Ontario Mathematics Gazette

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OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

Vol. 57 #3
March 2019

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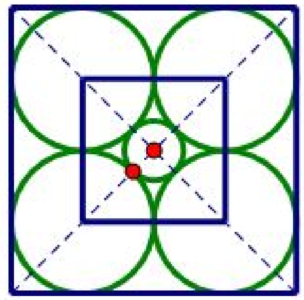

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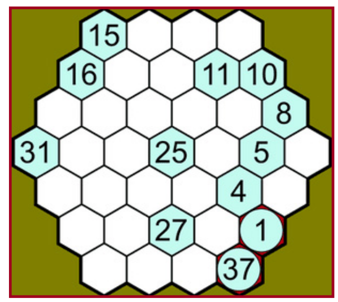

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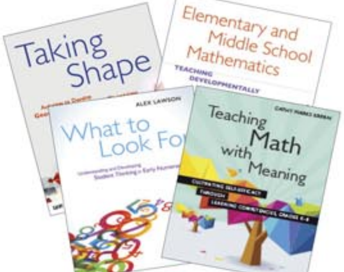

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Submission of Articles

The *Ontario Mathematics Gazette* (OMG) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *Gazette*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles are to be word-processed, MS Word is preferred, and prepared according to the *Publication Manual of the American Psychological Association*, Sixth Edition (2009). However, please use single-line spacing (not double) and only one space after each period. Articles should not exceed five numbered pages of text, and figures, images, and photographs should be placed in the text close to where they belong, with captions. The photographer's permission is required, and for photos of students under the age of 18, the written permission of a parent or guardian is required.

Please submit your article in one blind file (i.e., identity of author is not evident), and include author names, contact information including email and mailing addresses, photos, biographies, and all content removed for blinding in a second file. Please email these two files to Tim Sibbald at **gazette@oame.on.ca**.

Upon review, you will be notified whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected. The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME.

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▲ About the *Ontario Mathematics Gazette*

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Submission of Advertisements

Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** by email or at the above address. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

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▲ Editor's Report

TIMOTHY SIBBALD, OCT, PhD
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Tim Sibbald is the current Gazette *editor and a Past President of OAME. He is an associate professor in the Schulich School of Education, Nipissing University, with a focus on mathematics education in both pre-service and graduate programs.*

Imagine if a curriculum change could provide a net improvement in student learning for three percent of students, while leaving the other 97 percent unchanged. Without suggesting how one might measure such an improvement, hypothetically consider the impact of such a possibility. It would be an enormous change in the sense that the school system impacts huge numbers of students, and three percent of a huge number is also an enormous number. Now consider how much change you would be willing to make to your teaching practice to achieve that alteration.

The challenge of enacting a curriculum is partially a matter of whether teachers believe that it will make a difference compared to what they are already doing. Typically, when instituting change, there is a temporary reduction in teacher effectiveness because they are trying to do something new and do not know all the challenges that they should avoid. With time and perseverance, teacher effectiveness rebounds and, if there really is a three percent improvement, it is realized. The perception, however, is often that it made no significant difference because 3 percent of a class of 25 students is less than one student; i.e., the difference may be imperceptible.

Sometimes I wonder how much of the curriculum debate is about small margins of improvement. I do not suggest this is the case with all changes, and I am fully aware of the challenges that surrounded major curriculum changes in 1998 and 2003. However, there seems to be debate that I would characterize as being about emphasis within math.

To put this in concrete terms, consider being asked to determine the value of 456/2. Consider the following "solution":

```
2)456  223
  446
  ---
   10    5
   10
   --   ---
    0  228
```

In this approach, the initial estimate is 223, but that only provides 446, and so the additional 10 must also be divided. This gives 5 more that is added onto the estimate of 223 to arrive at the final result of 228. I would argue that this "solution" is valid and demonstrates an understanding of how numbers can be decomposed and reconstituted.

Moreover, I would argue this example could be modified to clarify the standard algorithm. Pedagogically, this would require some explanation, but it certainly highlights the emergence of each digit of the result.

```
2)456  200
  400
  ---
   56   20
   40
   --
   16    8
   16
   --  ---
    0  228
```

This example is not of my own making. I chose it to point out the shift in emphasis from 1972, when Routledge (1972) attributed this example to Dr. Fryer, who had referred to this as "silly division." Routledge also commented:

> Since the introduction of "new" mathematics, elementary teachers have [been] pressured into throwing out the textbook and workbooks and writing their own programs geared to the needs of individual children under their care… Ideas and processes must be discovered and explored, and the child's own methods accepted. Drill on basic facts and practice sessions for building and maintaining computational skills have been down-played, even outlawed (pp. 17–18).

The issue explained in the article (which you can find in the *Gazette* archive, located in the members-only part of the website) was that the method is accurate, but inefficient. The article says it might be used in a limited way, but that ultimately, the efficiency and value of a standard algorithm would outweigh the facilitation of a myriad of alternative and less efficient approaches.

What is particularly interesting about this example is that the commentary could very well have been written in the last five years. There has been increasing individualization through the proliferation of Individual Education Plans (IEPs), and discovery methods have been characterized by facilitating students to explore algorithms of their own creation. The debate today is very similar, but the emphasis placed on different aspects of the discovery and standard algorithms may have changed. Certainly there is more pedagogical emphasis on students comprehending the concept that results in the generation of the digits. This algorithm can be tailored to show this, and in that respect, this "silly division" addresses the primary issue with the standard algorithm—that some students do not understand why only part of the dividend is being examined at each step.

I will also confess to having used this approach, with a twist, to confuse teacher candidates into thinking about how they explain division. Consider the following version that contains negative numbers:

```
2)456  250
  500
  ---
  −44  −25
  −50
  ---
    6    3
    6
   --  ---
    0  228
```

Yes, that is "silly," but not without purpose. It does lead to discussions that induce teacher candidates to consider how well they understand what is taking place in long division. In theoretical terms, it causes them to take their implicit understanding of the algorithm and make it explicit so that they can articulate the conceptual details effectively.

So, while the approach has been described as "silly," there is no doubt that it has some instructional value. In my opinion, it can be used to highlight conceptual details and then introduce the standard algorithm. It is only "silly" if one wants

Penny Clemens (Graphic Designer)

In the words of Jack Weiner

When I became editor of the *Ontario Mathematics Gazette* (*OMG*) in 1992, during the tenure of President Mary Lou Kestell, Mary Lou and I, and indeed, the entire OAME Council, shared a vision: transform the *Gazette* into a fully peer-reviewed, professionally designed journal. This was a work in progress in our first year, a learning process. After a few stumbles, in 1993, I asked Penny Clemens to come on board. I had worked with Penny when she was part of the graphics team at the University of Guelph. By 1993, she had embarked on a career as a freelance graphics designer. Over three years, she molded the *OMG* into a beautiful journal, and in 2000, won the Mona McGregor award. She has continued to improve it with successive editors, such as John Egsgard, Louis Lim, and Marilyn Hurrell. (We miss you Marilyn. Rest in peace.) Penny, congratulations on, and thank you for, 25 years of service to OAME as the talented *Gazette* graphics editor.

to rapidly move to the standard algorithm. What makes me question whether it is "silly" is our emphasis on conceptual thinking and the implied change in how we migrate from conceptual understanding to efficiency—whether efficiency of the algorithm or student automaticity—which is at the heart of and muddies much of curricular implementation. This is why professional judgment is so important, but as teachers, we need to understand "emphasis" as a concept that is relevant to interpreting curriculum.

Women in Mathematics

The International Mathematical Union Committee for Women states that 25 percent of mathematicians are women and only 10 percent of leadership positions in mathematics are held by women. Note that this is in the field of Mathematics as opposed to Math Education. As part of their efforts to combat the discrepancy, they have released a video of women mathematicians from around the world, speaking about the importance of math. It is certainly good professional development and may suit some classrooms. See **www.youtube.com/watch?v=uNJ7riiPHOY**.

News Items

Early in January 2019, two OAME figures passed away. An obituary for Susan Stuart appears in this issue, and Jack Lesage will be remembered in the next issue. Both made significant contributions to math education and will be missed.

It is a distinct pleasure to extend congratulations to the Centre for Education in Mathematics, and Computing (CEMC) as the 2018 recipient of the Adrien Pouliot Award. The award was given in December at the Canadian Mathematical Society's (CMS) winter meeting. It is named after the second president of the CMS, who emphasized instructional methods over his 50-year teaching career.

In case you missed it, in October 2018, the UNESCO Executive Board moved "to recommend that the General Conference at its 40th session decide to proclaim 14 March of every year 'international day of mathematics'" (UNESCO, 2018, p. 37). The general conference takes place in the Fall of 2019 and potentially March 14, 2020 (i.e., Pi Day) will be the first international day of mathematics.

On December 21, 2018, the Great Internet Mersenne Prime Search (GIMPS) announced that $2^{82,589,933} - 1$ is a prime number. The discovery was actually made on December 7, but calculations to confirm the 24 862 046-digit number is prime took an additional two weeks. This is now the largest known prime number.

In This Issue

Our Executive Directors provide some thoughts about how their own children work with math as adults. President David Petro brings exciting news about OAME talks, an opportunity for members to engage with podcasts and webinars. Assessment Abby returns to address a question about how to pedagogically respond to diagnostic testing results. Angelica Mendaglio brings news of what has been going on at the Fields Institute. Jacqueline Hill writes about the benefits of NCTM membership.

David Costello suggests that linking literacy and mathematics allows teachers to leverage strategies from one to the other. Lynda Colgan has also returned (as we knew she would) with innovative ideas that she ties to historical developments. Jennifer Holm focuses on pi day and provides strategies for students to be introduced to the remarkable number. Carly Ziniuk presents a collection of different puzzles and problems—if you don't know what a "hidato" is, this is the column to show you. The provincial digital team brings us details of virtual Pattern Blocks. Mary Bourassa updates us on the latest Desmos features. Shawn Godin has us going around in circles, but not to worry, he brings in a parabola to give us a *focus*. Ann Arden interviews Richard Hoshino.

But wait, there is more! We have an invited feature to clarify the odd listing in the archives about *Math Man*. Featured articles focus on the following ideas: the power of asking 'What if?' questions (Irvine); the "E-Brock Bugs" probability teaching resource (Muller, Buteau, Chan, Calford); ways to utilize "cognitive dissonance" (Irvine); and adapting card games for fraction learning (Edwards)."

Clearly there is much on offer and I hope you find something to whet your appetite. I know that our columnists and authors appreciate your praises and constructive feedback. To that end, I welcome letters and we have the **@OAMEwrites** Twitter feed.

Lastly, I would like to extend thanks to Dan Jarvis for looking after the validation of the final proofs for this issue to facilitate my travel. I would also like to thank Anne, Jacquie, Gitta, and Penny. It really is a pleasure to work with such an effective team.

References

Routledge, J. (1972). Down with silly division. *Ontario Mathematics Gazette, 11*(1), 16–21.

UNESCO. (2018). Decisions adopted by the Executive Board at its 205th session. Retrieved from **http://unesdoc.unesco.org/images/0026/002659/265956e.pdf** ▲

▲ President's Message

DAVID PETRO
EMAIL: David.Petro@oame.on.ca

David Petro, the current President of OAME, is the math, science, and IT consultant at the Windsor Essex Catholic District School Board. He is a large proponent of exploiting technology for the educational benefit of students.

As I have mentioned in previous messages, part of my vision is to use online tools to enhance mathematics in Ontario. Something else we are working on is finding ways to bring added value to having a membership in this organization. One of our new initiatives tries to address both of those ideas. This is the creation of *OAME talks*.

One of the crown jewels of OAME is our annual conference in May of each year, but not everyone can physically attend. Our idea was to ask some of the popular speakers to repeat their sessions in an online environment exclusively for OAME members. This way, if you cannot attend the conference you can still experience some of the great math ideas that are discussed. We are starting with four presentations from OAME 2018:

- Kyle Pearce and Jon Orr – **Going Deeper with Memorable Math Moments**
- Marian Small – **Teaching with Intention: Focusing on What Matters**
- Fawn Nguyen – **What If We've Been Teaching Mathematics All Wrong?**
- Mark Chubb – **Building Your Students' Mathematical Intuitions**

Our plan is that for each session, we will have a podcast and a webinar. Each podcast will feature a brief conversation with the presenter(s) in which they will summarize their respective session These podcasts will be available for anyone to listen to and will usually be released on the first day of the month. These have been fun to record, but it's the learning I gained that came out of those short conversations that made them the most worthwhile.

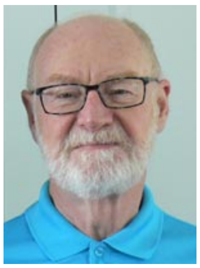
If you want to listen to the podcasts, you can do so in a couple of ways. One way is through your own podcast app. They are available now on iTunes and Google Play, so please subscribe and never miss an episode. It turns out that our first couple of episodes are popular in the K–12 podcast category, trending as high as fifth in the top 200. If you are not a regular podcast listener then you can hear the episodes any time you want on our *OAME talks* website (**www.sites.google.com/oame.on.ca/oame-talks**). Just click on the Season 1 tab to hear them all.

In addition to the podcasts, webinars are the main parts of *OAME talks*. This is where these speakers repeat their sessions from the OAME conference. These webinars are exclusively for OAME members, run on the second Wednesday of the month, and participants must register ahead of time on our MCIS registration site (**www.oame.on.ca/mcis/**). If you weren't at the conference, or if you were and you missed some of those sessions, you will have a chance to see the sessions in webinar form, although I think we can all agree that seeing a session in person is always better than seeing it in webinar format. As I write this, we just had our first webinar with Marian Small, and by all accounts, it was a great success. We had about 45 people attend, and many participated in the discussion. If you didn't get a chance to participate in any of the webinars, you will find that we have recorded them and they are available in the Members Only section of the OAME website.

Please note that we only had a podcast and no webinar with Kyle Pearce and Jon Orr. This is because they had already done webinars of that session for the Ontario Teachers' Federation and the Global Math Department; hence, we linked to those sessions instead. The plan, however, is for every future session to have a podcast (released on the first day of the month), followed by a longer webinar (taking place on the second Wednesday of the month). This year, our March webinar will be on the third Wednesday due to March break. Our hope is for this to become a regular feature after each year's conference. Therefore, you can look for Season 2 starting up in September, and keep up-to-date with the latest *OAME talks* schedules and news by checking out our website or following us on Twitter (@OAMEtalks). ▲

▲ Executive Directors' Corner

LYNDA AND FRED FERNEYHOUGH
EMAIL: eds@oame.on.ca



Lynda and Fred Ferneyhough have been the Executive Directors of OAME since 2010. They taught in the Peel District School Board for over 30 years and had a three-year stint in the United Arab Emirates. Both of them served OAME as Chapter Representatives for CHAMP, Directors, and Vice-Presidents. During their career, they were Department Heads, and authors for McGraw-Hill Ryerson. They are both certified as instructors for Texas Instruments (TI), and Fred continues to coach for TI in the United States.

Ten years ago (2005–2010), the television show *Numb3rs* used the tag line "We all use math every day." It's interesting that many people around us took exception to that statement. One of our neighbours tells me to be quiet each time that I try to explain things mathematically. And the worst reaction came from our children, one of whom still tries to convince us that "not everything is mathematical!" Hah! Little does she know.

Years later, I look at our children and realize how true the tag line was. Our oldest son, Drew, is an artist. His work requires the application of ratios and geometry more than he might admit. In fact, you have likely seen some of his work. When CHAMP was organizing the 2014 OAME Spring conference, Paul Alves was talking to him and discovered that he was an artist. Independent of us, Paul asked him if he'd be interested in designing a logo for the upcoming provincial conference. His submission was accepted and he has created the logo for several OAME events since then. Our son's understanding of mathematics involved in his graphic designs has contributed to his success in his chosen field.

Figure 1: *OAME 2014 logo*

Our other son, Owen, is a musician. He owns several guitars and, like many musical people, he understands fractions quite well. Having worked at a school that had a performing arts program, we have seen first hand how well a background in music helps students understand mathematics. When the 1999 curriculum was being designed, one proposal had four quarter-credit math courses, one of which was "Math and Music." Unfortunately, we didn't have the opportunity to teach that proposed course.

One of our daughters, Meredith, rolled her eyes anytime we talked about the connections between math and life. She now works in logistics and has come to understand that there is a mathematical order to matters that is tremendously helpful to her. She also does market analysis, complex crop forecasting, and mortality ratios for her employer.

Our youngest daughter, Bryanna, is in information technology, and we'd love to understand half of what she knows about computers and networks. She is the one who came home from studying computer science one day with the question, "Why don't my friends understand hexadecimal?"

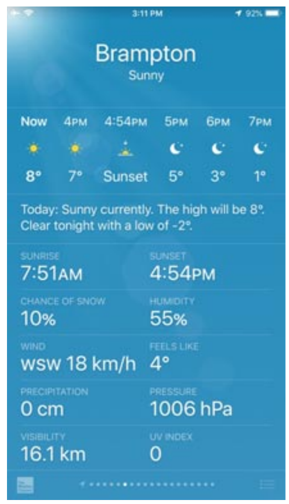


Figure 2: *Cellphone weather*

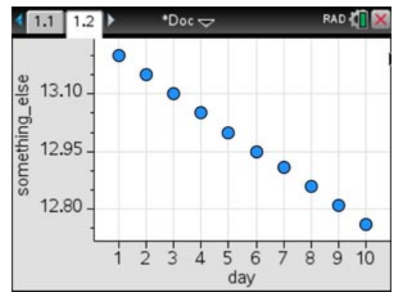


Figure 3: *A linear trend?*

A few years ago, we were in church on December 21. Our pastor made the statement, "Today is the shortest day of the year. Right, Fred?" And my reply was, "No, this day has 24 hours just like every other day." However, the comment segues to a lesson in pre-calculus that, to us, typifies an approach that shows students how daily events can be analyzed mathematically. The screen in Figure 1 is taken from a cellphone weather app showing the temperatures and weather data for Brampton on January 4, 2019. The line that is of interest is that of the sunrise and sunset times. This can be used to find the amount of daylight. What if we had students tracking and storing this data on a daily basis over the course of a semester? There is no reason why students could not set their phones for different cities around the world.

There are various expectations in our curriculum where students are expected to understand the differences between various functions. Figure 2 shows data where most students would identify the relationship as linear. The following screens are from compiling sets of data for five cities at five different latitudes. How could students apply their understanding of functions by observing the screens in Figure 3, all of which show the graphic representation of subsets of the hours of daylight data?

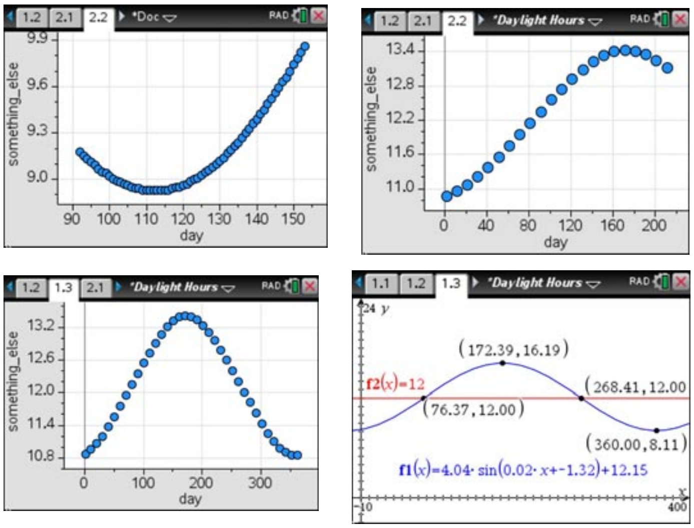


Figure 4: *Graphs of daylight hours*

Figure 4 shows the result after performing a sinusoidal regression to analyze the function.

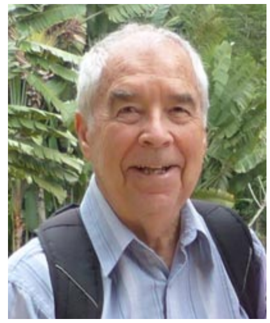
We added the horizontal line $y = 12$ to see when there were equal amounts of daylight and darkness. This occurs on day 76, which is March 17, very close to the March equinox. Day 268 is September 25, close to the September equinox. The maximum amount of daylight happens on day 172, or June 21, the summer solstice. The minimum amount of daylight occurs on December 26 in this model, which is close to the winter solstice.

The discrepancies could be due to the fact that we only used data points every ten days, rather than daily. But what we found really interesting was comparing the graphs of different cities. They all coincide on days 76 and 268, and all of them had the maximum and minimum amounts of daylight occur on the same days. The period was the same for each city. What differed was the amplitude of the sine waves, and that was due to the latitude of the cities. I'm sure that there must be a relationship there as well.

The point here is that by framing the context of daylight hours with mathematics, we can display mathematical properties in a manner that more students understand than by simply stating what is meant by phase shift and amplitude and vertical shifts and periods.

"We all use math every day" is more than a tag line to a popular show. It is a fact, and we can use that to find applications around us that we can bring into our teaching and conversations with neighbours. ▲

▲ Promoting Mathematics–Getting Involved with OAME



ERIC MULLER
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An invited opinion from a long-standing member of OAME.

As members of OAME, readers and contributors to the *Gazette* and *Abacus*, we join a community of mathematicians who, irrespective of their mother tongue, use and apply mathematics, sometimes referred to as "the universal language." Cultures around the world respect this language for its role in quantification and its ability to interpret science. At the start of the twenty-first century, our educational system in Ontario values mathematics to the same extent as English and French. Our challenge, as educators, is to help our students learn mathematics and enable them to use it in ways that they can appreciate its value, experience its role in society, and widen their employment horizon. This is a real challenge for many of our students who do not find it interesting to spend so much time learning the "rigid grammar" of mathematics but would be more engaged if they could tell, read, and write stories. My view is that we are now in a much better position to balance our mathematics teaching between "grammar" and "stories." For both the learner and the mathematician, technology opens a new world of mathematics "storytelling." In the early 70's, I started integrating technology in my mathematics courses at Brock, and it is my experience that evolving software offers interactive, dynamic, and colourful representations of mathematical concepts, and also provides an environment for exploring mathematical ideas. The student now has another creative avenue to engage in mathematics, for example, by simulating an application of mathematics in an area of personal interest.

I am a firm believer of sharing ideas and experiences both in mathematics as a subject and in its teaching and learning. OAME has provided an important avenue for me to do so. The Golden Section (OAME Chapter) was founded by a group of us so that we could host informal, but informative, meetings for mathematics teachers at all levels. Together we were motivated to develop and test, with school groups, mathematical activities in the Niagara Region. These included, among others, the Welland Canal Math Trail (**https://brocku.ca/cmt/English/index.htm**); the Niagara Falls Math Trail (out of print); the MDM4U projects (**https://brocku.ca/cmt/mdm4u/intro.htm**);

the Brock Bugs probability game, which is now available as an extended online game (**https://brocku.ca/mathematics-science/mathematics/brock-bugs-computer-game/**); and the "Mathematics and Visual Arts" celebration during the National Gallery of Canada exhibition of works by M.C. Escher at the Rodman Hall Arts Centre in St. Catharines.

Without OAME, my career as a university mathematician would not have evolved the way it did. I would not have developed a commitment to future mathematics teachers, and the Brock BSc/BEd program would not have been developed. It is important for all of us to share our mathematics teaching successes and stumbling blocks with members of the OAME community. ▲

▲ In Memory of Susan Stuart

Susan Stuart passed away Jan. 3, 2019 at the age of 72. She had a lengthy involvement with the OAME. She has been described as personable, supportive, and insightful.

She was an *Abacus* editor in 1992, President in 1995–1996, and a recipient of a Life Membership Award in 2004. She was involved with the NOMA chapter in its earliest stages and had a particular fondness for the Ontario Mathematics Olympiad (OMO).

During her tenure, there were substantial changes in education such as the cancellation of OAC courses (Grade 13). Susan was involved in consultations about curriculum and large-scale assessment leading up to Education Quality and Accountability Office (EQAO) being formally established in 1996. As the incoming president, she wrote "Education is changing, and so quickly. Your OAME executive and council try to keep ahead of things, but at times feel, as if they are barely keeping up" (*Gazette*, 33(4), p. 6). Within the OAME there were also significant changes that are reflected by changes in names of positions: "councillors" became "directors", and "directors" became "vice-presidents."

Susan was a classroom teacher and consultant for the Rainbow District School Board before becoming a professor at Nipissing University's Faculty of Education from 1990–2004. She was an author of many resources over her 36-year teaching career. In retirement she was involved in the Sudbury Teachers Lion Club and continued promoting education.

Susan will be missed by the mathematics education community.

▲ Who is Math Man?

GILL DUNN
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Editor Note: For many years, the OAME archives included an *Abacus* editor listed only as "Math Man." In this invited feature, Gill, a long-standing member of OAME, provides an explanation of who "Math Man" is and what he is all about.

Gill Dunn, aka *Marvin Mumbler*, was a mild-mannered math teacher, who tried to make mathematics more interesting, especially for those students who struggled with the subject. Since he retired from actively teaching in the classroom over ten years ago, he spends more time crusading around the planet (i.e., southern Ontario) as *Math Man*! Whenever he hears a student (or teacher) in distress, he quickly dons his golden Math Man cape, which has the digits from zero to nine on the right side of the cape, and all the mathematical operations on his left side. Math Man wears a cape, but doesn't fly. When asked, Math Man replies, "The cape is my Cloak of Logic. It's not for flying." But some students say, "Math Man flies around in my mind!"

Math Man with the Kid Zero twins, Zeke and Zena (Illustration by David Noble)

Upon entering the classroom, Math Man uses his S*uper Socks* to walk around the classroom without making a sound. When he spots a student who cannot solve the problem at hand, he'll whisper something in that student's ear. He's not telling the student the answer, but suggesting a strategy that may help to solve the problem. "It's not cheating," Math Man explains, "it's what teachers try to do every day."

Math Man has several enemies, but only one archrival—*GiveUp George*—who slips silently into students' heads, suggesting they cannot solve their math problems. GiveUp's methods are contagious, so Math Man must get to that student before other students around him begin to think the same way. If this situation gets out of hand, the whole class could give up, eventually telling the teacher the problem is *impossible* to solve!

One of Math Man's other superpowers is his *Math Vision*, also known as *D-Vision* (*division*). This *vision* allows him to see which students need his help, but you won't hear Math Man boast or brag about this or his other powers. He is very sensitive to those students who are intimidated by the class wizards who are excellent in mathematics. He would rather focus on a student's persistence to solve the problem. So when a student succeeds, Math Man delights in the student's improving self-confidence.

Like all superheroes, Math Man has one big weakness—chocolate pie. Whenever he sees chocolate pie, he is frozen in a sugar daze. Only the *Kid Zero twins*, Zeke and Zena, know how to snap him out of it by shouting out a mathematical word or phrase. If that fails, Zeke and Zena call on *Math Genie*, one of Math Man's closest allies—also known as Ms. Jeannie Tom—to rescue him from his frozen state.

Math Genie
(Illustration by David Noble)

If Math Man is not around, there is a special way that the students can get in touch with him. They simply hold their hand flat like a cellphone and dial "1–800–MathMan" in their head. (Please do not call this number on a real phone as it will not work—it requires imagination.) When called, Math Man will race to the rescue, provided the student has eliminated all the distractions that interfere with solving math problems: iPads, TV, video games, and even music. This way, students can hear Math Man's whispered suggestions that will help them solve the problem. ▲

(Illustration by David Noble)

▲ Mathematical Snapshots: A Lot of Math in a Parking Lot

RON LANCASTER
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Ron Lancaster is an Associate Professor at the University of Toronto, where he teaches mathematics courses for pre-service middle and high school teachers. He has over 20 years of experience teaching Grades 7–12 mathematics. Ron's professional activities include consultations and conference presentations in North America, Asia, England, the Middle East, Africa, India, and Europe. Ron is an author for the NCTM (The Mathematical Lens) and member of the Advisory Board for the Museum of Mathematics in New York. He is the recipient of the 2015 Margaret Sinclair Memorial Award Recognizing Innovation and Excellence in Mathematics Education, awarded by the Fields Institute.

This column takes us to the Glebe neighbourhood of Ottawa, where we will park for a while to notice, wonder, and question while we examine signs about parking. There is so much opportunity and, if you look for it, you will find it in your own neighbourhood.

Photograph 1: *Sign for a parking garage in the Glebe neighbourhood of Ottawa*

Photograph 2: *Another sign associated with the Glebe parking garage*

Notice that this is a four-storey parking garage in a residential area. Wonder why it is there, what purpose it serves, and how the residents reacted to having it built next to their homes. Wonder what was located there before the garage was built.

Notice there are 124 parking spaces currently available. Wonder how many spaces there are all together. Wonder if there are the same number of parking spaces available on each floor. Wonder what the relationship is between the area of a given floor and the number of parking spaces available.

Photographs 1 and 2: Mathematical Questions

Table 1 shows the total number of parking spaces available in the parking garage at 170 Second Avenue.

Table 1: *Parking Spaces Available*

Parking spaces for cars	146
Parking spaces for motorcycles	6
Accessible spaces	7
Parking spaces for electric vehicles	2

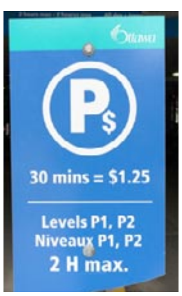
Source: **www.ottawa.ca/en/residents/transportation-and-parking/parking#short-term-and-monthly-rates**

(a) Determine the percentage of spaces available on the day Photographs 1 and 2 were taken for each of the following type of space.

Table 2: *Determining Percentage Availability*

	Percentage Available
Parking spaces for cars	
Parking spaces for motorcycles	
Accessible spaces	
Parking spaces for electric vehicles	

(b) What information and data would the city of Ottawa have used to decide upon the number of parking spaces for each type of vehicle? The parking garage opened in 2015, and the decision about the number of spaces for each type of vehicle was probably made several years beforehand. If the garage were being built today, what changes would you expect to see in the number of spaces for each type of vehicle?



Photograph 3: *Sign for the parking garage in the Glebe neighbourhood of Ottawa*

Notice the cost per 30 minutes. Wonder how much it would cost for parking 31 minutes or 59 minutes. Wonder if there is a daily maximum? Notice that nothing is said about the time of the day or the day of the week. Wonder if the cost of parking is still $1.25 for 30 minutes at 2 a.m. or on a Sunday. Note that people cannot park their cars for more than 2 hours on levels P1 and P2. Wonder if there are times when levels P3 and P4 are full and someone wants to park the car in the garage for more than 2 hours. Wonder if the person parks the car on P1 or P2, and takes a chance of not getting a ticket. Wonder what the fine is for parking more than 2 hours on levels P1 or P2. Wonder if it is less than, the same as, or more than paying for parking for the entire day.

Photograph 3: Mathematical Questions

The cost of parking a vehicle at the garage on any day of the week from 8 a.m. to midnight is $1.25 for 30 minutes. The maximum charge per day is $14. There is no charge to park in the garage from midnight to 8 a.m. A monthly pass costs $130. (Source: **www.ottawa.ca/en/residents/transportation-and-parking/ parking#short-term-and-monthly-rates**)

Table 3: *Cost of Parking*

Time	Cost
0.5 hour	
1 hour	
1.5 hours	
2 hours	
2.5 hours	
3 hours	

(a) Determine the cost of parking a vehicle at the garage for each of the following cases.

(b) Let *y* represent the cost of parking for *t* hours. Graph *y* versus *t* for $0 \le t \le 24$. Find a mathematical model for the cost as a function of the time.

(c) How would the graph and model change if:

(i) the cost of parking were changed to $1.50 for 30 minutes, with a daily maximum of $15 per day?

(ii) the cost of parking were changed to $0.75 for 15 minutes, with a daily maximum of $15?

(d) If someone paid $14 to park at the garage, what is the longest amount of time that the car was in the garage? For someone who parks the car in the garage on a regular basis, at what point does it make sense to buy a monthly pass?

Photograph 4: *Satellite view of the parking garage at 170 Second Avenue in the Glebe neighbourhood of Ottawa*

Note to teachers: If this image is not clear, access a real-time image, using Google maps, in your class so that students can view the digital image instead of a copy. Better yet, have students view the roof of the parking structure on their phones or tablets.

Notice the curves at both ends. Wonder if the curves are arcs of circles or part of a parabola. Wonder how a mathematical model for the curves could be obtained using the Geometer's Sketchpad®, Desmos, GeoGebra, or TI-Nspire™.

Notice the markings for the parking spaces on the roof (the fourth floor). Wonder how many parking spaces there are on the roof and if the number is less than, the same as, or more than the number of spaces on other floors.

Photograph 4: Mathematical Questions

Table 3: *Cost of Parking*	
Time	Cost
0.5 hour	
1 hour	
1.5 hours	
2 hours	
2.5 hours	
3 hours	

(a) Count the number of parking spots on the roof of the parking garage. Compare this number with the total number of parking spaces available in the garage.

(b) Do an Internet search for the average size of a parking space. Use this information to determine the length, width, and area of the roof of the parking garage.

(c) Figure 1 shows Photograph 4 (rotated clockwise through an angle of 30 degrees) superimposed on a set of coordinate axes. Describe the steps involved to determine the equation of the parabola shown in the figure.

(d) The parabola appears to be a good fit for the curved side of the building. How could we convince ourselves that the curve is in fact parabolic?

(e) Determine the equation of the parabola that contains the points D, E, and F.

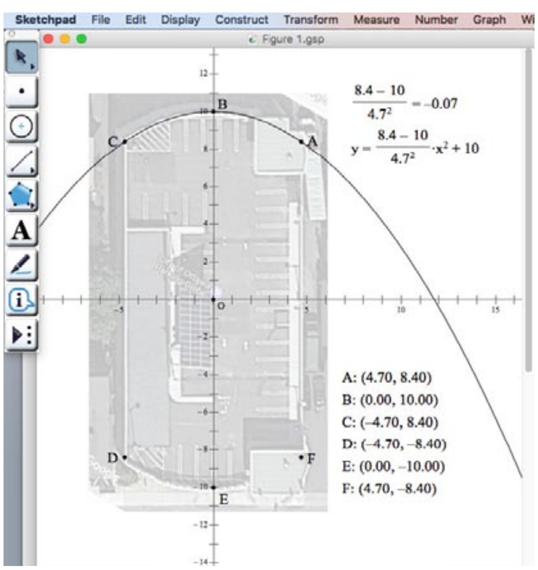
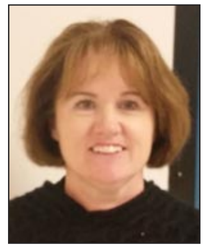


Figure 1: *Fitting a parabola to the side of the parking garage*

Final Thought

It would be interesting to have your students explore their neighbourhood and find situations that lend themselves to similar instances of noticing, wondering, and related mathematics questions. ▲

▲ OAME/NCTM Report: Classroom Enrichment, Using NCTM Membership



JACQUELINE HILL
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Jacqueline Hill is a Grades 9–12 teacher of mathematics, as well as an online instructor for York University. She is a Past President of OAME and OMCA, as well as the recipient of the award for Exceptional Teaching in Secondary Mathematics. She also wrote the "Director's Dialogue" for the Gazette *for a number of years.*

As a math teacher and mathematician, it is important to us here in Ontario to ensure that we reach every child in our respective classroom. It is our mission to make sure that, regardless of where students start when they enter our classroom for the first time, we help each and every one of them move forward in their learning. Being an NCTM member helps you fulfill your mission.

As a member of NCTM, you are able to pick the print journals that will help you the most in your classroom. The *Essential* membership tier includes one subscription, with print and digital access, to a school journal, while *Premium* members enjoy print and digital access to all three, as well as NCTM's research journal, the *Journal for Research in Mathematics Education*. Premium members also receive one FREE e-book per year after renewal. (Ed.: If you have received one, consider reviewing it for the *Gazette*!) All members enjoy a member-only discount at the NCTM Store (20 percent for Essential members and 30 percent for Premium members), and stock up on books, e-books, and other professional publications and products.

Personally, I enjoy the *Mathematics Teacher*, which focuses on Grades 8–12 and beyond. It has articles on research, pedagogy, and exciting student-relevant articles. Two recent articles that I have used for student engagement are "The Art of Mancala" (September 2018) and "Fishing" (February 2018). Each of these was given to interested students and they were asked for feedback, using two stem starters: 1) "What do you think?" and 2) "Where is the math?" If we consider parallel tasks that revolve around the big picture of mathematics all around us… these articles were used as one end of the spectrum for the question, while proportional reasoning was the other end.

When we think about differentiated instruction, we need to consider all students in our classrooms. It is often hard to find relevant meaningful points of entry for the top percentage of the student population. In my school district, these students (being in the top 2 percent of the population) are grouped together in separate classrooms for both mathematics and English in Grades 9–11 (as well as Grade 12 English). The mathematics teacher has a "Calendar Math" section, where thinking out of the box is sometimes required to obtain solutions. All Calendar questions have answers provided (so the teacher has at least one method of solution). Students as early as Grade 4 could use some of the Calendar Math as enrichment. (Note: Calendar Math can be used for all students who have an interest in mathematics.) One such question from September 2018 is:

"Each different letter in the alphanumeric puzzle represents a different digit. Find the numerical value of the word ME."

$$ME + ME + ME = AMA$$

The students "ink" their thinking with different points of entry, depending on the number of times they have tried the question. In my classroom, it is more about the journey of "inking their thinking" than the final solution, but one is always provided (and in this case, it is the only solution). Final solution: 47

NCTM important dates to note:

- April 1–3, 2019 – San Diego, California – NCTM Research Conference
- April 3–6, 2019 – San Diego, California – NCTM Annual Conference

Yours in mathematical fun,

Jacqueline ▲

Mathematical Wordsearch

M	R	G	E	O	B	O	A	R	D	A
U	A	T	D	I	V	I	S	I	O	N
L	T	G	T	S	H	E	D	P	B	A
T	I	N	N	S	S	I	T	R	T	L
I	O	O	Y	I	A	A	E	I	U	O
P	D	I	A	G	T	E	P	S	S	G
L	I	T	R	T	R	U	L	M	E	U
E	A	A	R	A	S	A	D	H	O	E
D	M	T	A	C	U	T	E	E	L	C
O	E	O	E	Q	S	D	R	C	O	L
M	T	R	E	G	N	A	R	N	O	O
D	E	G	R	E	E	I	E	B	U	C
N	R	B	E	N	C	H	M	A	R	K

Words to find: Acute, Array, Analogue clock, Benchmark, Compass, Census, Circle, Cone, Cube, Degree, Diagram, Diameter, Division, Equal, Geoboard, Least, Magnitude, Mode, Multiple, Obtuse, Prism, Range, Ratio, Rotation

Write the letters that are left over in the spaces below. But write them from left to right and top to bottom.

ALL FOUR SIDES ARE EQUILATERAL TRIANGLES:

"___ ___________"

Solution on Page 31

▲ Fields Institute MathEd Forum Report

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Angelica Mendaglio is an instructional designer at Vretta Inc. in Toronto, Ontario, where she helps to create interactive digital mathematics lessons and activities for middle school students.

The November meeting of the Fields MathEd Forum focused on the theme of *Computational Thinking in Mathematics Education*. The Forum included several talks from math and computer science education researchers on how coding and algorithmic thinking is being integrated into classrooms across the country. The day was an informative opportunity to see a variety of ways students are engaging with these topics, and how this can impact their mathematics education. The many different speakers throughout the day looked at the topic of computational thinking from several different perspectives, including what computational thinking looks like across K–12 classrooms, how some students are working with coding for the first time as undergraduates in math programs, how pre-service teachers perceive computational thinking, and how computational thinking was incorporated into the curriculum historically and in different provinces.

George Gadanidis' talk entitled *If You Build It, They Will Come* gave some examples of computational thinking tasks that he has observed being integrated into the classroom, and how they have led to new ways of thinking mathematically for students. George also identified four main goals for incorporating computational thinking tasks into the classroom: providing students with agency, while still addressing specific concepts; immersing students in a topic to allow them to spend time with ideas; modelling problems in order to bring the math to life; and creating math activities and experiences that students will want to share.

The power of agency and its role in many computational thinking tasks was also discussed in another talk given by Immaculate Namukasa, Derek Tangredi, Minakshi Patel, and Marja Miller. In this presentation, the speakers detailed a case study of a school incorporating computational and design thinking into Grades 3–8, using dynamic geometry environments. The researchers found that providing tasks with multiple pathways to learning provided students with agency, and was especially effective for typically reluctant learners and students with Individual Education Plans (IEPs).

Tasks with multiple possible approaches seemed close to being ubiquitous in computational thinking, where students are often presented with a problem and asked to find their own solution. In their talk, Brandon Dickson and Donna Kotsopoulos spoke about their research involving a classroom with students of different ages, where the older students were teaching the younger ones how to use programmable materials. This style of learning has been shown to aid the learning of both the older and younger students. Brandon observed that the younger students tended to be more divergent in their thinking than the older students. The older students seemed to be more focused on obtaining the solution they thought the teacher wanted, while a younger student insisted that his or her own solution (though perhaps more unorthodox) would achieve the same outcome.

On the topic of multiple solutions, Immaculate and her co-presenters also cited a case where students reported continuing to work on their programs at home to create more efficient code. The students in this case found intrinsic motivation to revisit a completed task to improve on their solution and explore other ways of reaching the same goal, thus deepening their understanding of the topics involved (in this case, the surface area of a pyramid).

Two of the speakers presented a close look at how computer science and computational thinking have been incorporated into curricula in Canada. The first talk, *An Overview of CT Integration in Canadian School Curricula*, given by Sarah Gannon, provided a picture of how each province has (or has yet to) incorporate computational thinking into its curricula. Sarah highlighted provinces such as British Columbia and Nova Scotia as leading the way toward integrating coding and other computational thinking concepts into their schools. In British Columbia, educators are endeavouring to teach all students the basics of coding by Grade 9, and in Nova Scotia, computational thinking was made a priority in their Education Action Plan, with coding being a mandatory part of the curriculum for Grades 4 to 6. In his talk *A Historical Look at Ontario's Computer Science Curriculum*, Steve Floyd looked at how computer science has been incorporated into the Ontario curriculum since its first appearance as a course called *Data Processing* in 1966.

The Fields MathEd Forum will reconvene on the last Saturday in January, February, March, and April at the Fields Institute in Toronto. If you are not able to travel to Toronto to join in person, you can watch the presentations remotely through the Fields website. ▲

▲ Provincial Digital Learning Resources – What's New? Pattern Blocks a Tool for All Strands and Divisions



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Ross (Near North District School Board), Agnes (Brant Haldimand Norfolk Catholic District School Board), and Markus (Near North District School Board) are Project Leads working on the development of the digital resources found at ***mathies.ca****.*

The Pattern Blocks+ tool at **www.mathies.ca/learningTools.php#Pp1** is an update to one of the very first tools in MathCLIPS. Even though the original Pattern Blocks tool had only four blocks and minimal functionality, it was one of the most visited tools on the **mathies.ca** site. Pattern blocks, physical or virtual, are very popular in schools because of the wide range of mathematical concepts that they can address. In this column, we describe some of the updated features and provide a selection of examples of how pattern blocks can be used to address mathematical concepts across all strands and divisions. Many of the following examples were inspired by explicit mentions of pattern blocks in the Ontario Curriculum.

This updated Pattern Blocks+ tool has the following features:

1. Improved functionality
    a. Fifteen blocks are available in a scrolling panel on the left edge of the tool—the four shapes from the original tool, the tan rhombus, the orange square, the boat shape that can represent ten-tenths, blocks to represent the remaining tenths, a double hexagon that can represent twelve-twelfths, a chevron, and a rectangle. Please let us know if there are commercially available sets of pattern blocks that include additional shapes.

b. Blocks snap to the triangular isometric grid and to each other to make visually appealing composite shapes. The snapping behaviour can be turned off to freely position and rotate blocks, using the magnet button.

c. A block or a group of blocks can be selected, moved, copied, reflected, rotated, or deleted.

d. The multiplier can be set to one, two, five, or ten, allowing multiple blocks to be dragged out from the selection panel.

e. The colour of a type of block can be changed by clicking the block in the panel and choosing a new colour.

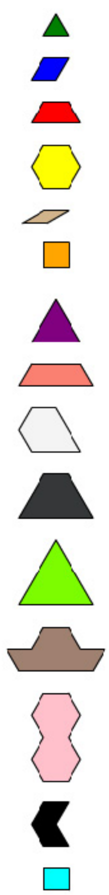
f. A block count is available.

g. The settings button can be used to determine the transparency of blocks, the transparency of the grid, the background colour, and whether the block's outline is visible.

h. The "Apply Template Settings" button in the settings dialog quickly changes your design to a grey "outline puzzle" with no internal divisions.

2. Common mathies tool features:

   a. Annotation
   b. Image import
   c. Undo and Redo—including for image manipulation
   d. Save and Load
   e. French and English vocalizations
   f. Available for:
      i. web browsers with the Flash plugin
      ii. iOS devices from the Apple App Store
      iii. Android devices from the Google Play Store
      iv. Windows or Mac as an installed application



It is expected that the update will be available by the time of publishing, together with an updated support page indexed at **https://support.mathies.ca/en/mainSpace/mathiesTools.php**. The revised support page makes explicit connections to the Fundamental Math Concepts and the Ontario Curriculum. It includes many more examples of how students can use the tool (or the physical manipulative) to learn mathematical concepts in addition to the ones given below.

Primary Division

As students analyze a design made of pattern blocks and recreate it, they increase their spatial awareness and improve their mathematical vocabulary. Students exercise their own creativity as they create original geometric designs or representations of familiar shapes, like a flower.

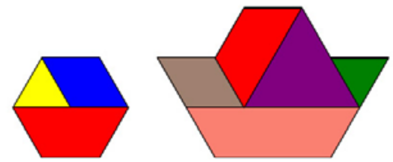


Figure 1: *Designs to create*

Figure 2: *Dinosaur outline puzzle*

Students could be asked to recreate the designs in Figure 1 (see the support page for a version in colour). Blocks in a design can be built next to each other or layered on top of each other. Looking carefully at the colours of the shapes will help reveal how a design was built.

When students are provided with an outline puzzle, like the dinosaur shape (Figure 2), they must recreate or cover the shape without any assistance from block outlines or colours. They can then count the various shapes. They can also be asked to find the fewest and most blocks necessary to cover the shape, with or without overlap. The Pattern Blocks+ tool allows outline puzzles to be created and solved (see the support page).

The curriculum not only expects students to identify shapes within other shapes, but also to identify symmetries in various designs. One of the Grade 2 Measurement curriculum examples asks students to estimate, measure, and record area, using Pattern Blocks (e.g., determine the number of yellow pattern blocks it takes to cover an outlined shape).

In addition to addressing measurement and geometry concepts, Pattern Blocks can be used to create and extend simple patterns as shown in Figure 3.

Junior Division

One of the Grade 4 Patterning and Algebra curriculum examples is: "Create a pattern block train by alternating one green triangle with one red trapezoid. Predict which block will be in the 30th place" (Ontario Ministry of Education, 2005, p. 73).

Students can work with patterns involving rotations and reflections (see Figure 4). Or you might have students identify the six congruent multi-piece shapes that make up the intriguing shape in Figure 5, and describe the transformations applied to one congruent shape to get the next.

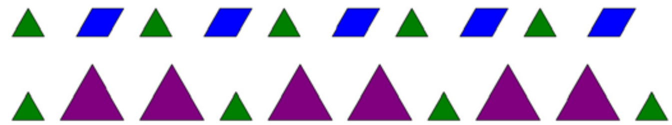


Figure 3: *Create and extend simple patterns*

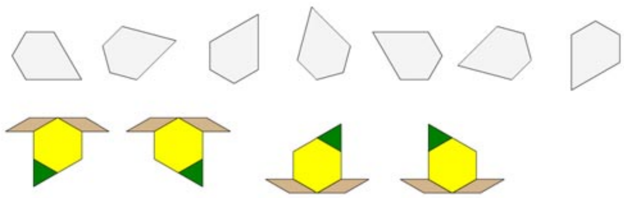


Figure 4: *Patterns with rotations and reflections*

One advantage of using virtual pattern blocks is the ease with which a group of blocks can be copied, rotated, or reflected to create the next term in the pattern or to extend a design.

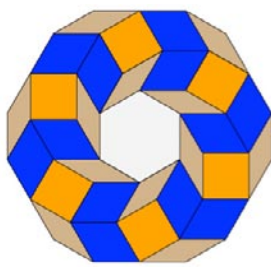
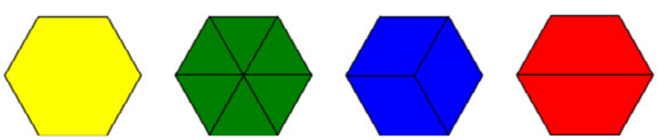
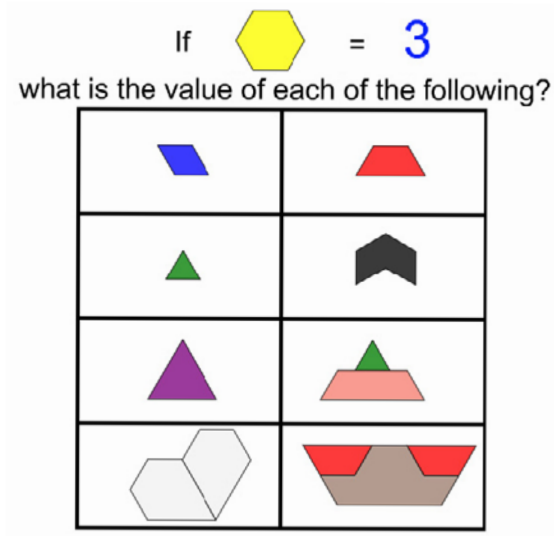


Figure 5: *Intriguing pattern from regelo54 on flickr and Twitter*

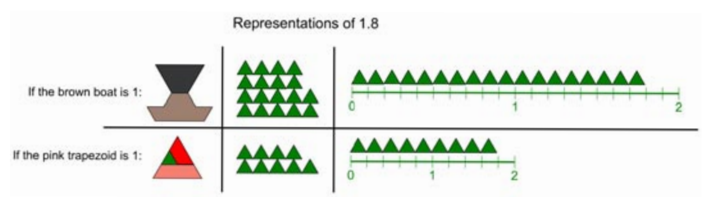
Pattern Blocks, and other manipulatives, are often used to provide opportunities to relate quantities. When students first use pattern blocks to represent simple fractions, the yellow hexagon's area is frequently considered to be one area unit, which allows the three other basic shapes to represent $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{1}{2}$.



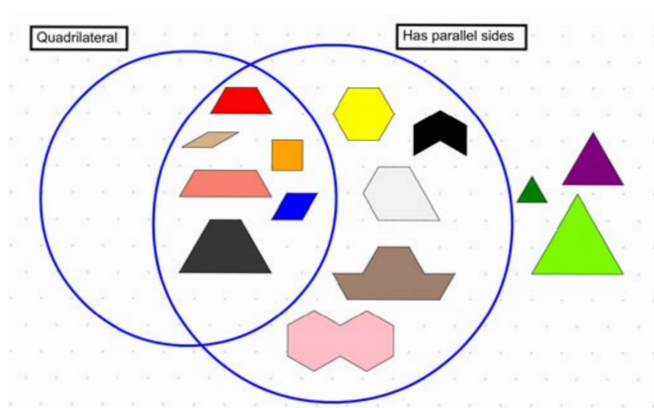
As students develop their number sense, other blocks (or groups of blocks) are considered to be one area unit or, as in the case below, considered to be a specific number of area units (see the *Changing Wholes with Pattern Blocks* task from Unit F of the Fractions Learning Pathways at **www.fractionslearningpathways.ca**).



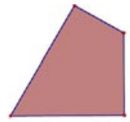
Similarly, decimals can be represented as areas or lengths. The brown boat is a convenient representation of a whole when representing tenths. It can be composed using ten green triangles, so each green triangle is one-tenth of the whole or (0.1). The pink trapezoid is a useful whole when representing decimals that are multiples of 0.2, since it is made up of five green triangles. As students work with fractions and decimals, it is crucial that they define the whole being used, and that they can adapt flexibly to changes to the defined whole.



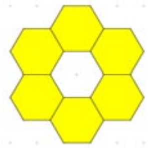
Pattern Blocks can also be used by students to identify and compare different types of shapes and sort them according to their geometric properties.



Students might be asked to design a block that fits in the empty region on the Venn diagram, i.e., a quadrilateral with no parallel sides (see right for one example).



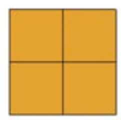
Junior students can be asked to create a composite shape, using six yellow hexagon blocks with the greatest possible perimeter. Independent of the "greatest perimeter," what discussion about perimeter might be sparked by the creation of this shape?



Intermediate Division

Students can identify, perform, and describe dilatations. Below, the grey boat is a dilatation of the brown boat. The centre of dilatation and the scale factor can be determined. How are the area and perimeter of the two similar shapes related?

A related sample problem from the Grade 7 Geometry and Spatial Sense curriculum is: A larger square can be composed from four congruent square pattern blocks. Identify another pattern block you can use to compose a larger shape that is similar to the shape of the block.



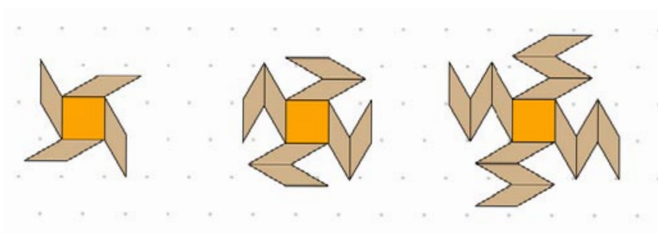


Figure 6: *Linear growing pattern*

Linear growing patterns can be created with Pattern Blocks (see Figure 6) so that problems related to them can be solved.

For example, PB Industries manufactures boat-shaped tables shown in Figure 7. Write an expression for the number of seats, *s*, for, *t*, tables. How many tables are needed to seat 100 people?

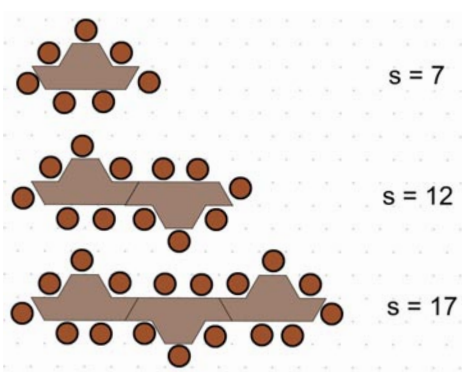


Figure 7: *Seats at tables*

A Grade 7 Geometry and Spatial Sense expectation is that students will determine, through investigation using a variety of tools (e.g., Pattern Blocks), polygons, or combinations of polygons that tile a plane, and describe the transformation(s) involved. Two interesting examples created in the Pattern Blocks+ tool are shown in Figure 8.

Senior Division

Students can express the area of various blocks, using the area of the orange square as 1 square unit. They might be surprised that the yellow hexagon is then less than 3 square units in area. Contrast this to the situation in which the area of the small green triangle is used as the area unit and the area of the hexagon measures 6 square units.

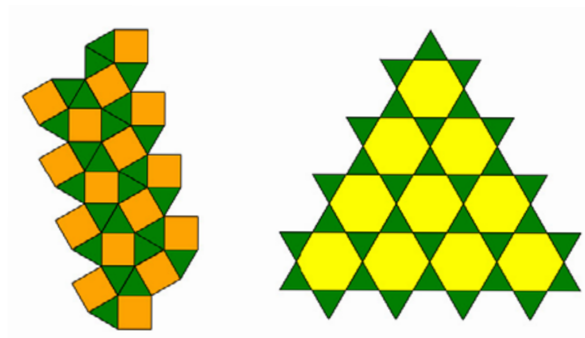
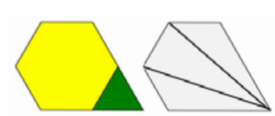


Figure 8: *Tiling the plane*

Since the hexagon is made up of six small green triangles, which have side length the same as the orange square, the hexagon's area can be calculated exactly as

$$6\left(\frac{1}{2}\right)(1)\left(\frac{\sqrt{3}}{2}\right).$$

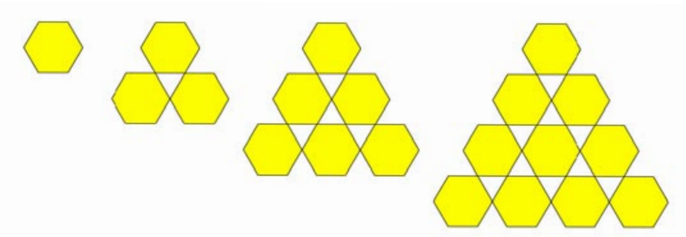
Students can use trigonometry to calculate lengths, angles, and areas of various shapes. For example, they could be asked to determine the length of the diagonals of the white pattern block (shown below on the right), and whether the two diagonals shown trisect the angle at the vertex at the bottom right.



If the length of the side of the small green triangle is 1, students could be challenged to create figures with sides or diagonals that measure $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}$, etc.

Students can determine shapes with the greatest possible perimeter for a given area, or the greatest possible area for a given perimeter or block combination.

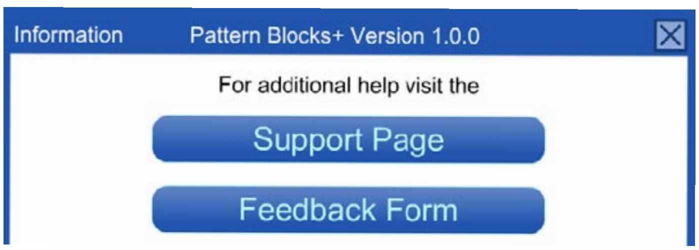
Students can create an expression for the general term in a pattern that is non-linear. Below, determine an expression for the number of hexagons in the *n*th term and the number of interior triangular gaps in the *n*th term. What is the difference between these two numbers in the *n*th term, and what does a graph of the value of the terms for the term numbers look like?



Pattern Blocks are versatile manipulatives that allow meaningful concrete and virtual representations of many varied mathematical concepts and relationships to be studied through all school mathematics.

Feedback and Future Requests

Please feel free to send us your feedback about any mathies tool, using the Feedback Form button inside the Information Dialog, accessed from the *i* button. Visit the support page for more examples and detailed descriptions of the functionality of the tool.

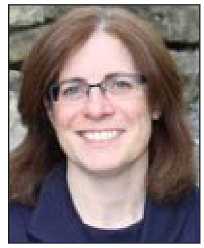


You can also send your comments to **WhatsNew@oame.on.ca**. You can share your experiences on Twitter, using the hashtag **#Onmathies**, and follow or message us, using **@ONmathies**. There is an increasing set of interesting posts of student and teacher work on Twitter. To be among the first to find out about the latest digital tool developments, sign up for our email list at **www.mathclips.ca/WhatsNewEmailList.html**. In fact, this past Fall, a draft version of the Pattern Blocks tool was made available to subscribers of the list.

Reference

Ontario Ministry of Education. (2005). *The Ontario curriculum, grades 1–8: Mathematics.* Toronto, ON: Queen's Printer for Ontario. ▲

▲ Technology Corner: Desmos Updates



MARY BOURASSA
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Mary Bourassa teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a Past VP of OAME, and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

Desmos continues to be an integral tool in many of our classrooms. Students can explore and investigate with it, they can check their answers, and reason through confusion. But Desmos developers are never content to rest on their laurels. They are always improving their product. They have introduced the ability to have answers displayed as fractions. You can scale one axis by holding the shift key as you move the axis. The recently released free scientific calculator app is a great addition to the Desmos family. They are also finding new ways of improving the student experience. Many of the activities on **teacher.desmos.com** are also available in French. Simply type "français" in the search field, and the activities that have been translated will appear. The hashtag **#ImproveMyAB** on Twitter was started by one of the Desmos fellows as a way of sharing an activity and soliciting feedback to make it as good as possible. Twitter has an incredible community of Desmos users who love helping each other out. Many have skills using the computation layer of Desmos, coding that allows control over many more aspects of a Desmos activity.

One of the latest activities to be released by Desmos is "Coin Capture: Lines." It incorporates some excellent features. The goal for students is to use equations to capture ten coins. The first screen is shown in Figure 1.

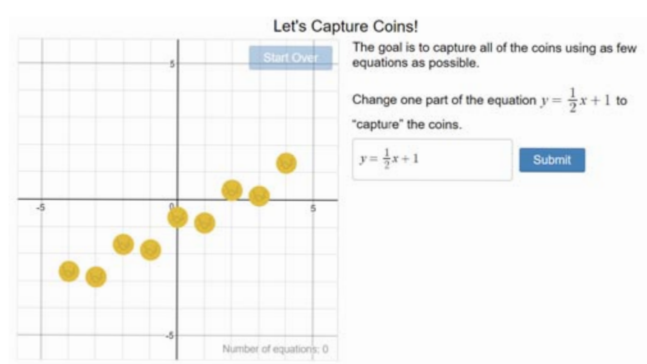

Figure 1: *Capture the coins*

When an equation is entered and the *Submit* button is pressed, the graph appears. As it hits coins, they disappear. If it does not hit all the coins, students can enter a new equation—the *Submit* button will have changed to “Add another line,” as shown in Figure 2. Note that the previously entered line still appears on the screen.

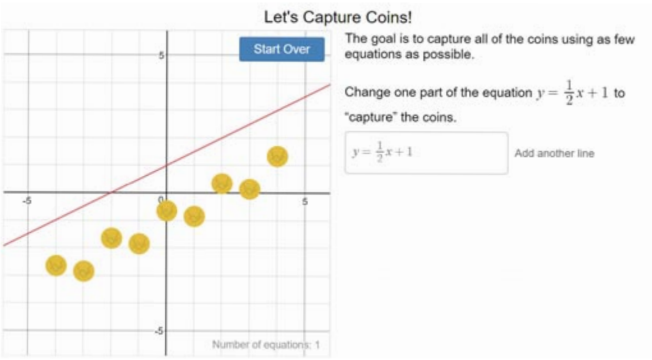

Figure 2: *After the first line has been entered*

Once students capture all the coins, there is a quick fireworks-like display, then students are shown the number of equations they used, and the fewest used by others in the class. They can start over if they think they can do it using fewer equations.

The next screen has the coins along a horizontal line and a vertical line, as shown in Figure 3.

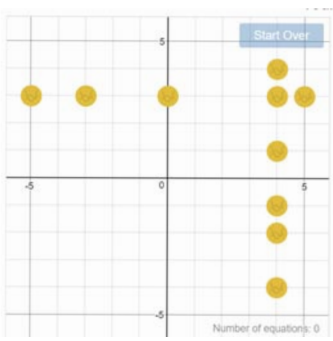

Figure 3: *Example with vertical and horizontal lines*

Although all the coins can be captured using two equations, students can use as many as they need. Teachers can receive a lot of feedback based on how their students approach this task. Figure 4 shows a student who likely doesn’t know how to write the equation of a vertical line, but is still doing well at capturing the coins.

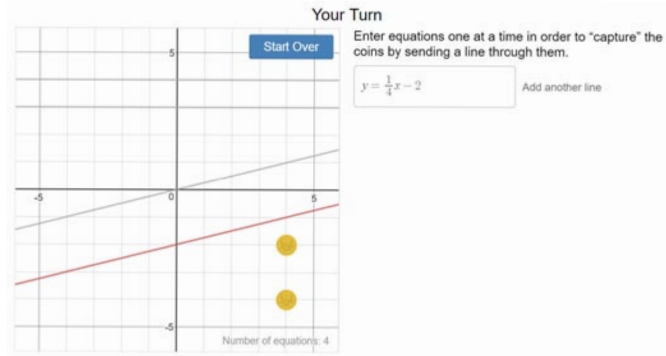

Figure 4: *Capturing a vertical line of coins—slowly*

The third screen, shown in Figure 5, sets up a challenge that can be solved in many ways. Again, you can see a lot of student thinking by how they choose to solve this challenge.

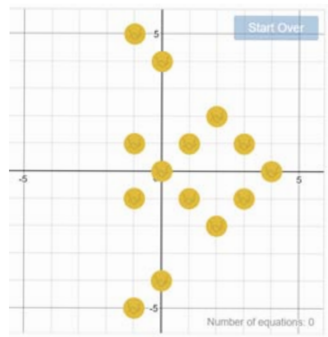

Figure 5: *Capturing a pattern of coins*

Once students have finished the first three screens, they can create their own. This functionality is known as a *Challenge Creator* screen. They can drag 10 points onto the grid and place them wherever they like, as shown in Figure 6.

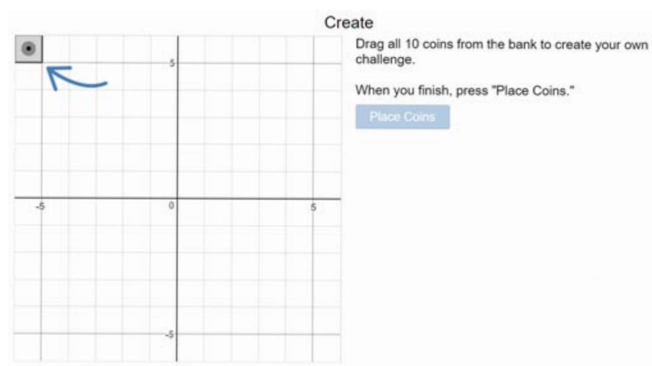

Figure 6: *The Challenge Creator screen, where students can create their own task*

Once students finish and press the *Place Coins* button, the points turn into coins, as shown in Figure 7, and the challenge is on. Students complete their own challenge first, then complete ones created by their classmates.

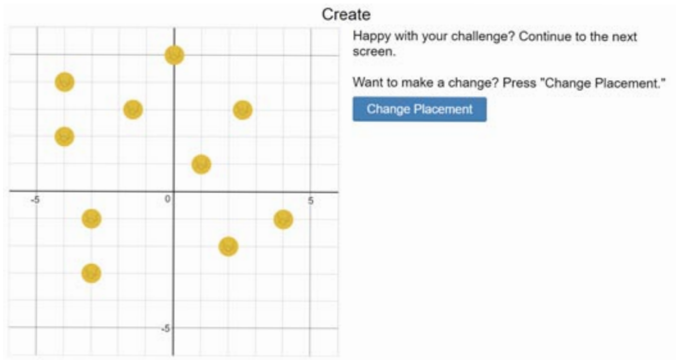

Figure 7: *Finalizing the constructed challenge*

Challenge Creator was created in part because the folks at Desmos, when creating activities, ask themselves, "Will this help teachers develop social and creative classrooms?" Giving students the ability to create their own challenges and try each other's, increases the engagement level, and shows a creative side that may not have otherwise come to light. Students care far more about their creations when they will be shared with their classmates than if they were only shared with their teacher. Furthermore, students interact with each other as well as with each other's creations. They loudly share their successes and want to know how to do things that they see other students doing. It raises the level of mathematics energy and understanding in the room.

There are *Challenge Creators* incorporated in many current activities, including *Parabola Slalom, Laser Slalom,* and *Point Collector: Lines*. The addition of tracking the number of equations and leveraging that information with students puts *Coin Capture* on a new level.

Another example of an activity that includes *Challenge Creator* is the *Two Truths and a Lie* series—Linears, Parabolas, Exponentials, and Conics. Each is only two screens long. The first screen familiarizes students with the concept, as they must choose the lie when presented with two statements that are true and one that is false. The first screen from *Two Truths and a Lie: Parabolas* is shown in Figure 8.

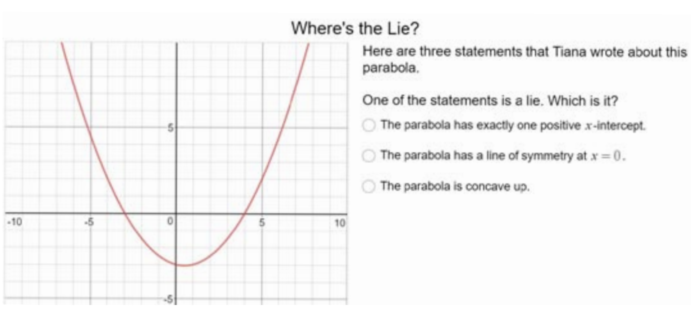

Figure 8: *Two truths and a lie example*

Once a choice is made, students are prompted for the reasoning behind their answer. This is an important opportunity for the teacher to use the *Snapshots* feature that allows you to quickly collect and display different student answers, as shown in Figure 9.

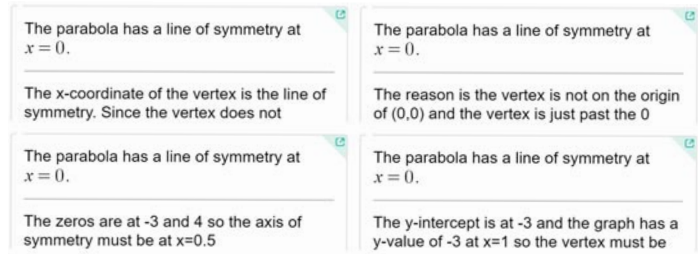

Figure 9: *The* Snapshots *feature that allows the teacher to see student answers*

When you click on the arrow at the top right of each response, you see the full response, as in Figure 10, including the graph that it is referencing. This allows for discussions and helps show student thinking, whether correct or incorrect.

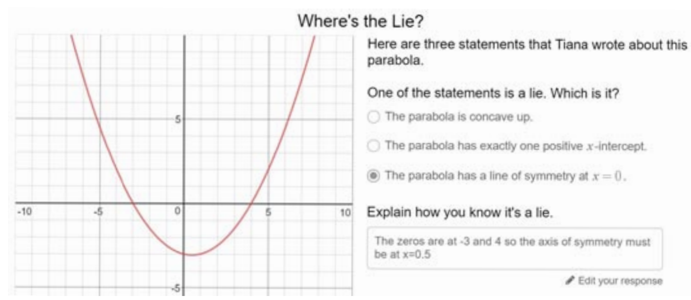

Figure 10: *Detailed student response*

The second screen of this activity is a *Challenge Creator*. Students drag points to move a parabola, then write three statements about their parabola—two true and one not. Students should be encouraged to not only try each other's challenges, but also to look at how their classmates have responded to the challenge they created. There are often unique answers that show variations in thinking that will help expand everyone's mathematical knowledge.

Desmos keeps improving their software and finding innovative ways in which to use it to help students learn and love (or at least like!) math. Check out the accessibility features for visually impaired students, and keep track of the latest news at **blog.desmos.com**. ▲

▲ Hey, It's Elementary: Turtles and Bluebirds and Bricks, Oh My!

LYNDA COLGAN
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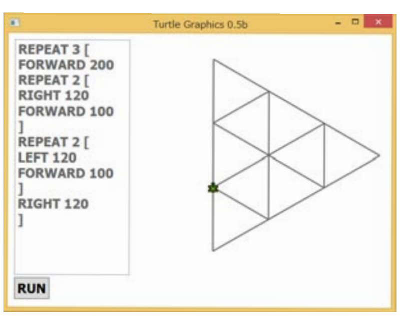
Dr. Lynda Colgan is a Professor at the Faculty of Education, Queen's University. In addition to her teaching responsibilities in the BEd and Graduate programs, she is involved in research and knowledge-mobilization projects with the Council of Ontario Directors of Education, TVO/TFO, the Ontario Ministry of Education, and the Mathematics Knowledge Network.

Many years ago, I participated in a unique program offered by the former Scarborough Board of Education called *Interchange*. This remarkable plan afforded the opportunity for elementary and secondary teachers to "trade places." I assumed the responsibilities for a Grade 5 classroom at Eastview Junior Public School, and that teacher took over my math sections at Sir Oliver Mowat Collegiate Institute. The "switch" was intended to last for two years, and was, of course, contingent on the approval of the math department head at the secondary school and the principal at the elementary school. Thanks to receiving the required approvals, the arrangement went ahead… an experience that changed my life and the direction of my career.

I remember that Grade 5 class with enormous affection and appreciation. We learned so much together, not the least of which was computer programming. Early in the year, buoyed by the energy of the 10-year-old, insatiably curious students, I applied for a microcomputer for my classroom through the Apple Education Foundation.



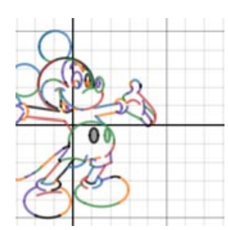
When it arrived, I received a monochrome monitor, an external drive for floppy disks, a printer, and an Allen key! The computer was equipped to use BASIC computer language… and that was it. A phone call to the curriculum department resulted in riches: programs for word processing, data management and, best of all… LOGO, the turtle-geometry programming language created by Seymour Papert and his team at the Massachusetts Institute of Technology (MIT).

What a year I had with that amazing crew of students at Eastview. For parent–teacher interview nights, we created images of waving robots, spinning squares, Haiku poems, and survey results to "wow" passersby with our digital discoveries. We posted printouts of illustrated short stories and the steps in our programming process, from graph paper to screen display. I learned as much as I taught from my Grade 5 gurus that year, regardless of the subject (except in basketball, where the scales were tipped greatly in their favour). Upon reflection, however, I recognize that together, Room 8 and I were pioneers in the integration of technology for learning, especially in mathematics.

That privilege led me to the role of Computers in Education consultant at the board level, and an opportunity to head up the creation of a microcomputer lab that students were bused to from across the city. They received lessons, best described as "computer literacy," that introduced spreadsheets, databases, word processers, paint programs, and, of course, LOGO.

For five years (excluding a one-year secondment to the Faculty of Education at York University), I was immersed in the world of educational computing, especially LOGO. I am grateful to Bill Higginson, Gary Flewelling, and Peter Skillen for their roles as mentors, teachers, and motivators—always pushing me to strive to use technology—not to do things that we could already do, but to push the boundaries of teaching and learning because of the uncharted opportunities that technology made possible, especially visualization and concretization of abstract concepts such as dynamic, visual proofs of trigonometric identities.

Fast-forward a few decades through my career path as PhD candidate, Mathematics District Coordinator, and Professor of Mathematics Education. Technology was always in my field of vision, from graphing calculators and Geometer's Sketchpad®, to Desmos and the Virtual Manipulative Repository. I have enthusiastically endorsed and supported efforts to bring high-quality software and ancillary resources into the mathematics classroom over the years.



Until recently, I have been saddened by the limited extent to which computer technology has impacted the mathematics education experience of most students. Interactive whiteboards get used as projection screens or large-screen worksheets; calculators are being used for rote computations; and software titles such as Desmos are sometimes being used as little more than electronic graph paper, as seen in this coordinate point-plotted image of Mickey Mouse.

That is why it has been so energizing to watch the "back to the future" popularity of coding in elementary schools, and in particular, the implementation of *Scratch—a* graphical programming language, developed by the *Lifelong Kindergarten Group* at MIT. Scratch is described as the programming equivalent of LEGO: children can drag and combine code blocks to make a range of programs, including animations, stories, musical instruments, and games. The widespread recognition of the value of coding has been nothing short of a transformation in terms of its global uptake and the number of large-scale events such as *The Hour of Code* and *Canada Learning Code*. The Kingston Frontenac Public Libraries run a highly successful *CoderDojo* for adults and children, and at Queen's, many student-led outreach teams are taking coding to regional schools, public libraries, and organizations such as The Boys and Girls Club.

Equally revitalizing recently was the experience of working with six extraordinary schools as part of an Ontario Ministry of Education-funded project, *Building Parent Engagement in Mathematics*. The project, one of many intended to achieve the goals of the *Renewed Mathematics Strategy*, has focused on involving parents and guardians as partners in their children's mathematics education journey throughout the elementary school years.

See Gazette December 2019 issue for explanation of this whitespace.

▲ MB4T (MATHEMATICS BY AND FOR TEACHERS): THE WONDERS OF PI

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Jennifer Holm is an Assistant Professor at Wilfrid Laurier University and works with primary/junior and junior/intermediate pre-service teachers, as well as in the field supporting current mathematics teachers. She is interested in developing mathematics knowledge for teaching and analyzing beliefs about mathematics with both pre-service and in-service teachers. Her current focus is on the beliefs and opinions that pre-service teachers hold about mathematics and teaching along with the connection they have to past experiences. She uses this research to support future teachers in developing beliefs and knowledge that will encourage and support effective teaching practices.

With Pi Day being this month (3/14), it seems like an opportune time to explore the magic and mystery that is the irrational number *pi* (π). It is defined as an irrational number because there is an ndefinite number of digits that continue without repeating, where they continue. *Pi* is the constant value that results when you look at the ratio of a circle's circumference and diameter $\left(\frac{\text{circumference}}{\text{diameter}}\right)$. At the elementary school level, it is important to have a discussion about approximation when calculating the circumference or area of a circle because a rounded value is always used to represent *pi* (usually 3.14 at this level). There are many activities that can be used for students to explore and experience *pi* and learn more about irrational numbers in the process.

An engaging way to start the discussion on *pi* is to focus on measuring the circumference of different circles and their diameters. First, have students measure the circumference and diameter of different circles and record this information in a table. Next, start a conversation about whether or not there is a relationship between the two measurements. Instead of telling students to divide the two numbers, it would be interesting to give certain circles where the ratio is obviously close to three when examining the values in the chart. Allowing students to discover that even with human measuring error or estimation, there is an approximate value of three when the circumference is divided by the diameter, and this can allow for them to experience *pi* instead of just worrying about using calculations to compute it. Many

▲

people round *pi* to 3.14 as a manageable number, but it is interesting to explore that there are more digits to *pi* than just these two decimal places. Many different websites show different numbers of digits for *pi* that can be explored. The "PiDay" website shows *pi* to one million places (**www.piday.org/million/**). It would be interesting to have students explore the digits and see what they notice about the numbers. Are there sections where the numbers repeat or show a pattern? You could also give out coloured circles, with each colour representing a digit, and have students "decorate" for Pi Day by putting the circles up along a hallway to as many places as possible. The colours for the digits would allow students to make some observations about what it means to be an "irrational" number.

Geometry software can be used to explore *pi* and tosee that for any circle this ratio exists between the circumference and diameter. Consider using GeoGebra (**www.geogebra.org/**) to illustrate that the ratio of *pi* exists in any circle without having to calculate multiple circles by hand. This software is freely available and allows for many different functionalities.

I will demonstrate this activity using the graphing app that is available on the GeoGebra home page. It is helpful before beginning to make some adjustments to the graph so that you can clearly see *pi* in the activity. If you look at Figure 1, there is a gear in the top right corner of the graph (as indicated by the arrow in the image on the left. When you click on the gear,) the menu on the right moves into your screen. Here you can set the number of decimal places for the formulas (e.g., mine is set to 13 places). You are going to want to have more than three decimal places selected. This is also where you can set the font size to a larger one, if you are needing to demonstrate the activity.

Figure 1: *Setting the graphing calculator in GeoGebra*

Now use the software to create a circle. First, you click on the "shapes" button at the top of the window (indicated by the horizontal arrow in Figure 2). Once you have selected this tool, a menu such as in Figure 2 appears. The arrow circle in the bottom right corner of the menu allows for the user to move the object around on the graph. To create a circle, choose the circle tool indicated by the vertical arrow on the bottom left. Once you click "Create a point on the graph," you can click to select the second point and create circles of any size, as shown in Figure 2.

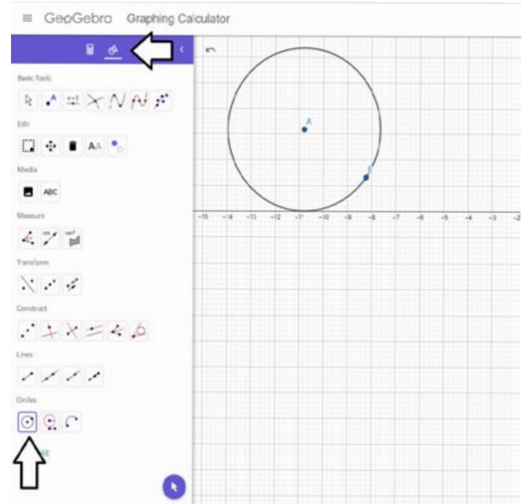


Figure 2: *Creating a circle in GeoGebra*

Next, create a line or ray as a step toward creating the diameter on the circle that you have created (we could use the radius, but the diameter is better for getting at the *pi* concept). I like to use a ray, rather than to a line, so that it only continues in one direction and does not distract so much from the circle and main focus of the activity. Figure 3 shows the tool to use to create a ray. To use the tool, once you have clicked the button, first click on the existing point on the outside of the circle (B). This will cause the ray to appear in the graph. Next, click the point in the centre of the circle (A), which will anchor the ray so that it starts at the outside of the circle and passes directly through the centre.

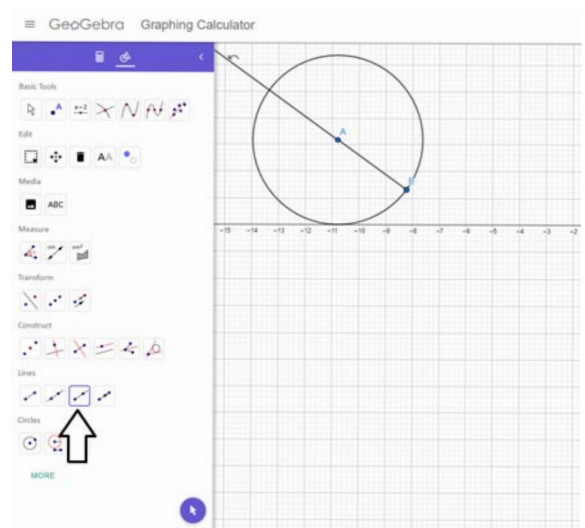


Figure 3: *Creating a ray on the circle*

Once the ray has been created, Figure 4 shows the tool for placing a point. Once you have selected the point tool indicated by the vertical arrow, you can select the point where the circle and the ray intersect (indicated by the horizontal arrow). You can test if you have chosen the right point by choosing the “arrow” tool (bottom right corner of the menu) and then change the size of the circle by dragging the point at the bottom right of the circle. The point at the top left needs to stay connected to the circle and the ray. This new point will be labelled as C on your graph, and you will now be able to use GeoGebra to measure both the circumference and the diameter.

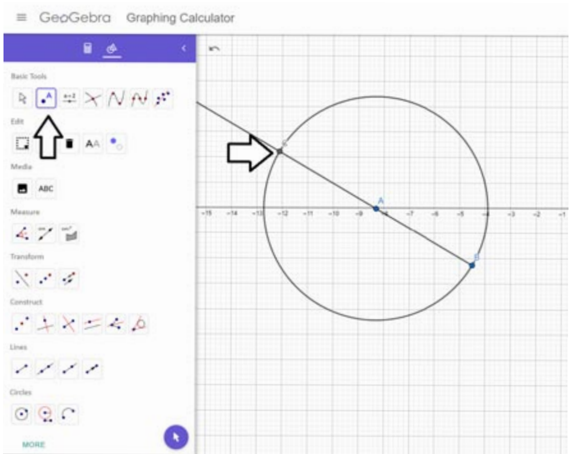


Figure 4: *Creating a point to calculate the diameter of the circle*

The tool for measuring a distance is shown in Figure 5, as indicated by the vertical arrow. Once you have clicked on this tool, you can now click on points B and C (the ones connected to the outside of the circle), and the measurement of CB (the diameter) is now printed on the graph. Next, click on the outside rim of the circle (not one of the points), and the circumference of the circle is now displayed on the graph. If you choose the “arrow” tool again (in the bottom right corner), you are able to adjust the size of the circle (by dragging the point on the outside of the the circle), and you will notice that the corresponding measurements also change with each increase or decrease of the circle size.

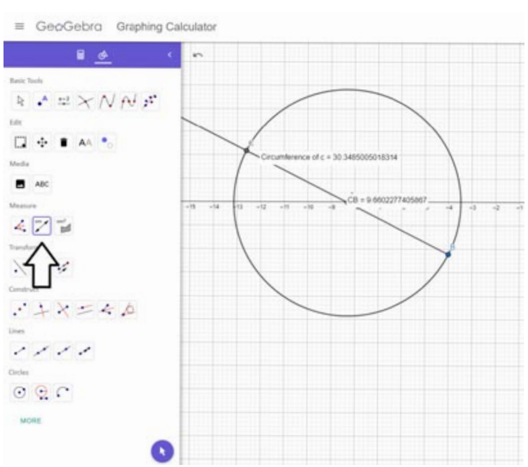


Figure 5: *Calculating the circumference and diameter*

Finally, go back to the original menu for the graphing program by selecting the calculator at the top of the menu. At the bottom, in the bar, you can type “circumference(c)/distance(C, B)” to create a formula to show the ratio between the two measurements. The arrow in Figure 6 shows the formula where it was created. (Note that you cannot click on measures to build the calculation; you must type in the formula directly. Also, if your measurement says B, C on the graph, you need to reverse the two letters in the formula for the distance.) With the formula in place, as students change the size of the circle, it is clear that the calculated ratio does not change, yet the two measurements you have calculated do.

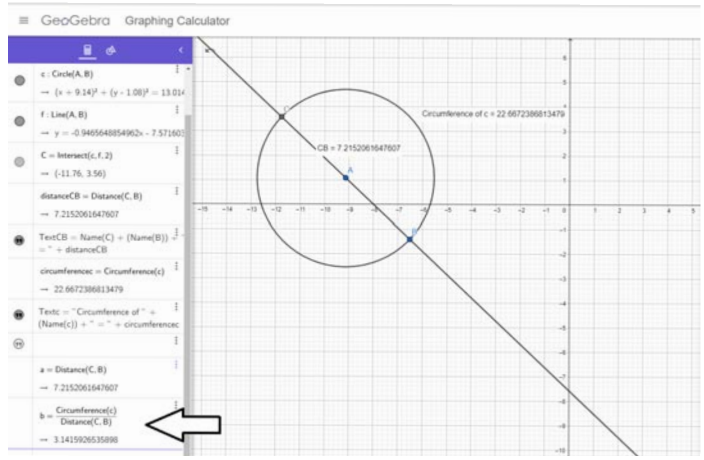
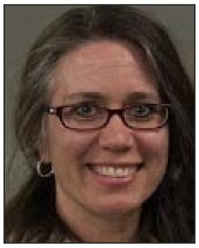


Figure 6: *Creating the formula to calculate* pi

Although it takes a bit of time to set up the circle activity, it now allows students to experience *pi* without having to labour through creating many circles and doing measurements or calculations (after the initial activity, where they have first seen *pi*). The idea with this is to, in a way, “prove” *pi* exists with all circles by easily testing all different sizes. This could be one of many activities to build the understanding of *pi* on Pi Day. Once students have explored *pi* and circles, an insight question to ask is, “If I give you the circumference, can you find the diameter?” What is fascinating about circles is that if you have any of the measurements (radius, diameter, circumference, or area), you can find all the other measurements, using algebra.

There are even some musical tributes to *pi* that can be explored to learn some of the sequence of digits or experience them in a new way. For example, *The Pi Song* (Memorize 100 Digits of π), created by AsapSCIENCE, is a catchy tune to help students remember more of the digits of *pi*: **www.youtube.com/watch?v=3HRkKznJoZA**. Another fun way to incorporate music and *pi* is to assign each digit to a specific Boomwacker or tone in order to play a tune. “aSongScout” has posted an idea of assigning each digit to a key on the piano to play the tune of *pi*: **www.youtube.com/watch?v=OMq9he-5HUU**. Have fun enjoying the wonders of the irrational number *pi* with your students! ▲

▲ In the Middle: Puzzles and Problems for All Ages



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Carly Ziniuk teaches Grade 9 Mathematics, Grade 12 Data Management, and Advanced Placement Statistics at the Bishop Strachan School in Toronto, Ontario, Canada. She is very active in adopting real-life data to engage her students in solving problems.

For many summers, I ran a one-week Math Camp for students from Grades 5 to 8. Rather than spending the week solely on content acceleration, we learned about topics that are not explicitly in the curriculum, but that were mostly historical and recreational mathematics problems. Once per week during the school year, I run a similar one-hour lunchtime Math Club with Grades 4 to 6. This weekly drop-in program has over 30 junior school students who are keen to learn more about math, beyond the specified curriculum. I usually focus on one or two classic problems, like the *four-colour map theorem*, the *traveling salesperson problem*, or the *prison cell problem*. This column presents some additional one-hour (or less) problems, puzzles, activities, and games that you can do with students of any age.

These problems span all areas of mathematics, require little background knowledge, and can be used for extension for a few students or for the entire class, after completing a lesson with time to spare. These problems may also be collected for use in a carousel or jigsaw activity to inspire problem-solving strategies. Many of these problems work well with the Thinking Routines that I discussed in the last issue of the *Gazette*. At the end of this article I have included additional sources for similar types of problems.

Perfect Party

Are there always two guests at a party who know the same number of people at the party?

Answer: No. This is my favourite introduction to proof by contradiction with any age of student: suppose the opposite and find a contradiction, to be used. I usually start with a small number of friends. If there are three people at the party they would need to have to know zero, one, and two people for them all to know a different number of people. However, if there is a person who knows two people (i.e., the other two people), then no one can know zero people (assuming knowing is mutual).

This can be generalized for *N* people at the party. You will need to know 0 to *N*-1 people if they are all going to know different numbers of people. However, this doesn't make sense since one person knows *N*-1 people (i.e., everyone), and one person knows 0 people at the party. If you don't allow people who know 0 people (Is that person actually going to be at the party?), then you don't have enough numbers to allocate a different number for each person at the party (this is known as the *pigeonhole principle*—see **www.mindyourdecisions.com/blog/2008/ 11/25/16-fun-applications-of-the-pigeonhole-principle/**).

Ten Questions for a Thousand Numbers

If someone chooses a number from 1 to 1000, can you use only 10 "yes or no" questions (or less!) in order to have you "guess" the number correctly?

Answer: It is possible if each question cuts the possible solutions in half. I often play this game on Halloween with a Mathemagician flair, starting with the question, "Is the number larger than 500?," and then cutting by half again in the appropriate direction. Generally, for ten questions, students can choose a number from 1 to $2^{10} = 1024$, but 1000 seems a more natural choice for an upper limit.

Fool's or Fair Gold

You are given nine golden rocks that appear identical. Eight of them are fool's gold. One rock is real gold, which is slightly heavier. Using only a balance scale that can compare one weight to another, what is the smallest number of comparisons you need to make in order to determine which rock is real gold?

Figure 1: *Fool's gold (pyrite) from* **https://geology.com/gold/fools-gold/**

Answer: Two. Divide the rocks into three groupings (ABC, DEF, GHI) and place two groupings on the balance scale. If they are equal, then the third grouping has the heavier item and can be divided similarly to determine the heavier rock. If the first comparison had a heavier pile, then take the heavier pile and divide it into three piles and weigh two of them. The Nrich Maths site features an interactive applet for this problem at **www.nrich.maths.org/5827**.

Dollar Words

If each letter has a value, in cents, equal to its value in the alphabet sequence (A=1¢, B=2¢,… Z=26¢), can you find words with a total value of exactly $1.00? This problem originated with Marilyn Burns' *The $1.00 Word Riddle Book* (1991), and has appeared in print as recently as the July 2018's *Mental Floss* **www.mentalfloss.com/article/518397/dollar-words-logophile-math-game**.

I enjoy starting this problem by showing students some hinting images: GRUMPY the dwarf, a Halloween

PUMPKIN, ELEPHANTS in a trunk and tail train, a map of Alberta with its capital EDMONTON marked, and a picture of a wall of TOILETS from a plumbing store. I often get the younger students to find other options in this problem by making a list of suffixes and prefixes including ED, ING, ION, and PRE. You can find a number of student-made apps on Scratch (or maybe your students can make one) and lists online. I also give the definition clues for some math-specific dollar words:

- Someone who uses a ruler or thermometer (*measurer*)
- The act of finding the biggest appropriate value (*maximize*)
- The ability to use and work with numbers (*numeracy*)
- A four-sided plane figure, especially a square or rectangle (*quadrangle*)
- An adjective for a solid having four plane triangular faces; a triangular pyramid (*tetrahedral*)
- The state or fact of something being likely; probability (*likelihood*)
- A structure of hexagonal cells of wax, made by bees to store honey and eggs (*honeycomb*)
- A direction to the typical forward movement of the hands of a clock (*clockwise*)

Olympic Rings

Consider the five Olympic rings in their classic formation. Place the numbers 1 through 9 in each of the sections so that the sum of the numbers in each ring is the same.

Answer: There are many ways to do this problem. In fact, Transum has an applet with increasingly harder starting versions of this problem. See **www.transum.org/Maths/Puzzles/Olympic/Rings.asp**.

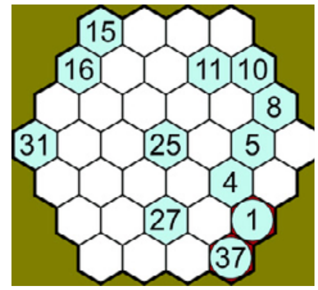
Chris Smith, a math teacher from Scotland, has a very charming visual for creating these intersecting rings from one piece of paper and a pair of scissors. Chris has a weekly newsletter and Twitter feed that may be of interest: **www.twitter.com/aap03102?lang=en**.

Tea with Milk or Milk with Tea

You have two mugs: one filled with milk, and one filled with tea. Each mug contains the same amount of liquid as the other. Take two spoonfuls from the milk mug, transfer them to the tea mug, and stir the resulting mixture well. Then take two spoonfuls from the now-mixed tea mug and transfer them back to the milk mug. Which is greater: the amount of tea in the mostly-milk mug, or the amount of milk in the mostly-tea mug?

Answer: My favourite answer to this problem comes from Plus Mathematics, **www.plus.maths.org/content/puzzle-page-36**: “Write *V* for the initial total volume in each mug. At the end, each mug still has a total volume *V*. Write *W* for the volume of milk in the second mug at the end (this means there is *V*–*W* of tea). By conservation of milk, since there was *V* to start with, there must be *V*–*W* of milk in the first mug and hence *W* of tea. So there is exactly the same amount of tea in the first mug as milk in the second.”

Hidato



This problem originally came to me via Alex Bellos in *The Guardian* (July 6, 2015). If you aren’t a regular reader of Alex Bellos’s Monday puzzle, it is something to look forward to every other Monday. Hidato is akin to Sudoku, in that it involves placement of numbers into a geometric arrangement, using logic rather than arithmetic. Hidato was created by Israeli computer scientist Gyora Benedek and is named after the Hebrew word *hida*, meaning “riddle.” The goal of the puzzle is to fill the cells with consecutive numbers from 1 to the highest number (the end and the beginning numbers are circled) so that the path created covers the beehive, honeycomb-shaped board entirely (1 next to 2, sharing a side with 3, and so on). See **www.theguardian.com/science/2015/jul/06/alex-bellos-monday-puzzle-can-you-solve-it-beehive-hidato-new-sudoku**.

There are increasingly difficult versions of Hidato boards available online, including at **www.hidato.com** and in the September 2012 issue of *Smithsonian Magazine*.

Telephone Pole Pacing

The Journal is an Irish news website that celebrates Ireland’s “Maths week” in the third week in October. They have published a math problem each day for that week, and have been doing so for years. Here’s one from 2016 that my students have always enjoyed:

> *There are 10 telephone poles along a road, each spaced 50 metres apart. If it takes you three minutes to make it from the first to the last pole, what is your average speed in km/h?*
> (From **www.jrnl.ie/3032917**)

Answer: Between the 10 poles there are 9 spaces, so 50 × 9 = 450 m for the entire distance from first to last pole. 450 m in 3 min. is equivalent to 150 m in 1 min., or 9000 m in 60 min., which is 9 km/h.

Why problems for any age?

At my school, the high school Math Society and middle school Math Olympics club occasionally use the same problems as the junior school Math club, regardless of student age, experience, or mathematical background. Considering one problem with a variety of ages has been rewarding for all of the teachers involved because the students have different perspectives on the same initial question. It leads to productive conversations about extensions to the problems and how they connect to the curriculum.

For example, for over ten years, our department has wrestled with the math behind the game "Spot It." Teachers from all three divisions start each new year off with the 1–100 challenge of the Year Game (see **www.mathforum.org/yeargames/**) when we return from the holiday break. The Junior Math Club created the 12 Days of Christmas pattern for number of gifts per day and number of gifts total, and connected it to the PNC Bank's annual posting of the inflationary costs. The Grades 9 and 10 teachers then applied modelling to the linear and quadratic functions for these patterns (see **www.pnc.com/en/about-pnc/topics/pnc-christmas-price-index.html** and NCTM's T*he Mathematics Teacher* – Media Clips, January 2005).

Problems like the 12 Days of Christmas and the Year Game can be used before the holidays or with a few students needing a challenge. Similar recreational math puzzles and problems for some students become lifetime recreations, like many people who do Sudoku regularly or play the game "Set." Although these problems are not often found in textbooks or explicitly in curriculum documents, they clearly support a wider understanding of mathematical reasoning and functional numeracy that the curriculum encourages among our students.

Here's hoping you enjoyed a few of these and are keen to go exploring for more!

Where to find more problems like these:

- Nrich: **www.nrich.maths.org/posters** and **www.nrich.maths.org/secondary**
- Plus Magazine: **www.plus.maths.org/content/Puzzle**
- NCTM Illuminations – Brain Teasers: **https://illuminations.nctm.org/BrainTeasers.aspx**
- Transum: **www.transum.org/Software/Puzzles/**
- Math Playground: **www.mathplayground.com/games.html**
- Think Fun: **http://info.thinkfun.com/stem-education/topic/brainteasers**
- Math Munch: **https://mathmunch.org/tag/puzzles/**
- Figure This!: **http://figurethis.nctm.org/index.html** ▲

▲ What's the Problem? Circles and Squares

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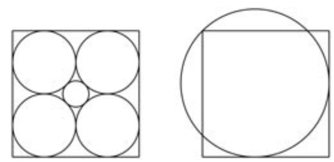
Shawn Godin is head of Mathematics, Business, Law, and Computer Science at Cairine Wilson Secondary School in Orleans. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: presenting workshops, writing articles, working on local projects, and helping create mathematics contests.

Welcome back, problem solvers. Last time, I left you with the following problem: Determine all you can about the squares and circles drawn in the figures below:



These diagrams appear on pages 45 (diagram on the left) and 50 (diagram on the right) from the book *Measurement* by Paul Lockhart (2012). Paul Lockhart is a mathematician who has also taught mathematics at a K–12 school. In 2002, he wrote an essay titled *A Mathematician's Lament* (available at **www.maa.org/external_archive/devlin/LockhartsLament.pdf**), critiquing mathematics education practices in the United States. The essay was circulated, Keith Devlin wrote about it in his monthly column, *Devlin's Angle*, for the Mathematics Association of America, and it was eventually expanded and published as a book (Lockhart, 2009). The essay and book suggest that students should encounter mathematics as mathematicians do: through creativity and discovery. I believe that *every* person who teaches mathematics should read this book or essay.

Both diagrams give us the opportunity to do some geometric reasoning. It is an interesting exercise to simply create the two diagrams. With the existence of dynamic geometry software, students can attempt to recreate the diagrams. Back in the Stone Age, when I was a student, we spent a fair amount of time doing constructions with compass and straight edge. I always enjoyed these exercises because they used geometric facts in a very specific way to create a particular figure.

In both diagrams, we are dealing with circles that are tangent to other objects, either line segments or other circles. Let's explore some easier, related problems before we attack the two main diagrams. To begin, try to construct, either with your favourite dynamic geometry software or compass and straight edge (i.e., ruler without using the scale), a square with an inscribed circle.

There are several ways that we can accomplish this task. We can do it by attaching a coordinate system. In Figure 1, points have been drawn at (0, 0), (6, 0), (6, 6), and (0, 6) and joined with segments to create a square with side length 6 units. The symmetry of the figures would suggest that the centre of the circle should be the centre of the square, (3, 3). Similarly, since the circle is tangent to each side, it suggests that it touches each side at its midpoint. So, for example, if points (3, 3) and (3, 6) were drawn, they could be used to draw the circle that has the desired properties.

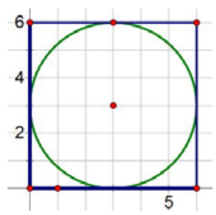


Figure 1: *A circle in a square, using a coordinate system*

Notice I used the words "draw," rather than "construct," in the last paragraph. That is because the circle we have drawn did not have the properties of an inscribed circle built into it. Since this is done within dynamic geometry software, I can move any of the points and, if I do, we will see that we no longer have a square, and the circle is no longer inscribed in the quadrilateral. Dynamic geometry software allows us to construct figures that maintain certain geometric properties, even if points are manipulated. This is what we want to do.

Let's start by examining some properties of circles. In Figure 2, we start by drawing a free segment *AB*. Using that segment tool, we will create two circles: one with centre *A* passing through *B*, the other with centre *B* passing through *A*. These circles intersect at two points, *P* and *Q*. Since the circles we created have the same radii, segments *AB*, *AP*, *AQ*, *BP*, and *BQ* all have the same length. Thus, Δ*ABP* and Δ*ABQ* are congruent equilateral triangles. We can also show that Δ*APQ* and Δ*BPQ* are congruent, which implies that *PQ* bisects angles ∠*APB* and ∠*AQB*. From there, we can show that Δ*APE* and Δ*BPE* are congruent, which implies that *PQ* is perpendicular to *AB* and that *E* is the midpoint of *AB*. This completes this part of the construction. If you move *A* or *B*, the rest of the construction will follow, and *PQ* will remain the perpendicular bisector of *AB*.

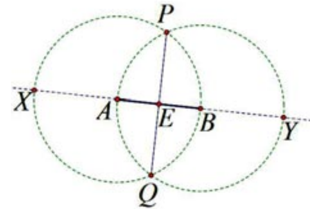


Figure 2: *The perpendicular bisector of a segment*

We can use the ideas from the first part of the construction to construct a square. We extend a line through *A* and *B*, and have the line intersect our two circles at *X* and *Y*, as shown in Figure 3. Segment *XB* is twice as long at *AB*, and *A* is its midpoint. Segment *MN* passes through *A* and is the perpendicular bisector of *XB*. If we label the point of intersection of *MN* and the circle centred at *A* as *D*, then *AD* is perpendicular to *AB* and the two segments have the same length.

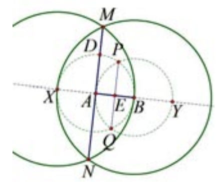


Figure 3: *Constructing a square*

If we go through the same process for segment *AY*, we can create a point *C* such that *ABCD* is a square. Now our square is constructed so that if any of the points are moved, *ABCD* retains the properties we used to construct it and, as such, remains a square. We can now hide the unwanted parts of the construction and keep just the parts that we want.

Dynamic geometry software offers us more options than we would have using compass and ruler. We can utilize the transformations menu to create a square by using rotations. We could also use the built in midpoint and perpendicular line constructions to save a few steps. Experiment by constructing objects, using different tools. Doing this will help form connections between various geometric concepts.

It would also be worthwhile to go through this process with a compass and straightedge. When doing this with a compass, it isn't necessary to draw the whole circles. Since we are only interested in their points of intersection, we can draw a couple of arcs in the approximate locations, to keep our construction neat.

Recall that our construction process has created the midpoint, *E*, of *AB*. If we draw the diagonals of *ABCD*, they will meet at *O*, the centre of the square. If we now construct a circle with centre *O* that passes through *E*, it will be our desired inscribed circle.

We can now almost create the first diagram shown at the beginning of this column. Start by constructing the midpoints of the sides of our square. Connecting opposite midpoints partitions the original square into four smaller congruent squares. If we construct inscribed circles in each of these squares, we have the four larger circles.

At this point, it is interesting to note a few measurements. If we have a circle of radius πR^2 inscribed in a square, the circle has area and the square has area $4R^2$ (see Figure 4 on the left). Thus, the ratio of the area of the circle to the square is π:4. If we have four congruent circles of radius r inscribed in a square, the total area of the circles is and the square has area $4\pi r^2$ (see Figure 4 in the middle). Thus, the ratio of the area of the circles to the square is again π:4. It turns out that if we take a number (must be a perfect square) of congruent circles and arrange them into a square, the ratio of the area of the circles to the area of the square is always π:4.

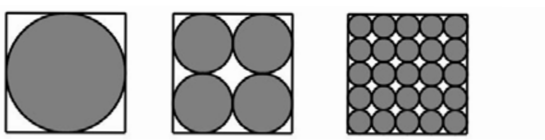


Figure 4: *Area of circles : Area of square =* π:4

To construct our fifth, smaller, circle, we need to know something about tangent circles. In a dynamic geometry software package, draw two circles and construct the segment joining their centres. Move the circles until they are tangent. You should notice that when two circles are tangent, the point of tangency lies on the segment joining their centres, as shown in Figure 5. When circles are *externally* tangent, as in Figure 5, their centres are separated by the sum of their radii (how far apart would their centres have to be for them to be *internally* tangent?).

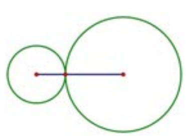


Figure 5: *Tangent circles*

Now we can complete the first diagram. If we draw the diagonals of the square, they will cross at the centre of the square, which is also the centre of the small circle. The diagonals will cross the four circles at the points of tangency of those circles with the little circle. We can now construct our small circle, since we have its centre and a point on the circumference (actually four points to choose from). Figure 6 shows the small circle with its centre and one of the points of tangency displayed.

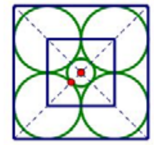


Figure 6: *The inner circle*

From Figure 6, we can also get an idea of the size of the small innermost circle. If s is the side length of the outer square, then the large circles have radius $\frac{s}{4}$, and the inner square has side $\frac{s}{2}$. Thus, the diagonal of the inner square is $\frac{s\sqrt{2}}{2}$ if we let r represent the radius of the little circle, then since the diameter of the little circle and two radii of the larger circles make up the diagonal of the small square, we get

$$2\left(\frac{s}{4}\right)+2r=\frac{s\sqrt{2}}{2},$$

which yields

$$r=\frac{s\left(\sqrt{2}-1\right)}{4},$$

Thus, the ratio of the area of all the circles to the area of the square is

$$\frac{\pi\left(7-2\sqrt{2}\right)}{16}:1,$$

or about 82 percent of the area of the square.

Proceeding to the second figure, we will start with a square, *ABCD* in Figure 7, and attempt to construct the circle. Notice that the circle is tangent to sides *AB* and *BC*. Thus, the centre of the circle must be equidistant from both of these sides. As such, the centre must be on diagonal *BD*. Diagonal *BD* is the bisector of ∠*ABC*.

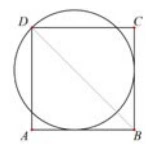


Figure 7: *The second problem*

Parabolas and Cones

A parabola is one of the so-called *conic sections*, curves that are formed by cutting a double cone with a plane in different ways. The conic sections was once part of the Ontario curriculum in the Grade 12 advanced course and the OAC Algebra and Geometry course, back when there were OAC courses (and earlier as well). To find out about conic sections, any Internet search will result in a multitude of websites, or check out some old textbooks from past curricula.

We will need another idea to complete our construction: the *locus* definition of a parabola as follows: A locus is a set of points that satisfy a particular condition. A parabola is defined to be the locus of points that are equidistant from a fixed point, called the *focus*, and a fixed line, called the *directrix*. For example, let $F(0,1)$ be the focus and let $y=-1$ be the directrix, as shown in Figure 8. Suppose $P(x,y)$ is a point of our locus. The distance from P to F is $\sqrt{x^2+(y-1)^2}$, while the distance from P to the directrix is $|y+1|$. Thus, the coordinates of P satisfy

$$\sqrt{x^2+(y-1)^2}=|y+1|,$$

which, after a little algebraic manipulation, yields

$$y=\frac{1}{4}x^2,$$

which describes a parabola with vertex (0, 0).

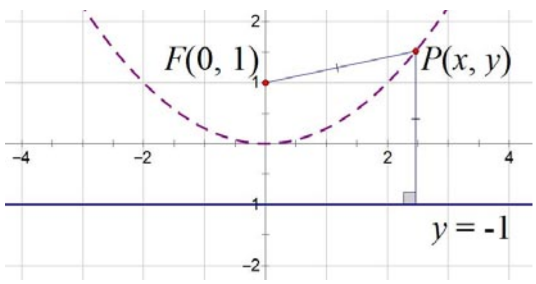


Figure 8: *The locus definition of a parabola*

In general, the locus of points equidistant from the point $F(0,p)$ and the line $y=-p$ is the parabola with equation $y=\frac{1}{4p}x^2$.

Looking back at our problem, the desired circle passes through vertex D. That means that the centre of the circle is equidistant from D and sides AB and BC. Thus, the centre of the circle is on diagonal BD; it is also on the parabola with focus D and directrix containing AB; and finally, it is also on the parabola with focus D and directrix containing BC.

Let the side length of the square be s and attach a coordinate system to our problem with A at the origin, as in Figure 9. The parabola with directrix containing AB then must have vertex $\left(0,\frac{s}{2}\right)$ and pass though C (why?).

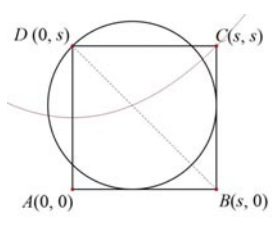


Figure 9: *Attaching a coordinate system*

We can then determine the equation of the parabola to be

$$y=\frac{1}{2s}x^2+\frac{s}{2}.$$

Combining this with the equation of the diagonal BD, $y=s-x$, we can determine the coordinates of the centre to be

$$\left(s\left(\sqrt{2}-1\right),s\left(2-\sqrt{2}\right)\right).$$

This tells us that the radius of the circle is $r=s\left(2-\sqrt{2}\right)$, from which we can get the ratio of the area of the circle to the area of the square to be $\pi\left(6-4\sqrt{2}\right):1$. In this case, the circle's area is almost 8 percent larger than that of the square.

The solution using the parabola was my first attempt. I knew that the construction starting with the circle was much easier, so that is why I looked at the one that started from the square, since it was more interesting. Even so, it was a little unsatisfying, since the use of a parabola is not a Euclidean compass and straightedge construction. When I sent my column to editor Tim Sibbald, he wondered if there was a “nicer” solution. So I went back, took my own advice, and thought about the problem some more.

In my further musings about our construction, I realized that since the centre, O, is equidistant from a vertex and two sides, it divides diagonal BD in the ratio $BE{:}ED=\sqrt{2}:1$. This is true, since BE is the diagonal of a square whose side length, through the property of our circle, is equal in length to ED (see Figure 10).

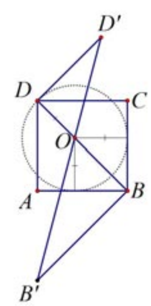


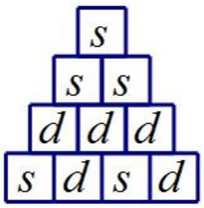
Figure 10: *A Euclidean solution*

Next we construct two points, B' and D' such that BB' and DD' are both perpendicular to diagonal BD; BB' is equal in length to diagonal BD; and DD' is equal in length to the side of the square. Segment $B'D'$ meets BD at the point O', and it is easily established that $\Delta BB'O'$ and $\Delta DD'O'$ are similar. As such, $BO':O'D=BB':DD'=\sqrt{2}:1$ which means that O' is our desired centre, O.

There are still lots of things we can explore with these two diagrams. In the first diagram, we can add more circles, which lead to the challenges of constructing these circles as well as trying to determine their area. We can also try to construct the diagram starting from one of the circles instead of the square (starting with the inner circle is more challenging). In the second diagram, we could start with the circle and construct the square (which is easier than what we did). Hopefully the diagrams will spark your creativity and you will come up with your own challenges in geometric reasoning.

Now it is time for your homework:

In a self-describing pyramid, a cell gets an "s" if the two cells below it are the same, and it gets a "d" if the two cells below it are different. The diagram below illustrates a self-describing pyramid with four levels.



How many possible ways are there to fill the four cells in the bottom row to produce an "s" at the top of the pyramid?

Until next time, happy problem solving!

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M	R	G	E	O	B	O	A	R	D	A
U	A	T	D	I	V	I	S	I	O	N
L	T	G	T	S	H	E	D	P	B	A
T	I	N	N	S	S	I	T	R	T	L
I	O	O	Y	I	A	A	E	I	U	O
P	D	I	A	G	T	E	P	S	S	G
L	I	T	R	T	R	U	L	M	E	U
E	A	A	R	A	S	A	D	H	O	E
D	M	T	A	C	U	T	E	E	L	C
O	E	O	E	Q	S	D	R	C	O	L
M	T	R	E	G	N	A	R	N	O	O
D	E	G	R	E	E	I	E	B	U	C
N	R	B	E	N	C	H	M	A	R	K

ALL FOUR SIDES ARE EQUILATERAL TRIANGLES:

" THE TETRAHEDRON "

▲ Linking Literacy and Math: Comprehension Strategies to Support Mathematical Understanding

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Dr. David Costello is a vice-principal who has previously held many roles. He taught in the primary division before assuming roles of numeracy interventionist, numeracy coach, numeracy leader, and curriculum consultant for Prince Edward Island. David has also instructed university courses focused on curriculum, differentiation, mathematics, and language arts. He facilitates professional learning sessions in the areas of mathematics and school development. His research situates mathematics within a literacy perspective and examines leveraging literacy-based instruction in mathematics.

A highly literate person is someone who can make meaning (Fountas & Pinnell, 2006; Keene & Zimmermann, 2007). When one is making meaning using in-the-head problem solving, it is called comprehension (Clay, 1991). For reading, in-the-head problem solving involves the cognitive actions applied by the reader as he or she reads the text. Therefore, reading is conceptualized as a cognitive experience, whereby the reader applies strategic actions.

While reading a text, the reader problem-solves by choosing which strategies he or she uses to make meaning (Clay, 1991). Being able to initiate or apply different strategies at different times enables the reader to exercise control over the strategies, thereby making him or her better able to read independently. While we cannot see the actions, we can look for evidence in the reader's behaviour as these actions are occurring in the reader's head. The teacher can observe the reader's behaviour to infer the strategies being applied (Clay, 1991; Pinnell & Fountas, 2007).

What is referred to as comprehension for reading is often referred to as conceptual understanding in mathematics (Small, 2008; Van de Walle, Lovin, Karp, & Bay-Williams, 2014). This thinking, a form of meaning making, can take the form of framing, supporting, and consolidating the processes involved in solving the problem. As the mathematician works through a problem, he or she problem-solves through the use of various cognitive actions. It is the development of a variety of strategies that strengthens the mathematician's ability to solve the problem. This is closely connected to reading: engage with a text, understand the context within which the text is written, and develop meaning as we read.

This suggests that leveraging comprehension strategies that students are already doing in their reading experiences to mathematics can be beneficial. Students are familiar with the comprehension strategies, what the strategies mean, and how to apply each strategy. It is not seen as something new or foreign, and the leveraging is a matter of having students apply a practice they already are engaged with, to mathematics content. Essentially, I am suggesting we consider comprehension strategies as thinking strategies instead of reading strategies. That broadening of scope facilitates applying the comprehension strategies to mathematics. Teachers do not need to teach conceptual understanding as a new direction or entirely new processes. Instead, teachers can leverage successful instructional practices from reading into mathematics.

Two examples of processes that students learn in reading are self-monitoring and paraphrasing. I will use these as examples of comprehension strategies that can be translated from the language arts classroom into the math classroom.

Self-Monitoring

Self-monitoring is a comprehension strategy through which the student monitors his or her comprehension while working through a problem. When working through a problem, a student who self-monitors can identify what part of the problem is causing his or her confusion. For example, a self-monitoring student can identify if the confusion stems from the problem as a whole, understanding only part of the problem, or being unsure of an approach to take when trying to solve the problem.

Being able to self-monitor implies that the student is continuously checking to see if his or her work makes sense. The student identifies ideas, concepts, and/or themes that do not make sense and then selects an appropriate strategy to address and solve these (Keene & Zimmermann, 2007). Self-monitoring is supported by a variety of "fix-it" strategies such as rereading, pausing, skipping ahead, and asking oneself what would make sense when encountering confusion with the task at hand. While in language arts, this task can be thought of as reading a text; in math, this task can be thought of as working/thinking through a problem. Too often students avoid self-monitoring by asking the teacher, "Is this right?" or "Can you help me with this problem?" If the teacher directly answers such questions, it is the teacher who is thinking.

When students read a text, we expect them to monitor their understanding—does this make sense? If it doesn't make sense, we expect them to employ self-monitoring strategies that address their confusion and work to make meaning. We can, and should, expect such self-monitoring for comprehension as students work through problems in mathematics. We want students to know when their approach and process choices make sense; when they do not make sense, the student needs to address what does not make sense and adopt strategies to repair the gap in comprehension.

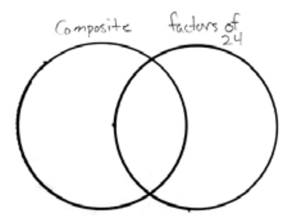
The following is an observation of an elementary-aged child applying the self-monitoring strategy when working through a problem. In this example, the teacher is sitting with Jonah (pseudonym) and asks him to think through the problem aloud. The teacher provides Jonah with a blank Venn diagram and asks him to sort the numbers from 1–24 into composite numbers and factors of 24.

Jonah: Okay… so I need to sort the numbers from 1 to 24 into either composite numbers or factors of 24. I know composite numbers are numbers which have more than just 1 times itself.

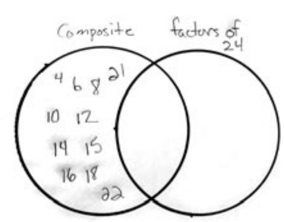
Jonah looks at the problem without beginning.

Jonah: Okay. So let's put this side (placing pencil above the left circle in the Venn diagram) as composite (Johan writes "Composite"), and this side (placing his pencil above the right circle in the Venn Diagram) as factors of 24.

Jonah pauses again after completing this step.



Jonah: Well, I know what composite numbers are, so I'll put those on the composite side. Jonah writes the following numbers in the left circle of the Venn diagram: 4, 6, 8, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24.



Jonah looks at the teacher and says:

Jonah: I don't remember what factors are.

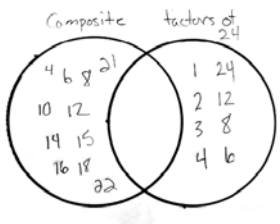
The teacher does not respond, as she is wanting to see how Jonah will tackle this stumbling block. Jonah looks back at his paper. He then says:

Jonah: I can look through my notes!

He pulls out his notebook and flips through a few pages, but is unable to locate an example of factors. He puts his notebook back in his desk and looks around the room. After another few seconds, Jonah takes his textbook out of his desk and flips to the glossary. He finds "factors" and reads the definition. He then turns to the section in the book and looks for examples.

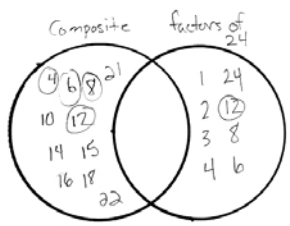
Jonah: Okay… so the factors of 24 are….

Jonah writes 1, 24, 2, 12, 3, 8, 4, and 6 into the right side of the Venn diagram. Jonah looks at the Venn diagram.



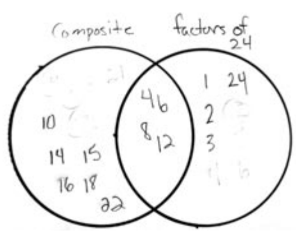
Jonah: This can't be right. I have 12 on both sides.

Jonah circles the 12 on both sides. He then circles 12, 4, 8, and 6 on the left-hand side.



Jonah: Oh yea… these go in the middle.

Jonah records these numbers in the centre of the Venn diagram, then erases the circled numbers on the left, as well as their matches on the right.



Jonah: Done!

In the above example of numbers including factors and composite, the teacher observes Jonah attempting various strategies to solve the task. The teacher witnessed Jonah encounter several stumbling blocks (how to approach the problem, forgetting the meaning of factor, and where to place numbers that fit both criteria of the Venn diagram), as well as employ strategies to overcome these stumbling blocks (looked for past tasks, used classroom resources, checked his work). In this situation, the teacher witnessed Jonah's flexibility in working through the problem, and can identify next steps for Jonah (placing unused numbers, 5, 7, 9, etc., outside the Venn diagram, and reviewing his work to ensure accuracy—9 was omitted from the composite numbers).

Paraphrasing

Often applied to reading, paraphrasing focuses on the reader's ability to digest the content of the text and articulate it, in his or her own words, to others. Being able to reword a text is a clear indicator that the text was understood by the reader. Paraphrasing entails the ability to review the content of the text, identify the highlights of the text, and then to rephrase this in one's own words. Multiple steps are involved, and it requires the reader to understand various aspects of the text to avoid losing the meaning when relaying the information to others.

For mathematics, paraphrasing can be applied by having students rephrase a problem into their own words. While this sounds direct, it presents its own challenges. To reword a problem implies comprehension of the problem context and the ability to communicate this context to others, while maintaining the various details involved. The essential question that paraphrasing addresses is: How can you understand a problem and be able to solve it if you cannot state it in your own words?

The following example demonstrates how children apply paraphrasing to support their understanding of the problem. The word problem below was given to a group of children (pseudonyms used):

> *There is a group of 62 children attending a 7-day summer camp. Of these children, 33 live in a city and the rest live in small communities. How many of the children are from small communities?*

The teacher reads the problem aloud to the class.

Teacher: Before we go right into solving this problem, I want you to read the problem again to yourselves.

Students are given a few minutes to silently read the problem.

Teacher: Can anyone tell me in their own words what this problem is about?

Amanda: It's about camping.

Teacher: Can you add to that?

Amanda: Yeah. It is about a group of children going to a summer camp.

Teacher: And, what is this problem asking you to do?

Amanda: Well, it is about 62 children going to a summer camp. Some are from a small community and 33 live in a city. You need to figure out how many are from the small community.

Charlie: What about 7?

Amanda: Well, 7 doesn't really matter because that is just the number of days, and that's not what the problem is asking you to figure out.

In this example, Amanda paraphrased the problem into her own words. Such paraphrasing enables the teacher to gauge the student's comprehension of the problem as the student works through the information. Initially, Amanda provided the setting of the problem, but did not indicate what was being asked. With prompting, Amanda was able to fully develop the word problem into her own words.

Final Thoughts

The two comprehension strategies provided in this column, self-monitoring and paraphrasing, can assist students in understanding a problem and working through it. Too many times it is the teacher explaining to the student what the problem is about and providing hints and supports when students are unsure of how to start or progress through the problem. Using self-monitoring and paraphrasing puts the onus on the student. These two comprehension strategies help students to engage with the task in a meaningful way and provide the tools necessary to work toward the solution independently.

In addition to the two comprehension strategies explored in this column, self-monitoring and paraphrasing, there are others where mathematics can take advantage of prior teaching in language arts. Making connections, synthesizing, and questioning all lend themselves to supporting student comprehension in mathematics. Such strategies will be explored in a future column. In the meantime, I look forward to continuing the discussion of linking literacy and math via Twitter (**@dr_costello**) or email.

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▲ Conjecture, Predict, and the Power of "What if?"

JEFF IRVINE

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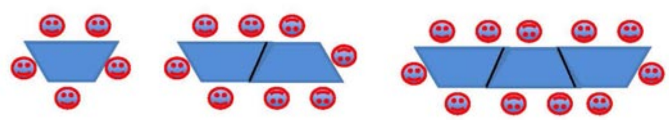
Jeff Irvine has been a secondary math teacher in Waterloo and Peel District School Boards, a secondary math department head, and a secondary vice-principal. Jeff has taught at three faculties of education and at Sheridan College. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. He is currently an instructor of mathematics education at Brock University, where he is pursuing a PhD.

Attending to the mathematical processes reinforces that mathematics is not only about procedures and definitions. When students are asked to extend and justify their thinking and reflect on alternatives, they are truly engaged in higher-order mathematical thought. Conjecturing, predicting, and posing "What if?" questions is an engaging way to push students' thinking to higher levels.

Most of us have seen the Trapezoidal Tables problem (typically used with Grades 7, 8, and 9):

Sally set up trapezoidal tables like this:



Sally says 39 people can sit around 12 tables that are pushed together like the pictures shown. Is she correct? Justify your choice. Show your thinking and your work.

This problem is a straightforward patterning problem that is accessible to most students in the intermediate grades. The real power lies in student conjectures and predictions in response to "What if?" questions posed by either the teacher or by students. For example:

What if…

- more students can sit on each side of the tables? (This changes the slope if the change affects the number of students on the long side and the intercept, if the number on the ends changes.)
- the tables were other geometric shapes (rectangles,

parallelograms, equilateral triangles, isosceles triangles, hexagons, circular, etc.)?

- rather than combining tables in a straight line, combining in other configurations (trapezoidal tables with the long sides pushed together, four tables arranged as a square with no one in the middle)?
- different configurations are combined with different geometric shapes?

I had some students go to the library on their own time to investigate, since the library had circular tables, leading to a discussion of how much space each student would need to be able to sit comfortably (sort of a "packing" problem).

The "What if?" questions lead to a richer discussion and promote higher-order thinking as shown below:

What If:						
Solve base problem	Extend	Conjecture	Justify	Predict	Verify	Explain

These "What if?" questions provide opportunities for students to think critically about a problem and to learn about how to justify or explain their thinking. EQAO defines *justify* as, "The response must give reasons, evidence, and/or calculations to show why an answer, argument, or conclusion is correct," and *explain* as, "The response must use mathematical symbols or words to make clear and understandable why a mathematical solution is correct." Posing and responding to "What if?" questions is an excellent way for students to develop these skills in an interesting way.

Some further activities I've used that lend themselves to "What if?" questioning include the following:

Going around in Circles: This is a standard activity that you may have used. (Grades 11–12)

Base Problem: A student walks around a hula hoop in front of a motion detector. Students conjecture the shape of the resulting distance–time graph (sinusoidal) and verify by experiment.

What if…

- the walker starts 5 m away from the motion detector rather than 2 m?
- the walker walks twice as fast?
- the walker starts at a different point on the hula hoop?
- the walker walks in the opposite direction?
- the hula hoop is bigger/smaller?
- you used a combination of several of the above modifications?
- the walker walks around an ellipse?

Penny Circles: This activity is best suited to using round discs, now that Canada has eliminated the penny. (Grades 7–8)

Base Problem: For circles with various diameters, determine the number of non-overlapping pennies required: (a) to go around the circle, and (b) to fill up the circle. Construct scatterplots and look for relationships. Justify your thinking, and explain your reasoning.

What if…

- you used dimes, quarters, loonies, toonies—or discs of various sizes, instead of pennies?
- you did this with a hula hoop?
- you used a combination of discs?

Vector Conjectures: This one is all about conjectures and proving. (MCV4U)

Formulate a conjecture about each of the following:

$$\vec{A}\times\vec{B}\times\vec{C}$$
$$\vec{A}\bullet\vec{B}\times\vec{C}$$

Some conjectures that students might come up with are as follows:

$\vec{A}\bullet\vec{B}\times\vec{C}$ is undefined, since $\vec{A}\bullet\vec{B}$ is a scalar, and a vector cross-product requires two vectors.

[If $\vec{A}\times\vec{B}\times\vec{C}=0$, then the three vectors are coplanar.]

A teacher can follow up with the following questions:

What if you insert a pair of brackets ( )? Does this change any of your conjectures? [Possible answers: $\left(\vec{A}\times\vec{B}\right)\times\vec{C}$ [no change], $\left(\vec{A}\bullet\vec{B}\right)\times\vec{C}$ is undefined, whereas $\vec{A}\bullet\left(\vec{B}\times\vec{C}\right)$ is a scalar.]

Demonstrate or refute each of your conjectures by choosing values for the components of the three vectors. Which conjectures does this process refute? Which conjectures does this process support? Does this process prove the truth of any of your conjectures? Prove each of your supported conjectures.

Slinky® Standing Waves: Your Science department may have Slinkys to lend. (Grades 11–12)

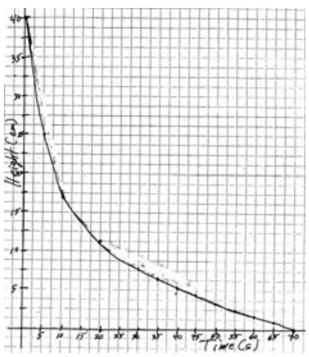
Base Problem: On a smooth floor, stretch a Slinky between two people and generate a *standing wave* (a wave that doesn't appear to be moving down the length of the Slinky). One person generates the wave and the other person doesn't move. Measure the amplitude and period, and write an equation for the standing wave. Generate standing waves with amplitudes one-half as large and twice as large as the first standing wave. How are the amplitude and period of standing waves related?

How would your results change if:

- the Slinky had a smaller/larger diameter?
- the length of the slinky were changed?
- both students holding the Slinky generate waves?

Leaking Cylinders: I got the base problem for this activity from Peter Taylor at Queen's University. He uses it in calculus to illustrate the difference between average and instantaneous rates of change. (MCV4U)

Base problem: A cylinder full of liquid has a hole at the bottom. Conjecture and sketch a graph of how the height of the liquid changes over time. (Note that we measure only the portion of the bottle that is a cylinder.) Now release the liquid at 10-second intervals, and graph the resulting data (height versus time). Describe how this compares with your conjecture, and reflect on your initial thinking. Connect the two intercepts with a line. What does the resulting line represent? (i.e., average rate of change). By sliding the line to the point of tangency on the graph, identify areas of the graph where the instantaneous rate of change is equal to/greater than/less than the average rate of change. [This is a fun activity for students that works best in groups of five. In addition to the usual roles of timekeeper, reader, and recorder, there is the catcher (who catches the leaking water in a pail or garbage can) and the "finger," a vital role where the student sticks his or her finger into the pole every 10 seconds, so that the reader can read the current height.]



What if…

- the cylinder has a different diameter/height?
- there are two identical holes at the bottom?
- there is one hole with twice the diameter of the original hole? Would this result be the same as the previous bullet? Why or why not?

Treat Time: A version of the base problem was first shared with me by Lionel LaCroix. Information on the box: total number of bars = 90; total mass of the treats = 924 g; 1 small Coffee Crisp® = 12 g; 1 small Kit Kat® = 12 g; 1 small Aero® = 7.5 g; 1 small Smarties® = 10 g. (Grade 7 through MCV as a real-life application of parametric equations).

Base problem: How many of each treat are in the box? Find as many different solutions as you can.

What if…

- there must be a certain number of one particular treat in the box?
- the masses of all the small treats were the same?
- the total number of treats were changed?
- the total mass of the treats was changed
- the ratios of masses for the small bars were changed; for example, Kit Kat : Aero : Smarties is 12 : 7.5 : 10; what if the ratio were 12 : 6 : 9?

Students who pose and investigate their own conjectures and "What if?" questions are engaging in reflecting, connecting, and reasoning and proving. For example, connecting previous "What ifs?" and current "What ifs?" leads naturally to reflecting on the key parameters of a problem. Justifying and explaining thinking follow from conjecturing and predicting. The power of "What if?" is increased active engagement and an emphasis on the mathematical processes as big ideas in mathematics education. ▲

▲ Assessment Abby
Dealing with Diagnostics

ASSESSMENT ABBY
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Ask Assessment Abby A³ is a regular column in the *OAME* Gazette, where teachers can share concerns and best practices about assessment, evaluation, and reporting of mathematics. Please send your questions to Ask Abby at **assessmentabby@oame.on.ca**.

Dear Assessment Abby,

I am a new teacher. I have a Grade 5 class. I have given my students a diagnostic Number Sense assessment, but many of them struggled. How do I figure out where to go from here? How do I get my students to the level they need to be at to effectively teach the Grade 5 curriculum?

Sincerely,
I.M. Stuck

Dear I.M. Stuck,

As Grade 5 teachers, we cannot focus solely on the Grade 5 mathematics curriculum, and we need to be familiar with what students have experienced in previous grades, as well as their prior developmental stages. An easy and recent starting point can be *Focusing on the Fundamentals of Math: A Teacher's Guide* (**www.edu.gov.on.ca/eng/teachers/teacher_guide_math_en.pdf**). In particular, there are useful charts on pages 5 through 7. For example, when working with numbers, you can see the continuum of expectations from Grades 1 through 6.

Using this, we can more accurately determine what level our students are working at right now. Moreover, it gives us a guide of what is to come next.

It is important to realize that students might not be ready for the Grade 5 expectations and in that case, it will be important to differentiate instruction to meet students where they are. For example, if a question involves fractions, we can look beyond students' responses as being right or wrong. Does the answer reflect an understanding that fractions represent equal parts (which would be a Grade 3 expectation), but the answer does not show an understanding of ordering fractions (which is a Grade 4 expectation)? Then we can "go back" to that concept in order to move students to order fractions as proper and improper fractions—the Grade 5 expectation. A Ministry resource that delineates the development of fractional thinking states:

Continued on page 44

CATEGORY*	GRADE 1	GRADE 2	GRADE 3
Working with Numbers	• Understand and use: ◆ whole numbers to 50 (i.e., 0, 1, 2, 3, … 50) ◆ anchors of 5 and 10 ◆ fractions, as follows: divide whole objects into equal-sized parts and identify the parts as unit fractions, e.g., $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, using various models, such as an area model, number line model, volume model, set model	• Understand and use: ◆ whole numbers to 100 (i.e., 0, 1, 2, 3, … 100) ◆ fractions, as follows: compose and decompose wholes using unit fractions, e.g., show that $\frac{9}{4}$ is the same as two wholes and one-fourth; compare and order unit fractions using various models	• Understand and use: ◆ whole numbers to 1000 (i.e., 0, 1, 2, 3, … 1000) ◆ fractions, as follows: divide whole objects and sets of objects into equal parts, and identify parts using fractional names

CATEGORY*	GRADE 4	GRADE 5	GRADE 6
Working with Numbers	• Understand and use: ◆ whole numbers to 10 000 (i.e., 0, 1, 2, 3, … 10 000) ◆ decimal numbers to tenths ◆ fractions, as follows: compare and order fractions with like numerators by considering the size and the number of fractional parts or by using benchmarks of 0, $\frac{1}{2}$ and 1; demonstrate and explain'the relationship between equivalent fractions, using concrete materials and drawings ◆ relationships between fractions and decimals to tenths	• Understand and use: ◆ whole numbers to 100 000 (i.e., 0, 1, 2, 3, … 100 000) ◆ decimal numbers to hundredths ◆ fractions, as follows: compare and order fractions with like denominators, including proper and improper fractions and mixed numbers; demonstrate and explain the concept of equivalent fractions, using concrete materials ◆ relationships between fractions and their equivalent decimal forms	• Understand and use: ◆ whole numbers to 1 000 000 (i.e., 0, 1, 2, 3, … 1 000 000) ◆ decimal numbers to thousandths ◆ fractions, as follows: compare and order fractions with unlike denominators, including proper and improper fractions and mixed numbers ◆ relationships among fractions, decimals, and percents ◆ composite and prime numbers, and the relationship between them

▲ E-Brock Bugs© Takes Flight

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Eric Muller is Professor Emeritus at Brock University and a Fellow of the Fields Institute for Research in Mathematical Sciences. During his many years at Brock, he was actively involved with OAME and was a founding member of the Golden Section. In 2005, he was honoured to receive the OAME Life Membership Award. It is very rewarding for him to see that E-Brock Bugs is being as well received in Ontario mathematics classrooms as was the original Brock Bugs board game.

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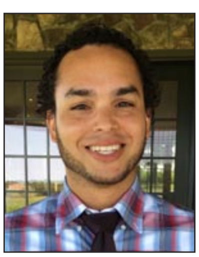
Michael Chan has worked as an Electrical Engineer after graduation. In 1990, he started his teaching career with the Toronto District School Board. He taught Mathematics and Technological Studies for over 26 years. He is a firm supporter of STEM. He retired from formal classroom teaching in 2017, but is still actively engaged in online teaching and developing projects/apps for the classroom.



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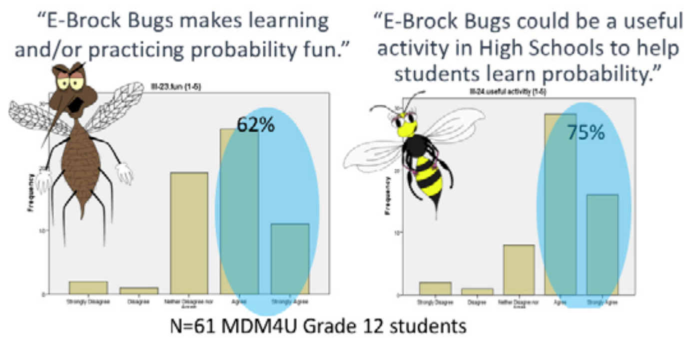
Timothy Calford has been teaching high school math since graduating from Brock's Continuing Education program in 2008. He innovates to help students connect with math concepts. His 5-to-9 passion is coding in ***VB.NET****. (Check* ***calfordmath.ca*** *for some recent creations.) When not teaching or coding, he enjoys time with his amazing wife and two sons.*

"The moment you doubt whether you can fly, you cease forever to be able to do it."
~ J.M. Barrie in Peter Pan

Of 61 students in three Grade 12 MDM4U classes, 62 percent of them agreed that the free, online E-Brock Bugs game (**www.brocku.ca/mathematics/brock-bugs**) made learning and/or practising of probability fun (yes, "fun"!), and 75 percent concurred that the game could be a useful activity in high schools to help students learn probability. What do you think? What is this all about?



Preparation for lift-off

"What if becoming a (better) mathematician were a fun-filled adventure?" was the title of our 2015 OAME *Gazette* article (Broley, Buteau, & Muller, 2015), describing the E-Brock Bugs game. Would players of the game in a mathematics class find it so? Would they judge that they have learned some new mathematics, or found understanding in the mathematics covered in class and, if so, was it fun?

We recall that in E-Brock Bugs, a player moves through six different districts, each ruled by a bully, and then to a Finale, where the mastermind, Dr. P, resides. The overall goal is to save Bug City by defeating each of the six bullies, as well as Dr. P, at their individual game. There are two ways of defeating a bully, either: 1) by winning six rounds at the probabilistic game (a mathematical strategy leads to a higher probability of winning, and as such, leads to quicker progress in the game), or 2) through a shortcut by correctly answering a related probability challenge question (available after having played against the bully). This is costly if the challenge question is answered incorrectly.

I, Timothy, had the opportunity to use E-Brock Bugs with my small MDM4U class of ten students prior to teaching them probability or distributions formally. I took them to a computer lab, and had them play the game individually. I noted that my students' time-to-task was very quick, and the game was intuitive to navigate. Despite working solo, they were quite conversational with each other about the strategies they employed to defeat the bullies for various districts. They joked with one student, who kept losing due to guessing, pointing out that winning was not random, but due to strategy. Their pace was perhaps slower than it would have been if I'd used the game as a review at the end of the unit, but it was informative to observe the authentic connections and understanding formed about theoretical probability, and discrete distributions. I encouraged them to use a spreadsheet (Google Sheets, or Microsoft Excel were their preferences) to efficiently tabulate the possible sums for two dice, etc.

My students found District 4 to be a challenge, with the sum of two non-uniform spinners. The explanation provided in the game was excellent, as it showed them how to make a sum table with repeated numbers, corresponding to their larger share of the circle on the spinner. There was a healthy dose of probability vocabulary embedded in the game as well ("theoretical relative frequency," "probability distribution," etc.), which I wouldn't expect a student to absorb upon first read, but early exposure provided a good foundation for reinforcement, as these terms were used in subsequent classes. There is a nice balance of familiarity in the games at each District, with slight twists that push students to extend their reasoning. I liked the multiple path options with which students are presented, giving them the chance to learn the theoretical explanations only if they want to look "under the hood." I found students chose a mix of game learning (trial and error with their strategies) and explicit instruction, so it was nice to have both options. I will use E-Brock Bugs again with an MDM4U class! I think I will create some worksheets to accompany their exploration, where they can communicate their strategies and record their reflections. This would give me more insight into their learning, and encourage them to formalize their mathematical approach to each bully. (I wouldn't want to take away from the natural "game-ified" learning experience, though perhaps I'd let them work through a District or two.)

I, Michael, have used E-Brock Bugs in my MDM4U classes for a review of probability concepts I covered in class. My students engaged in class and at home in different probability activities, which included playing E-Brock Bugs. I then scheduled a 75-minute class session, in which my students played E-Brock Bugs individually or in pairs and were graded on their game progress (my classroom included computers). One week later, students were required to submit a report about the mathematical theory that included: (a) an outline of probability theories involved, and strategies to play the E-Brock Bugs game, (b) a detailed analysis of an E-Brock Bugs District (randomly assigned to each individual or pair of students) with a solution for the assigned District Challenge Question, and (c) an anatomy of the Finale of the game. In short, all students were expected to play the whole E-Brock Bugs computer game and to engage in all of its probability concepts. Using the game as part of the teaching tools in the MDM4U classes for over two years, I found students favourably benefited from the game overall in areas of learning motivation and theory applications. At the high-level end, groups were excited in the working session, equipped with well-prepared charts and notes, while at the low-level end, groups would be resorting to moving up Districts by sheer trial-and-error strategy and learned as they went. Regardless of their levels, at the end of the session, a majority of students observed demonstrated a certain degree of pride of achievement.

One may ask, "To what extent do students actually engage in mathematics when playing E-Brock Bugs, and do they come to identify with their selected mathematically capable avatar?" We invite you to climb on board and explore how Chantal and Eric have examined these questions through a research study in Michael's classes.

Departure

In 2014–2015, students in Michael's three MDM4U classes voluntarily participated in the study, which was structured so as not to interfere with the way he taught his classes, while also ensuring he would not know whether a student was participating in the study. Students completed a questionnaire, which is reproduced in the Appendix of a scientific article (including a detailed discussion of the research), written by Buteau and Muller (2018).

We were pleased to have a participation rate of 85 percent, with 61 students (59 percent male and 41 percent female) completing the questionnaire. Of these, 50 percent played the entire game for the first time with a partner, 25 percent of them played the game alone, while the others played part of the game alone and other part with a partner. Of the participants, 67 percent played E-Brock Bugs without a break, and only 7 percent used more than two breaks. Participants reported playing slightly more than an average of 1 hour to save Bug City.

In-flight activities: Students' math engagement when playing E-Brock Bugs

To get some measure of students' math engagement, we used their responses to three questions. The first one focused on whether a player had an initial strategy to defeat a particular bully and Dr. P and, if so, whether it was built on mathematics they had learned in class.

The resulting data plotted in Figure 1 indicate that about 67 percent, of the participants reported having an initial strategy in each District, of which the first three Districts were built on mathematics learned previously. There is a clear distinction from District 4 onwards, where the probability distributions involve a randomly generated parameter. For these Districts, a significantly smaller proportion of respondents indicated that their strategy was based on mathematics. It is not surprising that even fewer indicated that they had an initial strategy for the Finale, as the game no longer concerns the direct computation of probabilities, but rather the number of trials of binomial experiments that meet a given probability. However, Michael had covered in class all of the probability concepts built into E-Brock Bugs. This may suggest that students had difficulty (or lacked motivation) in classifying the probabilistic experiments from District 4 onwards.

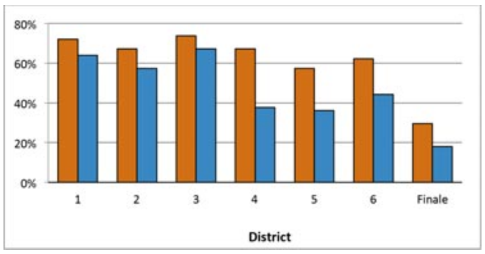


Figure 1: *Percentage of participants per District who had an initial strategy when they started playing against the bully (red), and of those whose strategy was built on some mathematics they had learned in their MDM4U class (blue)*

The second question we used to develop a measure of the students' math engagement focused on a player's use of the in-game mathematician mentor (Smarty) to help develop new strategies, and, if Smarty was used, whether they reported her to be helpful in developing a new strategy. Figure 2 provides a summary by District of the percentage of players who changed strategy in order to beat the bully, and of those who visited Smarty and found her help useful. A player who figures out a probability-based optimal strategy from the first encounter with a bully would likely not change strategy, and, as such, this indicator of mathematical engagement is also only partial. Nevertheless, we note that of those players who changed strategy when defeating a bully, a great proportion (from 59 percent to 92 percent) visited Smarty. Importantly, except for the Finale, which is of much higher mathematical difficulty, close to 100 percent of those who visited Smarty found her help useful, which could suggest that they then engaged in the probability concepts.

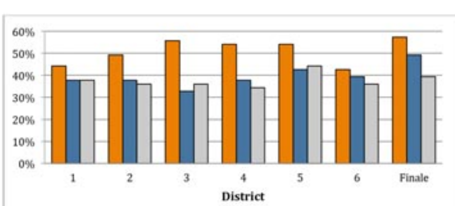


Figure 2: *Percentages of participants in each District who changed their game strategy (red), and if so, who visited Smarty (blue), and if so, if Smarty helped to develop the new strategy (grey)*

We note that the data in the bar graph shows that some participants responded having changed their strategy as a result of talking to Smarty, yet also indicated not having talked to Smarty. The bar graph faithfully reports the participants' responses. We interpret this as possibly slightly fewer participants having changed their strategy as a result of talking to Smarty. When we categorized participants who visited Smarty in at least one District or the Finale according to their mathematics achievement level, we found that over 80 percent of lower achievers (mathematics grades 60–69%) found her helpful, while 60 percent of the higher achievers (90–100%) found her helpful in developing a strategy in the Finale. We thus deduce that E-Brock Bugs might have mathematically challenged even the higher achievers. Overall, this suggests that players who decided to consult with the in-game mathematics mentor perceived her role as useful for their game progress in saving Bug City. In short, this could indicate that many players did in fact engage in mathematics as they played E-Brock Bugs.

The third indicator we used to measure participants' mathematical game play was the paths they chose to progress through each of the Districts, that is, whether they defeated the bully of a District by winning the probabilistic game at least six times (possibly without explicitly using probability), or by correctly answering the probability challenge question. Figure 3 summarizes how E-Brock Bugs players defeated the bullies per District. Because there is only one possibility to defeat Dr. P in the Finale, it is not recorded in this graph. A majority of players (48 percent to 79 percent) beat the bullies through the challenge questions; however, this proportion is smaller from District 4 onwards, which are more mathematically demanding.

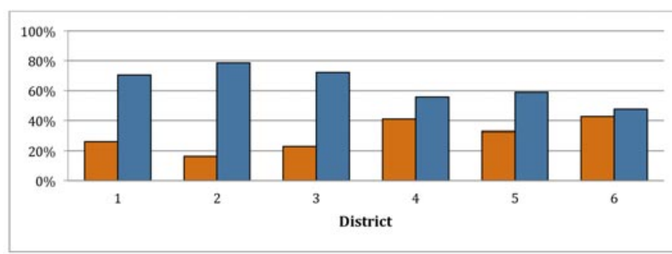


Figure 3: *Percentages of participants in each District who defeated the bully by either winning six times (red) the probabilistic game or through the challenge question (blue)*

Figure 4 summarizes Figure 3's data from individual participants' overall game play point of view in terms of the percentage of Districts they defeated by answering the challenge questions. A small proportion (16 percent) of players did not defeat the bullies (in none or only one of the Districts), using the challenge question, which may suggest their lack of explicit mathematical engagement in the tasks.

However, one higher-achieving participant (mathematics grade > 90%) indicated that she defeated all bullies by strictly playing the probabilistic games. This highlights the limitation of strictly using this measure (number of defeated Districts by the challenge question) for a player's mathematical engagement in E-Brock Bugs. Nevertheless, a high score for this measure conversely could strongly suggest a mathematical engagement during the game play. Of importance, 50 percent of the players progressed through answering the challenge question in five or six Districts. Not surprisingly, those who progressed with challenge questions for fewer than four Districts did so for the (mathematically) easier Districts, namely Districts 1, 2, and 3.

Figure 4: *Participants' percentage of Districts defeated by answering the challenge questions*

Overall, the results presented suggest that most players did in fact play E-Brock Bugs mathematically.

Preparation for landing: Student's in-game mathematical identity

Devlin (2011) reminds us that:

> Mathematics education, when it is successful, is only partially about people learning how to *do* mathematics; it is also helping them adopt a particular identity—that of *being* at least a mathematically able person, and possibly even a mathematician (p. 125).

Devlin argues that in a good mathematical game, "a player is not being asked to 'learn how to do X.' Rather, it is all about 'becoming a (better) X-er'" (p. 126). Players "should come to adopt the identity of 'being a mathematically able person' (i.e., being someone who can think mathematically) not only in the game but subsequently in the real world" (p. 127). In E-Brock Bugs, we attempted to make it explicit to players that they are doing (in other words, engaging in) mathematics. We hoped that when succeeding in the game, the player would identify as "being a mathematically able person." In E-Brock Bugs, the learning of mathematics or mathematical thinking is supported by an exploratory approach, which as Devlin maintains, "leads to ownership of what is learned. We appreciate, value, take pride in what we discovered for ourselves" (p. 79).

To build a quantitative perspective of the students' in-game mathematical identity, we developed a measure from their Likert scale (1 to 5) responses to the following statements:

A. I succeeded in E-Brock Bugs much more quickly if I used mathematics.
B. Beating a bully in E-Brock Bugs was challenging at times, but never impossible.
C. I was 'good', or becoming good, at knowing how to efficiently beat the bullies.
D. When playing E-Brock Bugs, I felt as though I understood the math.
E. I discovered (or learned) some mathematics when playing E-Brock Bugs.
F. I am proud of the math I believe I discovered (or learned or applied) when playing E-Brock Bugs.

To measure a student's perception of his or her mathematical in-game identity when playing E-Brock Bugs, we used the weighted measure

Math in-game identity: (A + ½ B + C + D + E + F)/5.5

We view the response to question B as only indirectly contributing to the student's math in-game identity and have therefore weighted it by a factor of ½. The percentage distribution of the math in-game identity data is shown in Figure 5.

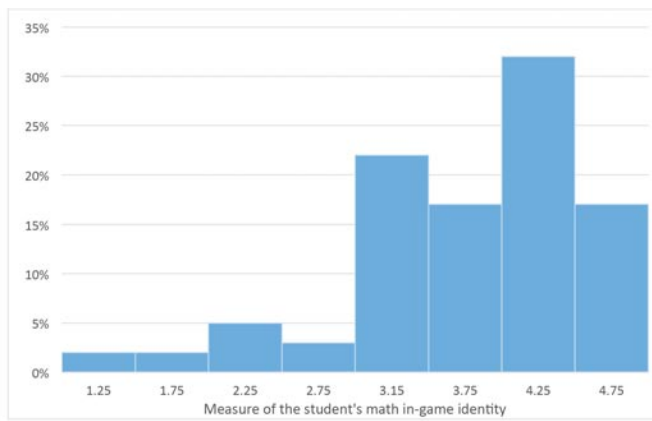


Figure 5: *Percentage distribution of E-Brock Bugs players' perception of their math in-game identity*

The data is positively skewed with a first quartile of 3.27, a median of 3.91, and third quartile of 4.36. This suggests to us that the majority of the students developed a mathematics in-game identity. Using J.M. Barrie's words, these students probably didn't doubt they could fly in Bug City, and indeed felt they brought together their mathematics knowledge to save it from bullies.

Landing… What about the next flight?

We would be delighted to communicate with teachers who are already using, or who are planning to use, E-Brock Bugs (**www.brocku.ca/mathematics/brock-bugs**) in their mathematics classroom. An online teacher document providing a summary of the mathematics in the game is available at **https://brocku.ca/webfm_send/32032**. In Timothy's class, we saw that students played the game to explore concepts of probability before they were taught, whereas in Michael's class, E-Brock Bugs was used to revise probability concepts covered in class. How would E-Brock Bugs fit best into your teaching of probability? Could some of the program be used in earlier classes? We are very much aware that this is an amateur game; however, it has been developed on principles that are proposed by a number of recognized mathematics educators and video-game designers. Experiences by teachers using E-Brock Bugs in their classes could provide useful information for those who develop and implement Serious Educational Games (Offenholley, 2011) in mathematics. In our study, 79 percent of the respondents reported that they like to play video games. After all, wouldn't we want to exploit this extensively used environment by our students to have them learn and engage in mathematics?

The study reported in this article was approved by the Brock University Research Ethics Board (#REB-14-072) and by the Research Ethics Board at the Toronto District School Board (#2014-2015-37E]).

The authors thank the Editor and the Reviewers for their constructive comments.

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▲ Cognitive Dissonance: Learning from Happy Accidents

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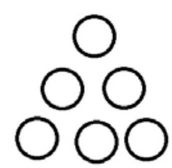
Jeff Irvine has been a secondary math teacher in Waterloo and Peel District School Boards, a secondary math department head, and a secondary vice-principal. Jeff has taught at three faculties of education and at Sheridan College. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. He is currently an instructor of mathematics education at Brock University, where he is pursuing a PhD.

Cognitive dissonance occurs when a situation or result clashes with students' already-formed beliefs. When something occurs that doesn't fit students' preconceived notions, they encounter psychological discomfort, which results in them questioning their beliefs. This type of situation is an opportunity for students to learn through what I call "happy accidents." For example, rather than telling students information, such as "division by zero is undefined," by allowing students to "discover" this for themselves, two things happen: (1) students are more likely to remember the information; and (2) students are more likely to understand what happened and why. In this article, you will find several examples of situations that cause cognitive dissonance, by students correctly performing a mathematical solution, but getting an unexpected result. Inserting situations that cause cognitive dissonance into our math courses is a productive method of advancing student thinking while encouraging social interactions and discussions.

Here is an example that I have used with students from Grades 3 through 9. I first got this problem from Don Fraser. The result clashes with our students' beliefs that in mathematics, a question can have only one correct answer or solution.

Using all the numbers 1, 2, 3, 4, 5, 6, fill in the circles so that all three sides of the triangle have the same sum.



This is a problem that every student can understand and be successful with, using trial and error. The "happy accident" occurs during whole-class sharing, when a student will confidently say, "I solved the problem. The total of each side is 9," only to have another student say, "I solved the problem. The total of each side is 10." Or 11, or 12. This is often the first time that a student is confronted by a problem with multiple valid solutions. By clashing with the usual situation in math, where problems only have one valid answer, the students are forced to reconsider their preconceived notions. When I use this problem in class, we then break into jigsaw groups, with each group being asked to find multiple solutions to the problem, using six different consecutive numbers. For example, one group might be given the numbers 17, 18, 19, 20, 21, 22. Most groups continue to use trial and error, but some groups will use one-to-one mapping to locate the positions of the numbers for the four different solutions, mapping 17 onto 1, 18 onto 2, and so on. Groups will sometimes also engage in "What if?" questions, such as: "What if each smaller triangle of three circles had to have the same sum as the sides?" or "What if there were four circles on each side of the triangle using numbers 1 through 10?" Will there be multiple solutions in these cases as well?

Here are some other examples where I have used cognitive dissonance to productively advance student thinking. Consider for yourself how these situations result in cognitive dissonance, forcing students to think critically about what was previously considered routine procedures.

What happened with the slope and why?

The teacher told her class, "Choose two points. Then, find the slope of the line segment between the two points."

Maria chose (3, 5) and (7, 5).

Ken chose (8, -2) and (8, 6).

Siobhan chose (3, -5) and (7, 5).

Each of them found the slope of the line segment between their two points. No one made any mechanical errors. But one of the students was confused by the results and asked the teacher for help.

Who asked for help? What happened? Why did it happen? Explain to a partner what happened and why.

This following two examples address students' preconceived idea that if a mathematical procedure is carried out correctly, there will be no surprises and nothing unexpected will occur.

What happened with the equation and why?

The teacher told her class, "Make up an equation and then write a full solution."

Tom's equation was $5x + 4 = 2(x – 1) + 3x$.

Cindy's equation was $3x + 4 = 2(x – 1) + 3x$.

Frahad's equation was $5x + 4 = 2(x + 2) + 3x$.

Each of them solved their equation. No one made any mechanical errors. But two of the students were confused by their results and asked the teacher for help.

Which two students asked for help? What happened? Why did it happen? Explain to a partner what happened and why.

What happened with intersections and why?

Students were given systems of equations to solve, using the method of their choice.

Carly's system was $2x + 3y = 6$ and $6x + 9y = 8$.

Tuvah's system was $3x – 2y = -2$ and $-9x + 6y = 6$.

Sandeep's system was $4x – 3y = 17$ and $2x + 6y = -14$.

Each of them solved their equations. No one made any mechanical errors. But two of the students were confused by their results.

Which two students were confused? What happened? Why did it happen? Explain to a partner what happened and why.

I use this example to introduce dependent and inconsistent systems of equations. Students will often assume that if something unexpected happens, an error has been made. However, once the "What happened and why?" became a regular feature in each of my classrooms, students began to critically reflect on almost everything we did, looking for exceptions or contradictions.

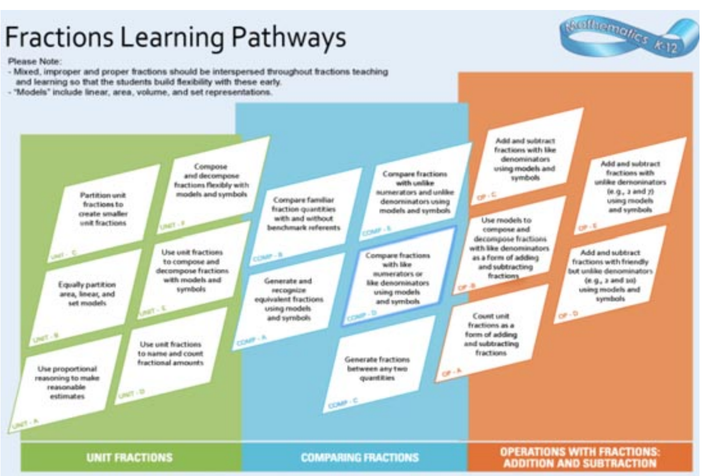
By providing situations in which students must confront their preconceived beliefs, resolve the conflict, and modify their previous position, students are given a chance to construct their own knowledge, and justify their position to others. If I tell students, "The slope of a vertical line is undefined," they will all believe it, and some may remember it. But if students are allowed to "discover" this fact for themselves, not only will the memory endure longer, the memory will be accompanied by understanding of why this fact is true.

Can you think of other situations in our mathematics curriculum that provide opportunities for us to use cognitive dissonance to advance student thinking? You might consider topics such as: restrictions related to functions involving square roots; restrictions on rational expressions; linear versus non-linear relations; asymptotes of graphs; the inverse trigonometric relationships (e.g., sine with cosecant); permutations with repeated elements; the relationship between permutations and combinations; limits in calculus; the relationship between tangents and secants; the relationship between functions and their inverses; angles inside circles. I encourage you to find other instances to cause cognitive dissonance for students. All that is required is that something unexpected happens. Formatting content in an activity such as "What happened and why?" is quite straightforward, but pays big dividends in advancing student thinking and presenting content in an interesting way. ▲

Continued from page 37

"The purpose of this interactive planning tool for teaching fractions is to provide educators with a research informed framework. It includes a range of field-tested tasks (Grades 3–10) that have proven to be effective in Ontario schools. The collection of tasks follows a logical sequence that can be modified and/or adjusted to fit teacher and student needs. Video and photos are also included to bring the learning to life. This interactive planning tool also includes one-page summaries of key fractions math ideas, as well as anticipation guides that feature Ontario students' thinking."

(**www.edugains.ca/newsite/DigitalPapers/FractionsLearningPathway/index.html**)



Our job as Grade 5 teachers is not to teach *just* the Grade 5 math curriculum—teaching students is our priority and we do that by moving them along the curriculum continuum. Remember, some students will have a short move, and for others, it will be a longer journey!

Keep Math Rich with Students in Mind,
Assessment Abby ▲

▲ Interview with Richard Hoshino

ANN ARDEN
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Ann Arden is a math teacher in the Ottawa-Carleton District School Board and is currently an instructional coach. She has also taught in the Faculty of Education at the University of Ottawa, and her son's teacher this year was a former student! Ann is very interested in assessment to improve learning, especially through the use of conversational and observational evidence. Ann is on Twitter as @annarden.

Ann Arden interviewed Richard Hoshino, a math professor at Quest University Canada in Squamish, British Columbia, on December 16, 2018 via Google Hangouts. Richard is the recipient of the 2017 Canadian Mathematical Society's Adrien Pouliot Award, honouring him for the impact his work has had within the Canadian mathematical community, and for his involvement with several important outreach activities in Canada. Richard's teaching is driven by the following question: "In what ways can a student experience mathematics to develop the confidence, critical thinking, and communication skills so important in life?"

Thank you for taking the time for this chat today! Can you describe a bit of your school experience, thinking back to your K–8 and high school days? What were your experiences with mathematics as a learner?

I'll share two experiences—the most profound in my K–8 years. I am an immigrant child with two Japanese parents, who came to Canada speaking very little English. They believed strongly, as many parents do, that education is the key to future success. Math didn't have the language barrier common to other subjects in school, so I found myself gravitating toward the subject as a young boy. My mother enrolled my brother and me in an after-school program called Kumon, and it was through this program that I developed self-esteem and self-efficacy. Math enabled me to stand out as a student. I was very behind in English class, especially as I didn't speak the language at home. Math gave me a sense of self-esteem and self-confidence that allowed me to own the subject as something that I could do—and do well.

The second experience was meeting Jean Collins, my Grade 7 math teacher. Mrs. Collins taught math to our Grade 7 class through puzzles and inquiry-driven problem solving. She went far beyond my belief that mathematics was only about calculations and memorizing formulas. I still keep in touch with Jean, and I look forward to seeing her the next time I'm back in Toronto.

What was it about Mrs. Collins in Grade 7 that inspired you?

This was the Fall of 1990. I still remember it very clearly; I was 12. In the very first class, she taught us perfect numbers, abundant numbers, and deficient numbers. The number 28 is a *perfect number* because all of its proper divisors (1, 2, 4, 7, 14) sum to 28. The number 15 is *deficient* because its proper divisors (1, 3, 5) sum to a number less than 15. The number 12 is abundant because its proper divisors (1, 2, 3, 4, and 6) sum to a number greater than 12.

We started with an unsolved problem in math. This was extremely risky for a teacher to do this with a bunch of 12-year-olds! The question was: *Does there exist an odd perfect number?* This question is still unanswered.

It was really exciting as a 12-year-old to be able to play around with this problem. At the time, I went to a school that had students from Grades 7–12. One Grade 12 student, named Ian Goldberg, had just returned from the International Math Olympiad and he had done very, very well. I had never heard of this competition, but it was an experience that I wanted for myself. And it was awesome, as a 12-year-old, to work on a problem that no one in the history of the world had solved previously. I didn't get it, but I was able to show that 945 was an odd abundant number, which was a new result to all of us, including Ian. Later, I remember being mesmerized as Ian showed us proof as to why 945 was in fact the smallest odd abundant number, and I recall thinking that I would never be able to do math at his level, that I would never come anywhere close to being a Math Olympian.

At first, I wasn't particularly strong at problem solving, and I finished 24th in my own school in the very first math contest I wrote in Grade 7. But five years later, I was on the Olympiad team, as one of the top six high school students in Canada. I'm grateful for the spark that Mrs. Collins gave me through really interesting problems, and I gained the skills of problem solving through practising old math contests. It was through this persistent desire to achieve mastery, combined with my desire to represent Canada at the Math Olympiad, that my high school years were filled with such focus, clarity, and purpose.

That is a wonderful story! And what a wonderful problem!

I am so grateful. Years later I wrote my book, The Math Olympian. The main character is a 17-year-old, who decides, as a 12-year-old, that she is going to go for the Math Olympiad. I found it quite easy to write that story because so much of her fictional story was my real story.

I know you use this text with your students at Quest University. How do you find your students connect with that life experience of pursuing a dream and encountering the issues that your protagonist encounters?

For the students I teach, none of whom have done math contests and none of whom are math majors, they can't relate to my story of pursuing the Math Olympiad, but they can all relate to something. One of them was an aspiring Olympian in snowboarding. When I met Darren in 2014, he had just missed the Olympic team by one spot and it was heartbreaking for him. He could relate so much to my protagonist's perseverance of working so hard to achieve a goal that was not guaranteed, regardless of one's effort. We connected and kept in touch after the course, and I was so happy that he made the 2018 Olympics! One of my happiest moments of 2018 was watching the Olympics and seeing his smile at the start line in Korea. It really meant a lot that a book that tries to share the message of "as you find your own voice, you inspire others to find their voice" has resonated deeply with my students, and that's been a privilege.

Can you explain a bit more about your problem-solving course at Quest University? The idea of a problem-solving course for students who are not choosing to major in engineering or mathematics is absolutely wonderful. What kind of spirit do you find first-year students have about math? When you encounter high school graduates, what's your sense of students' experience with math in your course, and what are your goals for the course?

Students have a choice of foundation math courses in first year. Quest intentionally doesn't have calculus as the first math course. They can take my problem-solving course, or a course in spherical trigonometry offered by my colleague, or a whole range of interesting courses that expose students to the diverse ways that mathematicians do mathematics.

Many students come to Quest absolutely terrified of mathematics, and are certain the subject has no relevance in their lives. Unfortunately, many students arrive thinking that math is all about reproducing the formulas and rules of seventeenth-century European men. And unfortunately, that is how the subject has been portrayed by so many well-meaning teachers—that all of the math that is "out there" has already been created, and it's the job of the student to reproduce existing knowledge, rather than have the opportunity to create his or her own. In my course, I give students the opportunity to create their own math and tie it to issues they care about.

The first problem we do is play a two-player game that Ed Barbeau from the University of Toronto showed me years ago. Students take turns picking an integer from 1 to 9. The first person who chooses three numbers that add up to 15 wins the game. It turns out this game is identical to tic-tac-toe and relates to magic squares. In the first hour of our course, the students acquire a key strategy for solving hard problems: to first convert them into equivalent simpler problems. We use that strategy as a launching pad to share real-life problems I have solved, as well as the questions the students themselves are passionate about pursuing.

I know you have published a paper with two undergraduate students on employee scheduling. The scheduling problem is an adapted version of the nurse scheduling problem and is really interesting. Can you talk about how these students have ended up with a mathematics publication and really engaged in real-life math?

One of the authors of that paper has a sister who runs a very popular restaurant in Victoria, British Columbia. It's a trendy restaurant, where she was the front-of-house manager for many years, managing 30 bartenders, hosts, and servers. Every week, she was creating the schedule by hand and it caused so much frustration at her end, as well as frustration and anger with her employees, who felt they weren't getting the shifts they wanted. So my student wanted to help her by creating an automated weekly shift schedule at this popular restaurant, based on techniques he learned in my Discrete Math and Linear Algebra courses. That's where many of the projects begin—with a student wanting to make a difference in a tangible way, based on an issue or question he or she cares about. This student is now in graduate school, and intends on completing a Ph.D. in mathematics. It's so exciting to see students' lives get changed because they had an opportunity to engage in mathematics personally, to create their own math, and use it to solve a real-life problem and implement its solution. I can share with you stories of 20 other undergraduate students who have done this, including seven who have published a paper with me. I think this is a really big deal—for an undergraduate student to actually use math to help someone. This is not an experience I had until my late 20's and I'm trying to give my undergrads that experience now, with the hope of inspiring the love of the subject and desire to keep learning.

The standard approach to undergraduate mathematics education is top-down, where the professor professes: "This is the theorem. Let me show you how to prove the theorem." As a student, my goal was to regurgitate the proof to an already-solved theorem. I never had the chance in my early 20's to do anything other than reproduce other peoples' math. I did really well in university, but never felt a sense of ownership. That's what I'm trying to change with my students.

You have a long history of engaging with your own students, with fellow mathematicians and professors, and with classroom teachers. It's not always the case that mathematicians do work with high school teachers. What is your approach to working with classroom teachers?

I just returned from a four-day trip, working with a group of students and teachers, so this question is quite fitting. I love visiting high schools, partnering alongside classroom teachers as we team up together to inspire the next generation. I don't see myself as either a mathematician or as a math educator. I think of myself as both. It's really exciting to go to schools and share beautiful problems with teachers, and learn as I watch them teach and interact with the students. A few days ago, I led a workshop for 20 high school teachers from the same K–12 school in Victoria. We looked at a whole bunch of beautiful problems that straddle different areas of the K–12 curriculum in British Columbia. For these problems, we showed how one part would be accessible to Grade 4 students, one part to Grade 8 students, and one part to Grade 12 students. There are certain rich, inquiry-driven problems that lend themselves to computational thinking, problem solving, and critical thinking. And it's a treat to share these problems with teachers.

I love how you speak to the beauty of math. That wonder and awe that you have is really important, and I don't think that's something that all of our students see.

You have written: "The way in which the problem-solving process is managed in class is sometimes more important than the problem itself" (Hoshino, Polotskaia, & Reid, 2016 CMESG proceedings, p. 159). I feel that's a very interesting statement. Can you say a bit more about this?

I love watching experienced teachers—high school, elementary and university—manage the classroom so they feel every single person has had a say in the solution, rather than just one "star" student who takes over the problem and solves it on behalf of everyone. I have found the very best teachers are masters at empowering students, and giving them a sense of ownership of the problem. I think that's where that comment came from. It's not the one person getting the answer, or the key insight on behalf of the team. It's often the teacher posing the right question, or giving just the appropriate coaching, to help students discover the key insights themselves. As my Ph.D. supervisor taught me years ago, "the key to mathematics is not getting the right answer, but asking the right question." And that's what I try to remember, both with my research projects and in my teaching practices. ▲

▲ Adapting Card Games for Fractions Learning: Practical Ideas for Differentiation and Flexible Learning Opportunities throughout the Year

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Beth Edwards is a Teacher Consultant with the Grand Erie District School Board working with teachers and students in K–8. Our RMS (Renewed Mathematics Strategy) Learning Goal has been "to build our capacity as educators of mathematics by deepening our understanding of Quantity Relationships in order to recognize, respond and develop this thinking in all students."

This year, I had the opportunity to work with a teacher in a Grades 3-4 classroom. Our focus was Fractions Learning, and I had recently purchased Fractions Tasks, Supports, and Additional Resources (2017), published by the Ontario Mathematics Coordinators' Association (OMCA), so I wanted to try out some of the ideas and tasks. The materials in this resource have been developed through the work of Dr. Cathy Bruce, Tara Flynn, and Shelley Yearley, and also link to the Fractions Learning Pathway (available online; see all resources and links listed at the end of this article.)

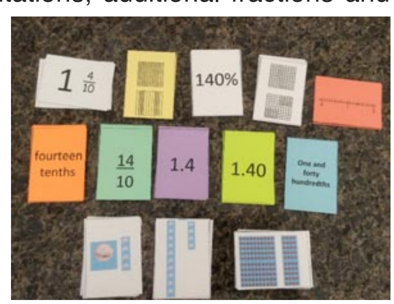
A few lessons into the learning, I wanted to find some cards for students to try some "matching" activities, something I have been using often with whole numbers this year. "Matching" is the first phase on the First Steps™ in Mathematics Developmental Continuum (see below); therefore, it is very important for students to do these sorts of activities, especially in the beginning of their learning of any number system.

Previously in the year, I had shared a game called "Tiny Polka Dots" with some Kindergarten and primary teachers. The box includes game ideas, which are all focused on the numbers 0–10, and there are six decks of cards. The decks are: number symbols, dots in ten-frames, dots in dice arrangements, randomly placed dots, dots of different sizes, and dots in circular arrangements. Each deck is a different colour for ease of selection. There is so much potential to differentiate for students by choosing one, two, or many decks, and by selecting a few, some, or all of the cards in each deck. One Kindergarten teacher has used these decks of cards almost every day with a variety of activities, such as provocations with manipulatives and small-/whole-group ordering/matching activities. I had also used the "Representation Match" game on **www.mathies.ca**, which is a virtual card-matching activity for whole numbers, fractions, and much more.

I saw the opportunity to create decks of cards for fractions tasks and games along the same lines as the Tiny Polka Dots game. I searched my consultant resources and the Internet and could not find any similar ideas, especially ones that included fraction words. (For Grade 3, *The Ontario Curriculum, Grades 1–8: Mathematics* (Revised), 2005, states: "divide whole objects and sets of objects into equal parts, and identify the parts using fractional names (e.g., one half; three thirds; two fourths or two quarters), without using numbers in standard fractional notation" (p. 55).)

I started with the images from the OMCA resource and found the BLM on the Fractions Learning Pathway (see below). I put the images on larger cards. From there, I created a deck of fraction words. I wanted to include images that were fractions of sets, so I included the Fractions Game cards from **www.mathies.ca** and created fraction word cards for those images. Later, I added symbols cards for the OMCA images and number-line cards for both sets. "Students benefit deeply from developing facility and flexibility with the use of and links between multiple representations" (OMCA, 2017, p. 22).

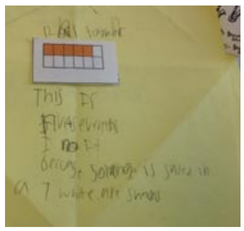
Students in the Grade 4 group showed me that they were quite fluent with fractions connected to money. Considering that the Grade 4 curriculum has many fractions and decimal learning expectations, I thought a separate set of cards would help these students make further connections. I created blank cards for all card decks so students could create alternate representations, additional fractions and images, their own decks of cards, etc. I decided to stay just with tenths and hundredths for the Grade 4-and-up decks. Students in Grade 4 are expected to be counting fractions beyond one whole, so I decided to use tenths up to 16 tenths for each deck.

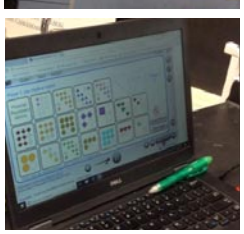


The decks for the Grade 4-and-up set of cards

So far, the Grade 4-and-up set of card decks includes:

- fraction words for tenths and hundredths
- proper and improper fraction notation
- mixed numbers
- decimal notation for tenths and hundredths
- tenths strips images
- 10x10 grids (composing/ decomposing, benchmarks and conceptual subitizing were considered here)
- number lines
- money images: pennies, dimes, loonies
- percent cards
- ideas for tasks and games
- blank cards for each type of representation
- each type of representation on a different colour of card stock for ease of differentiation



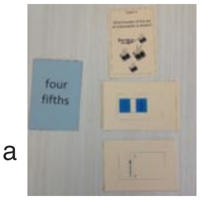


Examples of how the cards were used and ideas for how they could be used:

- Students randomly selected an image from the deck, named it, and explained how they knew the fraction name (exit card activity).
- Small groups of students selected a word or a symbol card from the deck and represented it in as many ways as they could on a concept circle (see Cathy Marks-Krpan's book, *Math Expressions*.)
- Small groups of students were given one, two, or many decks of cards (differentiated) to match, order, create a number line, play GO FISH. Some students played "Representation Match" on **www.mathies.ca**.

Other ideas:

- Use as flash cards to name, for example, how many are needed to make one whole, or a fraction that is more, less, or the same. Use a blank card to create an alternate representation. Use a blank card to create a new fraction/ extend the decks such as seventeen tenths. Create a number line on the floor with masking tape or hanging from a string with clothes pegs. Play a game where the person who lays down the greatest fraction is the winner (like WAR). Make up your own game. Represent the fraction as a set, volume, length (cm, m).
- Fractions of a set, area, length are shown, but there could also be volume, equivalent representations, and many others. Here, the idea is to build flexibility and connections for students. Also, there are so many opportunities for debate among students when they are using the cards for different activities. Teachers could be listening to student conversations as Assessment FOR Learning, which would mean the information would be used over the following days to plan instruction, or as Assessment OF Learning at the end of a learning term.



These card activities could be pulled out at any time throughout the year, used as a centre, used for one-on-one or whole-group tasks that would be differentiated. As stated in the OMCA resource: "We encourage educators to consider fractions not just as a discrete unit, but as a topic that can be woven into other strands and revisited through the year" (p. 3). Students enjoy the game formats and the opportunity for choice.

I realized the amount of time it took me to create the cards was a lot for a classroom teacher to invest on his or her own. This made me want to share the cards and ideas as a jumping-off point for the many creative ideas other teachers could come up with and be responsive to their students. The templates could be modified or added to as needed. Contact me by email if you would like a copy of the templates. Please try them out and share any ideas that can be added for others to use!

Resources

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