8-PAGE INSERT ELEMENTARY MATH
ABACUS

Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

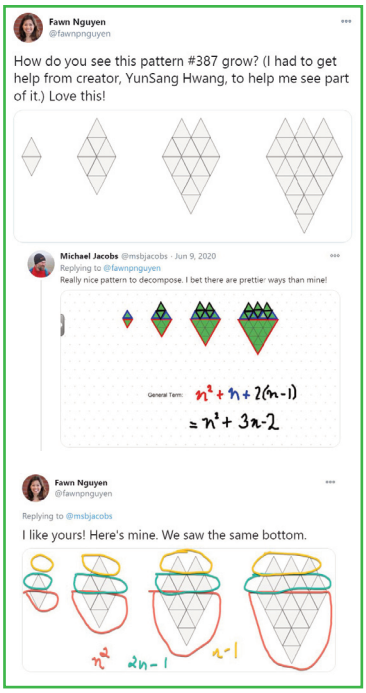
Vol. 59 #3
MARCH 2021

IN THIS ISSUE

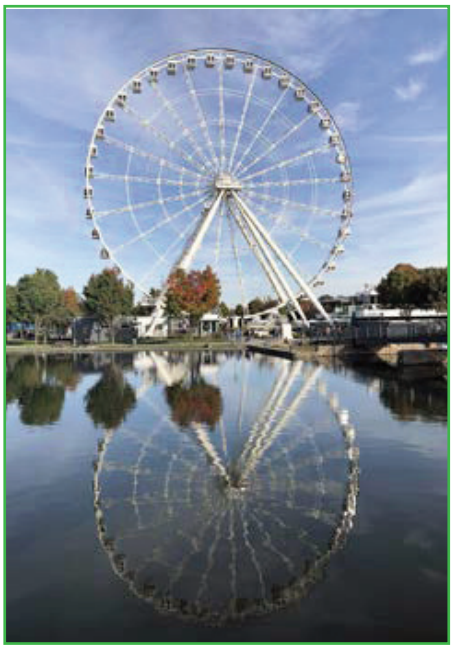
- Editor's Report
- President's Message
- Letter to the Editor
- The New Social–Emotional Strand in Elementary Mathematics—Part Two
- OAME/NCTM Report: SEL, That Is Social–Emotional Learning… Take Two. (That is, part 2 of 2, a bit more in depth.)
- Fields Institute MathEd Forum Report
- Technology Corner: Between 2 Numbers, Math Talks, and Chewable Math
- Linking Literacy and Math: Going Beyond Surface Level When Problem-Solving (Part Three)
- Mathematical Snapshots: Loving the Math in Kingston
- In the Middle: Making Choices to Create a Portfolio
- What's the Problem? Creative Counting
- Mb4T (Mathematics by and for Teachers): Focusing on Fractional Reasoning
- Reflections on Solving Equations: Transforming Equations into Equalities
- Elementary Math Matters: It's a bird, it's a plane… it's a math manipulative!
- Using Place Value to Add, Subtract, Multiply, and Divide with Whole Numbers
- Why We Need to Teach Metacognition in Our Math Classes
- The Power of Equivalent Families in Mathematics

See Linking Literacy and Math: Going Beyond Surface Level When Problem-Solving (Part Three)



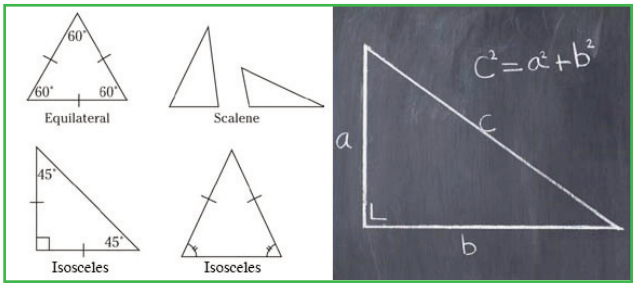


See Technology Corner: Between 2 Numbers, Math Talks, and Chewable Math

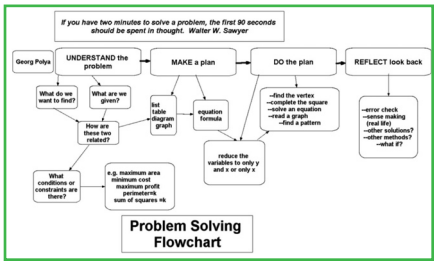


See President's Message

See Elementary Math Matters: It's a bird, it's a plane… it's a math manipulative!



See Why We Need to Teach Metacognition in Our Math Classes



OAME

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION



20... 21...

EQUITY COUNTS

NOW FULLY VIRTUAL!

NEW, MORE ACCESSIBLE
DATES AND TIMES
MAY 17-21, 2021

FROM 4:00 TO 9:00 PM
FULL REGISTRATION FEB 1, 2021

KEYNOTE SPEAKERS



ROBERT Q. BERRY III

EUGENIA CHENG





DAN MEYER

FEATURED SPEAKERS

GAIL BURRILL
GRAHAM FLETCHER
NANCY KAWAJA
RON LANCASTER
LISA LUNNEY BORDEN
KAREN MURRAY
JON ORR • KYLE PEARCE
MARIAN SMALL
CHRISTINE SUURTAMM

OAME2021.CA @OAMELEARNS

Are you looking for engaging math lessons activities and stories that support the new Ontario curriculum and are developed for in-class and at-home use

Have you tried Mathology.ca yet

mathology.ca

VALUE BY THE NUMBERS

Your on-line teacher hub for Early Math. Inside you have everything you need to search, plan, teach, assess & track and receive in-the-moment professional learning. Great value is built right in!

Each teacher license currently includes access to the following K–6 Mathology resources.

72 K–3 Mathology Little Books



OVER **380** Curriculum Focused Lessons from our Grades 1–3 Activity Kits

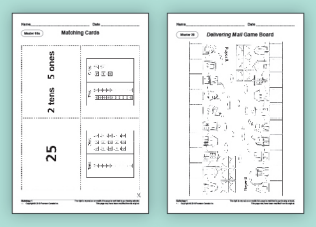




AND OVER **200** Professional Learning Videos

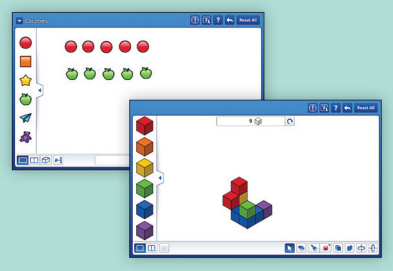


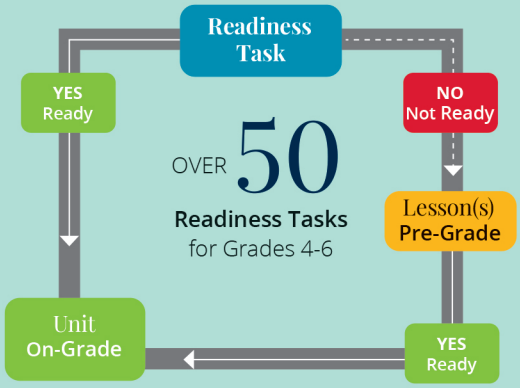
MORE THAN **140** Student Activities



MORE THAN **200** Editable Line and Assessment Masters
Great for Distance Learning!

OVER **20** Interactive Math Tools





"I have already approached my admin to encourage them to purchase Mathology for my classroom next year. I found it very easy to use and was super helpful with my planning for distance learning. I think it will be a great tool to use next year when it is likely we will have a combination of in-class and distance learning "

— Ontario Teacher



A Value of Over **$2500*** In 1 Teacher License!

*Based on value of Mathology print resources

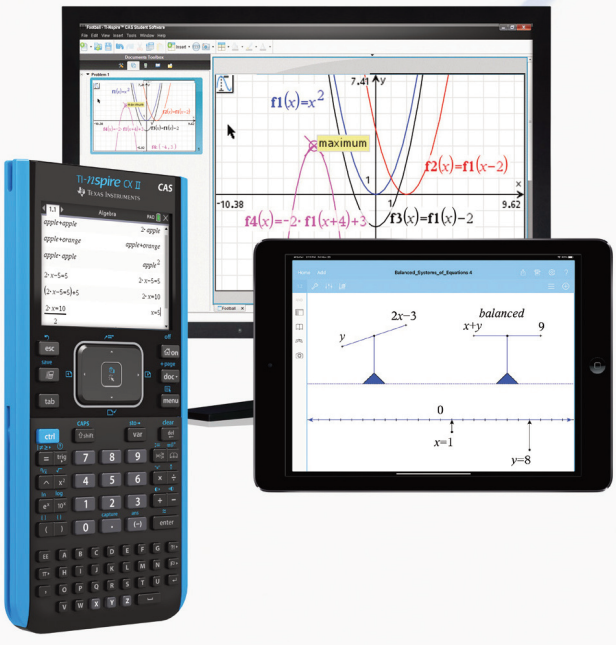
You pay only:
$140 for a 1-Year License or
$665 for a 5-Year License

Pearson

Try Mathology.ca for 1 month FREE at
pearsoncanada.ca/mathologydotca

Follow us on Twitter **@PearsonMathology**

FROM IMAGINATION TO THE REAL WORLD



Engage students with Computer Algebra System (CAS) technology in the classroom

The TI-Nspire™ CX II CAS graphing calculator:

- Supports the Ontario Math Curriculum
- Helps level the playing field in Grade 9 destreamed math
- Allows **all** students to access important algebraic ideas
- Engages **all** students in algebraic thinking and conversation

For activities mapped directly to the Ontario Math Curriculum, visit **education.ti.com/canada** and **stemsupports.ca**.

To get started, contact Tom Steinke:

tsteinke@ti.com | 613.601.6535

Texas Instruments

The platform bar is a trademark of Texas Instruments.
©2020 Texas Instruments Printed in U.S.A. AD21919

Submission of Articles

The *Ontario Mathematics Gazette* (OMG) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *Gazette*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles are to be word–processed, MS Word is preferred, and prepared according to the *Publication Manual of the American Psychological Association*, Seventh Edition. However, please use single-line spacing (not double). Articles should not exceed five numbered pages of text, and figures, images, and photographs should be placed in the text close to where they belong, with captions. The photographer's permission is required, and for photos of students under the age of 18, the written permission of a parent or guardian is required.

Please submit your article in one blind file (i.e., identity of author is not evident), and include author names, contact information including email and mailing addresses, photos—head and shoulders, biographies—less than 100 words, and all content removed for blinding in a second file. Please email these two files to Tim Sibbald at gazette@oame.on.ca.

Upon review, you will be notified whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected. The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME/AOEM.

PERMISSION TO REPRINT: No part of this publication may be reproduced, distributed, or transmitted in any form or by any means, including photocopying, recording, or other electronic or mechanical methods, without the prior written permission of the publisher, except in the case of brief quotations embodied in critical reviews and certain other non-commercial uses permitted by copyright law. Full credit must be given to the author and to the *Ontario Mathematics Gazette*. For permission requests, contact the Editor of the *Ontario Mathematics Gazette*.

The views expressed or implied in this publication, unless otherwise noted, should not be interpreted as official positions of OAME/AOEM.

▲ Table of Contents

Up Front

▲ Editor's Report
by Timothy Sibbald 2
▲ Letter to the Editor 4
▲ President's Message
by Judy Mendaglio 5

Regular Features

▲ The Abacus . . . . . 8-Page Insert
▲ The New Social–Emotional Strand in Elementary Mathematics—Part Two
by Jeff Irvine 10
▲ OAME/NCTM Report: SEL, That Is Social–Emotional Learning… Take Two. (That is, part 2 of 2, a bit more in depth.)
by Jacqueline Hill 16
▲ Fields Institute MathEd Forum Report
by Angelica Mendaglio 17
▲ Technology Corner: Between 2 Numbers, Math Talks, and Chewable Math
by Mary Bourassa 18
▲ Linking Literacy and Math: Going Beyond Surface Level When Problem-Solving (Part Three)
by David Costello 22
▲ Mathematical Snapshots: Loving the Math in Kingston
by Ron Lancaster 25
▲ In the Middle: Making Choices to Create a Portfolio
by Carly Ziniuk 28
▲ What's the Problem? Creative Counting
by Shawn Godin 32
▲ Mb4T (Mathematics by and for Teachers): Focusing on Fractional Reasoning
by Jennifer Holm 38
▲ Reflections on Solving Equations: Transforming Equations into Equalities
by Jérôme Proulx 40
▲ Elementary Math Matters: It's a bird, it's a plane… it's a math manipulative!
by Lynda Colgan 43

Articles

▲ Using Place Value to Add, Subtract, Multiply, and Divide with Whole Numbers
by Don Vander Klok 35
▲ Why We Need to Teach Metacognition in Our Math Classes
by Jeff Irvine 45
▲ The Power of Equivalent Families in Mathematics
by Jérôme Proulx 49

Notices

▲ OAME Graph Sticky Charts . . . . . .
▲ OAME Board Members 52

Advertisements

▲ Pearson Education . . . . . Inside Front Cover
▲ Neufeld Learning Systems Inc 4
▲ CCS 34
▲ Texas Instruments . . . . . Inside Back Cover
▲ OAME Conference 2021 . . . . . Back Cover

▲ About the *Ontario Mathematics Gazette*

The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

Editor
- ▲ **Tim Sibbald**
  gazette@oame.on.ca

Associate Editors
- ▲ **Anne Yeager**
  anne.yeager@rogers.com
- ▲ **Jacqueline Foster**
  jacquie.a.foster@gmail.com

Associate Editor (French articles) /Rédactrice adjointe (articles français)
- ▲ **Kyla Kadlec**
  kyla.kadlec@oame.on.ca

***Abacus* Co-Editors**
- ▲ **Mary Lou Kestell**
  marylou.kestell@oame.on.ca
- ▲ **Kathy Kubota-Zarivnij**
  kkz@oame.on.ca

Copy Editing
- ▲ **Gitta Berg**
  gitta.berg@sympatico.ca

Design and Production
- ▲ **Penny Clemens,** Graphic Designer
  pennyydezign@gmail.com

Printing & Binding
- ▲ **Pole Printing,** Box 69, 1-89 King Street East
  Forest, ON N0N 1J0 (519) 786-5112
  poles@xcelco.on.ca

Advertising Manager
- ▲ **Robert Sherk**
  4366 Snider Rd., Verona, ON K0H 2W0
  Home (613) 374-1515
  robert.sherk@oame.on.ca

Submission of Advertisements
Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** by email or at the above address. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

Advertising Rates
Advertising rates are available by telephoning, emailing, or writing to the Advertising Manager.

▲ Editor's Report



TIMOTHY SIBBALD, OCT, PhD
Gazette@oame.on.ca

Tim Sibbald holds the Universities Director position on the OAME Board of Directors. He is a former President of OAME. He is an associate professor in the Schulich School of Education, Nipissing University, with a focus on mathematics education.

Sometimes serendipitous events come our way. In my case, circumstances led to my teaching a philosophy of education course this term. It was timely to revisit the topic because I had been mulling some details of it with respect to mathematics, as a discipline, for several months. But let me backtrack briefly…

Mathematics as a discipline has many cliques. There are views about pure and applied, for example, and no matter how much one may argue any which way, there are distinctions. Similarly, in math education, there are several cliques of research where, as one portrayal, content, pedagogy, and psychology are somewhat supported through different collections of conferences. These veins of thought speak to different visions of what mathematics is and its utility—essentially philosophical views.

It is not enough to be an armchair commentator. If one really wants to understand the field of mathematics as a discipline, then one has to engage with it. That is the attitude I brought with me when I undertook graduate studies in mathematics. In spite of having a master's and doctorate in education, I have been pursuing a master's degree in mathematics. It is, finally, complete up to the final stages of awaiting convocation.

The reason for doing this was simply to better understand a field of interest. That interest is something that Judy Mendaglio speaks about in her President's Message. (Although in my case, I pursued chemistry and worried about how much of a hazard I might be to myself, so I switched to mathematics.) Having an interest and pursuing a degree are two very different things, especially when earning the degree would have no career implications.

The degree falls under a vision of self-improvement and personal growth or interest. It was also acknowledging a personal need for external oversight as a matter of obliging my ranging interests to remain focused. This speaks to the purpose of education that is discussed in the philosophy of

education. Teaching the course coincides with a need to personally reflect on the educational experience of graduate studies this second time.

Additionally, it facilitates thinking more deeply about the discipline of mathematics. I have at various times felt wonderment about mathematics, but at other times, felt it is an oppressive, exclusionary field. The *Gazette* errs on the wonderment side, and the publication is generally a constructive support for teachers. The oppressive exclusionary view comes from disconnections between the research side of mathematics and teachers—and I will not go far with this because I vacillate about the extent of that issue. However, I will say that there are very few options for pursuing graduate studies in advanced mathematics while working as a teacher.

There are options for graduate studies in education. Some feature math content, with an eye to deepening teacher content knowledge. However, to acquire a disciplinary taste of research mathematics, without it being lashed to education, is very difficult to do while teaching. It seems that mathematics for teachers is offered as a career support, but mathematics for *deepening one's understanding of the field* is not clearly available, except in a few rare circumstances.

I waiver in my feelings about this. Sometimes I am sympathetic to universities and argue it would not be economically viable. Other times, usually when reading science magazines relating what is going on in scientific research, my pendulum swings to feeling that teachers are unreasonably disconnected from the research side of their field.

With the master's degree in mathematics, I came to appreciate the problem of conveying mathematical research in a way that teachers could bring it into classrooms. I would like to know what mathematics research goal corresponds to a Mars mission? What mathematics research is analogous to mapping the human genome? Is there something in mathematics research that corresponds to finding large bodies of water under the ice sheets in Antarctica? I would be fine with explanations in general terms, just like popular science magazines. I would like conceptions from mathematical research that provide students with a hint of what is actively occurring. Yet, I came to realize that, in mathematics research, conveying the details is a problem in its own right.

This brings me back to why it is serendipitous to be teaching educational philosophy. The study of knowledge, epistemology, particularly in the field of mathematics, seems to be different from other disciplines. We use representations and have unusual abstractions (How can "0" represent the non-existence of something?). In essence, this speaks to having grown during the effort to make sense of the mathematics discipline.

This, by the way, mirrors what happened with my graduate studies in education. There, the courses helped me to comprehend a myriad of classroom experiences and demonstrated that theoretical ideas do draw together various conceptions within instruction and education. It is that sense of clarity that warrants pursuit of understanding and personal growth—in whatever way one chooses to pursue it. In my case, I doubt I will go further with formal education—pursuit of a doctorate in mathematics does not seem an option if one is working… there seem to be oppressive exclusionary forces at work…, but it might just be that there are too few people expressing an interest.…

In This Issue

Judy Mendaglio takes us on a math walk. **Angelica Mendaglio** reports on the Field's Institute MathEd Forum. **Jacqueline Hill** draws attention to how the various publications of the NCTM can provide additional support for teachers.

Lynda Colgan points out a possible benefit of the pandemic as students use diverse manipulatives along with odd camera angles. **Mary Bourassa** highlights three websites that are sure to support online teaching. **Jérôme Proulx** carefully navigates the distinction of an equation and equality, that can inform practices around establishing student understanding of the concept of a domain. **Jeff Irvine** addresses the "New Math" that was a strange time in the history of the Ontario math curriculum. **Carly Ziniuk** highlights online portfolios for assessment. **Jennifer Holm** orders fractions, with clarification of using fractional benchmarks along the way. **David Costello** explains several metacognitive strategies, and **Ron Lancaster** offers up a spoonful of math from Kingston.

Lastly, I would like to wish **Shawn Godin** well on his retirement from teaching. His column in this issue goes to pieces… perhaps I should say he takes partitioning apart.

As always there are several articles. Articles include the second part of **Jeff Irvine**'s detailed article about SEL. **Jérôme Proulx** examines equivalent families of problems. In another article, **Jeff Irvine** draws attention to metacognition. **Don Vander Klok** uses place value to get at arithmetic concepts.

Thank You

Special thanks to **Ralph Connelly** and **John Rodger**, who responded to a request for temporary associate editorial support. **Anne Yeager** has, as always, brought a careful eye

to the editorial process that I very much appreciate. **Judy Mendaglio** has taken it upon herself to translate the President's Message into French, as well as providing other French support, and **Kyla Kadlec** has provided a careful eye for the French content. In what has been a period with a very heavy workload, personally, it is reassuring to have **Gitta Berg** catching details and ensuring consistency within the publication. Lastly, but certainly not least, **Penny Clemens**, pulls everything together with little fanfare, but with a very professional result. ▲

▲ Letter to the Editor, OAME *Gazette*

The new Mathematics 1 to 8 guidelines include a strand called Social–Emotional Learning (SEL). I celebrate the inclusion of this strand in the curriculum. Over and above the research evidence that student-affective variables have a positive correlation with student achievement, it could be argued that the contents of this strand are in fact the core learnings that we want students to take away from their time in school. Developing a sense of identity, building relationships, thinking critically and creatively are clearly concepts that we want all students to develop. I anticipate that similar concepts will be included in the upcoming revised curriculum for secondary school mathematics.

I am concerned, however, with the format of the SEL strand as currently published. The strand contains a single overall expectation, followed by a list of skills not identified as specific expectations. To me, this is a mistake. As teachers, we are conditioned to teach overall and specific expectations for each strand of the curriculum. Calling the specific expectations of this strand "skills" may reduce the emphasis that teachers place on these critical components of students' development. In addition, when we evaluate overall expectations, we typically use students' performance on specific expectations to draw conclusions about our students' competence with respect to overall expectations. Since the SEL strand contains no specific expectations, such evaluation is not possible in this strand, unless we treat the "skills" listed as specific expectations. I urge teachers to adopt this philosophy.

Dr. Jeff Irvine, PhD ▲

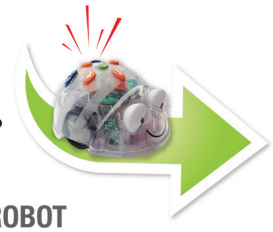
NEW ONTARIO MATH & CODING CURRICULA

INTRODUCTION to CODING through ROBOTICS & LOGO

Go to www.umathx.com
1. See Preview…sample of ebook
2. See Video



Grades 1 to 8
70 Reproducible Sheets



STUDENTS:
- LEARN by TEACHING the ROBOT
- BUILD KNOWLEDGE upon KNOWLEDGE

…and have fun doing it!

UMATH XI™
11th version of Understanding Math

UNDERSTANDING MATH
2020 - Access on all Devices
K to Algebra 1 (grade 10)

Guided Lessons for K - 10

Go to our website www.umathx.com
1. See Introductory Video - UMathX
2. Follow Steps to…Give UMathX a Try!
3. See Tests & Data Analysis



ONGOING SUPPORT
- Content Menu - fill gaps
- Curriculum Menus
- 3 Part Tiered Lessons

PROFESSIONAL LEARNING
- Webinars • Workshops

Neufeld Learning Systems Inc
Order Online at www.UMathX.com
519 657 9334
info@UMathX.com
See our OAME Sessions

President's Message



JUDY MENDAGLIO
judy.mendaglio@oame.on.ca

Judy Mendaglio is the President of OAME/AOEM. She has worked in education in Ontario in many capacities: college instructor, secondary teacher and department head, university instructor, elementary and secondary consultant, and writer. Currently, Judy devotes most of her time to her many volunteer activities.

It is early January. The new year has just begun. So much has happened since the writing of the last President's Message, it is very difficult to decide on a theme or topic to write about. The world may feel like a very different place by the time this column is read. At this moment, we are in a pandemic lockdown. Quebec has issued a curfew, and we are wondering what will happen here in Ontario. Teachers, students, and families are again facing so much uncertainty that sustained high stress levels are largely unavoidable. Thinking about this relentless uncertainty and stress has reminded me that there is something in my own life that has always allowed my mind and body to relax—thinking about the mathematics in everything that surrounds me. Yes, mathematics takes me to my happy place.

Why does mathematics mean calm and certainty to me? I have sat in on many seminars at which the question, "Why did you choose to study mathematics?," was asked. I have pondered that question for many years. The answer did not immediately jump out at me. For some people, mathematics is a subject for which they achieved high marks in school, and that was their impetus for continuing to study mathematics after high school. For some, they saw the value in studying mathematics because of their desire to pursue physics, or engineering, or computer science. Neither of these was true for me. While my marks in mathematics courses were certainly fine, they were not my best marks. I did not have a love of the natural sciences or other areas of applied mathematics. However, I did feel a passion for pure mathematics, and I knew that I needed to learn more about this mysterious and magical field.

I now believe that I chose to study mathematics because I came to understand that the essential nature of mathematics is that its truth is unwavering. It is always there for you when you need it. It is dependable. And, once you really look, you realize that it is everywhere—in our natural

Message de la présidente



JUDY MENDAGLIO
judy.mendaglio@oame.on.ca

Judy Mendaglio est la présidente de l'AOEM/OAME. Elle a exercé de nombreuses fonctions dans le secteur ontarien de l'éducation : formatrice au niveau collégial, professeure et chef de section du secondaire, chargée de cours au niveau universitaire, consultante pour le primaire et le secondaire et écrivaine. Judy se consacre actuellement à de nombreuses activités bénévoles.

Dans le présent document, le masculin est utilisé sans aucune discrimination et uniquement dans le but d'alléger le texte.

Nous sommes au début de janvier. La nouvelle année vient de commencer. Il s'est passé tellement de choses depuis la rédaction du dernier message du président, qu'il est très difficile de décider d'un thème ou d'un sujet à aborder. Le monde peut se sembler très différent au moment où cette colonne est lue. En ce moment, nous sommes dans un confinement pandémique. Le Québec a imposé un couvre-feu et on se demande ce qui va se passer ici, en Ontario. Les enseignants, les étudiants et les familles sont de nouveau confrontés à une telle incertitude que des niveaux de stress élevés soutenus sont en grande partie inévitables. Penser à cette incertitude et à ce stress incessant m'a rappelé qu'il y a quelque chose dans ma propre vie qui a toujours permis à mon esprit et à mon corps de se détendre — penser aux mathématiques dans tout ce qui m'entoure. Oui, les mathématiques m'amènent à mon bonheur.

Pourquoi les mathématiques signifient-elles, pour moi, le calme et la certitude ? J'ai participé à de nombreux séminaires au cours desquels la question « Pourquoi avez-vous choisi d'étudier les mathématiques ? » a été posée. J'ai réfléchi à cette question pendant de nombreuses années. Pour moi, la réponse n'était pas évidente. Pour certaines personnes, les mathématiques sont une matière pour laquelle elles ont obtenu des notes élevées à l'école, et c'était leur motivation pour continuer à étudier les mathématiques après le secondaire. Pour certains, ils voyaient la valeur de l'étude des mathématiques en raison de leur désir de poursuivre des études de physique, d'ingénierie ou d'informatique. Ni l'un ni l'autre n'était vrai pour moi. Bien que mes notes dans les cours de

world as well as our constructed worlds. We live in times of relentless change and face unprecedented challenges. How nice to have this wonderful, reliable, enduring mathematics surrounding you and be able to enjoy its unshakeable peace and comfort, secure in the knowledge the foundation of mathematics is unshakeable.

So, what math surrounded me when I was growing up in Montreal? What did I see that might have affected my choosing to study mathematics?

For me, Montreal is defined by its geography. It lies on an island, so it is surrounded by water. Its most prominent feature is Mont Royal, a three-head hill that provides park and recreation space as well as spectacular views. At its maximum, its elevation is 764 feet (233 metres). Have you ever looked at an old map in a museum and wondered how geographic maps of islands and their coastlines were made without the technological tools upon which we now depend? How were topological maps that accurately described the elevation of land masses made? Contour maps date back to the sixteenth century. How did they make these? Why did they want them? What questions were they seeking to answer? What I now know is that whoever mapped these coastlines or drew the topological maps knew about the power of mathematics.

Montreal is an old city with beautiful old buildings. As reported in the *New York Times* on December 10, 1881, Mark Twain spoke about Montreal in a speech at a banquet in his honour: "This is the first time I was ever in a city where you couldn't throw a brick without breaking a church window." Some of these churches, like Notre-Dame Basilica, date back to the 1600's. I never stop wondering how people, that long ago, with the instruments and tools they had, were able to go from two-dimensional designs to three-dimensional physical structures to create such beauty. For me, this is astonishing.

If you go into any place of worship, you are surrounded by geometry. Looking at this image taken inside the Notre-Dame Basilica (Figure 1) in Vieux-Montréal, one cannot help but notice the calming symmetry, the perfection of the curves of the arches, the exacting spacing of the spirals—all resulting from the careful application of mathematical relationships.

Walk toward the National Historic Site and Canada's largest church, St. Joseph's Oratory (Figure 2), and one is provided with another stellar example of the

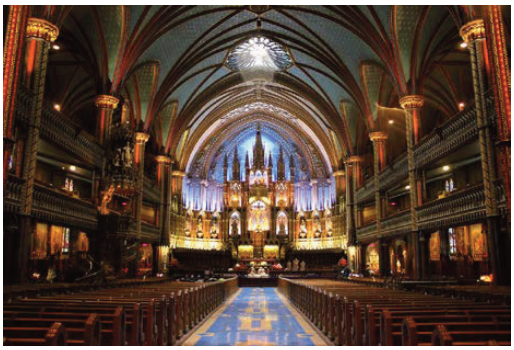


Figure 1: *Notre-Dame Basilica / La basilique Notre-Dame*

mathématiques étaient certainement bonnes, elles n'étaient absolument pas mes meilleures notes. Je n'ai pas eu un amour des sciences naturelles ou d'autres domaines des mathématiques appliquées. Cependant, j'ai senti une passion pour les mathématiques pures, et je savais que je devais en apprendre davantage sur ce domaine mystérieux et magique.

Je crois maintenant que j'ai choisi d'étudier les mathématiques parce que je suis venu à comprendre que la nature essentielle des mathématiques est que sa vérité est inébranlable. Il est toujours là pour vous lorsque vous en avez besoin. Il est fiable. Et, lorsqu'on regarde vraiment, on se rend compte que c'est partout — dans notre monde naturel aussi bien que dans nos mondes construits. Nous vivons une période de changements incessants et nous sommes confrontés à des défis sans précédent. Quel plaisir d'être entouré de ces mathématiques merveilleuses, fiables et durables et de pouvoir profiter de leur paix et de leur confort, en étant sûr de savoir que les fondements des mathématiques sont inébranlables.

Alors, quelles mathématiques m'entouraient lorsque je grandissais à Montréal ? Qu'est-ce que j'ai vu qui aurait pu influencer mon choix d'étudier les mathématiques ?

Pour moi, la cité de Montréal se définit par sa géographie. Elle est située sur une île, donc elle est entourée d'eau. Sa caractéristique la plus marquante est le mont Royal, une colline à trois têtes qui offre un parc et un espace de loisirs ainsi que des vues spectaculaires. À son maximum, son élévation est de 764 pieds (233 mètres). Avez-vous déjà regardé une vieille carte dans un musée et vous êtes-vous demandé comment les cartes géographiques des îles et de leurs côtes ont été réalisées sans les outils technologiques dont nous dépendons aujourd'hui ? Comment a-t-on établi des cartes topologiques qui décrivent avec précision l'altitude des masses terrestres ? Les cartes de contour remontent au XVIe siècle. Comment ont-elles été réalisées ? Pourquoi est-ce qu'elles étaient voulues ? Quelles questions cherchaient-ils à répondre ? Ce que je sais maintenant, c'est que celui qui a cartographié ces côtes ou dessiné les cartes topologiques connaissaient la puissance des mathématiques.

Montréal est une vieille ville avec de beaux bâtiments anciens. Tel que rapporté dans le *New York Times* le 10 décembre 1881, Mark Twain parlait de Montréal dans un discours prononcé à l'occasion d'un banquet en son honneur : « C'est la première fois que je me trouve dans une

beauty of symmetry. I am always stunned by the importance of symmetry in the design of old and even ancient structures all over the world. How careful was their planning, and how exact were their measurements to achieve such beauty? How can we use curved triangles to create a dome? Does a dome have the same kinds of symmetries as a hemisphere?

Look at any type of building (Figure 3)—homes, office buildings, government offices, museums, barns. How is symmetry manifested in these structures?

Perhaps it was growing up with the opportunity to visit Buckminster Fuller's geodesic dome (Figure 4) whenever I wanted that instilled love of the physical beauty of geometry. Despite the extraordinary exhibits that are found in museums such as this, I always find myself looking at the building itself, taking photographs of interesting scaffolding, buttresses, fixtures, staircases—all places in which geometric structures come to life. Function meets beauty.

Look at the outside staircases (Figure 5) that are so prominent in Montreal. Function meets beauty, with mathematics leading the way. Is the Olympic Stadium (Figure 6) just a sports venue, or does the geometry of this building capture your imagination as it does mine? I look at this structure and cannot help but wonder why early moviemakers designed spaceships to look this way? What geometric properties spoke to them and said, "This would be a space-worthy craft"?

We had two amusement parks in Montreal when I was



Figure 2: *St. Joseph's Oratory / L'oratoire Saint-Joseph*



Figure 3: *View of the city / Vue de la ville*



Figure 4: *Buckminster Fuller's geodesic dome / Le dôme géodésique de Buckminster Fuller*



Figure 5: *An outside staircase / Un escalier extérieur*

ville où on ne peut pas jeter une brique sans casser une fenêtre d'église. » Certaines de ces églises, comme la basilique Notre-Dame, remontent aux années 1600. Je ne cesse de me demander comment les gens, il y a si longtemps, avec les instruments et les outils dont ils disposaient, ont pu passer de dessins en deux dimensions à des structures physiques en trois dimensions pour créer une telle beauté. Pour moi, c'est étonnant.

Si vous entrez dans un lieu de culte, vous êtes entouré par la géométrie. En regardant cette image prise à l'intérieur de la basilique Notre-Dame (Figure 1) du Vieux-Montréal, on ne peut s'empêcher de remarquer la symétrie apaisante, la perfection des courbes des arcs, l'espacement précis des spirales, qui résulte de l'application minutieuse des relations mathématiques.

Marchez vers le lieu national historique et la plus grande église du Canada, l'Oratoire Saint-Joseph, et on découvre un autre exemple éclatant de la beauté de la symétrie. Je suis toujours abasourdi par l'importance de la symétrie dans la conception de structures anciennes partout dans le monde. Dans quelle mesure leur planification a-t-elle été minutieuse et à quel point leurs mesures étaient-elles exactes pour atteindre une telle beauté ? Comment pouvons-nous utiliser des triangles incurvés pour créer un dôme ? Un dôme a-t-il les mêmes types de symétries qu'un hémisphère ?

Examinez n'importe quel type de bâtiment (Figure 3) : maisons, immeubles de bureaux, bureaux gouvernementaux, musées, granges. Comment la symétrie se manifeste dans ces structures ?

C'est peut-être le fait d'avoir grandi avec la possibilité de visiter le dôme géodésique de Buckminster Fuller (Figure 4) chaque fois que je voulais qui a inculqué l'amour de la beauté physique de la géométrie. Malgré les expositions extraordinaires que l'on trouve dans des musées comme celui-ci, je me retrouve toujours à regarder le bâtiment lui-même, à prendre des photos d'échafaudages, de contreforts, de luminaires, d'escaliers intéressants — tous des endroits où des structures géométriques prennent vie. La fonction rencontre la beauté.

Regardez les escaliers extérieurs (Figure 5) qui sont si proéminents à Montréal. La fonction rencontre la beauté

growing up. Both were easy to get to, and lots of time was spent on rides. Rides are surprisingly mathematical! Examining the mathematics in the design of the rides in an amusement park is such a beautiful way to explore this wondrous subject. Along with the stunning symmetries that can be found in the Ferris wheel (Figure 7) at La Ronde, we could also think about how symmetry helps to ensure stability. Function meets beauty. What other properties of geometric structures can be harnessed to ensure stability and safety? Children love to build things. Could we help them to see the underlying mathematics in what they are building?

The older sister of a friend took an art class on Saturday mornings from Group of Seven artist Arthur Lismer. She would often share with us what she learned. I was always fascinated by her ability to create the illusion of three dimensions (Figure 8), as if the paper had depth. She taught us about perspective from an artist's point of view. It was years later that I learned that it is mathematics and mathematical relationships upon which this technique is based.

Montreal's Botanical Garden (Figure 9) is another place to explore the wonderful world of mathematics, but any garden, woodlot, or forest will do. While walking through a park, wooded area, or countryside provides an opportunity to experience nature, it also provides an opportunity to ask "how" and "why" questions that can be answered with mathematical modelling. Why do different plants grow in different places within the same natural space? Why do trees that are the same species and in the same place have different shapes? Why are some trees tall, while others hug the ground? How can you affect how



Figure 6: *The Olympic stadium / Le Stade olympique*



Figure 7: *The Ferris wheel at La Ronde / La grande roue à La Ronde*



Figure 8: *Arthur Lismer's HMCS Grilse on Convoy Duty / Le NCSM Grilse en mission de convoi d'Arthur Lismer*

avec les mathématiques menant la voie. Le Stade olympique (Figure 6) n'est-il qu'un lieu sportif, ou bien la géométrie de ce bâtiment capte-t-elle votre imagination comme la mienne ? Je regarde cette structure et je ne peux m'empêcher de me demander pourquoi les premiers cinéastes ont conçu des vaisseaux spatiaux de cette façon. Quelles propriétés géométriques leur ont dit : « Ce serait un vaisseau digne d'un voyage dans l'espace » ?

Nous avions deux parcs d'attractions à Montréal quand j'étais jeune. Les deux étaient faciles d'accès, donc on passait beaucoup de temps à faire des promenades. Les manèges sont incroyablement mathématiques ! Examiner les mathématiques dans la conception des manèges dans un parc d'attractions est une si belle façon d'explorer ce merveilleux sujet. En outre des symétries étonnantes que l'on retrouve dans la grande roue de La Ronde (Figure 7), nous pourrions également réfléchir à la façon dont la symétrie contribue à assurer la stabilité. La fonction rencontre la beauté. Quelles autres propriétés des structures géométriques peuvent être exploitées pour assurer la stabilité et la sécurité ? Les enfants adorent construire des choses. Pourrions-nous les aider à voir les mathématiques sous-jacentes dans ce qu'ils construisent ?

La sœur aînée de mon ami a suivi un cours d'art le samedi matin auprès l'artiste du Groupe des Sept, Arthur Lismer. Elle partageait souvent avec nous ce qu'elle avait appris.

J'ai toujours été fascinée par sa capacité à créer l'illusion des trois dimensions (Figure 8), comme si le papier avait de la profondeur. Elle nous a appris la perspective du point de vue d'un artiste. C'est des années plus tard que j'ai appris que ce sont les mathématiques et les relations mathématiques sur lesquelles repose cette technique.

Le Jardin botanique de Montréal (Figure 9) est un autre endroit pour explorer le monde merveilleux des mathématiques, mais n'importe quel terrain nivelé, boisé ou forêt fera l'affaire. En ce promenant dans un parc, une zone



Figure 9: *Montreal's Botanical Garden / Le Jardin botanique de Montréal*

quickly something will grow?

Artists, sculptors, architects, and engineers have long recognized the value of mathematics in achieving both function and beauty in their designs. Looking at beautiful things makes us smile. Smiling relaxes us, releasing stress naturally. Medical researchers, hospital administrators, public health officials, and government agencies have all turned to mathematics to help track, understand, control, and manage COVID-19. Mathematics, properly applied, will always give us reliable information, and so mathematics can offer hope in times of uncertainty. Helping our students understand these "big ideas" should always be part of our mathematics teaching, but especially now, when no one knows what waits for us around the next corner. Take any challenge and pull back far enough, you will find mathematics is essential to finding its solution, including managing change and uncertainty.

Perhaps you and your students would like to take a journey through your own pasts and your diverse cultures and explore the mathematics within? While I found inspiration in Montreal, consider what your students are mathematically seeing in the world around them.

Notes

All photos are from pixabay.com.

Figure 8 retrieved from www.gallery.ca/collection/artwork/hmcs-grilse-on-convoy-duty.

Mark Twain quotation retrieved from www.twainquotes.com/18811210.html. ▲

boisée ou une campagne, on peut faire l'expérience de la nature. Cela offre également l'occasion de poser des questions « comment » et « pourquoi » auxquelles on peut répondre grâce à la modélisation mathématique. Pourquoi différentes plantes poussent-elles à différents endroits dans le même espace naturel ? Pourquoi les arbres de la même espèce et situés au même endroit ont-ils des formes différentes ? Pourquoi certains arbres sont-ils grands tandis que d'autres épousent le sol ? Comment pouvez-vous influer sur la vitesse de croissance d'une plante ?

Les artistes, les sculpteurs, les architectes et les ingénieurs reconnaissent depuis longtemps la valeur des mathématiques dans la réalisation à la fois de la fonction et de la beauté de leurs créations. Regarder de belles choses nous fait sourire. Le sourire nous détend, libérant naturellement le stress. Les chercheurs en médecine, les administrateurs d'hôpitaux, les responsables de la santé publique et les agences gouvernementales se sont tous tournés vers les mathématiques pour aider à suivre, à comprendre, à contrôler et à gérer le COVID-19. Les mathématiques, correctement appliquées, nous donneront toujours des informations fiables, et ainsi les mathématiques peuvent offrir de l'espoir en temps d'incertitude. Aider nos élèves à comprendre ces « grandes idées » devrait toujours faire partie de notre enseignement des mathématiques, mais surtout maintenant, quand personne ne sait ce qui nous attend au prochain virage. Relevez n'importe quel défi et reculez suffisamment, vous constaterez que les mathématiques sont essentielles pour trouver sa solution, y compris la gestion du changement et de l'incertitude.

Peut-être que vous et vos élèves aimeriez faire un voyage à travers votre propre passé et vos diverses cultures et explorer les mathématiques qui s'y trouvent ? Bien que j'aie trouvé l'inspiration à Montréal, considérez ce que vos élèves voient mathématiquement dans le monde qui les entoure.

Notes

Toutes les photos, à l'exception de la figure 8, proviennent de pixabay.com.

Figure 8 extraite de www.gallery.ca/collection/artwork/hmcs-grilse-on-convoy-duty.

Citation de Mark Twain extraite de www.twainquotes.com/18811210.html. ▲

▲ The New Social–Emotional Strand in Elementary Mathematics—Part Two



DR. JEFF IRVINE
dr.jeffrey.irvine@bell.net

Jeff Irvine, PhD, has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

The new social–emotional strand in elementary mathematics has six expectations:

- Identify and manage emotions
- Recognize sources of stress and cope with challenges
- Maintain positive motivation and perseverance
- Build relationships and communicate effectively
- Develop self-awareness and sense of identity
- Think critically and creatively

The first part of this article discussed the importance and instructional strategies related to the first three expectations. Part Two will examine the final three expectations. This article gives the goal for each expectation, as indicated in the curriculum policy document; why each expectation is important; and some strategies teachers can employ to address the expectation.

Build Relationships and Communicate Effectively

> *Work collaboratively on math problems—expressing their thinking, listening to the thinking of others, and practising inclusivity—and in that way, fostering healthy relationships* (Ontario Ministry of Education [OME], 2020).

Socialization is often cited as an important dimension of schooling. In mathematics classrooms, respectful and productive math talk is critical for classrooms that employ student groups, and that expect students to justify, explain their thinking, and consider alternative solutions posed by others. The mathematical process of Communicating is central to much of what happens in today's math classrooms, but we cannot expect students to communicate respectfully and effectively without instruction. Further, fostering productive mathematical talk builds students' self-concept as positive contributors to math classes, and supports other affective dimensions related to emotions and attitudes.

Teaching Strategies to Support This Expectation

There are a number of instructional strategies that support this expectation. One major strategy is the formation of Math Talk Learning Communities (MTLC). This can be very effective, especially if there is whole-school commitment to supporting MTLCs (Irvine, 2017). Supporting an MTLC requires changes in teacher behaviours. In particular, an MTLC requires changes in teacher questioning. Rather than being the primary questioner, the teacher moves to a secondary role, utilizing questions in response to student statements (Gillies et al., 2012), and employing probing questions to move the student discussion forward (Hufford-Accles et al., 2004). Through questioning, the teacher scaffolds students' learning experience and provides guidance and direction for student thinking.

Another teacher function is to provide and clarify keywords, such as: *explain, justify, agree, compare, represent,* and *validate* (Brown, 2007). By providing such support, the teacher reduces extraneous cognitive load for the students, allowing them to focus working memory on the intrinsic cognitive load of the task (Irvine, 2016). In addition, the teacher models appropriate language and behaviours (Gillies et al., 2012), including appropriate social interactions.

Supporting appropriate student behaviours to achieve this expectation requires teachers to provide structures and supports for learning and assessment activities. For example, strategies such as *Bansho* (OME, 2011) and math congress (OME, 2010) can be used for discussing student work samples. Other strategies include *say something* (see Figure 6) and *save the last word for me* (see Figure 7). These last two strategies to support student talk can be employed in many subject areas, including mathematics and particularly mathematical problem solving.

In their book, *5 Practices for Orchestrating Productive Mathematics Discussions*, Smith and Stein (2013) provide a step-by-step approach to building students' communication skills in math classrooms.

Say Something!

Students are given a passage/passages/problem statement to read individually.

- If the passage is longer than one page, it should be "chunked."
- Strategy would then be repeated with each chunk.
- Each student identifies a brief (e.g., one sentence or less) segment that represents an important idea or statement that resonates with them.
- Students take turns outlining "what" and "why" for their segment.
- Other students then (round robin) ask questions or ask for clarifications from the student.
- Questioning students may "pass" if they have no questions.
- After everyone has had a chance to ask questions, the next student (round robin) identifies their segment and the questioning process repeats.
- After everyone has identified their segment and the questioning has occurred, students then read the next passage/page and the process is repeated.

Figure 6: *"Say something" strategy to support student talk*

Save the Last Word for Me

- Groups should have no more than five students.
- Each student is given an index card with a statement/question/problem.
- Reflect on your statement.
- Write your reflections on the back of the page.
- Sharing: In a round-robin format, hold up your statement, others contribute thoughts one at a time (not you).
- Participants may pass, rather than contribute.
- Once all participants have passed, you contribute final thoughts on your statement.
- The process is repeated for other statements.

Figure 7: *"Save the last word for me" strategy to support student talk*

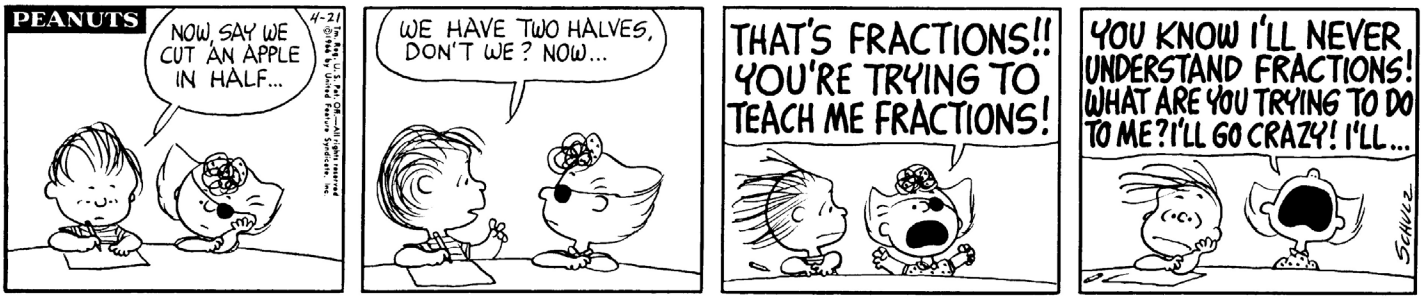
Develop Self-Awareness and Sense of Identity

> *See themselves as capable math learners, and strengthen their sense of ownership of their learning, as part of their emerging sense of identity and belonging* (OME, 2020).

Students are in the process of developing their possible selves. "Possible selves are one's ideas about what one wants to become (hoped-for self), wants to avoid becoming (feared self), and realistically expects to become (expected self)" (Zollman et al., 2011, p. 49). As teachers, we have the responsibility to assist students to "be all that they can be" in mathematics.

Schunk and Richardson (2011) found that teaching students' self-regulation through facilitating their choice of learning strategies not only improved their mathematics learning and achievement, it also enhanced motivation. One key component of self-regulation is self-efficacy, which is related to students' self-confidence. Self-efficacy (Bandura, 1997) refers to an individual's judgment about whether they are capable of accomplishing a task. Changes in self-efficacy can result in major changes in achievement. Ross (2008) used PISA 2003 mathematics data to find that a one-unit increase in self-efficacy was associated with a 32-unit increase in achievement. She showed that no other motivational variable that she studied (intrinsic motivation, goal orientation, instrumental versus relational view of instruction) was associated with as significant an impact on student achievement (Ross, 2008). Self-efficacy is a belief (Part 1, Figure 1), but when teachers continue to positively influence student emotions and attitudes, student beliefs can be modified. Improving students' self-efficacy in mathematics supports student growth in self-confidence and self-concept as a capable mathematics learner.

Developing students' self-regulation and metacognition has a significant impact on student success. Metacognition,



PEANUTS © 1966 Peanuts Worldwide LLC. Dist. By ANDREWS MCMEEL SYNDICATION. Reprinted with permission. All rights reserved.

defined as "the self-management of learning" (Hattie, et al., 1996), involves goal setting, planning, strategy selection, evaluating alternatives, monitoring accuracy, debugging, evaluating the reasonableness of results, considering implications of a solution, and performing "What if?" analyses (Marzano & Kendall, 2007; Meijer et al., 2006).

Metacognition arises in many of the mathematical processes, such as selecting tools and strategies, connecting, reflecting, as well as reasoning and proving. Teaching metacognitive skills has been shown to have a positive impact on student achievement in mathematics (Schneider & Artelt, 2010; Veenman et al., 2006), as well as supporting student self-regulation (Meijer et al., 2006). In a meta-analysis of over 800 studies involving over 2 million students, Hattie (2009) found that teaching metacognitive skills had an effect size of 0.69 on student achievement, which is considered large and could potentially move a student's achievement from the 50th percentile to the 74th percentile (Irvine, in press-b).

Teaching Strategies to Support This Expectation

Instructional strategies that support students' development of self-awareness and mathematical identity involve establishing a learning environment in which students are encouraged to seek alternatives; fostering a growth mindset and celebrating effort; recognition and praise of specific effort-related behaviours; accepting and encouraging multiple solutions and alternative computational algorithms; and co-operative learning strategies.

Explicitly teaching metacognitive and self-regulatory strategies can be accomplished through student goal-setting activities, as well as using graphic organizers to assist students in developing metacognitive activities, such as planning and evaluating alternatives (Irvine, in press-b), and teaching heuristics, such as Pólya's problem-solving strategies (Irvine, 2018).

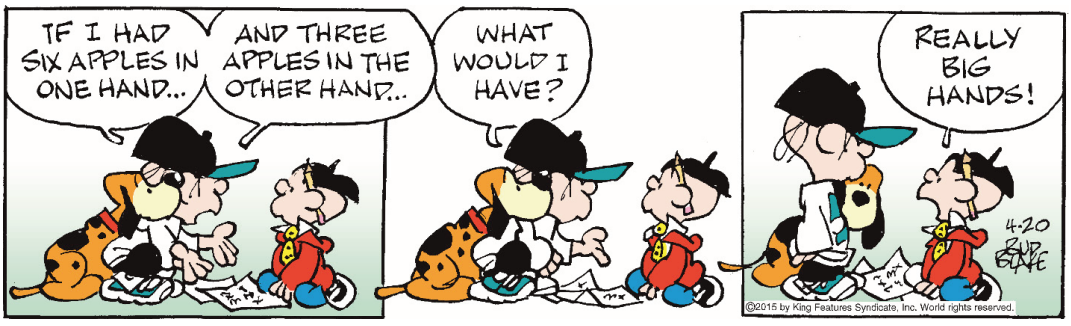
Students should be encouraged to see themselves as confident and mathematically skilled, flexible, and adaptive learners, and use mathematics to make sense of their world. They should monitor their own progress and develop self-regulation skills, reflecting on their accomplishments and sharing these accomplishments with peers and caring adults.

Think Critically and Creatively

> *Make connections between math and everyday contexts to help them make informed judgments and decisions* (OME, 2020).

Historically, teaching students to think has been the overarching reason for teaching mathematics (Phillips, 2016). In additon, critical thinking, together with higher-order thinking skills and integrative thinking, comprise deep learning (Campbell & Cabrerra, 2014). Deep learning contrasts with surface learning, which focuses on facts and basic procedures (Campbell & Cabrera, 2014). In deep learning, students make connections and integrate knowledge into internal cognitive networks. Teachers can address higher-order thinking through a focus on big ideas, and through taxonomies of learning, such as revised Bloom's taxonomy (Anderson & Krathwohl, 2001) or Marzano's taxonomy (Marzano & Kendall, 2007). Marzano's taxonomy includes a metacognitive dimension that also addresses critical reflection. Hattie (2003) found that expert teachers focused 72 percent of their instruction on deep learning compared to 28 percent on surface learning. Non-expert teachers' focus was exactly the opposite, with 28 percent on deep learning and 72 percent on surface learning. Deep learning is essential for students' enduring understanding of concepts, as well as for transfer of knowledge to new or different situations.

Creativity is embodied in the Tiger comic below. Creative thinkers often think divergently and pose questions or solutions that are unexpected or tangential to anticipated responses. Creativity is often associated with giftedness (Mann, 2006), but all students can exhibit creativity with appropriate instructional strategies. Teachers' views on creativity in mathematics are often seen as subjective (Leikin et al., 2013). However, there are taxonomies to assess student creativity in less subjective ways (see Lev-Zamir & Leikin, 2013, below).



Tiger © 2015 King Features Syndicate, Inc.

It is important to note that creativity for students is not necessarily the same as creativity for a professional mathematician (Liljedahl & Sriraman, 2006). A useful way

of looking at student creativity in mathematics is given by examining three dimensions: *flexibility*, when students generate different solutions to a problem; *originality*, when students have ideas, generate problems, discover facts, or suggest rare or insightful solutions to a problem; and *elaboration*, when students generalize ideas (Lev-Zamir & Leikin, 2013).

Teaching Strategies to Support This Expectation

Problem solving and mathematical modelling are important strategies to support this expectation, as well as open questions (Small, 2009) and problem posing (Irvine, 2017). To develop students' critical thinking, "What If?" questions should be encouraged (Irvine, 2019b, 2019c). Figure 8 shows a modelling activity for Junior or Intermediate math classes.

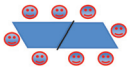
Trapezoidal Tables Task

Sally set up trapezoidal tables as shown below. The diagrams below show how many people can sit around one table, two tables, and three tables.

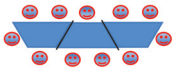
1 table



2 tables



3 tables



Sally says that 39 people can sit around 12 tables that are pushed together like the pictures show. Is she correct? Justify your choice.

Figure 8: *The Trapezoidal Table Problem (Irvine et al., 2016)*

This task is accessible to all students, with multiple entry points for a variety of solution methods. To encourage students to go deeper, "What If?" questions should be encouraged. The following discussion excerpt is from Irvine et al. (2016), to illustrate how the basic modelling task could be extended to develop students' deep learning.

> When the whole-class discussion began, the teachers posed this question: "How would your solution change if two people were able to sit at each end of the trapezoid?" Students were able to quickly identify how this modification affected their solutions. Students were then asked to pose similar questions to their peers, with correct results quickly offered.
>
> The teachers then posed this question: "What would change if the tables were not trapezoids, but some other geometrical shape?" Rectangles were quickly identified as simpler cases of the trapezoid problem, and solutions were offered to various rectangle scenarios. Students were able to generalize an equation for rectangles with any number of people on each side. However, most students were unable to make a similar generalization for the case of trapezoidal tables. This may have been due to having to consider three different possibilities for the number of people on each side of the trapezoid (long side, short side, ends).
>
> Equilateral triangles were the first situation in which some students resorted to drawing diagrams before offering solutions… Isosceles triangles led to students posing the question, "What would be the impact of changing the configuration?" For example, one student posed the following scenario: "What would happen if the tables are isosceles triangles, where one person can sit at the short side and two people at each of the long sides?" This led to students discussing possible configurations of the tables, rather than simply pushing the short sides together. The discussion then turned to other configurations, such as, "Rather than one long row, what about two rows of trapezoidal tables pushed together?" "What happens if the trapezoidal tables are combined into blocks of four tables each?" At this stage, most students resorted to drawing diagrams before offering solutions. The discussion naturally led to questions about possible configurations of hexagonal tables, circular tables, and even mixed configurations including several table shapes in combination.
>
> In one class, a discussion occurred when dealing with circular tables. There was no disputing that when circular tables were pushed together, each table lost one seat at the point of tangency. However, in considering the physical situation, one student pointed out that the tables might lose more than one seat, since positions near the point of tangency might not have sufficient room for people to be seated due to the circular shape of the table. A second student conjectured that this situation might be dependent on the diameters of the tables. When the diameters were fairly small, more than one seat would be lost. When diameters were larger, only the seats at the point of tangency would be lost. Since no actual circular tables were available in the classroom, the conjecture could not be tested. However, several students decided to go to the library after class to test the conjecture, since the library tables were circular. Since all the library tables were the same diameter,

only part of the conjecture could be tested (Irvine et al., 2016, p. 19).

Research by Kasmer and Kim (2011) found that using "What If?" questions prior to actually solving a problem activated prior knowledge and stimulated interest in solving both the original problem and the "What If?" scenarios. For example, suppose the original question posed were, "How many houses with three-digit street numbers have the property that the product of the digits is 24?" Before attempting to solve this problem, students were asked questions like "What if the product were bigger than 24?" "What if the product were smaller than 24?" "What if we included four-digit street numbers?" Prompts like these clarified students' understanding of the original problem, and motivated students to solve the basic problem as well as the conjectures.

Some strategies to encourage creativity by elementary students are:

- Notice-Wonder (Ruesch, 2020): Students are presented with a situation or potential problem without any numbers. They respond to two questions: "What do you notice?" and "What do you wonder?" Then additional detail is progressively provided, either by the teacher or by students, and the same two questions (notice-wonder) are asked again. The process continues until one or more problems are defined, and then solved. The *Mathematical Snapshots* column in *Gazette 57*(1), page 15, is a reminder of this strategy.
- Build on creativity (Johnson, 2020): This strategy builds on math content that has been taught. Johnson gives the example of a Grade 3 class that has been learning about polygons. Students are given sidewalk chalk and go outside to draw polygons on the pavement. The students then transform their polygons into "something they love," such as kittens, robots, dragons, and they then explain their drawings to their peers.
- Never-ending problems (Burkhart, 2020): These are open-ended, non-standard problems that provide rich ground for "What If?" questions and conjectures. Examples include the triangular number problem (Irvine, in press-a), patterning activities, geometric constructions, and fraction follies (Irvine, 2019a). These provide opportunities for students and teachers to learn together, since there is no way to anticipate which directions student inquiry will go, and so teachers do not need to be prepared with answers, but need to keep open minds and promulgate inquiry.

Conclusion

Social and emotional intelligence can be defined as "competencies linked to self-awareness, self-management, social awareness, and relationship management, which enable people to understand and manage their own and others' emotions in social interactions" (Salyers, 2017, p. 1). More and more research has confirmed that students' affective dimensions, such as attitudes and motivation, play a large role in student success in mathematics. Vandecandelaere et al. (2012) state that "attitude towards mathematics is a vital matter in mathematics education" (p. 107). The teacher plays a vital role in shaping student attitudes. Brahier (2011) suggests that "teachers are encouraged to write a motivational springboard or engagement activity for a lesson plan. If, in the first five minutes of a lesson, we can convince the students that the topic for the day connects with their interests, the class is more likely to become motivated to learn" (p. 5).

A large study by DiMartino and Zan (2010) asked 1500 students from Kindergarten through university to write responses to the topic "Me and Maths." They found that most students began school with positive feelings about mathematics and about themselves as learners of mathematics. However, for the majority of students, there was a *turning point*, one instance, usually involving a teacher, which resulted in the student becoming negative toward the subject and their own image as a mathematics learner. Their work implies it is imperative that teachers pay attention to both the mathematical content and the affective dimensions of learners. The Social–Emotional Learning strand affirms that need within the Ontario curriculum.

References

Anderson, L.W., & Krathwohl, D.R. (Eds.). (2001). *A taxonomy for learning, teaching, and assessing: A revision of Bloom's taxonomy of educational objectives* (1st ed.). Addison Wesley Longman.

Bandura, A. (1997). *Self-efficacy: The exercise of control.* Henry Holt and Co.

Brahier, D.J. (2011). Motivation and disposition: The hidden curriculum. In D.J. Brahier & W.R. Speer (Eds.), *Motivation and disposition: Pathways to learning mathematics. NCTM 73rd yearbook* (pp. 1–8). National Council of Teachers of Mathematics.

Brown, R. (2007). Exploring the social positions that students construct within a classroom community of practice. *International Journal of Educational Research, 46*(3–4), 116–128. doi.org/10.1016/j.ijer.2007.09.012

Burkhart, J. (2017). How open-ended math problems keep on giving. www.middleweb.com/35704/how-open-ended-math-problems-keep-on-giving/

Campbell, C.M., & Cabrera, A.F. (2014). Making the mark: Are grades and deep learning related? *Research in Higher Education, 55,* 494–507. doi.org/10.1007/s11162-013-9323-6

Di Martino. P., & Zan, R. (2010). "Me and maths": Towards a definition of attitude grounded in students' narratives. *Journal of Mathematics Teacher Education, 13,* 27–48. doi.org/10.1007/s10857-009-9134-z

Gillies, R.M., Nichols, K., Burgh, G., & Haynes, M. (2012). The effects of two strategies and meta-cognitive questioning approaches on children's explanatory behaviour, problem-solving, and learning during cooperative, inquiry-based science. *International Journal of Educational Research, 53*, 93–106. doi.org/10.1016/j.ijer.2012.02.003

Hattie, J. (2003, October). *Teachers make a difference: What is the research evidence?* Paper presented at the Australian Council for Educational Research Conference, Melbourne, Australia. http://research.acer.edu.au/research_conference_2003/4/

Hattie, J. (2009). *Visible learning.* Routledge.

Hattie, J., Biggs, J., & Purdie, N. (1996). Effects of learning skills interventions on student learning: A meta-analysis. *Review of Educational Research, 66*(2), 99–136. doi.org/10.3102/00346543066002099

Hufford-Accles, K., Fuson, K., & Gamoran Sherin, M. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education, 35*(2), 81–116. doi.org/10.2307/30034933

Irvine, J. (2014, May). The effect of attitudes and motivation on student achievement in mathematics. OAME Annual Conference, Toronto, Ontario.

Irvine, J. (2016). Fueling the mind: Maximizing student learning and performance. *Ontario Principals Council Register, February,* 6–14.

Irvine, J. (2017). Problem posing in consumer mathematics classes: Not just for future mathematicians. *The Mathematics Enthusiast, 14*(1), 387–412. https://scholarworks.umt.edu/tme/vol14/iss1/22

Irvine, J. (2018). A focus on student understanding: Georg Pólya's problem-solving heuristic. *Ontario Mathematics Gazette, 56*(4), 36–38.

Irvine, J. (2019a). Constructivism in secondary math education: A teacher's experience. *The Variable, 4*(2), 27–32. doi.org/10.2478/plua-2018-0001

Irvine, J. (2019b). Cognitive dissonance: Learning from happy accidents. *Ontario Mathematics Gazette, 57*(3), 42–44.

Irvine, J. (2019c). Conjecture, predict, and the power of What If? *Ontario Mathematics Gazette, 57*(3), 34–36.

Irvine, J. (in press-a). A little problem with big potential. *Ontario Mathematics Gazette.*

Irvine, J. (in press-b). Why we need to teach metacognition in our math classes. *Ontario Mathematics Gazette.*

Irvine, J., Telford, W., Anusic, V., & Alves, P. (2016). A straightforward model eliciting activity (MEA) and the power of What If? questions in supporting students' higher-order thinking. *Journal of Mathematical Sciences, 3*(1), 7–22.

Johnson, B. (2019). 4 ways to develop creativity in students. George Lucas Educational Foundation. www.edutopia.org/article/4-ways-develop-creativity-students

Kasmer, L., & Kim, O.K. (2011). Using prediction to motivate personal investment in problem solving. In D.J. Brahier & W.R. Speer (Eds.), *Motivation and disposition: Pathways to learning mathematics. NCTM 73rd yearbook* (pp. 157–167). National Council of Teachers of Mathematics.

Leikin, R., Subotnik, R., Pitta-Pantazi, D., Singer, F.M., & Pelczer, I. (2013). Teachers' views on creativity in mathematics education: An international survey. *ZDM Mathematics Education, 45*, 309–324. doi.org/10.1007/s11858-012-0472-4

Lev-Zamir, H., & Leikin, R. (2013). Saying versus doing: Teachers' conceptions of creativity in elementary mathematics learning. *ZDM Mathematics Education, 45*, 295–308. doi.org/10.1007/s11858-012-0464-4

Liljedahl, P., & Sriraman, B. (2006). Musings on mathematical creativity. *For the Learning of Mathematics, 26*(1), 17–19.

Mann, E.L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted, 30*(2), 236–260. doi.org/10.4219/jeg-2006-264

Marzano, R.J., & Kendall, J.S. (2007). *The new taxonomy of educational objectives* (2nd ed.). Corwin Press.

Meijer, J., Veenman, M.V.J., & van Hout-Wolters, B.H.A.M. (2006). Metacognitive activities in text-studying and problem-solving development of a taxonomy. *Educational Research and Evaluation, 12*(3), 209–237. doi.org/10.1080/13803610500479991

Ontario Ministry of Education. (2010). *Communication in the mathematics classroom.* Capacity building series. Queen's Printer for Ontario.

Ontario Ministry of Education. (2011). *Bansho (board writing)*. Capacity building series. Queen's Printer for Ontario.

Ontario Ministry of Education. (2020). *The Ontario curriculum, grades 1–8: Mathematics. Queen's Printer for Ontario.* www.dcp.edu.gov.on.ca/en/ curriculum/elementary-mathematics

Phillips, C.J. (2016). *The new math: A political history.* University of Chicago Press.

Ross, S.P. (2008). *Motivation correlates of academic achievement: Exploring how motivation influences academic achievement in the PISA 2003 dataset* [Doctoral dissertation]. University of Victoria, BC. https://dspace.library.uvic.ca/handle/1828/3209

Ruesch, J. (2018). Bringing wonder into mathematics. *George Lucas Educational Foundation.* www.edutopia.org/article/bringing-wonder-mathematics

Salyers, F.S. (2017). Social–emotional intelligence is missing from school. Here's why it matters. *Education Week Teacher*. http://blogs.edweek.org/teachers/teaching_ahead/2017/06/emotional_and_social_intelligence_is_missing_from_american_education.html

Schneider, W., & Artelt, C. (2010). Metacognition and mathematics education. *ZDM Mathematics Education, 42*(2), 149–161. doi.org/10.1007/s11858-010-0240-2

Schunk, D.H., & Richardson, K. (2011). Motivation and self-efficacy in mathematics education. In D.J. Brahier & W.R. Speer (Eds.), *Motivation and disposition: Pathways to learning mathematics. NCTM 73rd yearbook* (pp. 13–30). National Council of Teachers of Mathematics.

Small, M. (2009). *Good questions great ways to differentiate mathematics instruction.* Teachers College Press.

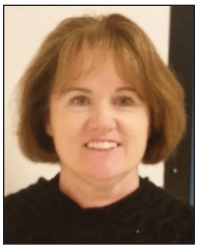
Smith. M., & Stein, M. (2011). *5 practices for orchestrating productive mathematics discussions*. National Council of Teachers of Mathematics.

Vandecandelaere, M., Speybroeck, S., Vanlaar, G., De Fraine, B., & Van Damme, J. (2012). Learning environment and students' mathematics attitude. *Studies in Educational Evaluation, 38*(3–4), 107–120. doi.org/10.1016/j.stueduc.2012.09.001

Veenman, M.V.J., Van Hout-Wolters, B.H.A.M., & Afflerbach, P. (2006). Metacognition and learning: Conceptual and methodological considerations. *Metacognition Learning, 1*(1), 3–14. doi.org/10.1007/s11409-006-6893-0

Zollman, A., Smith, M.C., & Reisdorf, P. (2011). Identity development: Critical component for learning in mathematics. In D.J. Brahier & W.R. Speer (Eds.), *Motivation and disposition: Pathways to learning mathematics. NCTM 73rd yearbook* (pp. 43–53). National Council of Teachers of Mathematics. ▲

▲ OAME/NCTM Report: SEL, That Is Social–Emotional Learning... Take Two. (That is, part 2 of 2, a bit more in depth.)



JACQUELINE HILL
jacqueline.hill@bell.net

Jacqueline Hill is a Grades 9–12 teacher of mathematics. She is a Past President of OAME and OMCA, as well as the recipient of the award for Exceptional Teaching in Secondary Mathematics. She also wrote the "Director's Dialogue" for the Gazette for a number of years.

Being a member of NCTM gives you access to their extensive research library. It gives you access to each of the books and journals that are within your membership scope. Here are five such articles to which you, as a member, can have access.

1) Mathematical Connections
   By Sandra M. Linder and Amanda Bennett

   Amanda Bennett is a doctoral candidate in learning sciences, with a cognate in early childhood education, at Clemson University in South Carolina. Her research centres on social–emotional development. The article appears in the April 2020 edition of *Mathematics Teacher: Learning and Teaching PK–12*. (Note that full references are provided at the end of this column.)

2) Teacher Care and Students' Sense of Connectedness in the Urban Mathematics Classroom
   By Tanya Maloney and Jamaal Sharif Matthews

   "…conversations with students to mindfully address social, emotional, and mathematical needs."

   In the July 2020 issue of *Journal for Research in Mathematics Education.*

3) Then: Equity and Access
   By John W. Staley

   "…social, emotional, and academic needs in a similar manner as we do when preparing to teach a new mathematics concept? What shifts in your instruction and language do you need to consider?"
   In the July 2020 issue of *Journal for Research in Mathematics Education.*

4) Chapter 4: Implementing Equitable Mathematics Instruction

   "…language and cognitive skills; educators' own experiences and cultural influences; and the cultural, social, cognitive, and emotional characteristics of the learning environment" (pp. 6–7).

   In the NCTM book *Catalyzing Change in Middle School Mathematics: Initiating Critical Conversations.*

5) Effects on Mathematics and Executive Function of a Mathematics and Play Intervention Versus Mathematics Alone
   By Douglas H. Clements, Julie Sarama, Carolyn Layzer, Fatih Unlu, and Lily Fesler

   "…challenge for parents and educators. In D.G. Singer, R.M. Golinkoff, & K. Hirsh-Pasek (Eds.), *Play = learning: How play motivates and enhances children's cognitive and social–emotional growth."*

   In the May 2020 issue of *Journal for Research in Mathematics Education.*

Part of my last column spent time discussing ways in which online teaching can provide meaningful connections to students. One such way for teachers to keep track is to use a notepad to jot down details, such as, "Luca's dog is named Bandit, and Clare's sick cat is Mr. Pepper." Such observations are conversation starters for "check-in" with students.

Keep your nctm.org web page as a favourite for cutting-edge research in mathematics education.

Respectfully submitted,
Yours in Mathematical fun,
Jacqueline Hill.

References

Clements, D.H, Sarama, J., Layzer, C. Unlu, F., Fesler, L. (2020). Effects on mathematics and executive function of mathematics and play intervention versus mathematics alone. *Journal for Research in Mathematics Education, 51*(3), 301–333. doi.org/10.5951/jresemtheduc-2019-0069

Lindor, S.M., & Bennett, A. (2020). Leveraging read alouds for *mathematical connections. Mathematics Teacher: Learning and Teaching PK–12, 113*(4), 317–321. doi.org/10.5951/MTLT.2019.0215

Maloney, T., & Sharif-Matthews, J. (2020). Teacher care and students' sense of connectedness in the urban mathematics classroom. *Journal for Research in Mathematics, 51*(4), 399–432. doi.org/10.5951/jresematheduc-2020-0044

National Council of Teachers of Mathematics. (2020). *Catalyzing change in middle school mathematics initiating critical conversations.* Author.

Staley, J.W. (2020). Then: equity and access. *Journal for Research in Mathematics Education, 113*(9), 759–761. doi.org/10.5951/MTLT.2020.0080 ▲

▲ Fields Institute MathEd Forum Report



ANGELICA MENDAGLIO
angelicamendaglio@gmail.com

Angelica Mendaglio is an instructional designer at Vretta Inc. in Toronto, Ontario, where she helps to create interactive digital mathematics lessons and activities for middle school students.

The November meeting of the MathEd Forum was an unusual one. Instead of a morning program where participants heard from a few presenters on their research or experiences on a theme, I was listening to a delightful conversation between Parker Glynn-Adey (a member of the Forum's Steering Committee), and Conrad Wolfram (a technologist known for his work on bringing technology into mathematics classrooms) on the topic "Thinking Computationally."

The conversation began with an introduction to Wolfram's new book, *The Math(s) Fix*, which he described as beginning to percolate following his popular 2010 TED Talk "Teaching Kids Real Math with Computers." He said that the 15- to 20-minute format was a great way to introduce a topic, but it did not afford him the opportunity to go over all of his thoughts on how technology could enhance the mathematics classroom. After the TED Talk, viewers continued to ask him questions, which he has since thought about deeply, and he created the book to respond to the questions in a format that would be accessible to a wide range of math education stakeholders, from policy-makers to mathematics enthusiasts.

Wolfram described his website, www.computerbasedmath.org, as a successor to Seymour Papert's work of bringing technology into the classroom. In particular, Papert co-invented the programming language Logo, which was created to help teach students to learn how to code. Wolfram spoke about how the availability of computers was a significant challenge in Papert's time, when a university might have access to only a single computer. However, now that computers are ubiquitous, it seems to Wolfram to be particularly absurd to have students completing computations by hand, especially in a Zoom classroom, where the student is attending the class, using a device that would be much better at performing that task. Additionally, coding languages have become "high level" enough that some languages require minimal training to use. For example, Wolfram Alpha (which solves various math problems with full solutions) is simple enough to use that middle school students can pick it up without prior training. It seems natural to teach students to make use of these tools, which are often readily available to them; however, this does mean a shift in thinking about what mathematics problems the students should be working on.

Wolfram promoted the use of "fuzzy problems" in the math classroom—problems that are not well-defined and must be interpreted by the student, such as "Am I normal?" These types of problems can be tackled using the four-step problem-solving process in computer-based math classes, where problem solvers must first define the problem precisely. The second step in the process is to abstract the problem into a mathematical form, so that all the powerful tools for solving problems in a mathematical form can be deployed. The third step is to use those tools to find the answer to the mathematical problem. The final step is to interpret the answer within the context of the initial question and reflect on whether this is a reasonable answer, or if it could be improved by making different considerations in the definition or abstraction of the problem. Wolfram sees the use of coding in the math classroom as bolstering both the second and third steps of this process. Students need to abstract their problem into a form that a computer can understand, which can sometimes be different and more flexible than what a student can achieve with a mathematical form alone, before having the computer perform the algorithms to produce an answer.

The "fuzzy problems" that Wolfram discussed can help students become more engaged with a problem (as they don't have the artificial flavour of many problems created to be easily represented in the mathematical form). They also give students a better opportunity to interpret the problem and make decisions as to what they think will be important before abstracting and solving the problem.

The conversation turned to self-taught coders, and Wolfram was asked how elementary and middle school teachers could foster what seems to be a natural curiosity of many students for programming. Wolfram said that starting with a problem that will interest the students is critical. This can be a challenge because many math problems and curricula are written by people who find math inherently interesting, and this can exclude those who would see math more as part of a toolkit for solving problems. He also recommended embedding coding as part of math and other subjects that the students are studying, rather than isolating it. Finding different hooks for different students is important, as each student may be interested in different aspects of coding—some may be interested in the abstract aspect of it, while others will appreciate the practicality of coding for solving problems.

The conversation was compelling and thought-provoking. I highly recommend looking up the video of the November meeting and experience it for yourself! Since COVID-19 has necessitated that the Forum meetings take place digitally over Zoom, all of the meetings have been recorded. They are included in a MathEd Forum playlist on the Fields Institute's YouTube channel.

The Fields Institute MathEd Forum will be convening on the last Saturday in January, February, March, and April. Visit the Fields Institute website to register. ▲

▲ Technology Corner: Between 2 Numbers, Math Talks, and Chewable Math



MARY BOURASSA
mary.bourassa@ocdsb.ca

Mary Bourassa teaches mathematics at West Carleton Secondary School in Ottawa. She is a strong advocate for the appropriate use of technology in the classroom. She has presented workshops internationally, authored mathematics resources, is a past VP of OAME, and a Past President of COMA. An award-winning teacher, Mary continually strives to learn new and better ways of helping students learn and love mathematics.

Visual Patterns (www.visualpatterns.org/) is an amazing site to help students and teachers alike build relationships between a pattern and its algebraic representation. I last wrote about Visual Patterns in 2013, and since then, the site has continued to grow. Its creator, Fawn Nguyen, is a remarkable educator in California. In addition to Visual Patterns, she has created three other websites with resources for teachers.

Between 2 Numbers (www.between2numbers.com/)

Between 2 Numbers features pairs of objects and asks a question relating the two. Most questions on the site use middle school mathematics, though all could be used in higher grades as warm-ups or starting points for discussions. Figures 1, 2, and 3 show examples of rate and ratio questions.





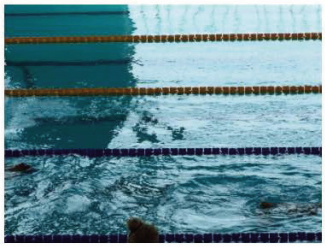
A garden snail's average speed is about 0.029 mi/h.

The North American X-15's top speed is about 4,520 mi/h.

How much time would it take for each to travel across the United States, a distance of about 2,800 miles?

Figure 1: *Examples of rate and ratio questions*





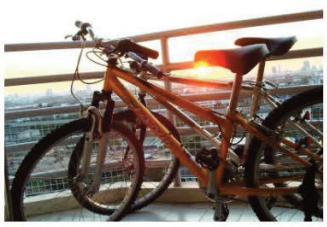
The average flow rate of a kitchen faucet is about 8.3 liters per minute.

An Olympic-sized pool can hold about 2,500,000 liters of water.

About how many *months* would it take to fill up an Olympic-sized pool with a kitchen faucet?

Figure 2: *Examples of rate and ratio questions*





The Bugatti Veyron's top speed is about 408.84 km/h (254.04 mph) and costs around $2 million.

A typical top speed of a bicycle is about 40 km/h (25 mph) and costs about $1,000 for a nice one.

What is the ratio of bicycle to Bugatti in terms of speed per dollar?

Figure 3: *Examples of rate and ratio questions*

Figures 4 and 5 involve unit conversions, a skill with which many high school math and science students could use extra practice.





The volume of Mt. Everest is about 2.1 trillion cubic feet.

A dump truck, on average, can haul 14 cubic yards.

How many dump trucks would it take to haul away Mt. Everest?

Figure 4: *Examples involving unit conversion*





Echolocation is the use of sound waves and echoes that bats and whales use to navigate and find food. A sperm whale can "see" objects via echolocation from 500 meters away.

A human with "normal" or 20/20 vision can clearly see something from 20 feet away.

How many times farther can a sperm whale "see" compared to a human?

Figure 5: *Examples involving unit conversion*

The clear, crisp images in Between 2 Numbers will capture students' interest, and the question may elicit responses from students who would not normally participate. Together, these mathematical fun facts, as Fawn puts it, will help develop numeracy skills. The comparisons made will also help students with their estimation skills and help them determine whether answers are reasonable. These numeracy skills are important, to develop and nurture to make our students mathematically literate members of society.

Answers to all Between 2 Number entries are available from Fawn upon request.

Math Talks (www.mathtalks.net/)

The primary goal of number talks/math talks is computational fluency. These are short, ongoing opportunities for students to think, reason, and communicate about mathematics. Students complete these using mental math strategies, without writing anything down. Fawn's site has 28 number talk prompts and another 28 pattern talk prompts, along with student responses to each one. These responses are particularly valuable to help teachers better anticipate their own students' responses. Figure 6 shows an example of a prompt and four student responses. Note the wonderful, visual way that Fawn demonstrated that $79 \times 25 \neq (70 \times 20) + (9 \times 5)$.

5. Which is greater, 79 x 25 or 75 x 29?

Mike Lawler shares a post and videos of his two sons doing this same problem.

Student 1

- 79 x 25 is greater. I remember a strategy that was shared last week, and that's what I did:
- 79 x 25 = (75 x 25) + (4 x 25)
- 75 x 29 = (75 x 25) + (75 x 4)
- So, clearly 75 x 4 is greater than 25 x 4.

The student realized that he meant to say 75 x 29 is bigger based on strategy that he'd used.

Student 2

- 79 x 25 is greater. I rounded the numbers.
- 79 x 25 -> 80 x 20 = 160 (I multiplied 8 times 2, then added a 0.)
- 75 x 29 -> 70 x 30 = 120 (I multiplied 7 times 3 to get… oh, wait, that's 21… it should be 210)

Other students pointed out to Student 2 that she didn't add enough zeros to her answers.

Student 3

- 75 x 29 is greater because…
- 75 x 29 -> 75 x 30 = 2250 (I multiplied 75 and 3, added a 0.)
- 79 x 25 -> 80 x 25 = 2000 (I know 4 times 25 is 100, so 8 times 25 is 200, then I added a 0.)
- I'm 99% confident my answer is correct.

Student 4

- I think they're both equal because…
- 75 x 29 = (70 x 20) + (5 x 9)
- 79 x 25 = (70 x 20) + (9 x 5)

A student reminded Student 4 that many students last week did break the problem down as she did, but they were wrong. Because of this recurring mistake, I explained to the class why breakdown of the product does not work or is insufficient via this rectangular model below.

$79 \times 25 \neq (70 \times 20) + (9 \times 5)$

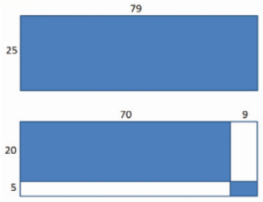


Figure 6: *Example of a prompt and four student responses*

Fawn has also written about the structure she uses when she does a math talk. Each math talk is done at the beginning of class, so the prompt is already projected as students walk in. When class begins, she sets a timer for 1 to 3 minutes, depending on the question. During that time, students quietly, using only mental math, estimate the answer. When the timer finishes, she resets it to allow students to discuss with their neighbour for 1 minute. Then she calls upon three students to share their thinking, which she and the other students all record. The class is then asked if anyone used a strategy that was not already shared.

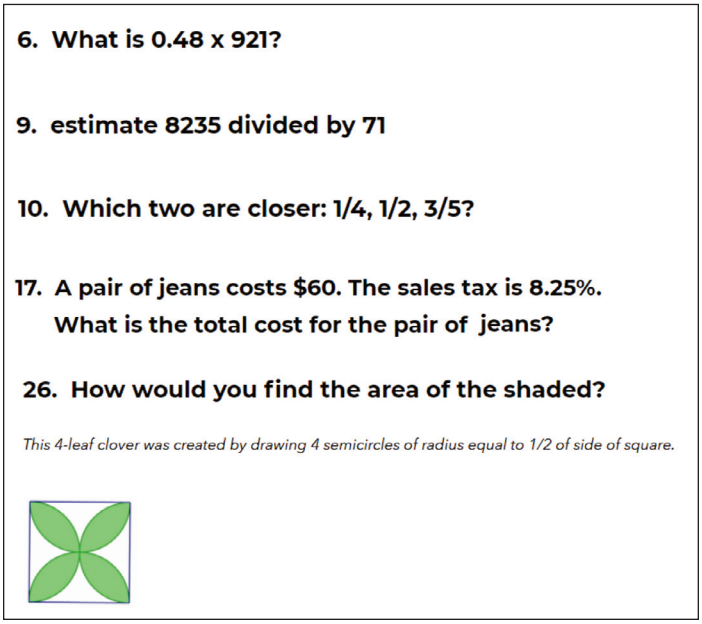


Figure 7: *Prompts from Math Talks website*

Figure 7 shows a few different prompts from the Math Talks site.

Chewable Math (www.chewablemath.com/)

Chewable Math combines mathematics with food. While students are doing mathematics, they are also learning some food facts, information about daily nutritional requirements, and some food science.

Figure 8 shows information about avocados. The first question (clearly in American dollars—if only I could buy a bag of 6 Hass avocados for $3.99!) has students determine the savings per avocado. This question alone would also make a good math talk. The second and third questions will allow students to practise unit conversions in a real-world context.

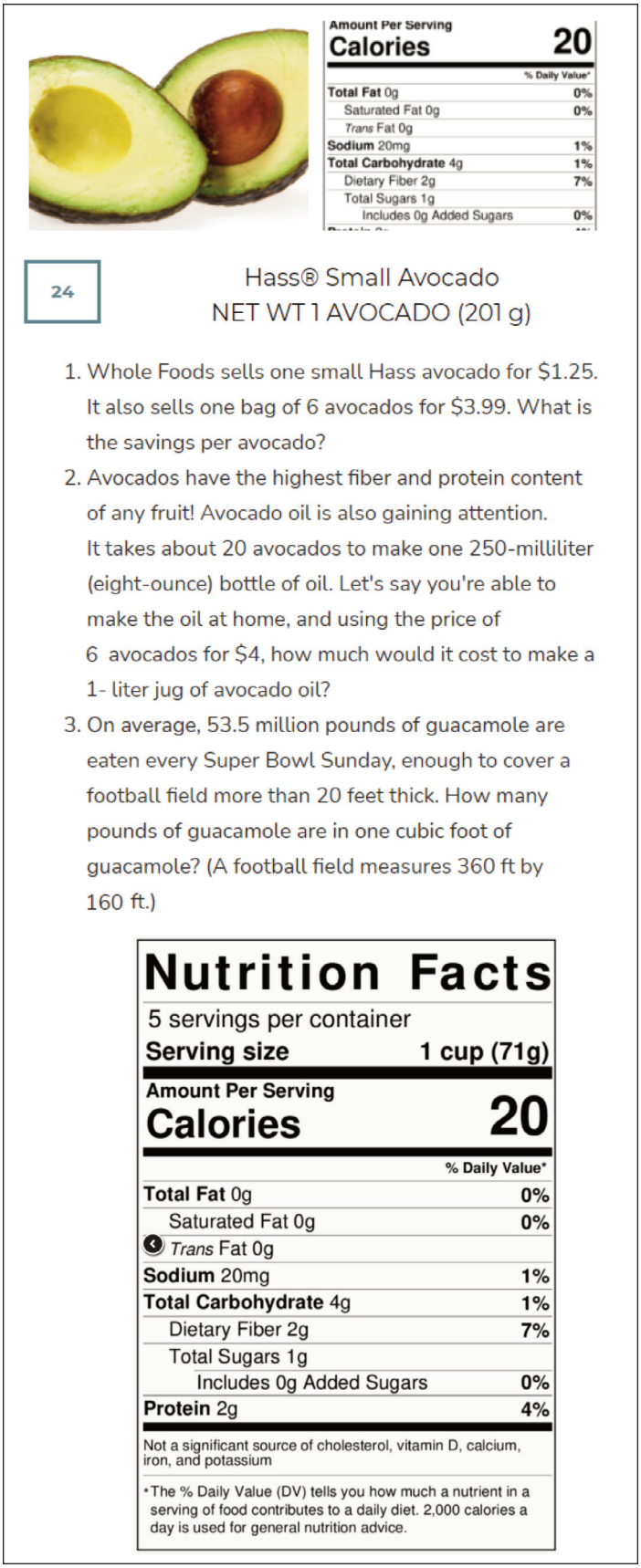


Figure 8: *Information about avocados that allows students to practise unit conversion*

Figure 9 has one question involving percent increase, followed by what the term "organic" means, and listing the benefits of flavonoids. This creates a very cross-curricular activity.



Amount Per Serving	
Calories	**140**
	% Daily Value*
Total Fat 0g	0%
Saturated Fat 0g	0%
Trans Fat 0g	
Sodium 0mg	0%
Total Carbohydrate 33g	12%
Dietary Fiber 4g	14%
Total Sugars 30g	
Includes 0g Added Sugars	0%

28

Natural Delights® Pitted Organic Fresh Medjool Dates
NET WT 12 OZ (340 g)

1. Whole Foods sells this package for $8.29. The store also sells organic Medjool dates in their produce section, without the packaging, for $8.29 per pound. What is the percent increase in cost when we buy this item packaged?
2. All dates are grown on date palm trees, similar to coconut palm trees. Many items sold at Whole Foods are organic, including these dates. What is meant by organic?
3. Most of the calories in dates come from carbs. Despite their calories, dates have a significant amount of fiber. Dates are also high in antioxidants, such as flavonoids. What are the health benefits of flavonoids?

Figure 9: *Questions regarding Medjool dates*



Amount Per Serving	
Calories	**170**
	% Daily Value*
Total Fat 0g	0%
Saturated Fat 0g	0%
Trans Fat 0g	
Sodium 0mg	0%
Total Carbohydrate 35g	13%
Dietary Fiber 2g	7%
Total Sugars 0g	
Includes 0g Added Sugars	0%

30

Bombay® Brown Basmati Rice
NET WT 10 LB (4.5 kg)

1. Amazon sells this sack of rice for $52.72. Costco sells its Kirkland Signature® Jasmine Rice in a 25-lb sack for $21.99. What is the unit cost ratio of the basmati to jasmine?
2. The two largest rice-producing countries are China and India. In the late 20th century, the world rice crop averaged 850 billion pounds annually and was cultivated on an average of about 358 million acres. How many pounds per acre is this?
3. Rice comes in *thousands* of varieties other than basmati and jasmine! You may have heard of sushi, arborio, long-grain vs. short-grain, white, brown, and others. Rice contains amylopectin starch and amylase starch. What is the difference between the two starches?

Figure 10: *Questions about rice*

Figure 10 asks about unit cost ratio between two types of bags of rice, followed by some history, math, and science.

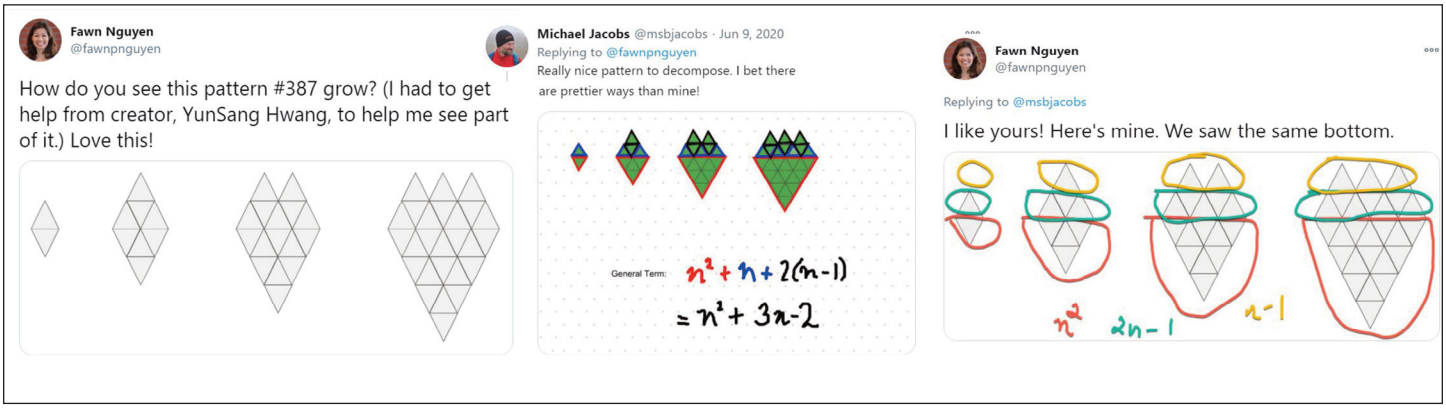


Figure 11: *Visual pattern and student responses*

A reminder to readers that Fawn's Visual Patterns site continues to grow with wonderful patterns like the one shown in Figure 11. Fawn occasionally shares new patterns on Twitter. Other teachers may respond with how they see the pattern growing. These lovely interactions are an example of the strong math education community on Twitter.

Each of these sites gives teachers resources to help students reason about mathematics and communicate their thinking. They provide great prompts that can inspire you to create your own. The student work included makes the resource that much more valuable, and Fawn models good teaching in her tips and the methodology she shares. If you enjoy using any of these sites, be sure to send a thank you Fawn's way (@fawnpnguyen on Twitter) for her time and dedication to help us and our students become better mathematicians. ▲

OMCA NEWS

The Ontario Mathematics Coordinators Association (OMCA) continues to offer exciting opportunities for mathematics coordinators, consultants, and coaches throughout the school year.

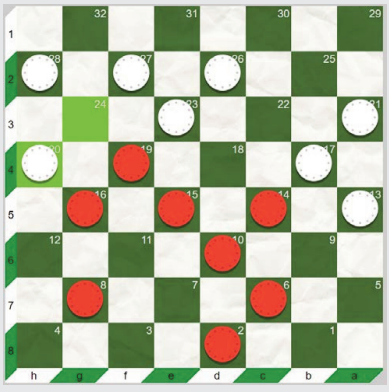
Recent members' meetings have included guest speakers Aleda Klassen and Carolyn Gingerich presenting *Assessment Practices to Keep After We're All Vaccinated*, as well as Dr. David Costello, presenting *Leveraging Literacy Practices in Math Instruction and Learning to Support the New Ontario Math Curriculum*.

OMCA's first virtual symposium on February 5 saw over 150 delegates from Timmins to Windsor come together to learn more about the day's theme of "Learning in a Virtual World." It was a full day of presentations by Mike Flynn, Faith Moynihan, Dr. Alex Lawson, Dr. Cathy Marks Krpan, Ryen Jackson, and Sandra Herbst, with music and door prizes to boot!

On March 1, in light of the new Grade 9 destreamed mathematics curriculum coming this fall, members were joined by Ministry representatives and Toronto District School Board educators in a special session to explore the "why" and the "how" behind destreaming.

For more details about OMCA, visit our website at www.omca.website (note the different domain), or contact us directly at OMCAmath@gmail.com.

Red to play and win (in several moves).
Solution page 51.



▲ Linking Literacy and Math: Going Beyond Surface Level When Problem-Solving (Part Three)



DAVID COSTELLO
dacostello@edu.pe.ca

Dr. David Costello is an author who has recently published Using What Works: Strategies for Developing a Literacy-Rich Environment in Math. *David is a vice-principal, who has held many roles. He taught in the primary division before assuming roles of numeracy interventionist, numeracy coach, numeracy leader, and curriculum consultant for Prince Edward Island. David has also instructed university courses focused on curriculum, differentiation, mathematics, and language arts. He facilitates professional learning sessions in the areas of mathematics and school development. His research situates mathematics within a literacy perspective and examines leveraging literacy-based instruction in mathematics.*

This is the third instalment of my focus on problem solving. The first instalment (September) examined the perspective of problem solving as a process, a verb, instead of a problem-solving question. I provided an overview of what the problem-solving process involved, and how it is necessary for students to understand and be able to apply problem-solving strategies. The second instalment (December) examined what it means to be teaching about problem solving, and some effective classroom strategies to support students in becoming problem solvers. This third instalment highlights the importance of metacognition in problem solving, and provides teachers with classroom strategies that will strengthen student metacognitive skills.

As described in the Ontario Math curriculum (Ontario Ministry of Education, 2020), the process of reflecting is a metacognitive process that occurs through the problem-solving experience. This article will provide four strategies to strengthen student metacognition as a way to support them as problem solvers. I will begin by explaining metacognition and why it is important to support this in students.

What Is Metacognition?

Metacognition is the ability to think about our thinking. Marzano (1998) referred to metacognition as the engine that drives thinking. In regards to student learning, metacognition will enable students to self-monitor their understanding as they engage with the task. They will be able to determine when and if they are making progress toward the solution. Without the ability to think about one's thinking, the focus is on doing math as opposed to thinking mathematically. In this way, Garofalo and Lester (1985) documented that mathematics instruction without metacognition is inadequate. It is imperative that students have opportunities to strengthen their metacognition and to be able to identify how such opportunities will assist them in their problem solving.

An important consideration in metacognition is feedback. Students must be provided with feedback so that they understand whether their thinking is correct or not. Such feedback will support students in having an accurate depiction of their thinking. This accuracy in metacognition will enable students to identify learning gaps and to be able to identify next steps in their learning journey.

Metacognitive Strategy #1: Bull's Eye

The Bull's Eye strategy involves students identifying where they are on the learning journey. It is not about whether they have solved a problem; it is about having students think about their thinking and effort, and recognize what they need to do to move forward as learners. It is about supporting students in understanding their thinking and having them be able to identify the next steps in their learning journey. Figure 1 shows an example of the Bull's Eye strategy that students can use as a visual to depict their placement on the learning journey.

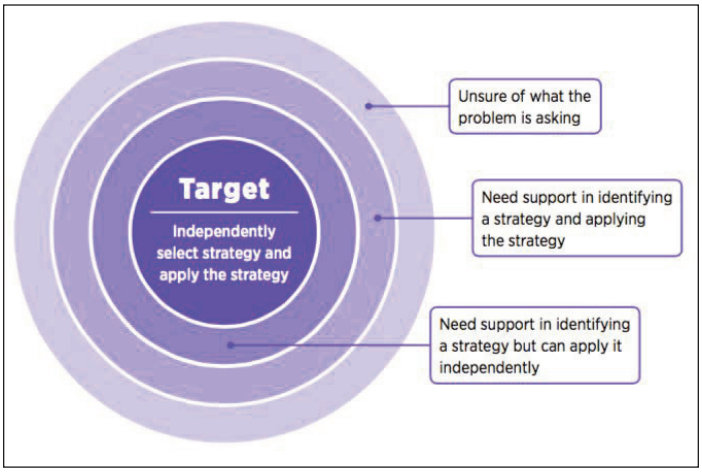


Figure 1: *Example of the Bull's Eye strategy (from Costello, 2019, with permission)*

In this metacognitive approach, students must be able to justify why they are depicted the way they are. The Bull's Eye strategy can be used as a whole-class visual, where students have a sticky note with their name on it and place this note in the appropriate band of the bull's eye. As an alternative, teachers may choose to provide individual copies that students can use to identify where they fit in, and their progress from their last self-assessment.

Metacognitive Strategy #2: Facilitating Discussion

Facilitating classroom discussions is a skill that continues to be explored in both literacy and mathematics. Although not a new skill, proficiency in facilitating classroom discussion can be a powerful tool in supporting instruction and learning. The four points outlined by Van de Walle et al. (2014, see Figure 2) support facilitating purposeful discussion. Recording these four points and then adding questions and comments are useful prompts to help ensure productive discussions.

Points	Notes
Allow time for students to share their work and thinking.	Provide wait time after asking for volunteers. Ensure that you have a time for sharing that will be uninterrupted. Don't rush to get as many students as possible to share.
Before sharing with the class yourself, take a step back and ask students to explain their ideas.	Don't say something yourself if a student could say it. Provide opportunities for students to share similarities and/or differences in their thinking.
Question student responses instead of answering student questions.	A few question stems: • What led you to think that? • Were there other strategies that you could have selected? • How does this relate to other problems we have solved? • How does this relate to what _____ shared?
Encourage student-tc-student dialogue as a form of learning.	Provide opportunities for students to work in small groups with assigned roles. Each role has a clear focus and can support discussion (cooperative learning). Provide sentence stems to students when engaged in dialogue: • _____'s idea is like my thinking because _____. • I like how you chose to _____. • Why did you choose to solve the problem that way? • What made you think of using that particular strategy?

Figure 2: *Four points facilitating classroom discussion (from Costello, 2019, with permission)*

Through the facilitation of discussion, students are provided opportunities to explore their thinking and to better understand the decisions they made when problem-solving. Each of the four points described can be used throughout the lesson to assist students in developing their metacognitive skills. Whether it be allowing time for students to share their thinking or asking students targeted questions, students will be engaged with activities that will require them to pause, think about their thinking, and communicate it with others, whether it be with the teacher and/or classmates.

Metacognitive Strategy #3: Think-Aloud

Think-alouds support metacognition. As students orally share what they are thinking, they are required to reflect upon the task, clarify what is being asked of them, and focus on the thinking points involved. Having students think aloud necessitates their having to go through the many decisions involved in the problem-solving process and to make those implicit aspects explicit. Think-alouds encourage students to organize their thinking because a sequential thought sequence supports explaining their thoughts to another student or to the teacher.

Students can be engaged in a think-aloud strategy at any point while problem-solving:

- At the beginning of the process, by outlining important aspects of the problem and sharing their plan for how they will work through the problem.
- While they are engaged with the problem, by slowing down and consciously thinking about the task they are involved with and the steps they are selecting to use to solve the problem.
- After the problem-solving process, while recounting their steps to consolidate their understanding and review their thinking points.

An important consideration of the think-aloud strategy is that it is not something that must be used every day by every student. Some days, it may be the teacher doing a think-aloud as an instructional practice to highlight a particular aspect of a concept or strategy. Other days, it may be one or two students engaged in a think-aloud. And still other days, there may not be the time or need for a think-aloud. What will dictate whether a think-aloud is necessary is if the teacher identifies an approach taken by a student that should be shared with the class, or if the teacher wants to strengthen the metacognitive skills of students as they engage with a problem.

The following points may be considered when employing think-alouds in the classroom:

- Select a problem that provides opportunities to encounter stumbling blocks.
- Select a problem that is open enough to require different strategies. For example: *There are two identical bookcases in the classroom of 15 students. Each bookcase has three shelves. The number of books on each shelf is more than the number of students in the classroom. How many books might there be in the two bookcases?*
- Frame your think-aloud in a way that highlights areas of understanding for students and common misconceptions within their work.
- Take frequent pauses to encourage student questioning of the process.
- Provide opportunities for students to talk with a partner before sharing a question.

Metacognitive Strategy #4: Journaling

Journals can be a metacognitive support as students work through a problem, whether used at the beginning, middle, or end of the problem-solving process. Such journals serve as a record of students' thinking, questions, and points of clarification, while helping to strengthen metacognition. Students can reflect on their thoughts, revise and edit their approach to the problem, and move their thinking forward as they record their thoughts. As students frame a problem, they can brainstorm a list of concepts or strategies applicable to the problem at hand and record their thinking in their journal. Journals are often used at the end of the problem-solving process, but they can also be a source of support for a problem at hand, and a source of help for problems that may arise in the future, when a journal can act as a reference source for past work.

Traditionally, students have a notebook where they write journal entries. Students regularly add their thoughts to this journal, which can be used as a tool while problem-solving. As an alternative, students could keep a digital journal, using a digital classroom application. With a digital journal, teachers can check in with students' writing on a regular basis because all of the work is easily accessible. Figure 3 shows an example of journaling from a Grade 4 student.

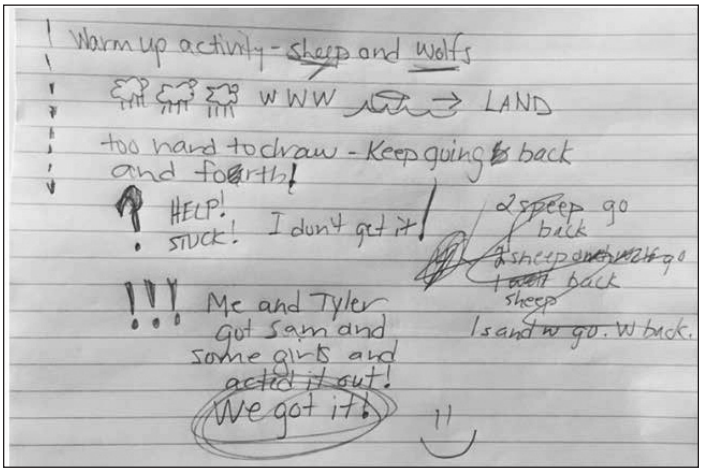


Figure 3: *Example of journaling (from Costello, 2019, with permission)*

This task required creative thinking, flexibility, and risk taking. This student used journaling to record his thinking so that his approach would be documented if he needed to return to a certain point in the problem. This specific journal entry involved students reviewing their thinking as they

engaged with the problem. While it has instances of reframing the language, it is primarily about metacognition. Within this particular journal entry, it is apparent how the student organized his thinking to approach and work through the problem. The student now has a record of his work and may be able to refer to his journal entry when he encounters similar problems in the future.

Final Thoughts

Problem solving is a process and an experience. It is not a question. For problem solving to go beyond the surface level, students must engage in metacognition, as it is this thinking that drives mathematical understanding. Going beyond surface level requires students to not only understand the problem in front of them, but also to recognize the many decision points within this experience. An important part of metacognition is that students have an accurate depiction of their thinking.

I look forward to continuing the discussion of linking literacy and math via Twitter (@dr_costello) or email.

References

Costello, D. (2019). *Using what works: Strategies for developing a literacy-rich environment in math.* Rubicon Publishing Inc.

Garofalo, J., & Lester, F.K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education, 16*(3), 163–176. doi.org/10.5951/jresematheduc.16.3.0163

Marzano, R.J. (1998). *A theory-based meta-analysis of research on instruction.* Mid-Continental Regional Educational Laboratory.

Ontario Ministry of Education. (2020). *The Ontario curriculum, grades 1–8: Mathematics.* Queen's Printer for Ontario.

Van de Walle, J.A., Lovin, L.H., Karp, K.S., & Bay-Williams, J.M. (2014). *Teaching student-centred mathematics: Developmentally appropriate instruction for grades pre-K–2.* Pearson Education, Inc. ▲

In the news… From the *Hamilton Spectator*, Nov. 18, 2020:

"The beaver actually knows mathematics," said Doug Dokis, director of the National Indigenous Youth in STEM (InSTEM) program for Actua, a charity that delivers science and technology programs to young people in Canada.

"When it builds a deep-water dam, it actually builds the foundation and then it puts wall joists up that are exactly 30 degrees."

When a tree is too big to drag down to the water, these industrious animals will build a log chute on a hillside, allowing the rain to wash the tree into the water below.

See www.thespec.com/news/hamilton-region/2020/11/18/what-can-beavers-teach-us-about-math-a-lot-say-indigenous-educators.html for the whole story.

▲ Mathematical Snapshots: Loving the Math in Kingston



RON LANCASTER
ron2718@nas.net

Ron Lancaster is an Associate Professor at the University of Toronto, where he teaches mathematics courses for pre-service middle and high school teachers. He has over 20 years of experience teaching Grades 7–12 mathematics. Ron's professional activities include consultations and conference presentations in North America, Asia, England, the Middle East, Africa, India, and Europe. Ron is an author for the NCTM (The Mathematical Lens) and member of the Advisory Board for the Museum of Mathematics in New York. He is the recipient of the 2015 Margaret Sinclair Memorial Award Recognizing Innovation and Excellence in Mathematics Education, awarded by the Fields Institute.

In 2012 and 2017, I was a speaker at the annual OAME Conference, which was held those years in Kingston. On both occasions, I drove from Hamilton to Kingston along back roads and stopped in small towns along the way to go for a walk, have something to eat, and take math photos. I took more math photos in Kingston and a few of them appear in this column. During the conference, I had a memorable dinner at Chez Piggy. Before entering the restaurant, I listened to two of my favourite songs—*Do You Believe in Magic?* and *You Didn't Have to Be So Nice*—but we'll come back to that.

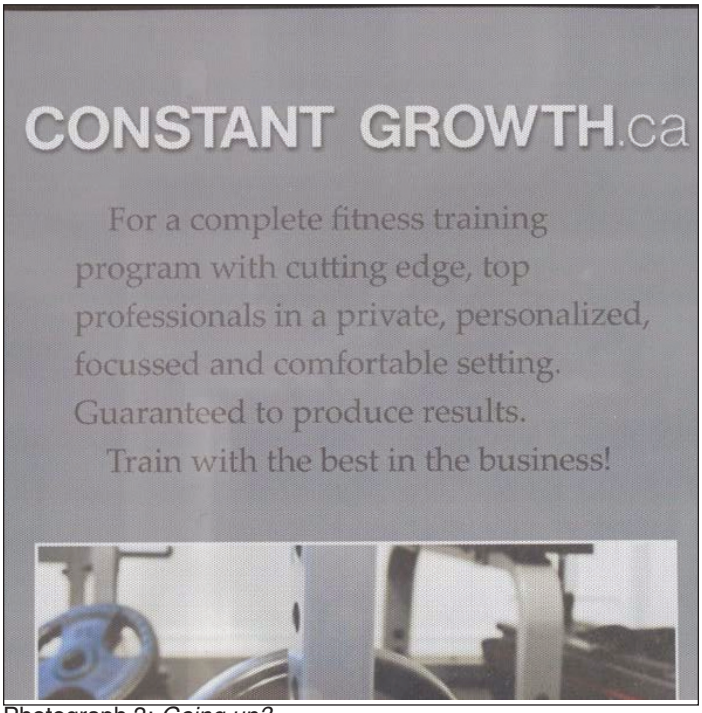
Before looking at the questions that follow, think of reasons why I took these photos. When you look at the photos, what do you notice? What are you wondering about?



Photograph 1a: *Dance on!*



Photograph 2: *What happened to curbside parking?*



Photograph 3: *Going up?*



Photograph 4: *Za in Kingston! Can that word be used in Scrabble?*

Question 1

(a) Without making any measurements or doing any calculations, put the areas of the five letters that spell the word “dance” (Photograph 1a) in order from smallest to largest.

(b) A close-up view of the letters is shown in Photograph 1b, and it appears that the underlying design used to make each letter is a square that is 5 units by 5 units. Assuming that this is the case, Ron Lancaster created each letter on a 5 x 5 square, as shown in Figure 1. Note that for the letter “d,” a second 5 x 5 square was used for the top part of the letter. Find the area of each letter, and rank the areas in order from smallest to largest. Is your ranking consistent with the estimate you made in part (a)?



Photograph 1b: *Close-up view of the letters*

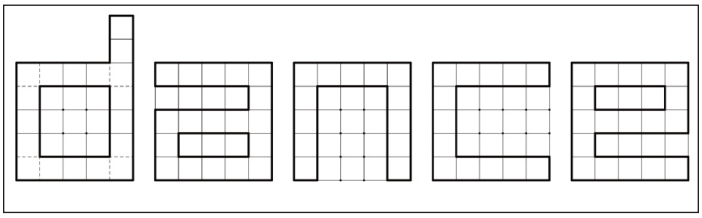


Figure 1: *Letters as squares*

(c) Create the rest of the alphabet using a 5 x 5 square, with an extra square, if needed, as was the case for the letter “d.” Which letter has the smallest area? Which one has the largest? What is the average area?

Question 2

(a) In Photograph 2, note that the cars are parked at an angle instead of parallel to the street. Estimate the angle between the side of a car and the curb.

(b) What are some advantages and disadvantages for using angled parking?

Question 3

Figure 2 shows the default settings for the Parking Planet Calculator. This calculator is used for determining the dimensions of parking lots where angle parking is used. Tables 1 and 2 show the same information, and have been provided, in case the image is not clear.

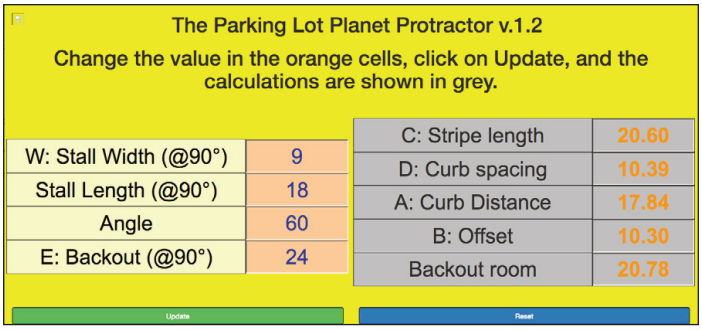


Figure 2: *Parking Planet Calculator*

Table 1	
W: Stall Width (@900)	9
Stall Length (@900)	18
Angle	60
E: Backout (@90°)	24
Table 2	
C: Stripe Length	20.60
D: Curb Spacing	10.39
A: Curb Distance	17.84
B: Offset	10.30
Backout Room	20.78

Source: www.store.parkinglotplanet.com/protractor/plp_protractor_1p2.htm

(a) Use the calculator to determine each of the quantities in Table 3. Analyze the data to determine how the six quantities in the table depend on the Stall Width. Suggest possible mathematical models for each quantity as a function of the Stall Width.

Table 3					
W: Stall Width (@90°)	C: Stripe Length	D: Curb Spacing	A: Curb Distance	B: Offset	Backout Room
9					
10					
11					
12					
13					
14					
15					
16					
17					
18					

(b) Carry out a similar analysis to explore how the remaining four quantities in Table 4 depend on the remaining three quantities in Table 1.

(c) Use the information on the website for the calculator, or other websites, if necessary, to find out the meaning and units of the quantities in Tables 1 and 2. Draw a diagram of an example of angle parking and identify each of these quantities.

Question 4

(a) In the context of the business whose sign is shown in Photograph 3, what does constant growth mean?

(b) In mathematics, what does constant growth mean? Provide at least five different examples where constant growth is mentioned in school mathematics.

(c) What does non-constant growth mean? Give an example where one would encounter non-constant growth.

Question 5

(a) Estimate the number of pieces of pepperoni on the pizza shown in Photograph 4.

(b) Use your estimate from part (a) to calculate the number of pieces of pepperoni there would be on a pizza twice as big. What assumptions have you made?

Question 6

The idea of placing pieces of pepperoni on a circular pizza is related to an area of mathematics called packing problems. In these problems, the goal is to try to cover as much of an area with a given number of circles (or other geometric objects). For example, Figure 3 shows the best-known way of packing 11 circles, each with a diameter of 1 unit, into a circle whose diameter is 4 units.

(a) Find the percentage of the large circle that is not covered by the 11 circles.

(b) The centres of the 11 circles are given in Table 4. It appears that 8 of these circles are tangential to the large circle. Pick one of the circles and determine if it is or is not tangential to the large circle.

(c) Can your work from part (b) be used to immediately determine if any other circles are or are not tangential to the large circle? Why or why not?

(d) Figure 4 shows two extra circles, with centres at A and B, added to the 11 circles in Figure 3. The circle, whose centre is at A, is tangential to the circle below it and to the big circle. The circle, whose centre is at B, is tangential to the neighbouring 4 circles.

Find the exact coordinates of A and B. Find the exact diameters of the circles with centres at A and B. How close are these diameters to the diameters of the 11 circles?

(e) Using a collection of circular objects (poker chips, coins etc.), what is the maximum number of these circles that you can pack into a circle whose diameter is equal to five times that of the small circles. Give it your best effort and then compare your result with the best-known result available at packomania.com.

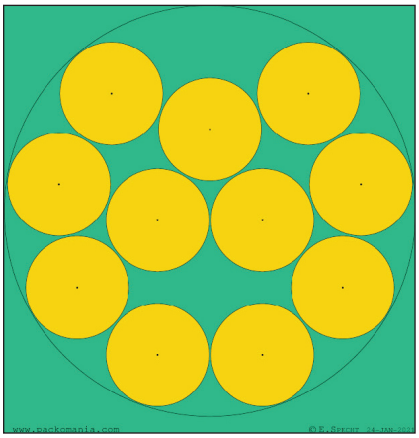


Figure 3: *The best-known way of packing 11 circles, each with a diameter of 1 unit, into a circle whose diameter is 4 units. (Source: www.packomania.com/)*

Table 4	
x	*y*
-0.50970940	-1.40041508
0.50970940	-1.40041508
-1.29062952	-0.7451453
1.29062952	-0.7451453
-0.50970940	-0.08987552
0.50970940	-0.08987552
-1.46764973	0.25878625
1.46764973	0.25878625
0.00000000	0.79296706
-0.95794033	1.14162883
0.95794033	1.14162883

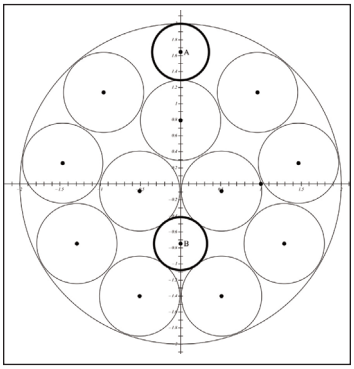


Figure 4: *Two extra circles added to the best-known way of packing 11 circles, each with a diameter of 1 unit, into a circle whose diameter is 4 units.*

Question 7

Why did Ron Lancaster listen to the songs mentioned in the introduction before he entered the restaurant? (Hint: See https://en.wikipedia.org/wiki/Zal_Yanovsky.) ▲

▲ In the Middle: Making Choices to Create a Portfolio



CARLY ZINIUK
carlyziniuk@gmail.com

Carly Ziniuk teaches Grade 9 Mathematics, Grade 12 Data Management, and Advanced Placement Statistics at the Bishop Strachan School in Toronto, Ontario, Canada. She is very active in adopting real-life data to engage her students in solving problems.

In June, 2020, I had my Grade 9 students create a portfolio of work to demonstrate their overall learning. Fundamental to this portfolio was giving students the opportunity to make choices for how and what to display that demonstrated their learning. I was surprised and impressed by the items they chose; the selected items in their portfolio revealed what often was not apparent in other types of assessment. With remote teaching leading to greater independence, providing students agency to decide emphasized this. With the new school year combination of online and in person, through both synchronous and asynchronous learning, I sought to embed the idea of choice throughout the course. This experience also provoked me to consider providing alternate opportunities for individual units, and throughout the course as a whole by having students curate both unit and course portfolios of work.

As an example, for a unit, last fall, I established a series of activities focusing on relationships. Some of these activities were done asynchronously, with others completed synchronously, and the students' choices exemplified their learning of distance–time relationships. They connected various representations—graphs, tables, slopes/speeds, stories—and used data management to investigate them. The goal was to select the items that illustrated students' overall understanding based on the success criteria for the unit. Figure 1 shows the components the students could complete, and how they recorded their chosen portfolio components. Many thanks to Dora Bellos, a technically gifted colleague, who created this checkbox list in Google Forms, keeping each student's work organized and in one place, readily documenting their progress. The 14 components of this portfolio, as listed in Figure 1, include the following areas: *Google Forms*, a series of skill multiple-choice quizzes; *Window Panes*, multiple representations

exhibited in different modalities; *Carolyn and Mary Story*, a written story submitted in drafts; *Card Sort*, a matching Desmos activity; *Beetle and Carolyn*, a technology activity done in Teacher Mode. These portfolio components and how they were implemented are the focus of this column.

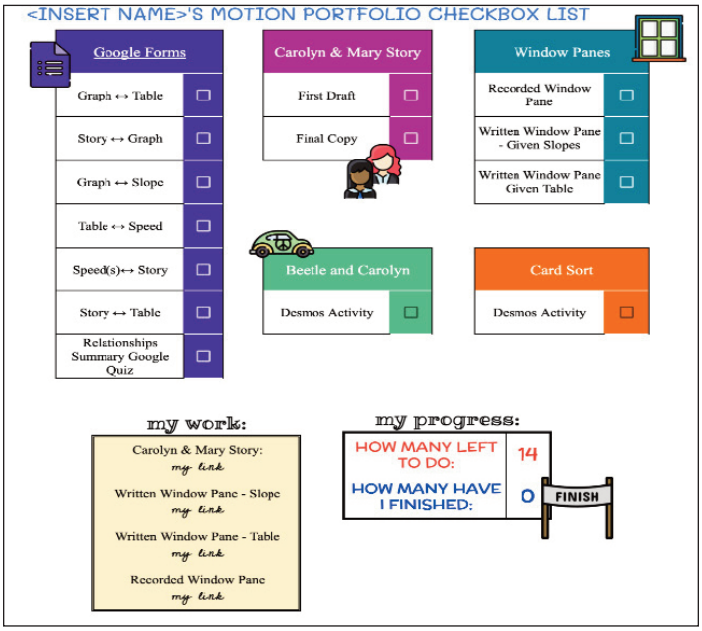


Figure 1: *Student's portfolio checkbox list in Google Forms*

On the left of Figure 1, the *Google Forms* component of the portfolio is a series of quizzes, generated using Google Forms. In these quizzes, students matched graph, table, slope/speed, and story, in pairs, with multiple-choice questions, and a comprehensive overall summary quiz. By choosing to complete these for their portfolio, students became proficient in the individual skills, and were able to go in both directions for each pair of representations. Figure 2 shows two examples from the "Graph ⟷ Slope" quiz, where the first section of the quiz had students choose the graph that matched the given slope, and the second section where the opposite occurred. Students could complete these Google Form quizzes asynchronously or synchronously in any order, with validity supported by the random selection of questions from a large bank of questions. Students used the quizzes to check their understanding; some students used them as assessment *for* learning, while others used them for assessment *of* learning. Any discussion between students about quiz questions constituted collaborative peer learning, while the random selection of questions ensured the students needed to focus on the concept, and not the specifics, of individual questions. Students felt emboldened to review the feedback and document their errors, and tried the quizzes repeatedly, in order to include their best work in their portfolio.

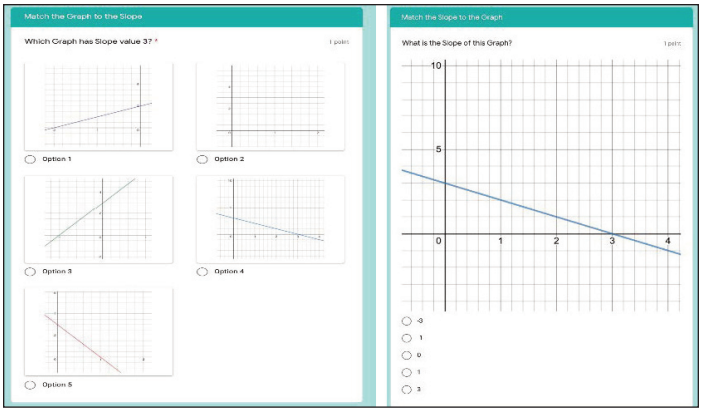


Figure 2: *Example questions from the graph ↔ slope Google quiz*

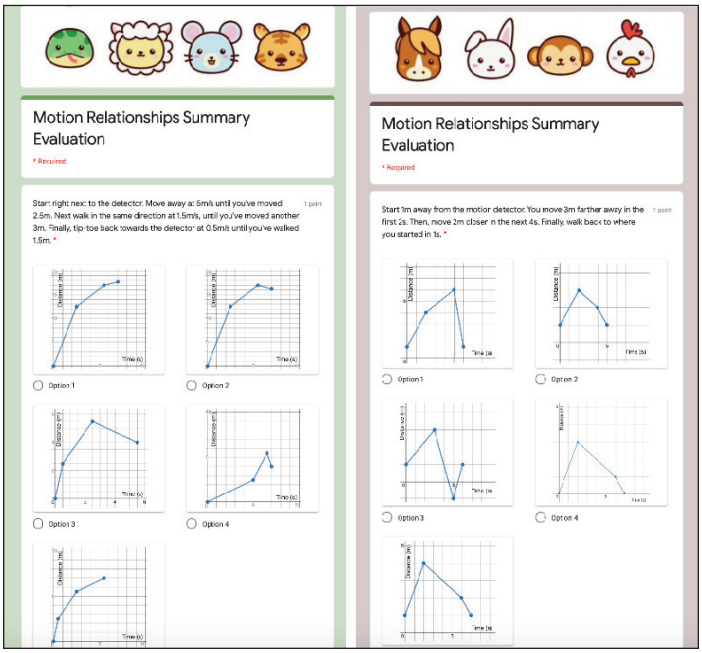


Figure 3: *Different quizzes originating from the large test bank*

When students felt ready, they chose to complete the overall summary quiz. Since this contained one randomly chosen question from each section, it was an ideal component for the portfolio, showing each skill once. With many different possible quizzes, students in the same class received a unique quiz, addressing the same content. Figure 3 shows two different summary quizzes starting with a different "Story ⟷ Graph" question. Figure 3 also illustrates changing the theme so the individual appearance of the quizzes is apparent if a student inadvertently looks at another's screen or does a summary quiz more than once.

The graphs in these "Story ⟷ Graph" quizzes, shown in Figure 3, were made with distance–time graphs in Desmos. I also included these piecewise linear graphs generated randomly, using Desmos, in other student portfolio options. Since this is such a useful feature for generating questions, let me go off on a brief tangent.

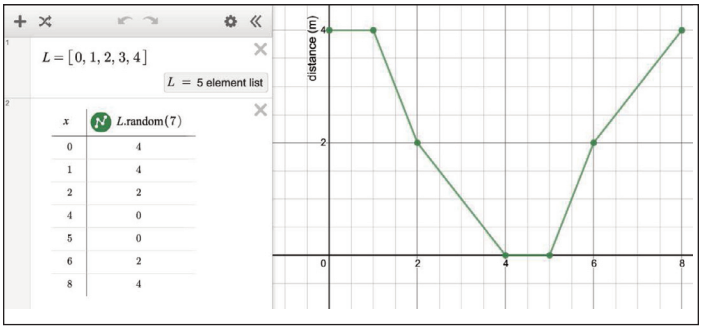


Figure 4: *A random distance–time graph generated by Desmos*

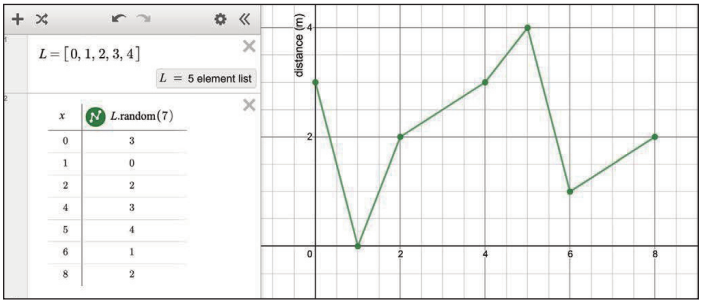


Figure 5: *Another distance–time graph generated randomly by Desmos*

Figures 4 and 5 show how Desmos randomly generates the connected scatterplots for these motion graphs. In Figure 4, I generated a random set of *y*-coordinates in a table from the list labelled "L." Figure 5 shows another completely random set, as generated by the shuffle feature noted in the upper left corner (with the red box around it).

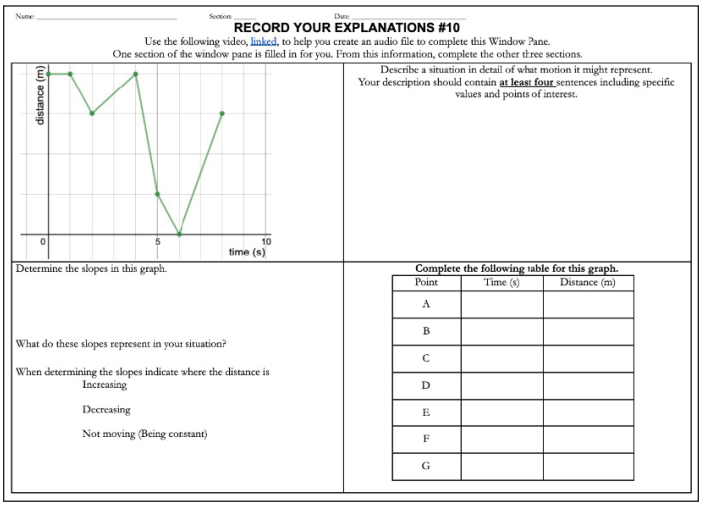


Figure 6: *Example of a Recorded Window Pane*

Another portfolio option was for students to use one of these unique graphs in their Recorded Window Pane. The Recorded Window Pane, shown in Figure 6, connects the various representations that arise from their distinct distance–time graph. (I explain my use of the Window Pane framework in more detail in the September 2016 column, "Linear Leaping—Window Pane Chart.")

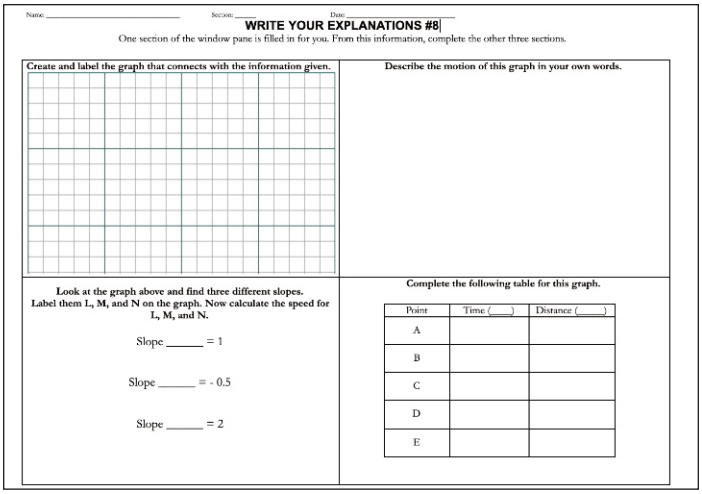


Figure 7: *Written Window Pane starting with a table of values*

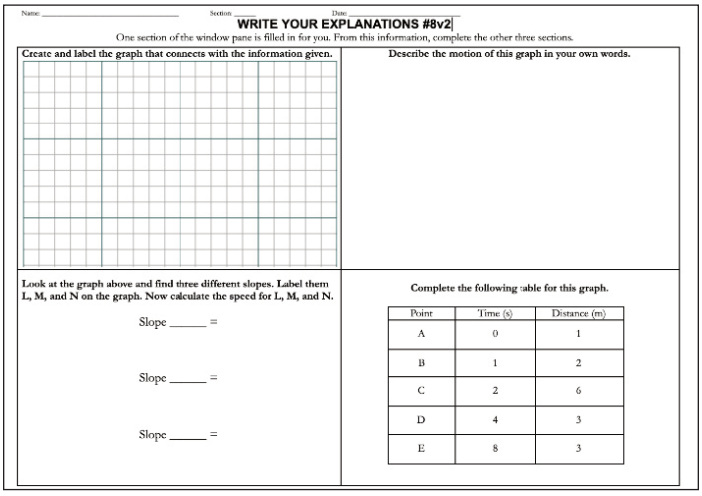


Figure 8: *Written Window Pane starting with three slopes*

The Recorded Window Pane explained the links between the representations. Asynchronously, students used Screencastify with Desmos for video explanations, or Google Slides, or with 123 Apps Voice Recorder for audio descriptions, or photos of their written paperwork with text connections. Doing this asynchronously allowed students to take the time to record their own stories creatively. Two other portfolio options—the Written Window Panes, shown in Figures 7 and 8—begin with different sections of the Window Pane, and are also generated randomly with ease, using the Desmos feature to give each student a novel option.

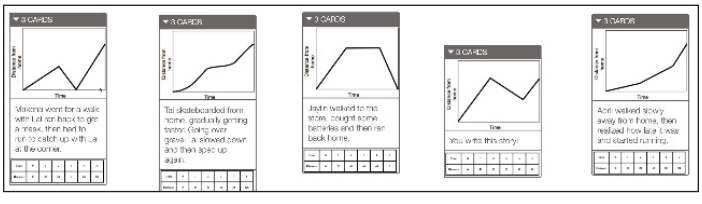


Figure 9: *Desmos Teacher Mode answer key for Card Sort with graph, table, and story*

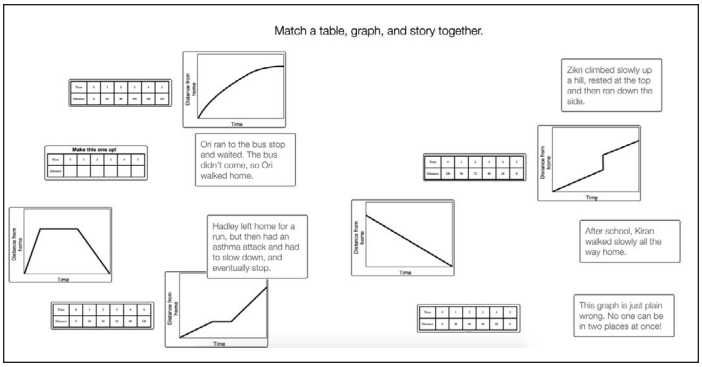


Figure 10: *Card Sort with graph, table, and story (the student screen preview seen in Teacher Mode)*

Figures 9 and 10 show another choice for the portfolio—the Card Sort—matching table, story, and graph, adapted from the EduGAINS TIPS4RM: Grade 9 Applied activity, "Graphs Tell Stories," found at www.edugains.ca/resources Math/CE/TIPS4Math/9Applied/LessonOverviews/Grade9Applied_Unit5_ModellingLinearRelationships.pdf. As explained in my last column, Desmos Teacher Mode's Card Sort can be done with the teacher synchronously watching the students' work, or asynchronously, with Desmos tracking which students have correctly made all the matches.

The final options for the portfolio were Desmos adaptations of "The Beetle and the BMW" distance–time graph presented in the EQAO Grade 9 Applied Practice Questions, 2000, and in the Carolyn and Mary distance–time graph presented in the MPM1D Public Profiles, 2000. I adapted these with both TI-Nspire™ and Desmos Teacher Mode for students to create scatterplots, lines of best fit, determining speeds/slopes and equations, and explaining the graph with a longer, more creative story, using Google Docs collaboratively with peer review. Using Teacher Mode in Desmos allows you to keep students on the same screen, by setting the pacing and pausing, as shown in Figure 11.

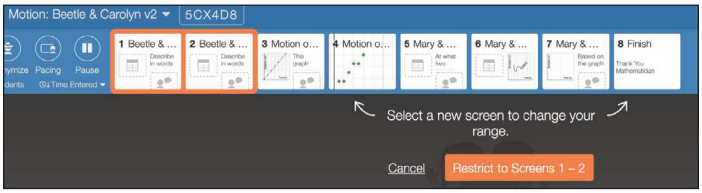


Figure 11: *Desmos Teacher Mode, where students only see specific questions, while the teacher checks progress/completion*

The overall goal for this portfolio curation was for the students to show multiple representations for distance–time relationships. The students were guided by the rubric (Figure 12) to choose their items. It was interesting to hear them discuss which items they felt showed each criterion and how to demonstrate them. Moving forward, I think students would benefit by explicitly reflecting on the reason for their choice of items, as well as noting the criteria. This would be particularly useful for assessing students who are remote or who complete the portfolio asynchronously.

The rubric made it apparent to students that the grading emphasized how often and consistently they had shown the criteria, rather than focusing on being error free. This marking scheme also allowed me to note, on the right side, misunderstandings or missteps, yet on the left side, I could note which criteria needed more evidence. It emphasized for all of us that this unit required multiple approaches and individual choices, rather than just right-or-wrong questions or a single test.

Students who had previously struggled with algebra-centred pencil-and-paper tasks were able to explain their thinking in a video, or use technology to create a graph with a line of best fit, and describe the slope correctly in the context of speed. Some students used matching cards without numbers to recognize and compare negative and positive slopes, followed by justifying their slopes with an audio file, rather than becoming bogged down by the integer operations. I was able to more clearly see when a student recognized the plotted points correctly, but had more difficulty plotting them. Alternatively, some students originally wondered aloud why they had to show things in more than one way, then later realized that the tasks could actually be easier with multiple representations, which improved their understanding of slope to represent it as a table with "skip counting" *x*-coordinates. Students' choices to approach the portfolio differed especially for multiple representations, which helped to establish linkages for future concepts in Physics (students asked how the changing velocity was connected to the second differences and non-linearity) and Calculus ("What happens if the table doesn't show a consistent slope?" "Doesn't that mean it is going to have a bunch of different small straight-line pieces, which could look like a curve?"). Incorporating choice throughout, and also not allowing students to choose items, even if I thought they should, heightened engagement and buy-in. The agency and independence I recognized in June were very powerful when cultivated throughout the learning process. I was able to see students' individual areas of strength and interest, and students were able to identify that for themselves as well. They engaged more deeply in what each portfolio piece was

demonstrating about their learning, and choice was foundational to this.

Knowledge – Slopes, Equations, First Differences – Algebra & Concepts			
• Calculates slopes and speeds correctly. • Correctly completes tables correctly. • Calculates the first differences correctly. • Correctly determines points in a given relationship. • Identifies slopes, speeds, and points correctly.	Level 1 – Rarely met Level 2 – Often met Level 3 – Regularly met Level 4 – Always met		
Application – Multiple Representations & Context			
• Correctly explains what a given point or slope means in the context of the problem. • Translates and represents graphs in table, slope, and word form. • Slope, points, direction, and choice of values make practical sense. • Word problems are concluded in context, explained with units where appropriate.	Level 1 – Rarely met Level 2 – Often met Level 3 – Regularly met Level 4 – Always met		
Thinking – Documentation of Mathematical Process, Connections, & Problem Solving			
• Written mathematical steps are easy to follow and clear to interpret. • Includes relevant, supporting ideas, and calculations, to explain thinking. • Problem-solving process clear, well organized, and easy to follow. • Stories accurately match all areas of mathematical expression. • Directions, speeds, and connections within problems make sense.	Level 1 – Rarely met Level 2 – Often met Level 3 – Regularly met Level 4 – Always met		
Communication – Graphing Conventions, Mathematical Form, Explanations, & Conclusions			
• Provides clear explanations of the situations, using correct mathematical vocabulary, values to justify conclusions in words. • Graphs are properly labelled, using a ruler, include labels and units on the axes, have appropriate scales. • Provides efficient explanation, no redundancy, flow of thought, independently without scaffolding. • Variables are defined, and dependent/ independent variables are properly chosen.	Level 1 – Rarely met Level 2 – Often met Level 3 – Regularly met Level 4 – Always met		

Figure 12: *Marking scheme for Relationship Portfolio*

▲

▲ What's the Problem? Creative Counting



SHAWN GODIN
shawn.godin@ocdsb.ca

Shawn is a former mathematics teacher, department head, and consultant. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: presenting workshops, writing articles, working on local projects, and helping create mathematics contests.

Welcome back, problem solvers. As I am writing this, I am in the last couple of weeks of the quadmester, teaching from home. As you are reading this, I will be a newly retired teacher, more than likely self-isolating until I can be vaccinated. I am looking forward to the time I can dedicate searching for nice problems for all of you to explore. Last issue, I left you with the following problem:

> *Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the people has at least 2 apples?*

This problem was question 25 from the 2019 AMC8 competition. The AMC8 is a multiple-choice competition for students up to Grade 8. The American Mathematics Competitions (AMC) runs the competition through the Mathematical Association of America (MAA). You can check out AMC competitions and other resources at the Art of Problem Solving Wiki: www.artofproblemsolving.com/wiki.

This is a nice problem because it is approachable by students from a large range of backgrounds. We will see some sophisticated mathematical tools that can be used to solve the problem later, but students can still solve it using properties of arithmetic and some patterning.

As in many problems, it is sometimes worthwhile exploring a simpler, related problem. Consider the problem of sharing 5 apples between 2 people, where each person gets at least 1 apple. Students could use manipulatives to represent the problem, for example, using red and green linking cubes (since we are representing apples), we could get something like Figure 1.

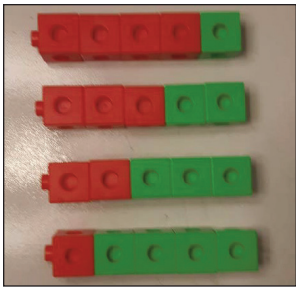


Figure 1: *Four ways to share 5 apples*

The manipulative may render some patterns more obvious. For instance, as the people are sharing the apples, if we want to give 1 person an extra apple, it must come from the other person. Looking at Figure 1 from bottom to top, we see that the numbers of green cubes goes down as the number of red cubes goes up. Looking from top to bottom, we see the opposite happening. Looking closer, we might see the symmetry in our solutions that is illustrated in Figure 2, where possible arrangements have been put into complementary pairs, that is, pairs where the reds and greens have been "swapped."

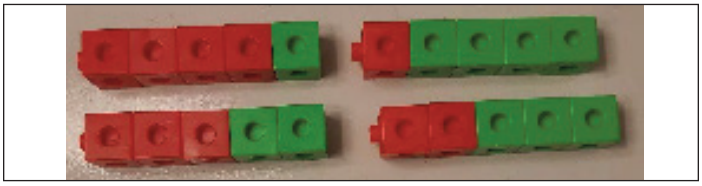


Figure 2: *Complementary arrangements*

These observations suggest certain shortcuts we can utilize in our solution. Representing the cases symbolically, we could have listed all the cases as: 1 + 4 = 2 + 3 = 3 + 2 = 4 + 1 = 5. Using the symmetry from Figure 2, we could have just looked at the cases where the first number is smaller, 1 + 4 = 2 + 3 = 5, and noted that we would have 2 × 2 = 4 solutions, since swapping the numbers for each solution creates a new solution (1 + 4 = 4 + 1), so the listed solutions actually count for two solutions each.

We might also have reasoned that since the smallest number of apples we could have is 1 and 1 + 4 = 4 + 1 = 5, then from the first pattern discussed above, the first person could have anything from 1 to 4 apples, so there must be 4 cases. We could generalize this approach to other situations. For example, if we were sharing 24 apples between 2 people, and each person must get 1, then since 1 + 23 = 24, we can reason that a person could have any number from 1 to 23, so there must be 23 cases. On the other hand, if each person must have at least 2 apples, and 2 + 22 = 24, each person could have between 2 and 22 apples, so there would be 21 cases.

Adding a layer of complexity to our problem, let's now consider 5 apples shared among 3 people, where each person gets at least 1 apple. We will use ordered triples (a, b, c) to represent the number of apples for Alice, Becky, and Chris. Proceeding systematically, assigning some apples to Alice first, then splitting the rest with Becky and Chris, we get (3, 1, 1), (2, 2, 1), (2, 1, 2), (1, 3, 1), (1, 2, 2), and (1, 1, 3), for 6 possible cases. A couple of patterns emerge from our solution.

First, we get the symmetry from Figure 2 extended to 3 people. As such, we get two types of solution, with three variants each. That is (3, 1, 1), (1, 3, 1), and (1, 1, 3) go together, as do (2, 2, 1), (2, 1, 2), and (1, 2, 2). So, if we could find all groups of numbers that add to our desired result, and then bear in mind the arrangements, we could solve our problem.

Returning to the original problem, we could start looking for numbers a, b, and c, such that $a + b + c = 24$ and $a, b, c \geq 2$. Adding the condition $a \geq b \geq c$, we can then count the arrangements of each case later. Looking at all possibilities, we might get something like

$$20 + 2 + 2 = 19 + 3 + 2 = 18 + 4 + 2 = \cdots = 8 + 8 + 8 = 24.$$

We then note that if all three numbers are the same, like 8 + 8 + 8 = 24, there is only one solution of this type. If there are two numbers the same, there are three solutions, such as

$$20 + 2 + 2 = 2 + 20 + 2 = 2 + 2 + 20 = 24,$$

whereas if all three numbers are different, there are six possible solutions. Students who have taken some counting, using permutations and combinations, will recognize these 3 cases numerically as looking for arrangements of three objects with 3, 2, or no repetitions, which have $\frac{3!}{3!} = 1$, $\frac{3!}{2!} = 3$ and $3! = 6$ arrangements, respectively. Using this technique, you can find the 37 different sets of three numbers that work, and calculate the final answer, 190.

The second pattern that we notice is that there was 1 case, (3, 1, 1), that started with a 3; 2 cases, (2, 2, 1) and (2, 1, 2), that started with a 2; and 3 cases that started with a 1. Think about what is happening. We see that if the first person gets 2 apples, it means the other 2 people must share 5 – 2 = 3 apples. Since they each must each get at least 1 apple and 1 + 2 = 3, each person's choice ranges from 1 to 2, so there must be two possibilities (1 + 2 = 2 + 1 = 3).

Returning to the original problem, the most apples a person could have is 20, and there would be 1 choice for the other 2 people, 2 + 2 = 4. Continuing, if the first person takes 19 apples, then we share the other 5 between the other 2 people, with at least 2 apples each. Since 2 + 3 = 5, their choices run from 2 to 3, giving 2 cases. Similarly, 18 apples for the first person leaves 6 = 2 + 4 for the other 2, and there are 3 choices running from 2 to 4. Hopefully, you can see that as the first person's number goes down by 1, the number shared by the other 2 goes up by 1, yielding 1 more choice. Hence, the lowest number for the first person is 2,

which leaves 22 = 2 + 20 for 19 cases. Thus, the total number of possibilities will be

$$1 + 2 + 3 + \cdots + 18 + 19 = 190.$$

The second solution allows us to find a pattern and arrive at the solution more quickly than in the first case. Both solutions should be accessible to middle school students (with some adjustment to the numbers, depending on the students' age).

For students with a bit more mathematical sophistication, we can look at the so-called *stars and bars* technique. This technique is used for finding partitions of a number into a fixed number of positive or non-negative integers. For example, if I am interested in calculating the number of ways to write 5 as the sum of three positive integers, there are six ways, as shown in Figure 3. The 5 *stars* represent the group of 5 objects that needs partitioning into 3 groups. The 2 *bars* represent where 1 group ends and the next one begins.

$5 = 3 + 1 + 1$	$\star\star\star\vert\star\vert\star$	$5 = 1 + 1 + 3$	$\star\vert\star\vert\star\star\star$
$= 2 + 1 + 2$	$\star\star\vert\star\vert\star\star$	$= 1 + 2 + 2$	$\star\vert\star\star\vert\star\star$
$= 2 + 2 + 1$	$\star\star\vert\star\star\vert\star$	$= 1 + 3 + 1$	$\star\vert\star\star\star\vert\star$

Figure 3: *Partitions of 5*

Therefore, we can think of our problem as placing 2 bars (separating the stars into 3 groups) into the 4 spaces between the stars. Thus, there are $\binom{4}{2} = 6$ ways to do that. In the featured problem, there are 24 apples to be sorted among 3 people, where each person gets at least 2. If we are creative, we can start by giving each person 1 apple, leaving 21, and then breaking that 21 up into 3 groups of positive size. Since there are 20 spaces between 21 stars, our answer would be $\binom{20}{2} = 190$. Isn't that slick!

We have a slight variation to our technique if zero is allowed as a choice. For example, there are 21 ways to write 5 as the sum of three non-negative integers, that is allowing zero as a choice. Some examples of sums that include zero, and their representations, are shown in Figure 4.

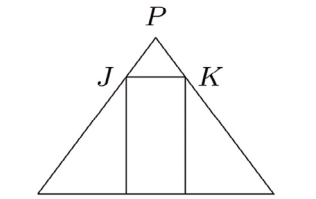
$5 = 3 + 0 + 2$	$\star\star\star\vert\vert\star\star$
$= 2 + 3 + 0$	$\star\star\vert\star\star\star\vert$
$= 0 + 5 + 0$	$\vert\star\star\star\star\star\vert$

Figure 4: *Some partitions of 5, where 0 is allowed*

In this representation, the stars still represent the number 5 and show how it was partitioned, and the bars separate the numbers, as before. However, we can have 2 bars in between the same 2 stars, as in the first case, or at either end, as the last 2 cases. As such, every possible distinct arrangement of the 5 stars and 2 bars (i.e., 7 items) gives us one of our partitions. There are $\binom{7!}{5!2!} = 21$ such arrangements (note that $\binom{7!}{5!2!} = \binom{7}{2} = \binom{7}{5}$, so we can also think of it as picking 2 of the 7 possible positions for the bars, or 5 of the places for the stars). Going back to our original problem, if we give each person 2 to start with, we have 18 left. We want the number of partitions when 18 is divided into three non-negative integers, which should be $\frac{(18+2)!}{18!2!} = 190$. It's interesting that a problem that could be solved with basic techniques by an elementary school student is also an interesting example of counting techniques for a Grade 12 University data-management class.

Now for your homework:

In the diagram, triangle PQR is isosceles with PQ = PR = 55 cm and QR = 66 cm. Rectangle JKLM is drawn such that J is on PQ, K is on PR, and JK is parallel to QR. What portion of the triangle does the largest such rectangle cover?



Until next time, happy problem solving! ▲

CCS *for your Math supplies!*

Calculators & Accessories

Graphing Calculator Sale

TI-83+ regular low price: $122.67 (limited quanties) on sale now for $107.67 each. Save $15.00 per unit.

Stericalc™ Your Calculators

To sterilize your calculators. Protect your students and you.

STEM Supplies & Accessories

TI-Rover, TI-Innovator and TI-Breadboard
Perfect for Coding and Robotics in the new curriculum

Calculator Repairs

We repair or replace your broken calculators

Cases for all Calculators

We sell Havilland & custom cases

SHARP · Texas Instruments · CASIO

24 Rogate Pl, M1M 3C3, Toronto 1-877-CCS-EDUC
info@ccseducational.com • www.ccseducational.com

▲ Using Place Value to Add, Subtract, Multiply, and Divide with Whole Numbers



DON VANDER KLOK
don.vanderklok@lambtoncollege.ca

Don Vander Klok has taught mathematics for over 30 years. Fifteen of those were in Ontario community colleges. He is a member of the Ontario College of Teachers. He is past president of the Ontario College Mathematics Association and has recently retired from Lambton College in Sarnia.

In a 2016 issue of the *Ontario Mathematics Gazette,* David Costello (2016) wrote about integrating place value with computation with whole numbers. This article fleshes out this approach in further detail.

Place value is foundational to our numerals. For example, for the number 313, we would say, "three hundred thirteen," to indicate that the two threes have different values. The 3 on the left represents 3 hundreds, or 300, while the 3 on the right stands for 3 ones, or 3. In terms of place value, 313 = 300 + 10 + 3. This is also called "expanded notation." The number 6687 would be "six thousand six hundred eighty seven." In expanded notation, 6687 = 6000 + 600 + 80 + 7. Both the names and the numbers help explain place value from different angles.

Using the place value strategy for adding, subtracting, multiplying, and dividing whole numbers begins with writing the numbers in expanded notation.

Addition

Write the numbers vertically in place value format to group like values together. The numbers are grouped so that hundreds line up with hundreds, tens with tens, and so on.

For example, let's add 527 + 92. You can start adding in any column. My students go left to right, the same as reading words.

In expanded notation:

$$\begin{array}{r} 500 + 20 + 7 \\ +\ 90 + 2 \\ \hline 500 + 110 + 9 \end{array}$$

This could also be done in word format:

5 hundreds + 2 tens + 7 ones
\+ 9 tens + 2 ones
5 hundreds + 11 tens + 9 ones

Or in table format:

Hundreds	Tens	Ones
5	2	7
	9	2
5	11	9

The final answer is 500 + 110 + 9 = 500 + 100 + 10 + 9 = 619.

This is another example: 863 + 249.

In expanded notation:

$$\begin{array}{r} 800 + 60 + 3 \\ +\ 200 + 40 + 9 \\ \hline 1000 + 100 + 12 \end{array}$$

This could also be done in word format:

8 hundreds + 6 tens + 3 ones
\+ 2 hundreds + 4 tens + 9 ones
10 hundreds + 10 tens + 12 ones

Or in table format:

Hundreds	Tens	Ones
8	6	3
2	4	9
10	10	12

The final answer is 1000 + 100 + 12 = 1000 + 100 + 10 + 2 = 1112.

Notice that you can start adding in any column. Starting with the left column makes sense just because left to right is the usual reading direction.

Using place value as a whole-number addition strategy is not a new approach. David Smith (1958, p. 91) notes that this approach is found in Bhāskara's book Līlāvatī, published in 1150. Also see Small (2010, pp. 52–54) and Carroll and Porter (in Morrow & Kenney, 1998, p. 108).

Subtraction

Subtraction using place value also starts with writing the numbers vertically in grouped place value format.

Let's subtract 487 – 235. Begin by writing these in expanded form vertically, grouping like values together, just like the addition.

$$\begin{array}{r} 400 + 80 + 7 \\ -\ 200 + 30 + 5 \\ \hline \end{array}$$

You can start subtracting in any column. As in addition, my students proceed from left to right in the usual reading direction.

$$\begin{array}{r} 400 + 80 + 7 \\ -\ 200 + 30 + 5 \\ \hline 200 + 50 + 2 \end{array}$$

The final answer is 200 + 50 + 2 = 252.

$$\begin{array}{r} 600 + 20 + 7 \\ -\ \quad 80 + 9 \\ \hline \end{array}$$

This is another example: 627 – 89. Again, begin by writing the numbers in expanded form vertically, grouping like values together.

This example again proceeds left to right. This is selected to illustrate "balanced borrowing." Whatever is borrowed (*subtracted*) from one column is *added* to the next column on the right, to balance things out.

$$\begin{array}{r} \quad 120\ \ 12 \\ 600 + \cancel{20} + \cancel{7} \\ -\ \quad 80 + 9 \\ \hline \cancel{600} \\ 500\ \ \cancel{40} \\ \quad 35\ \ 3 \end{array}$$

600 minus nothing is 600.

You "cannot" do the subtraction, 20 minus 80, so borrow something from the 600 to increase the 20 to at least 80. I will choose to borrow 100, which reduces the 600 to 500. The 100 is added to the 20 to balance. Subtract 80 from the 120 equals 40.

In the third column, you "cannot" do the subtraction, 7 minus 9, so borrow from the 40 to increase the 7 to at least 9. I will choose to borrow 5, bringing the 40 down to 35. To balance, 5 is added to the 7 to result in 12. Subtracting 9 from the 12 equals 3.

Alternatively, work with negative numbers.

$$\begin{array}{r} 600 + 20 + 7 \\ - \quad 80 + 9 \\ \hline 600 + (20 - 80 + (7-9) \\ = 600 + (-60) + (-2) \\ = 540 - 2 \\ = 538 \end{array}$$

Carroll and Porter (in Morrow & Kenney, 1998, pp. 109–110) refer to this as "partial differences."

You can borrow any number as long as you are trying to make the top number in each column equal to or greater than the number below. Working with negative numbers means there is no need to borrow at all.

See this pictured with Unifix® cubes in Small (2010, pp. 55–56).

Multiplication

Multiplying whole numbers using place value requires facility with multiplying numbers ending in zeros. These examples use the associative property of multiplication to rearrange the numbers after expressing them in expanded notation.

40×30	5000×7
$= (4 \times 10) \times (3 \times 10)$	$= (5 \times 1000) \times 7$
$= 4 \times 10 \times 3 \times 10$	$= 5 \times 1000 \times 7$
$= 4 \times 3 \times 10 \times 10$	$= 5 \times 7 \times 1000$
$= 12 \times 100$	$= 35 \times 1000$
$= 1200$	$= 35000$
In short, 40×30 is really 4×3 with two zeros at the end.	In short, 5000×7 is really 5×7 with three zeros at the end.

The total number of zeros in the problem is the same as the number of zeros in the answer.

Marilyn Burns (2018) cautions about being too glib about simply summarizing this process as "just add zeros at the end," since this does not apply with decimal numbers. For example, $36 \times 10 = 360$ (36×1 and tack on the zero at the end), but $3.6 \times 10 \neq 3.60$ (following the same pattern). Along the same lines, Marian Small (2009, p. 71) suggests thinking of this process in terms of shifting place values to provide a more expansive explanation:

36

Hundreds	Tens	Ones	Tenths
	3	6	

36 x 10 = 360

Hundreds	Tens	Ones	Tenths
3	6		

The 3 tens shift left to become 3 hundreds; the 6 ones shift left to become 6 tens, and then there are 0 ones.

3.6

Hundreds	Tens	Ones	Tenths
3	6		

3.6 x 10 = 360

Hundreds	Tens	Ones	Tenths
	3	6	

This explanation would apply in the same way to 3.6 × 10.

Now to carry on with multiplication using place value. Let's multiply 237 × 6. You begin with writing the numbers in place value format: (200 + 30 + 7) × 6. Using the distributive property, (200 + 30 + 7) × 6 = 200 × 6 + 30 × 6 + 7 × 6. Using a table helps visualize this, where the numbers in place value format are written on the perimeter of the table (my students need this set-up).

	200	30	7
6			

Multiply each column number times each row number to fill in the table. You can start anywhere.

	200	30	7
6	200 x 6	30 x 6	7 x 6

$\Longrightarrow$

	200	30	7
6	1200	180	42

Sal Khan (Khan Academy) calls this an "area model," since you are calculating length times width for each of the rectangles shown in the table. See also Van de Walle (2001, p. 183).

To calculate the result, add the products: 1200 + 180 + 36; in place value format:

$$\begin{array}{r} 1000 + 200 \\ 100 + 80 \\ + \quad 40 + 2 \\ \hline 1000 + 300 + 120 + 2 = 1422 \end{array}$$

Another example: let's multiply 49 × 27. Again, begin with writing the numbers in place value format. This becomes (40 + 9) × (20 + 7), and you can see that the multiplication to be carried out is the same as multiplying polynomials, applying the distributive property in a manner sometimes called FOIL (First Outside Inside Last) or "double distributive." Carroll and Porter (in Morrow & Kenney, 1998, p. 110) refer to this as "partial products."

$$(40 + 9) \times (20 + 7) = 40 \times 20 + 40 \times 7 + 9 \times 20 + 9 \times 7$$

	40	9
20		
7		

A table can help visualize this.

Multiply each column number times each row number to fill in the table, starting in any cell of the table.

	40	9
20	40 x 20	9 x 20
7	40 x 7	9 x 7

$\Longrightarrow$

	40	9
20	800	180
7	280	63

$$\begin{array}{l} 800 \\ 100 + 80 \\ 200 + 80 \\ \quad 60 + 3 \\ \hline 1100 + 220 + 3 \\ = (1000 + 100) + (200 + 20) + 3 \\ = 1323 \end{array}$$

To calculate the result, add the products, 800 + 180 + 280 + 63.

This is also called the "grid method" (illustrated in tonicbottle, 2017) and the "box method" (shown in Lorenzen, 2014).

Division

Dividing whole numbers using place value begins with writing the dividend in place value format. (This article deals only with one-digit divisors.)

For example, 8736 ÷ 7 becomes (8000 + 700 + 30 + 6) ÷ 7.

Like subtraction with place value, division involves "balanced borrowing," where numbers are subtracted in one place and then added in another place to balance things out and make the division simpler.

	1000	200	40	8
7	8000	+ 700	+ 30	+6
	-1000	+1000		
	7000	1700		
		-300	+300	
		1400	330	
			-50	+50
			280	56

Starting on the left, how many 7's in 8000? I don't know that answer readily, so let's change the 8000 to 7000. That involves subtracting 1000, and to balance that out, add 1000 to the 700 to make 1700.

There are 1000 7's in 7000. Write that on top.

Next, how many 7's in 700 + 1000, or 1700? I don't know that answer readily, so let's change that number to 1400 by subtracting 300. To balance things, add the 300 to the 30 to make 330.

There are 200 7's in 1400. Write that on top.

Next, how many 7's in 330? I don't know that answer readily, so let's change that number to 280 by subtracting 50. To balance things, add 50 to the 6 to make 56.

There are 40 7's in 280, and I write that on top.

Next, how many 7's in 56? There are 8, and again, write that on top.

The final answer is the sum of the individual top answers: 1000 + 200 + 40 + 8 = 1248.

Van de Walle (2001, p. 187) describes this strategy as "sharing pieces within each place value and trading leftovers for the next smaller piece," and he illustrates this process using Unifix cubes. The above example, 8736 ÷ 7, would be described as "sharing 8 thousands, 7 hundreds, 3 tens, and 6 ones equally among 7 sets."

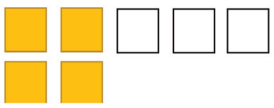
Each set would get 1 thousand, and that would leave 1 thousand left over.



Trade the leftover 1 thousand for 10 hundreds. Add that to the 7 hundreds in the 8736, and that totals 17 hundreds.

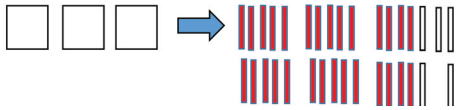


Plus the 700 from the 8763.

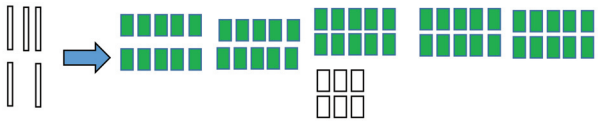


Share 2 of those hundreds in each of the 7 sets (for a total of 14), and that leaves 3 hundreds left over.

Trade those 3 hundreds for 30 tens. Add on the 3 tens from the 8736, and that sums up to 33 tens. Share 4 tens among the 7 sets for a total of 28 tens, and that leaves 5 tens left over.



Trade those 5 leftover tens for 50 ones.



Combining that with the 6 from the 8736 makes a total of 56 ones. And that can be divided 8 times in the 7 sets, using up all 56.

So each set would contain 1 thousand, 2 hundreds, 4 tens, and 8 ones, or 1248.

In Summary

Place value is foundational to our numerals. Explicitly using place value as a strategy to add, subtract, multiply, and divide whole numbers reinforces that understanding. David Costello (2016) was writing in response to an instruction set-up, where place value was taught in a four- week unit of study separately from the computation operations. He saw integrating those topics as the way to overcome a segmented instructional approach.

Using place value makes adding, subtracting, multiplying, and dividing whole numbers more intuitive. You carry out the operations in the same direction as reading – left to right. There are no seemingly meaningless rules that are involved in the "traditional algorithms," such as "begin on the right" (adding, subtracting, multiplying), "now carry one" (adding), "borrow one" (subtracting), "write down a zero first" (multiplying 2 two-digit numbers), and "bring the next number down" (dividing). The place value strategy explains those rules.

In Ontario's colleges, apprenticeship training and programs based on apprenticeship often are required to include a mathematics component, and that component usually begins with a review of arithmetic using no calculator, starting with whole numbers. Carpentry, welding, and boilermaker are examples of those apprenticeship programs. The arithmetic review tends to be a review of computation algorithms.

While most of the students in my classes first of all resort to the "traditional algorithms," they stumble on the procedures and have difficulty explaining why some steps are taken. Place value strategy helps explain steps, and many students in my classes end up making it their own go-to computation method. ***continued on page 42***

▲ Mb4T (Mathematics by and for Teachers): Focusing on Fractional Reasoning



JENNIFER HOLM
jholm@wlu.ca

Jennifer Holm is an Assistant Professor at Wilfrid Laurier University and works with primary/junior and junior/intermediate pre-service teachers, as well as in the field supporting current mathematics teachers. She is interested in developing mathematics knowledge for teaching with both pre-service and in-service teachers. She focuses on the beliefs and opinions that pre-service teachers hold about mathematics and teaching and the connection they have to past experiences. She uses this research to support future teachers in developing beliefs and knowledge that will encourage and support effective teaching practices in mathematics.

As I was deciding on this month's column, I came across a note from one of my students on their assignments, and her thoughts inspired this content. The assignment was focused on using reasoning only, to share how a student may order a list of fractions. Although the assignment did not require any reflection, one of my students shared how difficult it was to complete, to "unlearn" all she had put together in her years of working with fractions, and to think about where students may start. This made me wonder how many other topics in the curriculum we may take for granted as teachers because we have years of experience working with the content and forget how difficult it is for students to first grapple with these ideas. The new curriculum builds on the idea that we need to meet students where they are and support moving from there, so we, as teachers, need to remember where they are. This month, we are going to use the idea of fractions and go back to some of the seemingly "simple" ideas that are helpful for students to build upon.

The Ontario Math Support site has helpfully given us access to continua for each of the strands of the curriculum (www.ontariomath.support/?pg=resources&lang=EN), which allows a teacher to see how each strand builds across the grade levels. The new curriculum separates out the Number strand by topic to make it a lot simpler to locate different ideas, so instead of having all the expectations bundled together as "number," they are pulled apart by whole numbers, fractions, decimals, and integers, for example. Looking at the continuum for Number in the section on parts of a whole (Ontario Ministry of Education [OME], 2020a), one will notice that Grades 1 to 3 are focused on fractions; Grade 4 turns to fractions and decimals; and then Grades 5 to 8 look at fractions, decimals, and percentages. When looking at the *Key Changes* document (OME, 2020b), clearly, significant changes have occurred in the fractions section. See Figure 1 for a summary of the changes.

Topic	2005 Curriculum	2020 Curriculum
Fractions	• Gr. 1: Begin to look at halves. • Gr. 4: Writing fractions with standard fractional form, e.g., $\frac{1}{2}$ • Gr. 7: Add and subtract fractions with like and unlike denominators • Gr. 8: Represent multiplying and dividing fractions	• Gr. 1: Begin to recognize equivalent fractions • Gr. 3: Writing fractions with standard fractional notation, e.g., $\frac{1}{2}$ • Gr. 4: Begin to work with equivalent fractions • Gr. 5: Add and subtract fractions with like denominators, and multiplying and dividing one-digit whole numbers by unit fractions • Gr. 6: Add and subtract fractions with like and unlike denominators, and multiplying and dividing whole numbers by proper fractions • Gr. 7: Multiplying and dividing fractions by fractions using tools, applying equivalent fractions, and generating fractions and decimals between any two quantities • Gr. 8: Multiplying and dividing fractions and integers

Figure 1: *List of key changes in the fractions section of the curriculum (OME, 2020b, pp. 3–4).*

The changes mean that students will be working with fraction operations now in Grade 5 instead of Grade 7, where they were in the 2005 curriculum. For students to operate with fractions with understanding and not just a rote use of the formula, they need to be able to reason about and understand fractions themselves.

One way to develop some fraction understandings is to focus on reasoning about fractional amounts while comparing fractions, without using equivalent fractions or converting them into a different representation, such as percentages and decimals. Although as adults we have often internalized common equivalent fractions, when students are first talking about fractions, they have not. As well, we have often internalized the procedure of finding a common denominator, but students are in the process of learning what it means to make an equivalent fraction. For the remainder of this column, we will look at some different scenarios for reasoning about fractions and the understandings that would be helpful for students to learn as they are building their knowledge about fractions.

An introductory way to reason about fractions is to consider the difference between fractions with a common denominator. When comparing $\frac{3}{17}$ and $\frac{7}{17}$, how do you know which one is bigger? Although the denominators are daunting to some students, focusing on the number of same-sized pieces is the only part of the reasoning here. Since 7 pieces is more than 3 pieces when all pieces are the same size, $\frac{7}{17}$ is more and must be a larger fraction.

The second way to reason about fractions is to consider how to compare fractions with a common numerator. When comparing $\frac{3}{15}$ and $\frac{3}{20}$, how do you know which one is

bigger? This is a more difficult idea for students to grasp. They may say, "Well, 20 is bigger than 15, so $\frac{3}{20}$ must be bigger." Since both fractions have the same number of pieces (i.e., equal numerators), we need to look at the size of those pieces. In Figure 2, we see one whole divided into fifteenths and one whole divided into twentieths. When comparing the two fractions in the image, one can identify that the twentieths are smaller-sized pieces, so $\frac{3}{20}$ is the smaller fraction, since it will take up less of the area. The important understanding for students in this example is to consider how large the pieces are in each of the fractions.

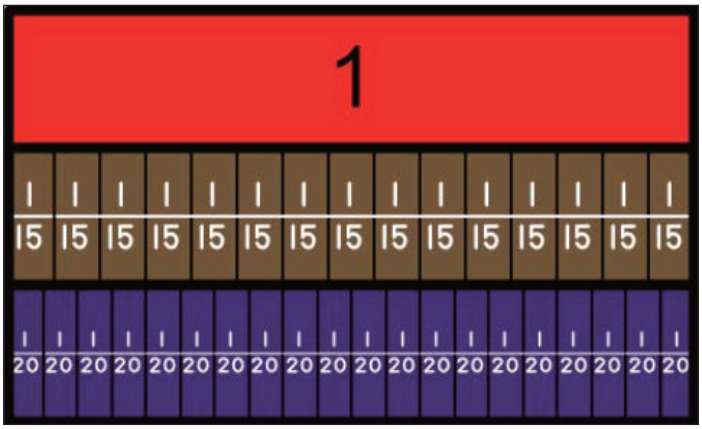


Figure 2: *Fifteenths and twentieths from a Mathies fraction-strips application*

Another important idea is reasoning about fractions, using benchmarks (0, $\frac{1}{2}$, and 1). Comparing to the $\frac{1}{2}$ benchmark when teaching is interesting and develops students' reasoning skills further. Consider these two fractions: $\frac{23}{47}$ and $\frac{27}{53}$. How could you decide which one is bigger? Students are not going to model these two fractions because the component values are too large for accurate models and would likely lead them to an incorrect answer. If you consider that half of 47 is 23.5 and half of 53 is 26.5, it is straightforward to reason about the fractions: $\frac{23}{47}$ is just under $\frac{1}{2}$ and $\frac{27}{53}$ is just over $\frac{1}{2}$. Students can even use estimates to help them with finding the $\frac{1}{2}$ point, such as half of 48 is 24 and half of 50 is 25.

Students can also compare the size of fraction pieces to decide how close to a benchmark the fractions are. An example is comparing $\frac{21}{23}$ and $\frac{25}{27}$. Once again, the denominators are difficult to draw and conceptualize, obliging students to focus on reasoning. Both fractions are missing two pieces relative to the benchmark of one whole; they are both close to one. When deciding which is larger, one can focus on the size of the pieces. Since cutting a whole into 23 pieces would generate larger parts than cutting the whole into 27 pieces, we can reason that $\frac{21}{23}$ would be farther from one whole, making it the smaller fraction. Helping students to compare fractions to the benchmarks of 0, $\frac{1}{2}$, and 1 can support a stronger ability to reason about fractions.

Next, we can have students put it all together and reason about a list of fractions. I am purposefully using different fractions because my goal is not to think about the fractions as equivalents or finding common denominators, but I would like students to think about what they know about fractions to help them. How might a student reason about the order for these six fractions: $\frac{27}{49}, \frac{7}{56}, \frac{48}{49}, \frac{7}{83}, \frac{17}{35}$, and $\frac{54}{55}$?

To start, we could notice that two of the fractions have the same numerator and are both small numbers: $\frac{7}{56}$ and $\frac{7}{83}$. Given they are such small portions of the whole, they are likely closest to zero and the smallest. Since they both have seven pieces, we can see that $\frac{7}{83}$ is smaller than $\frac{7}{56}$ because the pieces are smaller when the whole is cut into 83 pieces. We can notice that two of our fractions are almost equal to one whole: $\frac{48}{49}$ and $\frac{54}{55}$. Since they are both one piece away from one whole, we can once again consider the size of the pieces. Since cutting the whole into 55 pieces is going to result in smaller pieces, $\frac{54}{55}$ must be bigger, since it is a smaller amount from completing the whole. We can easily see that the two fractions with a numerator of seven are smaller than both, so we now have four of the fractions in order: $\frac{7}{83}, \frac{7}{56}, \frac{48}{49}$, and $\frac{54}{55}$. We are left with $\frac{27}{49}$ and $\frac{17}{35}$ to order and then place in the list. Well, neither of the fractions is close to one or to zero, and since both the numerators and denominators are different, we need to consider a different benchmark. If I think about how half of 48 is 24, then I can see that $\frac{27}{49}$ is going to be more than $\frac{1}{2}$. Since 17 is exactly half of 34, then I know $\frac{17}{35}$ is slightly less than equivalent to $\frac{1}{2}$. This can help me determine that $\frac{27}{49}$ is the larger fraction of these two, but since it is close to $\frac{1}{2}$, it is smaller than $\frac{48}{49}$. I can now place all six fractions in order: $\frac{7}{83}, \frac{7}{56}, \frac{17}{35}, \frac{27}{49}, \frac{48}{49}$, and $\frac{54}{55}$. Being able to reason about the fractions as parts of the whole allows for a deeper understanding of what the fractions mean.

Although it seems simple to create common denominators for all the fractions or to think of an equivalent way to write the fraction using decimals and percentages, it is important that students learn to reason about fractions so that they understand what they mean. Having this foundational understanding will help them with making estimates about operations and determining if their answers even make sense. As many researchers have pointed out, fractions are difficult for students to learn (Burns, 2001), and they need lots of experiences to reason about and discuss them as well as to concretely represent them. With the changes to the curriculum, moving a lot of fraction learning to earlier grades, now is a good time to consider what understandings of fractions your students have that they can

build upon, and if they can reason about fractions. Consider your own understanding of fractions: how much is based on the things you have memorized and now take for granted? How can you help students understand that diligent work with fractions can develop the ability to use them flexibly, which took you years to develop?

References

Burns, M. (2001). *Lessons for introducing fractions.* Math Solutions.

Ontario Ministry of Education. (2020a). *B. Number.* https://assets-us-01.kc-usercontent.com/fbd574c4-da36-0066-a0c5-849ffb2de96e/c2dea171-e219-4ec0-b097-24c959fcc924/Number_AODA.pdf

Ontario Ministry of Education. (2020b). *Key changes: Ontario mathematics curriculum, grades 1 to 8, 2020.* https://assets-us-01.kc-usercontent.com/fbd574c4-da36-0066-a0c5-849ffb2de96e/f2fad630-cd7b-402d-a40f-90959ad454d9/Key-Changes-in-Elementary-Math-Curriculum_Dec-4-2020.pdf ▲

Some Updates from the Centre for Education in Mathematics and Computing (CEMC)

All of the CEMC's 2020–2021 mathematics and computing contests are running, with students able to participate in school or at home. They may register through their school or as individuals. In December 2020, a team of more than 130 people marked roughly 15 000 full-solution Canadian Senior Math Contest and Canadian Intermediate Math Contest papers from their homes, using a hybrid online/on-paper model, following strict health and safety protocols.

The CEMC's free online courseware covers mathematics curriculum from Grade 7 to Grade 12. In 2019–2020 alone, the courseware had 11 million page views from students in 213 countries, who spent a combined 350 000 hours on the site.

The CEMC's Summer Conferences for mathematics and computing educators joined forces in summer 2020 and had more than 1000 teachers register for their free virtual sessions.

The CEMC's Problem of the Week is offered at five levels (Grades 3/4, 5/6, 7/8, 9/10, 11/12), in four languages (English, French, Spanish, Bahasa Indonesian), and currently has more than 45 000 subscribers, allowing it to reach more than 1 million students each week.

The CEMC's online Master of Mathematics for Teachers program for active secondary school teachers had more than 80 teachers start the program in Fall 2020. Currently there are 270 teachers in the program, and there have been almost 500 graduates in its short ten-year history.

The CEMC website is www.cemc.uwaterloo.ca.

▲ Reflections on Solving Equations: Transforming Equations into Equalities



JÉRÔME PROULX
proulx.jerome@uqam.ca

Jérôme Proulx is a professor of mathematics education (www.profmath.uqam.ca/~jproulx), and runs the Laboratoire Épistémologie et Activité Mathématique *(www.leam.uqam.ca). His research focuses on epistemological issues related to cognition and/in mathematics. His current research program is dedicated to mental mathematics and school mathematics content development. (Photo by Nathalie St-Pierre of Université du Québec à Montréal)*

It is a truism that students often come up with surprising ways of solving tasks we give them. Students even come up with strategies or ideas that in turn raise questions or shed light on mathematical aspects usually taken for granted or kept implicit.

This is particularly the case in the mental mathematics studies that my research team and I conduct in classrooms. Within the time constraints, students rarely refer to known procedures. These are often too time-consuming and hence impractical to employ in this context (see Butlen & Pézard, 2000). This leads many students to try a variety of strategies worth pondering. In particular, we conducted sessions on mental algebra, where the participating students (of various grade levels, from elementary to university) are given short amounts of time to solve some equations for x. The equations are posed orally or visually. The equations given are of the usual form: $Ax + B = C$, $Ax + B = Cx + D$, $\frac{Ax}{B} = \frac{C}{D}$, $Ax^2 + Bx + C = 0$, and their variants (see Proulx, 2013; Proulx et al., 2017).

In this third column, I am sharing examples of these strategies about equation solving that students engaged with. These strategies, albeit simple, often raise interesting mathematical questions.

The Proportion Strategy

Two examples of equations given in the form $\frac{Ax}{B} = \frac{C}{D}$ and its variants were $\frac{6}{x} = \frac{3}{5}$ and $\frac{1}{3x} = \frac{2}{5}$. One approach students used to solve these equations was to treat them as proportions. In the case of $\frac{6}{x} = \frac{3}{5}$, most students looked at how to maintain the proportion of 3/5 for $6/x$ (e.g., 3 is to 5

as 6 is to 10, or if 6 is the double of 3, then 10 is the double of 5).

This strategy, of working on the equations $\frac{6}{x} = \frac{3}{5}$ and $\frac{1}{3x} = \frac{2}{5}$ as proportions, raises interesting questions. Treating, for example, $\frac{6}{x} = \frac{3}{5}$ as a proportion is not so much about looking for the value of x that makes $\frac{6}{x} = \frac{3}{5}$ valid, because it is considered true right away, as it is about looking for the number that maintains the equivalent proportion. That proportion is 3/5; it exists, as do other equivalent proportions to be found for other sets of natural numbers that maintain this proportional relationship of 3/5 (e.g., 9/15, 18/30, 13.5/22.5).

This strategy points to one of the major differences between an equation and an equality in mathematics. Baruk (1995) traces an important distinction between the two by explaining that an equality is a statement that is true or false, like 7 + 2 = 9 is true and 1 + 3 = 6 is false. An equation, however, implicates unknown values. Its equivalency is conditional on the values of this or these unknowns. This makes salient what solving an equation refers to, that is, finding the value for x that verifies the equation, that makes it valid. The same is true for $x + y = -8$ or $x^2 + 7 = 21x$. Equations implicate unknowns and are conditional equalities, dependent on these values. (In this sense, one can almost say that calling $x + y = -8$ or $x^2 + 7 = 21x$ algebraic equations is redundant, because to be equations, they need to have unknowns. Algebraic equalities might be a more fitting term.) Equations are not stable, Baruk would add, because they depend on the value given to its unknowns. The equality 2 + 3 = 5 is stable, but $2x + 3 = 5$ is not an equality, by definition, and thus is conditional and not stable. The value of x obtained when solving the equation is what makes the equation stable. In short, it is what makes the equation an equality.

Therefore, treating $\frac{6}{x} = \frac{3}{5}$ as a proportion is not treating it as an equation, but directly as an equality, as a descriptive relation between established quantities. Because the proportion exists, because it is possible, the number x exists. When treated as a proportion, $\frac{6}{x} = \frac{3}{5}$ is no longer conditional, but becomes a statement about a relation between the established quantities 3, 5, 6, and x. The value of x, to some extent, is thus already predetermined as a number related and constrained by its relationship with 3, 5, and 6 in the proportion $\frac{6}{x} = \frac{3}{5}$.

The same could be said for students who treated $\frac{6}{x} = \frac{3}{5}$ as the quest for an equivalent fraction to 3/5. Doing this also transformes $\frac{6}{x} = \frac{3}{5}$ from an equation to an equality: there exists a fraction with 6 as numerator, for which 3/5 is equivalent.

One additional illustration of how treating $\frac{6}{x} = \frac{3}{5}$ as a proportion transforms it from an equation to an equality is related to the consideration of the constraint $x \neq 0$ in $\frac{6}{x} = \frac{3}{5}$. As an equation, $\frac{6}{x} = \frac{3}{5}$ implies that $x \neq 0$ be considered and not all algebraic manipulations are possible because of this constraint (e.g., multiplication by x is not directly admitted). However, this constraint fades in importance in a proportion, where the relationship between the numbers 3, 5, 6, and x simply does not allow for it. If the condition $x \neq 0$ were allowed, it would mean that the proportional relationship does not necessarily stand, that it is conditional, as well as the assertion of a relationship among numbers 3, 5, 6, and x.

In that sense, treating equations like $\frac{6}{x} = \frac{3}{5}$ or $\frac{1}{3x} = \frac{2}{5}$ as a proportion implies *transforming the equation into an equality*. This transformation to some extent leaves aside the quest for a value of x that makes the equation valid, since the equation is considered valid right from the start as a proportion.

The Impossible Strategy

In contrast to the above situation of treating equations as equalities, something different happened when the equation $5x + 6 + 4x + 3 = -1 + 9x$ was given to students. Some of them, after solving it, said that it was "false" and thus "impossible." This assertion of "falsehood" shows that this equation is treated quite differently from $\frac{6}{x} = \frac{3}{5}$. Both equations, however, have unknowns, but their status is not the same. The validity of $5x + 6 + 4x + 3 = -1 + 9x$ is dependent on the value for x; $\frac{6}{x} = \frac{3}{5}$ describes the relation between numbers of which x is part. For $5x + 6 + 4x + 3 = -1 + 9x$, one does not know if this x exists, so its validity is conditional and dependant on that value of x.

However conditional, questions arise in relation to these differences. When one solves $5x + 6 + 4x + 3 = -1 + 9x$, one acts "as if" it were valid, as if an x existed that verified the equation. But for this specific equation, that x does not exist. When attempting to solve it for x, one ends up with $-1 = 9$ or $10 = 0$, which are false statements, all false equalities, all stable. But to get there, the algebraic manipulations are made, thinking or assuming, even "knowing" to some extent, the equation $5x + 6 + 4x + 3 = -1 + 9x$ to be valid, to have an x that verifies it. When one ends up with $10 = 0$, it is not so much that $10 = 0$ is false, but that the *initial hypothesis was false*, that $5x + 6 + 4x + 3 = -1 + 9x$ is not equal and thus false, as students ultimately asserted. It says that having thought of $5x + 6 + 4x + 3 = -1 + 9x$ as valid was a false assumption.

However, what one student added about this $10 = 0$ conclusion is that "it's like if x is not worth the same on both sides of the equation." As Deledicq (2004) asserts, an

equality is the assertion that the first side of the equation represents the same quantity as the second one. This is also part of the starting assumption, that is, that both sides of the equation are equal. In the case of $5x + 6 + 4x + 3 = -1 + 9x$, this assumption is false, since it is conditional on values of x that were, however, shown to be impossible. Hence, this points to the curious idea that the expression on the left side, $5x + 6 + 4x + 3$, would have to be different from the member on the right side, $= -1 + 9x$, for it to be valid. Or, in other words, as Stacey and MacGregor (2000) point to, that the x in "$5x + 6 + 4x + 3$" would need to be different from that in "$-1 + 9x$." But if x is not the same on both sides of the equation, then weirdly, $9x - 9x$ would not be equal to zero.

Therefore, assuming $5x + 6 + 4x + 3 = -1 + 9x$ is valid *allows* for the operations and manipulations to be made on the equation. But these manipulations might have been prohibited by the fact that the values for x are not the same in the equation. Hence, the conclusion $10 = 0$ not only highlights how the validity of the equation is wrongly assumed, but also that for this assumption to be true then the values of x would have to not be the same. And, as mentioned, this would make the legitimacy of the algebraic manipulations undertaken problematic, where they also rest on this same hypothesis that makes $9x - 9x$ equal to zero.

Bridging these ideas with the previous point highlights the fact that solving an equation for x is not merely about solving for x: it requires *assuming* that the equation is valid, that it is an equality, which then allows for operating on it through algebraic manipulations. Conceiving it in this way makes the solving of an equation not only about finding the value of x that verifies the equation, but also about *attempting to disrupt and remove its conditional status*, for making it stable, making it an equality. Solving an equation for x is also confirming the initial hypothesis, to show it is true, in some sort of circular quest of finding the values that makes it warranted: (a) to have assumed that the equation was valid, and (b) to have manipulated it that way to find that value of x. As solving $5x + 6 + 4x + 3 = -1 + 9x$ shows, this might not always be possible, and so this initial assumption might be false, that is, shown contradictory.

Concluding Remarks

In line with the introduction, the issues raised about solving equations offer a salient illustration of how some mathematical insights can be gained from students' ways of solving, even when the strategies engaged with are quite simple. In this case, the proportion strategy provided the opportunity to reflect upon alternative ways of conceiving the usual quest of finding the values that satisfy the equation. Combined with the assertion of impossibility, it raised to the fore issues of the conditional status of equations, and how solving them makes it a quest to tame this conditionality for making the equation stable. By doing this, it brought forth the fundamental difference between an equation and an equality, and the interrelation that can be traced between them.

References

Baruk, S. (1995). *Dictionnaire des mathématiques.* Seuil.

Butlen, D., & Pézard, M. (2000). Calcul mental et résolution de problèmes numériques au début du collège. *Repères IREM n° 41*, 5–24.

Deledicq, A. (2004). *Maths – Lycée*. Éditions de la Cité.

Proulx, J. (2013). Le calcul mental au-delà des nombres : Conceptualisations et illustrations avec la résolution d'équations algébriques. *Annales de didactique et de sciences cognitives, 18*, 61–90.

Proulx, J., Lavallée-Lamarche, M.-L., & Tremblay, K.-P. (2017). Équations algébriques et activité mathématique en calcul mental : Regard sur les défis d'enseignement. *Annales de didactique et de sciences cognitives, 22,* 43–65.

Stacey, K., & MacGregor, M. (2000). Learning the algebraic method of solving problems. *Journal of Mathematical Behavior, 18*(2), 149–167. doi.org/10.1016/S0732-3123(99)00026-7 ▲ file

continued from page 37

References

Burns, M. (2018, October 1). When you multiply by 10, just add a zero? Horrors!?! *Marilyn Burns Math Blog*. www.marilynburnsmathblog.com/when-you-multiply-by-10-just-add-a-zero-horrors/

Costello, D. (2016). MB4T (Mathematics by and for Teachers): Teaching place value through the operations. *Ontario Mathematics Gazette, 55*(2), 20–21.

Khan, S. (n.d.). Multiplying with area model: 16 x 27. *Khan Academy.* www.khanacademy.org/math/4th-engage-ny/engage-4th-module-3/4th-module-3-topic-c/v/area-model-for-multiplication

Lorenzen, T. (2014, October 27). *Box method multiplication* [Video file]. YouTube. www.youtube.com/watch?v=ghssnpDtOlY.

Morrow, L., & Kenney, M. (Eds.). (1998). *The teaching and learning of algorithms in school mathematics.* NCTM. (This is the 1998 NCTM Yearbook.)

Small, M. (2009). *Big ideas from Dr. Small: Creating a comfort zone for teaching mathematics, grades 4–8.* Nelson Education.

Small, M. (2010). *Big ideas from Dr. Small: Creating a comfort zone for teaching mathematics, grades K–3.* Nelson Education.

Smith, D.E. (1958). *History of mathematics, vol. II.* Dover Publications. (See this classic for a thorough look at the history of methods of computation and terminology.)

[tonicbottle] (2011, January 25). Multiplying two 2 digit numbers using grid method KS2 KS3 [Video file]. YouTube. https://youtu.be/kGFZhdweTxI

Van de Walle, J.A. (2001). *Elementary and middle school mathematics: Teaching developmentally* (4th ed.). Pearson Education.

Also see www.explodingdots.org/ for a similar approach with activities for teachers and students. ▲

▲ Elementary Math Matters: It's a bird, it's a plane... it's a math manipulative!



LYNDA COLGAN
lynda.colgan@queensu.ca

Lynda Colgan is a Professor at the Faculty of Education at Queen's University. As a lifelong educator, Dr. Colgan constantly searches for creative avenues to engage students, parents, and educators in mathematics at home and in the classroom. She is inspired to do so by Seymour Papert, who once called upon his audience of "computers in education consultants" and mathematics teachers to bring about change by saying: "Instead of trying to make children love the math they hate, make a math they'll love."

> *It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair….*
>
> ~ Charles Dickens, *A Tale of Two Cities*, 1859

2020 was a year unlike any that preceded it. For those of us in Ontario, March 12, 2020 will stand out as *THE* day when the world as we knew it was turned upside-down and inside-out while time, it seemed, impossibly, accelerated yet simultaneously came to a crashing stop: COVID-19 was closing our schools.

Educators from Kindergarten to post-secondary graduate classes accepted the instantaneous challenge of a transition to remote teaching. Navigating multiple platforms and learning about new software and cloud-based sharing were no longer optional: virtually every sphere of educators' work and life came to be dominated by digital technology in that instant.

There are downsides to virtual classrooms, remote meetings, and online resource management. Those perspectives have been exhaustively documented, discussed, and deliberated. I will not revisit the variety of views here. Instead, I would like to focus on one upside that must be celebrated and shared because it has the potential to make enormous, positive differences to mathematics education in Ontario across panels.

In a recent *Pecha Kucha*[1] at Queen's Faculty of Education's *Homecoming*, I described how, when in my BEd mathematics education classes, I presented this image of a "Yield" sign (Figure 1) and asked the question, *What do you see?* The predictable response was that the photo was of an "upside-down triangle."



Figure 1: *What do you see?*

The first year that this happened, I was shocked, but I soon understood why this was the standard answer: textbooks of the day almost always presented triangles in the same orientation, as we see in Figure 2. Teachers almost always sketch triangles the same way on the blackboard/whiteboard/document camera, i.e., base running parallel to the bottom of the writing surface. Cognitive psychologists tell us that we understand what we see by deriving meaning from visuals. In other words, our brains use statistical regularities to form cognitive models based on the features that make "object recognition" possible (Clarke & Tyler, 2015). Thus, the more often we see a triangle in the same configuration, the more fixed our personal mental image of *triangle-ness* becomes.

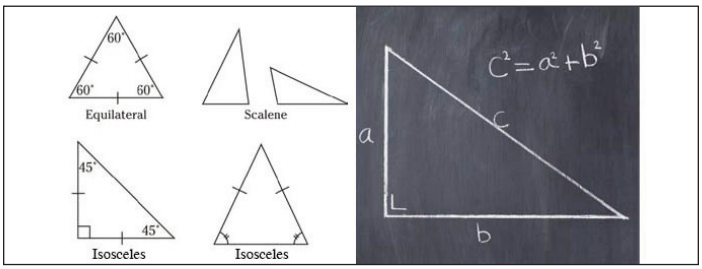


Figure 2: *Textbook triangles?*

Instant recognition of words on this page, the coffee cup on your desk, or the person who just walked into the room is based on our accurate and rapid extraction of object identity from familiar patterns that we see, all of which are stored in our neural networks.

More challenging is our ability to still detect and classify objects from among tens of thousands of possibilities—in other words, distinguishing a cup from a cup from a cup from a cup, but recognizing various shapes, colours, and sizes as "cups." "Core object recognition," i.e., the ability to rapidly recognize objects despite substantial appearance variation, is a complex neural processing function that is developed over time from basic building blocks—individual neurons

forming networks that form the circuitry driving our ability to generalize *cup-ness* (DiCarlo et al., 2012).

Of necessity, remote instruction sidelined traditional teaching tools. Students (including teacher candidates at Ontario's 17 Faculties of Education) had no access to commercial mathematics manipulatives, opening the door to improvisational "thinker tools" that were used by students at home to solve problems and concretize solution processes, then shared via various digital cameras: images of objects from around the home appropriated for use as math "objects." Not surprisingly, in addition to issues of focus, the screen images of the objects were likely to be rotated, inverted and dilated, obscured by shadows and bright light, or buried by a busy tablecloth pattern. In other words, none would have been "textbook," and therefore all became possibilities stored in the mental menu of objects to be classified as "triangles," for example.

The images in Figure 3 illustrate the representations shared in response to an introduction for a multiplication lesson, focusing on arrays. They showcase the fact that anything and everything—from baseball caps to action figures to hairbands, and face masks—can be a purposeful and illuminative mathematics manipulative. Note that while the student solutions showed different orientations, visual irregularities (e.g., imperfect rows and columns) and size of objects to model 3x3 and 3x4, all are correct, and, also importantly, personal. It is these visual differences that will generate the "cognitive dissonance" in our young mathematicians—stretching them to see the similarities among their peers' solutions. Surely it is a richer mathematical learning experience than plodding through endless "standard" triangle images.

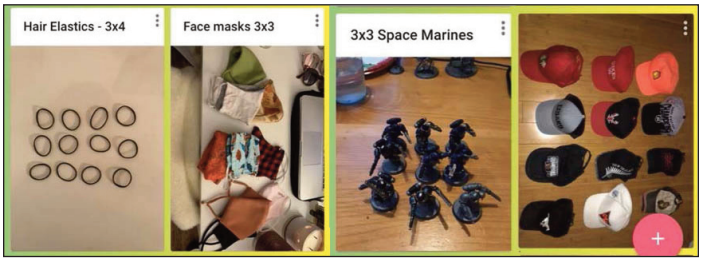


Figure 3: *Images of 3x3 and 3x4*

In remote teaching and learning environments, every task becomes a visuospatial task with increased cognitive demands: the arrangement of 3D action figures on the kitchen table becomes a 2D image on multiple digital screens and requiring not just identification of the representation, but also analysis because of transformations (e.g., changes in the object's position, size, pose, and background context). From a neural perspective, the task of the child's visual system is to somehow establish the equivalence of all of these response patterns (i.e., it is still an array) while, at the same time, not confuse any of them with mental images of any other possible objects (i.e., it is not a pictograph).

Many positive outcomes result when a student uses their own environment as an educational aid for learning and problem solving, such as choosing their own artifacts to communicate their thinking processes and procedural strategies. The notion of extracting the curriculum from the child's environment and learning as application of and reflection on their immediate world and experiences is not new: hands-on, self-activity directed learning was the hallmark of the very first Kindergarten, founded in 1837 by Friedrich Froebel[2].

The first manipulative set (Figure 4) was in fact a set of "gifts" designed by Froebel to support young students' learning. The first "gift" was a ball of yarn to be held, dropped, rolled, swung, hidden, and squished—activities that Froebel believed would lead the child to acquire knowledge of objects and spatial relationships, movement, speed and time, colour and contrast, and weights and gravity. Other gifts included wooden spheres, cubes, blocks of various shapes (including prisms), and proportional planks (twice as long, half as wide). The wooden shapes could be taken apart and reassembled, spun (after a dowel was placed through a hole in the solid), used in construction, and explored.



Figure 4: *A set of "gifts"*

Fast forward to 2020, when classrooms were abundant with commercial manipulative kits: plastic linking cubes, pattern blocks, base-10 blocks, colour tiles, plastic bears (in four sizes, weights, and colours), fraction circles, and more (Figure 5).

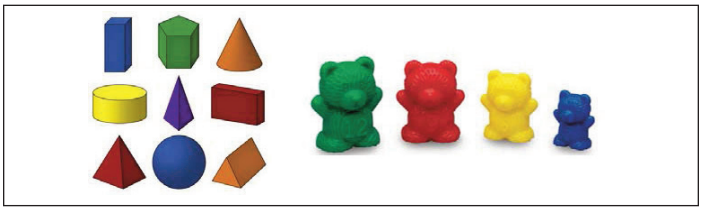


Figure 5: *2020 manipulatives*

There is no question that counters are essential resources for learning early addition and subtraction, or that

pattern blocks support conceptual development of fractions and area. The research on manipulatives and learning is unequivocal—concrete objects help children see mathematics. Abstract concepts can be difficult for young learners, but manipulatives make them relatable because children can make better sense of the concept, through sight and touch.

However, when manipulatives become the manifestation of *school* math because they are *school* tools, they become limiting. How many children, at first glance, will identify a pool noodle as a cylinder, or an interlocking paving stone as a hexagonal prism? At our peril, we seem to have achieved passive object recognition at the cost of the more intentional and robust core recognition. By homogenizing the objects our students see and use in math, we may have stripped them of the opportunity to identify objects over a large range of viewing conditions and possible objects.

Research tells us that neurons are activated by, at least moderately, complex combinations of visual features, and that core recognition is a function of tolerance to identity-preserving image transformations such as changes in object size, contrast, in-depth and in-plane rotation, and presence of background or clutter (DiCarlo et al., 2012). It is said that the obvious is difficult to spot and easy to overlook. The same can be said for the power of junk-drawer manipulatives.

Notes

1 Japanese for "chit chat"; it is a presentation format that uses 20 slides or images that are displayed for 20 seconds each. The slides will move automatically as the presenter is speaking.

2 Froebel founded his own school and called it "Kindergarten," or the children's garden. Prior to Froebel's Kindergarten, children under the age of seven did not attend school because at the time, it was believed that young children did not have the ability to focus or to develop cognitive and emotional skills before this age. Froebel labelled his approach to education as "self-activity." This idea allows the child to be led by their own interests and to freely explore them. The teacher's role, therefore, was to be a guide rather than lecturer (Scholastic Early Childhood Today, 2000).

References

Clarke, A., & Tyler, L.K. (2015). Understanding what we see: How we derive meaning from vision. *Trends in Cognitive Sciences, 19*(11), 677–687. doi.org/10.1016/j.tics.2015.08.008

DiCarlo, J.J., Zoccolan, D., & Rust, N.C. (2012). How does the brain solve visual object recognition? *Neuron, 73*(3), 415–434. doi.org/10.1016/j.neuron.2012.01.010

Early Childhood Today Editorial Staff. (2000). Pioneers in our field: Friedrich Froebel – founder of the first kindergarten. Scholastic Early Childhood Today™. www.scholastic.com/teachers/articles/teaching-content/pioneers-our-field-friedrich-froebel-founder-first-kindergarten/ ▲

▲ Why We Need to Teach Metacognition in Our Math Classes



DR. JEFF IRVINE
dr.jeffrey.irvine@bell.net

Jeff Irvine, PhD, has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

Metacognition is often described as "thinking about thinking" (Flavell, 1979). While a catchy tagline, this doesn't really tell us much about what metacognition is or why we should provide instruction about it. A better description of metacognition is "the self-management of learning" (Hattie, Biggs, & Purdie, 1996). Essentially metacognition is what students need to do before actually solving a problem: goal setting, planning, strategy selection, evaluating alternatives; as well as during problem solving: monitoring clarity, monitoring accuracy, debugging; and after problem solving: evaluating the reasonableness of results, considering implications of a solution, performing "What if?" analysis (Marzano & Kendall, 2007; Meijer, Veenman, & van Hout-Wolters, 2006).

Metacognition can be seen in several of the mathematical processes: selecting tools and strategies, connecting, reflecting, reasoning, and proving. While clearly important in problem solving, we also see metacognition involved in choosing the appropriate "tool" from the student's mathematical toolkit. For example, when solving a system of equations, the student has several possible strategies: substitution, comparison, elimination, graphing, using tables of values. Students need to learn to choose the best alternative based on *conditional knowledge*; i.e., are the equations in standard form, slope-y-intercept form, and so on?; what level of accuracy is needed for the solution?; does the solution need to be shown in algebraic form, graphical form, or tabular form? We cannot assume that these choices will be made automatically by the student unless we

consciously teach these alternatives and when they are most appropriate to use.

Teaching metacognitive skills has been shown to have a positive impact on student achievement in mathematics (Schneider & Artelt, 2010; Veenman,van Hout-Wolters, & Afflerbach, 2006), as well as supporting student self-regulation (Meijer, Veenman, & van Hout-Wolters, 2006). In a meta-analysis of over 800 studies involving over 2 million students, Hattie (2009) found that teaching metacognitive skills had an effect size of 0.69 on student achievement. Effect size is a way to compare studies and assign a numerical value to the study's impact.

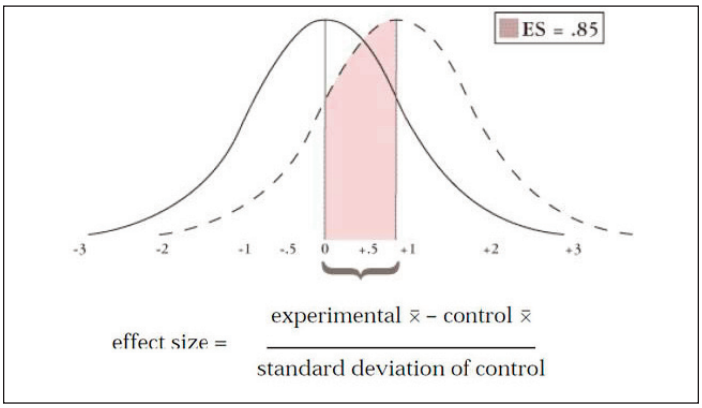
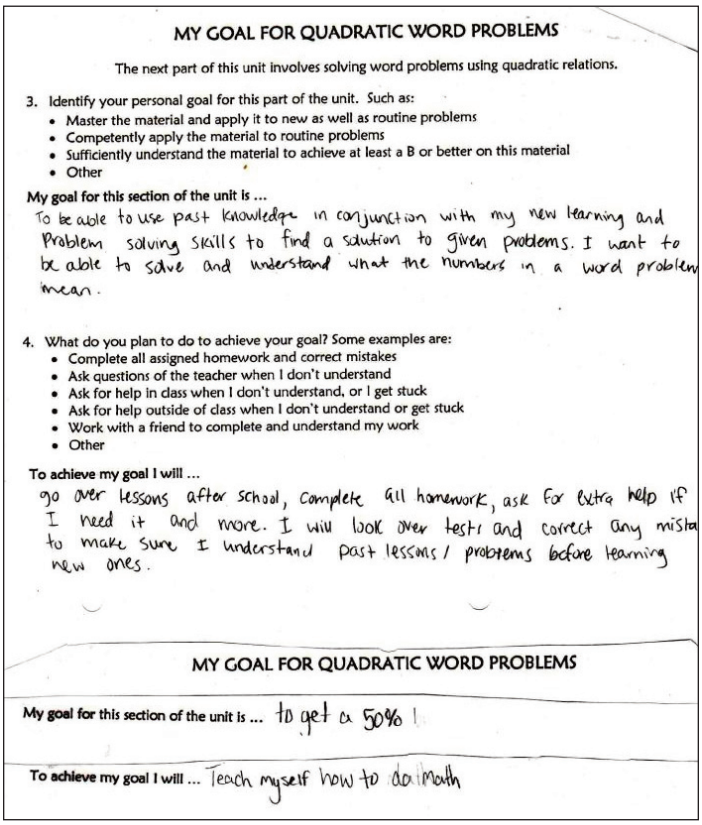


Figure 1: *Effect size computation; =mean score*


MY GOAL FOR QUADRATIC WORD PROBLEMS

The next part of this unit involves solving word problems using quadratic relations.

3. Identify your personal goal for this part of the unit. Such as:
   - Master the material and apply it to new as well as routine problems
   - Competently apply the material to routine problems
   - Sufficiently understand the material to achieve at least a B or better on this material
   - Other

My goal for this section of the unit is ...
To be able to use past knowledge in conjunction with my new learning and Problem solving skills to find a solution to given problems. I want to be able to solve and understand what the numbers in a word problem mean.

4. What do you plan to do to achieve your goal? Some examples are:
   - Complete all assigned homework and correct mistakes
   - Ask questions of the teacher when I don't understand
   - Ask for help in class when I don't understand, or I get stuck
   - Ask for help outside of class when I don't understand or get stuck
   - Work with a friend to complete and understand my work
   - Other

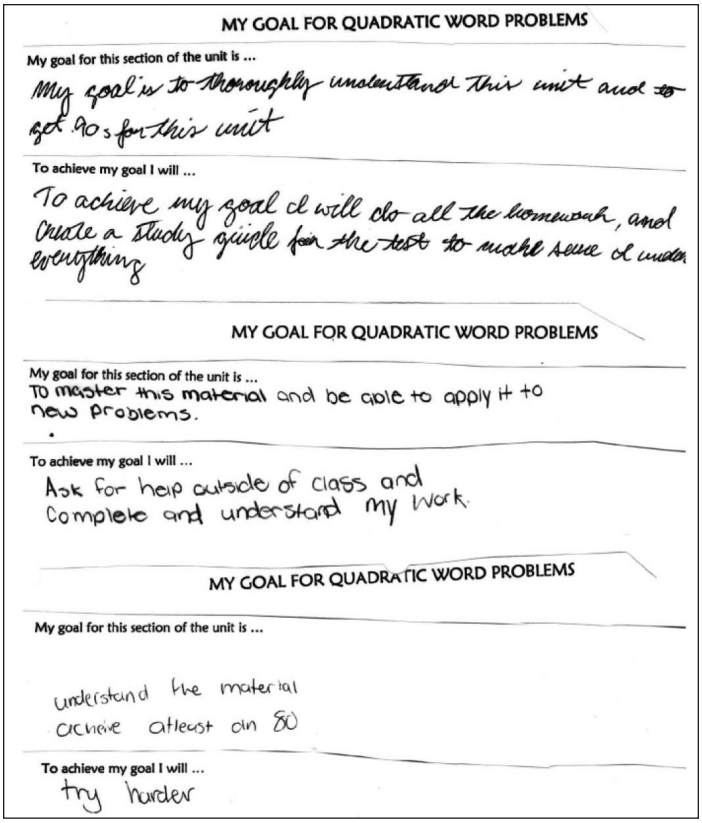
To achieve my goal I will ...
go over lessons after school, complete all homework, ask for extra help if I need it and more. I will look over tests and correct any mista to make sure I understand past lessons / problems before learning new ones.

MY GOAL FOR QUADRATIC WORD PROBLEMS

My goal for this section of the unit is ... to get a 50% !

To achieve my goal I will ... Teach myself how to do math

Figure 2A: *First example*


MY GOAL FOR QUADRATIC WORD PROBLEMS

My goal for this section of the unit is ...
My goal is to thoroughly understand this unit and to get 90s for this unit

To achieve my goal I will ...
To achieve my goal I will do all the homework, and create a study guide for the test to make sure I understand everything

MY GOAL FOR QUADRATIC WORD PROBLEMS

My goal for this section of the unit is ...
To master this material and be able to apply it to new problems.

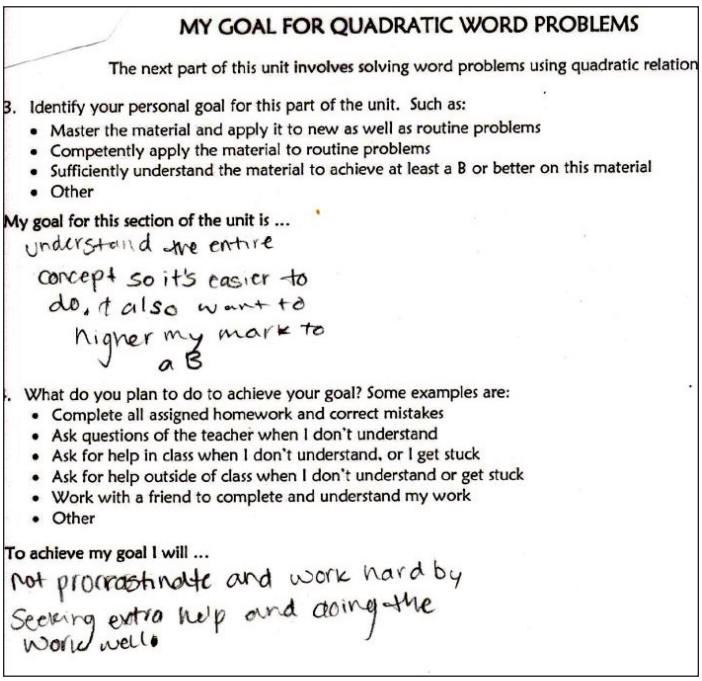
To achieve my goal I will ...
Ask for help outside of class and Complete and understand my work.

MY GOAL FOR QUADRATIC WORD PROBLEMS

My goal for this section of the unit is ...
understand the material
achieve atleast an 80

To achieve my goal I will ...
try harder

Figure 2B: *Second example*


MY GOAL FOR QUADRATIC WORD PROBLEMS

The next part of this unit involves solving word problems using quadratic relation

3. Identify your personal goal for this part of the unit. Such as:
   - Master the material and apply it to new as well as routine problems
   - Competently apply the material to routine problems
   - Sufficiently understand the material to achieve at least a B or better on this material
   - Other

My goal for this section of the unit is ...
understand the entire concept so it's easier to do, I also want to higher my mark to a B

4. What do you plan to do to achieve your goal? Some examples are:
   - Complete all assigned homework and correct mistakes
   - Ask questions of the teacher when I don't understand
   - Ask for help in class when I don't understand, or I get stuck
   - Ask for help outside of class when I don't understand or get stuck
   - Work with a friend to complete and understand my work
   - Other

To achieve my goal I will ...
not procrastinate and work hard by seeking extra help and doing the work well.

Figure 2C: *Third example*

Figure 1 illustrates effect size, using a value of 0.85. In the case of teaching metacognitive strategies, since for values close to the mean, 1 standard deviation = 34 percentile points; and effect size is measured in units of standard deviation; 0.69 standard deviations = 0.69 x 34 =

23.46 percentile points. This would move a student from the 50th percentile to above the 73th percentile in achievement. This is a significant improvement in student achievement.

There are many instructional strategies that can be used to teach metacognition. Direct instruction can be used, such as explicitly asking students to choose goals and ways of reaching those goals (e.g., "learning goals" and "success criteria"). Figures 2A, 2B, and 2C show examples of direct instruction with some student responses. Students were asked to identify their personal goal for this part of the unit, and select or create a strategy to achieve their goal.

This was the first time that students were asked about goal setting, and the responses varied from thoughtful to vague notions such as "teach myself how to do math" or "try harder." Over the course of the semester, as students were repeatedly asked about goal setting, their responses were increasingly thoughtful and their strategies more fulsome.

Student responses were anonymous. However, assuming that the responses were not facetious, the "try harder" comment may indicate student willingness, but needing focus. For students with this type of response, I would use heuristics (both teacher- and student-generated) to help the student focus on exactly what type of goal setting and planning is needed. For the "teach myself math" student, this could indicate a willingness to make an effort, but an assumption on the student's part that he or she is on his or her own as far as guidance. For these students, some work on goal setting and why it is important helps.

Another strategy for teaching metacognition is to use think-aloud. Figure 3 is an example of a think-aloud, where the teacher verbalizes his or her thoughts when solving a problem. Teachers often do this automatically, but to students, sometimes what we do looks like magic. Think-aloud exposes our thinking and allows students to learn our strategies. As students become comfortable with think-aloud, they can vocalize their own strategies with a partner, or use the timed retell strategy, where one student vocalizes his or her think-aloud, and the partner then demonstrates his or her understanding by retelling the strategy back to the other student.

A really powerful way of developing students' metacognitive skills is to have students (in groups) critique problem-solving plans of others. For an example of this, see "How Much Hot Chocolate" in Irvine (2019). As students gain confidence, they can critique problem-solving plans developed by other student groups in their classes.

Metacognition is very important in problem solving, especially for open problems or student-generated hypotheses. However, metacognitive skills can also be developed for a part of a unit, as shown in Figures 2A, 2B, and 2C. In my experience, asking students to set goals for an entire unit or an entire course is not productive, since the results are usually so vague that a plan to achieve the goals is not possible or effective.

In the THINK-ALOUD strategy, the teacher (or a student) vocalizes his or her thought processes while solving a problem.

PROBLEM: Find the equation of the line perpendicular to the line $y = -2x + 6$, and having an x-intercept of 8.

"What am I asked to find?

The equation of a line.

The equation of a line has the form $y = mx + b$.

So, I need the value of the slope and the y-intercept.

I don't have either one given to me.

I'll try to find the slope first.

What information do I have?

The line I want is perpendicular to $y = -2x + 6$.

I know the slope of this line is –2.

I know that the line I want is perpendicular to this one.

Perpendicular lines have slopes that are negative reciprocals of each other.

So the slope of the line I want must be $\frac{1}{2}$.

Now I know that my line is $y = \frac{1}{2}x + b$.

I still need to find the value of b.

What other information do I have in the problem?

The x-intercept of the line I want is 8.

But I need the y-intercept.

An x-intercept is a point, the point where the line crosses the x-axis.

Any point on the x-axis can be written as (something, 0).

So, this x-intercept of 8 is the same as the point (8,0).

How can I use this point to find the value of b?

The point (8,0) is on the line I want.

So, if I substitute 8 for x and 0 for y in $y = \frac{1}{2}x + b$, I can figure out the value of b.

$0 = \frac{1}{2}(8) + b$.

Solve this equation for b:

$0 = 4 + b$

$-4 = b$.

So, the y-intercept of my line is –4.

The equation of the line I want must be $y = \frac{1}{2}x - 4$.

How can I check to make sure I'm right?

I could graph both this line $y = -2x + 6$ and the line to make sure they're perpendicular.

Then I can verify that the x-intercept of my line is 8.

It would be easiest to do this on a graphing calculator or with graphing software.

If I don't have technology, I could find the x-intercept of my line by substituting 0 for y and solving, and make sure that the product of the slopes of the two lines is –1.

This would make sure the two lines are perpendicular and that my line satisfies both conditions in the problem.

Figure 3: *Example of a think-aloud strategy*

Another instructional strategy for teaching metacognition is to teach heuristics, such as the classic Pólya heuristic: "understand the problem; make a plan; carry out the plan; reflect/look back" (Pólya, 1945). This can be found in the front matter of all our mathematics curriculum policy documents (Ontario Ministry of Education, 2005, 2007). I have used this strategy frequently, in a more fulsome form (Figure 4). For best results, problem-solving flow charts should be constructed by students, not just provided by the teacher.

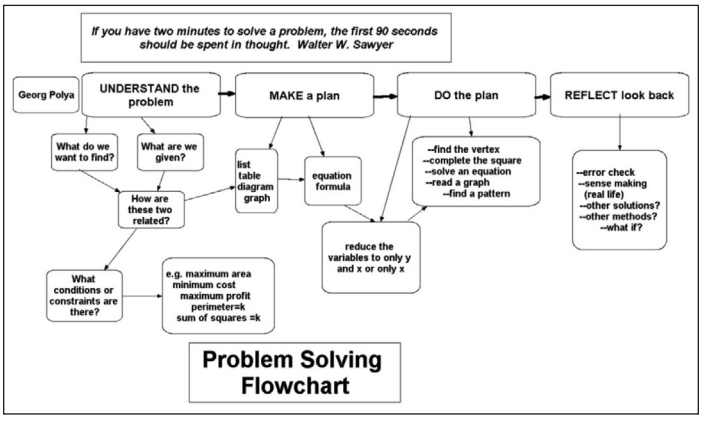


Figure 4: *A problem-solving heuristic for quadratic relations problems based on Georg Pólya*

In addition to think-aloud, here are some other instructional strategies that support teaching metacognition: open questions (for an excellent source for open questions, see Small and Lin (2010)); timed retell; decision trees (constructed by student groups); anticipation guides; what/so-what double entry. Details of these strategies can be found online. Metacognitive skills can be developed over the course of the semester, with focused activities where appropriate.

Teaching metacognitive strategies helps students in the critical "pre-cognitive" stages, planning, goal setting, selecting strategies, and evaluating solutions. We cannot assume that students will acquire these skills on their own, so we must employ intentional instruction to achieve this. Acquiring metacognitive skills improves student achievement in mathematics and builds students' self-regulation skills, which transfer to other areas of their learning. Students also develop increased confidence (Irvine, 2016) and become more motivated to learn mathematics (Irvine, 2015).

References

Flavell, J.H. (1979). Metacognition and cognitive monitoring: A new area of cognitive development inquiry. *American Psychologist, 34*(10), 906–911.

Hattie, J. (2009). Visible learning. Routledge.

Hattie, J., Biggs, J., & Purdie, N. (1996). Effects of learning skills interventions on student learning: A meta-analysis. *Review of Educational Research, 66*(2), 99–136. doi.org/10.3102/00346543066002099

Irvine, J. (2015). Problem solving as motivation in mathematics: Just in time teaching. *Journal of Mathematical Sciences, 2,* 106–117.

Irvine, J. (2016). Fueling the mind: Maximizing student learning and performance. *Ontario Principals Council Register,* February 6–14.

Irvine, J. (2019). Assessing the math processes and the learning skills. *Ontario Mathematics Gazette, 57*(4), 33–37.

Marzano, R.J., & Kendall, J.S. (2007). *The new taxonomy of educational objectives* (2nd ed.). Corwin Press.

Meijer, J., Veenman, J.V.J., & Van Hout-Wolters, B.H.A.M. (2006). Metacognitive activities in text- studying and problem-solving: Development of a taxonomy. *Educational Research and Evaluation, 12*(3), 209–237. doi.org/10.1080/13803610500479991

Ontario Ministry of Education. (2005). *The Ontario curriculum, grades 9 and 10: Mathematics* (revised). Queen's Printer for Ontario.

Ontario Ministry of Education. (2007). *The Ontario curriculum, grades 11 and 12: Mathematics* (revised). Queen's Printer for Ontario.

Pólya, G. (1945). *How to solve it.* Princeton University Press.

Schneider, W., & Artelt, C. (2010). Metacognition and mathematics education. *ZDM Mathematics Education, 42*(2), 149–161. doi.org/10.1007/s11858-010-0240-2

Small, M., & Lin, A. (2010). *More good questions: Great ways to differentiate secondary mathematics instruction.* Nelson.

Veenman, J.V.J., Van Hout-Wolters, B.H.A.M., & Afflerbach, P. (2006). Metacognition and learning: Conceptual and methodological considerations. *Metacognition Learning, 1*(1), 3–14. doi.org/10.1007/s11409-006-6893-0 ▲

OAME Sticky Graph Pads

Great for Student Use!



3 colours, 2.5" x 3"
50 sheets per pad

$10.00
for 10 pads

Price includes taxes.
Shipping and handling extra.

ORDER BY MAIL, FAX, OR ONLINE:
OAME, 4751 Mack Ave., Town of Plympton-Wyoming, ON NON 1J6
Phone/Fax: (519) 899-4203, or order online at www.oame.on.ca
Do not send cash or cheque. You will be invoiced, with shipping and handling costs added.

The Power of Equivalent Families in Mathematics



JÉRÔME PROULX
proulx.jerome@uqam.ca

Jérôme Proulx is a professor of mathematics education (profmath.uqam.ca/jproulx), and runs the Laboratoire Épistémologie et Activité Mathématique *(www.leam.uqam.ca). His research focuses on epistemological issues related to cognition and/in mathematics. His current research program is dedicated to mental mathematics and school mathematics content development. (Photo by Nathalie St-Pierre of UQAM)*

In his well-known book *How to Solve It*, Pólya (1957) discusses and outlines heuristics for/when solving problems in mathematics. A number of these are related to the simplification of the problem: *auxiliary problem* (pp. 50–57), *decomposing and recombining* (pp. 75–85), *related problem* (pp. 98–114), *specialization* (pp. 190–197), and *variation of the problem* (pp. 209–214). What all these heuristics have in common and imply is the transformation of the original problem into a more accessible form by modifying the problem itself (or parts of it).

In various studies that my research team and I are conducting in classrooms on mental mathematics, where numerous constraints come into play (e.g., time, no paper or pencil), students often engage in specific strategies that align with these more general heuristics. In other words, in these contexts, some students use strategies that at times transform the tasks given into more accessible forms in order to succeed in solving them.

In the following, three of these specific strategies are presented. These strategies are discussed in order to draw out one interesting and powerful mathematical element that appears to lie at their core, namely, the establishment of equivalent families. This article aims to illustrate the power of these equivalent families in mathematics.

Example 1: Arithmetic Mean and Families of Distributions

In one study, mental mathematics work involved statistics with Grade 9 students (Proulx, 2017). One of the tasks given to students was the following (inspired from the *Chenelière Éducation* textbook *À vos maths !*):

Here are four distributions:

(a) 41, 42, 44, 45

(b) 41, 42, 43, 44, 45

(c) 41, 42, 47

(d) 40, 42, 44, 45

Which distributions have the same mean?

To solve this question, some students explained that they paid attention only to the units in each distribution, taking out the 40 from each datum. They then added the units and divided them by the number of data, obtaining the same mean of 3 for (a) and (b), which after adding back the 40, became 43 for both.

This strategy leads to consideration that what is "common" in a statistical distribution can be set aside to analyze other elements, without changing the problem and its solution. Hence, by taking out the 40's, students transformed the problem to simplify the solving process. This strategy shows, for example, that both distributions "41, 42, 44, 45" and "1, 2, 4, 5" share the same structure, differentiated by 40. So the form "1, 2, 4, 5" has the same properties as "41, 42, 44, 45": the mean is the same (through adding 40 to it), and the same can be said of the median.

Both distributions also share the same standard deviation, and this time, 40 does not need to be added to it because it is numerically identical: the distances from each datum to the mean are the same. It is also the case for any other measure of dispersion in both distributions (e.g., variance, range). In this sense, it is as if one distribution were translated to the other, by 40. Figure 1 illustrates this translation.

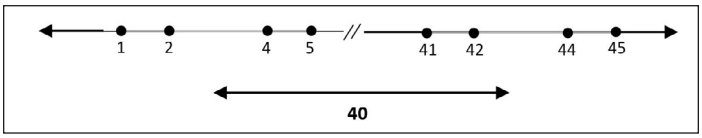


Figure 1: *Translation from left to right of the distribution by 40*

In this sense, this strategy can evoke an entire family of (connected) distributions that share the same structure, measures, and relationships between data, without having the same values for these data. Along the line from 3 to 43, there are numerous distributions, an entire family of those, that share the same structure and features: "8, 9, 11, 12"; "31, 32, 34, 35"; "3.5, 4.5, 6.5, 7,5"; etc. To go from one distribution to the next, all one has to do is to translate each datum by the number in the difference.

41, 42, 44, 45 ← 40 → 1, 2, 4, 5
3.5, 4.5, 6.5, 7.5 ← 37.5 → 41, 42, 44, 45
3.5, 4.5, 6.5, 7.5 ← 2.5 → 1, 2, 4, 5
8, 9, 11, 12 ← 33 → 1, 2, 4, 5
etc.

The equivalence of these distributions leads to the question, that there might be, in this entire family of equivalent distributions, one distribution that is simpler to work with and make computations with for evaluating its average, standard deviation, and so forth. Such a distribution could be used to infer back information on the initial distribution or on any other in the family. That distribution, as an example, might be the "1, 2, 4, 5," or it could be another like "–2, –1, 1, 2." Again, all one has to do in this case is to adjust the values of the measures obtained, e.g., for the mean or median, with the distance value between both distributions used, and then infer the properties aimed for. In other words, all these distributions are connected to one another in this family and are not isolated or independent.[1]

Example 2: Algebra and Families of Equivalent Equations

In another study, work was conducted on solving equations in mental mathematics environments with students taking mathematics at the university (Proulx, 2013). Students had to solve a variety of equations without paper and pencil. When students were given the task, "Solve $\frac{2}{5}x = \frac{1}{2}$ for x," one student explained first the aim to remove the $\frac{1}{2}$ by doubling the entire equation (multiplying it by 2 to create an equivalent equation). This gave him $\frac{4}{5}x = 1$, which he then multiplied by the inverse of $\frac{4}{5}$ to obtain the value of x in this equation.

This procedure to find the value of x raises questions about what we do when we solve equations. Somehow, the usual discourse is that we attempt to isolate the x in order to know what it is worth, and this is to be attempted in potentially the least number of steps possible. However, in this case, the first step did not aim to work on the x, and focused instead on the $\frac{1}{2}$ in the equation, in order to transform it. To avoid dealing with the $\frac{1}{2}$, the student got rid of it, obtaining another equation, conceived to be more accessible, which shared the same solution. This strategy of transforming the equation by doubling it, albeit perhaps a local action and working only for this specific equation, underlines the fact that solving an equation for x can be seen as a matter of establishing equivalent equations. We know that we produce equivalent equations when we solve equations. Here, however, producing equivalent equations can define the activity of solving itself. Through the various steps to solve for x, a family of equivalent equations, all connected to one another in sharing the same solution, is produced.

Also, it happens that one of these equivalent equations, one of the equations in this entire family of equivalent equations, is in the form "*x=something*." We normally see this one as the end point, as the final step that culminates in isolating the x, but it is in fact only one member taken from an entire family of equations; to some extent, it is merely that it is usually easier to read the value of x from it than from others (see also Arcavi, 1994, on this issue). Solving an equation for x is not about isolating x as much as it is about transforming the initial equation to obtain other equations, a series of them, that would be easier to read and from which to find the value of x.

In this case, $x = \frac{5}{4}$ is part of a host of equivalent equations, all in the same family: from $\frac{2}{5}x = \frac{1}{2}$ to $5 = 4x$, passing by $2x = \frac{5}{2}$ and $\frac{4}{5}x = 1$. At first sight, all these equations do not necessarily look equivalent (e.g., where $\frac{4}{5}x = 1$ is easily relatable to $= \frac{5}{4}$, $5 = 4x$ is harder to link directly to $\frac{2}{5}x = \frac{1}{2}$), but they do share the same solution. And there would be many others in this family, an infinity of them, all equivalent and connected to the more readable form $x = \frac{5}{4}$. Solving for x becomes a quest for finding equivalent equations, in more readable forms, that share the same properties and solution as the initial equation and that are part of an entire family of equivalent equations.

Example 3: Area and Families of Area-Equivalent Triangles

A third example of making transformations of the problem came from sessions conducted in Grade 8 and Grade 9 classrooms about area, perimeter, and volume. One of the tasks given, shown on the front board, was to explain why the shaded triangle represents half of the enclosing rectangle (Figure 2):

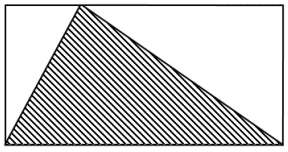


Figure 2: *"Explain why the shaded triangle has an area that is half the area of the rectangle" task*

Some students' strategies were to translate the summit along the side of the rectangle to produce a rectangular triangle (Figure 3).

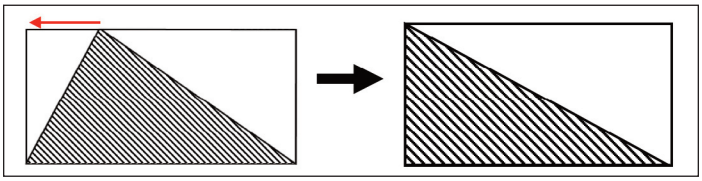


Figure 3: *Transforming/translating the triangle into a right-angle triangle*

At this point, depending on solvers, justifications varied from referring to issues of the formula for area (e.g., same base, same height, hence same area), to cutting and pasting, and even to Cavalieri's principle (see Proulx, 2008, on this issue).

What this strategy points to is how the problem was

transformed and connected to a more accessible one, that is, the right-angle triangle. This could be seen to offer the solution more easily: in relation to the fact that the diagonal cuts the rectangle in two parts of equal area, or that the formula of the triangle requires dividing the rectangle in two $\left(\frac{B \cdot h}{2}\right)$. This strategy also amounts to considering, as for the previous two examples, that there exists an entire family of connected triangles, sharing the same base and same height, so the same area (Figure 4).

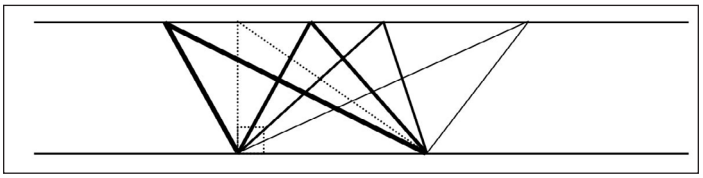


Figure 4: *An illustration of a family of area-equivalent triangles*

The right-angle triangle in dotted lines can be seen to represent the most accessible form of all the triangles in this family of equivalent-area triangles, where any triangle in the family can be linked back to this right-angle triangle in order to find its area. And this is the case for any triangle, where one only has to represent the family of equivalent-area triangles that this triangle is part of, for which all triangles share the same base and height, and then relate to the right-angle triangle.[2] In a nutshell, this strategy illustrates that *any triangle* is connected to an area-equivalent right-angle triangle, part of its family of area-equivalent triangles.[3]

Concluding Remarks on Families of Equivalences

These ideas related to families of equivalences have significant mathematical interest, and power: distributions are part of a family, which can be used to decipher some of its features; equivalent equations form an entire family of equivalent equations and lead to a specific way of understanding what solving an equation is about; triangles inhabit a continuum of area-equivalent family members, highlighting the links between various area-equivalent triangles (and a right-angle triangle). And one can then think of numerous other families of equivalences: families of equivalent fractions being one, families of equivalent systems of linear equations being another, and so forth.

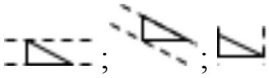
Pólya's (1957) heuristics related to simplifying a problem to find its solution are often referred to in mathematics problem solving. What is being brought forth here in relation to equivalent problems and solutions is that strategies aligned with this kind of heuristic open entire sets of equivalent families sharing the same structure and properties. These various equivalent families offer a way, as Pólya hints, of being able to solve the problem at stake more easily or at least to be able to engage with it. In addition, this view of families enables one to see how all distributions, equations, or triangles, as per the above cases, are linked and do not represent isolated forms: they are all connected to one another. In other words, these families of equivalences offer a dynamical entry into these well-known school mathematics concepts by linking mathematical forms together, through some transformations.

In this sense, these strategies do more than represent a way of attempting to solve the problem at hand. By bringing forth families of equivalences, these strategies enable the opening of an entire field of transformations and relations that are worth considering in (school) mathematics. It is in this sense that families of equivalences represent a powerful mathematical idea.

Notes

1 This can be linked to notions of scaling in advanced statistics to obtain distributions with a mean of 0 and standard deviation of 1, with $z = \frac{X - \mu}{\sigma}$.

2 It is to note that any right-angled triangle, because of its three sides, is simultaneously part of three distinct families of area-equivalent triangles, depending on which "base" is considered:



3 This idea of family of equivalent area also applies to other figures like parallelograms, rhombuses, trapezoids (see Proulx, 2008). Families of equivalences even applies to volume-equivalent solids like pyramids, prisms, cylinders, and so forth (see Proulx & Pimm, 2008).

References

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the learning of Mathematics, 14*(3), 24–35.

Pólya, G. (1957). *How to solve it* (2nd ed.). Doubleday.

Proulx, J. (2008). Establishing families of planar figures through the principle of Cavalieri. *Ontario Mathematics Gazette, 46*(4), 25–30.

Proulx, J. (2013). Mental mathematics and algebra equation solving. *Proceedings of CERME-8* (pp. 530–539), Antalya, Turkey.

Proulx, J. (2017). Le calcul mental en mathématiques : Quels potentiels pour l'activité mathématique ? *Canadian Journal of Science, Mathematics and Technology Education, 17*(4), 288–307. doi.org/10.1080/14926156.2017.1378833

Proulx, J., & Pimm, D. (2008). Algebraic formulas, geometric awareness and Cavalieri's principle. *For the Learning of Mathematics, 28*(2), 17–24. ▲

Checkers Solution (From page 22)

Checkers problem from the British Open 1996

From: www.checkercruncher.com/problems/start_next#

Problem: 11732

Solution: 8–11, 27–24, 2–7, 23–18, 14x23x30

▲ 2021-2022 OAME Board

DIRECT GENERAL INQUIRIES TO: EXECUTIVE DIRECTORS

Fred and Lynda Ferneyhough
4751 Mack Ave.
Plympton-Wyoming, ON N0N 1J6
Business & Fax: (519) 899–4203
eds@oame.on.ca

PRESIDENT

Judy Mendaglio
Mississauga, ON
judy.mendaglio@oame.on.ca

PAST-PRESIDENT

Paul Alves
Acton, ON
paul.alves@oame.on.ca

VICE-PRESIDENTS

Sandra Jean Price
Oshawa, ON
sandra.jean.price@oame.on.ca

Connie Quadrini
Stouffville, ON
connie.quadrini@oame.on.ca

EMERITUS

Bill Otto
Stillwater Lake, NS
bill.otto@oame.on.ca

CHAPTERS & REPRESENTATIVES

CHAMP: Kassandra Kobelak
Etobicoke, ON
kassandra.kobelak@oame.on.ca

GOLDEN: TBD

GVMA: Glen McMillan
Guelph, ON
glen.mcmillan@oame.on.ca

ISOMA: Cal Armstrong
Burlington, ON
cal.armstrong@oame.on.ca

MAC²: Paulene Washington
Barrie, ON
paulene.washington@oame.on.ca

NOMA: Heather Theijsmeijer
Kagawong, ON
heather.theijsmeijer@oame.on.ca

NWOAME: Corrine Russell-Pritoula
Thunder Bay, ON
corrine.russell-pritoula@oame.on.ca

$O_3$4ME: Jimmy Pai
Ottawa, ON
jimmy.pai@oame.on.ca

PRMA: Lori Yee
Pickering, ON
lori.yee@oame.on.ca

QSLMA: Siobhain Broekhoven
Kingston, ON
siobhain.broekhoven@oame.on.ca

SAME: Franca Genco
Scarborough, ON
franca.genco@oame.on.ca

SWOAME: Elizabeth Pearsall
Kingsville, ON
elizabeth.pearsall@oame.on.ca

TEAMS: Alvarine Aldridge
Toronto, ON
alvarine.aldridge@oame.on.ca

WOMA: TBD

Y⁴MA: Gina Micomonaco
Richmond Hill, ON
gina.micomonaco@oame.on.ca

DIRECTORS – TERM EXPIRING 2021

Freda Liu
Toronto, ON
freda.liu@oame.on.ca

Melissa Black
Kanata, ON
melissa.black@oame.on.ca

DIRECTORS – TERM EXPIRING 2022

Jane Silva
North York, ON
jane.silva@oame.on.ca

Liisa Suurtamm
St. Catharines, ON
liisa.suurtamm@oame.on.ca

Victoria Maystruk
Peterborough, ON
victoria.maystruk@oame.on.ca

Timothy Sibbald
North Bay, ON
tim.sibbald@oame.on.ca

NCTM REPRESENTATIVE

Jacqueline Hill
Lindsay, ON
jacqueline.hill@oame.on.ca

WEBSITE COORDINATOR

Greg Clarke
Orillia, ON
web@oame.on.ca

GAZETTE EDITOR

Timothy Sibbald
North Bay, ON
gazette@oame.on.ca

ABACUS CO-EDITORS

Mary Lou Kestell
Toronto, ON
marylou.kestell@oame.on.ca

Kathy Kubota-Zarivnij
Scarborough, ON
kkz@oame.on.ca

OAME WEBSITE WWW.OAME.ON.CA