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ABACUS

Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

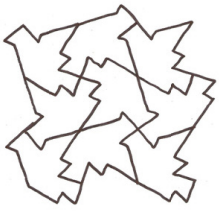
AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

Vol. 54 #4
June 2016

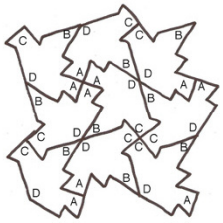
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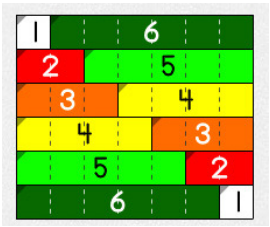
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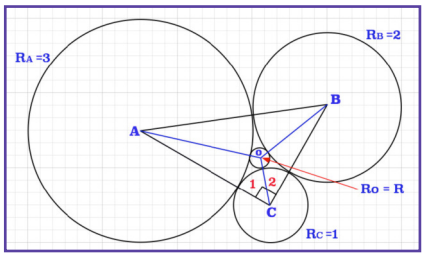
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See Provincial Digital Learning Resources – What's New? Relational Rods Plus (mathies.ca)



$$1 + 2 + 3 + \ldots + n$$
$$= \frac{n(n+1)}{2}$$
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See Three Spheres of Radii 3, 2, and 1 Mutually Tangent Problem (Part 1)

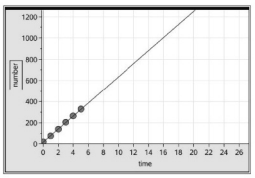


See Hey, It's Elementary: The Renewed Math Strategy –The Lead Teacher Dilemma

See In the Middle: Linear Leaping

Graphical
(Draw a sample sketch.)

Graph your data, showing the key points.



Logical
(Explain the starting value, and constant change rate in words.)

I started at 20 and performed at an average rate of 62 jumping jacks per minute.

62 is the average first difference, so the average number of jumping jacks I did per minute over the 5-minute period.

Algebraic
(State the variables and the equation.)

t represents the amount of time she has exercised, and n represents the number of exercises completed.

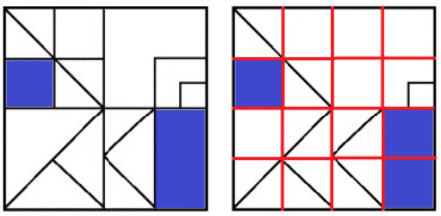
$n = 20 + 62t$

Numerical
(Complete the table, being sure to show important points.

Data collected from exercising.

t	n	first differences
0	20	X
1	77	57
2	140	63
3	202	62
4	263	61
5	328	65

See Technology Corner: Fraction Talks





Always Learning

With the recent **Ministry of Education** announcement, Math Lead Teachers will now have the opportunity to benefit from

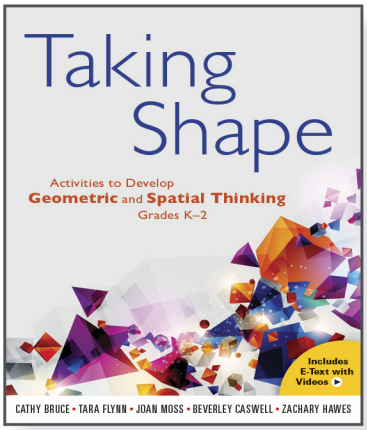
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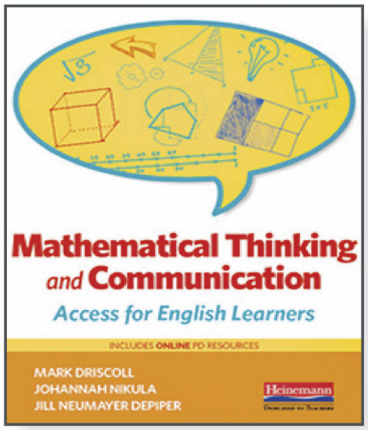
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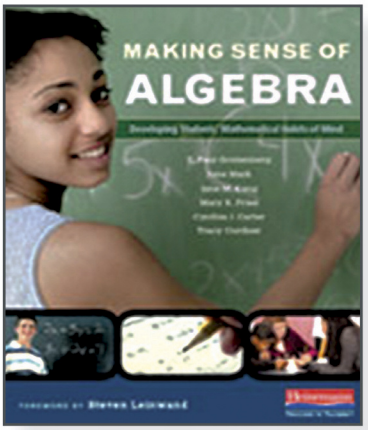
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Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Dan Jarvis, along with **separate files for each of the text, graphics, and/or photographs**. Please email all files to Dan Jarvis at **dan.jarvis@oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and email address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME.

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The views expressed or implied in this publication, unless otherwise noted, should not be interpreted as official positions of OAME.

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▲ Editor's Message



DANIEL H. JARVIS, O.C.T., Ph.D.
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Dr. Daniel Jarvis is Professor of Graduate and Mathematics Education in the Schulich School of Education at Nipissing University, North Bay, Ontario. His research interests include instructional technology, integrated curricula, and mathematics of the workplace.

It was unforgivably early on an overcast Wednesday morning that I found myself sitting next to a pleasant stranger on an Air Canada flight bound for sunnier southern climes. My new acquaintance and I eventually—as those confined to contiguous seating in small aircrafts are often wont to do—entered into polite conversation regarding travel destinations, weather, family, and employment. It was during the exchange on this final topic that the issue of Ontario's recently announced *Renewed Math Strategy 2016* (RMS) was brought up by my gregarious co-passenger. It was her opinion that while she certainly approved of the provincial government's investment in her children's education, she was also hopeful that this large sum of money ($60M) would be used wisely, and not just spent on encouraging even more "discovery methods" that had replaced, in her view, the more important "basics"-focused elements of the curriculum. Does this sound familiar to anyone? What are we to make of Canadian media coverage that pits "reformers" versus "traditionalists," revisiting the "math wars" (Wilson, 2003) controversy that lingers still in many parts of North America?

First, let's have a brief look at what the Renewed Math Strategy involves. In her April 4, 2016 news release, Education Minister Liz Sandals announced that "Ontario is dedicating more than $60 million to help support students across the province achieve better results in mathematics." This Strategy, beginning in September 2016, will include: a minimum of 60 minutes per day of protected "learning time for effective mathematics instruction and assessment" in Grades 1–8; up to three math lead teachers in all elementary schools; coaching for principals of select secondary schools to lead improvement; support for learning at home through parent resources; better access to online math resources; math support for Grades 6–8 students outside of the school day; and opportunities for educators to deepen their knowledge in math learning,

teaching, and leading, including a dedicated math Professional Development Day. Furthermore, Minister Sandals claims that the Strategy is "informed by research and the lessons learned from the education sector," and that it focuses on the "needs of students, their families, educators, schools, and district school boards" and "encourages shared responsibility."

After enjoying Canadian educator Marian Small's well-attended NCTM Conference Feature Presenter session entitled *Becoming the Math Teacher They Need!*, I caught up with her for a brief discussion about mathematics education in general, and the Ontario RMS more specifically. Later that evening, I received a requested email from Dr. Small, in which she shared the following related comments, and which I here present to you, with permission:

> I am very happy that the Ministry has decided to remain on the path it has. I believe we are going in the right direction, BUT it takes time and commitment not just from teachers, but from Directors of Education, Superintendents, and particularly principals. We need time and opportunity to help those critical individuals use their roles to move meaningful math, i.e., meaningful for the day and age in which our students live, forward. I believe that each child in Ontario deserves a teacher of math who has a passion and interest in math and who strives to develop a better understanding of how to interpret curriculum expectations to make the learning rich and deep. Having greater expertise in each school by supporting lead teachers is likely to help in this regard. I still believe that the "nut we have to crack" is consolidation of lessons. I remain very committed to the notion that consolidation is not just about sharing strategies. It is about ensuring that the mathematical ideas in which students have engaged are brought to the surface for all students. I also remain convinced that we need to stay the course with open questions.

As I've written in previous publications (Jarvis, 2008, 2016; Jarvis & Franks, 2011), I believe that a "balanced mathematics curriculum" should feature repetition and practice, as well as ample opportunities for hands-on manipulative and technology explorations, and for problem-based learning in both individual and collaborative contexts. Some researchers have insisted that number facts should not be "memorized," practised with flash cards and timed quizzes [Mad Minutes], or demanded through instant recall–common practices which they feel lead to increased student anxiety and disengagement–but rather should be explored through a flexible and rich number sense focus (Boaler & Williams, n.d.; Bratty, 2015). In other jurisdictions, such as the United Kingdom, Alberta, and Manitoba, math curricula explicitly required the memorization of number facts such as the multiplication table (Alphonso, 2014). I suppose I am somewhere in the middle of this particular debate. Knowing one's number facts, in terms of future facility and success in mathematics learning, seems to me to perhaps be more significant than Boaler maintains; yet I do agree that how one comes to understand number sense, and to learn to appreciate the relationships between numbers through modelling and flexible mental strategies, is indeed paramount (e.g., "Making 10" with one's fingers and then with the mind is something I've enjoyed doing with students in Canada, Kenya [see photo below], and Ecuador, and an activity that my teacher education students know I tend to frequently revisit as being foundational. Another one of my favourite activities is the use of non-timed triangle flash cards that allow students to understand the direct relationships between addends/sums and factors/products, as well as see the connections between add/subtract and multiple/divide operations, and the fact family relationships).



At the beginning of Winter term, I had sent an email to my Concurrent B.Ed. classes, asking teacher candidates for their input on what they were observing in Ontario classrooms during practicum placements, in terms of mathematics lessons. I shall just share a few comments here, that relate well to our topic at hand, again with permission. For example, Mary notes the following:

> I was in an FDK placement for three weeks, where math had a specific carpet-time lesson related to math centre activities 2–3 times a week. The teacher used a combination of mini-lessons and multi-modal learning activities to teach probability and ordinal numbers when I was collaborating with her. I am currently in a gifted Grade 4 program. My AT has a strong language and arts background… Though the school has frequent PD on strategies such as the 3-part math lesson, it appears that there isn't a lot of

> support that recognizes the experiences of mature teachers while, still helping them to re-imagine math in the classroom. I know she is frustrated by the types of PD being offered… Her approach has been to use a lot of authentic problem solving.

Another B.Ed. candidate, Sara, describes her observations:

> I attended a workshop along with my Associate Teacher about how to implement it in classrooms throughout all the elementary grades. . . . So for example, a 50-minute math period may begin with a Minds-On activity, followed by a short lesson introducing a new concept, and then students will participate in a 20-minute rotation through five centres: Guided Math/Problem Solving, Shared or Independent Problem Solving, Math Journal, Math Games, and Math Facts. Math Facts always involves numeracy skills of some kind, with the purpose of practising these skills all year, not just during that particular strand… It's a learning process, both for the students and the teachers.

Clearly, the mathematics experiences for students, teachers, and curriculum leaders vary considerably throughout our province, based even on these few brief practicum reports.

The National Council of Teachers of Mathematics (NCTM), in their publication *Principles to Actions: Ensuring Mathematical Success for All* (2014), based on "research-informed actions" (p. 4), argue that in order to move from "pockets of excellence" to "systemic excellence," the mathematics education community in the United States must address certain shortcomings including:

- too much focus is on learning procedures without any connection to meaning, understanding, or the applications that require these procedures
- too many teachers have limited access to the instructional materials, tools, and technology that they need
- too many teachers of mathematics remain professionally isolated, without the benefits of collaborative structures and coaching, and with inadequate opportunities for professional development (p. 3)

Ontario's Renewed Math Strategy has the potential to address the kind of mathematics being learned in classrooms (i.e., proficiency with understanding), the amount of time spent doing so, adequate resources, and professional development supports necessary for teacher and student success. I too remain convinced that we are on the right course of action here in Ontario, and am hopeful that teachers of mathematics, lead teachers, administrators, and coordinators will feel supported by the Ministry, by their respective District School Boards, and by parents/guardians, as the the RMS begins to see implementation this coming Fall.

The June 2016 final Volume 54 issue (V54N4) of the *Gazette* includes three articles, ten regular columns, the elementary school-focused *Abacus* insert, along with several special features.

In their article entitled *A Case for Humility in the Mathematics Classroom*, co-authors Ilona Vashchyshyn, Heidi Neufeld, and Egan Chernoff discuss the three characteristics of humility (a secure, accepting identity; emotional resilience; and openness to new information), two related classroom vignettes, and a series of reflections on why and how this virtue should characterize effective mathematics teaching and learning contexts.

Nipissing University's B.Ed. mathematics education instructor Tim Sibbald and B.Ed. candidate Miranda Wheatstone have collaborated on an intriguing article, *Advancing Escher Art through Generalization*, that extends the problem-solving component of Escher's famous tessellations and illustrates that, in terms of tiering instruction, "it facilitates a diverse array of opportunities for examining cases that show very different pattern behaviours."

Finally, in the first of a 2-part article, a retired teacher, pilot, and skydiver, André Lemaire, explores different mathematical methods for solving the ancient Greece classic "Three Spheres of Radii 3, 2, and 1 Mutually Tangent Problem," including the use of Pythagoras's Theorem, Heron's Formula, trigonometry, Cartesian coordinates, and analytic geometry.

Regular columns include the following highlights: OAME President, Tim Sibbald (in his final President's Message), discusses the NCTM Board of Directors meeting that was recently held in Toronto, and during which the strengths of, and current challenges faced by, Ontario mathematics education were discussed; Todd Romiens (OAME/NCTM Report) highlights NCTM's new Innov8 Conference designed to specifically help teachers engage struggling learners, and NCTM's STEM Project (science, technology, engineering, mathematics) which continues to inform the American Congress; Lynda Colgan (Hey, It's Elementary) discusses the Renewed Math Strategy, with special emphasis on what she refers to as the "Lead Teacher dilemma"; Ross Isenegger, Agnes Grafton, Markus Wolski, and Greg Clarke (Provincial Digital Learning Resources) continue explaining the various applications of the **mathies.ca** Relational Rods resource (e.g., reasoning and

proving, even/odd numbers, summing the natural numbers, partitions, and the ability to annotate within the software); and Mary Bourassa (Technology Corner) explores Nat Banting's website "Fraction Talks," and shares several examples of how others have used this helpful online resource in their mathematics teaching.

Carly Ziniuk (*In the Middle*) presents an investigation entitled "Linear Leaping," in which linear relationships connected to healthy movement are examined by students using available data and a 4-part template which includes graphical, logical, numerical, and algebraic thinking; Assessment Abby (eponymous) focuses on linking assessment and instruction; Shawn Godin (What's the Problem?) analyzes two rich problems involving card picking, probability, and the classic Monty Hall Problem with opened/closed doors and hidden prizes; Ann Kajander (MB4T) examines her fourth and final case involving division of integers, that of a positive divided by a negative number; and in the *Abacus* insert, co-editors Mary Lou Kestell and Kathy Kubota-Zarivnij complete their Volume 54 focus on relational thinking, here focusing on the operation of division.

Volume 54 Issue 4 also includes several special features: (i) a David Davidson Tribute (1940–2016), written by his close friends Dean Murray and Jack Weiner; (ii) a Letter to the Editor by Tim Sibbald, which raises awareness about a forthcoming publication and opportunities for participation; (iii) three further Ontario mathematics education researcher profile highlights; and (iv) some final excerpts from Lerman's *Encyclopedia of Mathematics Education* (2014) regarding a variety of interesting, math-related issues.

It was my great pleasure to attend the NCTM "Building Bridges to Student Success" Conference 2016 in San Francisco, California, and to accept, on behalf of OAME, the Publication Award from NCTM President, Diane Briars, during the annual meeting of affiliate delegates.



Once again, I would just like to take a final moment to thank OAME for allowing me this great privilege to serve as your Editor for the past two volumes. In my view, Presidents Paul Alves and Tim Sibbald have both been wonderful advocates for mathematics education, providing the organization with informed and indefatigable leadership. My thanks again to my gracious predecessor, Immaculate Namukasa, and I look forward to paying her kindness forward as I begin to work with my successor, Amy Lin, who will take over as *Gazette* Editor this summer.

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Errata: *In the approximately 200 printed copies of V54N3 (i.e., we were able to correct it in the digital version) paragraph three of the Editorial read as "(10 raised to the power of a googol, or a 1 followed by a 1000 zeroes)," but should have been "(10 raised to the power of a googol, or a 1 followed by a googol of zeroes)". Similarly, "(10 raised to the power of a googolplex, or a 1 followed by 10 000 zeroes)," should have read "(10 raised to the power of a googolplex, or a 1 followed by a googolplex of zeroes)".*

▲ Letters to the Editor

This letter is to draw attention to an opportunity for aspiring OAME writers. This particular call is for chapters regarding interdisciplinary mathematics from a mathematics point of view. The goal is to examine the combining of all school subjects from a mathematics perspective. The writing is to speak to combining subjects (as STEM and STEAM suggest) and specifically look at how these combinations impact the mathematics that is taught. Ideally, the resulting book will consider the teaching of mathematics in combination with a wide array of other subject areas. Interested potential writers should contact **timothys@nipissingu.ca** for the detailed Call for Chapters.

Tim Sibbald ▲

▲ Mathematics Education Researcher Highlights

Math Education Researcher: Dr. Chantal Buteau, Brock University

Projects: Dr. Buteau's research interests in mathematics concern mathematical music theory. She is particularly interested in deterministic modelling of music (melodic) analysis and structure. Her main research interests in mathematics education are the use of digital technology in teaching and learning mathematics, university mathematics education, and mathematics teacher education. Together with Dr. D. Jarvis (Nipissing) and Dr. Z. Lavicza (Cambridge), Chantal has taken part in an international study focusing on the integration of Computer Algebra Systems (CAS)-based technology in undergraduate mathematics teaching, as well as an analysis of sustained departmental shifts toward increased technology use in mathematics instruction. As co-researcher with Professor Emeritus, Dr. Eric Muller, she has also been studying diverse aspects of the programming-based mathematics course implementation at Brock University (Mathematics Integrated with Computers and Applications, or MICA). She has also recently initiated work with Muller on epistemic mathematics computer games.

Selected Publications:

Buteau, C., Muller, E., & Marshall, N. (2015). When a university mathematics department adopted core mathematics courses of an unintentionally constructionist nature: Really? *Digital Experiences in Mathematics Education, 1*(2-3), 133–155.

Buteau, C., Jarvis, D. & Lavicza Z. (2014). On the integration of computer algebra systems (CAS) by Canadian mathematicians: Results of a national survey. *Canadian Journal of Science, Mathematics and Technology Education, 14*(1), 35–57.

▲ President's Message



TIM SIBBALD

As many of our members realize, the OAME is an affiliate of the NCTM. There are several affiliates across Canada, and being an affiliate allows us to participate in the largest math-education organization on the planet. This is an opportunity for our members near the border, as well as those with the memberships in both organizations, but it is also important as an organization. The importance of this was reiterated when the NCTM held a board of directors meeting in Toronto last February. At that meeting, several provincial math associations spoke to the state of math education in their organization and their province. There were also some provincial representatives who spoke to the role of their respective Ministries of Education. A snapshot of mathematics education across Canada emerged.

It was a powerful event because it demonstrated very clearly that the affiliates in Canada have a wide range of experiences and relationships with their provincial counterparts. A remark that stood out was a statement that Nova Scotia has the same buying power as Peel District School Board. In terms of Ontario, the existence of the Ontario College of Teachers, as a professional governing body distinct from the government, and an acknowledgement by the Ontario Ministry representative that they work with partnerships because the Ministry of Education cannot do it all, brought a flavour of a fundamentally distinct approach to math education. The OAME is a self-governing partner in mathematics education.

The simple fact is we are doing well. The OAME is a strong organization that is an active partner within the Province. The organization provides a framework for grassroots growth and participation within the local and provincial communities. The chapters facilitate discussions across boundaries by including all school boards and schools. The organization is remarkable, and every member should engage some of his or her energy in continuing to help it to flourish. One example of engagement is the 21 years of success of the Math Challenge Western program in London, where the organizers have a long history with the OAME.

We are also actively engaging our partners in education—the Ontario Ministry of Education, of course, but OAME members are also engaged in the Fields Institute

MathEd Forums (Toronto), the Ontario College Mathematics Association annual conference (Orillia), and the Canadian Mathematics Education Study Group (Kingston this year). I am sure there are other events, but I chose these as representative of geographical diversity and different foci.

Even with such a positive image of our progress, there is work to be done. We are not without issues and points of discussion within the OAME. Consider, for example, that the increasing complexity of the annual conference has led to the board of directors taking the lead for 2018. This is the second time the board of directors has had to address the task, which reflects the challenge of chapters developing local leadership to the point of taking the three-year commitment for an increasingly complex event. Discussions about ways to address this will, doubtlessly, continue and, increasingly, it looks like a "conference master," somewhat analogous to a webmaster, may be a good idea to support local efforts. Perhaps, though, the lure of a "20-20 vision" and all the eyesight analogies that are possible will inspire a chapter to come forward for 2020 (2019 is taken).

Another issue that is actively percolating is a demographic issue. Our membership spans 50 years of teaching experience. This has led to questions about the interpretation of our mission and vision statement. Specifically, should the Ontario curriculum be a constraint on discussions, presentations, and articles. The difficulty is that for many of our members, their workplace is intrinsically tied to the curriculum, and having appropriate materials from the OAME is a direct support. However, the curriculum is the minimum we have to teach, and discussion about enrichment or going beyond the curriculum surely warrant some opportunities. (Couldn't Martin Gardner be a contributing OAME member?) What is more, our collective leadership role suggests that we need to actively think about the curriculum itself. As an example, conic sections (e.g., ellipses, hyperbolas) haven't fully been included in the curriculum in some time. However, they represent the full collection of quadratic forms and are not very distant from the current curriculum that includes some conics (e.g., the line, parabola, and, to some extent, the circle). Should there be discussion of the merits of conic sections? Certainly our most senior members can inform the most junior members of this rich vein of mathematics.

Another pair of issues is whether math-education discussions have swung excessively toward the use of technology and applications. The former is challenging because technology can support ways to examine the conceptual big picture without an exhaustive requirement for working out details. Increasingly, however, there seems to be a degree of technological malaise because of uneasiness about the relationship between technology and pedagogy. I can recall comments from retired, mid-career, and early-career teachers expressing concerns. A related issue is with the use of applications that help scaffold concepts and provide a meaningful purpose for mathematical concepts. This is not unreasonable, except that many feel the need for having the sheer beauty of mathematics for its own sake revealed. No one suggests this should always be the case; rather, it is a discussion about balance.

Fundamental to these issues is a perennial problem that we have in math education, which is how to infuse recent growth in the field of mathematics into K–12 education. Science has an ongoing popular media that facilitates the development of classroom updates that use developments from the research community within decades. This does not seem to be the case in mathematics, with a few exceptions; fractals come to mind as a specific example that did migrate from research to the classroom. However, there is no doubt that there are disproportionately fewer examples in mathematics than in science. In some ways, this seems to originate with the calculus gateway, but data management and the use of technology facilitate approaches to circumventing the need for calculus. To make this issue deeper is a general hesitancy to see the curriculum as a temporary bridge between current mathematical knowledge and what we will know in 50 years.

In short, we are in good shape, but need to continue discussing, presenting, volunteering, and writing in order to make sure that we navigate our way through the issues we continue to address. On that note, I would like to express my gratitude to the organizers of the OMO, the organizers of the annual conference in Mac^2 and the leadership in all 15 chapters. It is also a time to acknowledge Dr. Dan Jarvis for his editorial leadership of the *Gazette*, and to acknowledge his passing of that torch to Amy Lin. As Judy Mendaglio steps into the year of her presidency, I would like to welcome the new members to the executive and board of directors; however, at the time of sending in this message, the results are not known.

In my new role as past-president, I will be looking for award nominations and encouraging everyone to vote in next year's elections. The summer is a good time to consider whom you might nominate for an award (it's an honour, whether one wins or not), and a good time to consider encouraging local leaders to consider engaging in the election process. I can personally speak to it being a very rich opportunity and I am thankful that the membership put their trust in me. ▲

▲ Ask Assessment Abby: Linking Assessment and Instruction



ASSESSMENT ABBY
EMAIL: assessmentabby@oame.on.ca

Ask Assessment Abby A³ is a regular column in the OAME *Gazette*, where teachers can share concerns and best practices about assessment, evaluation, and reporting of mathematics. Please send your questions to Ask Abby at **assessmentabby@oame.on.ca**.

Dear Assessment Abby,

I was at a workshop presented by a representative of the Ontario Ministry of Education She said it was important to link assessment and instruction. What could this look like?

That is a good question, as many teachers have similar queries. Simply put, teachers gather information by assessing what students know before/during/after instruction and use this information to inform the instruction.

Growing Success (2010, p. 28) states: As essential steps in assessment for learning and as learning, teachers need to:

- plan assessment concurrently and integrate it seamlessly with instruction
- gather information about student learning before, during, and at or near the end of a period of instruction, using a variety of assessment strategies and tools

The following list contains some examples of strategies and tools that teachers employ to gather real-time information of student learning so that instruction can be adjusted based on the results of this assessment. (A short explanation of each and/or a website are provided at the end of this article, if you would like additional information.)

- Survey your class, using a polling device (e.g., Kahoot, Poll Everywhere, Nearpod, Bell Work, individual whiteboard, Clickers, Plickers) as a check-in anytime during a lesson.
- Circulate while students are working on a task (e.g., listen to their conversation, observe their work, ask probing questions regarding their understanding). Consider inviting a critical teacher/friend to assist with classroom management as you circulate.
- Pose a problem without a standard solution, and monitor/focus on student work and language as they solve it, to identify what they understand. Pay attention to the tools students choose. Remain open to a variety of solutions.
- Gather visual information of student understanding (e.g., Post It Parking lot, Thumbs up/down/sideways, Fist to five, Four Corners).
- End-of-class closure (e.g., journal entry, exit ticket to inform the follow-up instruction)

You may already be linking assessment and instruction, but perhaps you haven't recognized it as such. However, if this is a new concept, consider starting by gathering information on the learning of the class as a whole, then move toward examining individual assessment results. Some of the tools mentioned provide digital data that can be collected and broken down for individual student analysis.

More details on entries mentioned above follow here:

- Kahoot – **http://blog.getkahoot.com/post/49444981797/new-to-kahoot-quick-polls-and-surveys**
- Poll Everywhere – **www.polleverywhere.com/**
- Nearpod – **www.nearpod.com/**
- Bell Work – **www.calhounisd.org/downloads/dcia/Bellwork.pdf**
- Individual Whiteboard – **www.nwea.org/blog/2012/classroom-techniques-formative-assessment-idea-number-three/**
- Clickers – **www.engaging-technologies.com/classroom-clickers.html**
- Plickers – **https://plickers.com/**
- Teacher Observation – **www.ncca.biz/flashpaper/sect2/teacher%20observation.swf**
  Or for a more complete explanation, try **http://curry.virginia.edu/uploads/resourceLibrary/CASTL_practioner_Part4_single.pdf**
- Probing Questions – **www.edugains.ca/resources/LearningMaterials/MathProcesses/MathProcessessPackage.pdf**
  www.edugains.ca/resources/LearningMaterials/MathProcesses/MathProcesses_AfL-AaL.pdf (Bonus: The question stems provided are categorized by math processes!)
- Thumbs Up/Down/Sideways, Fist to Five, Four Corners – **www.teachingchannel.org/blog/2015/03/04/formative-assessment-resources/**
  (Bonus: Other quick assessment strategies are also included here!)
- Math Journal – **http://thecornerstoneforteachers.com/free-resources/math/math-journals**
- Exit Ticket – **www.nwea.org/blog/2012/classroom-techniques-formative-assessment-idea-number-two/**
- Post It Parking Lot – **www.thedailycafe.com/articles/Exit-slips-parking-lot** Students use a sticky note to identify what they understand/don't understand at that point in time. The teacher can then group similar notes together and differentiate accordingly. ▲

▲ Technology Corner: Fraction Talks

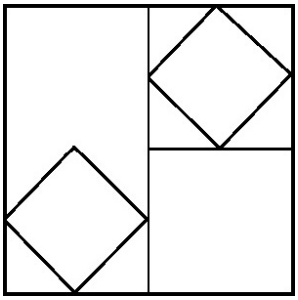


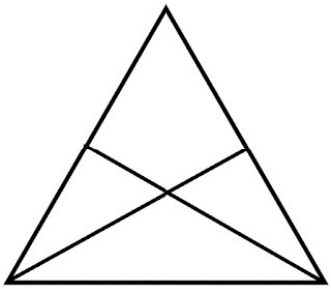
MARY BOURASSA
EMAIL: mary.bourassa@oame.on.ca

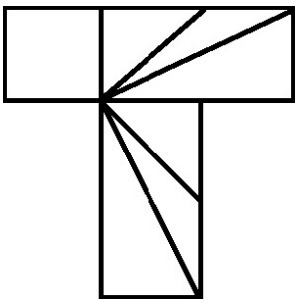
Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

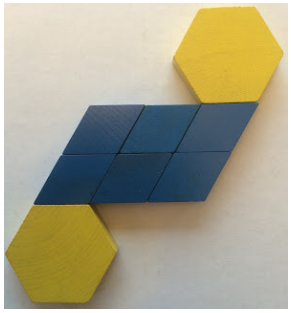
Earlier this year, Nat Banting (@NatBanting) started a fantastic website at **fractiontalks.com**, and his stated goal is "to develop a simple visual that could foster creative thinking around fractions." Many students struggle with fractions, so having a website with great images that spark conversations around fractions is definitely a step in the right direction. Fraction talks can be used as a daily warm-up or tied in to a particular topic to help strengthen students' number sense and foster better understanding of fractions. Fractions appear in the Ontario curriculum right from Grade 1, so Fraction Talks can happen in any classroom, not only when working with topics related to area and fraction operations.

Here are some examples of images from the website:

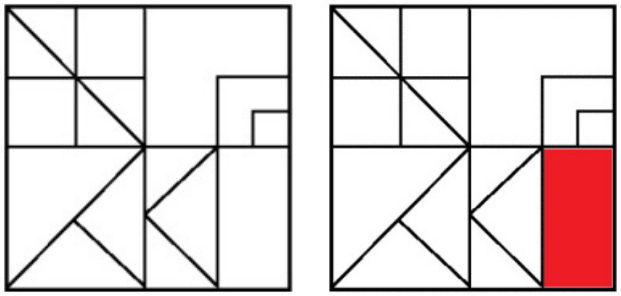






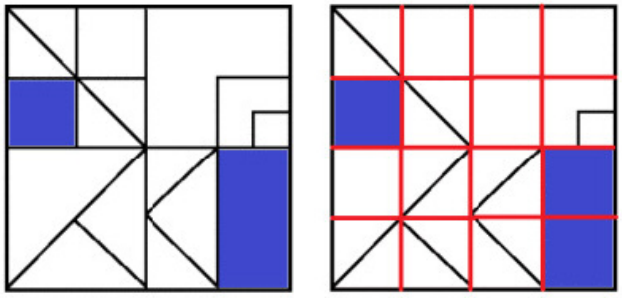


The idea is that you can shade one or more regions and ask what fraction it represents. Here is an example:



As students work through these, the questions can become richer and more open-ended. From Nat's website: "Can you shade a section with twice the area?" "Can you shade exactly one-quarter of the shape?" You could also ask students to shade a different section with the same area, or multiple sections with the same area. The student thinking and discussion that results from these questions, along with the multiple possible solutions, not only help students better understand fractions, but also strengthen their number sense and provide opportunities to construct viable arguments and critique the reasoning of others.

In the "How to…" tab on the website, there is a list of blog posts from teachers who have used Fraction Talks. Bryan Penfound's blog post includes how he uses Fraction Talks to deal with fraction multiplication and addition. The following images show a great strategy for adding fractional pieces of the large square.



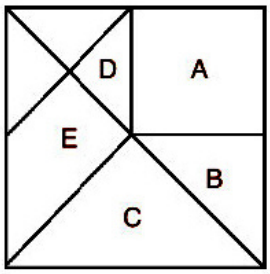
"Music is the arithmetic of sound, as optics is the geometry of light."

Claude Debussy



Amie Albrecht's post has examples of great questions with this annotated image:

Consider the diagram shown, which contains shapes of five different sizes.



(a) Without measuring, explain what fraction of the large square is represented by each of the shapes A, B, C, D and E.

(b) Shade exactly half of the large square. Explain how you know that this is exactly one half.

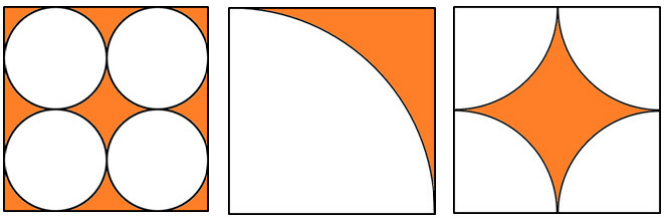
(c) Find four different ways of representing the fraction $\frac{1}{4}$.

(d) There are sixteen different fractions that can be represented using one or more of the smaller shapes. See how many you can find!

(e) If the entire area of the large square is 10 units^2, what is the area of C?

(f) If the area of E is 6 units^2, what is the area of the large square?

This submission, from Don Steward, asks which of the three squares is the most coloured, which would be great to use when investigating the area of a circle.



The website also provides a Fraction Talk handout to help students keep track of their reasoning. You can submit your own Fraction Talk image via the form on the website. Be sure to follow @FractionTalks on Twitter to keep up with all the latest submissions and conversations. ▲

▲ Mathematics Education Researcher Highlights

Math Education Researcher: Dr. Dragana Martinovic, University of Windsor

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Researcher Profile: Dr. Dragana Martinovic is a Faculty of Education Professor. She received a Ph.D. degree in Computer Applications in Education from the University of Toronto, and is a Fields Institute Fellow and a Director of the Human Development Technologies Research Group at University of Windsor. Dragana serves as a Co-Director of the Fields Cognitive Science Network and of the newly formed the Fields Centre for Mathematics Education. She is a Vice-President of SWOAME, a Co-Chair of the School Board University Research Exchange (SURE) network, and a Chair of the GeoGebra Institute of Canada/Institut de GeoGebra du Canada. Dragana is a Co-Editor of the peer-reviewed, online *Fields Mathematics Education Journal*, and of a Springer book series entitled *Mathematics Education in the Digital Era.*

Using her multidisciplinary background in mathematics, computer science, and education, Dragana investigates student learning and cognitive development in the presence of digital technologies, and the kinds of knowledge needed for teaching mathematics. She also explores the digital literacy skills needed for a successful learner and worker of the 21st century. With an expertise in research methods, Dragana is dedicated to supporting teacher-led research; access to and utilization of applied education research; and to providing opportunities for all involved in education to collaboratively work toward increasing student engagement, success, and love for learning mathematics.

Dragana is currently working on a number of national and international research projects, including: building research capacity of educators and students through collaborative inquiry; improving cognitive processing through playing computer games; and investigating benefits and constraints of Dynamic and Interactive Mathematics Learning Environments (DIMLE).

Selected Publications:

Martinovic, D., Winney, K., & Knight, C. (2015). Imagine the learning! Insights from a collaborative inquiry project in Southwest Ontario. *Learning Forward Ontario, 7*(1), 3–5.

Martinovic, D. (2015). Digital technologies and mathematical minds. In Mariana Bockarova, Marcel Danesi, Dragana Martinovic, and Rafael Núnez (Eds.), *Mind in mathematics: Essays on mathematical cognition and mathematical method.* Munich, Germany: Lincom Europa.

A Case for Humility in the Mathematics Classroom



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HEIDI L. NEUFELD

Heidi Neufeld is a recent Bachelor of Education graduate from the University of Saskatchewan. As a future teacher, she is interested in the field of parent engagement, specifically as it interacts with the current context of mathematics education in Canada.



EGAN J. CHERNOFF

Egan Chernoff is an associate professor of mathematics education at the University of Saskatchewan. He is an ardent user of social media for mathematics education.

"I'm out of moves."

"But you're the teacher!"

"Yes, I'm the teacher. And I'm out of moves."

Humility is not a virtue frequently associated with good teaching, and much less with mathematics—a subject considered by many, if not most, to be a serious and strict science that offers little room for "soft" values like humility. In fact, as Chancellor and Lyubomirsky (2013) write, humility may be the most overlooked and underappreciated virtue of all in any context. Perhaps this is due to the fact that humility is often conflated with low self-esteem, even in the psychology literature (see, e.g., Weiss & Knight, 1980), or that humility feels like a square peg in a round hole within a culture that shies away from the notion of fallibility, where boundless self-assurance is glorified as a means of achieving success. However, as counter-cultural as this virtue may be, it need not suggest weakness or docility. On the contrary, true humility requires emotional resilience and a secure sense of self. Moreover, as frightening as the prospect of admitting that one is "out of moves" may seem, humility on the teacher's part may have a critical role to play in enriching students' learning and enjoyment of mathematics, as well as in understanding the nature of mathematical activity itself. In this essay, we illustrate the point through two vignettes.

What is humility?

Given the misconceptions surrounding the virtue of humility, it is useful to first clarify what humility is—and what it is not. Given the psychological nature of this trait, it is natural to turn to the psychology literature. To start, humility must be distinguished from *humiliation*, which is associated with feelings of devaluation and loss of integrity or self-worth, and is conceptually related to social rejection (Chancellor & Lyubomirsky, 2013; Klein, 1991). As we shall shortly see, however, humility is *not* a detriment to self-worth, nor does it imply "spinelessness" or "submissiveness" (Axtel, 2000, p. 166). In their review of the research on humility, Chancellor and Lyubomirsky (2013) propose five main characteristics: an accepting and secure identity, freedom from distortion, openness to new information, other-focus, and egalitarian beliefs. Although they argue that each individual marker is not sufficient on its own to characterize humility, for the present discussion, we shall only consider the first three, inviting the reader to refer to Chancellor and Lyubomirsky (2013) for a more thorough analysis.

The first characteristic—a secure, accepting identity—may seem counter-intuitive, especially if one tends to identify humility with a lack of self-esteem or unfounded guilt (Daley, as cited in Klancer, 2012). However, a secure and accepting identity is not necessarily synonymous with pride or narcissism, because a true acceptance of one's identity involves being aware of one's weaknesses, deficiencies, and limitations (Chancellor & Lyubomirsky, 2013). In other words, it involves freedom from distortion—a recognition of the limitations to what one can do and know (Chancellor & Lyubomirsky, 2013; Paul, 2000). It is easy to see, then, that humility does indeed require emotional resilience, as Coulehan (2010) suggests. It takes a secure sense of self and, sometimes, no small amount of courage to admit one's ignorance, especially from a position of authority. However, as Graven (2004, p. 207) contends, "the insight to know when you don't know," coupled with "the confidence to admit to this," are key components of "mastery" in the practice of mathematics teaching. We must also note that an appropriate degree of self-confidence, which certainly has a

role to play in effective teaching (Graven, 2004; Sadler, 2013), need not be discarded when an attitude of humility is adopted. Indeed, the notions of self-confidence and humility in the classroom may be intertwined. As one teacher explains, "Confidence allows me not to have to know everything" (Graven, 2004, p. 203).

The third marker of humility characterized by Chancellor and Lyubomirsky (2013), openness to new information, is precisely what is necessary to avoid the trap of complacency. While the humble acknowledge the vulnerability of their claims to knowledge, this recognition does not lead them to hopelessness. Rather, it propels them to improve their present knowledge and understanding (Hare, 1993). They are open to learning from others and to exploring unfamiliar terrain, and they actively seek the truth (Chancellor & Lyubomirsky, 2013). In short, "the humble are teachable" (Chancellor & Lyubomirsky, 2013, p. 825), and the humble are curious. Although they understand and accept the fact that there are limitations to what one may know and can achieve, they are committed to continuous learning and growth and, as such, are not afraid of asking for help or exploring difficult questions.

Recognizing Humility: Two Vignettes

By now, perhaps it is easy to see how humility may be a useful virtue in the mathematics classroom… for the student. Certainly, all students are limited in what they know and can do, and teachers, even if they do not associate the term "humility" with certain attitudes and actions, surely hope that their students' awareness and acceptance of this fact stimulates them to expand their knowledge, as well as to gracefully accept and act on constructive criticism in their learning journey. Indeed, although the effects of psychological dispositions are difficult to quantitatively measure, the quality of humility has been associated with superior academic outcomes in several recent studies (see, e.g., Chancellor & Lyubomirsky, 2013; De Vries, De Vries, & Born, 2011). The necessity of humility for teachers, however, may be less clear, and perhaps even less so in the mathematics classroom.

Part of the reason may be that mathematics is often viewed as a strict, orderly, and primarily *linear* discipline, with results following from axioms and theories in a logical, perhaps even "obvious" fashion in the textbooks that many of us are familiar with from both our school and university- or college-level courses. (It is heartening to note some recently published mathematics textbooks, reflecting changes in pedagogy, do attempt to present content in a way that is more in line with an "inquiry" approach.) However, the precise and orderly exposition of an argument or proof on paper has little to do with true mathematical activity, and even the best textbooks often fail to deliver an authentic account of doing mathematics. According to Lakatos (1976, p. 42), the activity of doing mathematics develops as a process of conscious guessing about relationships, with proof following not a linear, but rather a "zig-zag," path that starts with "naïve" conjectures and moves to the examination of premises through the use of counterexamples. Naïve conjecture and counter-examples, however, are rarely recorded once this process has led to a resolution of the problem, and thus, "the zig-zag of discovery cannot be discerned in the end product" (Lakatos, 1976, p. 42). As Lampert (1990, p. 30) writes, although the *product* of mathematical activity might be justified with a deductive explanation or proof, "the product does not represent the process of coming to know." Moreover, even within the discipline of mathematics, knowing is not final or certain because the axioms on which proofs are based continue to be open to re-examination in the mathematical community (Lampert, 1990). This "vulnerability to re-examination," according to Lampert (1990, p. 30), is what allows mathematics to continue to grow and develop (consider, e.g., the history of the development of non-Euclidean geometry, which required a re-examination of the assumption that the parallel postulate must hold in a logically consistent geometry; Szudzik & Weisstein, n.d.).

Unfortunately, although students may have some understanding of this process through their own engagement in mathematical activity, the (commonly, although the list of exceptions is growing) deductive structure of many textbooks and of the carefully prepared lessons of some of their teachers do nothing to *legitimize* it. Perhaps the fear and anxiety that surrounds mathematics for so many students has something to do with the implicit belief in the linearity of mathematical activity—in other words, to quote Lakatos (1976, p. 2), the belief in a "formalist heaven […] purged of all the impurities of earthly uncertainty"—which serves to make students feel inadequate and hopeless when their first approach does not immediately lead them to the resolution of a given problem. Consider again, however, the opening vignette, recounted to the authors by a teacher of high school mathematics whom we shall call Mr. B:

Mr. B: "I'm out of moves."

Students: "But you're the teacher!"

Mr. B: "Yes, I'm the teacher. And I'm out of moves."

This short exchange between the students and the teacher in the heat of a high school lesson on proving trigonometric identities may have taught students much

more about mathematics than a perfectly constructed proof copied from the whiteboard—and, incidentally, took no small amount of humility on the part of the teacher in question. Coming to a seeming roadblock, a dead end: This is mathematics-in-action—as is retracing one's steps, questioning one's assumptions and conclusions, and asking for help, even if one is supposed to be the "expert." Clearly, this is a process that requires humility precisely of the kind described above. In this particular vignette, the teacher's humility is on full display. Aware that he has arrived at a dead end, he admits his limitations, although his identity is no less secure than it was before ("Yes, I'm the teacher," and not "Yes, I'm an inadequate teacher"); moreover, he is open to analyzing alternative viewpoints, inviting his students (by his admission of being "out of moves") to participate as equals in the resolution of the problem. By modelling and engaging students in the natural zig-zag process of mathematical activity, the teacher has legitimized it, offering it as an appropriate and justifiable means of coming to know mathematics both collectively and individually. Although taken aback, his students soon began to offer suggestions, eventually proving the identity in question.

A description of mathematical activity that involves only problem solving, however, is incomplete. Although problem solving in the mathematics classroom has garnered much attention in recent decades, one of the positive results being an increased interest in developing rich problems to foster conceptual understanding and engagement in the kind of mathematical activity described above, there has been much less interest in encouraging student problem *posing*: the reformulation of given problems and the formulation of new problems, which can occur before, during, and after the solution of a problem (Silver, 1994). Too often, students are only ever asked to solve problems that have been presented by the teacher or the textbook, and rarely given opportunities to pose their own mathematical problems in some public way (Silver, 1994)—this, despite the fact that student-generated problem posing has been recognized as an important component of mathematical activity, if not at the very heart of learning mathematics (see, e.g., English, 1997; NCTM, 1989, 1991; Nicolaou & Philippou, 2007; Silver, 1994). The benefits of fostering problem posing in classrooms include generating more diverse and flexible thinking and thus enhancing students' problem-solving abilities, correcting erroneous views on the nature of mathematics, dissipating common fears and anxieties about mathematics learning, and increased enjoyment of mathematics resulting from opportunities to pursue lines of inquiry that are personally satisfying (Brown & Walter, 1990; Silver, 1994). In short, incorporating problem posing into the mathematics classroom can breathe new life into a discipline which, according to Silver (1997), is intimately interwoven with creativity, but is ironically among the set of school subjects with which creativity is typically least often associated.

Fostering authentic problem exploration by encouraging problem posing in the classroom, in addition to problem solving, demands a great deal of humility on the part of the teacher. In doing so, teachers will almost certainly be put into situations where they have to deal with unfamiliar problems—and sometimes, they may not have an answer ready on demand. Some relinquishment of control and authority is inevitable, and the prospect may be frightening. However, the humble teacher will not associate her or his ignorance on a particular matter with her or his self-worth or effectiveness as a teacher. Moreover, recall that the humble teacher is "teachable" (Chancellor & Lyubomirsky, 2013). At least for a given time, she or he is willing to relinquish the role of the "teacher" and enter the role of the "student" so as to explore a new problem with her or his students with a willing and open mind—in other words, adopting the very attitude that she or he expects her or his students to embrace.

An incident observed by two of the authors in Mr. B's classroom serves to exemplify this humble attitude, which is foundational to fostering authentic student problem posing. During this lesson, the students were engaged in playing an iterated version of the Prisoner's Dilemma in partners, a game originally conceived for two players, with the goal of formulating a strategy to minimize one's time in prison (for a description and discussion of the Prisoner's Dilemma, see Kuhn, 2014; the details of the problem are not essential to the present discussion). Some time into the game, one of the students raised an unexpected question: "What if there were *three* prisoners?" The teacher admitted that he had not envisioned this situation. However, instead of dismissing the question and encouraging the student to return to the task at hand, the teacher accepted the challenge and at once began reformulating the game with the student. After they had conceived a plausible situation where three prisoners were involved, the teacher announced their reformulation to the class and encouraged the other students to analyze and compare the new problem to the original. In doing so, not only did the teacher facilitate a deeper analysis of the structure of the original problem, he also publicly validated the process of problem posing and presented an excellent example of mathematical creativity that is within all students' reach.

Reflection

Through the above vignettes, we have argued that humility has a critical role to play in the mathematics

classroom. With humility, the teacher can both recognize and embrace the limitations of her or his knowledge and is willing to open her or his mind so as to explore unfamiliar mathematical terrain. In doing so, the teacher may enrich students' understanding of true mathematical activity and increase their enjoyment of the subject. For instance, in allowing herself or himself to publicly admit to having come to a roadblock in a problem and inviting students to make conjectures, the teacher models and legitimizes the naturally non-linear process of doing mathematics, which is unfortunately often absent from textbooks and may contribute to the widespread dislike and fear of the discipline. In giving students opportunities to pose their own problems—even if it means, at times, admitting that she or he does not know—the teacher allows students to explore the full range of mathematical activity and to experience the creativity inherent in mathematics. In short, the humble teacher is willing to temporarily relinquish the role of "teacher" and enter the role of "student" so as to explore interesting, challenging problems *with* her or his students with a willing and open mind—in other words, adopting the very attitude that she or he expects the students to embrace.

The lessons of humility can be applied to other contexts in the mathematics classroom as well, such as assessment. As the research has shown (the research on people's conceptions of probability, for example, offers many relevant case studies), understanding cannot be evaluated on a binary scale, and even responses that, on the surface level, appear incorrect (i.e., those that would not be obtained by applying the "appropriate" or "relevant" procedure or theory) may not indicate faulty reasoning, but rather a communication breakdown between the student and the teacher (see, e.g., Abrahamson, 2009; Chernoff & Mamolo, 2015). In this context, then, humility would involve acknowledging the possibility of alternative interpretations of and responses to a given question, as well as the limitations of our ability to accurately evaluate students' understanding of a given concept. Encouraging students to justify their response and incorporating self-assessment into the mathematics classroom, then, may be seen as "humble" practices. (The centrality of justification in the mathematics classroom is certainly not a new idea, and has been embraced by the National Council of Teachers of Mathematics, among others [see, e.g., CCSSI, 2010; Lannin, Barker, & Townsend, 2006; Simon & Blume, 1996; Staples, Bartlo, & Thanheiser, 2012]. However, the role of humility is often overlooked.)

It is true that, as Gruppen (2014) writes, a values conflict may arise between humility and confidence, particularly within a society that glorifies certainty and self-assurance. Certainly, a healthy degree of confidence is necessary in effective teaching (see, e.g., Sadler, 2013). However, as we have argued, humility need *not* be synonymous with low self-esteem, and indeed a secure, self-accepting identity is one of its hallmarks. Rather, the value of humility lies in its ability to temper arrogance and to cultivate an open mind—one that is curious, and which welcomes criticisms and difficult questions. Such an attitude is essential not only in the context of teaching, but in the field of mathematics itself. After all, mathematics has grown and changed over time as a result of the willingness to wrestle with difficult questions and, to paraphrase Plato, to know what we do not know. Such growth could not have taken place without allegiance to humility. If we, as teachers of mathematics, are committed to our own growth as educators, and our students' growth as mathematicians, then certainly, we must do the same.

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"However impenetrable it seems, if you don't try it, then you can never do it."

Andrew Wiles

▲ Mathematics Education Researcher Highlights

Math Education Researcher: Dr. Christine Suurtamm, University of Ottawa

Projects: Dr. Suurtamm teaches in the areas of mathematics education, assessment, and qualitative research. Her research focuses on the complexity of teachers' classroom practice. She is particularly interested in teachers' formative assessment practices as these are opportunities of attending and responding to students' mathematical thinking. Dr. Suurtamm has been the lead researcher on several Ontario Ministry of Education large-scale projects to examine mathematics teaching and learning.

For the past two years, the OAME and OMCA have collaborated on a research project with a University of Ottawa research team, headed by Dr. Suurtamm. The project focuses on documenting the implementation of the Grade 9 Applied Mathematics curriculum in different ways through school teams in nine different school boards. Chris has overseen this project which, among other components, has offered Summer Learning Institutes in which school-based teams and the research team share their learning to facilitate a co-planning process to encourage other schools to begin their own journey of deeper implementation of the mathematics curriculum.

Other research projects in which she has been involved include: Observing Teachers: Mathematics Pedagogy in Regions of Canada; Curriculum Review: The Influence of Learning Trajectories in Curriculum Design; Curriculum Implementation in Intermediate Mathematics (CIIM); What Counts in Math: Teacher Inquiry in Classroom Assessment; Study of the Implementation of the Early Reading and Early Math Strategies; and Immersion in Mathematical Inquiry (Math Camp).

Selected Publications:

Suurtamm, C., & Koch, M.J. (2014). Navigating dilemmas in transforming assessment practices: Experiences of mathematics teachers in Ontario, Canada. *Educational Assessment, Evaluation and Accountability, 26*(3), 263–287.

Suurtamm, C., Quigley, B., Lazarus, J. (2015). Making space for students to think mathematically. *What Works: Research into Practice, Research Monograph 59*.

Suurtamm, C., & Roth-McDuffie, A. (Eds.). (2015). *Annual perspectives in mathematics education: Assessment to enhance learning and teaching.* Reston, VA: NCTM.

Hey, It's Elementary: The Renewed Math Strategy–The Lead Teacher Dilemma

LYNDA COLGAN



Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.

Leadership is about learning that leads to constructive change. (Lambert, 1998, p. 5)

On April 4, 2016, the Ontario Minister of Education, Liz Sandals, announced that the Province was dedicating $60 million for a Renewed Math Strategy, the purpose of which is to help more students achieve excellence in numeracy.

Beginning next September, key elements of the renewed strategy will be introduced, including:

- a minimum of 60 minutes each day of protected learning time for effective mathematics instruction and assessment for students in Grades 1 to 8
- up to three math lead teachers in all elementary schools
- coaching for principals of select secondary schools to lead improvement in math among their students
- support for learning at home through parent resources that provide helpful tips and information on the mathematics curriculum
- better access to online math resources and math supports such as *Homework Help* or *SOS Devoirs*
- math support for Grades 6 to 9 outside of the school day
- opportunities for educators to deepen their knowledge in math learning, teaching and leading, including a dedicated math Professional Development Day to further their school improvement efforts

Of most importance to elementary teachers will be the protected math time and access to the mentorship and support of in-house math lead teachers.



What is a math lead teacher and what are his or her responsibilities? If one assumes that the role is the same as that of the literacy lead teacher, the role requires an enormous breadth and depth of knowledge. I have adapted the requisites of the lead math teacher from the description that was outlined by the Ministry of Education for the lead *literacy teacher during the implementation of The Early Literacy Strategy* in an effort to underscore the extraordinarily high expectations of the role.

I think that it is important to understand the scope of the role in order to help Faculties of Education and school districts to support the development of such specialized educators, since these individuals shoulder an enormous responsibility in terms of achieving the goals of this renewed mathematics strategy.

As you read this *job description,* imagine yourself putting together an application package for the position, or envisioning an individual in your school or school district who could fill these large shoes. Think about the steps that your principal would have to take to enable the math lead teacher to have the opportunity to fulfill the expectations. A whole-school approach is required if the lead teacher is to have any job satisfaction and the latitude to act in this professional coaching service capacity. And teacher education would have a significant role to play to support the development of future leaders.

Role of the Lead Math Teacher

The lead math teacher has extensive expertise in mathematics instruction and staff development. The lead math teacher's main goal is to improve mathematics achievement by working collaboratively with teachers to deepen their understanding of mathematics processes and extend their repertoire of instructional strategies. Lead teachers support principals in ensuring effective mathematics instruction throughout the school.



To be effective, lead math teachers should have a thorough conceptual understanding about the development of mathematics learning, about how children and adults learn, and about how to create opportunities for them to learn effectively. Lead math teachers must have a clear and well-articulated vision of what is possible in student achievement in mathematics.

If lead teachers are to fulfill their unique and pivotal role, their classroom responsibilities must take into account the time needed within the school day to model lessons and collaborate with and mentor other staff.

Effective lead math teachers have the following knowledge and skills:

- Extensive knowledge of effective mathematics practice
- The ability to evaluate and interpret current research
- In-depth knowledge of how children learn to do mathematics
- Expertise in assessment as it relates to the mathematics processes and content
- Commitment to student improvement
- Successful experience in elementary teaching
- Demonstrated knowledge of the principles of adult learning
- Strong commitment to teamwork
- Strong interpersonal skills

The main responsibilities for lead math teachers include:

- observing, coaching, and mentoring other teachers
- promoting learning teams
- managing resources
- analyzing and interpreting student achievement data

Observing, Coaching, and Mentoring

Lead teachers help their fellow staff members to refine their instructional strategies by observing, coaching, and mentoring them in the classroom. The role of the lead teacher is to:

- model effective instructional strategies
- work with the principal and teachers to schedule the math block
- help teachers to reflect on their practice
- help teachers to establish routines that allow for effective math instruction
- model how to use professional resources to improve instructional strategies
- help teachers to make meaningful connections between mathematics theory and classroom practice
- work with new teachers and their mentors to establish a classroom mathematics program

Promoting Learning Teams

The lead math teacher supports the primary–junior teaching team in its ongoing development. The role of the lead teacher in professional development is to:

- consult with teachers on topics for ongoing learning
- lead professional training sessions in areas determined by the priorities and goals in the school math plan
- lead discussions among teachers on current mathematics practices and current mathematics research
- provide opportunities for teachers to share their own effective practices with each other
- work collaboratively with principals and other lead teachers in the board to develop and share effective practices
- engage in ongoing professional learning and self-reflection
- support the principal in scheduling regular in-school math meetings

Managing Resources

Learning materials (print and electronic) are essential resources for elementary mathematics instruction. Time and attention need to be given to selecting, maintaining, and distributing these resources. The role of the lead teacher is to:

- share appropriate professional resources with teachers
- establish a framework or process for tracking ancillary math resources in the school
- instruct teachers on how to use remediation and enrichment materials
- work with classroom teachers and the teacher-librarian to establish resource priorities

Analyzing and Interpreting Student Achievement Data

The ability to assess math is the foundation for systematically improving student achievement in mathematics. To help the teachers in their school to develop this ability, lead teachers:

- model how to use assessment tools effectively
- support teachers in assessing student mathematics achievement at specific points during the school year
- model how to use student achievement data to plan for instruction
- work with the principal to analyze student achievement data
- work with the principal to interpret aggregate data from classroom assessments in mathematics for Kindergarten to Grade 8
- participate in developing and revising the school math plan as part of school improvement planning

While it is true that I substituted "math" for language, reading, and writing in the job description for a lead literacy teacher as articulated by the Ontario Ministry of Education, there is no reason to assume that a math specialist should have fewer qualifications than a literacy specialist. And therein lies the problem.

There are 3974 elementary schools in Ontario. This means that as a profession, we need a minimum of 3974 math lead teachers and a maximum of 11 382. That is a tall order.

Based on my experience as an elementary mathematics educator at Queen's Faculty of Education for nearly two decades, I can identify (on the basis of my mental Rolodex) about 50 individuals with the expertise, enthusiasm and leadership skills to take on such a role. If other Faculties across the province are the same, then we can conclude that there is an enormous gap between the need for math lead teachers and the available pool of individuals equipped to assume the role.

Perhaps the Ontario Ministry of Education, the Ministry of Colleges and Universities, and CODE (Council of Ontario Directors of Education) should host a summit, inviting elementary mathematics leaders to weigh in on the issue and work toward designing a B.Ed./A.Q. program that has an appropriate suite of courses and adequate time to prepare future math lead teachers that reflect the complexity of the task as defined.

If we believe the results of TEDS-M, a recent international comparative study of elementary and secondary mathematics teacher education, we have much to learn and act upon. For example, when countries offered separate programs for generalist and specialist math teachers at the elementary level, there were empirical benefits. In many countries, generalist math teachers are not certified to teach beyond Grade 4. Countries that were leading in international mathematics achievement were those that offered separate teacher preparation programs for generalist and specialist teachers (Grades 4 to 8) at the elementary level. The TEDS-M study suggests that there are reasons to be concerned about the inadequate mathematics knowledge of North American teachers, both at the elementary and middle school level. Although some North American teacher preparation programs are among the best of the world (according to many criteria), the reality is that far too many teachers do not receive adequate mathematics preparation (particularly content knowledge) before they enter the classroom.

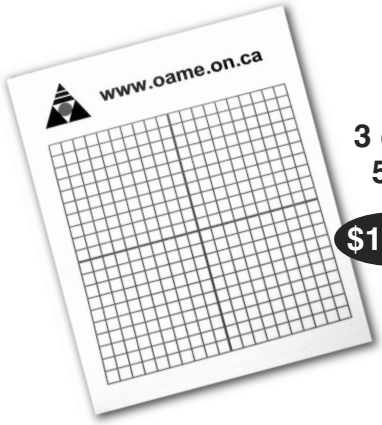
If ever there was a call to action, this is it. I'm in. Are you?

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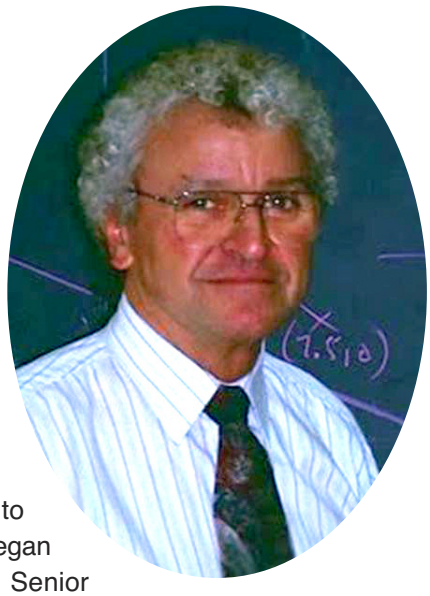
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▲ David Davidson Tribute (1940–2016)

BY DEAN MURRAY & JACK WEINER



Whether he was teaching a class, helping a student, coaching volleyball, playing a round of golf, taking a slalom run down the slopes, or travelling around the globe, Dave Davidson loved life. He was devoted to his family and to his church, and contributed to the greater good in many ways, such as through his active membership in the Kinsman Club.

From the one-room Edgecombe Public School, Dave graduated to Milverton High School, and then attended Stratford Teachers' College. He began his long and successful teaching career at age 20 in Guelph at Central Senior Public School, continuing at John F. Ross Collegiate and Vocational Institute. However, most of his teaching career was spent at Collingwood Collegiate Institute. He became department head of mathematics and acting vice-principal. During these teaching years, he found time to return to school, earning a degree from the University of Western Ontario and several mathematics credits from the University of Waterloo.

It bears repeating—he loved teaching; he loved helping. Beyond the classroom, Dave coached volleyball and mathematics contest teams, and helped in a myriad of other ways around the school. Dave satisfied a need to contribute beyond the classroom, beyond the school. He became involved first with the Grand Valley Mathematics Association (GVMA), serving from 1972 to 1990, and as Chair from 1990 until 1995. He was president of the Ontario Association for Mathematics Education (OAME) in 1983–84. Later, he was instrumental in establishing the Mathematics Association for Cottage Country (MAC^2). He was designated Life Member of OAME in 1995. As further acknowledgement of his contributions to mathematics education, he was awarded the prestigious Society of Descartes Medal by the University of Waterloo.

Not one to rest too comfortably in retirement, he continued his work with MAC^2, passed the Canadian Securities Course, and earned his CFP designation in 1999. He served as financial adviser to many, including the MAC^2 chapter.



GVMA colleagues of Dave commented on his superlative leadership skills and his ongoing mentorship of colleagues and students. One GVMA colleague recalls a valuable collaboration he had with TVO, that resulted directly from Dave bringing the two parties together. When GVMA friends visited Dave at the cottage, they would inevitably share a good Scotch, a Cuban cigar, good conversation, and a beautiful sunset.

Dave is mourned and missed by his family and his friends. We will long remember his hearty laugh, his joy in mentoring, and his welcoming spirit. ▲

▲ Fields Institute MathEd Forum Report

STEWART CRAVEN



Stewart Craven presently teaches courses for the Continuing Education Department and the Transition Years Program at York University. He is an active participant in the Fields Mathematics Education Forum.

The Margaret Sinclair Memorial Award is announced in February of each year, and the recipient subsequently gives a talk at the Fields Mathematics Education Forum. The following is an abridged description of the award:

> *The Margaret Sinclair Memorial Award* recognizes an educator in Canada who has demonstrated innovation and excellence in promoting mathematics education at the elementary, secondary, college, or university level.
>
> Candidates for the award will have shown an enthusiasm for enhancing the learning environment and displaying novel ideas, methods, or devices for teaching mathematics. They will also have supporting evidence of providing an opportunity for students to achieve, observe, and experience mathematics in a thoughtful and significant way. Creative leadership is also an important factor. Candidates must have served as a role model for other teachers and [have] had an inspirational influence on students and/or colleagues.

In April, we were honoured to hear *The Margaret Sinclair Memorial Award* (2015) Lecture given by **Ron Lancaster** of the Ontario Institute for Studies in Education of the University of Toronto:

> Title: *Martin Gardner & Recreational Mathematics*
>
> Abstract: By exposing our students to topics from recreational mathematics and the works of Martin Gardner, we can provide them with a lifelong interest in mathematics; a sense of joy and wonder for the subject; and a desire to play, tinker, and imagine.

For example, a remarkable magic square *f* has the sum of the four numbers in every row and column and is equal to 19 998, even when the square is turned upside down or held in a mirror. The sum of the four corner numbers is also equal to 19 998, as is the sum of the four numbers in the middle of the square… and so on.

Ron then proceeded to provide numerous other examples of "magical" Gardner problems.

The remainder of the meeting included the following presentations:

- Richard Barwell and Yasmine Abtah presented *Framing Mathematics Education in the News: The Problem, the Cause, and the Solution.*

  b) *Connecting to Math in Real Life*
- Math is not just number games played only by math nerds. In fact, mathematics closely connects to art, music, engineering, technology, and many other applications in real life. This session, consisting of three presentations, offered a series of math activities that focused on using *specific* real-life applications as a way of enhancing math learning:
  - i) Hongmei Zhu: "Get Inside Photoshop: Spy the Power of Math"
  - ii) Trang Dang: "Teaching Symmetry Through Dance and Kaleidoscope to Primary Students"
  - iii) Robert Jordan and Hamid Razaghi: "High School Modelling Contest in Canada"

Our next meeting will be in September 2016. Check the Fields website, **www.fields.utoronto.ca/programs/mathed**, for details.

The Steering Committee for the MathEd Forum seeks input for new themes throughout the year and would appreciate your suggestions. Drop me an email at **numeratecitizen@mac.com** and I will pass your ideas along! ▲

Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

"Critical mathematics education can be characterized in terms of concerns: to address social exclusion and suppression, to work for social justice in whatever form possible, to try to open new new possibilities for students, and to address critically mathematics in all its forms and application… Notions such as social justice, mathemacy, dialogue, and uncertainty together with many others are important for formulating concerns of critical mathematics education."

Skovsmose, O. (2014). Critical mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 116–120). London, UK: Springer.

▲ What's the Problem? Pick a Card, Any Card

SHAWN GODIN



Shawn is the head of Mathematics, Business, Law, and Computer Science at Cairine Wilson Secondary School in Orleans, Ontario. He is a past member of the OAME Board of Directors, and past Editor-in-Chief of Crux Mathematicorum. *He currently writes problems for the Fryer, Galois, Hypatia, and Canadian Open Mathematics Challenge contests.*

Welcome back, problem solvers. Last time, I left you with the following problem:

I have three cards: one has a circle on one side and a square on the other, the second has circles on both sides, and the third has squares on both sides. I put the cards into a hat, have you stir them up, and pick one. You show me one side, for example, a square. I say, "Since the other side is either a square or a circle, there is a 50 percent chance of me choosing correctly. Would you be willing to wager even money that I can guess what is on the other side?"

A good rule of thumb in life is to never trust a math teacher! This is a good one to play around with. This problem was inspired by the game three-card swindle that Martin Gardner talks about in his book *Aha! Gotcha*. The name of the game and the title of the book tell you something must be up, so let's dig in and find out what's happening.

Figure 1 depicts the outcomes from a simulation using *Fathom*. Out of the 50 trials, the circle came up on the "up" side 25 times and the square 25 times, 50 percent each. On the "down" side, circles and squares came up 27 and 23 times, respectively, corresponding to 54 percent and 46 percent of the time. These results shouldn't be surprising. We are dealing with two figures on the cards, and there is a symmetry in the situation, so we would expect roughly half the time, the "up" will be a circle (or square) and, similarly, half the time, the "down" side will be a circle (or a square). This may lead you to believe that there is no connection between the two, so there is a 50 percent chance of me predicting correctly, but what fun would there be in that situation?

There are 10 kinds of people: those who understand binary and those who don't.

Up	Down	Up	Down	Up	Down	Up	Down	Up	Down
circle	circle	Circle	circle	circle	square	square	square	circle	circle
circle	circle	Circle	circle	circle	circle	circle	square	square	circle
square	square	Circle	circle	circle	circle	square	circle	circle	square
circle	circle	Circle	circle	square	square	circle	square	square	circle
circle	circle	Circle	circle	circle	circle	square	square	circle	circle
square	square	Square	square	square	square	square	circle	square	square
circle	circle	Square	square	circle	square	square	circle	square	square
circle	circle	Square	square	square	circle	circle	square	square	square
square	square	Square	square	square	square	square	square	square	square
square	circle	Circle	circle	square	circle	circle	circle	circle	circle

Figure 1

If we look closer at the actual trials, a pattern begins to appear. The shaded cells in the table correspond to a trial where the "up" side and "down" side had *different* symbols. This corresponded to 14 of the trials, i.e., 28 percent of the total results. If I had predicted that the opposite side would be the same as the one showing, I would have won 72 percent of the time! In reality, using this strategy, I will only lose if the player picks the card that has different symbols on it, which will occur roughly one time in three. So if we were playing this game where I paid you $1 each time I was wrong and you paid me $1 each time I was right, for 300 plays, you would win about 100 times, for $100, but I would win 200 times for $200; thus, I would end up ahead $100 and you would be behind $100.

It turns out that human beings seem to have a very poor sense of probability. We feel we have an intuitive sense, but that intuition usually is wrong. A perfect example of this is the birthday paradox. Suppose you have a classroom with 30 students. What is the probability that two have the same birthday? Take some time to think about this. We will revisit it later in the column.

Another problem related to the card problem is the Monty Hall Problem. The problem is based on the "final deal" in the game show *Let's Make a Deal*, whose original host was Monty Hall. You are given a choice of three doors—one will have a prize behind it, the other two will not (on the show, there was usually a gag behind the losing doors, like a goat). After you have made your choice, the host opens one of the doors you did not pick and shows that it does not have a prize. You are then given the choice to keep the door you originally picked, or to switch to the other unopened door. The usual line of thought about the problem is to think that we now have two doors—one has the prize, one doesn't. The probability of the prize being behind either door is therefore 50 percent; thus, there is no real advantage to switching (or staying).

This is another nice problem to do as a simulation. You could have students hide something under a cup, have another student guess which cup it is under, flip over one of the cups with nothing under it, and give the student a choice to stay or switch. To make it interesting, have the student decide to switch or stay and do the same thing every time. Run the simulation 20 or more times, then have the students switch roles with the other student, using the opposite strategy. The solution to the problem is that in the beginning, there were three doors where one had a prize. Your chances of picking a winner is $\frac{1}{3}$, which doesn't change if you are shown what is behind the other doors. As such, if you switch, your probability of winning will be $\frac{2}{3}$, so you should switch.

The simulation of running the problem 20 times, where you use the same strategy (either always switch, or always stay), leads to a binomial distribution, where the probability of winning x times out of n trials is given by

$$P(x) = \binom{n}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{n-x}$$

for the case where you always keep your choice, and

$$P(x) = \binom{n}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{n-x},$$

where you always switch. The probability distributions are shown in the graphs created in *Fathom* in Figure 2. The left graph is for the strategy of never switching, while the one on the right is for the strategy of always switching. The graphs look like mirror images of each other, with the staying strategy having the most likely outcome of 6 or 7 wins, while the switch has the most likely outcomes of 13 or 14. These graphs show that a binomial distribution can look very much like a normal distribution. As such, the latter can be used to approximate the former in many situations.

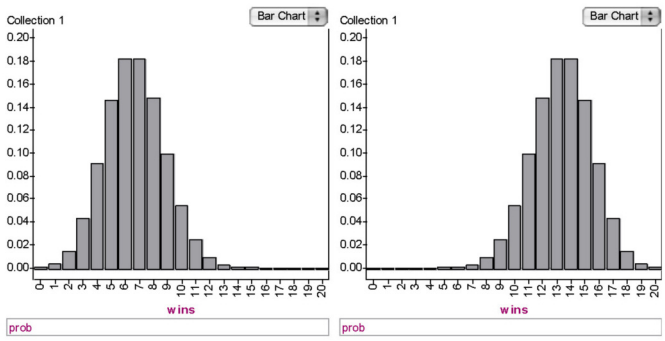


Figure 2

Let's return to the birthday problem that we mentioned earlier. Most people would reason that there are 365 days in the year, and since there are only 30 students, the chances that two share a birthday would be quite low (i.e., $\frac{30}{365}$). The complexity of the problem is that we don't want student 1 and student 2 to share a birthday, nor student 1 and student 3, nor student 1 and student 4, etc. We also don't want other combinations of more than two students sharing birthdays, for example, students 1, 3, and 17. The number of cases we have to consider grows astronomically. In this case, it is easier to determine the probability that nobody shares a birthday.

If we start with one student, he or she can have a birthday on any day of the year and there would be no match (since there is nobody with whom to match). When we add a second student, if we don't want a match, the student can have any birthday except that of the first, and this occurs with a probability of $\frac{364}{365}$ (we will ignore February 29 to simplify the problem). Similarly, when we add a third student, the probability that he or she does not share a birthday with either of the first two is $\frac{363}{365}$. Continuing this process down to the 30th student, and realizing that we want all these events to occur (i.e., the second doesn't match the first AND the third doesn't match the first two AND…), we get the probability of no shared birthdays as

$$p = \frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}\times\ldots\times\frac{336}{365} = \frac{365!}{335!\,365^{30}} \approx 0.294.$$

Therefore, the probability that nobody in the class shares a birthday is about 29.4 percent, which means that in about 70.6 percent of the times you do this, there will be a shared birthday. This is one of those situations that seem totally counterintuitive. If you have 23 students, the probability of a shared birthday is just over 50 percent; the more students, the quicker the probability climbs. This is a great activity to do with your class; it's always exciting when you have a pairing.

And now for your summer homework.

Alice and Bob are practising their free-throws. In the morning, Alice makes $\frac{1}{4}$ of her shots and Bob makes $\frac{1}{5}$ of his.In the afternoon, Alice makes $\frac{3}{4}$ of her shots and Bob makes $\frac{3}{5}$ of his. At the end of the day, which person makes the larger portion of his or her total shots?

Until next issue, happy problem solving. ▲

"I have had my results for a long time: but I do not yet know how I am to arrive at them."

Carl Friedrich Gauss

Advancing Escher Art Through Generalization



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Abstract

A generalization of Escher art is explored with many cases being able to support the teaching of geometry. One particular case is shown to lead to the production of holes or cracks in the tessellated pattern. This allows for enriched teaching and more effective differentiation within this classic activity.

"Escher Art" is an activity that many teachers are familiar with (Kerkusz, 2001; Pumfrey & Beardon, 2002). It provides a context for discussion of geometric transformations, develops patterning, and is easily tiered instructionally. While it is not explicitly mentioned in the Ontario curriculum, it is widely used because of its interdisciplinary connection to art and differing degrees of problem solving that arise from the creation of a template. This article extends the problem-solving component and illustrates that, in terms of tiering instruction, it facilitates a diverse array of opportunities for examining cases that show very different pattern behaviours. Thus, while it is not explicit in the curriculum, it provides a context for developing student comprehension of problem-solving strategies in a manner that can be differentiated according to student ability (or need for enrichment).

Students begin with a square of card stock and cut from one corner to a neighbouring corner with a continuous curve. The piece that is cut out, shown shaded lightly in Figure 1, is then translated (left in Figure 1) to the opposite side, or rotated (top in Figure 1) about one of the corners, where the cut began or ended. The process is then repeated with the other pair of sides. Once the template is made, students can trace the template and then transform it in order to trace it again, resulting in an endless chessboard-style pattern—known as a *tessellation*. Following the process, students can look for patterns in the design to draw out some sort of artistic component.

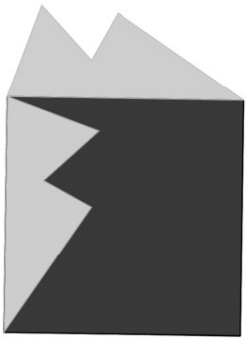


Figure 1: *Template preparation*

The activity is an opportunity to promote mathematical language around translations and rotations, as well as to revisit the directional language of clockwise and counter-clockwise. Often the tessellated pattern leads to artistic creativity and provides a context for discussing modelling using geometric shapes. Perhaps the template in Figure 1 suggests having mountains with a stream following the zigzag to the near side of the template.

As a teacher, I was introduced to this activity and provided a sheet of student instructions so that students could use square templates and would "get it right." This meant conforming to a particular method that would always tessellate. However, at some point, I misplaced the instructions and, in a moment of need, had a pair of classes simply attach the two cut-out pieces (one for each of two edges) to the other edges in whatever way they wanted. In addition, the use of equilateral triangles, which also tile the plane, was a variation I used for enrichment. The unanticipated consequences and subsequent pedagogically valuable findings are fascinating and the focus of this article.

To begin, an equilateral triangle is the simplest regular polygon shape that can be tessellated, and it provides an opportunity for exploration that simplifies the task. It has value for students who struggle with concepts and task completion. In this case, one may draw a continuous line between any two vertices, say from vertex A to vertex B (Figure 2), and cut out the resulting region. This scenario then yields two possible results, which are described together and shown in Figure 3. In the first case, the piece from side AB is rotated onto side AC (Figure 3,

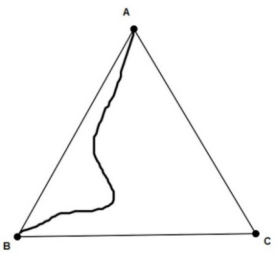


Figure 2: *Triangle template cut*

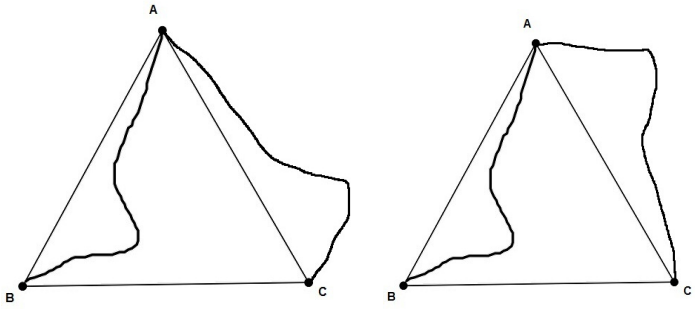


Figure 3: *Two possible triangle templates*

left). Alternatively, the piece is rotated onto side AC and then flipped so that it ends up on side CA (Figure 3, right). Other possibilities involving side BC are equivalent to one of these two possibilities, a fact that provides a good teaching opportunity with students.

The result of tessellating each of these two patterns is different and shown in Figure 4. In the first case, side AB only aligns with side AC, and the tessellation produces a

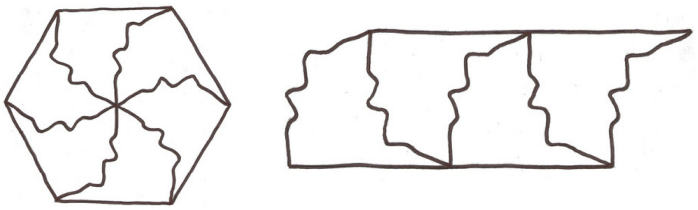


Figure 4: *Two possible triangle templates*

hexagon with six smooth exterior sides (Figure 4, left). These hexagons will then tessellate because all of the exterior sides of the hexagon are uncut edges corresponding to side BC. The second pattern has each triangle mesh with triangles that alternate their orientation (Figure 4, right). The tessellated result is a strip of triangles with smooth edges along the top and bottom. These strips will then tessellate with one another in one of two ways: the strips will stack one on top of one another, each strip merely being duplicated and translated upwards, or the second strip is a horizontal or vertical mirror image of the first strip, which may be thought of as duplicating the strip and then inverting the copy before translating it.

The value of these particular Escher designs is the opportunity for two very different outcomes to emerge in a classroom. Discussion about further pattern development, by tessellating the hexagons or multiple rows of the strip pattern, can be used as a step toward developing the idea that the pattern extends endlessly in two dimensions. It should be noted, as well, that the equilateral triangle is not the only triangle that can tessellate. Consider, as a simple example, cutting an equilateral triangle into two congruent right-angle triangles; those right-angle triangles would be able to tessellate.

A detailed examination of the square template requires making an abstract representation. As is done in geometry, the corners of the template can be labelled and the letters used to identify sides. Consider, for instance, the triangle on the left of Figure 3. Side AB is cut out and joined to side AC, which can be expressed AB-AC. As in other areas of geometry, this requires two letters for each side because the cut-out piece could be flipped, as it appears in Figure 3 on the right, resulting in AB-CA, which is not the same.

For the square template, a lettering scheme used throughout this article is shown in Figure 5. Using this notation, a component is cut out of a side, suppose it is cut out of AB, and then connected to another side. As an example, if it were joined to side AD, we can express this as (AB-AD), which corresponds to Figure 1. This means that segment AB will only join with AD in another piece. It has no implications for BC or CD, except that the components can only be connected to each other, either as BC-CD or BC-DC. To further clarify the notation, the pairing of the four sides is encapsulated as (AB-AD, BC-DC). This describes a template that one can attempt to tessellate. An example of (AB-AD, BC-DC) is shown abstractly in the left of Figure 6. The consequence of

A	D
B	C

Figure 5: *Abstraction of a square template*

C	B	D	C	C	B
D	A	A	B	D	A
B	A	A	D	B	A
C	D	B	C	C	D
C	B	D	C	C	B
D	A	A	B	D	A

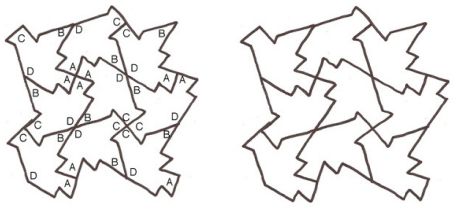


Figure 6: *A tessellation example*

tessellating an example is shown in the middle, with the corresponding letters overlaid on the template shapes. On the right of Figure 6, a line drawing resulting from the template is shown (where you might see the artistic element of a bird in the upper left corner).

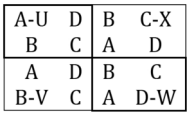
The abstract notation facilitates looking at different cases. In terms of making a template, the process can be viewed as having AB joined to one other side, and then the other pair of sides can be cut out and joined in essentially two different ways. This reduces the number of cases that can arise and facilitates looking at all possible behaviours.

The example (AB-DC, BC-AD) will be called the “simple tessellation” because it tessellates by translating the position of the template with no rotation or flips. The other tessellation cases that need to be considered are (AB-CD, BC-AD), (AB-CD, BC-DA), (AB-CB, CD-AD), (AB-BC, CD-DA), and (AB-BC, CD-AD). Rotating or flipping these templates can make all other tessellations possibilities. The interested reader is

challenged to consider the veracity of this claim. The (AB-DC, BC-DA) possibility has been left out because it is essentially the same as (AB-CD, BC-AD). Also, (AB-CB, CD-DA) has also been omitted because it is analogous to (AB-BC, CD-AD) if "B" and "D" are switched. This corresponds to turning the template upside down. Effectively, the tessellation is unchanged, but the tessellated pattern would be the mirror image.

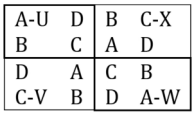
Each of the five distinct cases is explored using the notion that has been introduced. In four cases, the pattern is tessellated to an extent that creates a larger square, referred to as a "super template," which corresponds to one of the other template patterns. When this arises, additional letters U, V, W, and X are used in the corners of the super-template to facilitate discussion of the larger block.

(AB-CD, BC-AD)



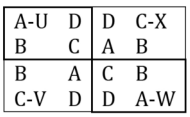
Note that, in this example, the additional letters U, V, W, and X, have been added to the corners of the larger block of four templates. This is to facilitate discussion about how the super template UVWX can be transformed. In this case, the combination of four small templates (i.e., ABCD) leads to a 4x4 super template that has corners labelled as both ABDC and UVWX. Since the original template includes AB-CD, a consequence, for a pair of templates, is ABAB-CDCD, so the left side of the super template (i.e., UV) can join the right side (i.e., XW). Similarly, the original template shows BCAD-ADBC, which means that the bottom of the super template (i.e., VW), can join to the top (i.e., UX). In effect, the super template block can be reproduced horizontally left and right repeatedly and similarly vertically. This can be articulated as (UV-XW, VW-UX), which is equivalent to (AB-DC, BC-AD). That is, the super template can be tessellated and corresponds to the simple tessellation.

(AB-CD, BC-DA)



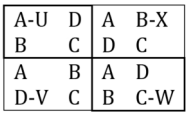
In this case, the 4x4 super template tessellates with horizontal and vertical reproductions of the super template. That is, (UV-XW, VW-UX) and this is equivalent to (AB-DC, BC-AD). In this case, the super template also tessellates as a simple tessellation.

(AB-CB, CD-AD)



Here the 4x4 template tessellates with the entire block being duplicated on any side of the given block, i.e., (UV-WX, VW-XU), which is equivalent to (AB-DC, BC-AD). Once again, the super template tessellates as a simple tessellation.

(AB-BC, CD-DA)

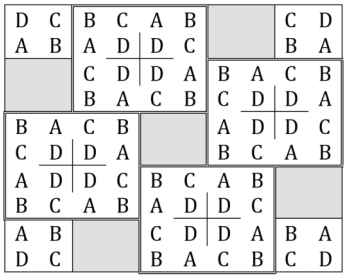


This super template is (UV-XW, VW-UX), which is equivalent to (AB-DC, BC-AD). This super template also follows a simple tessellation.

Having explained four of the five tessellations, one could have the impression that creating a 2x2 super template always leads to simple tessellation. Teachers who want simple variety can take advantage of that commonality. However, it is the final case that can provide an additional challenge for students who need some enrichment; it defies the conjecture and does not lead to the simple tessellation!

(AB-BC, CD-AD)

This tessellation is complicated by a characteristic that arises when four templates are assembled as a block. The characteristic, shown at the right, is that the "D" vertex appears only in the middle of the block of four templates. The group of four small templates only has three letters appearing on the outer edge. To complicate this, all four letters appear in the definition of the template pattern, and this leads to the formation of "holes" or "cracks." A hole is an area surrounded by templates, but which cannot actually be covered by the template. A crack arises when the tessellation has a persistent area that cannot be filled by the template.



The 2x2 template block can be used to construct a periodic tiling with holes. In the diagram, there are four sections composed of 2x2 templates that have a double border. The double-bordered block at the top is identical to the double-bordered block at the bottom, and similarly, the left and right are identical. The relationship between the left and top double-bordered blocks is a reflection. The four- corner templates are the corresponding components of the double-bordered blocks that will allow the pattern to tessellate. The collection of blocks within the double borders and required holes tessellates.

The recognition of the necessity of holes in this tessellation is unexpected. Holes are not the only unusual behaviour; a relatively simple pattern generates two parallel lines of templates where the gap cannot be filled. This is a crack because there is a gap with a width of one template that has periodic walls. The pattern is shown below and then explained.

The crack corresponds to the shaded column of six squares at the bottom of the diagram. On either side of the

B A	C D	A D	B C	A B	D C	B A	C D	A D	B C	A B	D C	A D	B C	D C	A B	B A	C D		
C B	D A	D C	A B													C B	D A		
		B C	A D																
A D	B C	A B	D C													A D	B C		
D C	A B															D C	A B		
B C	A D	C D	B A													B C	A D		
A B	D C	D A	C B	B A	C D	A D	B C	A B	D C	D C	A B	B C	A D	A B	D C	A B	D C		
		B A	C D			A D	B C			C B	D A	D C	A B						
		B	**A**	**C**	**D**		**C**	**B**	**D**	**A**		**D**	**A**	**C**	**B**				
		C	**D**	**A**	**B**		**B**	**A**	**C**	**B**		**C**	**D**	**D**	**A**				
A D	B C			B C	A D		B C	A D				D C	B A	B C	A D	A D	B C		
D C	A B			A B	D C		A D	B C	D C	A B		B C	A D			D C	A B		
				A D	B C		A D	B C				C B	D A	D C	A B				
		B	**A**	**C**	**D**		**D**	**A**	**B**	**A**		**C**	**B**						
		C	**D**	**A**	**B**		**C**	**B**	**C**	**D**		**D**	**A**						

shaded square at the bottom there are several templates with bold letters. These correspond to templates four rows above them that also have bold letters. Tessellating this section downward creates two parallel walls, where the interior can never be filled, i.e., a crack.

To fill the rest of the plane, there is a section in the upper half of the diagram with bold gridlines. That section has many blank templates that can be filled in a variety of ways. However, regardless of how the blank templates are filled, the row above and below the section are a reflection of one another, and that allows the tessellation to extend vertically upwards, using the reflection of the pattern. Similarly, the left column of templates in the diagram is identical to the right column of templates, which allows for the overall diagram to tessellate horizontally.

The area of blank templates (in the section with bold gridlines) can be filled in any manner. There are multiple solutions, which allows for unique choice in an otherwise deterministic pattern. The element of choice allows for this tessellation to be periodic or not, according to how repetitions of the blank templates are filled.

Conclusion to the Square Template

All of the tessellations explored have periodic solutions. While this implies they are not aperiodic (i.e., unable to have a periodic tiling), it does not imply that they must have periodic solutions. However, non-periodicity is tied to the occurrence of holes, and that restricts when it can arise. In the classroom, students should be forewarned of odd behaviour. When I have allowed the full range of templates without forewarning students, some assumed they had made mistakes—the possibility of a hole was assumed to be their error. That holes and cracks can emerge is a fascinating element of tessellation that likely relates to scientific concepts of porosity and crystal flaws. These sorts of macro-properties are the reason we should not avoid having students explore the larger possibilities of Escher Art.

Revisiting the Equilateral Triangle

This article began with the triangular template as a simple case that can allow younger students to explore two distinctly different outcomes. The process was constrained by only having two sides changed; one had a piece cut out, and the other had it added on (Figure 3). The third side was unchanged.

If the third side is reconsidered and the midpoint is used to allow a piece to be cut out of half of the side and connected onto the other half of the side, then a slightly more complex template is created. This type of triangular template can produce the hexagonal super template shown in Figure 7.

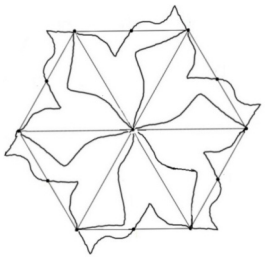


Figure 7: *Hexagon super template from trianlge template*

The detail that is intriguing about this super template is that it tessellates to produce a series of hexagons that will not tessellate freely with one another. What one obtains is a latticework with a series of holes. An example of a single hole is shown in Figure 8.

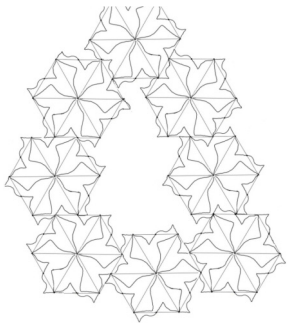


Figure 8: *A hole arising from the triangular template*

Conclusion

Escher art is well known, but used in a very limited manner in classroom settings (Kerkusz, 2001, p. 49). There is a very rich opportunity to develop this established activity in new ways that will encourage students to explore geometric properties further. The occurrence of holes and cracks is genuine and facilitates enrichment possibilities that can be brought into the classroom with minimal additional preparation. Why not use it to encourage students to see the full range of behaviours that simple geometric transformations can lead to?

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Mb4T (Mathematics by and for Teachers): Dividing Integers (Part 2)

ANN KAJANDER



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Are there any negatives in that positive number?!: Division of a positive by a negative

The previous column described modelling approaches to three of the cases of division of integers. The remaining case, the one of division of a positive number by a negative, is the most difficult combination to connect to a realistic context, or to model concretely. Hence, the power of concrete modelling is really pushed to the limits here as we attempt to concretize what is basically an abstract algebraic concept.

The partitive (splitting into groups) interpretation of division isn't much use in imagining dividing by a negative… It isn't very helpful to think of "splitting" an amount into, say, "negative two" groups. This leaves the measurement (how many groups are there) interpretation, as will be discussed here. The topic is interesting because it lies at the cusp of the power of concrete models in developing these ideas. The concrete models become more and more tenuous as we continue to move deeper into mathematical abstractions.

One way around this challenge of trying to model division by a negative is to avoid the issue altogether, by making use of ideas we already know. For example, we know from working with fractions, that 6/3 can also be thought of as equivalent to $6 \times 1/3$. So it makes sense that $6 / (-3) = 6 \times (-1/3)$, which might be solved by combining the ideas of fraction and integer multiplication, suggesting a negative product.

Still another interpretation, knowing about the relationship of multiplication and division, might be to explore the inverse statement to $6 \div (-3)$. If we are trying to solve $(6) \div (-3) = __$, then we do know that $(-3) \times __$ has to be 6, using the relationship of multiplication and division. This method also suggests the answer should be (-2).

These examples indicate that dividing a positive by a negative will result in a negative answer. Can we do better, and actually understand $6 \div (-3)$ directly?

Exploring Further

Let's explore further with counters. As mentioned, neither of our division models—measurement or partitive—are that immediately helpful. For example, neither question from "what is 6 split into (-3) groups," nor "how many (-3)'s are in 6," seems to make much sense. We will explore this example further with the coloured counters of "chips" model, to look for ideas, using the latter (measurement) approach.

However, looking at 6 (positive) counters, we can't find any (-3)'s:



Even if we try adding zero pairs, which was useful in other contexts, and then count the (-3)'s, we will still have a lot of positives left. Now what?

Some children might (quite reasonably) argue that the answer to $6 \div (-3)$ must be zero—by reasoning that there are *no* groups of (-3) in 6. It does seem that there is no positive number of groups of (-3) in 6.

Looking further at the model of 6, we do see 2 groups of 3. Perhaps we might predict then that the answer for groups of (-3) would have to be the *opposite* number of groups, so the answer might be (-2). Things are still unclear, however, as to why this really makes sense.

The Number-Line Model

How about the number-line model? We might ask, "Starting at zero, how many moves of (-3)... (each of which is 3 to the left)... would take us to 6?" Again the answer seems initially to be "none." The reasoning is a bit abstract this time, but still manageable. If we make one move of (-3), starting from zero, in our attempt to get to +6, that's clearly taking us the wrong way. We get even farther from 6 if we make another such move of (-3). This isn't helping! If we are lucky at this point, a student will say, "What if we go the *opposite* way?" We can think of this as a reverse or negative or even "backwards" move. In other words, we go *backwards* (-3), or the reverse of a move of (-3). Starting from zero, and going the opposite way to a move of (-3), we get *closer* to 6! What about another "reverse" move? In total, we might see two *reverse* or backwards moves, each the opposite way to a move of (-3). We find that two reverse or backwards moves, each of (-3), brings us to 6. Drawing heavily on previous notions of the meaning of – (-3) on a number line, such as "reverse and go backwards," we count a – (-3) move, and another – (-3) move, for a total of "two backwards moves of (-3)" or "(-2) groups of (-3)" in 6. This implies that $6 \div (-3)$ is

indeed (-2), as predicted previously.

It's fun to have students try this model out on a life-size number line drawn with chalk on a playground, or marked on the floor with tape. If a student makes repeated jumps of (-3) from zero, they will get farther and farther from 6. But if they make these jumps the *other* way—in a backwards or "negative" direction—they *will* reach 6. I have found this physical movement is very helpful to some students' understanding.

Back to the Counters

Attempting to more formally show a measurement (or repeated subtraction) interpretation, using the coloured counters does actually work as well, but might require a bit of guidance, as the model has to be thought of a bit differently. Let's start by reviewing the parallel reasoning with positive values.

Recall that in the whole-number repeated subtraction-interpretation of division, for a question like 6 ÷ 3, for example, that we start with the first number, 6, and then remove or subtract groups of the second number, the 3, until we reach 0. The answer is the number of subtractions, which tells us the number of groups. We might write 6 – (3) – (3) = 0. There are 2 three's in 6, so the answer is 2.

The above could also be written as a repeated *addition* statement. We can ask, "Starting with 0, how many 3's do we need to *add* to reach 6?" Numerically, 0 + (3) + (3) = 6, again explaining the answer of 2 as meaning "two groups of 3 in 6."

Let's return to the model of 6, and try again to think of dividing by (-3), using the repeated subtraction/repeated addition reasoning just illustrated. It turns out that the repeated addition version of the above reasoning is easier. Note that the reasoning aligns strongly with the reasoning provided with the number-line model. We can wonder, "Starting with 0, how many (-3)'s do we need to get to 6?"

A Model with Counters

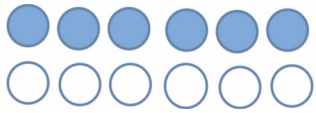
Start with one group of (-3):



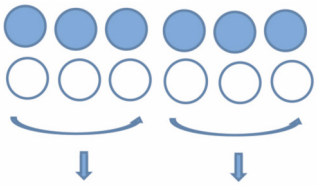
In trying to get to 6 (not (-6)) by adding (-3), we appear to be heading farther away from our target, rather than toward it! (Again, at this point, you might ask students, "What can we do?") The inspiration previously was to head the *other* way… or, as another metaphor here, let's try *removing* a group of (-3) from zero, which we know how to do from our work with subtraction using zero pairs. It is critical that students have the background to be able to use the reasoning related to subtracting negatives here, and are thus able to agree that *removing* a group of (-3) is analogous to "(-1) of a group of (-3)."

Let's try a model.

Here is zero, shown using zero pairs:



The question is, "How many sets of (-3) do we need to reach 6?" We have already tried adding in sets of (-3)… and found that that brings us in the wrong way. So what if we *subtract* or remove groups of (-3)?



If we remove two sets of (-3), we do get to the desired destination of 6. So, starting from 0, we see that 0 – (-3) – (-3) = 6. In total, we have (-2) groups of (-3) to reach the 6. So the answer to the "how many groups of (-3) in 6" question becomes (-2).

Generalizing

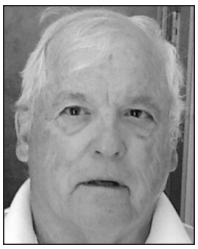
Using multiple lines of reasoning, as well as drawing heavily on the relationship between multiplication and division, we can in fact construct some meaning for the positive divided by a negative combination. Of the four possibilities, though, it is by far the most challenging.

After enough experience, students will begin to generalize the results of these operations. They might note that the quotient of two like-signed integers is always positive because we are counting groups of things that are the same (meaning they have the same sign, or are modelled using the same type of counter). And a negative divided by a positive is negative—it's just the number of negatives resulting in each newly made group. The discussion here provides the final combination, illustrating how we can think about the result that a positive divided by a negative gives a negative. Thus, in fact, the quotient of two unlike-signed integers is always negative. These problems provide rich sources of modelling, discussion, and debate for students, supporting the important processes of representing, reasoning, and proving as they are explored.

Acknowledgement

Of all the chapter sections in our book Mathematical Models for Teaching, this topic was the most controversial and challenging for us to write. My thanks to my book co-author, Tom Boland, for ultimately providing the conceptual idea described here. ▲

▲ OAME/NCTM Report



TODD ROMIENS
EMAIL: todd.romiens@oame.on.ca

Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a past president of OAME and a life member, as well as a previous editor of the Gazette*.*

In the last column, I talked about the CORE curriculum initiative currently being undertaken in the United States, and some of the controversies surrounding it. There are other large-scale ideas that are encouraged as projects for mathematics teachers as well.

NCTM has unveiled a new type of conference called "Innov8," the first of which will be held in St. Louis, Missouri from November 16–18, 2016. While it is too late to submit a proposal to be discussed and implemented for this year, it would be exciting to get involved in the 2017 conference. The focus of the conference is to find ways to "Engage the Struggling Learner." You will bring a team of teachers "to engage in a hands-on interactive, and new, learning experience for mathematics education."

Your team will work with other teams from different parts of the country, to navigate your experience through three pathways:

- Responding to intervention
- Supporting productive struggle
- Motivating the struggling learner

The format for proposals includes the following:

- Keynote and expert presentations
- Activities in the Learning Lounge, ranging from one-on-one time with other speakers to problem solving with peers
- Team time to map strategies
- A look at new technologies

According to the Innov8 Conference website (**www.nctm.org/innov8**), one can use Twitter and the hashtag **#NCTMinnov8** to "get the latest updates and engage with others about NCTM's new learning experience on social media."

A second large-scale initiative supported by NCTM is the STEM project. STEM stands for "Science, Technology, Engineering, and Mathematics." It has been a driving force in educational policy and funding decisions in the United States since 2012. It has been a prominent advocacy position for investing resources in Science, Technology, Engineering, and Mathematics for the purpose of keeping the United States globally competitive.

The STEM Education Coalition, an alliance of over 500 businesses, professional, and education organizations, raises awareness in Congress about the critical role that STEM education plays in maintaining the United States' global leadership in these areas. NCTM is a member of this coalition in an advocacy role.

Some projects that have been detailed on the NCTM website include:

a) an engineering design challenge for Grades K–5, where students design rubber band-powered vehicles and analyze the role of friction in motion
b) Grade 6 students using algebra and geometry skills to build rides for an amusement park at the school, designing the rides and calculating variables such cost effectiveness, ride capacity, and space.

These projects are very worthwhile, but there are some concerns in the math community. One such concern centres around making a STEM project the only manner in which meaningful funding can be made available to particular jurisdictions. Another is to ensure careful scrutiny of the substance of mathematics in each project. It is generally felt that each of the four areas should not carry equal weight, and that mathematics should play a larger, more important role. This is especially true in the early grades, which form the basis of other meaningful areas. Anyone implementing a STEM project needs to be able to identify its specific math focus.

NCTM has a number of new publications available for sale. Remember that there is a discount for current members. Here are a couple of examples:

Mathematical Understanding for Secondary Teaching: A Framework and Classroom-Based Situations (2015)

Editors: M.K. Heid, P.S. Wilson, & G.W. Blume

Publisher: Information Age Publishing, Charlotte, NC

This book "addresses the dynamic nature of mathematical understanding for secondary teaching."

Price: non-member – $59.90; member – $47.90

It's Elementary: A Parent's Guide to K–5 Mathematics (2015)

Authors: J.W. Whitenack, L.O. Cavey, & C. Henney

Publisher: National Council of Teachers of Mathematics, Reston, VA

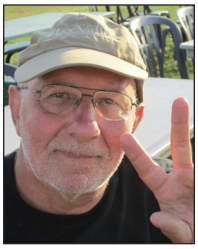
This book "decodes current teaching practices and demystifies the fundamental concepts children need to understand, to be successful math learners."

Price: non-members – $27.90; members – $22.30

Enjoy your summer!

Three Spheres of Radii 3, 2, and 1 Mutually Tangent Problem (Part 1)

ANDRÉ LEMAIRE



André is a retired mathematics and physics teacher, who taught at Garneau High School in Orleans, Ontario. Born in Belgium, André studied at the University of Ottawa, and is also a licensed pilot and active skydiver.

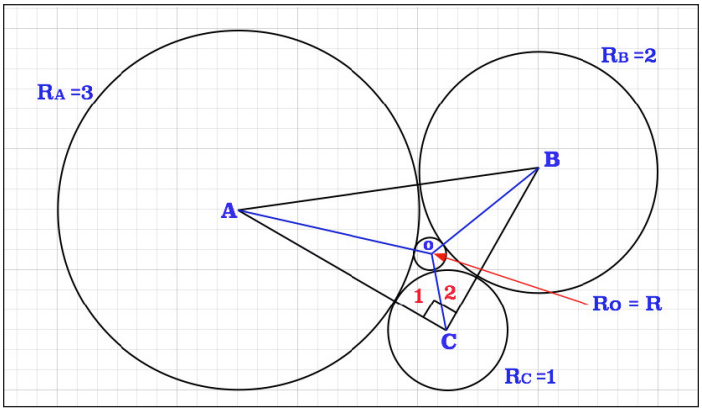
This problem is not a new one. Apollonius of Perga (262–190 BC), Descartes (1596–1650), and Soddy (1877–1956), to name only a few of the most famous mathematicians, have worked on it and have elaborated formulas to solve it. In this article, I have used mathematics studied at the high school level. These different approaches for this problem show how mathematics, used in different ways, are all in harmony in order to obtain the same result. These methods comprise trigonometry, area-of-triangle formulas, and analytic geometry of the circle. In each method, one has to find the mathematical expression of specific distances, two areas or two heights of a triangle, and an equation involving all of them. In certain cases, one gets indeterminate forms, but in others, well chosen, one gets an equality leading to a 2nd-degree or quadratic equation.

In 1971, someone challenged me to solve this problem. I had, at the time, tried to solve the general case, without any success. Later on, in 1973, I successfully solved this particular case. I was then at the University of Ottawa's Teacher's College, when Dr. Dupont, professor of didactic of mathematics, asked me if he could have the article published in the *Ontario Mathematics Gazette.* The article was subsequently published in Volume 12, Number 3 (Spring 1974), and this version can be found in the online archives of the *Gazette* on the OAME website.

Problem: Find the maximum radius *R* of a sphere able to pass in the interstice between spheres of radius 1, 2, and 3.

Method 1: Pythagoras, Heron's Formula, trigonometry, and 2nd-degree equation.

One can see that this small sphere of maximum radius will also be tangent with the three other spheres. Reminder: A tangent to a sphere, or circle, is always perpendicular to the radius in contact with the point of tangency. Therefore, if two circles are tangent, the point of tangency is in line with the two centres of those circles.



Centres A, B, and C form a plane. This can be considered a problem of three circles mutually tangent, and a circle of centre O in the plane, in the interstice, tangent to the three other circles.

1. Pythagoras: Triangle ABC is a right triangle with sides 5, 4, and 3; then angle C = 90°.
2. Heron's Formula for area S of a triangle with sides: a, b, and c, and $p = \frac{1}{2}$ perimeter

   $$S = \sqrt{p(p-a)(p-b)(p-c)}$$

   Note: Triangle BOC is involved in the right angle; therefore, $C = 90^\circ = C_2 + C_1$ **(A)**.

   $p = \frac{1}{2}$ perimeter of triangle BOC is: $p = \frac{3 + CO + BO}{2}$

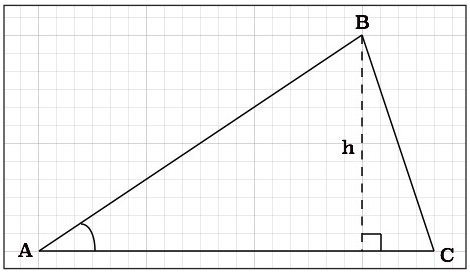
   (R being the radius of circle O) we have:

   $$p = \frac{3 + (1+R) + (2+R)}{2} \quad or \quad p = 3 + R,$$

   then $S_{BOC} = \sqrt{(3+R)(3+R-3)(3+R-(R+1))(3+R-(2+R))}$

   Simplifying, one gets: $S_{BOC} = \sqrt{6R + 2R^2}$ **(1)**
3. Area of a triangle with two sides and the sine of the angle located between those two sides

   Reminder:



$$area_{ABC} = \frac{AC \times h}{2} \textbf{ (B)}$$

$\sin A = \frac{h}{AB}$; *hence*, $h = AB \times \sin A$.

By substituting the value of h in **(B)**, one gets:

$area_{ABC} = \frac{1}{2} AB \times AC \sin A$ **(C)**

Triangle BOC: angle C_2 is a complement of C_1, since $C = 90°$.

From **(C)** $area_{BOC} \Rightarrow S_{BOC} = \frac{1}{2} CB \times CO \sin C_2$,

$\therefore S_{BOC} = \frac{1}{2} \times 3(1+R)\sin C_2$ **(2)**.

4. Let's consider triangle AOC, using the cosine law involving angle C_1 (see the main figure).

$AO^2 = AC^2 + CO^2 - 2AC \times CO \cos C_1$

$\therefore (3+R)^2 = 4^2 + (1+R)^2 - 2 \times 4(1+R)\cos C_1$ **(3)**

Note: cos C_1 = sin C_2 because C_1 and C_2 are complementary angles (see note **(A)**)

With (**1**) and (**2**), one gets:

$\sin C_2 = \frac{\sqrt{6R+2R^2}}{\frac{1}{2} \times 3(1+R)}$ Let's replace in **(3)** $\cos C_1$ by the value of $\sin C_2$ (see the note).

$$(3+R)^2 = 4^2 (1+R)^2 - 2 \times 4 \cancel{(1+R)} \left(\frac{\sqrt{6R+2R^2}}{\frac{1}{2} \times 3 \cancel{(1+R)}} \right)$$

Solving, one gets: $9 + 6R + R^2 = 16 + 1 + 2R + R^2 - \frac{16}{3}\sqrt{6R+2R^2}$

Simplifying: $-8 + 4R = \frac{-16}{3}\sqrt{6R+2R^2}$

Squaring: $64 - 64R + 16R^2 = \frac{256}{9}(6R + 2R^2)$

or $\frac{9}{256}(64 - 64R + 16R^2) = 6R + 2R^2$

Simplifying: $\frac{9}{16}(4 - 4R + R^2) = 6R + 2R^2$

Same denominator: $\frac{36}{16} - \frac{36}{16}R + \frac{9}{16}R^2 = \frac{96}{16}R + \frac{32}{16}R^2$

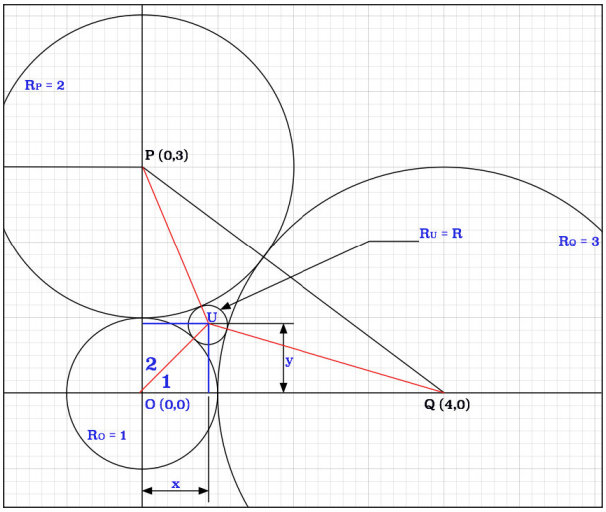
or $23R^2 + 132R - 36 = 0$

2nd-degree equation solving formula: $R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $b = 132$ $c = -36$ $a = 23$

Keeping the positive root, one obtains the maximum radius of the circle O to say $R = \frac{6}{23}$.

Method 2: Analytic geometry, coordinates of circle U (x, y), and the distances between this centre and the centres of the other circles.

It is Daniel Gauthier, one of my former students, who used this method. Judiciously, he chose to put the right angle of the triangle OPQ of sides 5, 4, and 3 at the origin (0, 0). Therefore, the centres of the circles all have at least one of their coordinates equal to zero.



Calculation of distance **OU**, using Pythagoras:

$x^2 + y^2 = OU^2$

$\therefore x^2 + y^2 = (R + 1)^2$ **(1)**

Calculation of distance **QU**, using the same approach:

$(4 - x)^2 + y^2 = QU^2$

$\therefore (4 - x)^2 + y^2 = (R + 3)^2$ **(2)**

Calculation of distance **PU**:

$x^2 + (3 - y)^2 = PU^2$

$\therefore x^2 + (3 - y)^2 = (R + 2)^2$ **(3)**

Developing (1), (2), and (3), one obtains:

$x^2 + y^2 = R^2 + 2R + 1$ **(A)**

$16 - 8x + x^2 + y^2 = R^2 + 6R + 9$ **(B)**

$x^2 + 9 - 6y + y^2 = R^2 + 4R + 4$ **(C)**

Let's solve the system (**A**) (**B**) (**C**).

Solving (**C**) – (**A**):

$9 - 6y = 2R + 3$ or $y = \frac{2R - 6}{-6}$, then $y = 1 - \frac{R}{3}$ **(D)**

Solving (**B**) – (**A**):

$16 - 8x = 4R + 8$ or $x = \frac{4R - 8}{-8}$ then $x = 1 - \frac{R}{2}$ **(E)**

Substituting (E) in (A) and (D) in (A):

$\left(1 - \frac{R}{2}\right)^2 + \left(1 - \frac{R}{3}\right)^2 = R^2 + 2R + 1$

Simplifying and using the same denominator:

$1 - \frac{2R}{2} + \frac{R^2}{4} + 1 - \frac{2R}{3} + \frac{R^2}{9} = R^2 + 2R + 1$

$36 - 36R + 9R^2 - 24R + 4R^2 = 36R^2 + 72R$

or $0 = 23R^2 + 132R - 36$

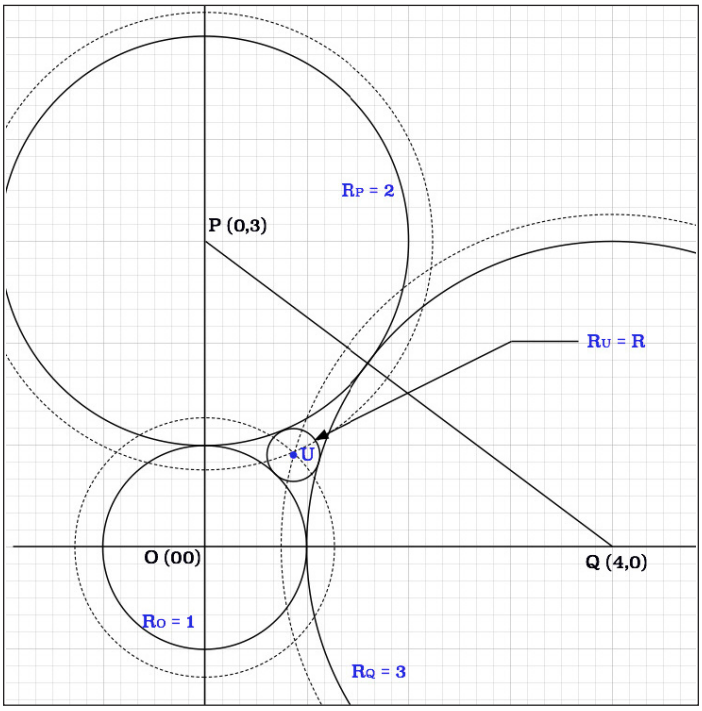
Using the 2nd-degree equation solving formula:

$R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $b = 132$, $c = -36$, and $a = 23$

one gets the positive root as: $R = \frac{6}{23}$

Method 3: Analytic geometry using the intersection of three circles O, P, and Q of increased radius by R, where R is the radius of circle U. These three new circles with

increased radii, O+, P+, and Q+, will obviously intersect at the centre of circle U.



Circle O+ of radius $1 + R$

Circle Q+ of radius $3 + R$

Circle P+ of radius $2 + R$

The equation of a circle of radius R and centre (x_0, y_0) is: $(x - x_0)^2 + (y - y_0)^2 = R^2$

Here are the equations of the three circles with increased radius:

circle O+: $(x - 0)^2 + (y - 0)^2 = (1 + R)^2$

circle Q+: $(x - 4)^2 + (y - 0)^2 = (3 + R)^2$

circle P+: $(x - 0)^2 + (y - 3)^2 = (2 + R)^2$

Simplifying and solving here are the equations of the increased radius circles (O+, P+, Q+):

circle O+: $x^2 + y^2 = 1 + 2R + R^2$ **(A)**

circle Q+: $x^2 - 8x + 16 + y^2 = 9 + 6R + R^2$ **(B)**

circle P+: $x^2 + y^2 - 6y + 9 = 4 + 4R + R^2$ **(C)**

Note: *Equations (A), (B), and (C) are the same as the ones derived in Daniel Gauthier's Method 2, thus the problem would, from this point on, be solved in the same way as Method 2 above.*

These three different methods of solving this classic problem illustrate the power of mathematics. It shows high school students that mathematics can be a sort of game, with difficulties, yes, but a process that can lead to successful results, if one is not afraid to work hard to solve problems. ▲

Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

"Lesson study is a Japanese approach to improve teaching and learning mathematics through a particular form of activity by a group of teachers. It provides teachers with key learning opportunities in working collaboratively with their colleagues to study subject matter, students' thinking and learning, and how to change classroom instruction... The activity of lesson study includes planning and implementing the research lesson as a core of the whole activity, followed by post-lesson discussion and reflection by participants. A lesson plan plays a key role as a medium for the teachers to share and discuss the ideas to be examined through the process of lesson study... Improvement of teaching and learning through lesson study over a long period of time can take place, in the Japanese education system, within the context in which clear learning goals for students are shared among teachers in relation to the national curriculum standards as well as teachers' voluntary hard efforts with the support of administrators. There are challenges to be resolved in practice, and research possibilities to be explored, in each context."

Shimizu, Y. (2014). Lesson study in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 358–360). London, UK: Springer.

Question: *Why did the chicken cross the road?*

Archimedes: *Give me a place to stand, and I can move the chicken across the road.*

Pierre de Fermat: *I have found an admirable explanation of this, but the margin is too narrow to contain it.*

Xeno of Elea: *He went halfway, then half again, then half yet again... but the chicken never got to the other side.*

Vennebush, G.P. (2010). *Math jokes 4 mathy folks.* Brandon, OR: Robert D. Reed.

Provincial Digital Learning Resources – What's New? Relational Rods Plus (mathies.ca)









ROSS ISENEGGER

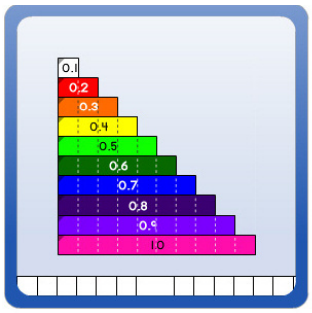
MARKUS WOLSKI

AGNES GRAFTON

GREG CLARKE

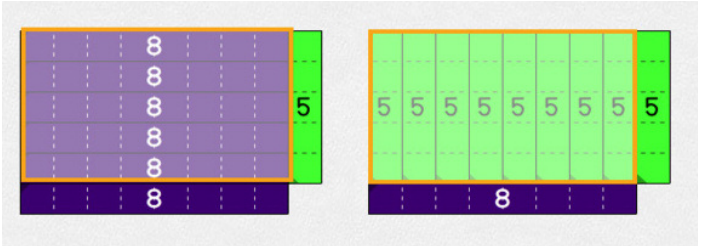
*Greg (Simcoe Muskoka Catholic DSB), Ross (Near North DSB), Agnes (Brant Haldimand Norfolk Catholic DSB), and Markus (Bluewater DSB) are all currently on assignment with the Ontario Ministry of Education as Provincial Math Leads, working on the CLIPS and **mathies.ca** projects.*

The new Relational Rods tool for desktop and mobile was made available at mathies.ca's Learning Tools page just after our last column was published in March. That column described the new features of the tool and gave some examples of the important math that can be explored with it. You might like to go back and have a look at that article, or use the i button in the tool and read over the Support Wiki page for Relational Rods+ to familiarize yourself with the tool. One of the delays in releasing the tool was the addition of Decimal Number Rods, in which the largest rod is labelled as 1 and the others as some decimal fraction of 1.



Decimal Number Rods

Another fun addition to the tool, unimagined in the previous article, is the ability to rotate rods from horizontal to vertical, which allows some interesting explorations, including explorations of an area model of multiplication.



If we were going to repeat our mistake from last issue, we would promise that a new image import feature would be available by the time this article is published.

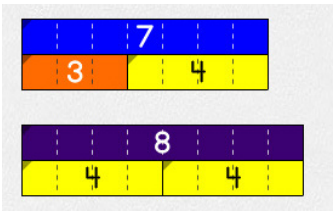
But we wouldn't do that—smart people learn from their mistakes.

Reasoning and Proving with Relational Rods

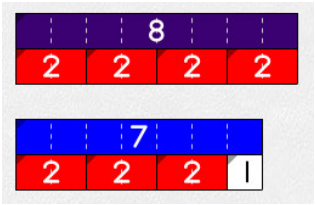
The Ministry of Education in Ontario has invested in the creation of these learning tools because of a strong belief in the power of visual representations to uncover important mathematical concepts. Even concepts of number are revealed and made stronger by relating them to visual representations, like those created within Relational Rods+. This article will provide some elementary and not-so-elementary examples of how the tool can be used to help students reason about number properties and how connections to algebraic thinking can be made.

Even and Odd Numbers

Ask students to represent a variety of numbers with the Whole Number Rods, like 7, 8, 17, and 24. Ask them whether the numbers are odd or even and how they could use the rods to support their conclusion.



One student might argue that 8 is even because it can be broken into 2 equal pieces of 4, while 7 is odd, since it cannot be broken in half by a whole number rod.

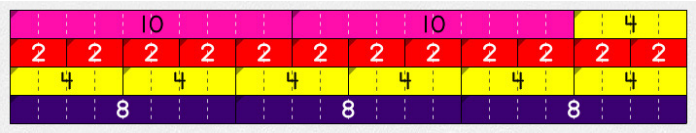


Another student might argue that 8 is even, since it can be broken into 2 evenly, with 4 equal pieces of 2, while with 7, there is a pesky 1 left over.

A comparison of the two methods raises some interesting properties of number:

- The commutative property: 2 pieces of 4 and 4 pieces of 2 have the same result.
- When dividing into groups of 2, the remainder will be either 0 or 1.
- 7 is a near double when thought of as 3 + 4 or as 3 + 3 + 1.

The copy feature of the digital tool makes it a lot more fun to create 12 twos by "wailing" on the copy button, rather than by dragging each rod out individually.

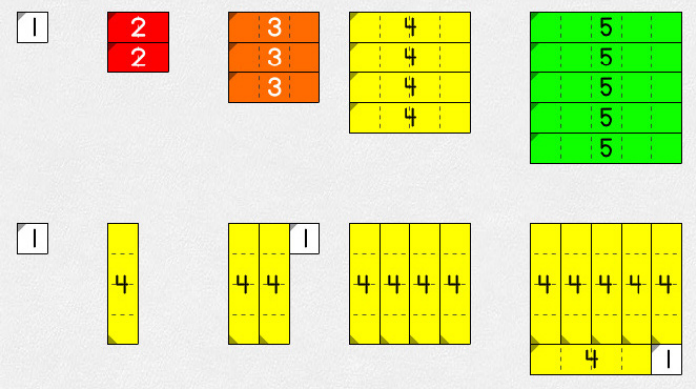


What questions might the representation above elicit after students establish that 24 is even?

Is every multiple of 4 also a multiple of 2? Is every multiple of 8 even? Is every multiple of 8 also a multiple of 16? If 4 goes into a number x times, how many times does 8 go into it?

Some examples of conjectures that students could generate or reason about include the following:

- An odd number plus an odd number is even.
- An odd number times an odd number is odd.
- The product of two consecutive numbers is even.
- Every perfect square is of the form $4k$ or $4k + 1$.

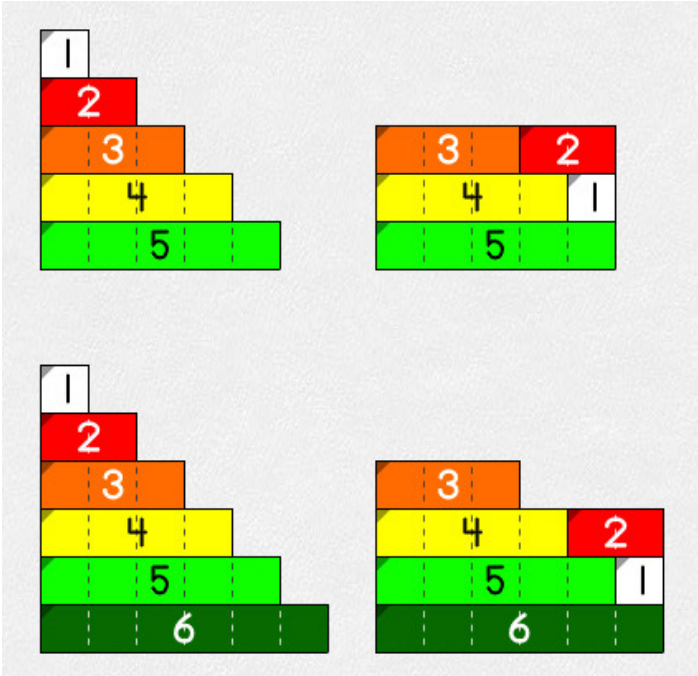


Doesn't this last statement and diagram make you wish we were promising an Algebra Tiles app, where you could create a representation for $(2n)(2n)$ and $(2n+1)(2n+1)$?

Summing the Natural Numbers

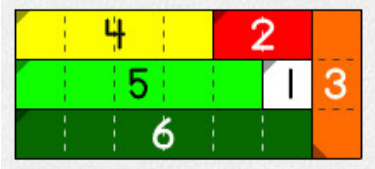
There is no story that inspires kids like the story of how a precocious Gauss embarrassed his teacher by determining the sum of all the natural numbers from 1 to 100 in seconds. Fortunately, there is no YouTube video of the encounter, and students can explore these sums without prejudice.

In the diagrams below, the sums from 1 to 5 and 1 to 6 are rearranged to show that the former is the same as 3 times 5 and the latter is the same as 3 times 6 plus 3.

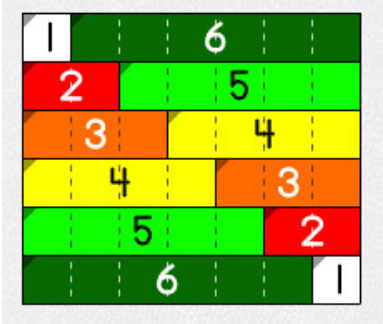


How would the pattern continue? Could this be used to determine the sum from 1 to 100 quickly?

How might it help to rearrange the last diagram as follows?



Of course, Gauss' trick was more subtle—he must have had the copy button as well!



The diagram above has two copies of 1 + 2 + 3 + 4 + 5 + 6 and forms a 6 by 7 rectangle. This means that 6 times 7 is twice the sum. If 42 is twice the sum, the sum must be 21.

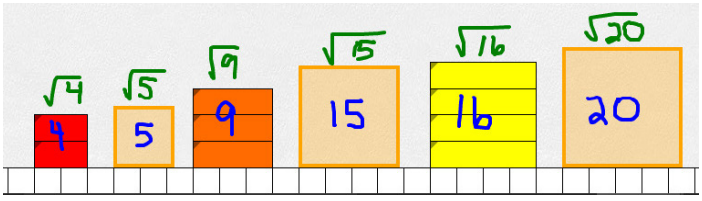
The Paying Attention to Algebraic Reasoning document discusses the importance of using algebra to generalize properties such as this. Here we see that:

$$1 + 2 + 3 + \ldots + n$$
$$= \frac{n(n+1)}{2}$$
$$= (\tfrac{n}{2})(n+1)$$
$$= (n)(\tfrac{n+1}{2})$$

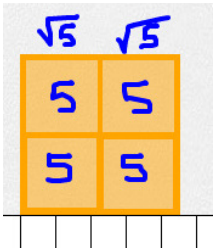
Having older students match the various equivalences above to the corresponding arrangement of the rods above would be most productive.

An Annotation Tool Example

All the recent *mathies* apps include the ability to annotate, which helps students explain and support their work. If we define the square root of 5 to mean the side length of a square with area 5 (and why wouldn't we?), then we can start to fill in some gaps.



Now we know that converting radicals to mixed and entire form is out of fashion, but how robust would a student's understanding of the meaning of square root, addition, and multiplication be to recognize that this:

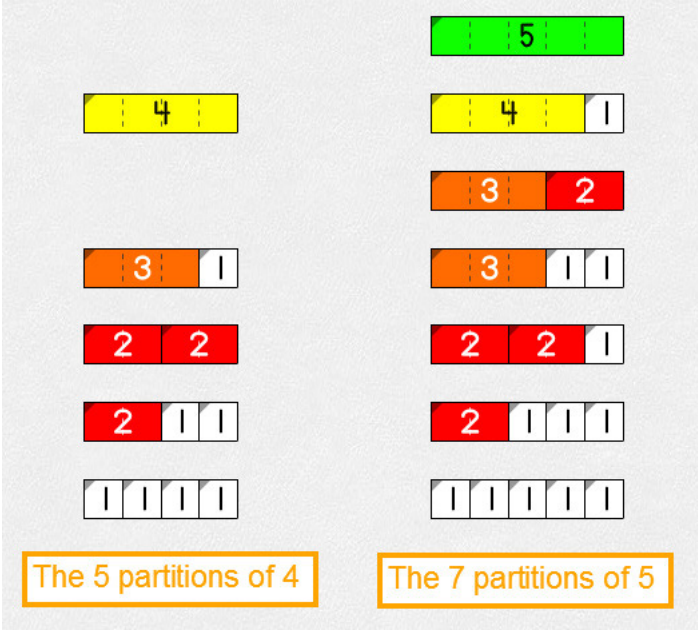


is a justification that

$$\sqrt{20} = 2 \times \sqrt{5}$$

Partitions

If you have been reading all the great Fractions work found on EduGAINS (**http://edugains.ca/newsite/math/fractions.html**), you will have run into the term "partitioning" and "equi-partitioning." In Number Theory, partitioning refers to the process of breaking a whole number like 5 into whole number addends, like 2 + 2 + 1.

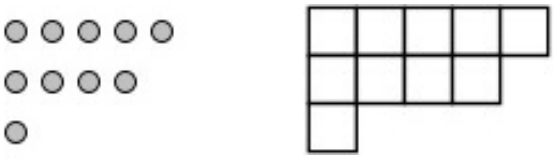


Even the simple diagram above brings out some interesting observations and questions:

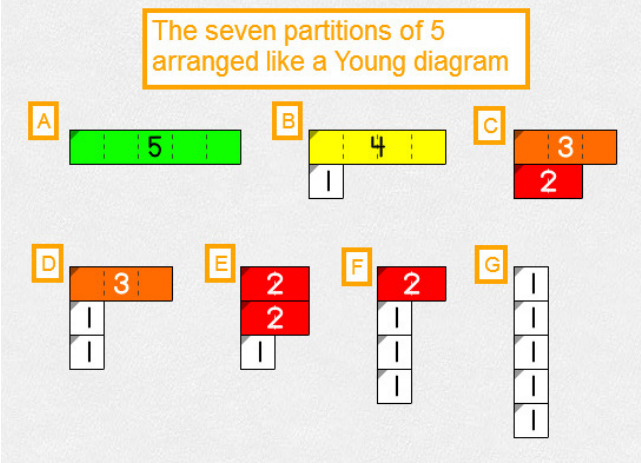
- 5 (i.e., the green rod) is considered to be a partition of 5. This is a pretty standard trick in Mathematics that can be somewhat bothersome.
- Order is not considered to be important, so 4 + 1 and 1 + 4 are not different partitions for the purpose of counting. If you do want to consider order to be important, you are talking about *compositions* of the number. Setting the ground rules for counting is crucial.
- Organized counting helps make sure you have found all the possibilities. Here, a convention of writing the addends in descending order is used to help organize the count.
- How many more partitions of 5 are there than partitions of 4?
- The diagram above pairs some partitions of 4 with partitions of 5 and not others. What rule is being applied? How might this help count the number of partitions of 6?
- How many partitions of 6 are there? 7? 8? *n*?

If you read the rather technical wikipedia article on partitions, **https://en.wikipedia.org/wiki/Partition_(number_theory)**, you will see two visual representations that mathematicians use to study partitions of whole numbers: the Ferrers diagram and the Young diagram. The Ferrers diagram uses an arrangement of dots, whereas a Young diagram uses an arrangement of squares. It is affirming to see how useful visual diagrams are to professional mathematicians studying Number Theory. We should feel confident promoting their use as thinking tools.

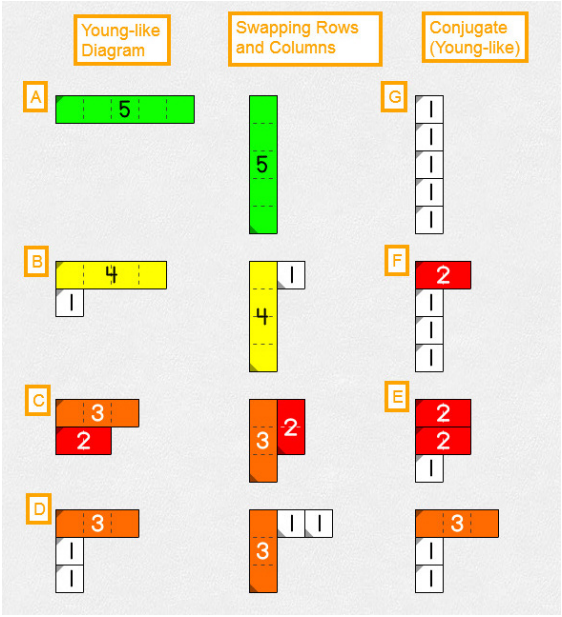
Here, the partition 5 + 4 + 1 of 10 is represented using a Ferrers diagram and a Young Diagram:



The seven partitions of 5 above can be easily rearranged to create Young-like diagrams for each, using relational rods:



It should not surprise anyone that mathematicians like to define operations on mathematical objects, like partitions. One such operation is conjugation (which is a good example of a word that is used multiple ways in mathematics, never mind in French grammar). A conjugate is created by swapping rows and columns in a Young diagram—we might have called it another popular math term: transposition, but were not consulted.



When you swap the rows and columns and turn something like B: 4 + 1 into F: 2 + 1 + 1 + 1, you still have the same number of squares, so each is a partition of the same number. B and F are called conjugate pairs. Students should convince themselves that F's conjugate is B, i.e., that the pairing works "backwards." When the seven partitions of 5 are paired up in conjugates, there must be one left unpaired. How is that possible?

It turns out that D: 3 + 1 + 1 is its own conjugate and so is referred to as a self-conjugate. An excellent activity for students would be to come up with lots of other examples of self-conjugates (for bigger numbers) and describe what geometrical properties the Young diagram must have for a partition to be a self-conjugate.

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION
ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

For the past two years, the OAME and OMCA have collaborated on a research project with the University of Ottawa to pilot and document the implementation of Grade 9 Applied Mathematics through school teams in nine different school boards.

There is an opportunity for some of your schools to attend a **Summer Learning Institute,** where this project's school teams and the research team will share their learning and facilitate a co-planning process encouraging schools to begin their own journey of deeper implementation of the curriculum. Over two days, the forum will provide a variety of plenary, workshop, and working-group sessions.

LIMITED SPACES ARE AVAILABLE.

Please pass this on to anyone willing to make up a team to take advantage of this opportunity.

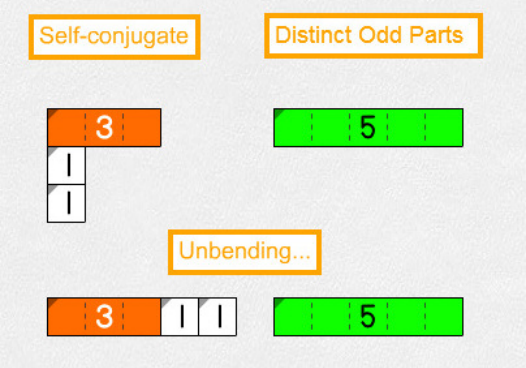
Find more information and the registration package at:
http://oame.on.ca/main/files/wshops/Gr9InstituteAug2016registration.pdf

Exercising Spatial Reasoning and Proving

One theorem that affords a visual justification is the following:

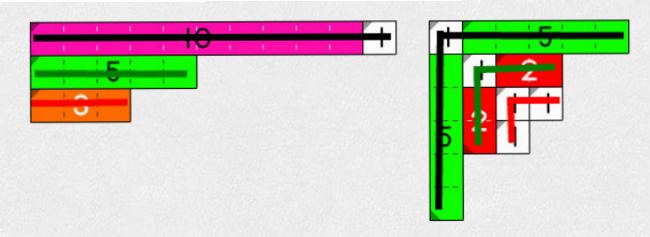
> The number of self-conjugate partitions of a number is the same as the number of partitions with distinct odd parts.

Above, we found one self-conjugate partition, 3 + 1 + 1. There is one partition with distinct odd parts, the bothersome one, 5. Note that 3 + 1 + 1 has only odd parts, but they are not distinct, since the 1 is repeated. We need a bigger number to obtain more interesting partitions with distinct odd parts, like the 11 + 5 + 3 used below.



This count of 1 in each case is not particularly convincing for the general result. Students could use their geometric understanding of self-conjugates and a hint about unbending to start playing with lots of other examples.

In designing student explorations, "spoilers" should not be provided too early. The Wikipedia article provides a very nice sketch of the proof that highlights another important counting idea: if you can make a one-to-one correspondence between two sets, they must have the same number of elements (cardinality). Suffice it here to provide a hint of how one partition with distinct odd parts, 11 + 5 + 3, could be bent using Relational Rods to get a self-conjugate.



Notice that 11 is represented by 1 plus 5 in the horizontal direction plus 5 in the vertical direction, which brings us full circle back to how to "see" odd numbers using the tool.

Conclusion

The Relational Rods+ tool for desktop computers and mobile devices provides a platform for rich investigation of number properties, using spatial reasoning. The support wiki provides other useful links to resources about Relational Rods, including a YouTube playlist with some other fascinating explorations, including one related to perfect squares.

Feedback and Future Requests

Please feel free to send us your feedback about the Relational Rods+ tool, using the Feedback Form button inside the Information Dialog. You can also send your comments to **WhatsNew@oame.on.ca**. You can share worked examples on Twitter, using the hashtag #ONmathies. To be among the first to find out about the latest digital tool developments, sign up for our email list at **http://mathclips.ca/WhatsNewEmailList.html**. If you have ideas about nice uses for Relational Rods, we would love to receive them and include them on the Support Wiki. ▲

Encyclopedia of Mathematics Education (S. Lerman, 2014) Excerpts

"Constructivism is an epistemological stance regarding the nature of human knowledge, having roots in the writings of Epicurus, Lucretius, Vico, Berkeley, Hume, and Kant. Modern constructivism also contains traces of pragmatism (Pierce, Baldwin, and Dewey). In mathematics education the greatest influences are due to Piaget, Vygotsky, and von Glasersfeld… There are two principle schools of thought within constructivism: radical constructivism (some people say individual or psychological) and social constructivism. Within each there is also a range of positions… Constructivism has clear and operationalized implications for the design of instruction (Confrey, 1990; Simon, 1995; Steffe & D'Ambrosio, 1995; Forman, 1996; Thompson, 2002) and assessment (Carlson et al., 2010; Kersting et al., 2012)."

Thompson, P.W. (2014). Constructivism in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 96–102). London, UK: Springer.

▲ In the Middle: Linear Leaping

CARLY ZINIUK



Carly Ziniuk teaches Grade 9 Mathematics, Grade 12 Data Management, and AP Statistics at the Bishop Strachan School in Toronto, Ontario, Canada. She is very involved in using real-life data to engage her students in solving problems.

Following on the heels of my past two articles discussing how much we can do with food data in a math classroom, especially fast food like Starbucks, in middle school, I thought it might be fun to consider how we can look at linear relationships connected to healthy movement. I was not surprised to read in the 2012 and 2013 Canadian Healthy Measures Survey that most school-aged children and youth are not getting enough physical activity, but I was shocked to read that only 3 percent of girls aged 12 to 17 meet the recommendations and only 18 percent of boys in the same age range.

Now, the 5-minute burst of physical effort that the students endure through this activity might not be the 10-minute recommended bout, but it's a start! This article includes samples of student work (thanks, Sarah and Thea, for letting me include their TI-NSpire files and their written answers) and some sample student scenarios. You might want to also notice the focus on multiple representations for this activity in what my students like to call "window-pane" charts (see downloadable file).

This is another two-part activity, with the first part originating from the students' own movement. The next issue will feature a more traditional assessment (like questions that could be done on a test), with the same framing and the same "window-pane" charts and scenarios that arose from doing these problems in my Grade 9 classroom.

LINEAR LEAPING

In Physical Education class, you often learn how to improve your physical fitness by repeating exercises with good form. These exercises might include squat thrusts (burpees), bench-ups, side-to-side jumps, and jumping jacks.

Group Discussion Question

1. Can you maintain a pace in your exercising for 5 minutes that keeps your rate of exercise constant throughout the entire time?

Group Investigation

- Choose one exercise where you will try to maintain a **consistent pace for 5 full minutes**.
- Each member of your group must complete a different exercise. In your group, you rotate through the following roles: **exerciser, counter, timer/recorder, and trainer (motivating and checking form)**.

Each exerciser will do the following:

- Warm up by completing your exercise from 5–20 times. Make sure that other members of your group count, record, and check the exercise's form. Use these warm-ups as practice for the roles.
- Once you have warmed up and you are consistently able to do your exercise with good form, and your group members can do their tasks efficiently, begin the experiment.
- *The final number of warm-ups you do is the starting value of the experiment.* The recorder/timer will measure over 5 minutes the **appropriate intervals of time**, while the counter keeps track of the total number of correct exercises you have completed. You should also keep track of how many exercises you have completed over that time. Throughout this experiment, you should attempt to keep your rate of exercise as consistent as possible. You should have at least five pieces of data beyond the initial warm-up value.
- Record your results in a table that shows the time elapsed and the total number of exercises completed.
- Rotate through the roles so that each person conducts the exercises and collects the data.

Individual Analysis – Part One

Be sure that each person has the data that he or she created for his or her exercise (not the data that was collected). The work will be done **on your own**.

Take a rest from the leaping and move onto to the linear…

1. [K] Is your data linear? Justify your response in at least two ways.
2. Assume the data does have a linear relationship (regardless of your preceding answer).
   a. [K] Determine the equation of the line (using technology) for this data.
   b. [K, A] What is the slope of this line? Comment on what this value means realistically.
   c. [A] What is the y-intercept of this line? Comment on what this value means realistically.
   d. [K, A] Using your linear model, how many exercises could you do in 20 minutes (whew!)? Confirm your response, using the equation and the graph.

e. [K, A] Using your linear model, how many minutes would it take to do 400 exercises? Confirm your response, using the equation and the graph.

3. [T, C] If you decided to exercise at your full potential, instead of trying to perform at a consistent rate, describe how you think the data and the linear model would change.

Ontario Curriculum Achievement Chart Categories: K: Knowledge/Understanding; A: Application; T: Thinking; C: Communication

Individual Analysis – Part Two

How would this scenario extend beyond your exercising experience? You might be curious how fancy and fit these students are if they were to do these too.

4. [A, C] Taylor is a regular exerciser and Lucy created his graph. Complete this chart.
5. [T, C] Sasha felt chipper and fit that morning during a long warm-up, but she was mindful that she had a recovering sore ankle and she didn't want to injure it again. So although Sasha conducted the Jumping Jacks, she was very gentle during her workout. Compare Sasha's workout to Taylor's in this chart.
6. [T, C] How would the situation differ for Avery, who is a tremendously fit, cardio-master, healthy exerciser, who had done 50 warm-ups ahead of time?
7. [T, C] If Matt decided to exercise at his full potential during the warm-up, then arrived at the experiment data-collecting time exhausted, describe how you think his data and his linear model would differ.

LINEAR LEAPING – SAMPLE STUDENT ANSWERS

Group Discussion Question

1. Can you maintain a pace in your exercising for 5 minutes that keeps your rate of exercise constant throughout the entire time?

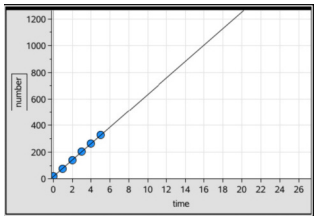
   I think the hard part is going to be constant and actually try hard to exercise. I think I will get tired and not do my best.

Individual Analysis – Part One

Sarah's Answers

Graphical (Draw a sample sketch.)

Graph your data, showing the key points.



Logical (Explain the starting value, and constant change rate in words.)

I started at 20 and performed at an average rate of 62 jumping jacks per minute.

62 is the average first difference, so the average number of jumping jacks I did per minute over the 5-minute period.

Algebraic (State the variables and the equation.)

t *represents the amount of time she has exercised, and* n *represents the number of exercises completed.*

$n = 20 + 62t$

Numerical (Complete the table, being sure to show important points.

Data collected from exercising.

t	n	first differences
0	20	X
1	77	57
2	140	63
3	202	62
4	263	61
5	328	65

2. *My data is approximately linear because it increases by a relatively steady rate of 62 jumping jacks per minute. Also when I graph the line of best fit, it almost goes through all the points.*
3. $Y = 17.381 + 61.7143x$
   a. The slope of the line I made by hand is 62. The slope of the line on the computer is 61.7143. They are both pretty much the same. It means I did 62 jumping jacks per minute.
   b. The *y*-intercept of the line I made is 20. The *y*-intercept of the line on the computer is 17.381, meaning when the time was 0, I had already done approximately 20 jumping jacks.
   c. At 20 minutes, I would have done 1250 jumping jacks. I know this from looking at the graph and following the line to 20 minutes. I also checked it using algebra on NSpire with my hand equation and got 1260.

d. To do 400 jumping jacks, it would have taken me approximately 6 minutes.

I found this by using the equation $20 + 62x = 400$.

$62x = 400 - 20$,
$x = 380/62$,
$x = 6.13$

Another way was to check the line where it goes to 400 on the graph, around 6.

4. If I exercised at my full potential and did not perform at a consistent rate, my graph would not be linear. My data would increase at a higher rate at the beginning, then die down a bit in the middle because of tiredness, and then increase again at the end to finish off the 5 minutes. This means the data would not be consistent. It might look more like a saggy line.

Individual Analysis – Part Two

Thea's Answers

5. Logical: Taylor started at 10, and performed at a rate of 15 jumping jacks every 30 seconds.

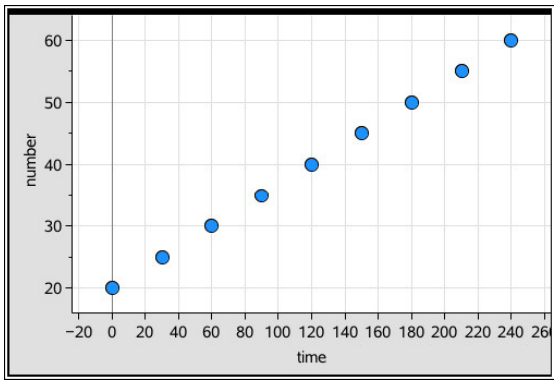
Algebraic: $N = 10 + 0.5T$

T	N	T	N	T	N
0	10	120	70	240	130
30	25	150	85	270	145
60	40	180	100	300	160
90	55	210	115		

6. Logical: Sasha started at 20, and performed at a rate of 5 jumping jacks every 30 seconds.

Algebraic: $N = 20 + 0.16T$

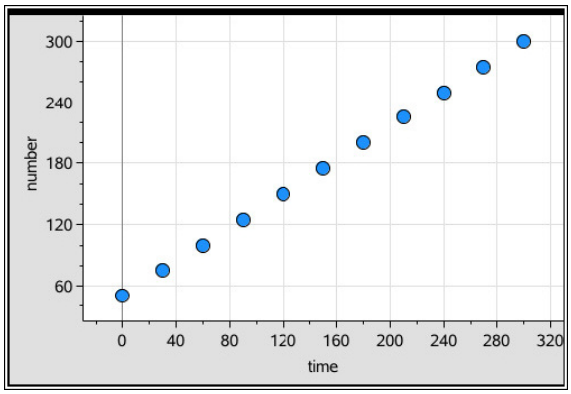
T	N	T	N	T	N
0	20	90	35	180	50
30	25	120	40	210	55
60	30	150	45	240	60



7. Logical: Avery started at 50, and performed at a rate of 25 jumping jacks every 30 seconds.

Algebraic: $N = 50 + 0.83T$

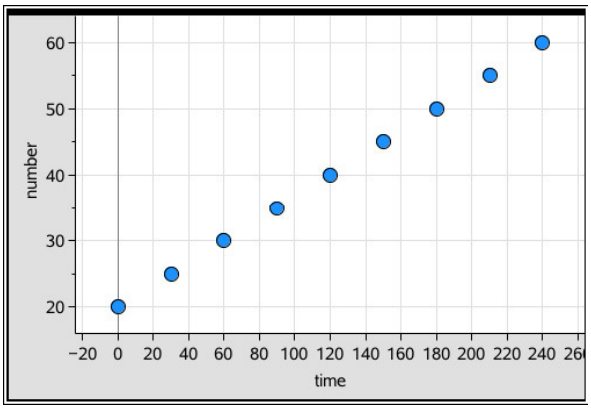
T	N	T	N	T	N
0	50	120	150	240	250
30	75	150	175	270	275
60	100	180	200	300	300
90	125	210	225		



8. Logical: Matt started at 30, and performed at a rate of 2 jumping jacks every 30 seconds.

Algebraic: $N = 30 + 0.06T$

T	N	T	N	T	N
0	20	90	35	180	50
30	25	120	40	210	55
60	30	150	45	240	60



The other option is that he might not be linear at all and then not have a constant rate. ▲

Click on the following link, or enter the URL into your browser, to access a separate PDF version of this column that includes a student worksheet and a marking scheme. Document URL: ***http://oame.on.ca/main/files/gazettelinks/ Gz54i4Ziniuk.pdf***