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# Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

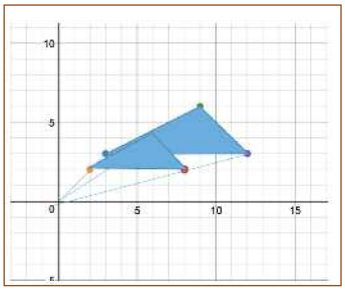
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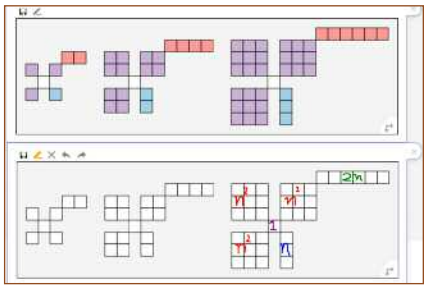
Vol. 55 #1
September 2016

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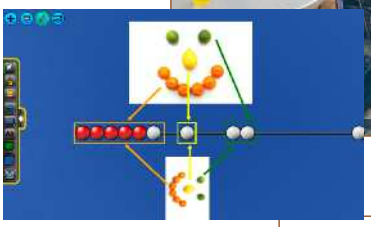


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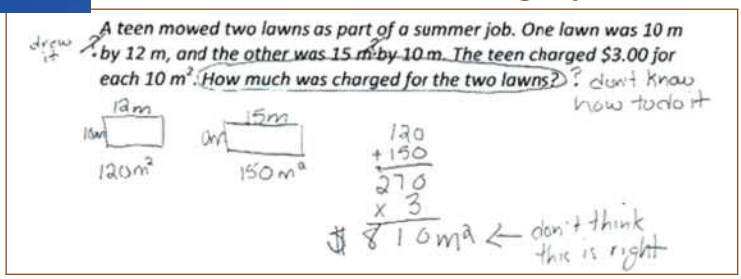
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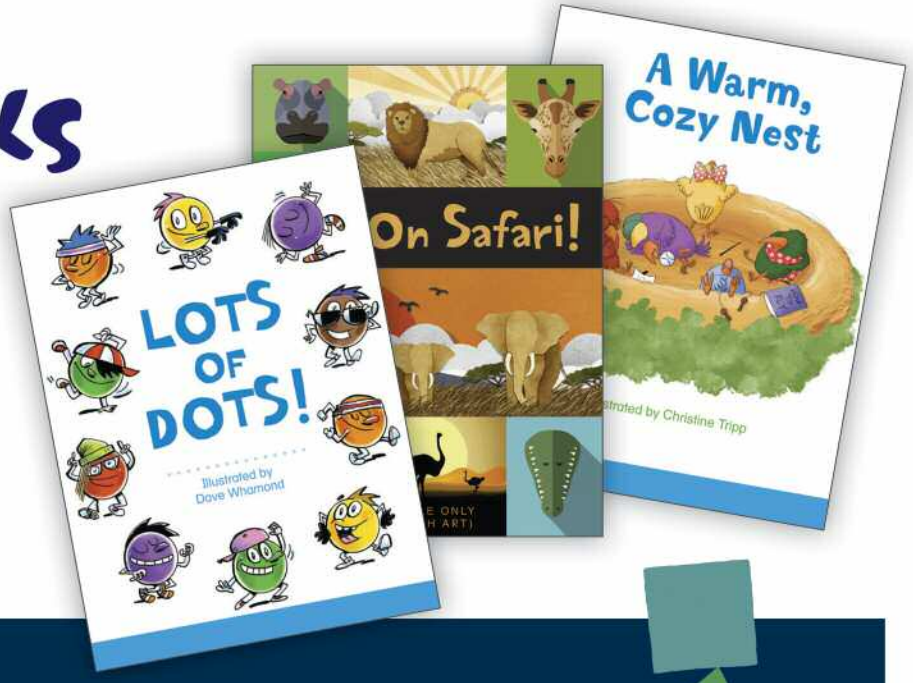
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Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Amy Lin, along with **separate files for each of the text, graphics, and/or photographs**. Please email all files to Amy Lin at **gazette@oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and email address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

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# ▲ 2016-2017 OAME Board

## DIRECT GENERAL INQUIRIES TO: EXECUTIVE DIRECTORS

**Fred and Lynda Ferneyhough**
eds@oame.on.ca

## PRESIDENT

**Judy Mendaglio**
Mississauga, ON
judy.mendaglio@oame.on.ca

## PAST-PRESIDENT

**Timothy Sibbald**
North Bay, ON
tim.sibbald@oame.on.ca

## PRESIDENT-ELECT

**Jill Lazarus**
Petawawa, ON
jill.lazarus@oame.on.ca

## VICE-PRESIDENTS

**Bill Otto**
Stillwater Lake, NS
bill.otto@oame.on.ca

**Sandra Jean Price**
Oshawa, ON
sandra.jean.price@oame.on.ca

## CHAPTERS & REPRESENTATIVES

**CHAMP: Amanda Laxton**
Brampton, ON
amanda.laxton@oame.on.ca

**COMA: Melissa Black**
Kanata, ON
melissa.black@oame.on.ca

**GOLDEN: Pamela Chun**
Grimsby, ON
pamela.chun@oame.on.ca

**GVMA: Andrew Skelton**
Guelph, ON
andrew.skelton@oame.on.ca

**ISOMA: Charlotte Aust**
Toronto, ON
charlotte.aust@oame.on.ca

**MAC²: Paulene Washington**
Barrie, ON
paulene.washington@oame.on.ca

**NOMA: Denise Filipovic**
Sudbury, ON
denise.filipovic@oame.on.ca

**NWOAME: Corrine Russell-Pritoula**
Thunder Bay, ON
corrine.russell-pritoula@oame.on.ca

**PRMA: Lori Yee**
Pickering, ON
lori.yee@oame.on.ca

**QSLMA: Kathy Pilon**
Smith Falls, ON
kathy.pilon@oame.on.ca

**SAME: Freda Liu**
Toronto, ON
freda.liu@oame.on.ca

**SWOAME: Christine Sasso**
Windsor, ON
christine.sasso@oame.on.ca

**TEAMS: Alvarine Aldridge**
Toronto, ON
alvarine.aldridge@oame.on.ca

**WOMA: Katelyn Beecroft**
London, ON
katelyn.beecroft@oame.on.ca

**Y⁴MA: Chi Hang**
East York, ON
chi.hang@oame.on.ca

## DIRECTORS – TERM EXPIRING 2017

**Jung Choi-Perkins**
Toronto, ON
jung.choi-perkins@oame.on.ca

**T. Anne Yeager**
Orangeville, ON
anne.yeager@oame.on.ca

## DIRECTORS – TERM EXPIRING 2018

**Pat Kehoe**
Kanata, ON
patricia.kehoe@oame.on.ca

**William Lundy**
Belleville, ON
bill.lundy@oame.on.ca

## DIRECTORS – TERM EXPIRING 2019

**Liisa Suurtamm**
St. Catharines, ON
liisa.suurtamm@oame.on.ca

**Bart Vanslack**
Toronto, ON
bart.vanslack@oame.on.ca

## NCTM REPRESENTATIVE

**Todd Romiens**
Windsor, ON
todd.romiens@oame.on.ca

## WEBSITE COORDINATOR

**Greg Clarke**
Orillia, ON
web@oame.on.ca

## *GAZETTE* EDITOR

**Amy Lin**
Burlington, ON
gazette@oame.on.ca

## *ABACUS* CO-EDITORS

**Mary Lou Kestell**
Toronto, ON
marylou.kestell@oame.on.ca

**Kathy Kubota-Zarivnij**
Scarborough, ON
kkz@oame.on.ca

# ▲ About the Ontario Mathematics Gazette

The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

**Editor**
▲ **Amy Lin**
Email **gazette@oame.on.ca**

**Associate Editors**
▲ **Anne Yeager**
Email **anne.yeager@oame.on.ca**
▲ **Marilyn Hurrell**

***Abacus* Co-Editors**
▲ **Mary Lou Kestell**
Email **marylou.kestell@oame.on.ca**
▲ **Kathy Kubota-Zarivnij**
Email **kkz@oame.on.ca**

**Copy Editing**
▲ **Gitta Berg**

**Design and Production**
▲ **Penny Clemens,** Graphic Designer

**Printing & Binding**
▲ **Pole Printing,** Box 69, 1-89 King Street East
Forest, ON N0N 1J0 (519) 786-5112

**Advertising Manager**
▲ **Robert Sherk**
Email **robert.sherk@oame.on.ca**

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Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

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# ▲ Editor's Message



**AMY LIN**
**EMAIL: amy.lin@oame.on.ca**

*Amy Lin's background in the sciences and the arts helps explain why her passion in mathematics education is to develop both the logical and creative aspects of the subject. She is currently working on her PhD at Brock University in math education and is a Vice-Principal at Applewood Heights Secondary School. She is often asked to speak on mathematics teaching and learning around the province and beyond, and she is an author of a number of mathematics resources for students and teachers at both the elementary and secondary levels. She has initiated and successfully led many professional development projects in school boards in Ontario.*

It is always an exciting time of year when we embark on a new school year and reconnect with our colleagues and students. I am pleased to take on this role as Editor of the *Gazette* and look forward to my new adventure.

After I had taken on this new position, I decided to peruse past issues of the *Gazette* and considered the content. There have been a variety of articles and columns that have adorned the pages of this publication. It solidifies my long-time knowledge that mathematics education in Ontario is made up of well-connected communities.

This past summer, I was fortunate enough to attend the International Congress on Mathematics Education (ICME 13) in Hamburg, Germany. If I walked away from this conference with one key piece of learning, it would be the attention to effective approaches to teaching mathematics that involves human complexity, and a definite realization and awareness of the influences that shape students' learning of mathematics, including the differences among people, schools, and communities.

I want to emphasize the importance of and my personal goal of linking research and practice in mathematics education and that this is the only way to addressing critical issues of mathematics teaching and learning.

> "Researchers and school personnel must work together to address and examine issues pertinent to the teaching and learning of mathematics. Together they need to consider ways to implement strategies across contexts that build on findings from research." National Council of Teachers of Mathematics. (2010). *Linking Research & Practice.* Reston, VA: NCTM. www.nctm.org.

Let's consider the purpose of mathematics education research. This research has the ultimate goal of enabling teachers, teacher educators, and school districts and institutions on all levels to make the right decisions about the mathematics education activities and experiences that will best serve students. It is exciting to see the range of research—from systemic matters such as instructional practices and evaluation, to individual decisions each teacher and teacher educator makes about his or her particular classroom.

The relationship between research and practice is a two-way street. It is about using ideas from both communities—meaning that the work is grounded in conceptual understanding, valid methodologies, is open to peer review, and that the reports of the findings should be shared among math education researchers, teachers, school and system leaders, policy-makers, and any other stakeholders. In Ontario, we have witnessed a growth in our research designs, both in and out of classrooms, including action research, collaborative inquiries, large-scale and small-scale designs, qualitative methods, and quantitative methods.

Bridging these communities also means having effective ways for disseminating the research, practices, and findings that meet the needs of both researchers and practitioners, resulting in a better understanding of its true impact on mathematics teaching and learning. Research then informs practice and vice versa, with continuous communications back and forth between the two.

I have been fortunate in my educational career to have the opportunities to work with our mathematics educational researchers, math teachers, leaders, and policy-makers. We have a unique situation in Ontario with communities and organizations such as OAME, OMCA, Fields Institute, Ministry of Education, Faculties of Educations and Department of Mathematics, and School Boards that have always contributed to, and committed to, understanding and investigating problems of practice, changing learning environments, and the use of technology.

I look forward to sharing these many perspectives on math education in *Ontario Mathematics Gazette*. It is a great forum to showcase our collaborative communities, and I hope that it will provide something for everyone.

In this issue, author David Costello has written an article about his collaborative project with a Grades 5/6 math teacher entitled *Math in the Margins – Making Student Mathematical Understanding Explicit.* Part Two of *Three Spheres of Radii 3, 2, and 1 Mutually Tangent Problem*, written by André Lemaire, can be found in this month's *Gazette*.

Tom Griffiths and Tim Sibbald collaborate on a feature article that bridges the long history of the OAME, and to address concerns about the extent to which applications and practical purposes of mathematics may have diminished the beauty that can be found in math that is not utilitarian.

OAME Co-Chairs of the very successful OAME Annual Conference this year, Greg Clarke and Bruce McKay, provide us with a glimpse of what went on in Georgian College, Barrie this past May. Also, OAME Award winners for 2016 are highlighted. There is a tribute to math educator Lloyd Auckland in this issue.

Our regular columns feature the following: our incoming OAME President's message from Judy Mendaglio; "Assessment Abby" (eponymous); Mary Bourassa's "Technology Corner"; Carly Ziniuk continues this year with her "In the Middle" column; Greg Clarke, Agnes Grafton, Ross Isenegger, and Markus Wolski update us on "Provincial Digital Learning Resources"; Lynda Colgan with her "Hey, It's Elementary" column; Shawn Godin with his "What's the Problem?" column; and Todd Romiens with his update report for OAME/NCTM. As usual, this issue also features the rich contribution of ideas for elementary math teaching as found in the Abacus insert, co-edited by Mary Lou Kestell and Kathy Kubota-Zarivnij.

We would like to thank Ann Kajander for her work on the column "MB4T." Ann has decided to take a break from the column. We will be featuring guest columnists as we move forward, and the first for this issue is Marian Small.

As we start this new school year, I would like to invite all of you to consider contributing to this publication, including submission of articles, or if you are interested in being a guest columnist for MB4T. I look forward to hearing from you and sharing our work, which actively supports our mathematics education community. ▲



# President's Message



JUDY MENDAGLIO

Welcome to a new school year. One thing I always did in September was vow to keep my files organized. That trig lesson that worked so well in Grade 10 Applied – I should save that in my 2D folder as well. And in my MBF3C folder. Or maybe I should just make a Trig folder as a repository for all things trig? The connections across many different courses, while clear in my mind, were an insurmountable challenge to my recordkeeping skills. Every year, I promised myself that this year would be different. Well, it never happened. In fact, I was never even able to keep my desk organized, let alone my files. I always had more to do than time in which to get it done. If anyone has any file maintenance tips to share with their teacher colleagues, perhaps you might like to submit an article to the *Gazette*.

As I move into the role of President of OAME, I have to thank the volunteers that make up the Executive of the Board, for their hard work and dedication this past year. First, I want to thank Tim Sibbald, who will now assume the role of Past-President. If OAME is a tall ship, then Tim is the rudder, steering us in the right direction. Tim has worked very hard to establish and grow connections between OAME and other organizations and communities interested in mathematics education, in Ontario as well as beyond our borders. He has also explored ways in which OAME, as a provincial organization, can best meet the needs of its local chapters. He has provided support and leadership to the Board of Directors and the Executive members of the Board. Paul Alves, who spent this past year as our Past-President, will be leaving our Board. Paul has been our hull, providing structural integrity to our organization. He has always been a valued mentor to new members of the Board, and his support and leadership will be greatly missed. Some of you may know that Paul will have served on one or another Annual Conference organizing committees 4.5 of the past 5 years and is still involved in 2018. Bill Otto and Sandra Jean Price round off the Executive team by bringing with them different experiences and perspectives, and their contributions are invaluable. Sandra and Bill are our sails, catching the wind and keeping us moving. Finally, we have our ship's ballast in the team of Lynda and Fred Ferneyhough. Without their skills and expertise, we would surely tip over as we sail the often choppy waters to support mathematics education in Ontario. I also recognize that it is the energy and talent of the members of OAME that drives us forward. You are all true leaders.

As we begin a new year, OAME is saying goodbye to our editor of the past few years, Dan Jarvis. Dan had the honour of picking up the NCTM *Publications Award for Outstanding Journal* that was awarded to the OAME *Gazette* at the NCTM Annual Conference earlier this year. We wish Dan all the best as he takes on new challenges. As we say goodbye to Dan, we welcome Amy Lin as our new editor. Everyone at OAME is very proud of the excellence of our publications, the *Gazette* and the *Abacus*, and we would like to acknowledge the work done by our regular contributors, our guest authors, and our fine editors.

During this past year as President-Elect of OAME, I had the great fortune of being invited to represent OAME at several events. In February, I attended the three-day Ontario Education Research Symposium. One key message that was repeated through many of the presentations was the desirability of engaging the student as a researcher, solving meaningful problems. The other message that one could not miss was the increasing importance of collaboration and building networks—in our classrooms, schools, and educational communities. Also in February, Tim Sibbald and I attended a meeting of representatives of ministries of education and math subject associations from across Canada. The meeting was organized by NCTM, and there was a large contingent of representatives from that association as well. It was wonderful to get updated about what is happening in math education in other jurisdictions as we heard from other provincial representatives. In April, I attended the three-day NCTM conference, went to a great many sessions, and came away with a renewed sense that we, in the Ontario math education community, are doing the right things. In May, Tim Sibbald, Paul Alves, and I gave a presentation at the Learning and Teaching for Tomorrow: Building Collaboration and Capacity forum that was co-hosted by the Ontario Ministry of Education and Faculties of Education. We spoke about the challenges facing us as a math subject association, and the conversation that was generated by our presentation was extremely rich. Also in May, I attended a session jointly sponsored by the Ontario Ministry of Education and the Ontario Teachers' Federation (OTF): the Subject/Division Association meeting, followed by the OTF's Curriculum Forum. As part of the two-day program, we saw clips from a wonderful documentary called *The Messenger.* I had dozens of lesson ideas while watching a preview of that film. Each opportunity to connect with other educators as a representative of OAME provides me with new insights about the very complex task of educating our children, but I also recognize that we are very well positioned

at OAME to support teachers in providing the very best math instruction.

In my quiet moments, I enjoy reading. I love to read about mathematics. I always have a stack of books waiting to be read, and the majority of the books in my stack are math related. Right now, I am reading Ian Hacking's *Why Is There Philosophy of Mathematics At All?*. In it, he speaks about the problem with the idea of "application" in reference to mathematics. Application, he says, only goes in one direction, as, for example, applying math to the study of climate change. However, what we might really be talking about are *connections between* math and some other thing. Connections go in both directions. Sometimes, we learn more about a real-life structure by looking at it through a mathematical lens, but other times, we learn more about a math structure by looking at it from a real-life perspective. It feels to me that our students are exposed to many instances of applying the math that they have just learned to some "real-life" scenario, but not much time at all looking at instances where real life generated a new understanding of mathematical structures. Perhaps more of the latter would help students to understand the connections between mathematics and the world in which they live, and allow them to see the side of mathematics that serves our human thirst for understanding. Perhaps they will understand that mathematics does not live outside of the real world, but is part of the real world.

I am very excited about this coming year. By the time you read this, we will have run Grade 9 Applied Summer Institutes, at which our teacher colleagues will have shared their work over the past two years. School teams that were fortunate enough to be able to attend these Institutes will arrive back to school with new research-supported strategies for improving the math experiences of our Applied students. As a Board, we will continue to build stronger partnerships with our regional groups as well as with other math education organizations. I look forward to seeing what will appear in this year's issues of the *Gazette* and *Abacus*. I am ready to welcome new and unanticipated invitations for OAME's involvement.

Try to find time this year to engage in your local chapter events and connect with your math colleagues. Perhaps you are considering submitting a proposal for OAME 2017 in Kingston in May. Maybe you are already planning to spend the day with Jo Boaler at November's Leadership Conference. Whatever your plans, have a great year!

> "Mathematics is a huge and highly interconnected structure. As one reads mathematics, one need to have an active mind, asking questions, forming mental connections between the current topic and other ideas from other contexts, so as to develop a sense of the structure, not just familiarity with a particular tour through the structure."
>
> ~ William Thurston



## ▲ In Memory of Lloyd Auckland

Lloyd was raised in a rural area outside of London, Ontario, and began his education in a one-room schoolhouse. This led to his studying mathematics at Western University and the Ontario College of Teachers in Toronto. He had a lengthy career in the St. Thomas and London areas, where his teaching career began in 1947. He was very involved in coaching athletics, was a vice-principal, and a math department head. In 1968, Lloyd was one of the original winners of the Descartes Medal. In 1970, he became a committee member of the Junior Mathematics Contest, which was the forerunner of the Canadian Mathematics Competition. His connection to Waterloo continued in retirement, and he volunteered for a further 26 years at the institution. Lloyd was a past-president of the OATM, which preceded the OAME, and a lifetime member of the OAME.

# ▲ MB4T (Mathematics by and for Teachers): Solving Proportions

MARIAN SMALL



*Marian Small is Professor and Dean Emerita of the University of New Brunswick. She lives and works in Ontario with K–12 teachers in many school districts and has written a number of resources.*

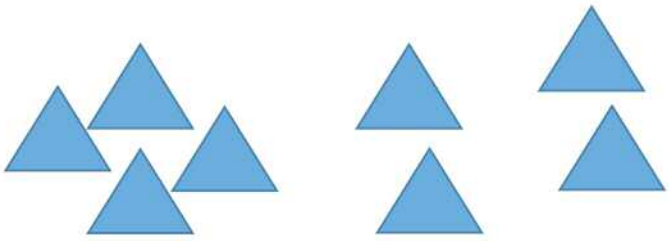
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## Re-thinking Multiplication

Many teachers in the primary years, as suggested by the curriculum, introduce multiplication as the result of counting the total of a set of equal groups. This leads, quite quickly, to the notion of multiplication as repeated addition. There is nothing wrong with this approach, but it has limitations.

One limitation is that it does not generalize well to work with fractions, decimals, and sometimes negative quantities. For example, what does 3/4 x 2/3 mean? How or what do you repeatedly add or count when looking at 3/4 of a group of 2/3, or 0.8 groups of 3.2 or –4 groups of –8.

Another limitation is that too tight a definition sometimes precludes flexibility. For example, does the picture below show multiplication?



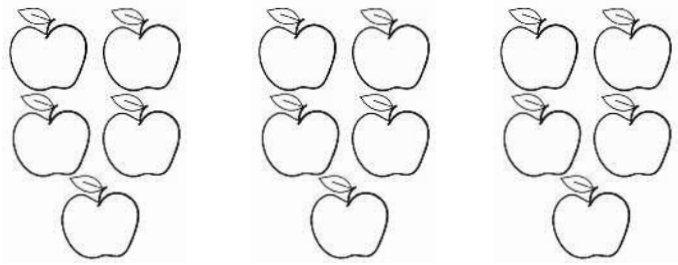
Initially a student might argue that there is no multiplication here, since he or she was taught to look for equal groups, and he or she sees 4 and 2 and 2, not all 4's or all 2's. However, we want flexible students who see that actually there are 2 groups of 4 (if the 2 and 2 are put together) or 4 groups of 2 (if the 4 is separated into two 2's) or 8 groups of 1, or later, 2 2/3 groups of 3.

Yet another limitation is that not all real-life situations translate well into equal groups or repeated addition meanings. For example, if the problem asks: *Jennifer had 8 times as much money as Lia. Lia had $1.25. What does Jennifer have?*, it is often difficult for students to envision the "groups" or know what to add repeatedly. It is not impossible to make sense of this problem with those meanings, but it's not clear enough to all students.

So perhaps we should start considering more meanings of multiplication early on. Think of the material below not as instructions for a laundry list of what you should do, but as something to reflect on in terms of when you would want to raise these ideas and how.

## Multiplication as a Unit Change

One way to think about multiplication is as a unit change. Just as we think of writing 200 cm as 2 m as a unit change, we can think of a situation where there are 3 piles of 5 apples as a unit change; the unit changes from piles to apples. 3 piles = 15 apples.



When the unit is a pile of 5 apples, there are only 3 copies of the unit, but when the unit is individual apples, there are 15 copies of the unit.

Even though metric measurement conversion (other than centimetres to metres in very simple situations) usually happens after the introduction of multiplication, this notion of unit change in multiplication can happen first. A student can think, for example, as the value of 4 nickels as moving from a unit of nickels, in which case there are 4 copies, to a unit of cents, in which case there are 20 copies.

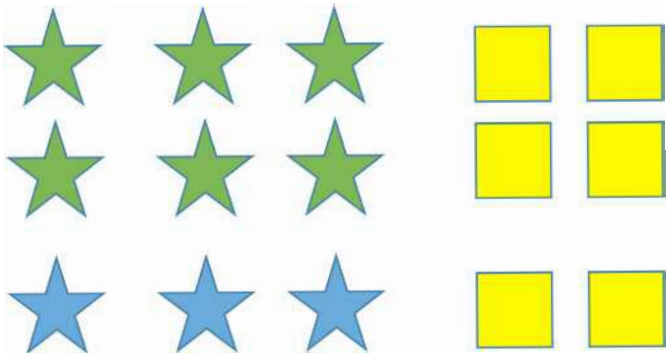


is the same as



The notion of unit change is particularly valuable when multiplication of fractions is introduced. For example,

2/3 x 3/5 means that there is 2/3 of a unit if the unit is 3/5 of an amount (the stars only), but there is 6/15 of a unit if the unit is the whole amount (stars and squares).



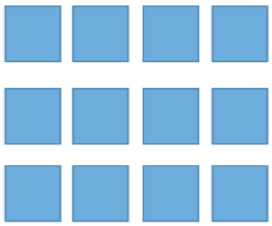
## Multiplication as a Comparison

Thinking in terms of units helps students use multiplication to describe comparisons, thinking of one number as so many copies of another, e.g., four times as much as another or half as much as another,

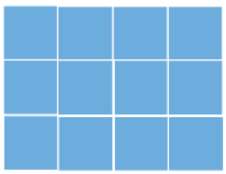
For example, you might think of the number 10 as two and a half units of the number 4. Or you might think of the number 4 as 2/5 of a unit of 10. Every number can be thought of as units of another number. For example, – 0.6 is 3 units of –0.2 or 0.6 units of –1 or 4 units of –0.15. Thinking of one number as units of another is the basis for proportional reasoning.

## Multiplication as Area of a Rectangle

If students use square tiles as an early model for multiplication, e.g., showing 3 x 4, as

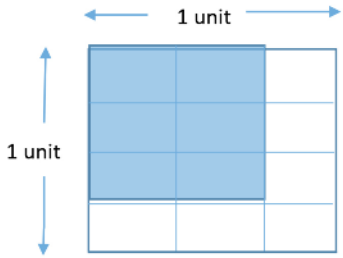


it is a quick transition to multiplication as area of a rectangle.



This is helpful, since later, 4.2 x 3.4 can easily be interpreted as the area of a rectangle that is 4.2 units across and 3.4 units deep. Or 2/3 x 3/5 can easily be interpreted as the area of a rectangle that is 2/3 of a unit wide and 3/5 of a unit deep.

Seeing a rectangle as 2/3 of a unit wide and 3/5 of a unit deep helps a student make sense of why the product (or area) is 6/15.




## Multiplication as an Operator

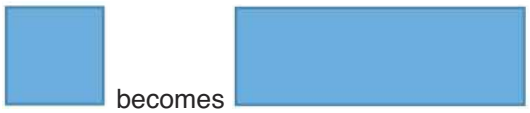
One of the ways to think of multiplication is as an "operator." It operates on an amount. Technically, an operator turns one set of numbers into another. For example, the operator x 2 makes whatever there already is twice as much, whether it is twice as long, twice as heavy, twice as many, etc., depending on which attribute is being considered. So, for example, if we only multiplied whole numbers by 2, the operator x 2 changes the set of whole numbers to the set of all the even whole numbers.

Multiplication by 3 can be modelled in these ways.

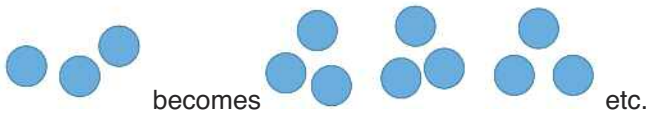
A length can triple, so

_____ becomes _____

An area can triple, so



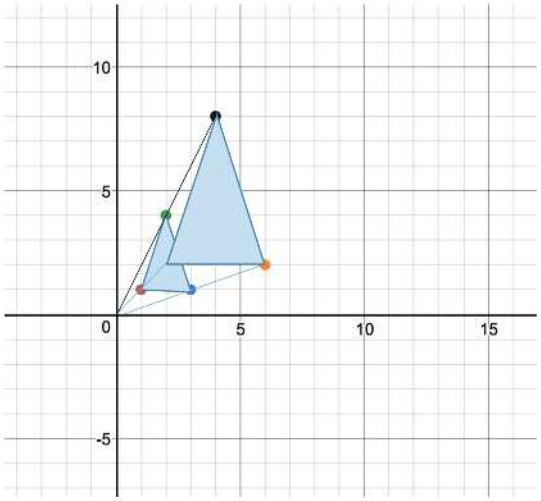

A number can triple, so



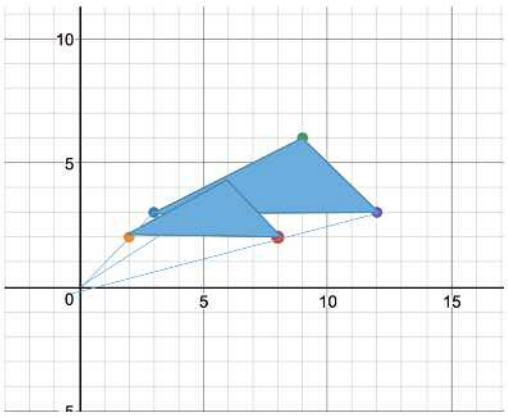

## Multiplication as a Dilatation

The operator meaning can eventually be related to dilatations.

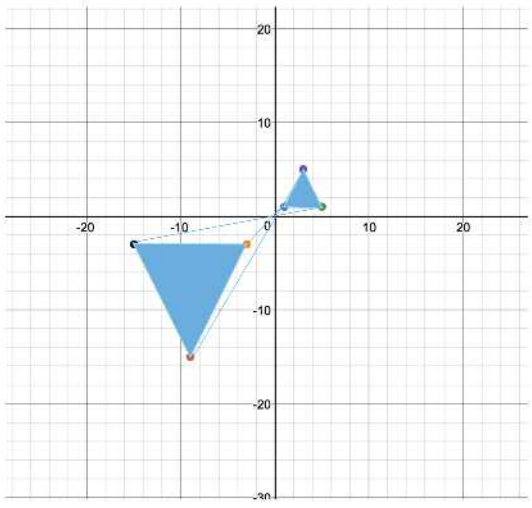
For example, a dilatation of x 2 doubles all lengths in the shape being dilated (although area is quadrupled).




This works well with multiplying fractions or decimals, so that a dilatation of x 2/3 makes all lengths 2/3 as long, or a dilatation of 1.4 makes all lengths 1.4 times as long.




It even works with negative values, since a dilatation of –3 makes all lengths 3 times as long, but places them on the opposite side of the dilatation centre.




## Multiplication as Counting Combinations

Another way to think of multiplication when talking about whole numbers is as the number of combinations when items in two sets are combined. For example, 3 x 5 tells the number of unique ice cream cones that can be created if there are 3 kinds of cones and 5 flavours of ice cream.

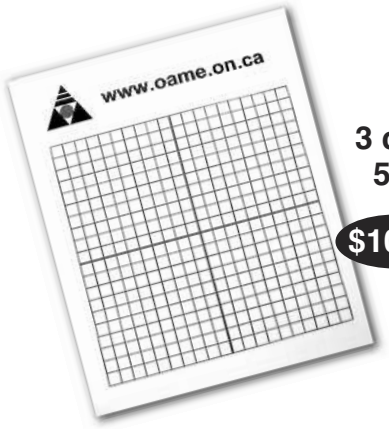
| Cone Types: | Waffle | Sugar Cake | |
|---|---|---|---|
| Kind of Ice cream | Vanilla<br>Hazelnut | Chocolate<br>Butter rum | Strawberry |

| Kind of cone | Kind of ice cream | |
|---|---|---|
| Waffle cone | Vanilla<br>Chocolate<br>Strawberry<br>Hazelnut<br>Butter rum |  |
| Sugar cone | Vanilla<br>Chocolate<br>Strawberry<br>Hazelnut<br>Butter rum |  |
| Cake cone | Vanilla<br>Chocolate<br>Strawberry<br>Hazelnut<br>Butter rum |  |

## Summary

A broader view of multiplication will lead to a richer understanding. Teachers need to decide at which point which meaning might feel right to introduce. ▲

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# ▲ Ask Assessment Abby: Linking Assessment and Instruction

**ASSESSMENT ABBY**
**EMAIL: assessmentabby@oame.on.ca**



*Ask Assessment Abby* A³ is a regular column in the OAME *Gazette*, where teachers can share concerns and best practices about assessment, evaluation, and reporting of mathematics. Please send your questions to Ask Abby at **assessmentabby@oame.on.ca**.

Dear Assessment Abby,

How do I make my student's thinking more visible? I am struggling with the documentation piece!

**Using Technology to Make Student Thinking More Visible**

"Using various forms of documenting student learning challenged us to see children differently. Different kinds of demonstrations of learning moved us all beyond what we had come to expect, and led us to a place of valuing each child's contribution. What was made visible was the learning process of children, their multiple languages, and the strategies used by each child." – Educator, Early Primary Collaborative Inquiry (Pedagogical Documentation)

We need to tap into the expertise of the trailblazers in our province. Our Early Learning Teams (Kindergarten teachers and Early Childhood Educators) have changed the face of documentation. They are using technology to capture authentic learning and student thinking. The work they are doing can be extended into any learning space K–12 and beyond!

We now know that it is beneficial to document through the eyes of the learner. Even young students take responsibility for some of their own documentation. We have witnessed 4-year-olds grabbing an iPad to take a photo of their work, record their explanations, and drop it into their virtual portfolio.

We know many teachers are using video, voice recording, whiteboard, portfolio, and interactive web activities to extend their documentation beyond paper and pencil.

There are so many apps available, we thought we would just highlight a few to get you started.

Why not pick one and give it a try?

| If you want to see how students... | Try… |
|---|---|
| • solve a problem or explain a solution | • Explain Everything<br>• Educreations<br>• Showbie<br>• iMovie/Camera (photo/video) |
| • organize their work in a portfolio, as evidence of their learning journey | • SeeSaw<br>• Showbie<br>• D2L |
| • explore concepts and think about them (and you want the ability to see the evidence later) | • Prodigy<br>• Desmos<br>• D2L |

Yours in Math,
Assessment Abby ▲

# ▲ OAME/AOEM Annual General Meeting

The Annual General Meeting (AGM) will occur in conjunction with the October board meeting on **October 15**. The AGM will be held Saturday morning and will be made available using a virtual meeting. Registration is required for attendance purposes, and any member who wishes to attend will be able to sign up on MCIS. If chapters choose to hold an event in conjunction with the AGM, please forward attendance to the executive directors. Should voting be required, this will be set up for all registered participants and done digitally with instructions provided at the time. Members of the board of directors are reminded that the October board meeting will run until 4:00 p.m. to accommodate the AGM.

It is anticipated that the AGM will be approximately half an hour. Combining this with chapter events is encouraged.

## Summer 2016 Math Adventure #1 – Math AQ Course

BY RUTHIE SLOAN, PEEL DISTRICT SCHOOL BOARD

Ruthie Sloan, a teacher with the Peel DSB, shares her summer experience of taking a Math AQ course.

The first time it occurred to me was during our teacher-reflection time on Day 1 of our Math AQ. What became obvious to me that first day was the theme of fear and disconnect that resonated through each of the educators' reflections. What surfaced quite clearly revolved around fear of evaluation, fear of changing our pedagogy, but deeper still, the obvious disconnect between the learning ideals of fostering inquisitiveness (the learning ideals that we preach as valuable for our students) and the extent to which we as "teacher-learners" were willing to engage in our own professional learning. My curiosity of examining the differences in teacher and student learning has lead me to unpack and reflect on what possibly needs to happen in order to enable professional learning to move beyond a "Pinterest" and "uploading of information" mentality to cultivating changes in our pedagogy. What has surfaced so far in this journey can be summed up in thinking about professional learning as a collection of *intentional* choices:

- A choice to engage in reflection on pedagogy and ask "what if" questions that move beyond buzzwords
- A choice to allow ourselves to be uncomfortable, to acknowledge our fears, without allowing them to control us. It is an understanding that the refining process is essential on the path of transformation and when we lean into this discomfort, we choose to craft the outcome of our own learning story.
- A choice to be inspired and curious about new implications of research; looking for connections in current practice
- A choice to have a voice and enter into horizontal conversations that sound like a symphony of questions and have the power to ignite independent learning that continues to feed back into our own teaching practices

In the simplest way, it is a choice to authentically put into practice the learning ideals that we set for our students, understanding that in order to cultivate curiosity, creativity, innovation, and transformation within our pedagogy, we must first choose to be vulnerable.

What started as a list of characteristics has slowly evolved into a personal professional learning manifesto. It is in no way a "how-to guide" of navigating the waters of professional learning, but rather, a working document of wrestling with my reflections from this summer. This thinking does beg the question on how the province, boards of education, and schools themselves can create structures that go beyond the current model of AQ courses and annual conferences—structures that consider continuously working more intimately to cultivate authentic reflection that leads to transforming pedagogy, while at the same time holding teachers accountable for intentional choices. In closing, I will leave you with a quotation from Thomas Paine, who suggests that, *"The harder the conflict, the more glorious the triumph. What we obtain too cheap, we esteem too lightly; it is dearness only that gives everything its value. I love the man that can smile in trouble, that can gather strength from distress and grow."*

## ▲ In the Middle: Linear Leaping – Window-Pane Chart

CARLY ZINIUK



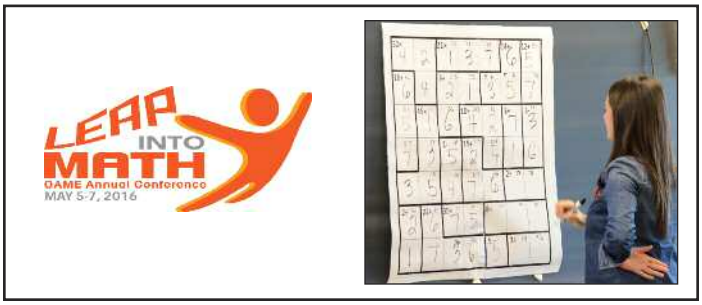
*Carly Ziniuk teaches Grade 9 Mathematics, Grade 12 Data Management, and AP Statistics at the Bishop Strachan School in Toronto, Ontario, Canada. She is very involved in using real-life data to engage her students in solving problems.*

Multiple representations of mathematics can be a great way to differentiate in a classroom. Having students look at multiple ways of expressing mathematical thinking is helpful to extend their learning as all forms of assessment. This activity extends from the more active (pun intended!) learning last issue, but in this case, this allows students to decide independently how they want to begin their learning and then connect the multiple representations.

I highly recommend these "window pane" charts as an opportunity to have students collect and connect their thinking with this activity, but also with linear and non-linear scenarios. These were collected from the previous Linear Leaping activity. When I used these with my students as all three forms of assessment, I was able to observe how much they were able to understand from just the context. You may notice the opportunity to reinforce and rehearse these at the very end, where the students can create their own window-pane charts.

Try this! Use these window-pane charts to interpret this very different scenario.

Reading experts in education use this handy formula for children's language acquisition: *A child normally starts Grade 1 with a vocabulary of 3000 words, and learns 1000 words every year.* Use this example to help you define what a linear relation is and demonstrate how you would use it mathematically and practically. ▲



**Graphical (Draw a sample sketch.)**

*Graph your data, showing the key points.*

**Algebraic (State the variables and the equation.)**

t *represents the amount of time she has exercised, and* n *represents the number of exercises completed.*

**Logical (Explain the starting value, and constant change rate in words.)**

*Somerset has a warm-up of 10 burpees and then does 4 burpees every 30 seconds. She stays quite linear for the entire 5 minutes.*

**Numerical (Complete the table, being sure to show important points.)**

*Data collected from exercising:*

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**Graphical (Draw a sample sketch.)**

*Graph your data, showing the key points.*

**Algebraic (State the variables and the equation.)**

t *represents the amount of time she has exercised, and* n *represents the number of exercises completed.*

**Logical (Explain the starting value, and constant change rate in words.)**

*Onmi has a warm-up of 20 bench-ups and then does 25 bench-ups every 30 seconds.She stays quite linear for the first 2 minutes, but then reduces her rate to 20 bench-ups every 30 seconds.*

**Numerical (Complete the table, being sure to show important points.)**

*Data collected from exercising:*

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**Graphical (Draw a sample sketch.)**

*Graph your data, showing the key points.*

**Algebraic (State the variables and the equation.)**

t *represents the amount of time she has exercised, and* n *represents the number of exercises completed.*

**Logical (Explain the starting value, and constant change rate in words.)**

*Alison has a warm-up of 5 jumping jacks and then does 1 jumping jack per second. She jumps using this pace, rather than exercising her full potential for the full experiment.*

**Numerical (Complete the table, being sure to show important points.)**

*Data collected from exercising:*

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**Graphical (Draw a sample sketch.)**

*Graph your data, showing the key points.*

**Algebraic (State the variables and the equation.)**

t *represents the amount of time she has exercised, and* n *represents the number of exercises completed.*

**Logical (Explain the starting value, and constant change rate in words.)**

*Grace knows she did 17 side-to-side jumps for warm-up. Her partner didn't keep track of the middle values and only counts for the 5-minute mark, recording a total completed of 242.*

**Numerical (Complete the table, being sure to show important points.)**

*Data collected from exercising:*

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# Three Spheres of Radii 3, 2, and 1 Mutually Tangent Problem (Part 2)

ANDRÉ LEMAIRE



*André is a retired mathematics and physics teacher, who taught at Garneau High School in Orleans, Ontario. Born in Belgium, André studied at the University of Ottawa, and is also a licensed pilot and active skydiver.*

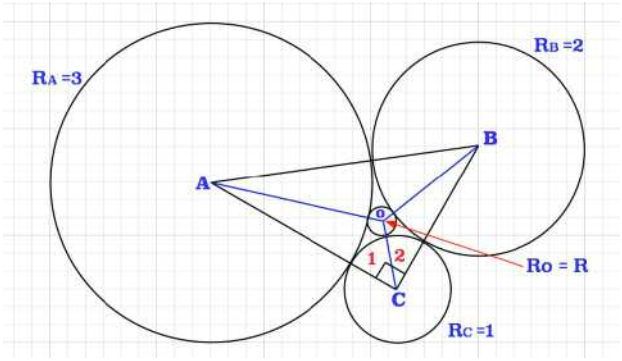
Continuing from V54 N4, I here present three other methods for solving the same classic problem: Find the maximum radius *R* of a sphere able to pass in the interstice between the three other spheres.

Centres A, B, and C form a plane. This can be considered a problem of three circles mutually tangent, with a circle of centre O in the plane, in the interstice, tangent to the three other circles.

**Method 4:** Descartes and Soddy's Formula.

If four circles are mutually tangent with radii *RA, RB, RC*, and a circle of radius *R* occupies the interstice between the three other circles (our problem), then the following relation exists (*Soddy's Formula:* **www.pballew.net/soddy.html**):

$$\frac{1}{R_A^2}+\frac{1}{R_B^2}+\frac{1}{R_C^2}+\frac{1}{R^2}=\frac{1}{2}\left(\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}+\frac{1}{R}\right)^2$$




From the formula above and the data from the sketch:

$$\frac{1}{3^2}+\frac{1}{2^2}+\frac{1}{1^2}+\frac{1}{R^2}=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{1}+\frac{1}{R}\right)^2$$

$$\Rightarrow \frac{1}{9}+\frac{1}{4}+\frac{1}{1}+\frac{1}{R^2}=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{1}+\frac{1}{R}\right)^2$$

Using the same denominator, one gets:

$$\frac{4}{36}+\frac{9}{36}+\frac{36}{36}+\frac{1}{R^2}=\frac{1}{2}\left(\frac{2+3+6}{6}+\frac{1}{R}\right)^2$$

$$\therefore \frac{49}{36}+\frac{1}{R^2}=\frac{1}{2}\left(\frac{11}{6}+\frac{1}{R}\right)^2 \Rightarrow \frac{98}{36}+\frac{2}{R^2}=\frac{121}{36}+\frac{22}{6R}+\frac{1^2}{R^2}$$

$$\Rightarrow \frac{-23}{36}=\frac{22}{6R}-\frac{1}{R^2} \text{ or } \frac{-23}{36}=\frac{22R-6}{6R^2} \textbf{ (1)}$$

From (1) and a cross-multiplication, one gets: $-138R^2 = 792R - 216$ or $\mathbf{0 = 138R^2 + 792R - 216}$

With this 2nd-degree equation, the root is given by: $d=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where $b$ = 792, $c$ = –216, $a$ = 138, keeping the positive root: $\boldsymbol{R}=\frac{6}{23}$.

**Method 5:** Analytic geometry, Heron Formula, and height of the triangle.

Consider the triangle PUO, cosine law, and point U being the centre of circle U or (*x,y*)

$$\therefore PU^2 = OU^2 + PO^2 - 2OU \times PO\cos O_2$$

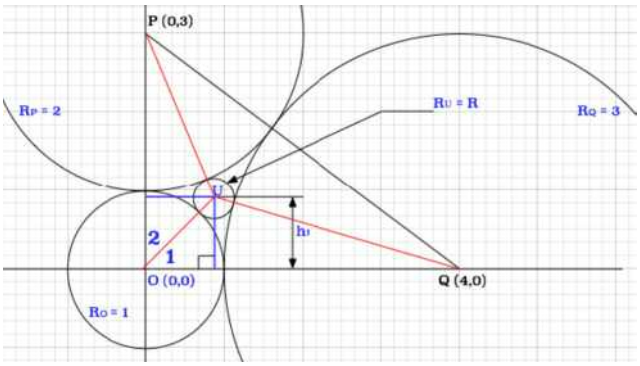
$$(2+R)^2 = (1+R)^2 + 3^2 - 2(1+R)\times 3\cos O_2$$

Developing and simplifying, one gets:

$$\cos_0.(6 + 6R) = 6 - 2R$$

$$\therefore \cos O_2 = \frac{6-2R}{6+6R}$$

$$\text{or } \cos O_2 = \frac{1}{3}\left(\frac{3-R}{1+R}\right) \textbf{ (1)}$$




Triangle OUQ and Heron Formula: perimeter of triangle OUQ = (3 + *R*) + (1 + *R*) + 4 or 8 + 2*R*

$p=\frac{1}{2}$ perimeter is then 4 + *R*

Area of triangle OUQ or

$$S_{OUQ}=\sqrt{(4+R)(4+R-(3+R))(4+R-(1+R))(4+R-4)}$$

Reducing, one gets: $S_{OUQ}=\sqrt{(4+R)(1)(3)(R)}$

$$\therefore S_{OUQ}=\sqrt{12R+3R^2} \textbf{ (2)}$$

But $S_{OUQ}=\frac{OQ.h_1}{2}$ *then* $S_{OUQ}=\frac{4.h_1}{2}$ *or* $S_{OUQ}=2h_1$ but from

(2), one gets: $h_1 = \frac{1}{2}\sqrt{12R+3R^2}$ **(3)**

Since $\sin 0_1 = \frac{h_1}{1+R}$ and angle $O$ = 90° and from (1):

$$\cos O_2 = \sin O_1 = \frac{1}{3}\left(\frac{3-R}{1+R}\right) \textbf{(4)}$$

and from (4): $h_1 = \frac{(1+R)(3-R)}{3(1+R)}$

Hence, $h_1 = \frac{1}{3}(3-R)$. From (3), one gets:

$$h_1 = \frac{1}{2}\sqrt{12R+3R^2} = \frac{1}{3}(3-R)$$

Squaring the equality:

$$\frac{1}{4}(12R+3R^2) = \frac{1}{9}(9-6R+R^2) \Rightarrow \frac{108R+27R+4R^2}{36} = \frac{36-24R+4R^2}{36}$$

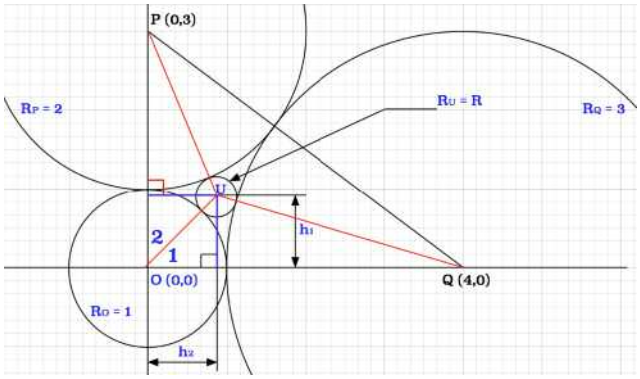
Simplifying, one obtains the following 2nd-degree equation: $23R^2 + 132R - 36 = 0$

And the solution is given by: $R = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, where $b$ = 132, $c$ = –36, and $a$ = 23.

Hence, $R = \frac{6}{23}$.

**Method 6:** Heron's Formula, analytic geometry, heights of two triangles, and Pythagoras.

**Calculation of $h_1$**, using Heron's Formula and the classic triangle-area formula:




Area of triangle OUQ

$S_{OUQ} = \frac{h_1 \times 4}{2}$; hence, $h_1 = \frac{1}{2}S_{OUQ}$ **(1)**

Calculation of $S_{OUQ}$, using Heron's Formula:

p is the $\frac{1}{2}$ perimeter; one has: $S = \sqrt{p(p-a)(p-b)(p-c)}$

$$p_{OUQ} = \frac{(1+R)+(3+R)+4}{2} = 4+R$$

$$S_{OUQ} = \sqrt{(4+R)(4+R-(1+R))(4+R-(3+R))(4+R-4)} = \sqrt{(4+R)(3)(1)(R)}$$

*or* $S_{OUQ} = \sqrt{12R+3R^2}$ from (1), one gets: $h_1 = \frac{1}{2}\sqrt{12R+3R^2}$ **(2)**

In the right triangle, with $h_1$ and angle $O_1$, one gets: $\sin O_1 = \frac{h_1}{1+R}$ = hence, $h_1$ - (1 + $R$) sin $O_1$ (3).

Calculation of $\cos O_2$: Using cosine law in triangle OUP: $PU^2 = PO^2 + OU^2 - 2PO \times OU \cos O_2$, then $(2 + R)^2 = 3^2 + (1 + R)^2 - 2 \times 3(1 + R) \cos O_2$. Developing and simplifying, one gets: $\cos O_2 = \frac{6-2R}{6+6R} = \frac{2}{3}\left(\frac{3-R}{1+R}\right)$, knowing that $\cos O_2 = \sin O_1$, and with (3), one has: $h_1 = (1+R) \times \frac{2}{6}\left(\frac{3-R}{1+R}\right)$.

Simplifying by $(1+D)$: $h_1 = \frac{1}{3}(3-R)$ **(4)**

**Calculation of $h_2$**

Triangle PUO $S_{PUO} = \frac{h_{2\times 3}}{2}$; *hence*, $h_2 = \frac{2}{3}S_{POU}$ **(5)**

$p$ is the $\frac{1}{2}$ perimeter $\therefore p_{PUO} = \frac{PU+OU+PO}{2}$ *or* $\frac{(2+R)+(1+R)+3}{2}$

$p = \frac{6+2R}{2}$; hence, $p$ = 3 + $R$

$$S_{POU} = \sqrt{(3+R)(3+R-(2+R))(3+R-(1+R))(3+R-3)}$$

(Heron's Formula)

$S_{PUO} = \sqrt{(3+R)(1)(2)(R)}$ or $S_{PUO} = \sqrt{6R+2R^2}$ **(6)**

Hence, from (5) and (6): $h_2 = \frac{2}{3}\sqrt{6R+2R^2}$ **(7)**

On the other hand: $S_{PUO} = \frac{1}{2}OU \times PO \sin O_2$ which gives: $S_{PUO} = \frac{1}{2}(1+R) \times 3\sin O_2$

$\therefore S_{PUO} = \frac{3}{2}(1+R).\sin O_2$ **(8)**

**Calculation of cos $O_1$**

cosine law: $QU^2 = OU^2 + OQ^2 - 2OU \times OQ \cos O_1$ $(3 + R)^2 = (1 + R)^2 + 4^2 - 2(1 + R) \times 4\cos O_1$ Solving and simplifying, one gets: $\cos O_1 = \frac{8-4R}{8+8R}$ $\therefore \cos O_1 = \frac{1}{2}\left(\frac{2-R}{1+R}\right) = \sin 0_2$ because $\cos O_1 = \sin O_2$ (complementary angles) (9)

From (8) and (9), one gets: $S_{PUO} = \frac{3}{2}(1+R) \times \frac{1}{2}\left(\frac{2-R}{1+R}\right)$ **(10)**

In the right triangle containing $h_2$ and angle $O_2$ plus (9), one gets: $\sin O_2 = \frac{h_2}{1+R}$; *hence*, $h_2 = (1+R) \times \frac{1}{2}\left(\frac{2-R}{1+R}\right)$

*simplifying by* (1 + R) $\Rightarrow h_2 = \frac{1}{2}(2-R)$ **(11)**

We now have two formulas for $h_1$ and also two for $h_2$. See (2) and (4) plus (7) and (11). Here they are:

$$h_1(i) = \frac{1}{2}\sqrt{12R+3R^2} \qquad h_2(i) = \frac{2}{3}\sqrt{6R+2R^2}$$

$$h_1(ii) = \frac{1}{3}\sqrt{3-R} \qquad h_2(ii) = \frac{1}{2}(2-R) \textbf{(12)}$$

By establishing these four formulas, we obtain four possibilities for finding $R$.

(I) With the equality of the two values of $h_1$ and squaring, one gets: $23R^2 + 132R - 36 = 0$

(II) With the equality of the two values of $h_2$ and squaring, one gets: $23R^2 + 132R - 36 = 0$

(III) Using Pythagoras' theorem for the right triangle containing angle $O_1$, $h_1$, and $h_2$, and using them (see (12)), one gets: $h_1^2(i) + h_2^2(i) = OU^2$ and $h_1^2(ii) + h_2^2(ii) = OU^2$ as well,

hence: $\frac{1}{4}(12R + 3R^2) + \frac{4}{9}(6R + 2R^2) = (1+R)^2$ **(III) (i)**

or $\frac{1}{9}(9 - 6R + R^2) + \frac{1}{4}(4 - 4R + R^2) = (1+R)^2$ **(III) (ii)**

Those two equalities, by solving and simplifying, both yield: $\mathbf{23R^2 + 132R - 30 = 0}$

By solving this 2nd-degree equation, one obtains: $R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $b = 132$, $c = -36$, and $a = 23$, and the result is: $R = \frac{6}{23}$.

Once again, these extra methods, in addition to the first three featured in *Gazette* V54 N4, illustrate the power of mathematics and the satisfaction that comes from persisting with a challenging problem. ▲








# ▲ Provincial Digital Learning Resources – What's New? Upgrades and Features (mathies.ca)










**MARKUS WOLSKI**

**AGNES GRAFTON**

**ROSS ISENEGGER**

**GREG CLARKE**

*Greg is from the Simcoe Muskoka Catholic District School Board and Ross is from the Near North District School Board. Both are currently on assignment with the Ontario Ministry of Education as Provincial Math Leads, working on the CLIPS project, in conjunction with their colleague, Agnes Grafton, from the Brant Haldimand Norfolk Catholic District School Board. Markus Wolski is a teacher with the Bluewater District School Board and has been working on the CLIPS project for the past two years.*


mathies
interactive electronic supports

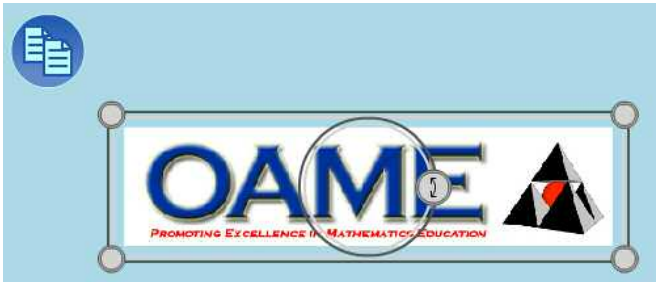
Regular *Gazette* readers will know that over the last two years, we have been busy developing learning tools for desktop as well as iOS and Android devices. In addition to programming the actual tools, we also have the opportunity to facilitate various workshops that introduce people to the tools and the mathematics that can be explored using them. As we watch participants interact with the mathies tools, we take note of any challenges or suggestions for improvement. We also value any feedback that is sent to us via the feedback button. Based on these observations and feedback, we have recently engaged in a round of upgrades to make the tools more powerful, convenient, and intuitive to use. These updated features have opened up new possibilities when exploring mathematical ideas within the tool. This article will showcase the latest updates and share some ideas about how these features might be exploited.

## Importing Images



In the last *Gazette* article, we promised this new feature, and now we are pleased to announce the inclusion of it in all available versions of Number Line, Rekenrek, Relational Rods+, Money, and Notepad. Users now have the ability to import as many pictures into the upgraded tools as they would like.

When the icon is clicked, you get a dialog that allows you to choose an image file from your device. Selected images will be imported to the tool and can then be manipulated in various ways.




To **duplicate** an image, select it and click the copy button. The duplicate will be selected so that it can be manipulated.

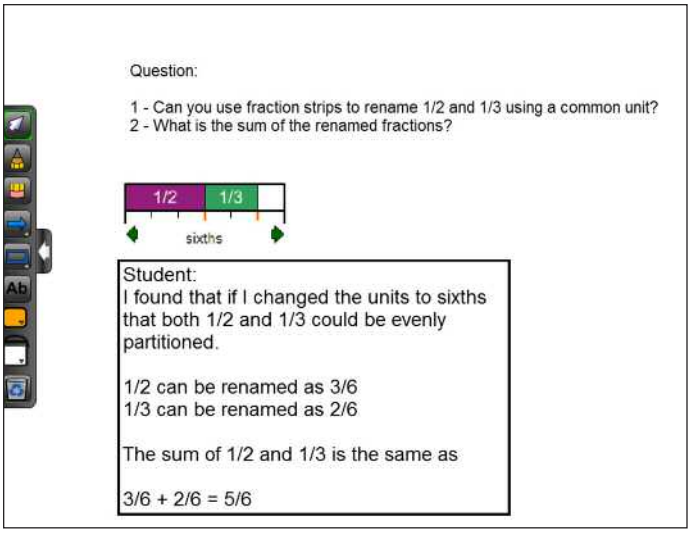
To **resize** a picture, click and drag one of the resizer handles at the four corners of the image. The image will maintain its proportions and will not shrink past a certain point.

To **rotate** an image 90°, 180°, 270°, or 0°, click and drag the rotation handle near the centre of the image.

Image import is available on all platforms (desktop, iOS, and Android) and can be used in all kinds of creative ways. You can now:

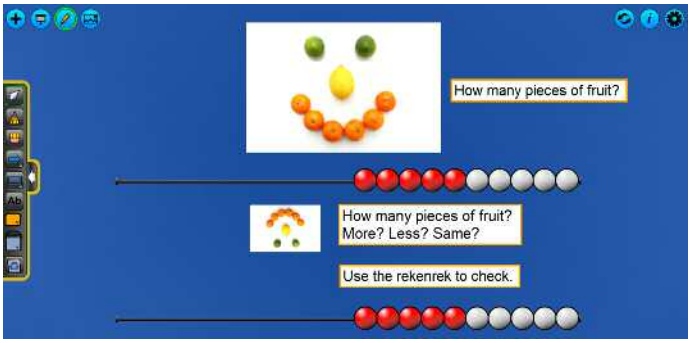
- explore a picture of a real-life situation
- take a photograph of some work with a physical manipulative and use it in a virtual tool
- import a picture of the question that the student is working on
- take a screenshot of some work with one digital tool and import it into another tool (See the link at the end of this article for how to take screenshots on various devices and operating systems.)

While the older Mathies tools do not include the annotation feature, it is possible to take a screenshot of your work and then import this image into the Notepad. The example below uses a picture taken from the Fraction Strip tool:



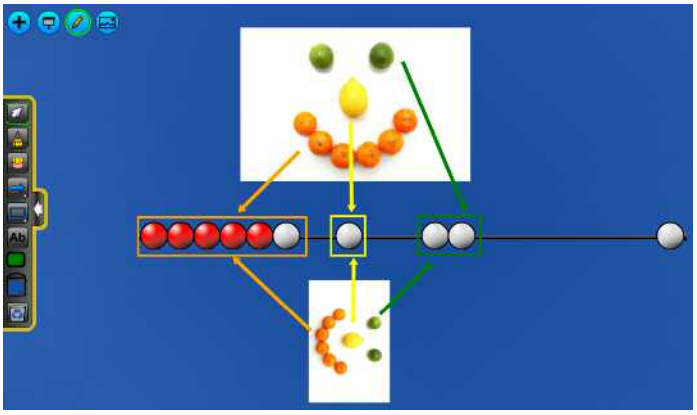

## Investigating Conservation of Number, Using Image Importing

Conservation of number and one-to-one correspondence are two important counting principles that children need to master. Young students often struggle to understand that the number of items in a set is not affected by changing the size, position, or amount of space between objects. Below is an example of how this concept could be posited to the student.




As always, teachers are encouraged to have the students use the annotation tool to explain their thinking by pressing the button. Many improvements have been made to the annotation layer, including the ability to erase across all drawn layers and to manipulate all objects, imported images, and annotations while in (selection) mode. In the example above, the student is able to interact with any of the images, rekenreks, or annotation text, without having to change modes.

By using the annotation tool, the students can also practise mapping particular fruit by colour coding the beads to convince themselves they have the right count. They can also use this technique to further convince themselves that, regardless of the orientation or size of the picture, the number of objects within it will stay the same.



As a next step, ask students to import their own sets of objects, including a picture of a set they may have created using manipulatives, and then rotate and resize the image themselves to investigate what other attributes stay fixed when the image is manipulated.

## Using Imported Pictures to Represent Data – Preserving the Average

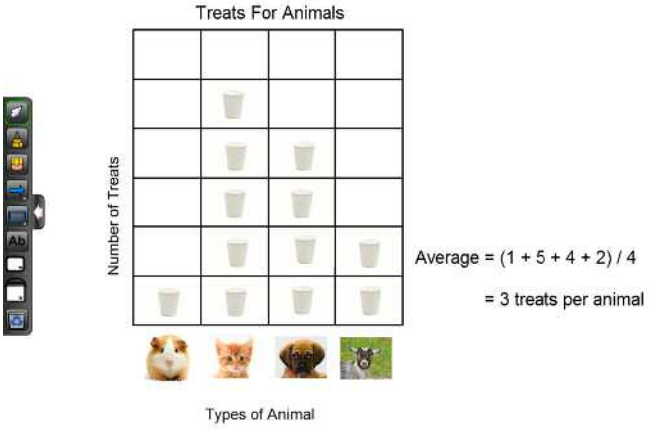
Another fun and useful way to use multiple image importing is to create pictographs. The *Mathies* Notepad tool now allows the importing of multiple pictures and has a handy opening dialog that allows the user to choose the background with which he or she would like to start. It also allows the user to choose whether to have the images sit on top of the grid (like in the example below) or underneath the grid so the grid lines explicitly partition the image.

In the example below, the windowed grid has been selected. A pictograph is used to show one solution to this problem:

*You are at a petting zoo, where it costs the same amount to give one of four animals a treat. You have enough money for 12 treats.*

*a) Distribute the treats among the animals any way you like.*

*b) Calculate the average number of treats per animal.*




Ask each student to distribute the treats in a different way and then calculate the average. As a whole class, share several different solutions and discuss the following:

1) *What do you notice about the average?*

   *It is always 3 treats per pet! Is that surprising to you? Can you explain why the average is always 3? What average would you expect if you had 24 treats to distribute? 30 treats?*

2) *If you tell your friend that you gave out an average of 3 treats per pet, could your friend tell you how many treats you gave to the kitten?*

3) *Does the average let you know how fair the distribution was?*

This activity will reinforce the idea that a total can be decomposed into many subsets, and that the average is preserved throughout this decomposition and redistribution. The task might also show that using averages can strip important details from data that provide interesting and useful information (such as a bias toward kittens and puppies).

## Importing Images for Fun and Profit – Comparing and Preserving Monetary Amounts

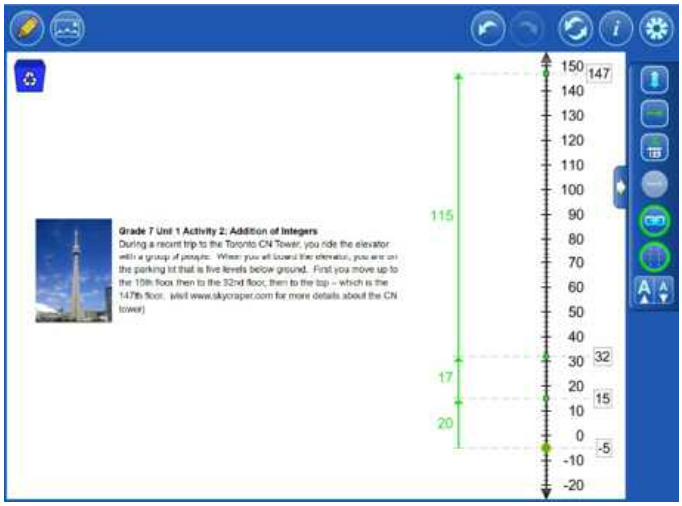
Here is an example of using the image import to compare the costs of two stuffed bears within the *Mathies* Money tool. Of course, to make it more fun in class, and to put a personal touch on the task, you could take pictures of objects in the classroom and have students pretend to sell them




Students should be encouraged to compare the values, using different strategies. Possible solutions could include using a take-away strategy, or alternately, mapping the currency values to each other (mapping a $20 bill to two 10s for instance). Other scenarios, like ordering food at a restaurant or shopping for sale items, can also be used. How much for those snazzy shoes you're wearing?

### Importing Images – Real-World Examples and Text Questions

Another advantage of being able to import images is the ability to take a picture of a question being worked on and import it right into a tool. In this way, the tool remains the site of the problem solving. After the student is done, a screenshot can be taken to preserve and share evidence of his or her work.




These are just a few of the possibilities that the upgraded functionality in the tools offers. For more detailed information, please visit our wiki.

For all functionality pertaining to image importing: **http://mathclips.wikispaces.com/How+to+Insert+Pictures**

For information about how to take screenshots on various devices: **http://mathclips.wikispaces.com/Screenshots**

For information on how the upgraded version of the annotation tool works: **http://mathclips.wikispaces.com/Annotation**

As always, please feel free to send us your feedback, using the Feedback Form button inside the Information Dialog. You can also send your comments to **WhatsNew@oame.on.ca**. You can share worked examples on Twitter, using the hashtag #ONmathies. To be among the first to find out about the latest digital tool developments, sign up for our email list at **http://mathclips.ca/WhatsNewEmailList.html**. ▲




## ▲ Hey, It's Elementary: Play and Learning: The Current Kindergarten Conundrum

LYNDA COLGAN



*Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.*

Anyone who has received an email from me over the past few years will have noted the quotation that I include in my electronic signature: *The understanding of atomic physics is child's play compared with the understanding of child's play* (David Kresh).

It generates many questions and a wide spectrum of reactions from recipients: "So true!," say some; "How ridiculous!," say others; and, "Who is David Kresh?," ask the curious.

The question is easy to answer. David Kresh (1940–2006) was an American poet, who penned three volumes of poetry, was a poetry reference specialist at the Library of Congress in Washington, DC, and was a poet-in-residence at an elementary school for 11 years. His wisdom about the complexity of play is, I am certain, attributable to his keen observation of elementary school-aged children engaged in learning and living in classrooms and playgrounds at the school in which he played a special role.

To address the reactions by readers to Kresh's quotation in my signature, I need to turn to the research on play and learning. As a committee member for a recent Queen's PhD graduate, Alexandra (Sandy) Youmans, and examiner for another, Jeffrey MacCormack, I had the privilege to read and learn much about play and its role in learning. Sandy's research examined high-fidelity Kindergarten classes (meaning that they were implemented according to the criteria delineated by the Ontario Ministry of Education) and math and reading achievement as well as self-regulation among children. Jeff's study examined the role of structured play in supporting social interaction among autistic children. My presentation of the literature on play and learning is

attributable to what I learned from their important dissertations.

In the Ontario context, play is most closely affiliated with Kindergarten. Kindergarten has a long and laudable history: founded by Friedrich Froebel in Germany during the early 1800s, using the metaphor of teachers as gardeners and children as the plants that need to be nurtured (*Kinder* meaning *children* and *Garten* meaning *garden*). Froebel's Kindergarten was based on play-based learning through the use of play materials ("gifts") and hands-on activities ("occupations") to capitalize on children's inclination toward being active. Children's early sensory experiences with play materials (blocks, balls of yarn, and geometric shapes) were believed to promote early language and spatial development because children were actively involved in understanding and describing material properties (Froebel, 1887). Froebel asserted that learning in Kindergarten needed to be valued and respected as being important in its own right, and emphasized that Kindergarten learning should not be viewed merely as "preparation" for their future schooling.

The philosophies, purposes, and goals of Kindergarten changed over time: Kindergarten became integrated into public schools in the early 1900s, and placed greater demands on students because the Kindergarten curriculum became more unified with the Grade 1 curriculum. In the mid-1950s, a distinct philosophy around Kindergarten emerged, i.e., that children needed more rigorous academics in Kindergarten. Consequently, Kindergarten became increasingly more academic in North America, with students spending less and less time involved in free play or "choice time."

Recently, this long-standing and more constrained view of early learning has been replaced by other stances. For example, in 2012, the Council of Ministers of Education for Canada released a statement on play-based learning in full support of using purposeful play to promote positive outcomes for children; and, in Ontario, the Full-Day Early Learning – Kindergarten (FDELK) Program, implemented in 2010/2011, adopted a play-based curriculum.

In its FDELK program document, one of the fundamental principles asserts that *play is a means to early learning that capitalizes on children's natural curiosity and exuberance*. While the Ontario Ministry of Education does not provide an explicit definition of the term "play-based learning," it does provide examples: pretend play (e.g., acting out roles or scenarios), socio-dramatic play (e.g., storytelling), and constructive play (e.g., drawing, building, and painting). The FDELK document adopts some oft-cited, research-based tenets of play: (a) it is self-directed; (b) it is intrinsically motivating; (c) its structure and rules are defined by the players; (d) it is imaginative and non-literal; and (e) it requires active engagement, but is stress-free.

And therein lies the problem. These underpinnings assume that *all* play involves learning. They fail to address the underlying cognitive processes by which play can actually translate into learning. In 1962, Piaget emphasized that most child-directed play involves what children already know and, as a result, is assimilative in nature—it does not result in new knowledge, skills, and understanding. Many contemporary cognitive psychologists concur, contending that if play is to advance learning, children must encounter new ideas and problems that do not fit with their prior knowledge, experience uncertainty, and work toward accommodating the novel information with prior knowledge.

To transform simple *play* into *play-based learning*, research on play-based learning has underscored the importance of adult interaction to support children's learning during play. Studies on *guided play* (sometimes called *structured play*) suggest it may promote more favourable academic outcomes than free play because it involves adult-scaffolded instruction in a child-led context. In *guided play*, adults initiate and participate in play with clear learning goals, ask open-ended questions, co-play, confirm children's discoveries, and encourage the use of materials in new and innovative ways.

However, the balance is fine: when too much or too little adult interaction occurs in a guided play, play-based environment, learning is impeded. In guided play, research has shown that teachers must present materials strategically, give timely and appropriate feedback, and probe for understanding, without taking control of the discovery aspect of learning. And there is consensus: ultimately, the effectiveness of play-based learning hinges on the knowledge, support, and guidance of educators. In terms of mathematics education, research suggests that this calls for a Kindergarten classroom teacher who has: (a) an appreciation of the discipline of mathematics itself, and of what it means to do mathematics; (b) an understanding of how children learn mathematics; (c) nurtured a problem-solving environment for learning mathematics; and (d) an extensive menu of strategies for integrating assessment into teaching to enhance both learning and instruction.

In spite of the subtleties around the concept of play itself and the complexities of the teacher's role in orchestrating guided play in a natural context, Full-Day Kindergarten (FDK) for 5-year-olds has been implemented in many parts of Canada. The majority of provinces (6 of 10: British Columbia, Ontario, Prince Edward Island, Quebec, New Brunswick, Nova Scotia) and one of three territories (Northwest Territories) offer FDK programs, while

Newfoundland and Labrador is preparing for implementation in September 2016.

As a country, we are forging ahead with play-based full-day early learning Kindergarten programs, but are we doing so at our peril?

The quantitative part of Dr. Youman's study, *An Investigation of the Features and Effectiveness of the Full-Day Early Learning Kindergarten Program in Ontario*, used hierarchical regression analyses of 32 207 students' self-regulation, literacy, and numeracy outcomes to test the hypothesis that FDELK would improve students' achievement in these three areas. Unfortunately, Sandy's study concludes this way: *If anything, children in the FDELK program performed more poorly on self-regulation and numeracy outcomes. These were all small effects, but significant because of the large sample size*. This finding was more negative for children from low socio-economic backgrounds—a demographic long assumed to benefit from FDELK. The qualitative complement of her study suggests that many of the FDELK teachers in her sample self-reported that they did not understand how to implement play-based learning, nor did they have the knowledge about or strategies for using play-based learning to support academic outcomes in Numeracy. Could this confusion and lack of preparedness contribute to the fact that many children are not actually learning anything new with respect to mathematics in FDELK?

As a researcher, I know the dangers of speculation; however, as I read and reread Sandy's dissertation, and reflect on the state of the curriculum in Ontario over the past few years, especially as it relates to Kindergarten, I must ask myself: Could the less than promising results from her study (especially in terms of numeracy) be due to the fact that expectations outlined in the Ontario Kindergarten curriculum did not change along with FDELK program implementation? While the amount of time FDELK students spent in school and the focus of the program changed, numeracy expectations around what students should know and be able to do by the end of the program did not change.

Essentially, the FDELK program provides more time for students to learn the same things expected of their peers who were in half-day programs. Not promoting a greater focus on numeracy in the FDELK program is disappointing, given that studies on school readiness skills stress the critical role of early numeracy skills in preparing students for later success.

Without having an age-appropriate rigorous curriculum for mathematics in the FDELK context, children may not be "stretched" to achieve the mathematics learning milestones appropriate for their age.

In a must-read document for all early mathematics classroom teachers and curriculum leaders, *Foundations for Numeracy: An Evidence-based Toolkit for Early Learning Practitioners* (**www.eyeonkids.ca/docs/files/foundations_for_numeracy.pdf**), we read that for the average Kindergarten child (aged 4 to 6 years), counting and quantity skills develop quickly and become more abstract. Children can count up to 100, skip count first by 10's (10, 20, 30, etc.) and later by 5's and then 2's. They begin to count up, starting with numbers other than 1 ("8, 9, 10..."). They starting to think about the relationship between individual items (like wooden sticks) and groups of ten items (e.g., a bundle of ten wooden sticks). They are forming the foundation for learning place value, and reliably produce the correct answers to addition and subtraction word problems with sums to about 5 (e.g., I had three dinosaurs and I got two more for my birthday. Now how many do I have?). Kindergarten children know the doubles up to 10 (1 + 1 is 2, 2 + 2 is 4, etc.) and can divide 100 things equally among ten children. They can quickly tell that there are five objects on the table without counting them, and recognize common patterns up to 10, like dots on number cubes or dominoes. They can use several different ways to measure, compare, and reproduce the length of objects, and talk about comparisons of quantities by using words such as *taller, shorter, skinnier, fatter,* and *wider*.

Although early mathematics and cognitive researchers concur that the paragraph above describes the kind of knowledge, skill, and procedural fluency that Kindergarten children should be demonstrating, the Ontario Kindergarten mathematics expectations for numeracy are notably discrepant, requiring children to meet a much lower standard than that which the research shows to be possible. A review of the Number Sense and Numeration expectations for Ontario children progressing through the Kindergarten years shows that our 4- to 6-years-olds are required to work with smaller numbers and simpler concepts. As the Ontario document states, by the end of Kindergarten, children will:

a) recognize some quantities without having to count, using a variety of tools (e.g., dominoes, dot plates, dice, number of fingers) or strategies;

b) begin to use information to estimate the number in a small set;

c) use, read, and represent whole numbers to 10 in a variety of meaningful contexts;

d) use ordinal numbers in a variety of everyday contexts;

e) demonstrate an understanding of number relationships for numbers from 0 to 10, through investigation;

f) investigate and develop strategies for composing and decomposing quantities to 10;

g) explore different Canadian coins, using coin manipulatives;

h) demonstrate understanding of the counting concepts of stable order (that is, the concept that the counting sequence is always the same: 1 is always followed by 2, 2 by 3, and so on;

i) begin to make use of one-to-one correspondence in counting objects and matching groups of objects; and

j) investigate addition and subtraction in everyday activities through the use of manipulatives, visual models, or oral exploration.

Lev Vygotsky of the Zone of Proximal Development (ZPD) fame said that children learn best when new material is not too easy, and just challenging enough that, with a little help from a more learned individual, they can master the material and move their learning forward.

We hear less about the positive impact of challenging students to meet Vygotsky's *potential developmental level*, the high achievement that can be attained as the learner validates and extends knowledge in response to challenges that generate dissonance, i.e., "stretching" them in response to new tasks.

In sharp contrast to studies about the value of positive, challenging expectations, research shows that when learners are confronted by tasks that are *too easy* or not challenging enough, they experience boredom and underachievement (i.e., staying at the surface level of understanding). It appears that when learners are not forced to confront their own knowledge gaps via challenging new material, and are not required to reconcile those gaps, they simply recognize the material and assume they know it; thus, investing and learning less.

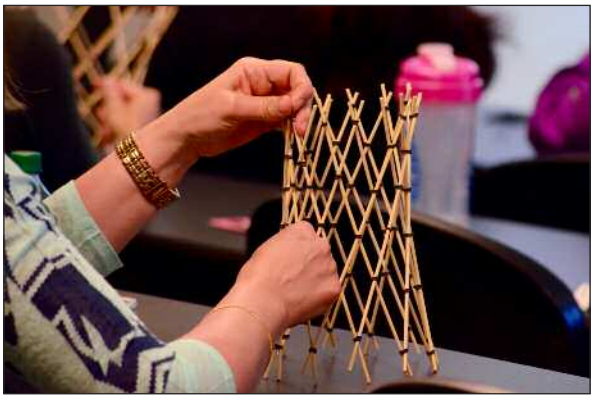
However, real learning happens when students wrestle a bit with content, while being supported by the help of (in Vygotsky's words) a *more knowledgeable other*. I believe that these two conditions define important foundations for our Kindergarten program in Ontario.

First, the Kindergarten curriculum must be challenging enough to allow the learner to expand his or her knowledge, moving well beyond pre-school expectations. Second, we must invest heavily in professional development for our Kindergarten teachers, for they are the individuals charged with empowering children to solve a task or achieve a goal that would be beyond their unassisted efforts—controlling those elements of the task that are initially beyond the students' capability and ultimately orchestrating the context so that learners concentrate upon and complete only those elements that fall within their comfort/confidence level and scope of knowledge. A tall order indeed.

As Vygotsky wrote: "Through others, we become ourselves." That may become the new quotation on my email signature: a constant reminder of the fact that when teachers are equipped with a developmentally appropriate curriculum and an internalized suite of teaching/assessment strategies, *all* students, including Kindergartners, can become the mathematicians they have the potential to be. ▲







# ▲ What's the Problem? Free-Throw Fractions

SHAWN GODIN



*Shawn is the head of Mathematics, Business, Law, and Computer Science at Cairine Wilson Secondary School in Orleans, Ontario. He is a past member of the OAME Board of Directors, and past Editor-in-Chief of* Crux Mathematicorum*. He currently writes problems for the Fryer, Galois, Hypatia, and Canadian Open Mathematics Challenge contests.*

Welcome back to another school year, problem solvers. Last time, I left you with the following problem:

> Alice and Bob are practising their free-throws. In the morning, Alice makes $\frac{1}{4}$ of her shots and Bob makes $\frac{1}{5}$ of his. In the afternoon, Alice makes $\frac{3}{4}$ of her shots and Bob makes $\frac{3}{5}$ of his. At the end of the day, which person makes the larger portion of their total shots?

I must be losing my edge. Alice makes 25% of her morning shots and 75% of her afternoon shots, while Bob makes 20% of his shots in the morning and 60% in the afternoon. Alice makes a larger portion of her shots in the morning and a larger portion of her shots in the afternoon, so she must make a larger portion of her shots…, mustn't she?

Fortunately, it isn't that easy. As usual, a little computation will help to see some patterns and may lead to a method to solve the problem.

Let's start with Alice. Since she makes exactly $\frac{1}{4}$ of her shots in the morning and exactly $\frac{3}{4}$ of her shots in the afternoon, we notice that the number of shots taken in both the morning and the afternoon must be multiples of 4. The table in Figure 1 looks at some possibilities for Alice.

| | | AFTERNOON SHOTS | | | | |
|---|---|---|---|---|---|---|
| | | 3/4 | 6/8 | 9/12 | 12/16 | 15/20 |
| MORNING SHOTS | 1/4 | **4/8 = 50%** | 7/12 ≈ 58.3% | 10/16 = 62.5% | 13/20 = 65% | 16/24 ≈ 66.7% |
| | 2/8 | 5/12 ≈ 41.7% | **8/16 = 50%** | 11/20 = 55% | 14/24 ≈ 58.3% | 17/28 ≈ 60.7% |
| | 3/12 | 6/16 = 37.5% | 9/20 = 45% | **12/24 = 50%** | 15/28 ≈ 53.6% | 18/32 ≈ 56.3% |
| | 4/16 | 7/20 = 35% | 10/24 ≈ 41.7% | 13/28 ≈ 46.4% | **16/32 = 50%** | 19/36 ≈ 52.8% |
| | 5/20 | 8/24 ≈ 33.3% | 11/28 ≈ 39.3% | 14/32 ≈ 43.8% | 17/36 ≈ 47.2% | **20/40 = 50%** |

Figure 1

A couple of things should start to become clear. First, if you were tempted to average the rates for Alice, you would get $\frac{\frac{1}{4}+\frac{3}{4}}{2}=\frac{1}{2}=50\%$. This only occurs if Alice took the same number of shots in the morning as in the afternoon. These are the boldface entries on the diagonal of Figure 1. Next you will notice that if an entry is below the diagonal, since Alice took more shots in the morning than in the afternoon, then the total portion of made shots is less than 50% (yet larger than 25%). Similarly, if Alice took more shots in the afternoon, her total portion of shots made will be larger than 50% (but smaller than 75%). In general, the larger the ratio of morning shots to afternoon shots, the closer the overall portion is to 25%, and the larger the ratio of afternoon shots to morning shots, the closer the overall portion is to 75%.

This leads us to an interesting aside. If we have two unequal fractions and we add them incorrectly by just adding the numerators and adding the denominators, the resulting fraction is actually *in between* the original two fractions. For example, starting with $\frac{2}{3}<\frac{3}{4}$, adding these two fractions improperly gives $\frac{5}{7}$. Yet, we have $\frac{2}{3}=\frac{56}{84}$, $\frac{3}{4}=\frac{63}{84}$, and $\frac{5}{7}=\frac{60}{84}$, so $\frac{2}{3}<\frac{5}{7}<\frac{3}{4}$. The new fraction is called the mediant of the other two and in general, if $\frac{a}{b}<\frac{c}{d}$, then $\frac{a}{b}<\frac{a+c}{b+d}$. This can be proved with a little algebra, which I leave as an exercise to the reader.

An interesting property of the mediant is that if we start with the fractions $\frac{0}{1}$ and $\frac{1}{1}$ and successively create the mediants, keeping only the new fractions with denominator one more than the largest denominator at the last stage, we end up with a *Farey sequence*, which contains all reduced fractions with denominators up to a certain point, arranged from smallest to largest. For example, the fifth Farey sequence would be

$$F_5=\left\{\frac{0}{1},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{1}{1}\right\}$$

Farey sequences have all kinds of interesting properties that can be explored.

Let's return to the original problem. Without doing any further calculations, we know that Bob's final portion of made shots will be somewhere between 20% and 60%, depending on the ratio of shots in the morning to shots in the afternoon. Thus, it should be possible for Alice to have made the greater portion, or for Bob to have made the greater portion or, possibly, for them to have made the same portion.

| MORNING SHOTS | AFTERNOON SHOTS | | | | |
|---|---|---|---|---|---|
| | 3/5 | 6/10 | 9/15 | 12/20 | 15/25 |
| 1/5 | 4/10 = 40% | 7/15 ≈ 46.7% | **10/20 = 50%** | 13/25 = 52% | 16/30 ≈ 53.3% |
| 2/10 | 5/15 ≈ 33.3% | 8/20 = 40% | 11/25 = 44% | 14/30 ≈ 46.7% | 17/35 ≈ 48.6% |
| 3/15 | 6/20 = 30% | 9/25 = 36% | 12/30 = 40% | 15/35 ≈ 42.9% | 18/40 = 45% |
| 4/20 | 7/25 = 28% | 10/30 ≈ 33.3% | 13/35 ≈ 37.1% | 16/40 = 40% | 19/45 ≈ 42.2% |
| 5/25 | 8/30 ≈ 26.7% | 11/35 ≈ 31.4% | 14/40 = 35% | 17/45 ≈ 37.8% | 20/50 = 40% |

Figure 2

We saw in Figure 1 that if Alice took the same number of shots in the morning as the afternoon, it seems she would always have made 50% of her shots. We can prove this by letting the number of shots she took in the morning and afternoon be $4k$ for some positive integer $k$. Then in the morning, she made $k$ shots, and in the afternoon, she made $3k$ shots, so her total portion of shots made is $\frac{k+3k}{4k+4k}=\frac{4k}{8k}=\frac{1}{2}=50\%$.

Examining the final portions for Bob in Figure 2, we see that if Bob made 1 out of 5 shots in the morning and 9 out of 15 in the afternoon, he made 50% of his shots during the day. So it is possible for Alice and Bob to make the same total portion of shots over the course of the day. You can prove that if Bob takes three times as many shots in the afternoon as morning, he will make 50% of his shots, using a method similar to the one we used for Alice.

Just examining the two tables we can find other instances of equal final ratios:

- Alice makes 5 of 20 in the morning and 3 of 4 in the afternoon; Bob makes 2 of 10 in the morning and 3 of 5 in the afternoon (or 4 of 20 in the morning and 6 of 10 in the afternoon); both make one-third of their total shots.
- Alice makes 4 of 16 in the morning and 3 of 4 in the afternoon; Bob makes 5 of 25 in the morning and 9 of 15 in the afternoon; both make 35% of their total shots.
- Alice makes 3 of 12 in the morning and 6 of 8 in the afternoon; Bob makes 3 of 15 in the morning and 15 of 25 in the afternoon; both make 45% of their total shots.

We found that if Alice takes an equal number of shots in the morning and afternoon, it is possible for Bob to match her total portion for the day. If Bob takes an equal number of shots in the morning and afternoon, is it possible for Alice to match Bob's total portion for the day? Since Bob would make 40% of his shots and $25\% < 40\% < 75\%$, it seems that it may be possible for Alice to match his total portion. Let's see if we can determine when that would occur.

Let's let Alice take $4m$ shots in the morning and $4a$ shots in the afternoon. Then, her total portion is $\frac{m+3a}{4m+4a}$. Matching her portion to Bob's 40% gives $\frac{m+3a}{4m+4a}=\frac{2}{5}$ which, upon multiplying both sides by $20(m + a)$ and simplifying, yields $5m + 15a = 8m +8a$. Isolating the two variables gives us $3m = 7a$ which, since the variables are both integers and since 3 and 7 have no common factors, tell us that $m$ must be a multiple of 7 and $a$ must be a multiple of 3. Hence, $m = 7k$ and $a = 3k$ for some integer $k$. Checking the easiest case, $k = 1$ yields that Alice made 7 of 28 shots in the morning and 9 of 12 shots in the afternoon, so her total portion of shots is $\frac{16}{40}=\frac{2}{5}$ = 40% just like Bob.

We can take things one step further and determine under what conditions the two will have equal portions of made shots by the end of the day. If we let $4m_A$, $5m_B$, $4a_A$, $5a_B$ represent, respectively, the number of shots Alice and Bob have taken in the morning and the number of shots they've taken in the afternoon, then setting the total portions equal yields

$$\frac{m_A+3a_A}{4m_A+4a_A}=\frac{m_B+3a_B}{5m_B+5a_B}$$

which, after a bit of algebra, gives

$$m_Am_B-7m_Aa_B+11a_Am_B+3a_Aa_B=0$$

We seek solutions to the above equation in integers. We saw earlier that the ratio of morning to afternoon shots was important in the solution, so we can rewrite the equation as

$$\frac{m_A}{a_A}=\frac{11\left(\frac{m_B}{a_B}\right)+3}{7-\frac{m_B}{a_B}}$$

Since all our variables are positive integers, the denominator of the right side tells us that we must have $7-\frac{m_B}{a_B}>0$. A quick check (I will leave the details to the interested reader to verify) reveals that if $7-\frac{m_B}{a_B}=0$ that is, if Bob takes 7 times as many shots in the morning as afternoon, then Bob's overall portion of made shots would be 25%, which would be impossible for Alice, since she takes some shots in both the morning and afternoon. We could do a similar analysis on the ratio of Bob's shots and find that $\frac{m_A}{a_A}>\frac{3}{7}$, or else the portion of Alice's total shots will be at least 60%, which is impossible for Bob to match.

We have started with a problem involving simple fractions, which was very open ended. There were things to

be discovered by the student who could just do a few calculations, and yet it was open enough that, with some algebra and determination, lots of generalizations could be made and we can see into the heart of the problem.

And now for your homework.

If $x$ and $y$ are real numbers, determine the minimum possible value of the expression

$$(x+3)^2 + 2(y-2)^2 + 4(x-7)^2 + (y+4)^2$$

Until next time, happy problem solving. ▲

## Summer 2016 Math Adventure #2 – Educator and Leadership Institute, Haiti

**BY JHONEL MORVAN, ONTARIO MINISTRY OF EDUCATION**

Summer is generally a time when educators choose to replenish their energy by spending time with their families or by disconnecting from teaching-related activities. A select group of these hard-working professionals sometimes elect to engage in professional learning, service learning, or volunteer activities. I have been involved in all of the above this summer. The following is a summary of one of these activities:





**ELI 2016**

After close to three years of planning, the Educator and Leadership Institute (ELI) was launched in Cap Haitian, Haiti this summer. This project, a five year commitment, is an initiative of a Wilfrid Laurier University professor, a mathematics Education Officer, and a former Senior Policy Adviser from the Ontario Ministry of Education. At the request of local partners, the Institute offered training sessions to more than 200 participants in mathematics, science and technology, critical literacy, special education, early childhood education, and leadership. Each of these six subject areas was facilitated by a pair of Ontario certified and experienced teachers, who dedicated their time and resources to give a helping hand to their Haitian colleagues over the course of the first week of August. The project included a camp in the morning for more than 100 children aged between 4 and 15. These children became students in the afternoon to allow a practicum for the Haitian teachers, who had the opportunity to put into practice what they have learned in the morning.

For more information, please contact **morvanja@gmail.com**.

## ▲ Mathematics More Than a Utility



**TIM SIBBALD**

*Dr. Timothy Sibbald is an assistant professor at the Schulich School of Education, Nipissing University. His interests focus on classroom instructional issues, content development and delivery, as well as teacher development. He is currently Past-President of OAME.*



**TOM GRIFFITHS**

Tom served as President of OAME in 1975, and was awarded a Lifetime Membership in 1994 at the Kingston OAME Conference. Tom is a Descartes medalist (1984) and was a Coach of the Canadian IMO team in 1985. Tom is retired, has taught mathematics for 54 years, and is still loving it. He has been involved in the Maths Challenge at Western University since 1996. Tom is currently enjoying being a very proud Great Grandpa.

Mathematics, as taught at the elementary and secondary levels of education, is often considered a utility subject. The goal of the elementary curriculum is to provide the knowledge and skills needed for basic numeracy encountered in daily living and courses in secondary school. The curriculum in secondary school emphasizes the provision of skills and knowledge needed in the workplace, college, or university. While universities facilitate learning mathematics as a subject in its own right, it is more often taught as a service subject for other disciplines, such as engineering or sciences in general. This was the case, as the first author can attest through his involvement in provincial curriculum committees from the 60's to the 80's. There were some exceptions, such as the then new topic of transformation; however, the main direction was still as preparation for future courses.

The same concern is valid today, with applications being a significant category of achievement in the current curriculum. In addition, many articles in the *Gazette* in the last few years tie to practical or contextual implementations. The inclusion of technology has been primarily a tool for

practical skills. Calculating mortgages comes to mind, whether with a Time-Value-of-Money calculator or using a spreadsheet.

The curricula over the years have become more directed, and prescriptive. There has continued to be an allowance for the teacher to teach ideas as he or she chooses, but the specifics of teaching some topics is now prescribed in a manner that affects pedagogical choices. One example of this is trigonometry, with the introduction through the ratios of sides in a right-angled triangle in Grade 10, which is then extended to obtuse triangles, using the sine and cosine law, and finally connected to the unit circle in order to explain continuous graphing in Grade 11. There are benefits to this, but the prescriptive elements added to the curriculum have essentially removed the option of teaching using the unit circle from the beginning.

Another example of how the curriculum has changed is the removal of math history. This is an ongoing challenge to teachers because there is a rich history, but how to use it for teaching is problematic, the issue being that it is gender biased and largely a Eurocentric history. However, a subject without a history seems to have a diminished capacity to give students something to which they can aspire. In this particular case, the curriculum is prescriptive, but with a clear social mandate to promote equity. To the extent one can include equitable history of mathematicians, it may be beneficial to do so. In the case of trigonometry, the reason for the level of prescription is far less clear.

Without a doubt, mathematics has been a means to develop technology throughout history. Often mathematical research has created the knowledge needed for technological innovation long before the need arose. The mathematical ideas were typically not developed for the practical purpose, and actually were developed as a mental exercise. An example of this is the Boolean algebra, which was in place in the late nineteenth century, and which was subsequently an enormous asset for those developing electronic computers, but Boole developed it simply as an interesting algebra.

This suggests that it may be important to instill in our students a recognition that some mathematics may simply be for a cognitive challenge without application. This is to suggest that sometimes abstraction may be a refinement to a "simple" form that realistically portrays the refinement of a task through the removal of contextual factors. That potentially has benefits that require a careful weighing of how much context and utility teachers should offer. Note that it is not to suggest that scaffolds or useful purposes should be eliminated: only that now and again we should encourage some math for its own sake, without promoting it as necessarily always having a specific utility.

Mathematics has also been a main area of development of ideas and logical thought throughout history. The development of music has given excitement to hearing, painting to sight, dance to motion, the culinary arts to taste, and mathematics has given joy to the brain. Mathematics is an endeavour in which the participant can achieve the joy of total understanding and complete logical insight, and it may be the only subject that allows this. When one solves a problem in mathematics, and completely understands it, the appreciation is far more complete and specific than in any other area of study. Do we give our students enough opportunity to experience and appreciate this? Perhaps the increased emphasis on utilitarian mathematics has contributed to students increasingly expressing a disinterest in math. When mathematicians are asked why they do what they do, they say that it is for fun and satisfaction. They do not always say that they do it for some useful purpose, unless they are working on a problem associated with a "real life" scenario.

Some examples of this fun with the mind for our students can be found in number theory and especially in Euclidean geometry. In number theory, the formula for the sum of the first $N$ natural numbers (i.e., $1+2+3+\ldots+N$) can be explained in several different and simple ways, as can the sum of the first n odd natural numbers. These can be approached in different ways, and the student can follow an inquiry approach to both discover the formulae and investigate the history. A similar problem that also demonstrates the applicability of patterns to more than one problem is the number of handshakes possible with $N$ people. While this sounds practical, it is not for larger values of $N$. It is accessible to students by considering the number of line segments possible in a polygon with $N$ vertices. Modular arithmetic, based on remainders when dividing by a specified value, also provides opportunities, particularly when combined with the Fibonacci numbers. These are problems that young people can solve themselves, as well as experiencing a total understanding of the solution.

Euclidean geometry is possibly the richest area for this kind of understanding, and has been a joy to mathematicians for thousands of years. Yet, with geometry having a reduced role in the curriculum, and the increased use of dynamic geometry software, there is a need to consider if an unintended consequence has been a diminishing of the experience of students achieving logical insight. The days of accurately constructing a geometric picture, using a straightedge and compass, have clearly been left behind, and the question is whether a wealth of student opportunities may also have been left behind.

The goal of this opinion piece is to promote discussion of some of the changes that have taken place over the last half-century. We have seen significant changes, but sometimes it is worthwhile to take a long view and ask if the incremental decisions have led to a sound overall decision. The uncomfortable sense that brought this article into being is that mathematics has migrated toward a utility-based approach and diminished the beauty and personal sense of accomplishment that arises when one finally finds a way to solve a purely mathematical problem. ▲











# ▲ Technology Corner: Dudamath



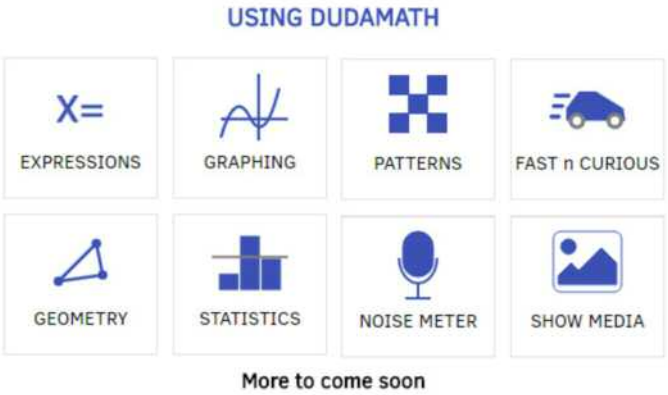
**MARY BOURASSA**
**EMAIL: mary.bourassa@oame.on.ca**

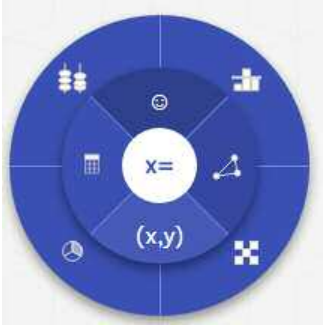
*Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.*

Dudamath (**www.dudamath.com**) is very powerful, free, online software that is useful to both students and teachers. It allows students to explore mathematical concepts at the elementary and secondary levels, and helps teachers quickly create graphs, geometric sketches, patterns, and more. It was created by Ethan Hall, an engineer-turned-teacher, who is committed to improving this software with feedback from you. From the website: "Dudamath is an integrated environment for interactive exploration of mathematical concepts and problem solving."

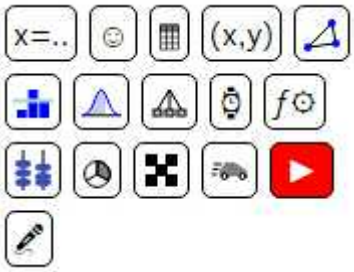
Clicking on the question mark brings up the menu shown to the right, which allows you to access to great, short videos on each of Dudamath's widgets. Each video walks you through the main features so that you can quickly start exploring on your own.

To start exploring, click anywhere on the blank canvas, and the tool shown below will pop up. Each section is a different widget.

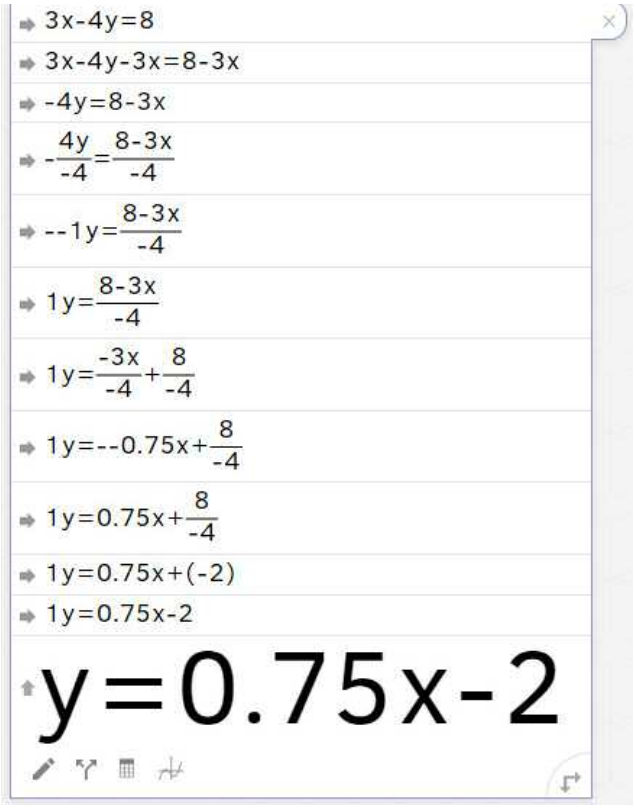




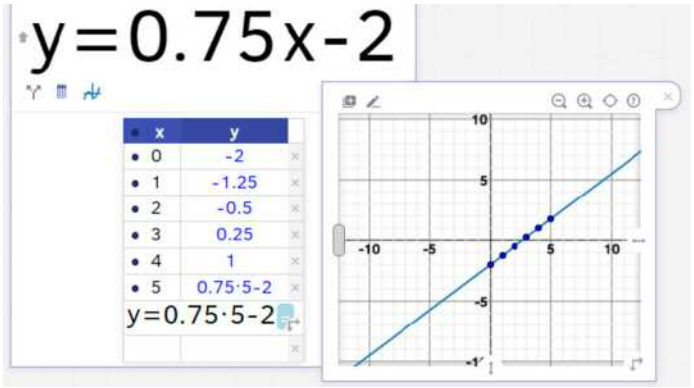

You can also click on the chevron at the right of the screen to pull up icons for each of the widgets, along with many pre-made formulae, such as the quadratic formula, cosine law, and equation of a circle.

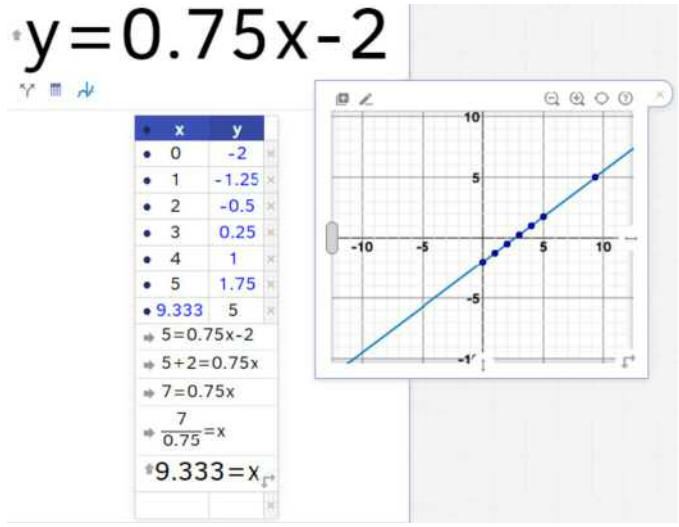



Let's start our exploration of Dudamath with algebra. Dudamath has a good computer algebra system (CAS) built in. Although it can do the work for you (or your students), you can input each step (for example, subtract 3ˆ from both sides) as shown below.

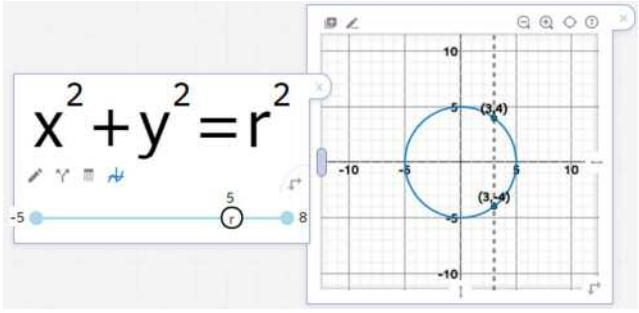



Clicking on the table icon produces the associated table of values. I really like that it shows the substitution of the value of the variable into the expression so that students can see where the results are coming from. For the example shown below, if you enter a value for *x*, you can click on the equal sign to have the expression evaluated. If you enter a value for *y*, you have to work through solving for *x*. You can choose whether to plot the points as you add values, and can draw the line representing the function with a simple click on the graph icon.

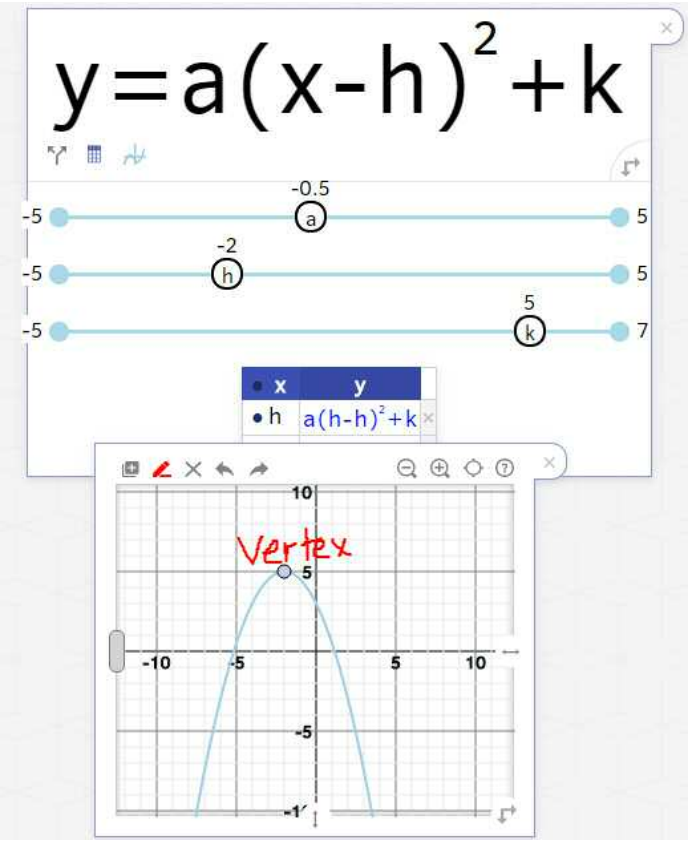



Dudamath will also work with relations, and adds sliders automatically. This example shows the equation of a circle with the radius controlled by a slider. For any graph, you can pull a vertical line from the left side of the widget, and it will show the coordinates of intersection with the graph.

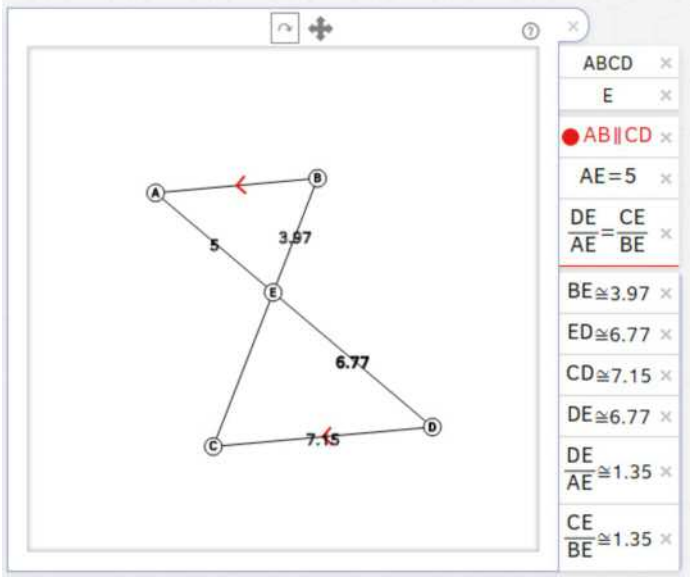


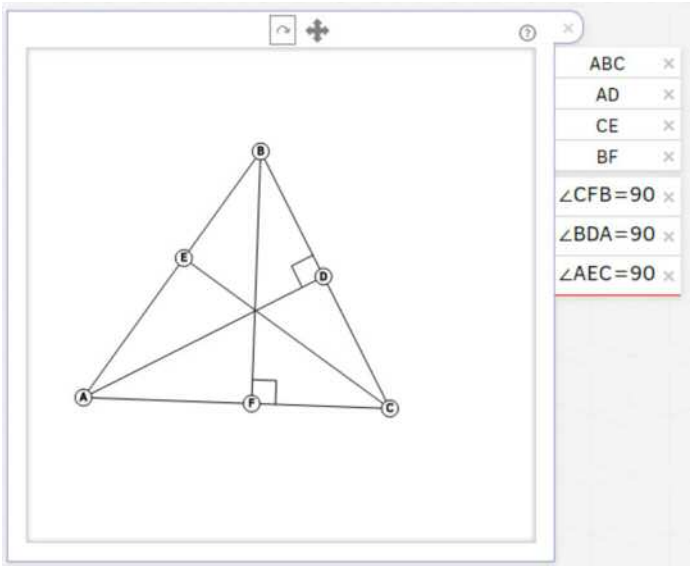
Here is another example with sliders. They automatically pop up when you click on the graph icon. You can adjust the end values as needed. You can also write on your graph, using the pencil icon.




There are so many more features for you to explore in the graphing menu, including series graphs and derivatives.

The geometry tools are very good, although they are currently limited to polygons. You can draw a shape freehand and it will make your lines straight and label the vertices. You can easily set the measure of angles and the length of sides, including defining sides or angles to be equal. All other side lengths and angle measures will be based on the ones you defined. You can define sides to be parallel and it adds the arrows to the segments to show that they are parallel. You can also enter expressions for it to calculate.



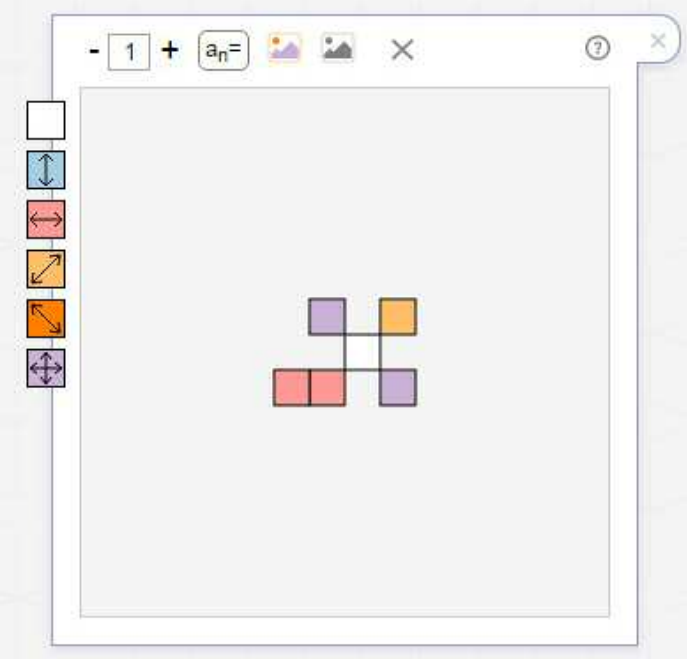



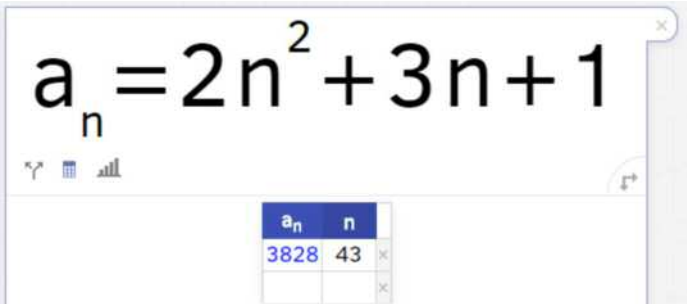

If you love using visual patterns (**www.visualpatterns.org**), Dudamath is the fastest way to make your own. Click on the pattern widget and then add blocks to make step 1 of



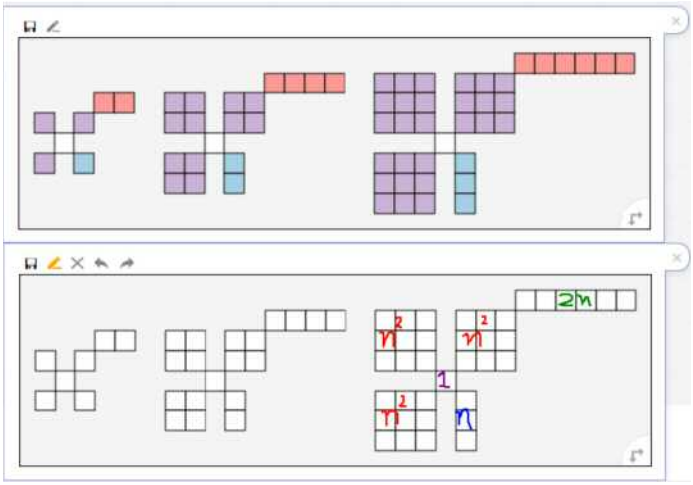


your pattern. Clicking on the + icon will show you steps 2, 3, 4, etc. You can also easily see the algebraic representation of your pattern and can pull up a table to calculate the number of blocks at a particular step.

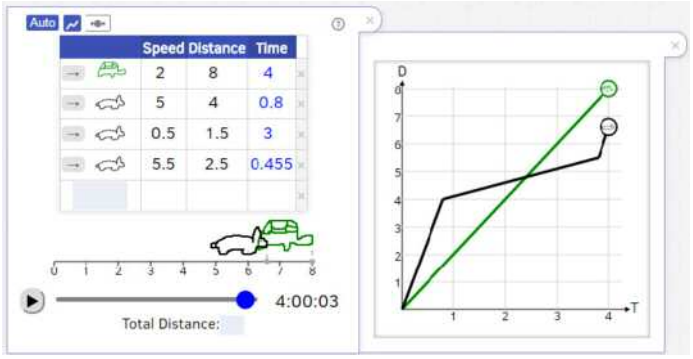





You can see the first three steps in colour or black and white with the click of a button, which you can also save as a .png file. These can also be annotated using the pencil tool.



The latest addition to Dudamath is called Fast & Curious. This widget lets you create your own characters that travel according to the speed, distance, or time you specify—you enter two, and the third can be automatically calculated or you can work it out. Here is a little example I created for a tortoise and a hare.




There is so much more to explore here, such as fractions and even a sound meter! I encourage you to take some time to watch the videos and see all that Dudamath has to offer. ▲


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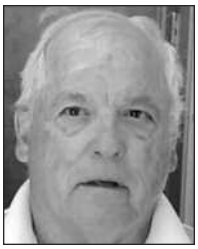
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# ▲ OAME/NCTM Report



**TODD ROMIENS**
**EMAIL: todd.romiens@oame.on.ca**

*Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a past president of OAME and a life member, as well as a previous editor of the* Gazette*.*

Welcome back to your teaching. My hope is that you have an exciting and inspiring year with your students.

As you may know, NCTM has a new president, Matt Larson. He is a K to 12 curriculum specialist for mathematics in Lincoln Public Schools, Nebraska. In his application for the presidency, he described his position in the mathematics community and the role NCTM must play to enhance mathematics education around the world. Here are some of his comments:

> NCTM faces two significant challenges: growing while retaining membership, and regaining its position as the go-to authority on mathematics education issues. My goals as NCTM president are threefold:
>
> 1. Increase and improve the quality, diversity, and delivery platforms of NCTM professional-development offerings to enhance members' access to mathematics content and effective instructional practice
> 2. Increase NCTM's engagement in public and political advocacy for high-quality teaching and learning of mathematics
> 3. Increase and sustain the membership of NCTM
>
> To support these goals and its strategic priorities, NCTM must undertake to develop a specific and coherent professional-development plan to support teachers' implementation of effective instruction and assessment practices, while supporting Common Core State Standards for mathematics. New professional-development options must also emphasize the use of cutting-edge technology. This professional development will support the implementation of practices outlined in *Principles to Action: Ensuring Mathematical Success for All*.
>
> NCTM must also develop partnerships to assist in launching a campaign to place NCTM at the forefront of improving the public's perception of the importance and accessibility of mathematics, as well as generate support for high-quality mathematics' curriculum, instruction, and assessment practices.
>
> As a beginning, NCTM has formed a Classroom Resources Committee to create supplemental classroom activities called ARCs (Activities with Rigor and Coherence). Some of these activities have already been field-tested and are being produced for teachers. Google "ARC activities" for access.

As you return to school, your students' parents (and guardians) will invariably have many questions, to try and make sure their child succeeds in his or her mathematics' courses. NCTM puts forth many suggestions to assist them in ensuring their children have a very positive classroom experience. Some ideas to pass on to parents are outlined in the following two publications—*A Family's Guide: Fostering Your Child's Success in School Mathematics* and *Figure This! Math Challenges for Families*—which can be ordered through NCTM.

These resources include a variety of ideas for you to suggest to parents, ideas that may help them to understand what you are doing in your classroom to benefit their child in mathematics education. As you know, many parents have beliefs about mathematics learning and instruction that may be at odds with current content expectations and effective teaching practices.

Following are some ideas included in the two resources that may provide clarification:

- Parents need to know that preparing for the twenty-first-century workforce requires more than being able to simply compute and follow procedures. Children need conceptual understanding as well as procedural fluency. They need to know how, why, and when to apply their knowledge. They also need to reason mathematically and communicate their reasoning effectively to others.
- Parents want to know the specific mathematics that their children will be learning. Provide them with curriculum expectations that you will be covering in class.
- They need to know what their children will be actively engaged in when doing mathematics—solving unfamiliar problems alone and collaboratively, analyzing alternate solutions, and generalizing their solutions.
- Have parents remember that their role at home is to support homework, not do it. Connect the math to real-life situations. Include places where students actively use mathematics and how important it is for them.

As you begin the school year, keep in mind that there are many resources available through NCTM that can assist you with your teaching. Many are free through the Internet. In addition, two NCTM regional conferences will be taking place in the Fall, which you may find beneficial:

- October 26–28 in Phoenix, Arizona
- October 31 – November 2 in Philadelphia, Pennsylvania ▲

## Summer 2016 Math Adventure #3 – ICME-13, Hamburg, Germany

BY AMY LIN, PEEL DSB/BROCK UNIVERSITY

This summer, along with several of my Canadian colleagues, I attended the International Congress for Mathematics Education (ICME-13) in the pretty and historic city of Hamburg. The numbers were impressive, with 3486 math educators and researchers from 109 countries taking part in workshops, discussion groups, and poster sessions.

The opportunity to share and learn from others from around the world was invaluable. There were 54 Topic Study Groups (TSGs) that covered various topics from calculus, algebra, and geometry to digital technology, task design, and pre-service teaching. I presented in the TSG on Visualization, and I took part in six sessions throughout the week on discussions and presentations.

There were plenary lectures at the start of each day on different topics that would appeal to the general audience. My personal favourite was the plenary by Bill Barton from the University of New Zealand on "Mathematics, education, and culture: A contemporary moral imperative." His presentation was thought provoking, and he was passionate about the role of mathematics education in building a more just and beautiful world.

I also found the plenary by Günter M. Ziegler, entitled "What is Mathematics?," interesting, and in particular, his presentation on how public perceptions of mathematics have changed over time. He also emphasized the importance of learning mathematics in view of Klein's "double discontinuity"—the first gap being the disconnect between what you learn in school math and what you experience at university, and the second gap being when math teachers return to the school classroom and fail to see any place for the content they learned in university.

There was a gathering of the Canadians at ICME-13 one night, resulting in our group photo.



*Photo credits: A. Lin and F. Glanfield*

# OAME Award Winners 2016

## Elementary School Staff Award

Recipient: **Roselawn Public School**



Roselawn Public School is the recipient of the 2016 OAME award for an Elementary School Staff. Roselawn Public School was nominated for its outstanding collegiality and teamwork, involving the spectrum of staff members in co-planning, co-teaching, and co-debriefing. The extra focus on students with learning disabilities was of particular merit in the opinion of the selection committee. Roselawn's video resource at the *Learn Teach Lead* website was specifically cited as a positive contribution to the mathematics community.

## Award for Outstanding Contribution to OAME



Recipient: **Rod Yeager**

Rod Yeager was the unanimous choice to be the 2016 recipient of the Award for Outstanding Contribution to OAME. The strong support given him by his nominators highlighted the many contributions he has made to OAME in particular and to mathematics education in general throughout his career. In particular, his work with developing and administering MathGAINS, as well as the OAME eConference, was cited as the pinnacle of his many achievements.



## OAME Lifetime Membership

Recipient: **Dr. Lyn Vause**

Dr. Lyn Vause exemplified the qualities sought after when deliberating the recipient of the 2016 OAME Lifetime Membership. The support we received from many individuals, and in particular Lyn's chapter, for the nomination, are testament to the many

strong contributions Lyn has made to OAME in particular and mathematics education in general. Lyn was particularly cited for skillful leadership and organization in having MAC$^2$ host two very successful annual conferences, chapter roles including chapter president, and involvement as a director of OAME.

## Secondary School Department Award for Exceptional and Collaborative Mathematics Teaching

Recipients: **Brid MacDonald** and **Carrie Stewart**




The exceptional work being done with all students at St. Patrick's High School is the reason Brid MacDonald and Carrie Stewart were selected to be the recipients of this year's Secondary School Department Award for Exceptional and Collaborative Mathematics Teaching. From their innovative and creative uses of modern technologies, to day-to-day professional learning and dialogue, to the strong support they have demonstrated for each other, as well as that afforded by all stakeholders in their program, it was clear they clearly merited this recognition. The eloquent letter of support provided by a student, Eli Yammine, reiterated what their many other supporters had to say about their remarkable efforts.



## OAME Award for Exceptional and Creative Teaching in Secondary Mathematics

Recipient: **Amy Klose**

The 2016 OAME Award for Exceptional and Creative Teaching in Secondary Mathematics has been deservedly given to Amy Klose. The many, very strong and detailed nominations we received from her colleagues at all levels of her involvement were strong testimony as to how she has positively and profoundly affected the learning of her students in creative and innovative ways. The highlight of her nominations was certainly her many successes with Grade 9 Applied students, from her alterations of the classroom environment to the incorporation of "FitBit" technology. When these achievements were coupled with her work elsewhere in the school and beyond the reaches of her district (with Trent University), she is certainly very deserving of this accolade. ▲

LEAP INTO MATH
OAME Annual Conference
MAY 5-7, 2016







# Conference Report – OAME 2016

GREG CLARKE
BRUCE MCKAY





*Greg is from the Simcoe Muskoka Catholic District School Board, currently on assignment with the Ontario Ministry of Education as a Provincial Math Lead, and also the OAME webmaster. Bruce is retired from the Simcoe County District School Board, where he was a secondary math teacher and math coach, as well as long-time MAC² executive member.*

The OAME 2016 Annual Mathematics Conference was held this past May in beautiful Barrie, Ontario, at the Georgian College campus over three days, May 5–7, 2016. The theme was "Leap into Math," and it was a great and successful fun-filled leap indeed!

The host chapter, MAC² (Mathematics Association for Cottage Country), had two and half years to plan the conference, and though small in number, they pulled off the mighty task with great results. Huge thanks go out to those who agreed to serve as committee chairs and co-chairs, and to others who helped out during the planning meetings as committee members, and also many thanks to the many who volunteered during the conference itself.

OAME 2016 had over 1700 people involved, including the attendees, presenters, organizers, and volunteers. With three keynote addresses (Robert J. Lang, Steven Strogatz, Sarah Greenwald) and 282 workshops, including a double Ignite session on Friday afternoon, attendees had many choices. Once again, an eConference was part of the event so that teachers who could not travel the distance to Barrie could take part. An expanded menu of options this year allowed 440 teachers to view the keynotes and featured speakers from afar.

The willingness of individuals and teams to lead workshops is key to the success of any OAME Conference. We would like to express our thanks to the many individuals who shared their learning, knowledge, and experiences with their colleagues. Perhaps you might consider presenting at OAME 2017 in Kingston.

Each conference inevitably boasts about its firsts. One of our firsts for OAME 2016 was the first time an OAME Conference had a virtual speaker, Al Richards, who delivered his workshop via Google Hangouts from Ignace, ON to attendees both on site in Barrie and as part of the eConference. At OAME 2016, delegates were also treated to a Flash Mob, a Ken Ken Tournament, and a STEAM room.

The OAME 2016 Banquet, which began with Robert Lang leading the group in creating their own personal origami, honoured the Past Presidents of OAME, as well as this year's OAME Award Winners. (For full details, see the article on award winners elsewhere in this issue). The evening was special for MAC² members, since our own Lyn Vause was presented with the OAME Lifetime Membership Award.

The OAME 2016 conference was a huge success, thanks to a fantastic organizing team, who worked tirelessly and covered all the bases so that things ran like clockwork. Whether your contribution was big or small, as co-chairs, we'd like to thank all who helped out before, during, and/or afterwards with the conference. You took a leap of faith, and we made a great LEAP INTO MATH! We are now all looking forward to enjoying OAME 2017 in Kingston… and relaxing… for something completely different!





# ▲ Math in the Margins: Making Student Mathematical Understanding Explicit

DAVID COSTELLO



*David is the elementary numeracy leader for the province of Prince Edward Island. In this role, he supports teachers, coordinates elementary math services, leads a team of math coaches, and is responsible for professional development and learning for teachers of Kindergarten through Grade 6. Previous to this, David worked as an elementary teacher, math coach, math consultant, and elementary curriculum leader for the English Language School Board of Prince Edward Island. David also instructs university courses in the areas of mathematics methods and problem solving, facilitates professional learning sessions for schools and school boards, and shares his research and expertise in understanding math as a literacy at various conferences locally, nationally, and internationally.*

I received an email from a teacher of a combined Grades 5/6 class, who was struggling with math. This teacher was in a small, elementary rural school in Prince Edward Island, Canada. The school was identified as having below average scores on the provincial math common assessments for primary (Grade 3) and junior (Grade 6). The teacher had a group of 25 students in the classroom, with the majority of students being identified as below grade level. This group of students had struggled with math for quite some time and many were grade levels behind. When I asked what interventions were applied as forms of gap closing, I was told that previous classroom teachers relied on extra small-group instruction and homework.

The teacher had a solid understanding of the math as well as the curriculum for her students, but she was having difficulty filling the gaps while trying to move forward. She had discussed her issues at her school-based case management meetings, as well as with the student support team. Her resource teacher suggested she contact me, as I was the Elementary Curriculum Consultant with a specific focus on mathematics.

## Supporting Schools as Consultant and Coach

As consultant, I was tasked not only with coordinating Kindergarten through Grade 6 math, but I also had coaching responsibilities at numerous elementary schools. The focus of my role was to strengthen student mathematical understanding through supporting teachers with effective instructional practices (Small, 2010). To achieve this goal, I coordinated training for math coaches assigned to groups of schools, provided training opportunities for resource teachers, and facilitated workshops and professional development opportunities for teachers.

One of my biggest messages during any of my training or workshops was the importance of mathematical comprehension. I stressed the value of thinking over memorizing, and concept over construct. I shared that, by only looking at the final answer, we miss important aspects of student thinking. This was the basis of my conversation with the Grades 5/6 teacher as she sought ways of strengthening student understanding.

We met to discuss her class as a whole and for me to gain a basic understanding of where students were functioning. Although she was able to identify students' areas of struggles, she was struggling to figure out why they were struggling and therefore how to appropriately support their learning. As we chatted, we realized we needed to install a practice that would both strengthen student understanding, through the use of metacognition (Rhodes & Dudley-Marling, 1996) and provide her, as teacher, with insight into student thinking. We wanted this practice to be embedded into daily practices and to be meaningful and provide value added, without taking a lot of time—effective and efficient.

Our discussion continuously returned to the importance of comprehension. She was not interested in having the students just "know how to do it"; she wanted them to have a solid conceptual understanding of the concepts addressed in the curriculum. Knowing the importance of communication in mathematics, the classroom teacher and I decided to emphasize communicating mathematical understanding to students (Small, 2013). We brainstormed a list of strategies she could use, but none of them seemed to be what she was looking for. She felt many of the strategies we discussed would not give her insight into students' thinking. She wanted to "get inside their heads."

## Margin Notes in Language Arts

We talked a lot about how this parallelled what our school board had been doing in Language Arts (LA) over the past few years. We discussed the practices put in place to support student comprehension in LA, and talked about ways to incorporate those into her math class. She was really interested in using margin notes in her math class, as

this practice had been quite successful in her LA class.

When students read in LA, they used sticky notes to record their thinking. As students read texts, both fiction and non-fiction, they used a variety of symbols to represent their thinking. For example, students would record a "?" when they had a question about a particular aspect. This question could relate to content, questions for the author, or a stumbling block related to comprehension. An "!" represented student thinking surrounding an important part of the text. Students would use this symbol to identify parts of the text that were important to the meaning and as a reminder to come back to this in their future reading, if necessary. An "*" represented a connection to the text. This "*" would then be supplemented by either a circle (representing a globe), meaning *connection to the world*; a square, *meaning connection to other text*; or a stick figure, meaning *connection to self*. Finally, a "←" meant that the student had to stop and reread a previous portion of the text, as there was a question surrounding comprehension.

These margin notes in the LA classroom proved to be beneficial in relation to students self-monitoring comprehension when reading. Prior to this, students would ask the teacher general questions for support when reading. When responding to the teacher regarding specifics leading to the questions, students were not able to provide specificity. Also, these margin notes supported students in documenting their reading experience with a brief notation versus written notes, so as to not impede fluency and/or comprehension. In addition to this, the teacher was provided a form of documentation into student reading, and this was then used to monitor progress (as a formative assessment tool).

The classroom teacher and I had a discussion with students regarding their use of margin notes for reading, and how they felt this practice supported their learning. Students shared that they didn't mind recording margin notes, and asked what they were used for by the teacher; and, if they needed to use them all. The teacher explained that she used the margin notes to monitor students' reading progress and identify trends (how many times students reread, make connections, have questions about the text being read, and if students are able to identify important parts of the text). When students were probed about the number of symbols used in margin notes, they responded that the connections symbol "*" did not really add to their comprehension, as they found this was just "something extra" for the teacher, not for them.

## Margin Notes in Mathematics

Seeing the progress in terms of student reading achievement in LA, we decided to apply the use of margin notes to mathematics. This closely related to the work of Mason, Burton, and Stacey (2010) when they discussed writing keywords on the page as students problem-solved. Writing in mathematics is a form of reflection and can be used to support students in consolidating their thinking (National Council of Teachers of Mathematics, 2000).

Initially we focused on using margin notes to document when students had questions. Students would record a "?" at the specific part of the problem with which they had difficulty. It was important to emphasize to students that the "?" had to be placed properly at the point of the question so that they and the teacher could easily identify what the student found difficult. In addition to the "?," students were to write a keyword to describe the question. This could be "operation" (if students were unsure of the operation) or "start" (if students were unsure as to how to start the problem). Similar to LA, a "*"represented a connection, and "←" represented instances when the student realized there was an error in their work and had to go back to the start of the problem to trace their work and identify the point of error (whether this be a calculation error or omission). The "*" would then be supplemented by either a square, meaning connection to other problem they experienced in the past; or a stick figure, meaning connection to self (lived experience—aiding in understanding how to approach the problem). A "√" indicated that students reviewed their completed work to ensure that there were no errors and that they addressed the question being asked of them.

Once we finished discussing how we would use margin notes in math, the teacher and I modelled the process through a think-aloud. The classroom teacher completed a problem on the whiteboard and recorded her margin notes as I probed her with questions. The next day, the classroom teacher modelled the process on her own. On the third day, the teacher invited students to the front of the room to model the process. The students worked through problems and recorded their thinking, using the margin note symbols, while the teacher asked probing questions as needed. Students were then given the opportunity to work in partners—one student solving a problem, while the other watched, questioned, and probed. Students then switched roles.

After the partner practice was complete, the teacher asked the students what they had learned about their partners' thinking and learning. Then she, the teacher, connected this to the insights she would be able to gain through the use of the margin notes. The teacher subsequently had the students work individually, using the margin notes to explicitly demonstrate their thinking, while she circulated through the class, discussing, probing, and supporting as needed. Figure 1 illustrates an example of a

Grade 5 student using margin notes as he worked through a problem involving two-digit by two-digit multiplication.

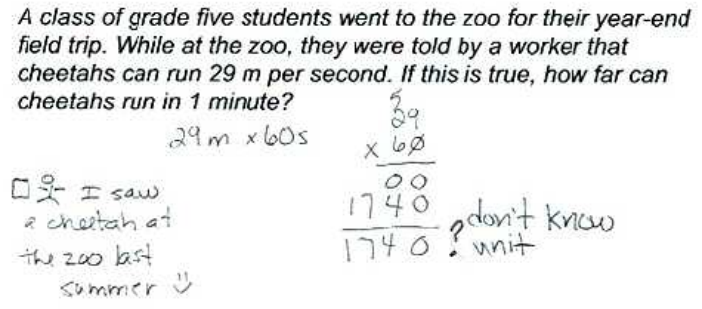



Figure 1: *Grade 5 student work sample*

We had students occasionally complete their problem solving and margin notes using pen, so as not to erase their work. Their margin notes were reflective of the scratches on the page. We discussed with students how all their margin notes were important for us to see and how they told a story about what the student was thinking. For example, a "?" with a strike through it would suggest that the student was able to figure out a question, whereas if the "?" was erased, we would have no way of knowing that the student had experienced this success. One student suggested it might be useful to record a note beside anything that had been struck out, to explain how the student had figured things out. We thought this was a great idea and the students agreed to incorporate the strike-out "/" and explanation to their margin notes. Figure 2 illustrates an example of a Grade 6 student using margin notes to record her thinking as she worked through a problem involving area. In this example, the student used the strike-out "/" to indicate that she figured out the meaning of by through a visual representation of the problem.

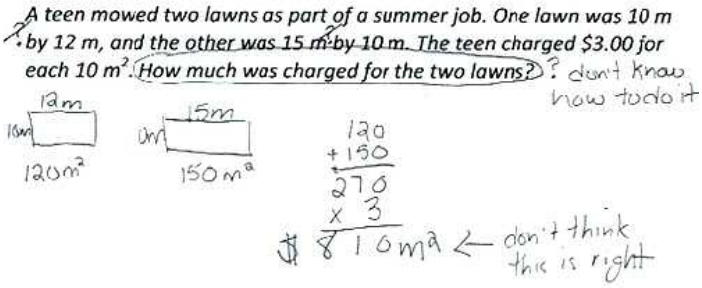



Figure 2: *Grade 6 student work sample*

The teacher used the margin notes for formative assessment. She felt that they gave her the insights she was looking for, and she was able to understand where students were struggling. She used this information to target her instruction individually, in small groups, and for the whole class, if/when appropriate. She also used the margin notes to provide feedback for students, and to develop individual goals with students, which she monitored during her conferences. The teacher used this format to monitor progress throughout the year and felt it had been very successful in moving her class forward.

Once the students were comfortable using margin notes in the classroom, the teacher started using these as a means of math conferencing (a practice also used in LA). She would have the students bring a work sample to her desk and describe what they had been thinking as they had completed the problem. This provided valuable information for the teacher, but also allowed the students to reflect on their own thinking. Teachers, coaches, and administrators were engaged in professional learning opportunities offered by Sandra Herbst for the past two years on meaningful assessment. Included in this professional learning was the importance of student self-assessment. Teachers of mathematics were looking for ways to provide authentic opportunities for students to self-assess instead of relying on the teacher for assessment. Margin notes provided an avenue for students to assess their work, in terms of working through a problem, making connections, revising their work, and examining their work errors.

Students became proficient at using margin notes and shared how this made them aware of their thinking in a way they had not been before. Students shared that they were now thinking about their thinking. One student told the teacher that she was now able to "pre-think" about the problems because she knew what she usually thought and why she usually thought that way. The teacher was thrilled with this level of cognition.

The teacher felt so comfortable with the students' ability to think about their own thinking and learning that she invited students to parent–teacher interviews. She had the students explain to their parents what margin notes were and how they worked. They then chose some examples of their work to show their parents how they had "thought through" the problem. Many of the parents were impressed with this part of the interview and shared that they noticed an increase in their child's confidence in relation to math.

What had begun in the narrow margins of the page had spread to conference and interviews and had become the foundation for learning in this teacher's combined Grades 5/6 classroom.

## Recommendations

It is important to recognize that what we are seeing in schools and classrooms is not "new math," but rather, a focus on student mathematical understanding. To achieve this goal of understanding, it is imperative that teachers have effective instructional practices available to them so that they can recognize patterns in student thinking, identify

misconceptions, and plan instruction accordingly. One such effective instructional practice is the use of margin notes.

Margin notes provide opportunities for students to explicitly communicate their understanding, sometimes making visible actions that may otherwise go unnoticed. It is important to note that instruction and instructional practices will vary based on the needs of the students and the experiences of the teacher. For example, a symbol could be added into margin notes for students to indicate that they used concrete materials to assist in solving the problem. However, this was not included in the classroom I worked with, as the teacher did not feel it was necessary for her students.

This process of focusing on student mathematical understanding does not need to be started from scratch, give that there are numerous effective instructional practices used in the LA classroom that relate to mathematics. We need to take a pause, identify some of these practices, and problem-solve ways for them to be included in mathematics instruction, assessment, and learning. The more we can provide connections between subjects for students, the more they can see the importance of thinking holistically as opposed to thinking in silos.

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## Summer 2016 Math Adventure #4 – Book Review

**BY MATTHEW OLDRIDGE, PEEL DISTRICT SCHOOL BOARD**

Matthew Oldridge, from Peel District School Board, decided to spend some time this summer reading, and shared with us a review of a book called Taking Shape, by Catherine Bruce, Joan Moss, Beverly Caswell, Tara Flynn, and Zachary Hawes.

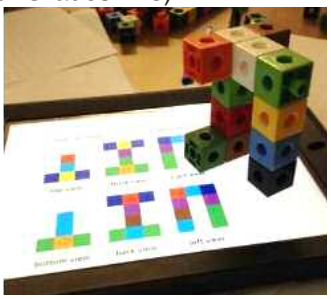
Spatial reasoning, to judge from this Spring's OAME conference, is one of the hottest topics in K–12 mathematics education. People I know treated the arrival of Pearson's Taking Shape with the same rapture that others treat a new Harry Potter book release. Seriously, they would have dressed up like tangrams and lined up at midnight, if they could have!

The Ministry monograph, Paying Attention to Spatial Reasoning, is an important new document, but, thus far, underserved by professional materials to support it. Into this space steps Taking Shape.

Researchers have long known of a link between early spatial ability and later mathematics achievement. "Spatial reasoning and mathematical thinking are intimately linked," and there is no reason to even doubt the link any more, as the authors state,

The extensive Math for Young Children (M4YC) research project underpins the work of the book. The book provides a much-needed "expanded vision of geometry"—a topic which is often reduced in the early years to naming shapes. The five aspects of geometry for which the authors provide background and tasks are: composing, decomposing and transforming 2D and 3D figures; locating, orienting, mapping, and coding; and perspective taking. The activities are practical, user-friendly, and ready to use. Hands-on tools like snap cubes play a large part throughout. The emphasis on early coding through the body is particularly relevant, with how popular coding is in education right now.

If you are like me, you had a hard time coming up with interesting Geometry tasks for your classroom. It was my toughest area by far. Taking Shape is a great new resource for your K–3 classroom (and many activities can be adapted for Grades 4–8).



Interlocking cubes are one of the best thinking tools to support spatial reasoning!



Perspective taking is important in spatial reasoning.

What does the building in shadow look like from above?

## Summer 2016 Math Adventure #5 – Toronto OTF Summer Institute Curious Math / Apple Institute 2016, Berlin, Germany

BY JON ORR, LAMBTON KENT DISTRICT SCHOOL BOARD

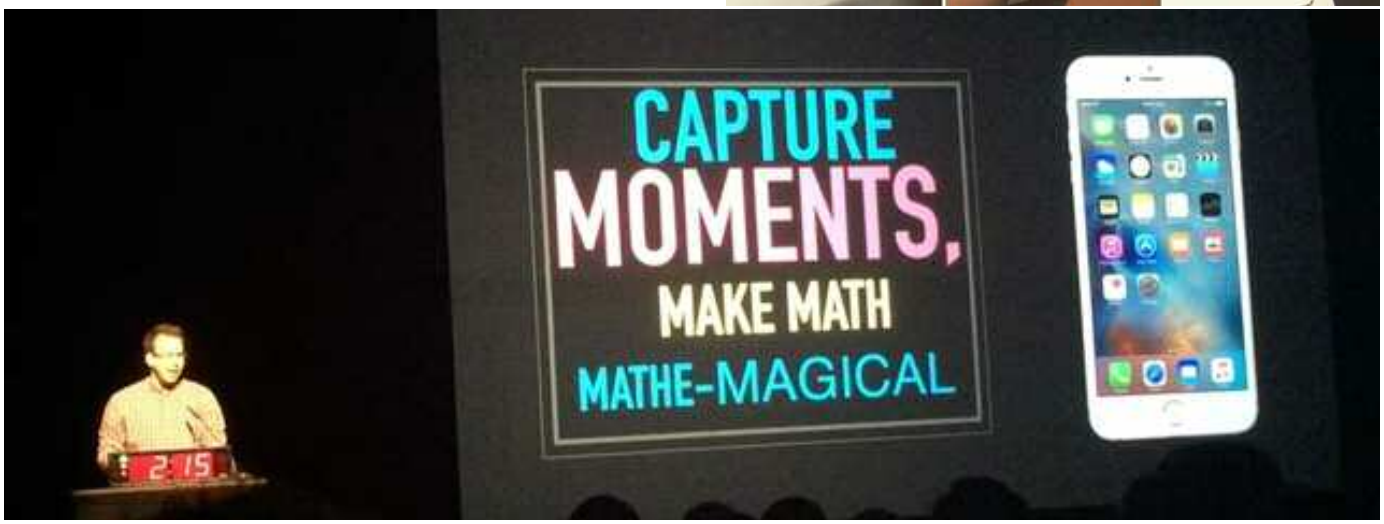
**July 12–15 – Toronto OTF Summer Institute Curious Math**

Kyle Pearce (Greater Essex County DSB) and I co-hosted a three-day workshop in Toronto for math teachers.








**Apple Institute 2016, Berlin, Germany**

I also travelled to Berlin to participate in the Apple Global Institute for Apple Distinguished Educators.

That's about it for my summer!

Thanks for thinking of me!