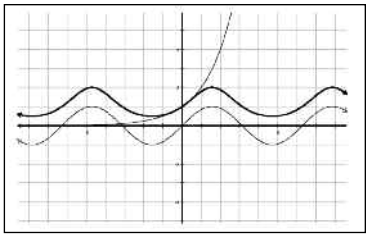


Figure 2.

The graph of this composition is shown in Figure 2 (note that radians are being used). In this example, the restricted range of the sine function restricts the range of the composition. In addition, the composition exhibits the same periodicity as the sine curve. The exponential function does not affect the periodicity (generally, if $f(x)$ is periodic, so is $f(g(x))$, as explained later), but it does alter the shape of the curve, making the peaks a little narrower and the troughs a little broader. Note that using a small base, such as the 1.1 value used earlier, does not show the alteration of shape very well. Students need to be encouraged to explore, even after they achieve an initial understanding. You might ask students to think about how the composition would change if the base were 3? If the students have been taught how to solve exponential equations, they could be asked what the exact range of the composition is for a general example. In simpler cases, such as the example here, exponent properties suffice to see that $-1 \le \sin(x) \le 1$ so that $\frac{1}{2} \le 2^{\sin(x)} \le 2$.

What if the composition order were reversed in this particular case? Consider $g(f(x))$, which applies the sine function to an exponential. The graph is shown in Figure 3. This outcome is an attention grabber! Note, however, that some unusual graphing effects can arise, depending on the software or calculator you use—we are obliged to encourage technological literacy. Students realize that rate of oscillation is increasing rapidly as x increases. As a teacher, it is beneficial to ask students about the negative values of x and why there are apparently no oscillations—a slightly misleading question, but beneficial for students to think through. Students typically comprehend the limited range of the composition fairly quickly. However, the distinctly different character of $f(g(x))$ compared to $g(f(x))$ gives an opportunity to address those students who are hesitant about the order of the functions when using the common $f^o g$ and $g^o f$ notations.

With the two examples presented, a teacher can ask students which one they would like to see with a tangent function instead of a sine function. Can they predict what

will happen? In my experience, encouraging exploration allows the teacher time to discuss the underlying properties with students who need assistance, while the strongest students will sometimes challenge themselves by looking at reciprocal trigonometric functions. They may arrive at new questions, which is good, but not something you have to resolve. Often it is sufficient to refocus student questions on the key characteristics of the functions and allow them to work out the details. I have had students come in a few days later, excited to explain the details, not because it was homework, but to celebrate their personal victory.

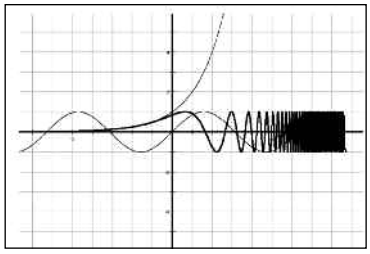


Figure 3. *Sine of an exponential.*

Logarithm of a Rational Function

This composition example combines a domain that is bounded below with a domain that has a hole. There are also both vertical and horizontal asymptotes. In general, I find it beneficial to have students work with base-2 logs because they are not as slow growing as base-10, and have specific cases that students can work with (such as the log, base-2, of 8 being 3). The specific example used here is:

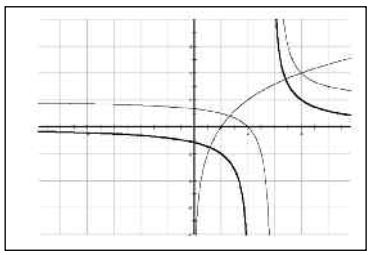


Figure 4.

$$f(x)\log_2(x) \quad g(x)=\frac{x-2}{x-3} \quad h(x)=f\left(g(x)\right)=\log_2\left(\frac{x-2}{x-3}\right)$$

This particular choice was made so that the asymptote $x = 3$ and zero $x = 2$ of the rational function would fall within the domain of the log function. A graph of the resulting function is shown in Figure 4.

The graph, in this instance, appears fairly bland. However, looking at the characteristic features of the graph shows that this is not the case. The horizontal asymptote of the rational function is $y = 1$ and when the logarithm is taken yields a horizontal asymptote, for the composition, of $y = 0$. The vertical asymptote of the rational function gives a vertical asymptote in the composition. However, if you simply go along with that, you miss the important detail that the asymptote in the composition is one-sided! Teachers often worry that they may miss such details in exploratory classes; however, you would realize it when you asked about the impact of the one-sided domain of the log function. The domain of the log function excludes negative and zero values. As a result, the domain of the composition excludes any values where the rational function is negative. In this example, that occurs when $x\epsilon[2,3]$ and one-sided asymptotes, when $x = 2$ and $x = 3$.

Polynomial of a Polynomial

Unlike the foregoing examples, this section will delve into some technical characteristics. It can be done in an exploratory fashion, but there are many connections to the polynomial unit of the Advanced Functions course that provide teachable moments. This section also provides a specific area where a deeper understanding of technical methods is feasible.

To start with, consider the composition of two lines: $f(x) = ax + b$ and $g(x) = cx + d$, where a and c are the slopes, while b and d are the respective y-intercepts. The composition is:

$$f(g(x)) = ag(x) + b = a(cx + d + b = acx + ad + b$$

The composition of two linear functions is a linear function. The new slope is the product of the original slopes, and the y-intercept has become a more complicated combination of three parameters. It is noteworthy that this demonstrates that, in general $f(g(x))$ is not equal to $g(f(x)) = acx + bc + d$, which has the same slope, but a different y-intercept. These sorts of algebraic experiments represent one approach to investigating the process and can clearly be augmented by graphing, but it is unclear how instructive such graphing would be.

Polynomials in general are examined in two forms in the Advanced Functions course: standard form and factored form. Considering a quadratic, as an example, in standard form:

$$f(x) = ax^2 + bx + c \text{ and } g(x) = dx^2 + ex + f$$

If one proceeds using substitution, then:

$$f(g(x)) = a(dx^2 + ex + f)^2 + b(dx^2 + ex + f) + c$$

$$= ad^2x^4 + 2adf + ae^2 + bd)x^2 + (2aef + be)x + (af^2 + bf + c)$$

What this example provides is an opportunity to examine what happens to the degree of the composition as the degree of the original polynomials changes. You are not likely to do this in your classes with all the letter coefficients, but the general approach is used here to advise on suitable choices for demonstrations. We already saw that two linear functions compose to give a linear function. If we set $d = 0$, then we see that a quadratic composed with a linear function gives a

quadratic function. It is also notable that, when $d = 0$, the quadratic composition has a leading coefficient with a sign determined by "a," and therefore opens in the same direction as $f(x)$. This is not true for a linear function composed with a quadratic (i.e., if one sets $a = 0$ instead of d). In this case, the direction of opening is determined by bd, and the slope of the line may leave the opening direction the same (positive slope) or change it (negative slope). This type of analysis, supported with practical examples, is appropriate to the Advanced Functions course.

It is insightful to contrast composition using standard form with composition using factored form. Suppose $f(x) = a(x-b)(x-c)$ and $g(x) = d(x-e)(x-f)$, then the composition is: $f(g) = a(d(x-e)(x-f)-b)(d(x-e)(x-f)-c)$, which is not nearly as useful for graphing as factored form usually is. It is curious, however, that proving the property that, when $f(g(x))$ is the composition of two quadratics, the resulting axis of symmetry corresponds to the axis of symmetry of $g(x)$ (test it out!), is most readily done by having $g(x)$ in vertex form and $f(x)$ in standard form (an explanation appears later).

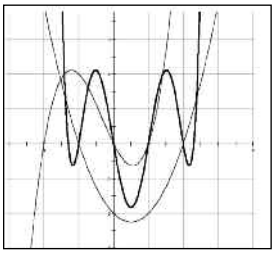


Figure 5. *Cubic composed with a quadratic.*

What is instructive to students, and presented here, more abstractly, is the general nature of the polynomials that arise. In general, if $f(x) = axn+\ldots$ and $g(x) = bxm+\ldots$, then the composition of these two polynomials is of the form:

$$f(g(x)) = a(bx^m + \cdots)^n + \cdots = ab^n x^{mn} + \cdots$$

That is, the degree of the composition of two polynomials has a degree equal to the product of the degrees of the two original polynomials, and this dictates the end effect behaviour and the maximum number of roots that are possible. Note also that the sign of the leading coefficient, ab^n determines the sign of the positive end effect for the composition. This could be demonstrated using specific examples with quadratics and cubics, for example, Figure 5 shows the composition of $f(x) = x(x-1)(x+2)$ and $g(x) = (x+1)(x-2)$, which is a degree six polynomial. Some students will be able to generalize the results from a few examples. I will leave the reader to ponder the shared zeros between the original functions and the composition. Is it an artifact due to choices in the example, or an underlying property awaiting your discovery?

A small detail that is quite important, but glossed over in the above, is how the number of real zeros in the composition relates to the original pair of functions. Consider the pair of quadratics $f(x) = x(x+2)$ and $g(x) = x^2-1$ (shown in Figure 6), since $g(x) \geq -1$, the argument of $f(x)$ never reaches $x = -2$ and so only one zero of $f(x)$ arises, albeit it does arise twice because $g(x)$ is zero for two different values of x. If this idea is generalized, it becomes apparent that the number of real zeros in the composition has to do with how many numbers in the range of $g(x)$ are roots of f(x). Consider, for example, whether a quadratic with two distinct real roots is composed with a cubic. The range of the cubic is all real values; therefore, the composition will exhibit both zeros of the quadratic once (i.e., composition has a minimum of two zeros). However, it could happen that the minimum and maximum of the cubic are on alternate sides of one or both of the quadratic zeros. Given the odd order of the cubic, this would mean that there would be three values of the cubic that produce one or both of the zeros of the quadratic. This means that the composition has two zeros, four zeros, or six zeros. Other numbers of zeros could only arise if the maximum or minimum of the cubic happened to correspond to a zero of the quadratic. In these cases, one might only have two values of x, where the cubic produces a root value of the quadratic, and this could lead to three, four, or five zeros.

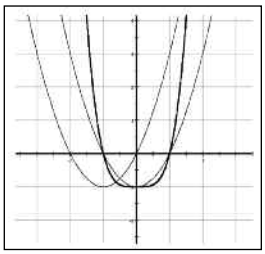


Figure 6. *Composition of two quadratics.*

Closing Thoughts

In writing this article, I was careful to avoid the use of theorems and proofs. The concern was that they can detract from exploratory and student-centred approaches to teaching composition. However, one of the reviewers suggested a set of theorems that assist in understanding some of the ideas that are touched on in this article and are amenable to easy proofs, if that is desirable. The theorems are:

Domain of $f(g(x))$ is a subset of the domain of $g(x)$.

Range of $f(g(x))$ is a subset of the range of $f(x)$.

If g(a) = g(b), then $f(g(a)) = f(g(b))$.

A consequence of this last theorem is that any periodicity in $g(x)$ is preserved in the composition $f(g(x))$, as suggested in the "exponential with trigonometric functions" example. A second consequence is that if $g(x)$ has an axis of symmetry, then $f(g(x))$ will preserve that

axis of symmetry, as suggested in the "polynomial of a polynomial" section. I am appreciative of the suggestions made by this reviewer because rather than detract, these suggestions provide some consolidation for the classroom.

The objective of this article was not to resolve all the details of composition of functions that might arise in the Advanced Functions or Calculus courses. Rather, the goal was to highlight particular ways of looking at the deeper concepts that are based on function properties. While there is much more that can be done to develop these ideas, the best approach is to have your students explore the varied combinations. In my classroom, such explorations led to considerable excitement and discussion of properties. Student presentations also led to active sharing of discoveries of similar properties for different function combinations. This is more encouraging for the teacher and students than simply working out mechanical details. ▲

Acknowledgement: *Thanks to Sonia Ellison, Andrew Dmytriw and to Reviewer 3, whose reviews added to this article.*

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▲ "Does Anyone Have Another Way?": Patterning, Algebra, and Inquiry in the Intermediate Classroom

ANN KAJANDER



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ABSTRACT

Preparing students for early high school algebra involves the intentional link between concrete and more generalized understandings of patterns and variables. This article explores appropriate classroom approaches to patterning, based on students' own visualizations of geometric patterns. This approach allows for a conceptually-based understanding of algebraic function rules to be subsequently developed by students. The pitfalls of providing a prescriptive format for the development of linear pattern rules at the elementary level is discussed in the context of the problem-based learning paradigm.

INTRODUCTION

Learning through problem solving is central to post-*Standards* curricula, and provides the central paradigm for the Ontario Elementary Mathematics curriculum (Ministry of Education [MOE], 2005). Given that problem solving "should be the mainstay of mathematical instruction" (MOE, 2005, p. 11), this approach should be fundamental to classroom mathematics learning. The openness of problem solving, by its very nature, means it is not fully prescriptive, yet some basic assumptions do apply. In particular, the recommendation that "students' problem-solving skills are enhanced when they reflect on alternate ways to perform a task" (MOE, 2005, p. 14) applies to all aspects of the mathematics curriculum. In this article, we focus on the strand of patterning and algebra, a critically important strand especially for the Grade 8 to Grade 9 transition. We explore how learning opportunities in this strand can be enhanced by supporting students' alternate ways of solving problems.

BACKGROUND

Problem-Based Learning and the Mathematical Processes

One of the tenets of problem-based learning is that teachers must structure learning based on students' developmental level. Knowing that not all students develop at the same speed and in the same way, and aligning instruction to their level, is a basic goal of differentiated instruction (Hume, 2008). Teachers must provide problems which students are, in fact, capable of exploring, and then provide support for them to do so in ways which make sense to them. Small (2013, p. 1) further asserts that:

> There is a strongly held belief in the mathematics education community that mathematics is best learned when students are actively involved in constructing their own understandings. This is only likely to happen in classrooms that emphasize rich problem solving and the exchange of many approaches to mathematical situations, and that give attention to and value students' mathematical reasoning.

The "exchange of many approaches" is particularly challenging for teachers to manage in that students' "many approaches" are not always predictable. In our own classrooms, we often find that a student discovers a new approach to a problem with which we thought we were familiar; and therein lie the challenge—and excitement—of teaching mathematics.

Given then that the challenge of a problem-based classroom is to respond appropriately to sometimes unexpected student responses which are not always predictable in the way that they might be in a teacher-driven, traditional classroom, teachers must work to be very open to student thinking and alternate solution approaches. Since "the value of students solving a problem in more than one way cannot be overestimated" (VandeWalle & Folk, 2005, p. 52), teachers do need to encourage and be prepared for (and open to) multiple solution approaches, and be willing to support and encourage class discussions in which students compare and debate multiple strategies and solutions. If a student solution method has been deemed correct by the class, then teachers should not force students to adopt a different method provided by the teacher (VandeWalle & Folk, 2005). Instead, students should be encouraged to understand and compare the merits of multiple approaches, and make their own decisions about which methods to use and when. Initially, students may choose solution methods that align with their own ways of "seeing" a pattern. Visualizing a pattern differently often highlights different aspects of the pattern composition (Boaler & Humphreys, 2005). Later, students may gradually move toward more computationally efficient pattern rules, but such a goal should not be the purpose of lessons centred around constructing algebraic expressions that connect to visual representations (Boaler & Humphreys, 2005). Similarly, computational efficiency should not be the primary goal of early patterning rule development; rather, the purpose of such activities is to help students link the visual pattern growth to a generalized way of describing it—in whichever way makes sense to them.

The literature is rich in description of the importance of students sharing their multiple approaches as a way to support many of the mathematical processes, in particular, reasoning and communicating. The NCTM *Standards* (2000), for example, include in the description of the "communications" process that "Conversations in which mathematical ideas are explored from multiple perspectives help the participants sharpen their thinking and make connections" (p. 60). As well, "a student who has one way of seeing a problem often can profit from another student's view" (p. 62). Connections are also important, and the *Standards* go on to say that "middle-grades students [should] recognize and connect multiple representations of the same mathematical idea" (p. 65). The Ontario elementary curriculum reflects this same vision, as does the description of the learning processes in the Grades 9/10 mathematics curriculum, which states, "Students' problem-solving skills are enhanced when they reflect on alternative ways to perform a task, even if they have successfully completed it" (MOE, 2005b, p. 14). The content expectations also support this idea of multiple approaches. For example, specifically in the linear relations expectations in the Ontario Grade 9 curriculum (MOE, 2005b), which are the ones that most directly follow earlier explorations of patterns rules, a number of different ways to express linear functions are mentioned; the "$y = mx + b$" version of a linear function is *no longer* the only one studied. It is indeed important that students compare and contrast different ways of describing patterns, such as visual, arithmetic, graphical, tabular, and algebraic (Beatty, 2010; Boaler & Humphreys, 2005). The particular focus of the current article is the connection between the different visual structures as seen by different students, and how these can be differently described arithmetically and ultimately algebraically.

Mathematical Learning and Teachers' Knowing

Standing back and supporting students as they debate and explore their own ideas, while reserving judgment, is something we all find challenging as teachers. It can be so tempting to "help" a struggling student by offering a possible solution method, even though we know such interventions can actually rob the student of the opportunity to problem-solve. As well, according to VandeWalle and Folk, "Students who sense that the teacher has a preferred method or approach are extremely reluctant to use their own strategies" (2005, p. 59). While the authors do encourage teachers to share information, this should only be done when information does not solve the problem or take away from students the need to develop solution methods they understand.

Furthermore, research explains that methods that seem "easy" to the teacher may not be helpful to students. In order to further understand this idea, we need to also understand a bit about the development of mathematical ideas in human minds. We as teachers are continually challenged by operating at a different developmental level from our students.

Our mathematical minds are uniquely programmed to work toward increasingly higher-order abstractions (Skemp, 1986). Once we understand the underlying mathematical constructs, these higher-order abstractions (such as mathematical generalizations) will seem "easier" to us as ways to describe mathematical ideas than working with individual concrete examples (Skemp, 1986). Yet this is not the way mathematics is initially constructed or understood—such generalizations make sense only *after* experiencing enough examples that our own minds are able to construct them (Skemp, 1986). In other words, "telling" students a "rule" before they have constructed the generalization in their own minds, based on enough experience, tends to make it just that—a meaningless "rule" to "remember," often without understanding. This simple idea is so easy to forget during the day-to-day pressures of classroom teaching—it is so tempting (not to mention easier) to simply "give" students a method, template, rule, generalization, or formula in an attempt to make it "easier" for them. While the "do it this way" method may show short-term gains in highly narrow and restricted task sets (such as quizzes or tests consisting of rote methods using the same format), they do not prepare students to use the ideas in new situations or contexts—the hallmark of problem-based learning—nor do such methods support the development of conceptual understanding.

To use a contrived example, supposing you wanted your students to determine the perimeter of a rectangle. In the past, in a traditional classroom, we might have told students that the "way to do it" is to find the two numbers that fit in the formula 2 x __ + 2 x __ . Since a given rectangle would have at most two different values labelling its lengths, it doesn't take much luck to guess the values correctly. However, inserting the two values into such a template does *not* mean that the student has realized that the perimeter is really, for example, 3 + 5 + 3 + 5, and that this can *also* be calculated as 2 x (3) + 2 x (5). While the latter calculation method may

appear simpler to the *teacher*, who already understands the underlying generalization, it is not usually easier for the *learner*, who likely has not yet constructed this higher-order abstraction of pairing up the equal sides as a way to calculate the perimeter more efficiently. This subtle-sounding difference is at the heart, we believe, of many ineffective mathematical lessons.

It can be very difficult to stand back and not "help" a student when he or she is working to construct his or her own representations. An important pedagogical aspect of teachers' knowing, therefore, includes knowing when *not* to intervene, and what *not* to say. It is not necessary to be familiar with every possible outcome or solution; in a community of learners, the teacher's role is to help the student explain his or her reasoning so that everyone (sometimes even the teacher!) can follow and understand the student's thinking. Teachers can also help build connections among multiple solutions during the consolidation phase of the lesson, and help students gather and organize the necessary conceptual ideas needed to build new levels of understanding. Such a process truly reflects the idea of a community of learners—and sometimes we as teachers are very much a part of that learning community.

NAVIGATING PATTERNING IN THE CLASSROOM

An Early Classroom Example

In Ontario, the elementary curriculum was fundamentally changed in 1997 to reflect the learning processes just described, among others, and then slightly revised again in 2005. In the early years of this implementation, teachers had to learn what it meant to support student's own solution methods, often with little guidance.

Consider the following task, in which a junior-level student was asked to find a "rule" to connect the input number to the output number. (Note that a visual version of this task might have been more helpful for a number of reasons.)

Input	Output
5	10
6	13
9	22
11	28

Take a moment to find a rule to determine the output number, given the input number. Test your rule, and reflect on your thinking. Now let's examine a student's thinking in a classroom in which this task was used.

A researcher (the first author), working with a student who was given this task in his junior-level classroom shortly after the 1997 curriculum revision, asked the student to "think aloud" while he worked. The student noticed that the output values were "always one more than multiples of 3," but that they were the "wrong" multiples (not multiples of the input value). When asked, the student was able to correctly produce other output values for other input numbers, using his method. He was able to explain that the "right" input number was 2 less than the one given each time ("right" meaning what you had to multiply by 3). After a great deal more effort, he was able to state his rule as follows:

> *Subtract 2 from the input number, multiply that new number by 3, and add 1.*

Before reading on, examine the student's rule and reflect on his thinking. (How would you respond to this student?)[1] How did your own rule compare to the student's rule?

In fact, this student is using a highly sophisticated mathematical idea—the notion of adjusting the starting value, such as is important in later secondary study of, for example, sequences and series. The student's idea would have provided ripe fodder for a rich classroom discussion. Instead, the "times 3 minus 5" version was provided as the "right" answer.

For us as teachers, as explained earlier, the temptation to "simplify" a student's solution to something that is simpler for us must be avoided at all costs. Imposing another way of stating the rule as the "right" way was confusing for the student in the previous example. The student did not "see" the rule as "three times the input number minus 5." And while he might have benefited from another student's explanation of that latter rule, his own was perfectly correct, and in fact quite mathematically sophisticated. VandeWalle and Folk explain further the importance of students sharing their own rules, rather than the teacher "correcting" rules to align with a certain format, explaining that "students will usually question their peers when an explanation does not make sense to them, whereas explanations from the teacher are usually accepted without scrutiny (and hence without understanding)" (2005, p. 60). Sharing and defending their own rules empowers students' in their learning.

Another Classroom Example

The powerful researcher and classroom teacher team of Boaler and Humphreys provide a compelling description and video example of what we *should* be doing in intermediate classrooms in patterning and

[1] *Sadly, this student solution was marked "wrong" by the teacher, as it was not the "simplest solution."*

algebra (2005). In the "border problem" example they provide, students are asked to find a way to count the number of squares on the border of a ten by ten grid, "without counting them one by one." Take a moment to examine Figure 1, and see how many "ways" you can find.

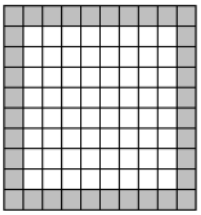


Figure 1: *The Border Problem Grid*

In the classroom video provided with this problem, students are able to share and defend half a dozen solutions to this problem. Some examples of these are:

2 x 10 + 2 x 8

10 + 9 + 9 + 8

4 x 10 – 4

4 x 8 + 4

10 x 10 – 8 x 8

Take a moment to explore what each student was "seeing" as they worked through the problem and organized the counting in different ways. Note that most of the examples above do not adhere very well to the traditional simplified "$y = mx + b$" formulation, although the 10 x 4 – 4 version of the rule looks fairly similar. While all of the versions calculate the same answer as a rule in the traditional format, none of these solution examples reflect the students visualizing the rule that way. In fact, with this pattern, thinking of the pattern using a $y = mx + b$ format is awkward because the version 4 x 8 + 4 is really 4 x (10 – 2) + 4, and the version 4 x 10 – 4 requires four squares being *removed* to account for the overlap, rather than being directly identified and counted. Different again is the 10 + 9 + 9 + 8 solution, which becomes $n + (n - 1) + (n - 1) + (n - 2)$, and 10 x 10 – 8 x 8 (a very elegant rule to which the students ooh'd and ah'd in the video) becomes $n \times n - (n - 2) \times (n - 2)$.

In other words, this discussion suggests that there are patterns which simply do not lend themselves to a direct "$y = mx + b$" or "slope-intercept" rule visualization, and thus the "$y = mx + b$" format is not a good way to record one's thinking. Often, many other ways of visualizing patterns—and subsequently translating them to algebra—are possible and interesting. In an earlier paradigm, students might have been asked to present the rule using such a format (as in the earlier Input/Output chart example), but looking at the examples of student thinking in the Border problem, it makes it clear how restrictive such a teacher-mandated solution template is.

Finding Other Helpful Classroom Examples

The Border problem provides a great classroom example, and the video provided with the book (Boaler & Humphreys, 2005) makes it clear in subsequent lessons how this example can be used to introduce the idea of "any sized square," i.e., the notion of a variable. Still other examples of problems are found in VandeWalle and Folk (2005, pp. 405 and 419). The latter example involves the perimeter of a shape created by lining up hexagons, and the authors provide a number of different possible pattern rules, explaining that students should then be asked to share and defend their different formulas.

Yet another example is suggested in the Ontario curriculum (MOE, 2005), the "how many matchsticks in each picture" problem (see Figure 2). Students can be asked to find a way of counting the matchsticks to find the number of them for any picture number.

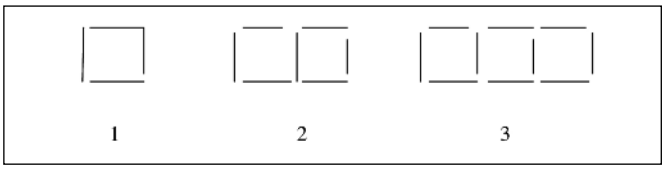


Figure 2

In our own classrooms, we always find several typical formulations of the associated pattern rule for the matchsticks problem. Some students "see" the pattern as "C" shapes, one for each picture number, plus one more at the end, resulting in the rule 3 x n + 1, which aligns with the traditional slope-intercept ($y = mx + b$) form. Others "see" it as one matchstick at the beginning, plus "C" shapes facing the other way, or 1 + 3 x n. And still others "see" the pattern as an initial square with four matchsticks, and then three more matchsticks for each additional box. Since the "added" boxes number one *less* than the picture number, the algebraic rule—often after some deliberation—becomes $4 + 3x(n - 1)$. We note that the technique of adjusting the picture number aligns with the thinking of the student described in the first example in this article. And occasionally, students see the pattern in still other ways. For example, students might see the perimeter of the shape plus the enclosed vertical matchsticks. This rule could be written in a number of ways, and one way, for example, is $2 + 2 \times n + (n - 1)$. A little exploring makes it clear that the "2 + 2 x n"

is the perimeter, while the "(n – 1)" term counts the remaining enclosed vertical matchsticks. Yet another way (and there are indeed more) is to think of "n" separate squares (4 x n matchsticks required), and then when we slide the separate squares together into a row, we have to remove the overlapped vertical edges (n – 1). So the rule becomes 4 x n – (n – 1).

Of course, all these (correct) pattern rules do simplify to the same algebraic rule. Should students be asked to "simplify" their rules initially? In discussing the Border problem, master teacher Cathy Humphreys explains why she does not want her Grade 8 students working to simplify rules, when she is trying to help them understand the connection of patterns, rules, and ultimately algebraic expressions with variables:

> I did not want to focus on simplifying expressions (yet), nor did I want students to determine the "most efficient" way of finding the number of squares on the border (efficiency is often in the eye of the beholder when it comes to mathematical thinking)... I set out to draw upon students' natural abilities to visualize and, in doing so, to breathe life into symbolic representation.
>
> (Boaler & Humphreys, 2005, p. 16)

In other words, if students visualize a rule in a certain way, an algebraic representation that supports their own mental image will be the most useful to them. They can then compare their images and rules with one another. The *Standards* document echoes the same idea, stating that relationships "can often be expressed symbolically in more than one way, providing opportunities for students to examine the equivalence of various algebraic expressions" (NCTM, 2000, pp. 225–226). As explained earlier, it is only through their own constructions that students are able to generalize and ultimately construct their own higher-order abstractions.

Choosing Good Problems

We would recommend the geometric examples provided in the previous subsections as useful in helping students construct an understanding of the idea of a linear pattern and ultimately the notion of a variable. Patterns which do not lend themselves to multiple ways of constructing the pattern rules may be less useful in that they do not promote the connection of alternate visualizations to their representations. For example, consider the pattern in Figure 3.

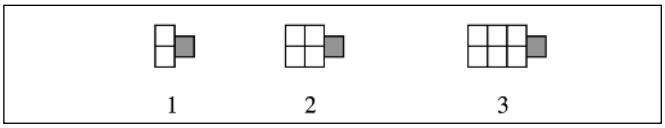


Figure 3

The two colours in the pattern might be intended to lead or scaffold students, by making it clear which part is constant and which parts changes with each picture. As well, the pattern is set up carefully so that the pattern rule "2 times the picture number plus one" is the most obvious formulation of the rule. While including a few patterns of this type may not be problematic, if students see *only* patterns set up in this prescriptive way, they will most likely miss out on the potential for exploring multiple representations (which might include multiple algebraic representations, as well as different visual and verbal descriptions) which can lead to rich discourse.

An extreme case of this over-simplification (which may possibly remove the problem-solving aspect of the activity nearly altogether) would be found in classroom materials that have students characterize such patterns by formatting their rules, using a template such as the following:

___ x *picture number* + ___

Such a rule template, we argue, does not allow for the kind of rich pattern exploration described earlier in this article. At worst, it may even encourage students to simply use trial and error to find which numbers "work." Most realistic problems are not set up in such a prescriptive manner. As was illustrated in both the Border problem and the Matchstick-squares problem, most pattern-problem solutions can be formulated in a number of ways. The discussion and explanation of these ways allows for rich classroom discussion and the development and meaningful connection of visualisation, representation, communication, and reasoning, as was discussed. As Boaler and Humphreys explain, "The point is not to find the rule—especially a simplified rule! —but rather to represent symbolically what they *see*. The different visualizations provide a springboard from which students can grapple with equivalence..." (2005, p. 25). The authors go on to state that "algebraic expressions of different forms give insights into different aspects of mathematical relationships" (p. 23). Hence, problems which have the greatest potential for the type of experiences we describe in the Border problem and Matchstick-squares problem are the most fruitful choices for classroom explorations.

The Grade 8 to Grade 9 Transition

Some teachers in our regional intermediate years mathematics professional learning groups (Kajander & Mason, 2007) have recognized that there may be some potential gaps in the Ontario mathematics curriculum documents between the elementary and the secondary curricula, not just in content understanding, but in transfer of knowledge from elementary to secondary, and the ability to problem-solve outside of a given strand or topic. We have also noticed that without rich problem solving and exploration embedded in lessons regularly, students lose the ability to tackle problems independently, using a variety of tools that they have developed previously. For this reason, both teachers of Grade 8 and Grade 9 must be particularly cognizant of what students learn in adjacent grades, and patterning and algebra in particular is a critical strand for success at the early secondary level. As the *Standards* explain, "Most students will need extensive experience in interpreting relationships among quantities in a variety of problem contexts before they can work meaningfully with variables and expressions (NCTM, 2000, p. 225). Working flexibly with algebraic expressions that have been constructed in ways which have meaning for students, and contrasting geometric, arithmetic, algebraic, tabular and graphical representations are critically important activities for intermediate students (Boaler & Humphreys, 2005; NCTM, 2000; Vandewalle & Folk, 2005).

SUMMARY

Patterning and algebra in the upper elementary years is a critical strand for success at the secondary mathematics level, in that the high school curriculum draws heavily on this strand. Our secondary colleagues report noticing an increase in student confidence and skills when students have been exposed to problem-based, constructive learning environments. In algebra, a fundamental understanding of the notion of a variable as a *quantity that can vary*—rather than just a placeholder for a single number—is a conceptually crucial but challenging notion. Providing multiple opportunities for students to visualize geometric patterns and describe them arithmetically, and then generalize these descriptions to algebraic formats, is an important area of study for the intermediate and even junior classroom. In particular, we feel that using students' own representations and descriptions based on their own visualizations—rather than using a prescriptive format for the rule—is critically important in the development of conceptual understanding.

REFERENCES

Boaler, J., & Humphreys, C. (2005). *Connecting mathematical ideas: Middle school video cases to support teaching and learning.* Portsmouth, NH: Heinemann.

Hume, K. (2008). *Start where they are: Differentiating for success with the young adolescent.* Toronto: Pearson Education Canada.

Kajander, A., & Mason, R. (2007). Examining teacher growth in professional learning groups for in-service teachers of mathematics. *Canadian Journal of Science, Mathematics, and Technology Education, 7*(4), 417–438.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Ontario Ministry of Education. (2005a). *The Ontario curriculum, grades 1–8: Mathematics* (Rev. ed.). Toronto: Queen's Printer for Ontario.

Ontario Ministry of Education. (2005b). *The Ontario curriculum, grades 9 and 10: Mathematics* (Rev. ed.). Toronto: Queen's Printer for Ontario.

Skemp, R. (1986). *The psychology of learning mathematics.* Markham, ON: Penguin Books Canada.

Small, M. (2013). *Making math meaningful to Canadian students, K–8.* (2nd ed.). Toronto: Nelson Education.

Vandewalle, J., & Folk, S. (2005). *Elementary and middle school mathematics.* Toronto: Pearson Allyn and Bacon. ▲

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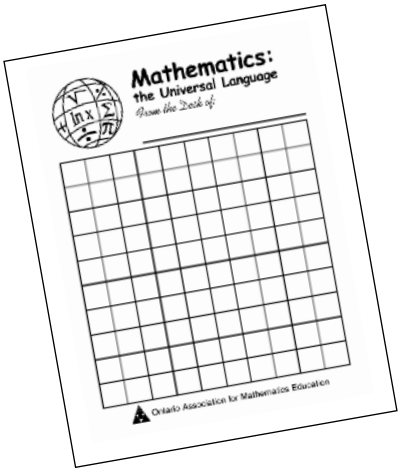
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▲ Data Rocks – New Resources and Census Data Available from Statistics Canada for Teaching MDM4U

JOEL YAN



Joel Yan, a volunteer education consultant for Statistics Canada, is a mathematician who is keen to help educators use Canadian data in mathematics teaching and learning. In 2008, Joel helped negotiate a provincial licence with the Ontario Ministry of Education for the Canadian Community Health Survey CCHS microdata. While Joel retired in 2009 from the Statistics Canada team responsible for the education outreach work, he maintains, independently from Statistics Canada, the website http://teacherweb.com/ON/Statistics/Math/ to provide support based on Canadian data for mathematics teachers. Joel received an award from the OAME in 2009 for his contribution to mathematics education.

In the last edition of this column, I encouraged teachers to switch from using the E-STAT database to using the master CANSIM database as a rich source of Canadian data for their teaching. CANSIM contains over 42 million time series on every aspect of the Canadian economy and society, with links to related articles and surveys. Also, CANSIM is updated overnight as soon as new data are available from over 350 different surveys (e.g. health, demographic and business surveys) and statistical administrative data sources (e.g. crime data and tax records). CANSIM is the primary source used by journalists to track the Canadian economy and demographic trends and should be available to serve the data needs of your students for many years.

Readers are reminded to try out the excellent starter activity on the use of CANSIM, developed by Angela McCanny, that was included in the last edition of the Gazette as a way of introducing your classes to CANSIM.

Even though the Education Outreach team has been shut down due to government cutbacks, Statistics Canada continues to produce a range of data-based resources that can be very useful in teaching. This column highlights two such resources: Visual Census and Census Profiles.

Visual Census – A Graphing Tool for Displaying Trends from the Latest Census

Visual Census module is an interesting source of the latest data from the 2011 Census and National Household Survey (NHS). Every time new data are released from this census and survey, new datasets, easy-to-use graphics, and downloaded data will be provided here.

To access this resource, go to **www.statcan.gc.ca**.

- At the top on the right side, click "Census."
- Midway down on the right-side column, select "Visual Census" beside the bar-chart icon.

Then you can very easily create bar graphs generating the distribution of the population for a number of variables for a range of geographies across Canada.

Just select your geographic areas from the list, then select the data topic of interest and click on the ***submit*** button.

For example, below is the resulting graph showing the rapid increase in those over 65 in Canada, compared to other age groups.

Figure 2.2 Population change by broad age groups, Canada, 1996 to 2001, 2001 to 2006 and 2006 to 2011

Figure description

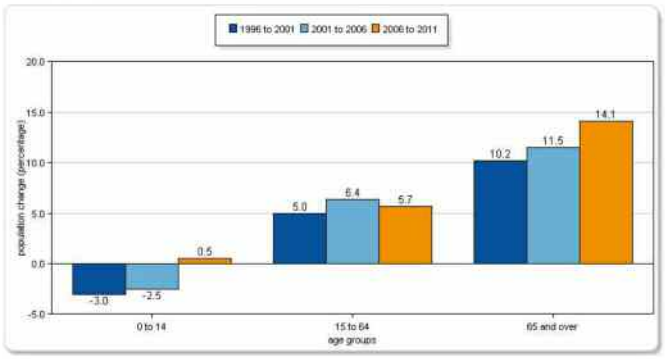


Figure 1: *One of the many graphs available from the Visual Census module – a Graph of Population change in Canada by broad age groups since 1996*

Another interesting graph (Figure 2) below shows how the population of those age 65 and over (the light line below) is rapidly increasing and will soon cross the population of those age 14 and under (the darker line).

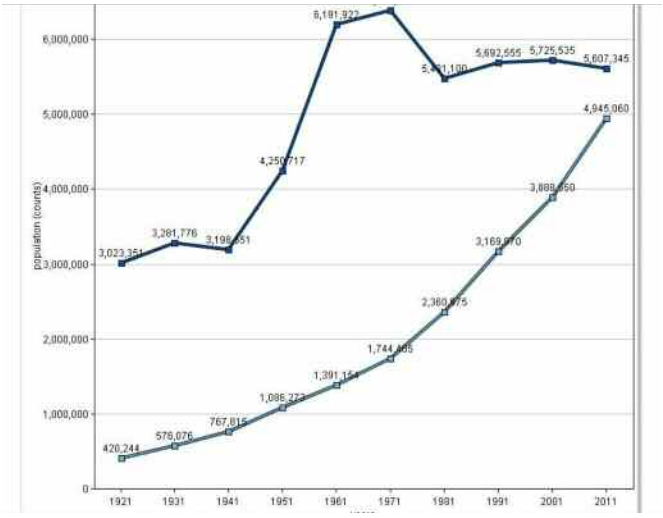


Figure 2: *One of the many graphs available from the Visual Census module – Number of children aged 14 and under and of persons aged 65 and over, Canada, 1921 to 2011*

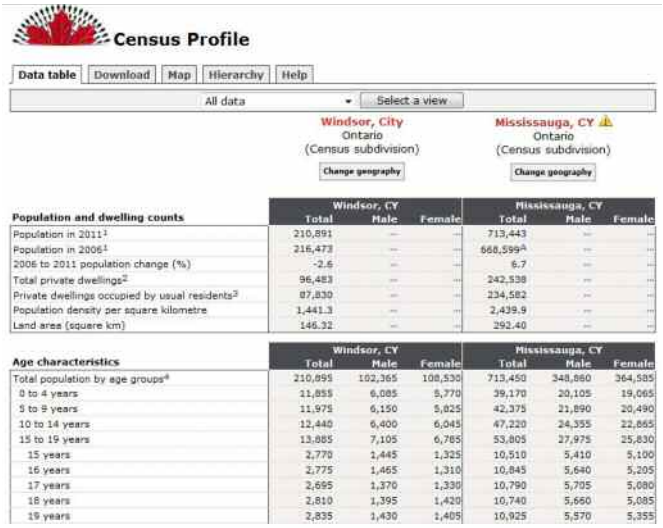
To download any data values from the Visual Census module, click on the CSV or TAB buttons at the top of the screen to generate comma-separated or tab-separated data outputs.

Census Profiles – A Range of Topical Datasets from the Census

One of the best ways to find interesting data for your class is to use the Census Profiles module.

To access this resource, go to **www.statcan.gc.ca**.

- At the top on the right side, click "Census."
- Under 2011 Census on the left sidebar, select "Data products."
- Select "Census profiles."



Census Profile

Data table | Download | Map | Hierarchy | Help

All data — Select a view

Population and dwelling counts	Windsor, CY (Ontario, Census subdivision) Total	Male	Female	Mississauga, CY (Ontario, Census subdivision) Total	Male	Female
Population in 2011[1]	210,891	...	...	713,443	...	...
Population in 2006[1]	216,473	...	...	668,599[A]	...	...
2006 to 2011 population change (%)	-2.6	...	...	6.7	...	...
Total private dwellings[2]	96,483	...	...	242,538	...	...
Private dwellings occupied by usual residents[3]	87,830	...	...	234,582	...	...
Population density per square kilometre	1,441.3	...	...	2,439.9	...	...
Land area (square km)	146.32	...	...	292.40	...	...

Age characteristics	Windsor, CY Total	Male	Female	Mississauga, CY Total	Male	Female
Total population by age groups[4]	210,895	102,365	108,530	713,450	348,860	364,585
0 to 4 years	11,855	6,085	5,770	39,170	20,105	19,065
5 to 9 years	11,975	6,150	5,825	42,375	21,890	20,490
10 to 14 years	12,440	6,400	6,045	47,220	24,355	22,865
15 to 19 years	13,885	7,105	6,785	53,805	27,975	25,830
15 years	2,770	1,445	1,325	10,510	5,410	5,100
16 years	2,775	1,465	1,310	10,845	5,640	5,205
17 years	2,695	1,370	1,330	10,790	5,705	5,080
18 years	2,810	1,395	1,420	10,740	5,660	5,085
19 years	2,835	1,430	1,405	10,925	5,570	5,355

Figure 3: *Sample output from 2011 Census Profiles comparing the population in the cities of Windsor and Mississauga, Ontario*

This Census profile presents information from the 2011 Census of Population for various levels of geography, including provinces and territories, census metropolitan areas, communities and census tracts. Users can search for an area of interest by typing its "place name" or by browsing a list, by entering a postal code or a geographic code.

A sample output table from Census Profiles is shown in Figure 3. All the same data can also be downloaded directly and then graphed using spreadsheet software.

Suggested Activity for Grades 7–8

- Use the Census Profiles module at **www.statcan.gc.ca** to extract data for your community or a community of interest, and compare it to the similar data for your province or another community.
- Write a paragraph describing how your community is near, below or above the average in terms of several characteristics including median age, percent males versus females your age, average number of children at home per census family, and two other characteristics.
- Prepare a bar and circle graph showing the relative value for your community, your province, and Canada for two variables of interest.

Indicate the reason why the type of graph you have chosen is most appropriate for graphing the selected data. ▲

OAME Sticky Graph Pads

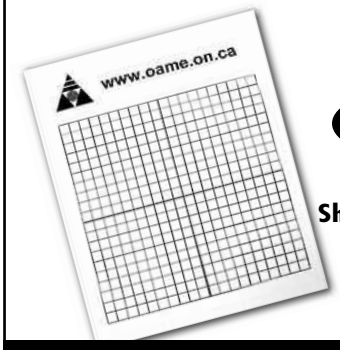
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What's New with CLIPS? Clips Tools!

GREG CLARKE

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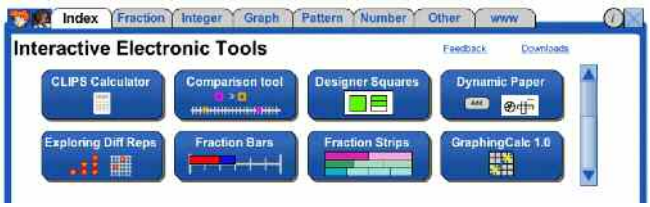
Agnes Grafton, Greg Clarke and Ross Isenegger are currently on assignment with the Ontario Ministry of Education as Provincial Math Leads, working on the CLIPS project.

Agnes Grafton is from the Brant Haldimand Norfolk Catholic District School Board, Greg is from the Simcoe Muskoka Catholic District School Board and Ross is from the Near North District School Board.

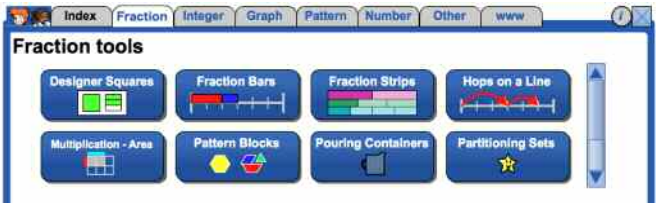
Have you visited the MathCLIPS.ca website lately? One of the biggest changes that has happened over the past year has been the introduction of the CLIPS Tools pod. This is accessible within CLIPS by clicking on the wrench icon.



The Tools pod opens to the Index tab, which lists all of the available tools in alphabetical order. Use the scroll bar on the right to see all the tools.



You will note the title "Interactive Electronic Tools." Some of the tools in the collection are electronic versions of common classroom manipulatives, such as the Fraction Strips tool. Others are newly designed specifically for CLIPS, such as the Hops on a Line tool. There are also links to a few select high-quality web tools. All are dynamic, interactive tools that allow you to explore a math concept or mathematical operation.

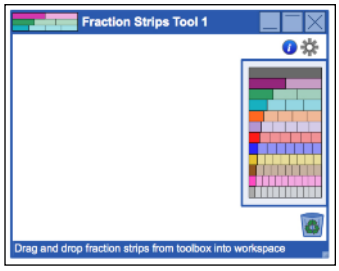


To focus on tools designed for specific types of number operations or math strands, click on the appropriate tab. Notice when you hover your mouse over a tool button, the white text that appears in the status bar at the bottom of the Tools pod gives you a brief descriptor of the tool and its usage.

The Index tab has two links that don't appear anywhere else. Use the *Feedback* link to report bugs, ask questions of the programmers, suggest improvements or other tools that should be developed. Use the *Downloads* link to open the latest version in a new browser window or to save individual tools for use offline, or for embedding in lessonware, such as SMART™ Notebook or ActivInspire Flipchart. Notice that you can download a .zip file containing the whole Tool Container.

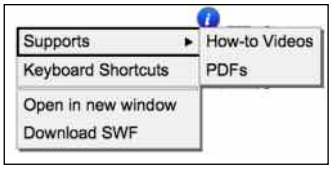


When you click on a tool button in the Tools pod, you will either be taken to a new window (if the tool is a link to an external website, or to a standalone CLIPS activity tool), or the CLIPS tool will load inside its own pod, which can be minimized, or resized and moved around the screen. Here is an example of what the Fraction Strips tool looks like when it is first loaded.



The Strips toolbox is on the right. To model fractions, drag strips into the white workspace in the main body of the pod. This tool could be used to represent or compare fractions as well as to model fraction addition or subtraction, or multiplication of a fraction by a whole number.

This tool demonstrates some of the common features of many of the CLIPS tools:



- The top half of the (i) button menu provides information about how to use the tool. In the Supports submenu, for example, both links will take you to a wiki page, where you will find a variety of video and print resources. These supports are currently being developed. The lower part of the (i) button menu provides two alternatives for using the tool—either in a new browser window, or to save it directly to your computer for use independently or offline.

- The gears icon will open up a Settings pod that allows you to customize the way the tool works and/or the way some of its components are displayed



- The recycle bin icon is used to remove any objects in the workspace that are currently highlighted. If none are highlighted, it will remove all the objects to return to a blank workspace.

- The bottom right corner of the Tools pod itself is a resizer. If you click and hold, and drag the corner around, the Tools pod itself will stretch and resize. This is useful if you wish to make more room for your set of fraction strips or if you wish to shrink it so that you can see other things on the screen at the same time.

- You can use the camera icon at any time to take a picture of a whole Tools pod, or just a piece of what you have modelled on it.
- You can have multiple tools on the stage at the same time. Just click the wrench icon to bring up the main Tools pod and select another one.

For more information about CLIPS Tools, and ideas about how to use them in your classroom, check out the following wiki that houses all the supports for these tools. This is a work in progress, so if you have a great idea about how to use one of these tools, please consider sharing it with the authors.

http://mathclips.wikispaces.com/MathCLIPS+Tools

▲

▲ Voices on CLIPS

KERRY-LYNN MARTEL
TINA FRANZEN



Kerry-Lynn Martel is in her fifth year of teaching. She currently works for the Rainbow District School Board. She has taught Mathematics, Physical Education, and Science. Her interests include playing baseball.



Tina Franzen has been teaching for 13 years. She has specialists in both Special Education and Integration of Information and Computer Technology in Instruction. She currently teaches Grades 9 to 12 in the Trillium Lakelands District School Board. She enjoys puzzles, reading, and running.

As the third and final article on CLIPS, this piece will focus on the intermediate/senior perspective and applications of the CLIPS activities.

In an ongoing attempt to convince all educators of the benefits of broadening their use of technology in their classrooms, this article will highlight two interviews of secondary teachers, Kerry-Lynn Martel and Tina Franzen, who describe their reflections on their use of CLIPS in their intermediate and senior classes. These teachers approached the activities in different ways. Here is what they said.

Kerry-Lynn's Experience

Please briefly explain how you used CLIPS in your classroom.

I used CLIPS in my Grade 11 University math class for the trigonometry unit. I have also used CLIPS in my science class, when we were studying sound waves.

Why did you decide to use CLIPS?

CLIPS are a great interactive learning tool. Students find the activities engaging, and CLIPS allow them to work at different paces.

What were your students' reactions to CLIPS?

Most of the students enjoyed using CLIPS. They felt the activities explained the concepts very well and they were a hands-on way to learn. They also enjoyed how I allowed

them to work independently and at their own pace.

> "I really liked it because if you didn't understand something, you could always just go back and relearn it. Also, you could redo the quizzes as many times as you want, so when you got an okay mark, you could do it again to improve, but this also gave you the opportunity to learn from your mistakes… and better understand the material. The learning material presented to us was really well presented and easy to understand."
>
> (Quotation from one of Kerry's students)

How did you use CLIPS?

I have used CLIPS as an introduction to periodic motion in trigonometry. I have used them in a variety of ways, but more recently, I began the unit by having the students work through the activities independently. We utilized a computer lab, and all the students completed the work on their own, with some teacher guidance when needed. At the end of the set of tasks, students were expected to complete the quizzes. I encouraged students to repeat the quizzes until they felt they understood the concepts. Following the CLIPS activities, I would teach the remainder of the trigonometry unit. I noticed that I was able to refer back to examples from the CLIPS to reinforce the concepts and overall, students gained a deeper understanding of the concepts. I also was able to complete the unit of study in less time because of this independent work.



What barriers did you encounter while using CLIPS? How did you overcome them?

Since students worked at their own pace, not all students finished the activities at the same time. This meant that some students had to complete the activities from home or during their free time. That being said, this demonstrates how useful this tool is, given that is can be accessed 24/7 by anyone.

Many of the students wanted to listen to audio in the activities, and due to different pacing, this created a noise problem. The next time we used the activities, I asked them to bring in their headphones, and this made a considerable difference.

In your opinion, what has been the greatest benefit?

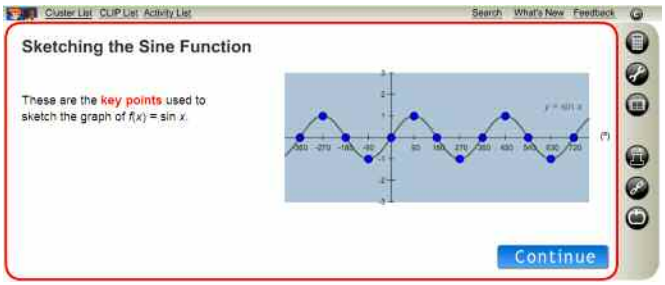
The greatest benefit is the opportunity for students to engage in interactive learning.

What would you recommend for first-time users or people unfamiliar with the CLIPS learning tool?

Go through the activities yourself and get a feel for them to see where you can apply these activities to your daily lessons/units.

Is there a specific CLIPS activity that you have found to be particularly powerful?

The Sine function section has been particularly powerful for both my students and me. I have used it in many of my Grade 11 classes.



Tina's Experience

Please briefly explain how you used CLIPS in your classroom.

I use CLIPS as part of my Minds On, Action, and/or Consolidation portions of my lessons. I have also used them in a computer lab.

In both situations, I integrate CLIPS within my lessons. I have found that this works best, given my teaching style. I have had my students work independently on CLIPS in a computer lab, but I have had greater success when I integrate CLIPS into my lessons.

Why did you decide to use CLIPS?

I don't recall how or when exactly I found out about CLIPS, but when I did see them, it became obvious right away that this was "quality stuff"! I am a big "techie" and I am always looking for ways to integrate technology into my lessons to help benefit my students. CLIPS do that. My students, whether they are 14 or 18, are amazed that there are activities out there (that are free) to help them learn mathematical concepts.

What were students' reactions to CLIPS?

The most common reaction I get from students when

working in a computer lab on some of the CLIPS activities is that they like having control, meaning that they can replay a scene as often as they like to make sure that they understood a concept before moving on to the next scene. I've actually had a student comment on how she wishes at times that she could "replay" my lessons/lectures like she could the CLIPS scene(s).

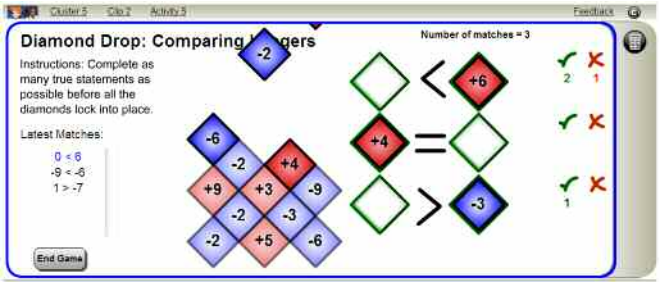
How did you use CLIPS?

I integrate CLIPS into my lessons. Depending on the topic, I may use a CLIPS video as my Minds On activity. Also, I have students work on some activities in a computer lab or in stations for the Action portion of the lesson. There are also great Consolidation activities and "Show What You Know" activities, such as FLASH matching games or PDF handouts/cards that can be printed off and easily integrated into the lesson.

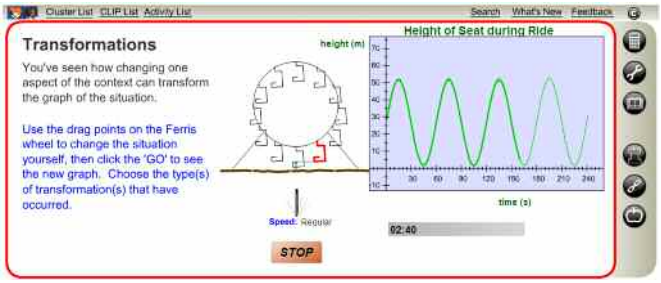
I love using CLIPS! I love that they are free and that they make it so easy for me to use them as I like. Everything that is on the CLIPS site is easy for me to download and use later, without having to be online. CLIPS are clearly created to benefit both students and teachers. Did I mention that I love CLIPS?! ;)

Is there a specific CLIPS activity that you have found to be particularly powerful?

For my Grade 9 classes, I like to use the activities from the Fractions & Integers CLIPS.



For my Grade 11 University/College (MCF3M) and Grade 12 Mathematics in Technology (MCT4C) classes, I use the Periodic & Sine Functions & Their Transformations CLIPS.



What barriers did you encounter while using CLIPS? How did you overcome them?

The biggest barrier that I've had to deal with is not being able to give students a direct URL to a CLIPS activity that I wanted them to use.* To overcome this barrier, I'd give a handout with specific instructions on how to navigate the CLIPS site to get the activity that I wanted them to work on. Another way of getting around this issue is to download the SWF and import it into a Notebook file. I am more likely to do the latter if only using the activity in the classroom.

***Note:** CLIPS developers have shared how to navigate to a specific activity online, using the numbers provided at the bottom of the wrapper. Locate the following information—Cluster number, Clip number, Activity number—and place the numbers in the appropriate place in the URL.

Address for activity shown below **http://oame.on.ca/CLIPS/index.html?cluster=5&clip=3&activity=3&scene=6**

In your opinion, what has been the greatest benefit?

There are many benefits to using CLIPS in the classroom, but I would have to say that the best one is giving students the control of going through the activities at their own pace. This gives them control over their own learning.

What would you recommend for first-time users or people unfamiliar with the CLIPS learning tool?

For teachers who are new to CLIPS, I would highly recommend going through the activities on your own before introducing them to your students.

Thank You

We would like to thank both Kerry-Lynn and Tina for sharing their insights and experiences with CLIPS. It is our hope that this three-part series has been helpful and will inspire others to investigate the powerful possibilities that CLIPS can have in all classrooms. ▲

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