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Ontario Mathematics Gazette

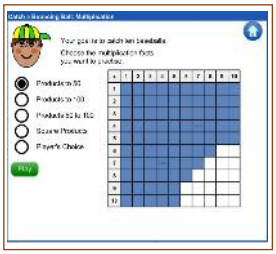
OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

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Vol. 55 #4
June 2017

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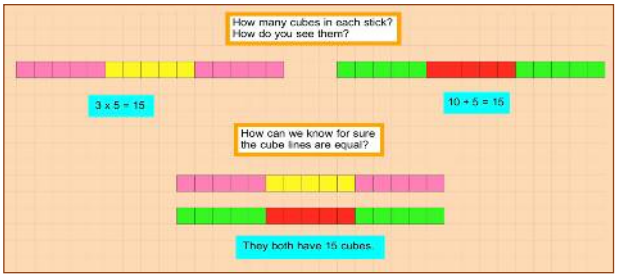


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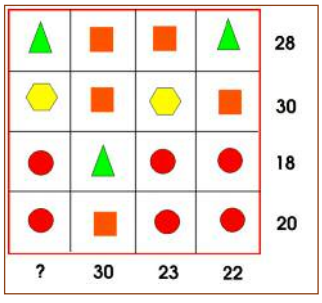
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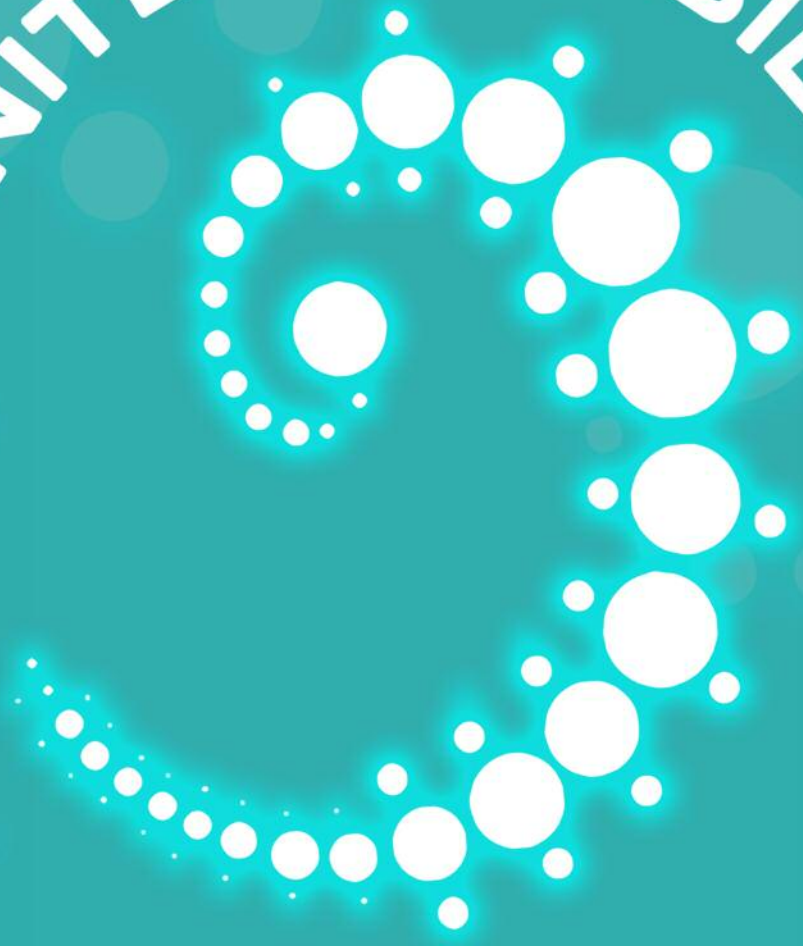


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Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *OMG*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles must be word-processed in MS Word, double-spaced with wide margins, not exceeding 10 numbered pages of text, and prepared according to the *Publication Manual of the American Psychological Association, Sixth Edition*. Figures and diagrams should be drawn by computer, if possible, or drawn in black ink in camera-ready form. Embedded images must also be submitted separately in jpeg or tif format. Proof of the photographer's permission is required, and for **photos of students** under the age of 18, the written permission of a **parent or guardian is required**.

You must submit **one complete copy** of your article, embedded with any tables, figures, and captioned photographs or graphics, to the Editor, Amy Lin, along with **separate files for each of the text, graphics, and/or photographs**. Please email all files to Amy Lin at **gazette@oame.on.ca**.

Your name should not appear anywhere in your article, including websites, so that your article can be sent out for blind review. Your name, full mailing address, and email address must be included on a separate sheet. Upon review, you will be notified as to whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected.

The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME.

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The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

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▲ Editor's Message



AMY LIN
EMAIL: amy.lin@oame.on.ca

Amy Lin's background in the sciences and the arts helps explain why her passion in mathematics education is to develop both the logical and creative aspects of the subject. She is currently working on her PhD at Brock University in math education and is a Vice-Principal at Applewood Heights Secondary School. She is often asked to speak on mathematics teaching and learning around the province and beyond, and she is an author of a number of mathematics resources for students and teachers at both the elementary and secondary levels. She has initiated and successfully led many professional development projects in school boards in Ontario.

As we approach the end of the school year, I have become immersed at my school in the routines such as timetabling, staffing, exam preparations, committee and council meetings, and a review and wrap-up of our theories of actions for the school and across departments and groups. The flurry of activity generates thinking of what will best serve our modern learners and how can we cultivate curiosity, engage them in creative and critical thoughts, and help them realize and meet their potential. However, the more that I go down this road with my staff, the more I start to wonder if we are truly allowing our students to be innovative learners or if the result is less of what we had set out to do.

Schools and districts are inundated now with experts and innovative thinkers who have ideas that permeate teachers' collaborative inquiries, staff meetings and PD days. There are endless books, articles, websites, and handouts shared, and teachers are expected to implement these great ideas in the classroom and bring back "artifacts" to further share with the school. These strategies do not necessarily promote the types of innovative learners we want, but rather just result in measurable increases in test scores or other yearly progress reports. It's the experts that can build teacher capacity by coaching them and helping them understand the thinking processes that have helped us go in the right direction.

Theories of action seem to pop up everywhere, and these ideas have been widely broadcasted on websites, during PD sessions and committee and council meetings. Through the sharing, collaborating and working together, the result can sometimes end up standardizing instruction across classrooms – which is counterproductive and makes

it difficult for the learning to be innovative and ultimately messy. Research-based strategies, ministry expectations, and district policies become more important, and the curious, creative and innovative student falls by the wayside.

In *Curriculum 21: Essential Education for a Changing World*, Heidi Hayes Jacobs reminds us that "curriculum should not only focus on the tools necessary to develop reasoned and logical construction of new knowledge in our various fields of study but also should aggressively cultivate a culture that nurtures creativity in all our learners" (Jacobs, 2010, p. 17). In times of open discussions, social media, and learning platforms it is surprising how structured our learning communities have become. Math teachers are involved in collaborative inquiries, action research, and professional learning communities that are overly rigid and can end up with an instructional design that is uniform across classrooms and teachers. To encourage innovation teachers should be working in a climate where learning happens because teachers, students, and the content are carefully considered and not just a result of some formulaic response after disaggregating the data.

With a focus on grades and report cards, it is difficult for teachers and students to engage in messy learning and to highlight creativity. An assigned letter grade or percentage has become more of the focus than on the learning. The grade holds little meaning when it comes to innovation, curiosity, and perseverance.

It comes down to students feeling that their classroom is different every day and teachers are reinventing lessons and instructional strategies to meet the needs of all their students. Teachers spark dialogue and interest in everything they try. A culture of innovation means there is trust and teachers can allow students to be creative, to experiment, and to make mistakes. The trust goes two ways – meaning students trust their teachers to do their job and to help them learn. Teachers who work on generational characteristics such as student collaboration, research, and critical evaluation are more likely to inspire learning that leads to innovative thinking.

This issue features articles by Andre Lemaire, Jennifer Holm and Ann Kajander, and a follow-up article in the last issue from Edward Schroeter. A feature article informs us of the important work being done involving the newly formed Mathematics Knowledge Network. In our regular columns, Mary Bourassa shares her findings on the NRICH website, Lynda Colgan on early math learning, problem-solving with Shawn Godin and the usual updates from Mathies.ca, the NCTM report, and the Fields Report. I would like to thank Krista Sarmatiuk who was our guest columnist for M4bt this month and has written about equality.

In the March issue of the Gazette, Mary Bourassa wrote about the usefulness of blogs as resources for teachers. Throughout this issue, I have included some excerpts from blogs from inspiring mathematics educators. Thanks to all of those who have given me permission to share. Have a wonderful summer and hope you have time to embark on some math teaching and learning adventures!

Hayes Jacobs, H. (2010). *Curriculum 21: Essential Education for a Changing World.*

Blog Post

Dave Richeson: divisbyzero.com

Zip-Apart Möbius Bands

Posted by DAVE RICHESON *on* APRIL 11, 2016

I've taught topology many times. One of the highlights for the students (and for me) is the investigation of the Möbius band—the one sided, one edged, non-orientable surface with boundary. On the day we introduce the Möbius band I bring many strips of paper, clear tape, and scissors and have the students make conjectures about what would happen if we taped and cut apart various topological shapes. Here are some activities that are fun to do:

1. Twist the paper zero times, and tape the ends (making a cylinder). Cut down the midline.
2. Give the paper one half-twist, and tape the ends (making a Möbius band). Cut down the midline.
3. Give the paper two half-twists ,and tape the ends. Cut down the midline.
4. Give the paper three half-twists, and tape the ends. Cut down the midline.
5. Twist the paper zero times, and tape the ends. Cut into thirds.
6. Give the paper one half-twist, and tape the ends. Cut into thirds.
7. Give the paper two half-twists, and tape the ends. Cut into thirds.
8. Give the paper three half-twists, and tape the ends. Cut into thirds.

In fact, these activities are fun for people of *any* age. My senior math majors enjoy it, and my kids' kindergarten classes have too.

Last week I attended the 12th biennial Gathering 4 Gardner conference—a wonderful meeting of people interested in mathematics, puzzles, games, magic, and skepticism. One of the speakers (Iwahiro Hirokazu Iwasawa) suggested making zip-apart Möbius bands. Genius! And perfect timing (since I'm teaching topology this semester).

When I got home I bought zippers and Velcro and used them to make reusable strips. Half of them were just one zipper with Velcro on the ends. These can be used to do activities 1-4 above. Then I took identical pairs of zippers and joined them side-by side to make strips for activities 5-8.



President's Message



JUDY MENDAGLIO
EMAIL: judy.mendaglio@oame.on.ca

Yesterday, CHAMP conducted their regional trials for the OMO. For those of you who are not aware of the Ontario Math Olympics, this is an annual event in which Grades 7 and 8 students come together as a team and engage with rich and interesting math problems. The provincial finals are organized by a Chapter of OAME, and local Chapters organize their own regional trials to determine which team they will send to the provincials. You can find out more about the OMO on the OAME website.

I have been thinking about some of what I heard as I walked around as students worked with their partners on the wonderful challenges written for them by our CHAMPions. (Yes, assessment for learning through observation and conversation—and on a Saturday morning!) Many things I overheard caused me to reflect on how we teach children mathematics. I would like to share some of my thoughts.

Students were given a photo of an actual roped-off swimming area at an Ontario park. They were asked if the park authorities had created the largest possible swimming area with that length of rope. Rather quickly (way too quickly), I heard:

"The largest area is always a square."

"It has to be in a ratio of 2:1 because the shoreline is one side."

What do these two statements have in common? Lots. For one thing, neither gave the correct solution to the problem. For another, these were obviously "rules" that students had learned. I put the word "rules" in quotation marks deliberately. What is a "rule"? Definition of rule: *"One of a set of explicit or understood regulations or principles governing conduct within a particular activity or sphere."* (Thank you, Google.)

To me, there are two important pieces in this definition: *principles governing conduct* and *within a particular activity or sphere*. We all know that there are different principles governing conduct in different spheres. You are usually free to get up from your desk when you are at work, but not when you are writing an exam. You can drive 100 km/h on the highway, but not on a residential street. Students understand that. Moreover, they can probably tell you why that rule makes sense. Why do they not understand that rules in mathematics also have spheres (environments, contexts) in which they apply, and spheres in which they do not? Somehow, students view mathematical rules as absolute—they always work. Since they always work, there is no need to pay attention to the circumstances under which they are applying the rule. I have to think that somewhere, they were misled by an adult in their lives. Maybe it was a parent, maybe a teacher, maybe a tutor or an older sibling. I don't know. But what I do know is that we can alleviate a lot of frustration that students experience when doing mathematics if we are very clear about when a "rule" can be used, and when it can't. I could probably treat all the readers of this President's Message to a nice meal if I had a dollar for each time a student had used a "rule" when that rule was not available to use. "$A^2 + B^2 = C^2$?" No, that is not the rule. The rule is: In a right triangle,…." This is the sphere in which this rule is valid. "Squaring always makes a number larger." Don't tell that to ½. A vertical stretch is the same as a horizontal compression. Not if you are studying periodic behaviour. These are all examples of rules that my own students came into my class carrying around with them. Students who had foregone thinking in favour of rule-remembering were at a distinct disadvantage, not only because they got questions wrong, but because they had not learned how to persevere and try different strategies to solve problems that did not easily fit into one of their rules. I am guessing that many students are not even aware that mathematical rules, like life rules, have a sphere. I know that we can help with this. When students begin learning "rules," maybe we can use simple language to describe the sphere: "When we have a right triangle, we can…," "If our shape is a quadrilateral, then…." Even those "rules" that we know don't last very long in a student's academic life, like "You can't take a bigger number away from a smaller number" and "You can't take the square root of a negative number," have spheres that students can understand, if we are thoughtful about how we present them. I am certain, based on many years of asking my students as well as my colleagues, that students rarely understand why a rule makes sense. It's no surprise that students feel as if someone has pulled the rug out from under them and then say something like "Math is stupid" when they are inevitably told that they cannot use that rule in this circumstance.

We saw a few students dividing by zero. I wonder if students understand what we mean when we tell them, "Division by zero is undefined." As with the rules working within their spheres, operations have spheres. Divide 4 by 2 works everywhere. However, divide 4 by 3 doesn't work when you only have the counting numbers 1, 2, 3,…. Mathematicians are creative, but they like consistency. The operations were all created by people. They didn't come pre-

made. They were defined to have a particular meaning. They are well defined within their sphere. However, no one ever came up with a definition for dividing by zero that didn't upset some other part of the well-defined world of mathematics that was already set up, so everyone agreed to leave that undefined. Division by zero has never been given meaning. It is undefined; rather like the word "kerthancular."

So, what have I been reading? Well, it seems I am a bit behind many of you. I have been reading a lot about number talks, and most recently read *Making Number Talks Matter: Developing Mathematical Practices and Deepening Understanding, Grades 4–10*. No wonder so many people are reading and talking about number talks. In this particular book, I saw how rich investigations, vibrant conversations, and deep and meaningful learning could come from some well-selected, but basic, computations. While I have long valued the area model for multiplication for the way it provides an organizer to deconstruct and reconstruct numbers and algebraic expressions, I had never thought much about the extraordinary variety of ways that students might bring their own understanding and curiosity to other operations. However, the true joy of this book, for geeky me, is the way the authors build student understanding of the properties of the mathematical operations (commutativity, associativity, distributivity) through these number talks. The talks, and the consolidation that follows them, provide students with a way to understand (and remember) some very abstract mathematical concepts, all while allowing them to become more fluent with numbers and operations. What more could you ask for? Imagine how this exchange with a Grade 12 OMO volunteer would have been different if this student had had the benefit and joy of number talks:

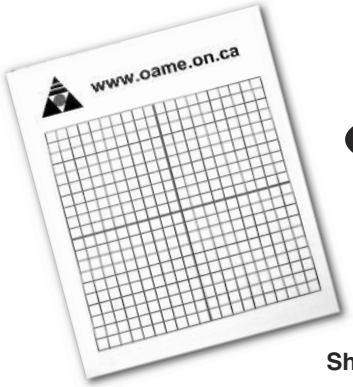
Student (to me): How many students are we expecting today?

Me: Nineteen teams, four students a team, so….?

Student: I can't do that without a paper and pencil or a calculator. ▲

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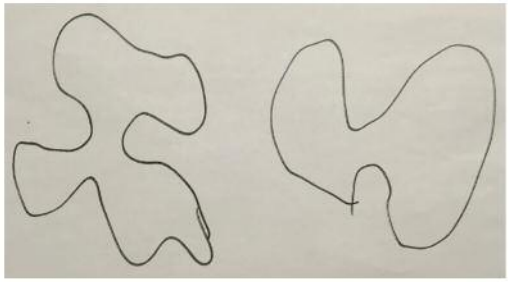
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Blog Post **Mark Chubb:buildingmathematicians.wordpress.com**

Many grade 3 teachers in my district, after taking part in some professional development recently (provided by @teatherboard), have tried the same task relating to area. I'd like to share the task with you and discuss some generalities we can consider for any topic in any grade.

The task:

As an introductory activity to area, students were provided with two images and asked which of the two shapes had the largest area.

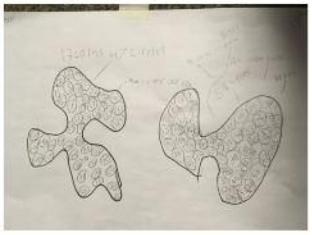


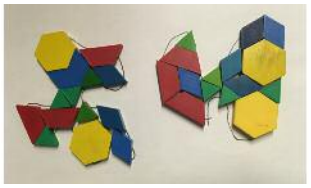
Student ideas

Given very little direction and lots of time to think about how to solve this problem, we saw a wide range of student thinking. Take a look at a few:

Some students used circles to help them find area. What does this say about what they understand? What issues do you see with this approach though?

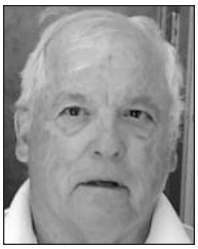








▲ OAME/NCTM Report



TODD ROMIENS
EMAIL: todd.romiens@oame.on.ca

Todd recently retired from teaching K–12 methodology in the Faculty of Education at the University of Windsor. He is a past president of OAME and a life member, as well as a previous editor of the Gazette*.*

In my last column, I provided NCTM president Matt Larson's view of the issues surrounding the purposes of homework and in general, the approach to teaching that needs to be taken to enhance student learning. Even though he is talking about education in the United States, it is clear that the issues apply to Ontario education as well.

The question lies in whether we should be teaching the "how" to do mathematics versus "why" we are doing the mathematics. It seems that in the beginning, mathematics was to be taught by teachers stating a rule, providing examples of the rule, and assigning practice problems to reinforce the understanding of the rule. This technique has been around for quite some time and certainly reflects how many of us experienced mathematics classrooms. It may still be prevalent in many classrooms today, even though there have been movements to have students "discover" the rule by working on their own with manipulatives or guided instruction to lead them to the desired result.

Herein lies the rub. Which is best, a back to the basics focus with procedural skills and direct instruction or guided questions allowing students to construct their own learning?

The answer, it seems, is a balance between these two approaches. The problem, of course, is how to maintain a balance and what it looks like.

The NCTM publication *Adding It Up* (2001) lists five components to having our students attain mathematical proficiency:

1. Conceptual understanding – The comprehension of mathematical concepts, operations, and relations
2. Procedural fluency – The skill in carrying out procedures flexibly, accurately, and appropriately
3. Strategic competence – The ability to formulate, represent, and solve mathematical problems
4. Adaptive reasoning – The capacity for logical thought, reflection, exploration, and justification
5. Productive disposition – The ability to see mathematics as sensible, useful, and worthwhile

My friend, Dick Seitz, who is a long-time math teacher in Detroit, said it best: "Powerful teachers of mathematics work daily to provide these five components to a core of big ideas. It is important to remember that the procedural skills for today's students span the gamut from mental math to the use of technology."

As a final note, NCTM's Illuminations has produced a series of pre-K to Grade 2 student-friendly activities that are mainly interactive and that can be accessed free of charge at **illuminations.nctm.org**.

The first is a pre-K–2 activity called "Five Frame." There are four games that can be used to help develop counting and addition skills.

Go to **http://illuminations.nctm.org/Activity.aspn?id=3564** to open the window. The first game is called "How Many?," where students are asked to determine the number of circles required to complete a frame of squares, or to determine the number of empty squares. The second game is called "Build," where students move circles to the square frame to complete addition problems. The third game is "Fill," where students determine the number of circles needed to complete a frame of squares, using adding and subtracting for sums up to 10.

The Grades 3–5 game is "Tesselation Creator." Students create their own tessellations by dragging shapes to make it on a board. There are shapes with 3 to 12 sides. Students are asked to determine which shapes can be used to tessellate the plane and what allows a shape to tessellate. The game can be accessed as above, using number 3533.

The Grades 6–8 activity is "Cube Nets." Here, 24 nets are displayed, and students are asked to find the ones that will fold up to make a cube. Once a net has been selected, students decide whether or not it can be folded to make a cube. A visual folding shows them if they were correct. Students are timed to see how long it takes them to go through all 24, and they are given the opportunity to try again. Use 3544 to access the activity.

The Grades 9–12 activity is "Advanced Data Grapher," where students analyze data with box plots, bubble graphs, scatter plots, histograms, and stem-and-leaf plots. Students can enter data and select the method to display it. The activity can be accessed with number 3476.

I hope you all have a great summer! ▲

▲ What's the Problem? Turning Addition into Multiplication

SHAWN GODIN



Shawn is the head of Mathematics, Business, Law, and Computer Science at Cairine Wilson Secondary School in Orleans, Ontario. He is a past member of the OAME Board of Directors, and past Editor-in-Chief of Crux Mathematicorum. *He currently writes problems for the Fryer, Galois, Hypatia, and Canadian Open Mathematics Challenge contests.*

Welcome back, problem solvers. Last time, I left you with the following problem:

Determine all functions that turn addition into multiplication; that is, determine all functions f such that

$$f(x + y) = f(x)f(y).$$

This problem was inspired by the book *Burn Math Class: And Reinvent Mathematics for Yourself* (2016) by Jason Wilkes. In his quirky book, Wilkes suggests that mathematics education has things backwards. He argues that the conceptual tools of calculus are needed to really understand many of the "pre-calculus" topics that are in the traditional mathematics curricula. The book is a journey through mathematics from first principles. New ideas are developed as needed or as questions arise in the exploration process. The book does have some unorthodox elements, including the author carrying on conversations with the reader and mathematics itself within the text of the book. Overall, his treatment of the exploration of the mathematics makes this a worthwhile read.

Returning to the problem, we may have no idea where to start. This is a good case where it makes sense to consider a simpler problem. Instead of looking at the problem of determining which functions turn addition into multiplication, let's look at functions that turn addition into addition and ones that turn multiplication into multiplication.

The first case that we will consider is functions that turn addition into addition; that is, which functions f satisfy

$$f(x + y) = f(x) + f(y)$$

for all real numbers x and y?

Again, this may seem a dauntingly abstract task. To make things a bit more concrete, let's look at some specific values. Starting with the easiest possible case, we will use $x = y = 0$, which leads to

$$f(0 + 0) = f(0) + f(0),$$

or $f(0) = 2f(0)$, which leads to $f(0) = 0$. We can convince ourselves that this must be the case through some other examples, or by noticing

$$f(x) = f(x + 0) = f(x) + f(0).$$

So we know one particular value of the function that we will continue to explore by setting $y = 1$ and seeing what happens for various values of x.

x		
1	$f(2) = f(1) + f(1)$	$f(2) = 2f(1)$
2	$f(3) = f(2) + f(1)$	$f(3) = 3f(1)$
3	$f(4) = f(3) + f(1)$	$f(4) = 4f(1)$
4	$f(5) = f(4) + f(1)$	$f(5) = 5f(1)$

Figure 1

The relations in Figure 1 let us see that the values of $f(x)$ are increasing at a constant rate. Looking at it a different way, we know that since $f(x + 1) = f(x) + f(1)$, then $f(x + 1) - f(x) = f(1)$ for all values of x. This would lead us to believe that f is linear, and since $f(0) = 0$, we would expect f to be of the form $f(x) = kx$ for some number k, which is supported by the findings in the third column. If we use this definition, we see that

$$Ef(x + y) = k(x + y) = kx + ky = f(x) + f(y),$$

so the original condition is satisfied and we have found the desired functions. Note that in this case, there wasn't a single function that satisfied the original relation, but a family of functions.

The second case that we will consider is functions that turn multiplication into multiplication, that is, functions that satisfy

$$f(xy) = f(x)f(y)$$

for all real numbers x and y.

Again, if we consider the easiest possible case, we will set $y = 0$, which yields

$$f(0) = f(x)f(0).$$

This leads to three possibilities: either $f(x) = 0$ for all values of x, $f(x) = 1$ for all values of x, or $f(0) = 0$. The first two possibilities are solutions, but are a bit boring. We will explore the third to see what we get.

If we again try $y = 1$, we get $f(x) = f(x \times 1) = f(x)f(1)$ for all values of x. Notice that our boring cases satisfy this as well, so we can assume that f takes on a value other than 0 or 1 for some value of x, which means that we must have $f(1) = 1$

We may continue trying other values of x and y to see what happens and eventually we may notice some of the patterns in Figure 2.

$f(4) = f(2^2) = [f(2)]^2$	$f(9) = f(3^2) = [f(3)]^2$
$f(8) = f(2^3) = [f(2)]^3$	$f(27) = f(3^3) = [f(3)]^3$
$f(16) = f(2^4) = [f(2)]^4$	$f(81) = f(3^4) = [f(3)]^4$

Figure 2

So our function preserves powers. If we recall our power laws, we know that $(x^a)^b = x^{ab} = (x^b)^a$. Thus, if we set $f(x) = x^k$, then $f(xy) = (xy)^k = x^k y^k = f(x)f(y)$.

Now that we have seen how some of these types of functions behave, let's attack the original problem: $f(x + y) = f(x)f(y)$. As with the lead-up problems, let's try some specific values for y. Setting $y = 0$ yields $f(x + 0) = f(x) = f(x)f(0)$, which tells us that $f(0) = 1$ (or either of the two boring cases encountered earlier). Substituting $y = 1$ gives $f(x + 1) = f(x)f(1)$, which we can use to generate the table in Figure 3.

x	$f(x)$
0	1
1	$f(1)$
2	$f(2) = f(1)f(1) = [f(1)]^2$
3	$f(3) = f(2)f(1) = [f(1)]^3$
4	$f(4) = f(3)f(1) = [f(1)]^4$

Figure 3

This leads us to believe that $f(x) = k^x$. When we check it out, we see that

$$f(x + y) = k^{x+y} = k^x k^y = f(x)f(y)$$

and our conditions are satisfied.

In the book, the author was using his development of functions and calculus to lead naturally to these types of questions, that is, what function behaves in some particular way. As such, he defined functions by their behaviour. If we had used calculus to attack the original problem by differentiating with respect to x, it would have yielded

$$f'(x + y) = f'(x)f(y).$$

Then, setting $x = 0$ in this equation, we get $f'(y) = f'(0)f(y)$, which means that if $f'(0) = 1$, then f is its own derivative. Otherwise, it is almost its own derivative, with an extra scale factor added in, a well-known property of derivatives of exponential functions.

You may have fun exploring the other variations of the function, using calculus as well as exploring one that I left out that turns multiplication into addition, that is

$$f(xy) = f(x) + f(y).$$

Students who have been introduced to function notation should be able to play with some of these function families. It would be interesting to see how they would attack the problem.

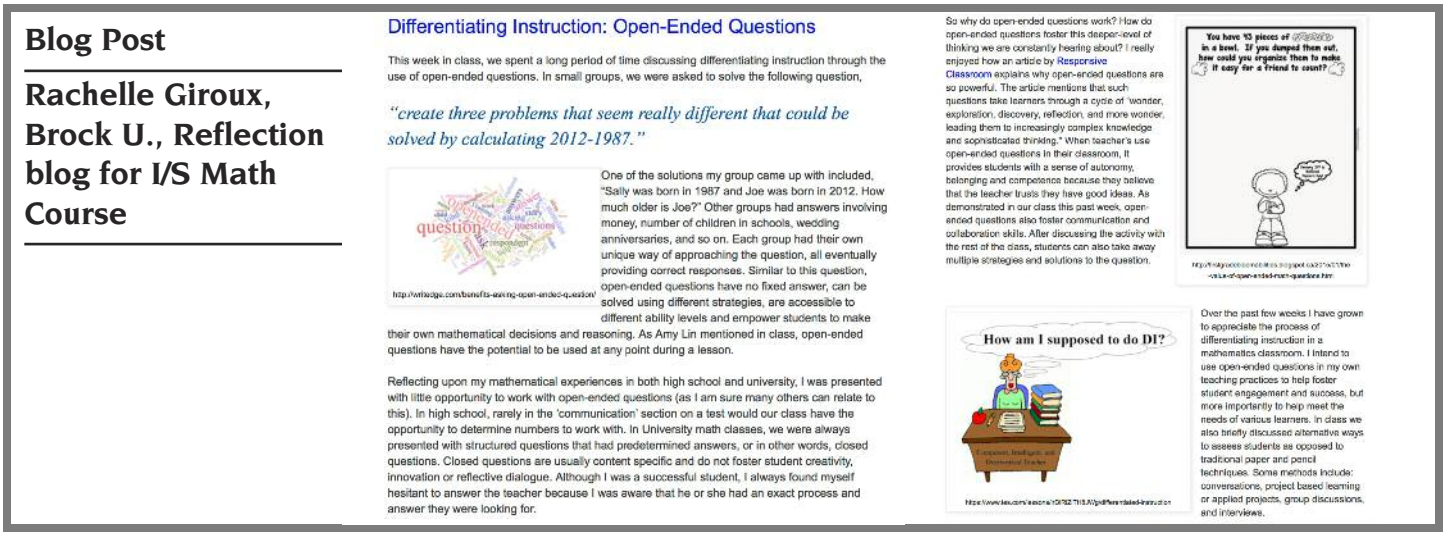
Remember that a great strategy in problem solving is to solve an easier problem. This often leads to patterns that we can exploit to solve the original problem. Alternatively, if you try to treat "the answer" to a problem as a starting point to other investigations, you will often be led to some very interesting mathematics. Now it is time for some homework:

A band of eight pirates are going to disband. They have divided up all of their gold, but there remains one GIANT diamond that cannot be divided. To decide who gets it, the captain puts all of the pirates (including himself) in a circle. Then he points at one person to begin. This person steps out of the circle, takes his gold, and leaves. The person on his left stays in the circle, but the next person steps out. This continues with every second pirate leaving until there is only one left. Whom should the captain point at if he wants to make sure he gets to keep the diamond for himself?

Until next time, happy problem solving! ▲

Blog Post

Rachelle Giroux, Brock U., Reflection blog for I/S Math Course



Differentiating Instruction: Open-Ended Questions

This week in class, we spent a long period of time discussing differentiating instruction through the use of open-ended questions. In small groups, we were asked to solve the following question,

"create three problems that seem really different that could be solved by calculating 2012-1987."

One of the solutions my group came up with included, "Sally was born in 1987 and Joe was born in 2012. How much older is Joe?" Other groups had answers involving money, number of children in schools, wedding anniversaries, and so on. Each group had their own unique way of approaching the question, all eventually providing correct responses. Similar to this question, open-ended questions have no fixed answer, can be solved using different strategies, are accessible to different ability levels and empower students to make their own mathematical decisions and reasoning. As Amy Lin mentioned in class, open-ended questions have the potential to be used at any point during a lesson.

Reflecting upon my mathematical experiences in both high school and university, I was presented with little opportunity to work with open-ended questions (as I am sure many others can relate to this). In high school, rarely in the 'communication' section on a test would our class have the opportunity to determine numbers to work with. In University math classes, we were always presented with structured questions that had predetermined answers, or in other words, closed questions. Closed questions are usually content specific and do not foster student creativity, innovation or reflective dialogue. Although I was a successful student, I always found myself hesitant to answer the teacher because I was aware that he or she had an exact process and answer they were looking for.

So why do open-ended questions work? How do open-ended questions foster this deeper-level of thinking we are constantly hearing about? I really enjoyed how an article by Responsive Classroom explains why open-ended questions are so powerful. The article mentions that such questions take learners through a cycle of 'wonder, exploration, discovery, reflection, and more wonder, leading them to increasingly complex knowledge and sophisticated thinking." When teacher's use open-ended questions in their classroom, it provides students with a sense of autonomy, belonging and competence because they believe that the teacher trusts they have good ideas. As demonstrated in our class this past week, open-ended questions also foster communication and collaboration skills. After discussing the activity with the rest of the class, students can also take away multiple strategies and solutions to the question.

Over the past few weeks I have grown to appreciate the process of differentiating instruction in a mathematics classroom. I intend to use open-ended questions in my own teaching practices to help foster student engagement and success, but more importantly to help meet the needs of various learners. In class we also briefly discussed alternative ways to assess students as opposed to traditional paper and pencil techniques. Some methods include: conversations, project based learning or applied projects, group discussions, and interviews.

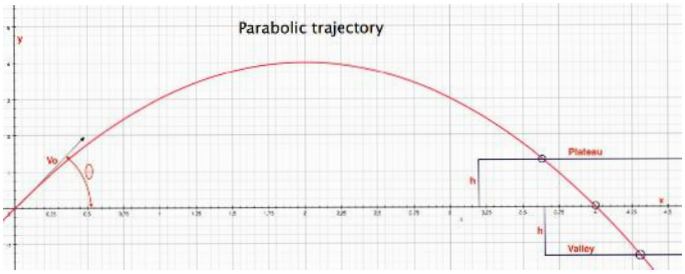
▲ Calculation of the Shooting Angle for a Maximum Range

ANDRÉ LEMAIRE



André is a retired mathematics and physics teacher, who taught at Garneau High School in Orleans, Ontario. Born in Belgium, André studied at the University of Ottawa, and is also a licensed pilot and active skydiver.

Note: Here, we are supposing that the earth is "locally" flat, and we also neglect the friction of the air.



PARAMETERS			
V_0	initial speed	g	acceleration of gravity here, $g = -10$ m/s^2
x	horizontal distance	h	target height/depth (plateau/valley)
y	vertical distance	θ	shooting angle
t	trajectory time		

Note: When confronted by a complex problem, it is useful to give the students a checklist or chart of operations that need to be done. For instance, I have used this approach successfully in calculus with problems about optimization, since they require many steps. This problem is of the same type.

Calculation checklist:

A) Establishing equations $x(t)$ and $y(t)$
B) Resulting 2nd-degree equation
C) Establishing $t(\theta)$
D) Substitution to get $x(\theta)$
E) Grouping constants together
F) Differentiating $x(\theta)$ or getting $x'(\theta)$
G) Establishing condition to have $x' = 0$
H) Use of the trigonometric identity: $\sin^2\theta + \cos^2\theta = 1$
I) Algebraic handling (factorizing, squaring, simplifying, reducing, etc.)
J) Bringing back the original constants
K) Finding θ for the maximum range

Problem: A cannon is shooting at a target located on a plateau or in a valley of height/depth *h*.

Note: The addition of a parameter (here, $h \neq 0$) generates a lot of difficulties in order to find the angle for the longest range x. It takes imagination to reduce long equations where trigonometry is a must.

Calculation of the Shooting Angle θ

A) Establishing equations *x*(*t*) and *y*(*t*)

From kinematics: In constant-speed motion, we have *d = vt, or distance = speed x time*. The horizontal distance component, here $x(t)$ is performed at a constant speed (neglecting air friction), then $\therefore x(t) = V_0\cos\theta \times t$(1).

From kinematics: In a constant-acceleration motion, for the distance, we have:

$$d = \frac{at^2}{2} = V_0 t + d_0 .$$

The vertical distance component, here $y(t)$ is submitted to the gravity acceleration g. *Here, d_0 is the initial distance* = 0,

$$\therefore y(t) = \frac{gt^2}{2} + V_o \sin\theta \times t \qquad (2),$$

but the range is reached when $y = h$ when the shell lands.

B) Resulting 2nd-degree equation

From (2), one has:

$$y(t) = h = \frac{gt^2}{2} + V_o \sin\theta \times t \text{ or } 0 = \frac{gt^2}{2} + V_0 \sin\theta \times t - h \quad (3)$$

C) Establishing

(3) is a 2nd-degree equation, with solution being:

$$t = \frac{-V_o \sin\theta \pm \left(V_o^2 \sin^2\theta - 4 \times \frac{g}{2}(-h)\right)^{\frac{1}{2}}}{\frac{2g}{2}} \qquad (4)$$

Because *g is negative* in order to keep the positive root, since the firing of the cannon moves toward the right (first quadrant), one keeps the sign "–" before the square root and later factorizes (–1) to incorporate it with a constant.

D) Substitution to get $x(\theta)$

By putting (4) in (1), simplifying and factorizing, one gets:

$$x(\theta) = \frac{V_o}{g}\cos\theta\left(-V_o\sin\theta - \left(V_o^2\sin^2\theta + 2gh\right)^{\frac{1}{2}}\right)$$

$$or\ x(\theta) = (-1)\frac{V_o}{g}\cos\theta\left(V_o\sin\theta + \left(V_o^2\sin^2\theta + 2gh\right)^{\frac{1}{2}}\right)$$

$$\therefore x(\theta) = -\frac{V_o}{g}\cos\theta\left(V_o\sin\theta + \left(V_o^2\sin^2\theta + 2gh\right)^{\frac{1}{2}}\right)$$

Multiplying the first factor by V_0 and dividing the second factor by V_0, the value of the expression doesn't change, and one gets:

$$x(\theta) = -\frac{V_o^2}{g}\cos\theta\left(\frac{V_o}{V_o}\sin\theta + \frac{1}{V_o}\left(V_o^2\sin^2\theta + 2gh\right)^{\frac{1}{2}}\right)$$

Simplifying and introducing $\frac{1}{V_0}$ in the inside parenthesis, one gets:

$$x(\theta) = -\frac{V_o^2}{g}\cos\theta\left(\sin\theta + \left(\frac{V_o^2}{V_o^2}\sin^2\theta + \frac{2gh}{V_o^2}\right)^{\frac{1}{2}}\right)$$

$$or\ x(\theta) = -\frac{V_0^2}{g}\cos\theta\left(\sin\theta + \left(\sin^2\theta + \frac{2gh}{V_o^2}\right)^{\frac{1}{2}}\right)$$

E) Grouping constants together

Keeping in mind that *V_0, g, and h are constants*, let's put

$$A = -\frac{V_0^2}{g}\ and\ C = \frac{2gh}{V_o^2}$$

The whole expression becomes:

$$x(\theta) = \underbrace{A\cos\theta}_{u}\underbrace{\left(\sin\theta + \left(\sin^2\theta + C\right)^{\frac{1}{2}}\right)}_{v} \qquad (5)$$

F) Differentiating $x(\theta)$ or $x'(\theta)$

The expression (5) will be used to optimize $x(\theta)$. The curve $x(\theta)$ has a maximum corresponding to the longest range for a specific value of θ.

Reminder: If $x = u \times v$, its derivitive is $x' = u' \times v + u \times v'$

Let's differentiate $x(\theta)$: The derivative of $x(\theta) = x'(\theta)$

$$x'(\theta) = -A\sin\theta\left(\sin\theta + (\sin^2\theta + C)^{\frac{1}{2}}\right) + A\cos\theta\left(\cos\theta + \underline{\frac{1}{2}}\left(\sin^2\theta + C\right)^{\frac{-1}{2}} \times \underline{2}\sin\theta \times \cos\right)$$

or simplifying by 2 and factorizing A:

$$x'(\theta) = A\left[-\sin\theta\left(\sin\theta + (\sin^2\theta + C)^{\frac{1}{2}}\right) + \cos\theta\left(\cos\theta + \frac{\sin\theta.\cos\theta}{\left(\sin^2\theta + c\right)^{\frac{1}{2}}}\right)\right] \quad (6)$$

One now gets a more complex expression for $x'(\theta)$ than for $x(\theta)$. $x(\theta)$ is maximum for a value of θ, requiring that $x'(\theta) = 0$, which corresponds to the summit or maximum of the curve $x(\theta)$. At the summit of a curve, the slope or the derivative at that point = 0.

Let's solve (6):

$$x'(\theta) = A\left[-\sin^2\theta - \sin\theta\left(\sin\theta - (\sin^2\theta + C)^{\frac{1}{2}}\right) + \cos^2\theta\left(\cos\theta + \frac{\sin\theta.\cos^2\theta}{\left(\sin^2\theta + c\right)^{\frac{1}{2}}}\right)\right.$$

G) Establishing the condition to have $x'(\theta) = 0$

Let's find the condition for $x'(\theta) = 0$. A is a constant $\neq 0$. Then one needs the following equation:

$$\cos^2\theta + \frac{\cos^2\theta.\sin\theta}{\left(\sin^2\theta + C\right)^{\frac{1}{2}}} = \sin^2\theta + \sin\theta\left(\sin^2\theta + C\right)^{\frac{1}{2}}$$

H) Use of the trigonometric identity: $\sin^2\theta + \cos^2\theta = 1$

Reminder: $\sin^2\theta + \cos^2\theta = 1$, then $\sin^2\theta = 1 - \cos^2\theta$ and $\sin\theta = \sqrt{1-\cos^2\theta} = \left(1-\cos^2\right)^{\frac{1}{2}}$.

One proposes to keep the expression $\cos\theta$. Let's then change expressions using $\sin\theta$ by their equivalent in $\cos\theta$ as indicated in the reminder above. This then yields:

$$\cos^2\theta + \frac{\cos^2\theta\left(1-\cos^2\theta\right)^{\frac{1}{2}}}{\left(1-\cos^2\theta + c\right)^{\frac{1}{2}}} = 1 - \cos^2\theta + \left(1-\cos^2\theta\right)^{\frac{1}{2}}\left(1-\cos^2\theta + c\right)^{\frac{1}{2}} \quad (7)$$

I) Algebraic handling (factorizing, squaring, simplifying, reducing, etc.)

By grouping and changing terms from right to left and vice versa of equation (7), it becomes:

$$2\cos^2\theta - 1 = (1-\cos^2\theta)^{\frac{1}{2}}\left(1-\cos^2\theta + c\right)^{\frac{1}{2}} - \frac{\cos^2\theta(1-\cos^2\theta)^{\frac{1}{2}}}{(1-\cos^2\theta + C)^{\frac{1}{2}}}$$

Factorizing the right member, one gets:

$$2\cos^2\theta - 1 = (1-\cos^2\theta)^{\frac{1}{2}}\left(1-\cos^2\theta + C\right)^{\frac{1}{2}} - \frac{\cos^2\theta}{(1-\cos^2\theta + C)^{\frac{1}{2}}}$$

In the right member, multiplying the first factor and the denominator of the second factor (part between brackets) by $(1-\cos^2\theta + c)^{\frac{1}{2}}$, the value of the right member is unchanged. Then one obtains:

$$2\cos^2\theta - 1 = (1-\cos^2\theta)^{\frac{1}{2}}\left(1-\cos^2\theta + c\right)^{\frac{1}{2}}\left[\frac{(1-\cos^2\theta + C)^{\frac{1}{2}}}{(1-\cos^2\theta + c)^{\frac{1}{2}}} - \frac{\cos^2\theta}{(1-\cos^2\theta + C)}\right]$$

$$2\cos^2\theta - 1 = (1-\cos^2\theta)^{\frac{1}{2}}\left(1-\cos^2\theta + c\right)^{\frac{1}{2}}\left[1 - \frac{\cos^2\theta}{(1-\cos^2\theta + C)}\right.$$

Multiplying one factor of the twomembers by –1 one obtains:

$$1 - 2\cos^2\theta - 1 = (1-\cos^2\theta)^{\frac{1}{2}}\left(1-\cos^2\theta + c\right)^{\frac{1}{2}}\left[\frac{\cos^2\theta}{(1-\cos^2\theta + C)} - 1\right]$$

Putting the factor between brackets to the left, one gets:

$$\frac{1-2\cos^2\theta}{\frac{\cos^2\theta}{(1-\cos^2\theta+C)}-1}=(1-\cos^2\theta)^{\frac{1}{2}}\left(1-\cos^2\theta+c\right)^{\frac{1}{2}}$$

With the same denominator for the left member, one gets:

$$\frac{\left(1-2\cos^2\theta\right)\left(1-2\cos^2\theta+C\right)}{\frac{\left(2\cos^2\theta-1-C\right)}{1-\cos^2\theta+C}}=(1-\cos^2\theta)^{\frac{1}{2}}\left(1-\cos^2\theta+c\right)^{\frac{1}{2}}$$

or by simplifying:

$$\frac{\left(1-2\cos^2\theta\right)\left(1-\cos^2\theta+C\right)^{\frac{1}{2}}}{\left(2\cos^2\theta-1-C\right)}=\left(1-\cos^2\theta\right)^{\frac{1}{2}}\left(1-\cos^2\theta+c\right)^{\frac{1}{2}}$$

and dividing each member by $\left(1-\cos^2\theta+C\right)^{\frac{1}{2}}$ and changing members from left to right and vice versa, one gets:

$$\left(1-\cos^2\theta\right)^{\frac{1}{2}}=\frac{\left(1-2\cos^2\theta\right)\left(1-\cos^2\theta+c\right)^{\frac{1}{2}}}{\left(2\cos^2\theta-1-C\right)};$$

by cross-product:

$$\left(1-\cos^2\theta\right)^{\frac{1}{2}}\left(2\cos^2\theta-1-C\right)=\left(1-2\cos^2\theta\right)\left(1-\cos^2\theta+c\right)^{\frac{1}{2}}$$

P Q R S

And here we are, but we will have a happy ending, provided one puts the two members to the square, solving and reducing:

By squaring, we get:

$(1-\cos^2\theta)(2\cos^2\theta-1-C)^2=(1-2\cos^2\theta)^2\,(1-\cos^2\theta+C)$

Calculation of Q^2: $(2\cos^2\theta-1-C)(2\cos^2\theta-1-C)=4\cos^4\theta-2\cos^2\theta-2C\cos^2\theta-2\cos^2\theta+1+C-2C\cos^2\theta+C+C^2$ or $Q_2=4\cos^4\theta-\cos^2\theta-4C^2\cos^2\theta+1+2C+C^2$

Calculation of R^2: $R^2=1-4\cos^2\theta+4\cos^4\theta$

Final calculation is: $P^2\times Q^2=R^2\times S^2$ becomes: $(1-\cos^2\theta)(4\cos^4\theta-4\cos^2\theta-4C\cos^2\theta+1+2C+C^2)=(1-4\cos^2\theta+4\cos^4\theta)(1-\cos^2\theta+C)$

It will be long, with a lot of terms *(12 at the left and 9 at the right)*. But victory is coming.

$4\cos^4\theta-4\cos^2\theta-4C\cos^2\theta+1+2C+C^2-4\cos^6\theta+4\cos^4\theta+4C\cos^4\theta-\cos^2\theta-2C\cos^2\theta-C^2\cos^2\theta=1-\cos^2\theta+C-4\cos^2\theta+4\cos^4\theta-4C\cos^2\theta+4\cos^4\theta-4\cos^6\theta+4C\cos^4\theta$

Here are the terms which cancel each other from both sides of the equation:

$4\cos^4\theta-4\cos^2\theta-4C\cos^2\theta+1+2C+C^2-4\cos^6\theta+4\cos^4\theta+4C\cos^4\theta-\cos^2\theta-2C\cos^2\theta-C^2\cos^2\theta=1-\cos^2\theta+C-4\cos^2\theta+4\cos^4\theta-4C\cos^2\theta+4\cos^4\theta-4\cos^6\theta+4C\cos^4\theta$

This yields: $1+2C+C^2-2C\cos^2\theta-C^2\cos^2\theta=1+C$ or: $C+C^2-2C\cos^2\theta-C^2\cos^2\theta=0$ $\therefore C+C^2=\cos^2\theta\,(C^2+2C)$, which becomes;

$$\cos^2\theta=\frac{C^2+C}{C^2+2C}$$

J) Bringing back the original constants

Finally, $\cos^2\theta=\sqrt{\frac{C^2+C}{C^2+2C}}$ (8), knowing that

$$C=\frac{2gh}{V_o^2},\ then\ \frac{C^2+C}{C^2+2C}=\frac{\frac{(2gh)^2}{V_o^4}+\frac{2gh}{V_o^2}}{\frac{(2gh)^2}{V_o^4}+\frac{4gh}{V_o^2}}.$$

With the same denominator for numerator and denominator, one obtains:

$$\frac{\frac{(2gh)^2+2V_o^2gh}{V_o^4}}{\frac{(2gh)^2+4V_o^2gh}{V_o^4}}$$

Simplifying top and bottom, one gets: $\frac{2gh+V_o^2}{2gh+2V_o^2}$

K) Find θ for the maximum range

Then from (8), for a *maximum range x,* one gets:

$$\cos\theta=\sqrt{\frac{2gh+V_o^2}{2gh+2V_o^2}}\ or\ \theta=ar\cos\sqrt{\frac{2gh+V_o^2}{2gh+2V_o^2}}$$

Using this formula, when $h=0$ (cannon and target are on the same horizontal), we have a particular case of the problem; therefore, $\vartheta=ar\cos\sqrt{\frac{1}{2}}=45^\circ$. Elementary, my dear Watson!

Actual 2D Function ($y(x)$Graph) of the Shell Trajectory

Let's find $y(x)$. From (1), we get $t=\frac{x}{V_0\cos\theta}$ and by substitution in (2), one obtains:

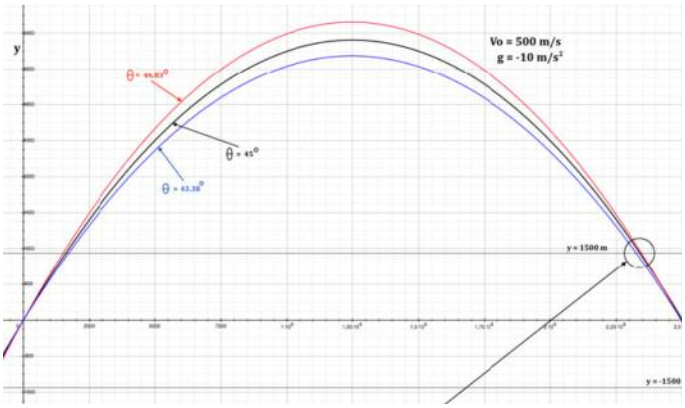
$$y(x)=\frac{g}{2V_0^2\cos^2\theta}x^2+V_0\sin\theta\times\frac{x}{V_0\cos\theta}$$

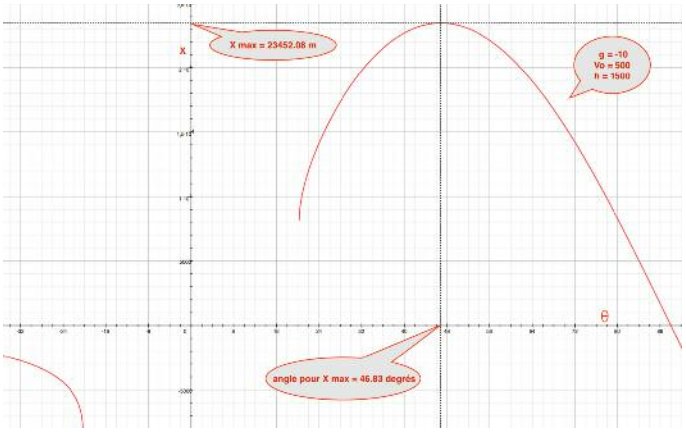
$$or\ y(x)=\frac{g}{2V_0^2\cos^2\theta}x^2+\tan\theta\quad(9)$$

Using curve (9) with constants

$$V_0=500\frac{m}{s},\ g=-10\frac{m}{s^2}$$

and three different shooting angles chosen to allow for the maximum range in cases of horizontal shooting (black), shooting on a plateau (red), and shooting in a valley (blue), one obtains the following graph:





References:

1) Range of a projectile: **en.wikipedia.org**
2) How to find the derivative and the extrema: **an.answers.yahoo.com**

Note on references: Reference 1) shows the equivalent to $x(t)$, $y(t)$, $t(\theta)$, $x(\theta)$ functions and the final answer, but no derivative of $x(\theta)$, no $x'(\theta) = 0$ long process of handling mathematical expression, and no graphs. Reference 2) shows the equivalent of $x(\theta)$, the derivitive of $x(\theta)$, and three suggested *big* steps (not obvious at all) to "help" handle mathematical expressions. There are no details, no answers, and no graphs. No wonder, this is typically what you get on the Internet. Information is there, but is not always very clear and complete. Be ready to guess a lot, to use your imagination, and be challenged.

Two Samples of Curves for Maximum Range

1. Shooting on a Plateau

The curve illustrated below is:

$$x(\theta) = A.\cos\theta.\left(\sin\theta + \left(\sin^2\theta + C\right)^{\frac{1}{2}}\right)$$

See (5), where

$$A = \frac{-V_o^2}{g} \text{ et } C = \frac{2gh}{V_0^2}$$

For the following parameters

$$g = -10\frac{m}{s^2}, V_o = 500\frac{m}{s}, h = 1500m \text{ (plateau)}$$

the expression of the curve:

$$x(\theta) = -\frac{500^2}{-10}\cos\theta\left(\sin\theta + \left(\sin^2\theta + \frac{2(-10)\times 1500}{500^2}\right)^{\frac{1}{2}}\right)$$

or: $x(\theta) = 25000\cos\theta\left(\sin\theta + \sqrt{\sin^2\theta + 0.12}\right)$

The discontinuity of the curve comes from the zone where the expression below the radical is negative (first quadrant). One can see that for shooting on a plateau, for a maximum range, one has to increase the angle of shooting θ with respect to 45˚. This reduces the range for a shooting on the same horizontal, but will increase somewhat the trajectory time (since the vertical speed is higher) and will cause the horizontal speed to influence the trajectory for a longer time, thus resulting in a longer range.

2. Shooting in a Valley

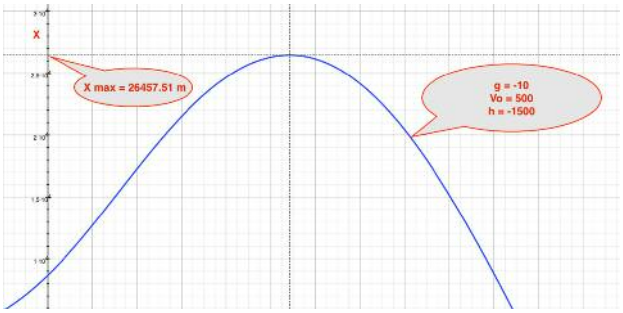
The curve illustrated below is:

$$x(\theta) = A\cos\theta\left(\sin\theta + \left(\sin^2\theta + C\right)^{\frac{1}{2}}\right) \text{ See 5, where } A = \frac{-V_o^2}{g} \text{ et } C = \frac{2gh}{V_0^2}$$

$$\text{For the following parameters}: g = -10\frac{m}{s^2}, V_o = 500\frac{m}{s}, h = 1500m \text{ (valley)}$$

$$x(\theta) = -\frac{500^2}{-10}\cos\theta\left(\sin\theta + \left(\sin^2\theta + \frac{2(-10)(-1500)}{500^2}\right)^{\frac{1}{2}}\right)$$

$$\text{or } x(\theta) = 25000\cos\theta\left(\sin\theta + \sqrt{\sin^2\theta + 0.12}\right)$$



Here, the expression under the radical is always positive (first quadrant), then the curve has a continuity. One can see that for shooting in a valley, for a maximum range, one has

to decrease the shooting angle with respect to 45˚. This would reduce the range for a shooting on the same horizontal, but will increase the horizontal speed. The part of the trajectory between the horizontal and the bottom of the valley increases the time of the trajectory, plus with higher horizontal speed, influencing the range, which will be increased. ▲

Blog Post

Mary Bourassa: marybourassa.blogspot.ca

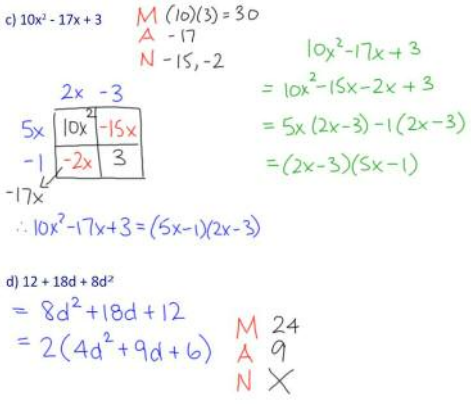
MPM2D - Day 57: Factoring & Speed Dating



It is interesting to watch a class of teenagers arrange the desks given only verbal instructions. They eventually got them into two rows, facing each other. I told the class that they would be speed dating this morning, which caused a few raised eyebrows. Then I said factoring speed dating. I gave each student a piece of paper (half of an 8.5" by 11" sheet) upon which was written a quadratic expression. Their first job was to factor that, below the fold line. They also had to check their work by expanding and simplifying their answer. They were now each the expert on factoring their quadratic.

Once everyone was ready (and I took care of the one quadratic that didn't factor - oops) they got ready. They each factored the quadratic opposite them. I started the timer at 1 minute which turned out to be too short - we settled on 1:30 for each round, then cut back to 1 minute once they were all sure of what they were doing. I also got some background music going as it seemed too quiet. They each ended up factoring 13 quadratics in a much more fun way than just giving them a worksheet and saying factor.

We spent the remainder of the period working through examples together.



▲ Provincial Digital Learning Resources – What's New?









AGNES GRAFTON

ROSS ISENEGGER

MARKUS WOLSKI

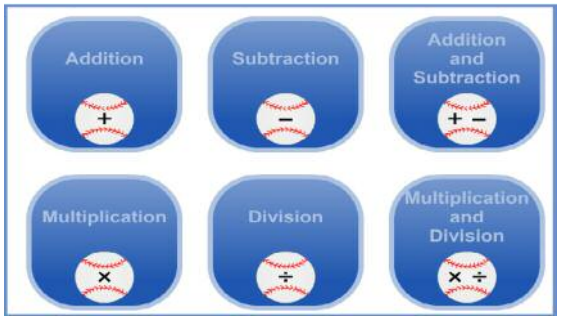
GREG CLARKE

*Ross (Near North District School Board), Agnes (Brant Haldimand Norfolk Catholic District School Board), Markus (Bluewater District School Board), and Greg (Simcoe Muskoka Catholic District School Board) are all currently on assignment with the Ontario Ministry of Education as Provincial Math Leads, working on the CLIPS and **mathies.ca** projects.*

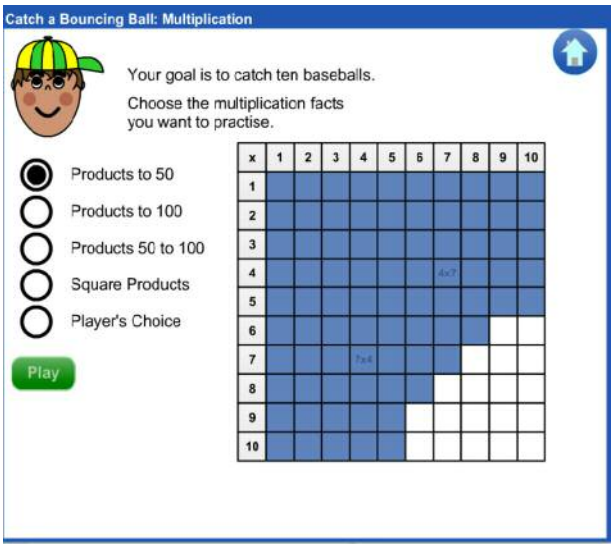
The team has been busy over the last few months, working on updates to a number of apps, the support wiki, as well as adding new information and functionality to the **mathies.ca** website itself. Here are some of the highlights of that work.

Catch a Bouncing Ball – Operations

This Operations game joins the family of mathies Catch a Bouncing Ball games and is the first game to be released on multiple platforms: Windows, Mac, iOS (on the App Store), and Android (on Google Play). Players decide which of six operation levels they wish to play. For single-operation games, the unknown value is the result (e.g., 3 + 5 = ?). For mixed-operation levels, the position of the unknown value changes (e.g., 3 + ? = 8).

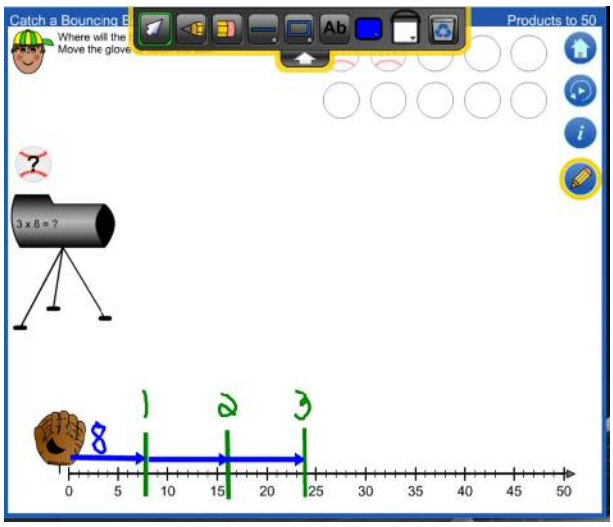


Each operation level offers players a choice of the types of numbers they wish to practise. For example, this multiplication level provides some pre-set product maximums, as well as the ability for the player to select only certain facts or fact families.

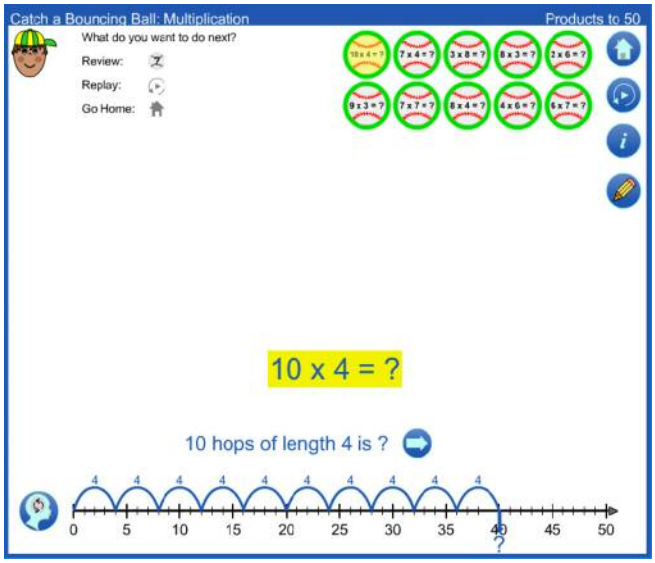


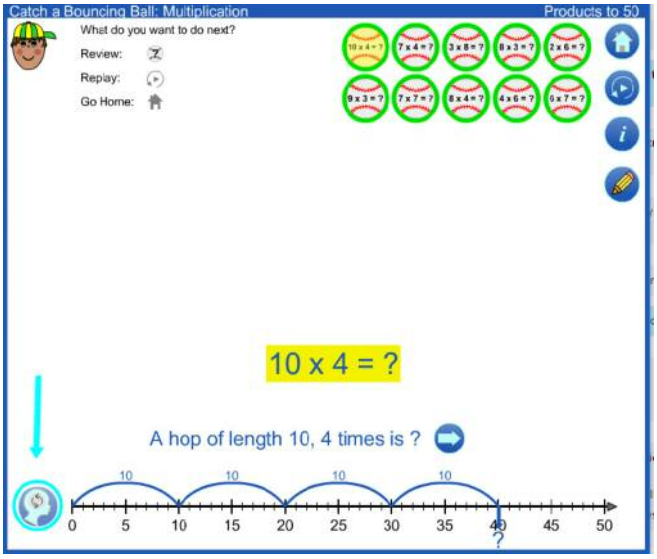
Players are presented with a series of ten questions. They determine the answer and drag the glove to the correct answer on the number line. The annotation tools are also available so that players can use the number line as a thinking space prior to placing the glove.

When appropriate, related questions appear back to back to help students notice certain properties. For example, in the multiplication game, facts such as 3 x 8 and 8 x 3 might appear consecutively to help reinforce the commutative property, while a subtraction game might show questions from the same fact family, such as 10 - 3 and 10 - 7.



At the end of the multiplication and division games, players have the opportunity to view a representation of the question on the number line. Players switch between questions by clicking a baseball, and switch between two different representations by pressing the "thinking head" button. This feedback is intended to deepen conceptual understanding of the operation by making connections between the symbolic and visual representations.





Current programming work includes incorporating feedback for the addition and subtraction levels of the game. Watch for it as well as the following annotation improvements in the next update!

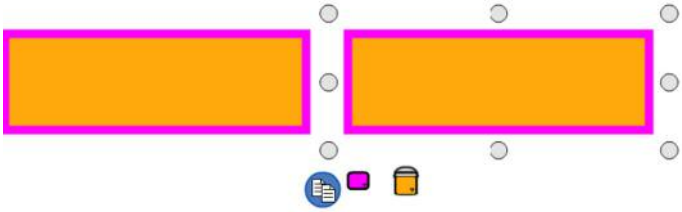
Annotation Updates

Our apps share functionality and benefit from an update to any one of them. This is especially true of the built-in Annotation tools. As part of recent updates:

- usability has been improved with text boxes and selection mode
- the default text colour is black

- lines that are close to being horizontal or vertical snap into position

The ability to copy has also been added. Create any annotation object, whether a free-hand drawing, arrow, line, shape, or text box. Change to Selection mode and notice that the copy button appears. When you click it, an exact duplicate of the annotation object is created.



This is useful for ensuring that labels have the same formatting or that shapes are congruent.



Venez ici pour vos apps en français

On the Learning Tools page, you will notice that some of the apps now sport a stylish Ouvrir button to go along with their OPEN button—our not-so-subtle way of indicating that this app is available in French.

In response to requests from the French mathematics community (French Immersion as well as French-language school boards), we have a team of educators who have assisted us in translating the text in some of our learning tools. As of the date this column was submitted, we have French translations for the Money tool, Notepad, Rekenrek, and Relational Rods tool.

If you click the OUVRIR button from the Learning Tools page, the tool will automatically open up en français. Regardless of how you open up the tool though, for those that have French versions, you will see a language-switcher button that displays the current language. Clicking on it swaps to the other one: en or fr

Downloads and Installers

On the Games or Learning Tools page at **mathies.ca**, the function of the Download button has changed. Previously, it automatically started to download the SWF file (Flash file) for the specific game or learning tool so you could run it in your browser. However, web browsers are continuing to become more hostile to raw Flash files (especially ones saved locally), and we thought it would be helpful to give some additional instructions on how to deal with this. Now, when you click on the Download button, you will be taken to a new page for that specific game or learning tool, with a number of options available.



In addition to the Flash SWF file, some tools now have the option to download an installer, for Mac or Windows, that will allow you to install the game or learning tool just like any other full-fledged computer program. Beside each button is some help text to lead you through the process, regardless of which option you choose. Using a downloaded version of a learning tool or game means that you are not reliant on an Internet connection.

Feedback and Future Requests

Please feel free to send us your feedback about any mathies tool, using the Feedback Form button inside the Information Dialog, accessed from the i button.



You can also send your comments to **WhatsNew@oame.on.ca**. You can share your experiences on Twitter, using the hashtag #ONmathies, and follow or message us at @ONmathies. There is an increasing set of interesting posts of student and teacher work on Twitter. To be among the first to find out about the latest digital-tool developments, sign up for our email list at **http://mathclips.ca/WhatsNewEmailList.html**. ▲

▲ Fields Institute MathEd Forum Notes

STEWART CRAVEN



Stewart Craven presently teaches courses for the Continuing Education Department and the Transition Years Program at York University. He is an active participant in the Fields Mathematics Education Forum.

The annual **Margaret Sinclair Memorial Award** is announced each February. This award recognizes an educator in Canada, who has demonstrated innovation and excellence in promoting mathematics education at the elementary, secondary, college, or university level.

This year, the recipient of the award was Jean-Marie De Koninck, a professor at Université Laval. Mr. De Koninck's acceptance speech was given at the Fields Institute on April 19 and was entitled "The Human Part of the Equation."

De Koninck posed a number of questions, beginning with, "Are we good at math?" He cited research findings that humans are hard-wired to understand the concept of number, but acknowledged that as we grow older, many, if not most of us, become anxious about anything mathematical. He suggested that many young people struggle with the abstract nature of our subject and are not introduced early enough to the humanistic aspects of mathematics. By this, he refers not only to the meaningful applications of mathematics with respect to how we live our lives, but also to learning about the history of mathematics and the very human stories about mathematicians that are typically missing from the schooling experience.

De Koninck provided a number of stories about mathematicians, their work, their emotions, and their inspirations. For example, Sophie Germain, French mathematician, physicist, and philosopher (1776–1831), was inspired to do mathematics after having read the account of Archimedes' tragic death. Using photographs, De Koninck demonstrated that mathematicians are not necessarily dour individuals, but rather, people who express joy and have other interests outside of mathematics. For example, Einstein played the violin, and Turing was a marathon runner (the best in England in the late 1940s).

De Koninck also spoke about how mathematicians solve problems and make discoveries, emphasizing that the process is far from linear. In most cases, mathematicians start by taking time to deeply understand the essence of the problem (the "preparation stage"). Next comes the "incubation stage," where the subconscious takes over. Finally, the "illumination stage" happens as the solution emerges.

Sometimes, problems take centuries to solve; Fermat's Last Theorem is a great example. De Koninck told this story to highlight the human struggles and triumphs that led to the Wiles Proof. Along the way, there were rivalries and jealousies on one hand and co-operation and mutual support on the other. He traced through the story by highlighting the roles of various mathematicians, including Germain, Euler, Kummer, Wolfskehl (he willed 100 000 marks to be a prize for the proof in1906), Faltings, Frey, Ribet, Taylor, and of course, Wiles.

De Koninck illustrated how mathematics is presently being popularized through literature and cinema. Simon Singh not only wrote about the Wiles Proof in *Fermat's Last Theorem*, but also about the mathematics buried in the *Simpson's* comedy TV series. Furthermore, recently released movies such as *The Man Who Knew Infinity, The Imitation Game,* and *Hidden Figures* have all received critical acclaim.

In closing, De Koninck talked briefly about a program he initiated called "Thriving in Math." The goal, simply put, is to help teenage students become enthusiastic about math. Mr. De Koninck suggests that by making the study of mathematics more humanistic in its approach, we might well make a difference in achieving this goal.

The audience was very receptive and seemed to greatly enjoy Mr. De Koninck's talk.

Our MathEd Forum meetings will start again in September and you are welcome to attend. More information can be found at: **www.fields.utoronto.ca/programs/mathed/forum/**. ▲

Ask Assessment Abby: What are some ways I can promote discussion in my classroom, using student work?



ASSESSMENT ABBY
EMAIL: assessmentabby@oame.on.ca

Ask Assessment Abby A³ is a regular column in the OAME *Gazette*, where teachers can share concerns and best practices about assessment, evaluation, and reporting of mathematics. Please send your questions to Ask Abby at **assessmentabby@oame.on.ca**.

Dear Assessment Abby,

What are some ways I can promote discussion in my classroom, using student work/exemplars?

Facilitating discussions to deepen mathematical understanding can be very challenging and involves a lot of planning, but the rewards far outweigh the challenges. In order to have rich meaningful discussions, we need to move beyond just putting up work, or having students share at the board, and having a "show and tell." As with everything else that makes for effective teaching, we need to have a clear and concise goal in mind. This goal should be considered when selecting the task, as well as when choosing the work to be discussed.

Two valuable resources that may really help you structure and conduct these discussions are *5 Practices for Orchestrating Productive Mathematics Discussions* (Smith & Stein, 2011) and *Intentional Talk: How to Structure and Lead Productive Mathematical Discussions* (Kazemi & Hintz, 2014).

Smith and Stein identify five practices that will help orchestrate productive discussions: anticipating, monitoring, selecting, sequencing, and connecting. Before giving students a problem or task, you should spend some time *anticipating* various ways the students might solve it, and some possible challenges or struggles they may encounter. When students are working, you should be *monitoring* how they are solving the problem. You will have a clear focus of what you are looking for that is connected to your goal, and will have questions and prompts ready because you have taken the time to anticipate the struggles. *Selecting* and *Sequencing* the work also links directly with your goal for the discussion. This may involve having students compare similarities and differences among strategies, or examining a piece of work that clearly communicates the strategy used. Other times, it may involve sequencing from most common strategy used to progressively more complex/sophisticated strategies. You may even choose incorrect work if that will allow the students to better understand the mathematics that you are trying to have the students understand. When students make *connections* between solutions and/or strategies, it deepens their understanding of the mathematics in the lesson, as well as creating a stronger foundation for problem solving in general.

Students need to learn strategies for sharing their ideas, and developing their expertise in sharing will take time. The classroom climate needs to be one where students will take risks, disagree and question each other, and be engaged in exploring mathematics. Students should co-construct anchor charts and develop common vocabulary (language) to look at student work and discuss it (Kazemi & Hintz, 2014).

Using the following "talk moves" during discussions will help you engage all students:

- Prompt (teacher prompts students to answer a question or explain their thinking)
- Wait Time (provide adequate time to respond… this allows students to reflect on the question and formulate an answer)
- Revoice (teacher repeats student responses to emphasize and/or clarify what the student said)
- Restate (teacher asks another student to restate what was said, directing them to listen and respond to each other)
- Apply Reasoning (teacher allows students to engage with each other's ideas)
- Turn and Talk (teacher allows students to clarify and share their ideas)

Ideas and resources to foster discussions:

1. Have your students solve an EQAO open-response question (in partners). **www.eqao.com/en**
2. Group discussion: Strategies used to solve the problem. **www.math4thenines.ca/professional-learning.html** (Five Practices)
3. Look at exemplars of student work from EQAO website (without the scoring code identified).
4. Create an anchor chart that identifies strategies, mathematical models, keywords, and vocabulary.
5. Have students do a gallery walk to provide feedback to each other, using the anchor chart to help guide their feedback. **www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_Communication_Mathematics.pdf**

Free Ontario Ministry of Education sources that might be useful:

Student Interaction in the Math Classroom: Stealing Ideas or Building Understanding **www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/Bruce.pdf**

Asking Effective Questions **www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_AskingEffectiveQuestions.pdf**

Communication in the Mathematics Classroom **www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_Communication_Mathematics.pdf** ▲

Blog Post

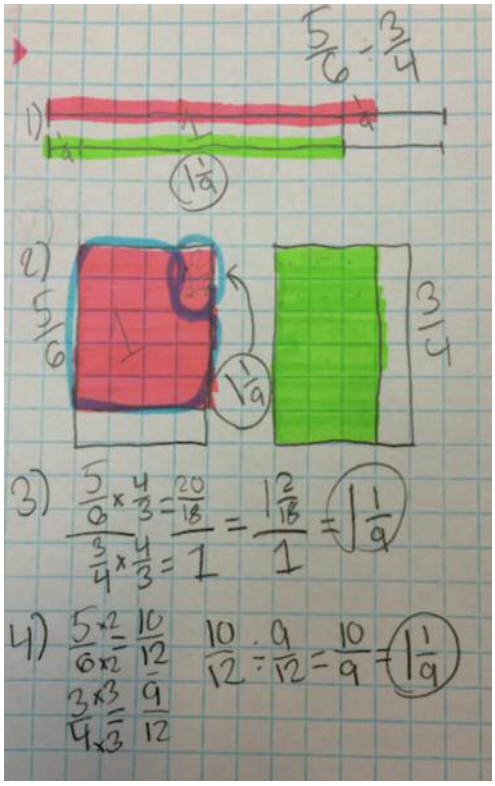
Fawn Nguyen: fawnnguyen.com

Dividing Fractions

My 6th graders have been working with dividing fractions for the last two weeks. We explore these four ways, in this order:

1. Number line
2. Rectangles — I wrote about this here.
3. Dividing by one
4. Common denominator
5.

It's completely intentional that we work with the number line and rectangles first. I want my kids to see the answer and that it should match their intuition and understanding.



▲ MB4T (Mathematics by and for Teachers): Building Equality One Cube at a Time

KRISTA SARMATIUK



Krista is the K–12 Board Mathematics Facilitator for the Sudbury Catholic District School Board and the President of NOMA. In this role, she supports educators and instructional leaders in math learning and teaching. Recently, she spent three years teaching Grade 3 French Immersion, and has previously been a Learning Strategies Teacher, a Numeracy Facilitator, and an Acting Vice-Principal.

I was hanging out with my math buddy, Jenni, today and we got into a conversation about the big idea of equality. Jenni and I have had some great math conversations over the past year. She's jumped into math with both feet, taking P/J Math Part 1 while participating in the Math Leadership Network, the North East region's big bold adventure. Her favourite saying is, "My brain is growing!," referencing Jo Boaler's work in growth mindset for mathematics.

"What do you think 'equal' means?," I ask her.

"The same," she says, "but not the same…."

When pushed to further explain her thinking about equality, she gestures with her hands to indicate balance.

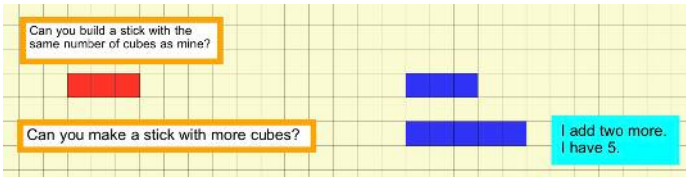
The idea that two things can look different, but have the same value, is very important. This is something that seems to be missing for many of the students I've been working with this year. Understanding equality lays the foundation for their understanding of equations. An equation is a pair of expressions that have the same value, even though they don't look the same.

Lately, I've been thinking a lot about equality and the misconceptions that surface in Grade 3. These misconceptions impact student thinking across all five strands. Many educators argue that students don't understand the language used in word problems. However, it's worth considering the possibility that students have merely constructed a limited definition of equality, effectively putting the brakes on their algebraic reasoning. How can we help students to see Jenni's visual understanding of this relationship?

In an effort to explore and correct these misconceptions, my colleagues and I have been engaging in discussion and exploration, using cube lines as concrete models with groups of students in Kindergarten to Grade 3.

Young learners start to build the idea of equality by comparing quantities and building an understanding of more, less, and the same. Students can be observed comparing measurements or quantities while at play. Recently, we used interviews like those presented at **www.mathforyoungchildren.ca**, asking students to demonstrate these concepts on a 5-frame.

In several cases, the students were puzzled by this prompt, and struggled to use counters to show appropriate responses. Our observations led us to wonder about our choice of models. Though the students had been using 5- and 10-frames in their classroom regularly, we wondered if this model was concrete enough for this group of students. This led us to try using linking cubes to model the same.



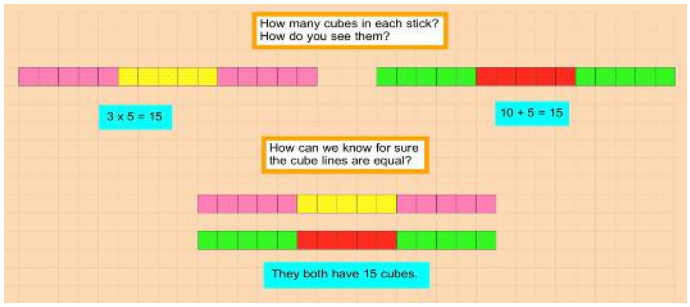
The students who had experienced difficulty with the representation on the 5-frame successfully built sticks of cubes that had more, less, or the same number of cubes as my stick. Some students went further, identifying how many more cubes there were on a longer stick, and how many more they would need to add to a stick to make it the same as another. When asked how they knew, they simply put the two sticks together to compare their lengths. One student stated he had measured them. This experience really underlined the importance of the concrete linear model as a starting point to explore equality. We also noticed that students independently used comparison as an efficient strategy when determining equality, and they were confident that their answer was correct.

At the same time, working with Grade 3 students in two schools, we uncovered an important misconception. When asked to choose a number to complete the equation 24 = 17 + 3, almost every student selected the number "7" from a set of possible answers. When asked to explain their thinking, students indicated that they counted back from 24 to 17 and so they knew this was the answer. This activity exposed a key misconception among students in this age group.

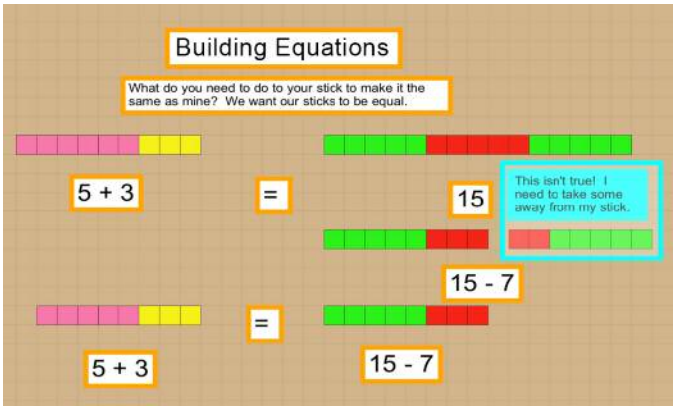
A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 3: Patterning and Algebra (Ontario Ministry of Education [OME], 2007) describes the importance of building an understanding of the equal sign in the development of algebraic reasoning. Often, we find that students consider the equal sign as "a signal to perform an operation" and simply provide a number that answers the operation prior to the sign. See page 27 of *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 3: Patterning and Algebra* for a more detailed explanation. Our observations of students solving 24 = 17 + 3 suggests that students recognized 24 – 7 = 17, completely ignoring the + 3. We wondered how to get them to see both sides of an equation, and realized that perhaps we should just try to build both sides and then compare to prove they are equal.

Cathy Fosnot encourages us to think about the importance of a linear model when considering quantity (see **http://thelearningexchange.ca/videos/young-mathematicians-models-powerful-tools-for-thinking/**).This led us to wonder if a linear model might help students to build equations of their own, thus enabling them to think explicitly about two different expressions.

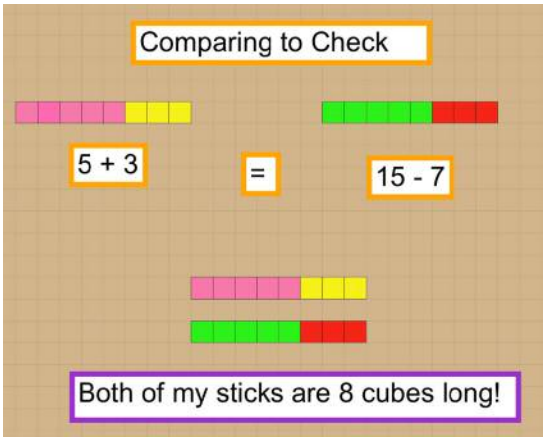
We began with a simple example to activate prior knowledge, building two sticks of 15 linking cubes (in alternating groups of 5 in two colours). Students were asked to determine the length of each stick visually. When we asked how we could prove the sticks were equal, they suggested we put them on top of each other to compare them. This sounded familiar: using measurement and comparison as strategies to determine equality. We then had the opportunity to spread our two sticks apart, and the students noticed that the two sticks formed a familiar symbol—an equal sign.



Our next step was to use the two equal lines to provoke further visual connections. I broke my stick into two pieces, hiding some of my cubes and presenting my smaller stick to the students. I asked them how many cubes they saw now and recorded the way they saw them as an addition statement. My co-teacher, Denise, held up her stick of 15 again. The students were not happy with our equation. They told us that the two sides weren't equal, and we were able to record their suggested solution, using a subtraction statement.



Our final step was to check to see if the two sides of the equation were equal. By holding our two sticks on top of one another, we proved they were the same length.

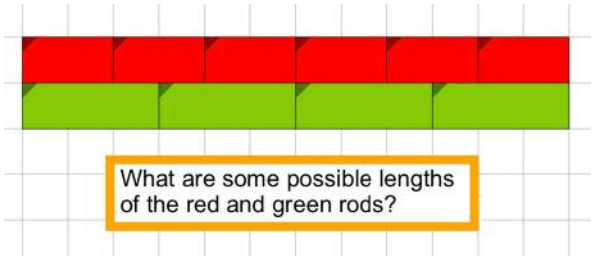


This lesson can be extended using an activity provided in *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6: Patterning and Algebra, Grades 4 to 6* (OME, 2008) (**www.edugains.ca/newsite/math/guides_effective_instruction.html**).

In Grades 4 to 6, students build on these experiences of finding missing numbers in simple equations, and using the associative property as they apply the concept of equality with multiplication and division. This helps them to deepen their understanding of operations, and to build more complex visual models when thinking mathematically. Seeing relationships between numbers and using these relationships to solve problems is important for every young mathematician.

Consider using cube lines or relational rods to continue the conversation in junior classrooms.

Cube lines can be used to think about decimals as well as whole numbers. Relational rods allow students to explore multiplicative relationships, fractions, and ratios.



In the end, equality is determined by what you're measuring. The key is building an understanding of what equality means, whether we are building and comparing lengths, amounts of money, congruent shapes, or scale diagrams. As students build visual models, using concrete materials, they will to be able to see the same thing in different ways, which is what equations are all about.

Ontario educators are incredibly fortunate to have a wealth of learning materials and tools at their fingertips, created for and by teachers from our province, informed by research. Here are a few helpful links related to the ideas presented in this column.

Cathy Fosnot: *Models and Powerful Tools*

http://thelearningexchange.ca/videos/young-mathematicians-models-powerful-tools-for-thinking/

Jo Boaler on growth mindset

http://thelearningexchange.ca/projects/jo-boaler/

EduGAINS: **www.edugains.ca/newsite/math/guides_effective_instruction.html**

- *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 3: Patterning and Algebra,* 2007
- *A Guide to Effective Instruction in Mathematics, Kindergarten to Grade 6: Patterning and Algebra, Grades 4 to 6,* 2008

Math Leadership Network (Learn more about the learning happening in the North East Region.) **http://mathleadershipnetwork.ca/**

Math for Young Children: Playful Mathematics **http://mathforyoungchildren.ca/playfulmath.html**

All of the images shared here were created using the free mathies coloured tiles and relational rods apps. For a collection of free apps and activities, please visit **http://oame.on.ca/mathies/index.html**. ▲

Mathematics Knowledge Network



GEORGE GADANIDIS

Dr. Gadanidis is an author of math and science stories for young children and a professor and researcher at the University of Western Ontario.The math and science stories he has authored with his daughter Molly, have been widely used in classrooms in Canada and beyond. George holds many scholarly and research awards, including a Lifetime Fellowship to the Fields Institute for Research in Mathematical Sciences. His research and teaching involves seeing mathematics through an Arts lens and using technology to bring mathematics to the wider world.



DONNA KOTSOPOULO

Dr. Kotsopoulos is a professor in the Faculty of Education, cross-appointed to the Department of Mathematics in the Faculty of Science. Dr. Kotsopoulos is the director of the Mathematical Brains Laboratory. She is an Ontario Certified Teacher (OCT). She was recently awarded an OCUFA teaching award. She was recognized in 2016 for her outstanding research and outreach activities over her career with the national Partners In Research Mathematics Ambassador Award. Dr. Kotsopoulos's research focuses on early mathematical cognition and learning. She has an emerging interest in post-secondary leadership.



DRAGANA MARTINOVIC

Dr. Martinovic is a Professor of Mathematics Education at University of Windsor, where she leads the Human Development Technologies Research Group. In her research, Dragana explores ways in which technology can improve teaching and learning outcomes, and the digital literacy skills needed for a successful learner and worker of the 21st century. She is dedicated to supporting teacher-led research and providing opportunities for all involved in education to collaboratively work toward increasing student engagement, success, and love for learning.

The Mathematics Knowledge Network (MKN) is a four-year KNAER Phase 2 project, funded by the Ontario Ministry of Education and hosted at the Fields Institute's Centre for Mathematics Education. The MKN is a partnership of nine Ontario organizations (including OAME) and twelve Ontario universities (see the list of partners at the MKN website: **mkn-rcm.ca**). The mandate of the MKN is to develop self-sustaining communities of practice on key areas of mathematics education, to engage educators and share practice-based and research knowledge across Ontario, in support of Ontario Ministry of Education policies and initiatives.

In this brief report, we outline: (1) the core principles of the MKN, (2) its four communities of practice, and (3) how you can get involved with the MKN.

Core Principles

Three principles guide the work of the MKN.

Addressing educator-identified needs for improving student learning

Scholars such as Fullan (2007), Hess (1998), and Elmore (1990) note that many, if not most, innovations have little impact on teaching and learning. A key goal of the MKN is to empower educators in a reform process, where they are active agents in their professional development, in the same way that teachers endeavour to empower students to be active agents in their mathematics learning. Mathematics teaching and learning needs identified by teachers will be at the core of our work, to engage educators in reform that is personally and professionally meaningful and rewarding. Ministry directions, particularly the Renewed Math Strategy (RMS, 2016) and evidence- and practice-based knowledge, will frame how we collaborate to address these needs in pedagogically and mathematically powerful ways that produce actionable, and shareable resources to improve student learning.

Changing attitudes toward mathematics

There is a long-standing problem with mathematics that we can no longer ignore: for generations, young children have entered school mathematically curious, enthusiastic, and capable, only to develop as adults who typically dislike and avoid the subject. Humans are biologically wired to seek, attend to, learn from, and take pleasure from surprising events. The MKN will engage educators in co-designing learning experiences that offer students mathematical surprise and conceptual insight, and opportunities to share their learning with family and friends, and with the wider community (Gadanidis, Borba, Hughes, & Lacerda, 2016). We plan to share our work through the MKN website (**mkn-rcm.ca**), through our partner mathematics education organizations (such as AFEMO and OAME), and through online documentaries and networks at TVO's **TeachOntario.ca**.

Fostering inclusion

Some groups are more marginalized than others when it comes to mathematics education, including Indigenous learners, newcomers to Canada, girls, and children with learning and socio-economic challenges. In alignment with priority areas of Ontario's RMS (2016) for improving student engagement, and equity of outcomes and well-being for marginalized students, we will endeavour to enhance learning and participation opportunities for marginalized learners. The goal is to work collaboratively to identify

and enhance strategies to strengthen marginalized students' performance in mathematics, and the practices of teachers, including special education teachers, who support these students. By designing experiences that allow for differentiated learning, using Seymour Papert's (1980) low-floor and high-ceiling approach, students can engage with big math ideas with minimal prerequisite knowledge, while having opportunities to extend their learning to more complex relationships and varied representations.

Communities of Practice

There are four communities of practice (CoPs) in the MKN. The initial formation of these CoPs stemmed from a variety of sources including research, policy initiatives, and reconciliation. The CoPs typically include representation from university partners and administrators and teachers from multiple school boards, French-language representation, and other professional or parent organizations. The four CoPs are:

1. *Mathematics Leadership.* A major emphasis of the RMS (2016) is on increasing professional learning opportunities for teachers and principals. The RMS requires that each elementary school have at least one math lead teacher to support mathematics education in their respective schools. These new math lead teachers, each with differing experience and expertise in mathematics content and pedagogy, are also simultaneously in a new leadership role/partner for school administrators. While researchers still must characterize "the mathematical knowledge and skills that leaders require" (Borko, Koellner, & Jacobs, 2014, p. 150), the goal of this CoP is to develop powerful practice-based models for supporting school-based mathematics leaders in their role. This CoP will also mobilize other leaders in the Province, including those who are school-board-based and faculty of education-based, with the idea that working collectively, we could provide adequate support to teachers, thus reaching every classroom and every child. Co-leading this CoP are Dr. Dragana Martinovic (University of Windsor) and the Greater Essex County District School Board.
2. *Critical Transitions in Student Mathematical Development.* The objectives of this CoP are to focus on transitions proven to be challenging for students, and to enhance the potential of "schools and people to function as communities which build bridges between students, parents, teachers, and communities" (Tilleczek, 2012, p. 17). Critical transitions for the purpose of this CoP is broadly defined and includes grade transitions (into formal schooling, Kindergarten to Grade 1, Grade 8 to Grade 9, etc.), transitions into learning environments (i.e., for new Canadians, language learners), and transitions into learning contexts that enhance opportunities for special-needs learners. The Co-lead for this CoP is Dr. Lynda Colgan from Queen's University.
3. *Indigenous Knowledge in Mathematics Education. The recent Truth and Reconciliation Commission of Canada: Calls to Action* (2015) makes recommendations for ways in which educators and school boards can contribute to the decolonization of Indigenous peoples. The need to adapt these recommendations is underway, and some progress has been made for most subject areas. The goal of this CoP is to promote and advance Indigenous knowledge in mathematics learning for FNMI (First Nations, Métis, and Inuit) students (on and off reserve) and other students, building on ways that have been identified through Indigenous scholarship and community practice. The Co-leads for this CoP are: Ms. Jodie Williams (FNMIEAO [First Nations, Métis, and Inuit Education Association of Ontario]) and Ms. Colinda Clyne (FNMIAEO).
4. *Computational Thinking in Mathematics Education.* Computational thinking (CT) in its various forms—such as screen-based coding, unplugged procedures, and digital tangibles—is growing as a pedagogical interest within and outside of Ontario schools, and there are pressures to include CT in curriculum. In January 2016, Prime Minister Justin Trudeau said, "We need to do a lot better job of getting young people to understand what coding is and how it's important, how to program, how to problem-solve, how to create the most elegant algorithm possible." In the same month, British Columbia announced that computer coding (one example of CT) will be added to all grades of the K–12 curriculum. CT assists students in developing their thinking in ways that support problem-solving, analytical thinking, and technology integration. This CoP will mobilize evidence-based and practice-based knowledge for integrating of CT to support mathematics instruction. The Co-leads for this CoP are Dr. Chantal Buteau (Brock University), Dr. Bev Casswell (Robertson Group, OISE/UT), Dr. George Gadanidis (Western University), and Bryan McMillan (Wellington Catholic District School Board).

How to Get Involved

Inclusion is the core principle of the MKN and its CoPs. Since most of the professional educators' organizations in Ontario have already partnered with MKN, it is most natural that educators contact their organizational representatives to learn how they could get involved and benefit from this partnership. Updates about activities of the MKN and the CoPs will be available at **mkn-rcm.ca**, through Twitter @mknrcm, and through TeachOntario. We will organize webinars, workshops, and participate in virtual and face-to-face professional gatherings across the Province (e.g., OAME conference), which should provide ample opportunities for engagement. Throughout the school year, the Centre for Mathematics Education organizes monthly MathEd Forums (**www.fields.utoronto.ca/activities/workshops/mathed- forum**) that are open to the public. Educators can attend

these meetings, which are excellent sources of informal professional development, and those who cannot attend, can watch the videos of presentations, either as live-streamed during the talk or as permanently available at a link below the abstract of each talk.

To address teacher-identified needs, we will conduct annual surveys and interviews with teachers and educators. Please respond to our calls and/or write to CoP leads to share your professional needs and observations, your successes and wanderings, and the results of your collaborative inquiries and professional development activities. Let us know if and how the work of the MKN has affected you, your school, students, and parents. We will be happy to respond and to embed your story in the documentary of our journey toward excellence in mathematics education.

We are inviting all educators to help us in changing attitudes toward mathematics among our students and general public. Send us the artifacts demonstrating your students' love for mathematics and how you have created conditions for your students to share their mathematics with parents, guardians, caregivers, and community members.

To achieve our goal to reach every teacher in Ontario, any educator can join the CoPs of their main interest to directly follow the announcements and utilize resources. Every year, the MKN will organize an Institute, and a series of meetings and workshops, which will be documented and posted on the MKN website. In October 2017, the MKN will organize a Computational Thinking Symposium, with guest speakers and working groups. The CoPs will create related research briefs, and pedagogy-, leadership-, and content-related artifacts to show what mathematics educators are doing to improve mathematics outcomes for students. The contact person for inquires is MKN Coordinator Melanie Langemeyer (mlangeme@fields.utoronto.ca or (416) 348-9710, x3042).

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▲ Technology Corner: NRICH – Enriching Mathematics



MARY BOURASSA
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Mary teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

The NRICH website can be found at **http://nrich.maths.org/**. It is a treasure trove of rich thinking activities for students from Grades 1–12.

From their website,

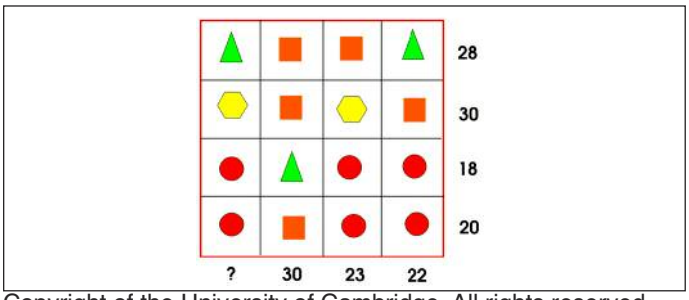
"NRICH aims to:

- enrich the experience of the mathematics curriculum for all learners
- offer challenging and engaging activities
- develop mathematical thinking and problem-solving skills
- show rich mathematics in meaningful contexts
- work in partnership with teachers, schools, and other educational settings"

The website is organized into "homes" for students and teachers in both primary and secondary. If you click through to the teacher home, there are a number of featured activities along with collections which are hand-picked activities organized by topic. For example, there are 130 activities dealing with trigonometry. There is also a strong search feature to help you find what you are looking for.

Here are a few examples of activities you can find on the NRICH site.

What's It Worth? (http://nrich.maths.org/1053)



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Students need to figure out the value of the question mark, given that the numbers at the end of each row and column represent the sum of the numbers that each symbol represents. These types of questions allow students multiple approaches. Students can use manipulatives (pattern blocks and pennies, for example), make picture equations, or replace shapes/colours with variables to create more abstract equations. There is more than one path to the solution, so students who finish quickly can see if they can find an alternate solution. I really like that the answer is not the value of each shape, but a sum of four shapes. You can quickly assess whether a student has the correct answer and if not, he or she has to go back through the work to find his or her mistake(s), which is a really worthwhile exercise.

How Much Can We Spend? (http://nrich.maths.org/6650)

A country has decided to have just two different coins.

It has been suggested that these should be 3z and 5z coins.



The shops think this is a good idea since most totals can be made.



$2 \times 3z + 1 \times 5z = 11z$ $7 \times 3z + 2 \times 5z = 31z$

Unfortunately some totals can't be made, for example 4z.
Which totals can be made?

Is there a largest total that cannot be made?
How do you know?

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They have decided that they will definitely have 3z coins but can't make up their minds about the other coin.

Experiment with other pairings containing 3z, and explore which totals can be made.

Can you find a relationship between 3z, the second coin, and the totals that can and can't be made?

In other countries they have also decided to have just two coins, but instead of the 3z coins they have chosen a different prime number.

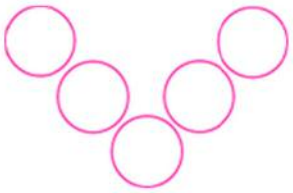
Can you find a relationship between pairs of coin values and the totals that can and can't be made with them?

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This activity is truly low threshold, high ceiling, like many of those featured on the NRICH site. Students all have an entry point as they can write down combinations of coins and their totals to see what values are possible and not possible. The extensions provide lots of rich mathematical exploration.

Magic V's (https://nrich.maths.org/6274)

Place each of the numbers 1 to 5 in the V shape below so that the two arms of the V have the same total.



How many different possibilities are there?
What do you notice about all the solutions you find?

Can you explain what you see?

Can you convince someone that you have all the solutions?

What happens if we use the numbers from 2 to 6? From 12 to 16? From 37 to 41? From 103 to 107?

What can you discover about a V that has arms of length 4 using the numbers 1 – 7?

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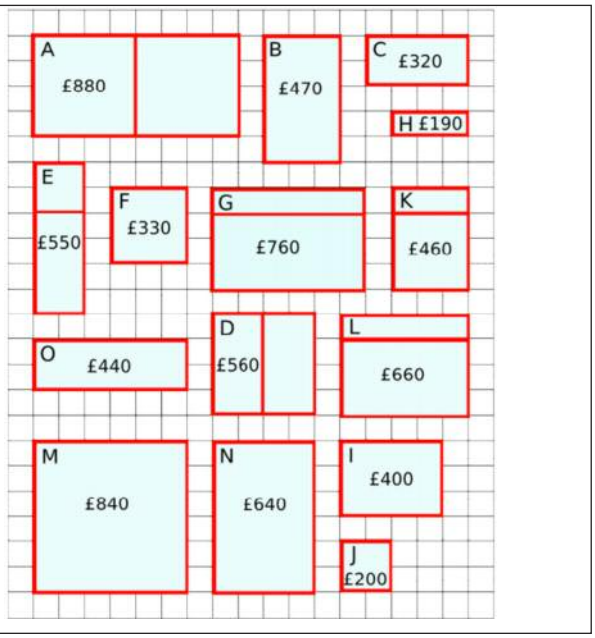
This activity starts off quite simply, but builds in such a way to really help develop students' number sense. Students will develop flexibility with numbers and notice patterns emerging as they work through this activity.

Warmsnug Double Glazing
(https://nrich.maths.org/4889)

Warmsnug calculate the prices of their windows according to the area of glass used and the length of frame needed.

Can you work out how Warmsnug arrived at the prices of the windows below?

Which window has been given an incorrect price?



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This activity allows students to solidify their understanding of area and perimeter along with how the required calculations may change when the window frame is divided. Although the answer is definitely not obvious, there are many paths that students can follow to get started. One strategy could be to find two windows with the same perimeter, but different areas, and compare their prices. Having students present the various strategies used to the class could foster some rich discussions. Once students figure out the pricing system, they must find the window with the wrong price, thus providing them with more practice and a sense of accomplishment when they find the error.

I highly recommend taking a look at the STEM tab on the NRICH site, which features connections among mathematics and science and engineering. They have sections for “Maths Meets Chemistry,” “Maths Meets Biology,” “Astronomy for Beginners,” “Maths Explores the Planet,” and “Engineering for Beginners.” Shown is the list of activities in the Engineering section.

Engineering for Beginners

Engineers design, make and test things to see how they can be improved and made fit for purpose. Try some of these ideas, and see how maths can help.



Make Your Own Robot
Stage: 2 and 3 ★
A simple robot to make, plus robots in everyday life to investigate.



Helicopters **Stage: 2, 3 and 4** ★
Design and test a paper helicopter. What is the best design?



Flip Your Mat! **Stage: 2 and 3** ★
What shape and size of drinks mat is best for flipping and catching?



Straw Scaffold **Stage: 3** ★
Build a scaffold out of drinking-straws to support a cup of water



Cool as Ice **Stage: 3 and 4** ★
Design and construct a prototype intercooler which will satisfy agreed quality control constraints.



How the Velodrome Found Its Form **Stage: 4 and 5**
The Velodrome was one of the most striking buildings in the London 2012 Olympic Park. This article explores how mathematics helped design the iconic building and its track.

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Along with the activities are suggested readings that relate to the topics at hand. These are on the **https://plus.maths.org** site and are a fantastic place to direct those students who always want to learn more. I highly recommend that you bookmark this site, as it is full of great information, in a variety of forms, about mathematics.

▲ Putting the Shape of Kindergarten and Geometric & Spatial Learning in Ontario into Perspective



EDWARD SCHROETER
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Edward Schroeter, B. J., B. Ed., OCT, is a Reading Specialist and Kindergarten Specialist, with qualifications and experience in Special Education, French as a Second Language, and Teacher Librarianship. He was trained and worked as a newspaper reporter and documentary video producer before he began his teaching career. He has taught Grade 4, Core French, French Immersion Senior Kindergarten and Grade 3, Kindergarten Years 1 and 2, and has been a Special Education Resource Teacher and Teacher-Librarian for the Kawartha Pine Ridge District School Board or its predecessor board since 1992. His current plan is to learn more about teaching and learning early math.



At first young children learn to identify shape using visual memory. Gradually they begin to identify them by their properties: number of sides, angles, and vertices.

Introduction

A recent unpublished online survey of Ontario's approximately 19 500 Kindergarten educators seems to corroborate a large body of early mathematics research which suggests that they are under-prepared to teach geometric and spatial reasoning to their students. The cause appears to be insufficient professional training and subject knowledge.

I designed and conducted the anonymous survey in April 2017 for an 11-member, tri-district Kindergarten educator Professional Learning Community (PLC), to which I belong. We are studying effective spatial reasoning and geometry learning in Kindergarten. We consider this an important area of children's mathematics education and wanted a snapshot of the situation in Ontario.

Spatial and geometric reasoning consists of many skills including perspective taking, visualizing, locating, orienting, dimension shifting, path-finding, sliding, rotating, reflecting, diagramming, modelling, symmetrizing, composing, decomposing, scaling, map-making, and designing (Davis, 2015).



Kindergarten students aged 4 and 5 learn the properties of geometric shapes (e.g., number of sides, angles, and vertices) through play, building complex structures.

They are considered essential life skills, necessary, for example, to efficiently pack the trunk of a car, cut equal-sized slices of pieces, use a mall map, correctly install a baby car seat with a diagram, or for jobs such as air traffic controller and engineer. They are often seen as essential skills for 21st-century STEAM jobs—Science, Technology, Engineering, Architecture, and Mathematics (Verdane et. al., 2014). Geometric and spatial reasoning skills are also considered by some early mathematics researchers to be "a gateway skill to the teaching of higher-order mathematics thinking skills" (Clements & Sarama, 2011).



Research suggests that jigsaw puzzles help children develop spatial thinking and reasoning.

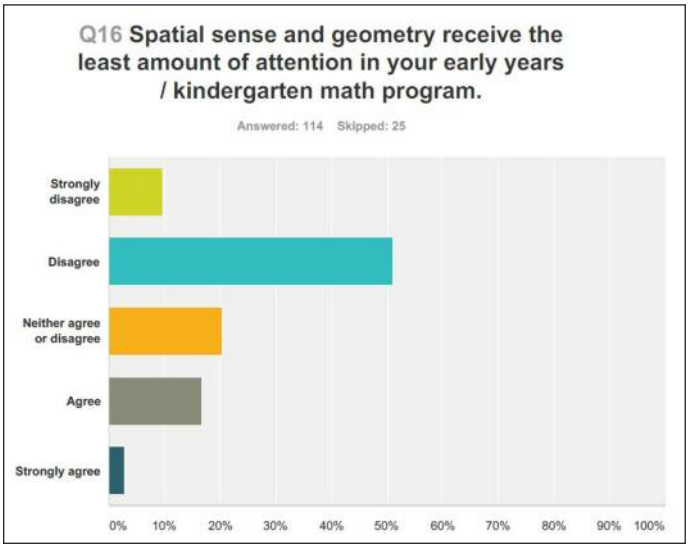
The importance of early spatial and geometric education in mathematics learning

In their informative and important 2011 article, "Early Childhood Teacher Education: The Case of Geometry," published in the *Journal of Mathematics Teacher Education*, Douglas H. Clements and Julie Sarama advance a compelling argument that geometry and spatial reasoning is an important area of mathematics learning for early childhood. The authors cite numerous studies which suggest that geometric and spatial thinking are not only important in and of themselves, but also support number and arithmetic concepts and skills. They also quote research which suggests that the ability to represent magnitude is dependent on visual–spatial systems in regions of the parietal cortex of the brain. They produce their own research, as well as evidence from others which suggests that geometry education may contribute to a

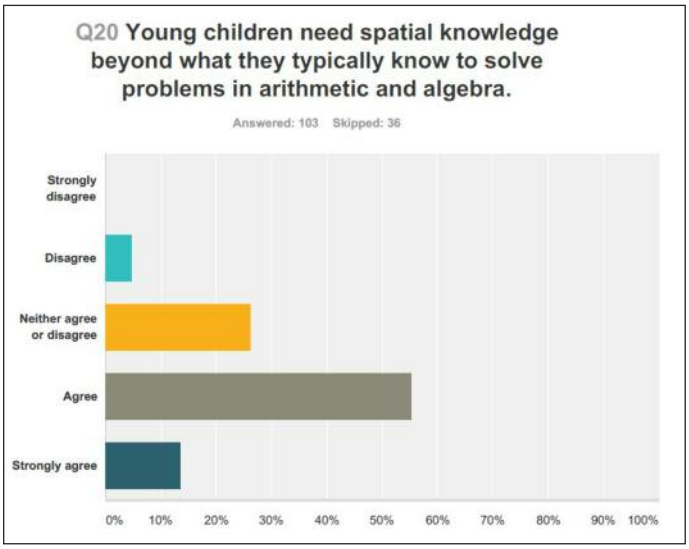
growth in mathematical competence and in other cognitive abilities, including IQ. They also point out that other studies indicate that geometric knowledge is highly related to mathematical reasoning and a host of other mathematics concepts and skills, such as proportional reasoning, judgmental application of knowledge, concepts and properties, and managing data and processing skills (Clements & Sarama, 2011). The U.S. National Research Council's (NRC) report, *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, has examined the above information and indicates that learning spatial skills is so important for children aged 3 to 6 years of age that it should be a top priority in early childhood and Kindergarten mathematics experience and education. The report points out that some children's mathematics skills are hindered by a lack of attention to spatial skills, and that they would benefit from increased geometry and spatial education (NRC, 2009).

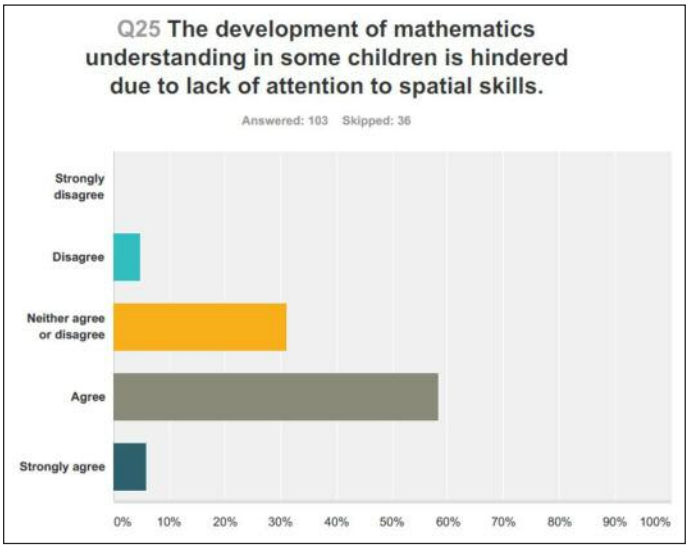
Geometry learning in Kindergarten and early childhood settings today

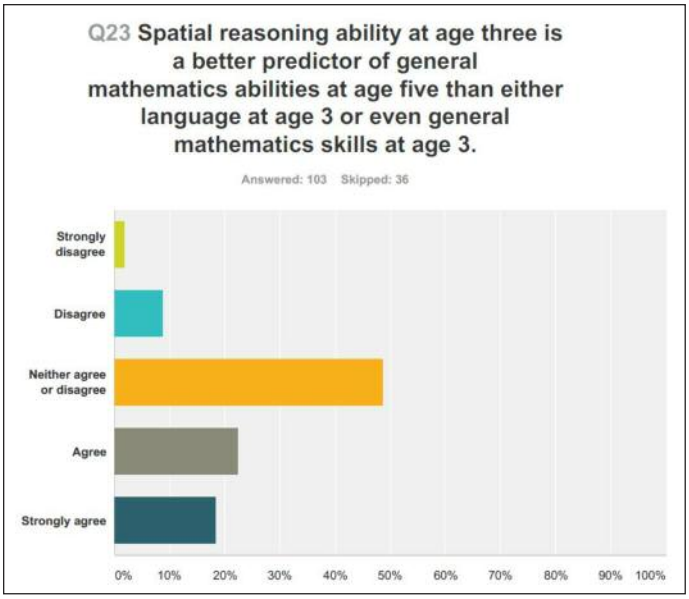
I had been wondering wonder whether the wealth of information about geometry and spatial reasoning is making its way into Kindergarten classrooms and early childhood education settings. Clements and Sarama suggested in 2011 that it is not. That year, they conducted a literature review and survey of 3000 U.S. preschool and Kindergarten educators. From their investigation and 400 completed questionnaires, they determined that spatial and geometric reasoning learning is "often ignored or minimized in both early education and in the professional development of early childhood teachers" (Clements & Sarama, 2011). After conducting their own survey of K–2 educators in Canada, Cathy Bruce, Joan Moss, and John Ross concluded that the situation in Canada is similar (Bruce, Moss, & Ross, 2012).



My April 2017 survey in Ontario seems to substantiate the results of the two national surveys quoted above: some Kindergarten educators might be minimizing spatial and geometric instruction. For example, almost 19.5%, or one-fifth, of 114 Kindergarten teachers (OCTs) and early childhood educators (RECEs) who completed this survey question either agree or strongly agree with the statement that "spatial sense and geometry reasoning receive the least amount of attention" in their classrooms. This is a surprisingly candid admission. It is also interesting that 20% of the educators "neither agree nor disagree" with that statement. Their reluctance to make a firm commitment seems to indicate some uncertainty or concern about spatial mathematics experiences in Kindergarten or the survey question itself. It is also noteworthy that 25 of the 114 chose to skip the question entirely (Schroeter, unpublished raw survey data, 2017).

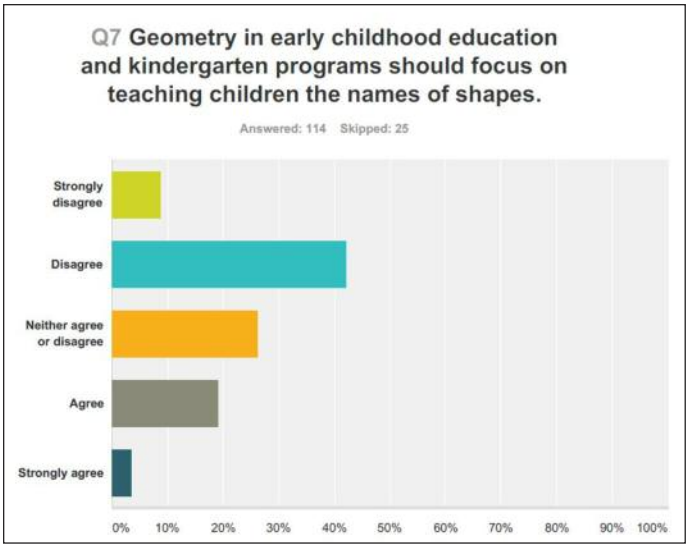


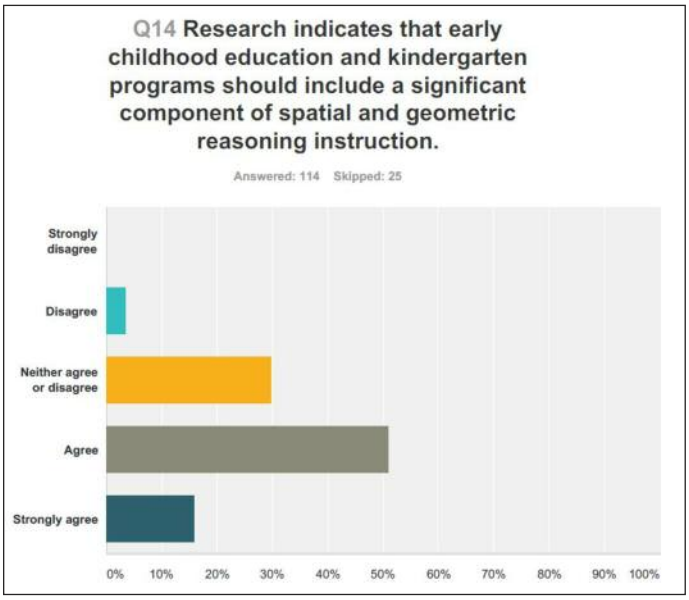




Are educators uncertain about the importance of learning geometry and spatial skills?

Another cause for concern are the educator responses to half a dozen research-based true-or-false statements about the importance of spatial reasoning and geometry in Kindergarten mathematics. They seem to suggest that Kindergarten educators are uncertain to some degree about the importance of geometry and spatial skills to young children. Approximately 23% to 34% of the survey participants—one-quarter to one-third—frequently chose non-committal answers ("neither agree nor disagree") to these questions. I justify my interpretation of the data with three examples. First, 26% of 103 respondents gave a neutral response to the true statement that "young children need spatial knowledge beyond what they typically know to solve problems in arithmetic and algebra," while only 69% correctly agreed or strongly agreed with it. Thirty-six skipped the question. Second, 31% of 103 respondents neither agreed nor disagreed with the true statement that "the development of mathematics understanding in some children is hindered due to lack of attention to spatial skills." In this example, only 64% correctly agreed or strongly agreed. Thirty-six did not answer the question. Finally, only 41% of Kindergarten educators surveyed agreed with the true, but admittedly obscure, statement that "spatial reasoning ability at age 3 is a better predictor of general mathematics abilities at age 5 than either language at age 3 or even general mathematics skills at age 3." Almost half the educators, 49%, neither agreed or disagreed with this true statement. Thirty-six educators did not respond to it.





Do educators understand the nature and degree of geometry and spatial learning required by early learners?

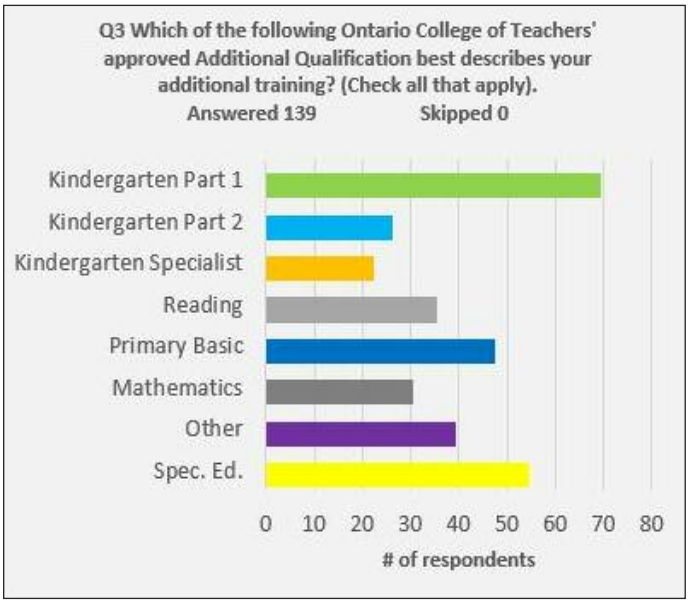
Another series of educator responses to a half dozen other statements about the optimal amount of geometry and spatial reasoning learning experiences for Kindergarten classrooms initially seemed to bear out my assumption, but ultimately turned out to be ambiguous evidence. In one example, only 51% of 114 survey participants disagreed or strongly disagreed with what seemed to me an obviously false statement: "Geometry in early childhood education and Kindergarten programs should focus on teaching children the names of shapes." Kindergarten students are capable of far more sophisticated spatial thinking than the rote memorization of shape names. Nevertheless, almost one-quarter of the survey respondents (23%) agreed or strongly agreed with the incorrect statement. Slightly more (26%) were non-committal in their answer. In another instance,

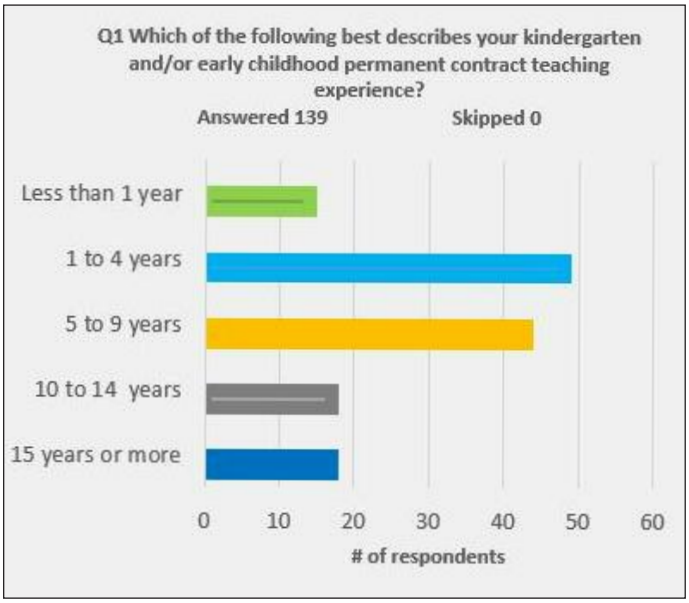
almost 30%—one-third—of 114 educators neither agreed nor disagreed with the true statement that research indicates that "early childhood education and Kindergarten programs should include a significant component" of spatial and geometric reasoning experiences. Twenty-five educators (26%) chose not to answer the question, and only 66% correctly agreed or strongly agreed with it (Schroeter, unpublished raw survey data, 2017).

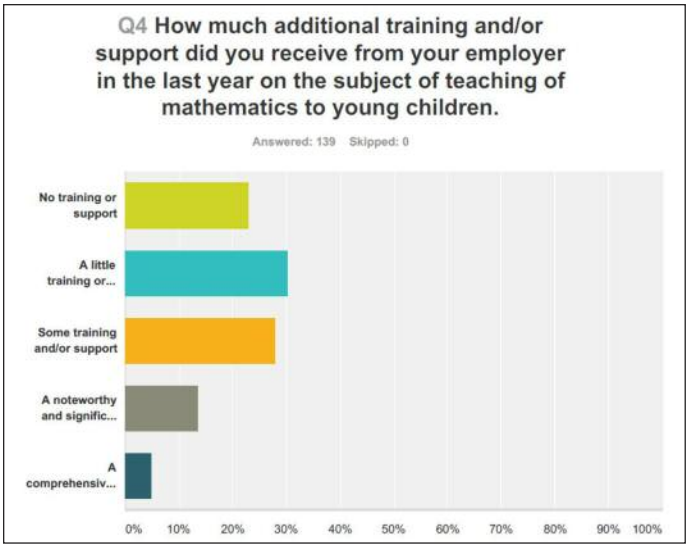
Initially it seemed to me that numerous survey participants had a lot to learn about spatial reasoning and geometry in Kindergarten. However, when I showed these responses to my PLC colleague, Kayla Ingram, RECE, she pointed out that the replies may have been influenced by my language, particularly the use of the terms "teaching" and "instruction." To Kindergarten educators, my terminology might conjure up a vision of direct, whole-group instruction for 3-, 4-, 5-, and 6-year-olds. This is developmentally inappropriate. I should have used a neutral term such as mathematics "experience" or "inquiry." She also said that my questions did not always specify that I was referring to a significant component of the Kindergarten mathematics curriculum as opposed to the overall program (Schroeter, interview, 2017), which consists of four frames of learning: Belonging & Contributing, Self-Regulation & Well-Being, Problem-solving & Innovating, and Literacy & Mathematics Behaviour (Ontario Ministry of Education, 2016). Armed with these insights, it seems reasonable to disregard the survey responses about whether Ontario educators know how much time to spend facilitating spatial and geometry learning in their math programs.

Reasons contributing to the situation

If Kindergarten educators in Ontario do tend to "ignore or minimize" spatial and geometric reasoning, it is important to understand the reason why, so that the situation can be improved. In their 2011 article mentioned earlier, "Early Childhood Teacher Education: The Case of Geometry," author-researchers Clements and Sarama cite more than a dozen studies that identify the cause of this situation as two things: inadequate educator professional learning in the teaching and learning of early geometry, and very low levels of elementary educator geometric knowledge. The authors point out that many early childhood and Kindergarten educators tend to score very low on the Van Hiele Geometry Test (© 1980 by the University of Chicago), a recognized geometry assessment. Low levels of elementary teacher geometry knowledge are a problem for our education system because educator content knowledge is just as important to improving student achievement as is knowledge of instructional techniques (Clements & Sarama, 2011). The two researchers write: "Although data on teachers of the youngest children is rare, our research with hundreds of pre-school teachers… suggests that most early childhood teachers also have not attained adequate levels of geometric knowledge." This is consistent with the finding [of Herbert P. Ginsburg and colleagues in 2006] that teachers of young children are provided with very limited professional development in mathematics" (Clements & Sarama, 2011).







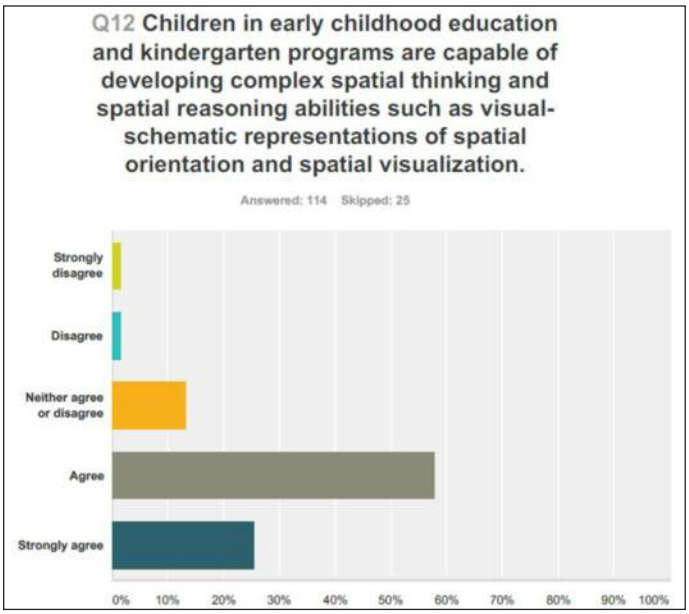
It would make sense to me to discover that Ontario Kindergarten educators lacked perspective on spatial reasoning and geometry purely based on the demographic characteristics of the Ontario Kindergarten educator community. As a group, they appear to be relatively inexperienced and to have little mathematics training. The results of the 30-question survey suggest that:

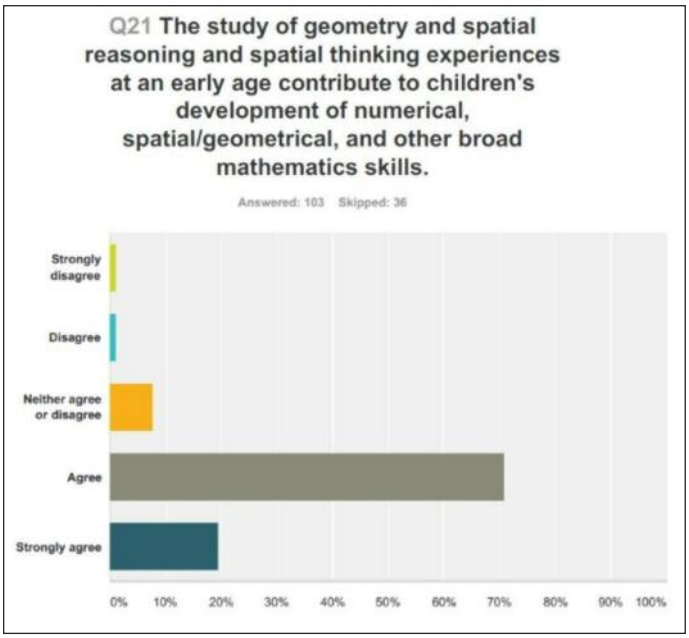
- 34.5% of 139 survey participants (48) who began the survey have from one to four years contract experience
- 10.07% (14) have less than one year contract experience
- 44.6% (62) have *less than one year to four years* contract experience
- 48.92% of (68) of 139 people surveyed have the Kindergarten Part 1 additional qualification, while 33.09% (46) have at least Reading Part 1, and 34.5% (48) have at least Special Education Part 1
- only 20.86% (29) have Mathematics Part 1 or greater
- 53% (74/139) received *"very little" or "no" training or support in mathematics* for young children from their district school board last year

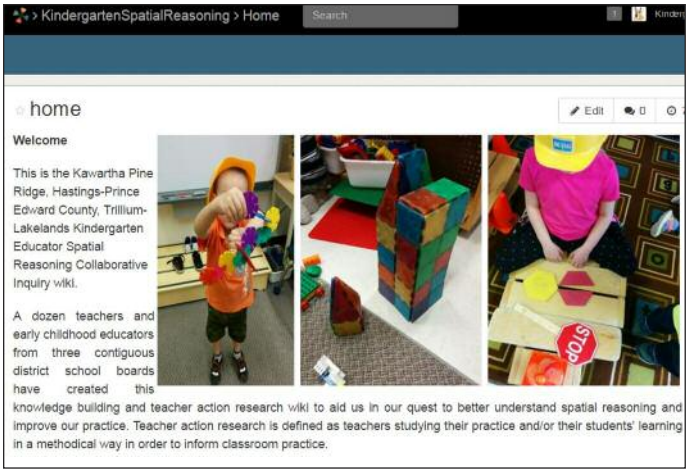
The survey data also provides cause for optimism

Although the research literature and my 2017 Ontario survey together raise questions about the quality of spatial reasoning and geometry practice in Ontario Kindergarten classes, there is cause for optimism. The survey results themselves provide the reason. Educator responses to a dozen questions seem to cement the argument that Kindergarten educators can implement effective spatial and geometric reasoning learning activities in their classrooms under the right conditions. They are committed educators, who understand the general importance of spatial reasoning and geometry (Schroeter, unpublished raw data, 2017). My inference is based on the following survey findings:

- 96% of the participants understand the concept of spatial and geometric reasoning
- 96% recognize that spatial skills are malleable and can be improved with exposure and tasks
- 91% rightly believe in the importance of spatial and geometric reasoning learning for the development of Kindergarten children's math and life skills
- 90% realize that these are developmentally appropriate skills for the age group
- 86% have great faith in their students' abilities to achieve in this area of mathematics (Schroeter, unpublished raw survey data, 2017)







Indeed, some of these educators are making changes on their own initiative to remedy the situation. The Kindergarten PLC to which I belong, a group of 11 Kindergarten educators consisting of teachers (OCTs) and early childhood educators (RECEs), is a case in point. We formed our PLC a year ago to improve our understanding of early spatial and geometric learning and to develop more effective instructional strategies for our young students. Calling ourselves *The Kawartha Pine Ridge, Hastings-Prince Edward County, Trillium-Lakelands Kindergarten Educator Spatial Reasoning Collaborative Inquiry*, we created and are posting resources to advance our understanding on our wiki: **https://kindergartenspatialreasoning.wikispaces.com/**. Our PLC applied for and received a Teacher Learning and Leadership Program (TLLP) grant from the Ontario Ministry of Education and Ontario Teachers' Federation to help support our professional learning. We are now devising our own professional learning to improve our teaching practices.

Conclusion

Our sample size of 99 completed surveys may well be too small to draw any definitive conclusions, but the data is sufficiently compelling to warrant further study. Given that both our survey results and the research literature suggest that the development of geometric and spatial reasoning is of vital importance to young children, and teacher knowledge of both content and pedagogy is fundamental to their learning, it would be prudent for educational leaders to make the mathematics professional learning needs of Kindergarten educators a high priority, and to create an action plan to address the issue, using the highly effective, research-based TRIAD professional development intervention model (Technology-enhanced, Research-based, Instruction, Assessment, and Professional Development), created by Clements and Sarama (2011). It would also seem to make sense to reform Ontario`s professional learning and pre-service educator training programs to provide more robust mathematics content and instructional techniques. All basic P–J teacher education and early childhood education courses should contain a strong geometry component along the lines recommended in the research literature.

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Blog Post

Fawn Nguyen: fawnnguyen.com

My own kids tell me they will stock up their dorm rooms and apartments with junk food and soda when they move out to make up for these years of deprivation. (And this is supposed to make me feel bad.) So when I intentionally bring home snack, like this bag of kettle corn, I usually find it empty within 24 hours. But seeing the empty bag made me think of a volume activity that I could do with my 6th graders with all these other bags of Orville Redenbacher's popcorn.

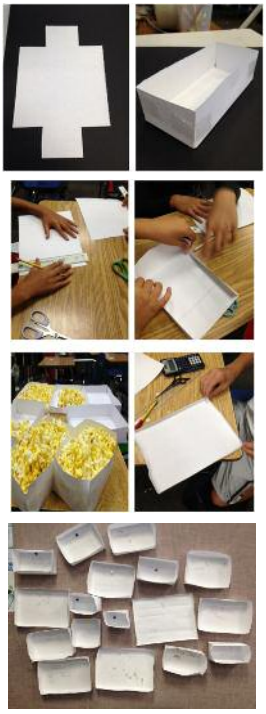




But the activity I had in mind — **maximizing the volume of a box** — is commonly done in a pre-calc or calculus class. These are my 6th grade babies. But didn't we do okay with approximating the volume of a torus via my doughnut lesson? So, why can't we do this too? I have to get rid of the popcorn.

I randomly assigned the kids in pairs, gave each pair two sheets of white copy paper. I told them to use one paper at a time to make a box — **the goal is to make the box as big as possible so it'll hold the most popcorn**. But the box must be made simply like this: cut off 4 corners from the the paper, then fold up the sides and tape them together. I used a half-sheet (so they couldn't duplicate mine) to demonstrate what I meant.

They quickly went to work. A few students were NOT cutting off square corners, so the top edges of the sides didn't line up. Two groups folded in their papers, in addition to cutting off corners, so they had to re-do.



▲ Hey, It's Elementary: They Took Away WHAT???

LYNDA COLGAN



Lynda Colgan's career has included roles as a classroom teacher, a university professor, and newspaper columnist. Her contributions to mathematics and its teaching have been recognized through awards such as the Marshall McLuhan Foundation Distinguished Teacher Award. Lynda always exhibits a passion for mathematics and views her professional mission as dispelling the myth that math is the bad guy.



Lately, I have had the opportunity to speak with many Kindergarten teachers and Early Childhood Educators because of a new project that I have undertaken as part of the Mathematics Knowledge Network, an initiative funded by KNAER and the Ontario Ministry of Education. As the leader of one of four Communities of Practice (CoP), *Critical Transitions,* I have the privilege and opportunity to work with a variety of gifted and caring individuals from universities, school districts, and organizations (such as the Association of Early Educators of Ontario) to design and plan the implementation of projects about the critical transition periods between preschool and Kindergarten and Kindergarten and Grade 1.

Our CoP will support a number of complementary initiatives over the next four years, with a goal of:

1. facilitating capacity building
2. creating opportunities for information and knowledge exchange
3. providing access to tools and resources

Over the full time frame of the project, we will host conferences, webinars; celebrate and share high-quality resources such as Foundations for Numeracy: An Evidence-based Toolkit for Early Learning Practitioners (**http://eyeonkids.ca/docs/files/foundations_for_numeracy.pdf**); and respond to provincial needs identified in a comprehensive environmental survey of early childhood educators, Kindergarten and early Primary teachers.

I will report regularly on the progress of the CoP and the activities of its membership in this column over the next few years… so watch this space!

Initial informal conversations have been focused on teachers' perceptions of the "removal" of the calendar from the new Kindergarten curriculum. "Why?," they ask. "What are the alternatives to calendar time?," many wonder.



If you are like me, you use your calendar to schedule appointments, remember birthdays, and anticipate upcoming special events (e.g., summer vacation, a niece's wedding). As adults, we use the calendar to mark and measure time, using many units such as hours, days, weeks, months, and years. Many families have their own calendar routines: noting ballet lessons, hockey games, birthday parties, and sleep-overs in different colours to denote the activities of individual members. The family calendar often occupies a prominent central location in the home, and many parents and guardians who are reluctant to assume an active role in their child's mathematics learning at home, find that the calendar is a "natural" tool to support vocabulary development and simple numeracy concepts.



However, there are many "anti-calendar" proponents who cite both empirical and putative evidence that "calendar time" is tedious and repetitive for young children—a prolonged classroom routine that is developmentally inappropriate for the young children who are required to gather around it every day.

According to the experts, children of Kindergarten age have limited temporal understanding, meaning that much of the measurement vocabulary associated with the calendar, especially those words that measure extended periods of time, such as week or month, are beyond their comprehension. Early childhood researchers contend that prior to Grade 1, the most developmentally appropriate lessons should focus on temporal vocabulary words like *later, before,* and *after* in the context of their daily activities.

These researchers argue that standard calendar routines include many concepts that are not developmentally accessible to Kindergarten children, specifically the sequential nature of time, a concept inherent in the design of the calendar, for example, *yesterday, today,* and *tomorrow; morning, afternoon,* and *evening; Monday, Tuesday, Wednesday;* and so on. According to empirical studies, "young children can talk about things that have happened or will happen, but they cannot yet understand or talk about these events in terms of units of time (days, weeks) or sequence" (Etheridge & King, 2005).

Basing their arguments on these and many other studies, child development researchers have brought into question the usefulness of calendar activities for children under age 6, using familiar examples to illuminate the point. For example, when a 4-year-old learns that there will be a field trip in five days, he or she will not judge the temporal distance of this event any differently than if he or she were told it is in eight days. In fact, a typical 6-year-old is unable to judge the length of time even within a given day, such as "in two hours" versus "in four hours." This will make sense to anyone who has ever been on an extended car trip with a young child who repeatedly asks, "How much longer 'til we get there?"

According to Friedman (2000), the ability to judge the relative time from a past event or until a future event in terms of the calendar year is not in place until sometime between 7 and 10 years of age. The following anecdote is used to illustrate the point:

> *As Joey's grandparents arrive for his birthday, Joey runs to greet them, saying, "I can't believe I'm gonna be 6." "So, you're going to be 6. Six what?," his grandmother asks. Joey responds, "It's my birthday. I'm gonna be 6." "Yes, I know," she replies, "but six what? You're not six books." At that point, Joey's 9-year-old brother whispers in his ear, "You're gonna be 6 years old!," and Joey says, "I'm gonna be 6 years old." Three days later, as Joey's friends assemble for the traditional noisy birthday party, a discussion begins about who is already 6 and who is not. Marta states, "Well, I'm 6½." Joey asks her, "Six-and-a-half what?" Marta responds, "I don't know." Another child says to 6½-year-old Marta, "Wait a minute. When were you a baby?" She hesitates and then answers, "I don't know, maybe 10 years ago."*

While early childhood specialists have cited numbers, spatial reasoning, patterning, logical relations, measurement, and early algebra as key components of young children's mathematical growth (for example, Greenes, 1999; NCTM, 2000), they also point out that most Junior Kindergarten children are not ready to grasp the complex concepts involved in dates (Etheridge & King, 2005).

It was likely research findings such as these about the level of thinking required to comprehend the time concepts of the calendar and the developmental abilities of young children that led to the reduced focus on the calendar in Kindergarten, a staple in Kindergarten classrooms for decades.

So what do the experts suggest as meaningful mathematics alternatives to calendar time? One idea is the picture schedule, or poster with illustrations, or photos of the day's activities in sequence. Although young children have difficulty judging the length of time between events (for example, how long the time between snack and outside play will be), they can understand a sequence of events (for example, snack comes after circle time).

A complementary suggestion is the classroom journal, a timeline that explicitly documents the sequence of classroom events using dated digital photographs: "On Monday, our butterfly kit arrived, we fed our caterpillar, and we read *The Very Hungry Caterpillar* by Eric Carle."

Further to this is the classroom documentation board. For example, when looking at a dated pictorial record about the class construction of a giant papier-mâché butterfly, one child said, "See, there's the butterfly we made that other time." Her teacher responded, "Yes, we made the giant butterfly two weeks ago. Here [pointing to a photograph on the display] is a picture of the frame we built the first day, and the picture next to it shows you adding the papier-mâché on the second day" (Beneke, Ostrowski, & Katz, 2008, p. 14).

There are many developmentally appropriate inquiry-based activities that involve time in context, such as measuring the time in minutes that it takes for a snowball to melt or to walk to the neighbourhood library. The measurement of "days" and "weeks" is made meaningful by keeping tallies during a classroom project to hatch incubated chicken eggs or document the life cycle of a butterfly.

It is likely that many Kindergarten and early primary educators already use many of these natural and appropriate routines to introduce and scaffold the complex constructs related to time and its many units. Calendars are an important part of our culture and they are familiar objects that link the often distinct and separate worlds of school and home, so they are not to be abandoned. They are important arbiters of concepts other than time, including numeracy, vocabulary (month, year, weekend), sequencing (yesterday, today, tomorrow), and patterning (Monday, Tuesday, Wednesday).

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▲ Teachers' Responses to Perceived Curricular Gaps: Voices and Views



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Jennifer Holm works at the University of Alberta with elementary pre-service teachers, as well as in the field, supporting current mathematics teachers. She is interested in developing mathematics knowledge for teaching with both pre-service and in-service teachers. She has begun to focus on the beliefs and opinions that pre-service teachers hold about mathematics and teaching, and how to support future teachers in developing beliefs and opinions that would encourage effective teaching practices within the confines of an education program.



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Ann Kajander works with teachers at all levels and is particularly interested in the development of knowledge for teaching. Her book Mathematical knowledge for teaching: Reasoning without memorization *with Tom Boland supports such content knowledge development. Recently she has begun a project supporting Grade 9 Locally Developed Math courses.*

This study took place in a medium-sized city within a school board that supported professional learning-group meetings of intermediate teachers. As part of the drive to increase graduation rates, the Ontario Ministry of Education began the "Student Success" or "Learning to 18" initiative (**www.edu.gov.on.ca/eng/teachers/studentsuccess/strategy.html**). This initiative was put into place to give resources to Grades 7 to 10 teachers to provide support for their students. These grade levels were specifically chosen as critically related to graduation rates. The funding for the professional learning group of this research came from this initiative to facilitate a positive transition for students from

elementary to secondary school. The following discussion focuses on one teacher group of this initiative in our school district, and explores the views of teacher-participants on how perceived curricular gaps in the Grades 8 to 9 shift should be addressed—and by whom.

Context

The teacher-participants in this study were members of a professional learning group, who met for at least a half day, approximately six times a year. The first author observed the group for a three-year period, as well as conducted interviews with teacher-participants during the study. The professional learning group was composed of Grades 9 and 10 teachers at a secondary school, together with their Grades 7 and 8 teacher counterparts at the feeder schools (as well as one Grade 6 teacher). While some movement in the group occurred during the study, most teachers remained part of the group for all three years of the study.

Methods

The first author observed the conversations of participants in the approximately monthly meetings of the professional learning group for the three-year period of the study. As well, semi-structured interviews were conducted at the end of the study. Most of the meetings consisted of teacher discussions; however, the professional learning group occasionally visited one another's classrooms, and twice they travelled to the local university to participate in professional development opportunities there.

The current paper focuses on data around the Ontario curriculum; in particular, what gaps in the curriculum teacher-participants perceived in the Grade 8 to Grade 9 transition, and how they felt these identified gaps should be addressed in their classrooms practice. Meeting and interview transcripts were analyzed by these themes; more details can be found in Holm (2014; 2016).

Observations and Results

There was general agreement among the teacher-participants that the Ontario elementary mathematics curriculum (Ontario Ministry of Education [OME], 2005a), and the Grades 9/10 curriculum (OME, 2005b) did not flow seamlessly from one to another. Teachers teaching in the elementary panel were observed to have different responses from those of their secondary counterparts for some topics, particularly algebra. As one secondary teacher noted, for some of the topics, "sometimes for us, it seems like they've never seen this stuff before," but he knew that was not a true statement and that something else was the problem. The following section focuses on the perceived gaps in the areas of algebra, fractions, integers, and geometry.

Algebra

Based on the comments made by the secondary teachers, the elementary teachers in the group were united in the view that there was not enough algebra specified in the elementary curriculum to prepare their students for success in Grade 9. Grades 9 and 10 teachers noted that algebra was the topic that decided student success in secondary school, so this area was decided to be of critical importance for the teachers to ensure student success. In general, the elementary group members acknowledged some need for responsibility to help address the problem by augmenting the written curriculum expectations from Grades 6 to 8. The strand of algebra was noted a number of times in group deliberations as a particularly problematic area. In fact, of the 17 meetings observed, 8 of them included conversations related to the topic of algebra.

The elementary teachers were interested in learning about how much emphasis is placed on algebra for success in secondary school, and developed a deeper appreciation of how their curriculum expectations did not adequately prepare their students for where they needed to be. Through their discussions, the teachers became aware that the Grade 8 curriculum focuses on solving algebra by inspection as being an adequate strategy, yet students who could do only this method would struggle in Grade 9. In order to narrow this gap, the Grades 7 and 8 teachers increased their algebra discussions with students to help better prepare students for Grade 9. One Grade 8 teacher made the decision to move the topic of algebra to earlier in the year, and she noted that the "discussions about algebra are more frequent and they're deeper and they're better" (meeting transcript). As the teachers noted in one meeting, the leap from elementary school, where they mainly talked about expressions in algebra, to creating equations and graphing in secondary school, was too large for most of the students. As one elementary teacher mentioned in her interview, "I have pushed kids further in Grade 8 than I ever had before because seeing where they wanted them to be in Grade 9… if you leave Grade 8 and all you can do is solve by inspection, you're not going to be able to survive what they ask you to do in Grade 9" (see Figure 1 for an illustration). A concern was expressed by an elementary teacher that often, questions in Grades 7 and 8 algebra lead students to believe that the answer needs to be a "nice" whole number, and that something was wrong if the answer was a decimal or fraction (meeting transcript), which would have an impact on secondary school. Another noted that, "I've always spent a lot of time on algebra, but what I downsized was the

geometry" based on the discussions with the secondary teachers on what was needed in Grade 9 and beyond (interview transcript). As she said, "The things they [secondary teachers] spend more time on, I spend more time on" (interview transcript).

Solve by inspection: $2x + 6 = 14$
Students most likely can determine $x = 4$ by looking at the equation or by using a guess-and-test method without having to manipulate the variables.

$6x + 2.5 = 2x + 10.5$
Students most likely will determine that $x = 2$ by manipulating the variables to get x by itself on one side of the equation.

Figure 1: Illustrative example of "solving by inspection" created by the author

Some of the secondary teachers, on the other hand, felt generally that in terms of algebra, it was the elementary teachers who needed to address this problem. As one noted in one meeting, it was really important in Grade 8 for students to "see lots that they can't do in their heads"; meaning that she felt it was important for the Grade 8 students to have to manipulate algebra to solve, instead of just determining the value of x in their heads (see Figure 1). One participant specifically argued that if elementary teachers taught students more "rules" for algebraic solving, students would be better prepared for Grade 9. One very experienced secondary teacher commented that the "teeter-totter method" [balance metaphor for solving linear equations] was something he did not have time to deal with, and students should have moved past that by Grade 9 (see Figure 2 for image of the balance method). However, some elementary panel members were vocal in the belief that students needed a deeper conceptual understanding prior to being taught formal rules for simplifying and solving algebraic expressions, and were very cautious about too much formalism being added to Grades 7 and 8.

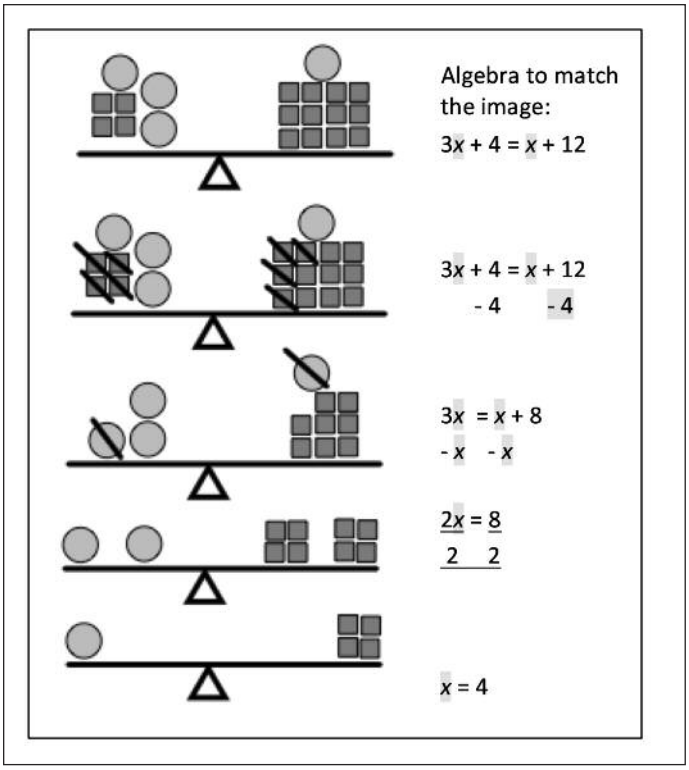


Figure 2: Illustrative example of the "teeter-totter method" created by the author

Fractions

Another area of concern in terms of the curriculum connection was that of fractions. The secondary teachers noted that an introduction or review of fraction operations was not in the Grade 9 curriculum, so they would expect incoming students to be fluent with fraction operations. They were surprised to learn that in Grade 7, students were first introduced to numeric methods for addition and subtraction of fractions, and only in Grade 8 were they introduced to multiplication and division procedures, so the majority of students would not have mastered the skills by Grade 9. As one elementary teacher noted, "fractions and division and they're a bit shaky on that" and "they're coming to [Grade 9] without a lot of comfort" (meeting transcript). As a result, the Grade 9 teachers found they needed to spend time addressing the deficits in fluency in these topics before moving on to more complicated concepts involving fractions. Some of the secondary teachers struggled with the fact that reviewing fraction operations was not part of their curriculum, and yet the students had not mastered these topics, even though they were expected to know them in Grade 9. The elementary teachers shared how new those concepts were to their students, and as one elementary teacher said, "if you think about even if we spend a fair amount of time on fractions, then we're worried about other things, so they're really like renting the ideas" in Grade 8 (meeting transcript).

The conversation about simplifying fractions was one topic of discussion when a secondary teacher mentioned that her students would be "wrong" if they did not simplify the fraction (see Figure 3). An elementary teacher pointed out that she stresses with her students that they are not "wrong," but that bringing to the lowest terms allows for "ease of use," and that she tells her students they "need a different version" of the fraction.

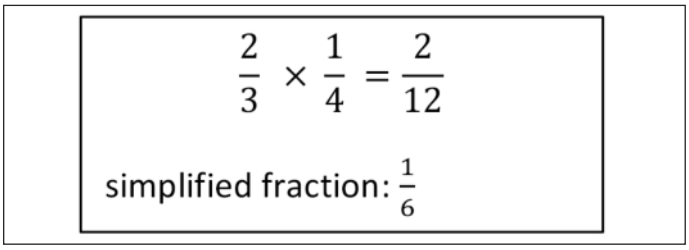


Figure 3: Illustrative example of "simplified fraction" created by the author

Integers

Integers posed similar concerns for the secondary teachers who realized that their students had yet to master the concepts. By secondary school, the teachers expected that students would be able to flexibly use their knowledge of integers in order to start mixing it with algebra. One secondary teacher stated that "the negative integer and subtracting that… I'm finding there's a huge deficiency there" (meeting transcript), even though "it's actually in Grade 7 and we revisit it in Grade 8" (meeting transcript). The example was given that with -4 + 2, students would answer that it would be -6 based on knowing 4 + 2. One secondary teacher made the comment that "in my Grade 12, the way it manifests itself is that they get a complicated algebraic expression and they just lose the negative" (meeting transcript). The example she gave was in $-x + 3$ being part of the expression; the "-" before the x would be lost when working with other portions of the expression (see Figure 4). Elementary teachers, on the one hand, felt that exploring integers with manipulatives, and gaining a conceptual understanding of the concepts, was very important, and did not want to inundate their students with too many rules. On the other hand, there was a tendency for some secondary teachers to view the manipulatives as "toys," with one referring to them as being "chips for dummies" to be used only by weaker students. With integers only being first introduced in the Grade 7 curriculum, the elementary teachers were concerned that not all students would have an understanding or mastery of integers before leaving elementary school.

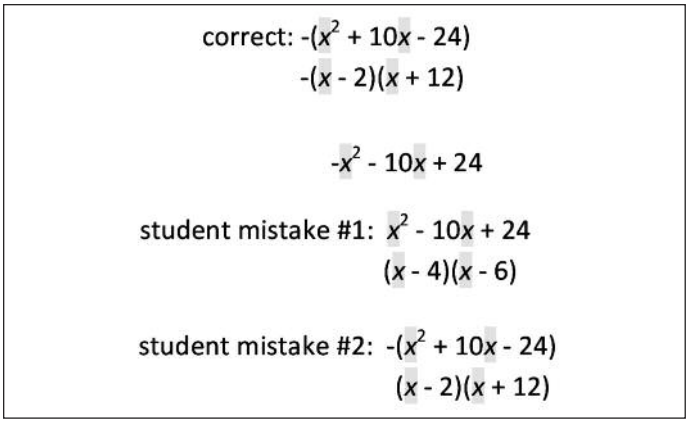


Figure 4: Illustrative example of "losing the negative" in factoring $-x^2 - 10x + 24$ created by author

Geometry

The teachers in both panels discussed how, after Grade 9, geometry had been pulled out of the secondary curriculum with the new curriculum changes, so spending an exorbitant amount of time on these topics was not as vital to future success as other areas. As one secondary teacher noted, "the problem with measurement and geometry from our end is that after Grade 9, we don't do it again," so it became more important for the emphasis in Grade 9 to be on number sense and algebra concepts, to prepare students for Grade 10. Some of the secondary teachers mentioned that when they were discussing angle theories in their classes, they did not enforce the use of the proper names for the theories because it was knowledge that would not be used again after Grade 9 (see Figure 5). All the teachers agreed that the students needed to develop an understanding of what the different theories were and how to apply them, but they were divided on whether students needed to remember the exact names. One elementary teacher mentioned that he emphasized to his students that they would be responsible for knowing the names in secondary school, which sparked the debate on whether it was necessary (meeting transcript). Another secondary panel group member noted that the Pythagorean theorem would be more beneficial for the students to spend time on because of trigonometry, so angle theory could be given less time.

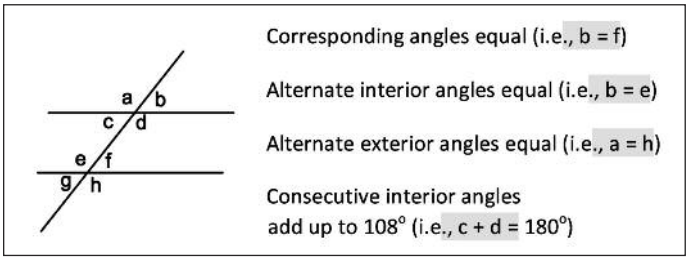


Figure 5: Example of angle theories discussed in the professional learning group

Discussion

The study specifically points to gaps between the elementary and secondary mathematics curricula, which, if not addressed by classroom teachers, could be particularly problematic for students. The teachers' responses, however, were varied in terms of how the identified issues should be addressed, and by whom.

The data highlight the fact that teachers need to know not only their own curriculum, but where students are heading or have come from in their mathematics journeys. Shulman (1986) describes this as vertical curricular knowledge and supports its importance in effective teaching. The identified gaps also suggest a need to revisit the curriculum or provide materials for teachers that share gaps between elementary and secondary curriculum Expectations.

We support the identified teacher concerns by specifically juxtaposing some Expectations from both of the relevant curriculum documents. This information is provided to further illuminate aspects of the challenges being faced by teachers in these grades.

Algebra

The Patterning and Algebra strand of the elementary (OME, 2005a) curriculum states that students should learn to "solve and verify linear equations, using a one-variable term and having solutions that are integers, by using inspection, guess and check, and a 'balance' model" (p. 117). The "balance" model would in fact evolve directly into a more algebraic treatment in Grade 9, but some Grade 9 teachers felt this method was not sufficiently formal. In fact, one of the Expectations in the Grade 9 Academic curriculum (OME, 2005b) explicitly mentions "using a variety of tools (e.g., computer algebra systems, paper and pencil) and strategies (e.g., the balance analogy, algebraic strategies)" (p. 31), with the goal of the mathematics curriculum to integrate specific mathematical processes throughout, including "select and use a variety of concrete, visual, and electronic learning tools" (p. 29). The secondary curriculum specifically mentions the balance analogy and using tools to support understanding, yet the secondary teachers in this study felt the Grade 9 curriculum should be more formalized. The focus of the discussions of the secondary panel members was much more around the algebraic techniques. In Grade 9, students are expected to use exponents with algebra, so the explicit stop in algebra at only linear systems in Grade 8 also seemed problematic to the teachers in the study in positioning students to be successful with non-linear systems in Grade 9, having never been exposed to them.

Fractions and Integers

Similar concerns were voiced around both fraction and integers, so they will be discussed together. In Grade 8, the Expectations include strong emphasis on the use of concrete models, tools, and strategies, such as the area model for multiplying fractions and coloured counters for integer operations. Following several Expectations of this style for each topic, there is one Grade 8 Expectation referring to fraction operations, namely "solve problems involving addition, subtraction, multiplication, and division of simple fractions" (OME, 2005a, p. 111), and one referring to integer operations, "evaluate expressions that involve integers, including expressions that contain brackets and exponents" (OME, 2005a, p.112) that do not specifically mention "using a variety of tools." However, in Grade 9 (Academic) Expectations, fractions and integers are referred to as computations embedded within algebraic manipulations. As one elementary teacher commented, "fractions with integers with exponents and all in one group is new, separately no worries, but combine them together" and this is where the problem starts for Grade 9 students (meeting transcript). There are no Expectations in the Grade 9 Academic curriculum that specifically address or review fraction or integer operations as topics on their own.

Implications and Recommendations

Teachers within the professional learning group in the study were well aware of the gaps between the Grades 8 and 9 Ontario curriculum documents based on their professional learning-group discussions, and were doing their best to deal with these gaps. An examination of the documents underscores the challenges being faced in doing so.

In general, it appeared that in cases where Expectations mentioned both concrete and more formalized methods, elementary panel members were particularly concerned about aspects relating to conceptual understanding, while some of their secondary counterparts focused more on the need for formalisms.

While the gaps between Grades 8 and 9 may well be reasonably easily bridgeable (for example, the balance analogy directly supports formal algebraic equation solving), it was unclear to many participants whose responsibility this was. Thus, an unevenness in Grades 8 to 9 programming may well exist in some jurisdictions; teachers in the current study were very appreciative of the regular time they had to coordinate their content, and expressed that they would not be able to do so in isolation.

In the short run, that is, until the Ontario curricula are more seamlessly aligned, such regular conversations

between elementary and secondary panel members are of vital importance in minimizing the occurrence of learning gaps for students. In particular, guidelines around algebra, and related fractions and integers concepts, in the Grades 8 to 9 transition, might be helpful resources for teachers as they engage in such deliberations.

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Blog Post

Amy Lin: amylin.me

Grade 9 and 10 Applied Final Performance Task

We made it through Semester 1 and we just completed week one of our new semester. Whew! I wanted to summarize and share our final days in our applied classrooms. This year I was responsible for creating a final performance task for the grade 9 applied and grade 10 applied classes. A few weeks ago, a team of teachers came together do discuss what this might look like. The teachers were creating a task for the applied and college classes (grade 9, 10 and 11). Much of the plans for this task was based on the work done by Alex Overwijk and what I could gather from his blog postings and a few inquiries directly made to him.

We didn't have as many days dedicated to the performance task – and the Grade 11 teachers wanted to include other assessments on those days. On Day One, we posted three themes to create stations that included four pictures/videos each. Videos could be viewed on our Chrome books. Chart paper was posted on the walls and sticky notes were provided for their questions. The students were paired up and generated questions for each of the pictures/videos for a time of 10 minutes at each station. They rotated and completed the same time at each station. This prevented from having too many repetitive questions – although we did see some of course.

When they had returned to their original station, we had them, as a group, decide and sort each suggested question into one of the topics we learned about in the semester. For Grade 9's we sorted into Proportional Reasoning, Linear Relations, Measurement and Geometry. For Grade 10's we sorted into Linear Relations, Quadratic Relations, Measurement, Trigonometry and Similar Triangles. This also eliminated the "unrelated" questions that were proposed.

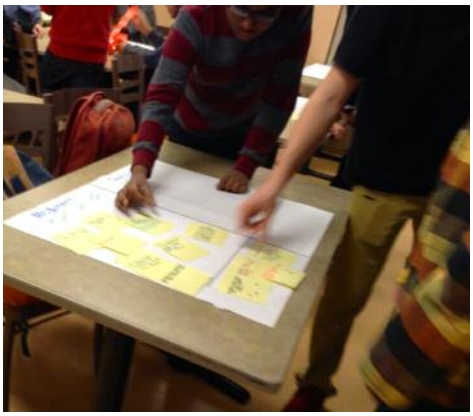
Once this was completed (about 15 minutes), the groups decided on the best five questions placing accompanying reasons why they thought so on smaller sticky notes beside them.

The rotations began again – and students moved to each station to decide, using "dotmocracy" which questions of the best five they preferred to see on their final task on Day 2 and 3. I gave each student 10 dots to vote with.

This was the end of Day 1, which worked out well – but left a large task for the teachers to sort through and decide on questions for the final two days. We honored their choices and if possible – used the questions they voted on. Some of the questions needed clarification or added parts to it – but most were usable in the form they provided. This was encouraging. We separated questions into "easy", "medium" and "hard". Then we provided all the necessary data for the students to complete the questions.

Day 2 and Day 3 were the same – with students completing the questions which we compiled into many tasks to choose from. There were three parts – and they had choice within each part. Overall, this went well – with most students performing well in both my Grade 9 and Grade 10 classes.

This type of summative task near the end of the semester allowed for students to tell you what they took away as the most important concepts and ideas from each of the topics they studied. The conversations were rich and some of the arguments over the "best" questions were interesting to listen to.





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