8-PAGE INSERT ELEMENTARY ABACUS

Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

Vol. 61#2
December 2022

IN THIS ISSUE

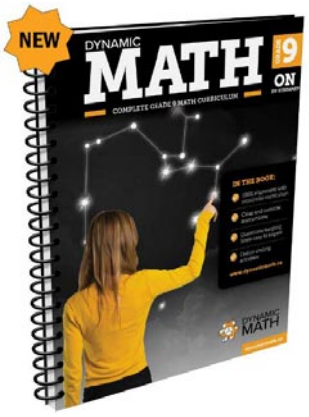
- Editor's Report
- President's Message
- Blast from the Past! John Charles Fields
- OAME/NCTM Report: Happy Fresh New School Year!
- What's the Problem? Fun with Factorials!
- Linking Literacy and Math: Using Ill-Defined Problems to Remove the Barrier Between School and Lived Experiences of Students
- In the Middle: How Teacher Talk Meets the Thinking Classroom
- Dynamical View of Proportionality: Families of Ratios
- Elementary Math Matters: The Anchor, the Deputy Prime Minister, and the Math Ed Professor
- OMCA News: December 2022
- Coding in the Classroom: The Destreamed Grade 9 Mathematics Curriculum
- A Deep Dive into a Desmos Activity
- Mb4T (Mathematics by and for Teachers): Number Patterns
- Book Review: How dynamic is Dynamic Math 9?
- Family Math Nights: Engaging Students and Their Parents Outside of Class
- Solving Linear Equations: Metacognition and the Birthday Present Principle
- A Lesson in Culturally Relevant Curriculum: Focusing on Locally Developed Mathematics



See Family Math Nights: Engaging Students and Their Parents Outside of Class

See In the Middle: How Teacher Talk Meets the Thinking Classroom





See Book Review: How dynamic is Dynamic Math 9?



See Blast from the Past! John Charles Fields

ACCELERATE LEARNING KITS

Helpful kits to assist students with READING & MATH RECOVERY



ADDITION & SUBTRACTION KIT

RECOMMENDED AGE
5-7 years | Kindergarten-2nd grade



TT855

FLUENCY & COMPREHENSION KIT

RECOMMENDED AGE
7-9 years | 2nd-4th grade



TT868

MULTIPLICATION & DIVISION KIT

RECOMMENDED AGE
8-10 years | 3rd-5th grade



TT861

WINTERGREEN
LEARNING MATERIALS

P 1-800-268-1268 F 1-800-567-8054 W wintergreen.ca S sales@wintergreen.ca

Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *Gazette*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles are to be word–processed, MS Word is preferred, and prepared according to the *Publication Manual of the American Psychological Association*, Seventh Edition. However, please use single-line spacing (not double). Articles should not exceed five numbered pages of text, and figures, images, and photographs should be placed in the text close to where they belong, with captions. The photographer's permission is required, and for photos of students under the age of 18, the written permission of a parent or guardian is required.

Please submit your article in one blind file (i.e., identity of author is not evident), and include author names, contact information including email and mailing addresses, photos—head and shoulders, biographies—less than 100 words, and all content removed for blinding in a second file. Please email these two files to Tim Sibbald at gazette@oame.on.ca.

Upon review, you will be notified whether your article has been accepted for publication (as is, or pending minor or major revisions) or rejected. The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME/AOEM.

PERMISSION TO REPRINT: No part of this publication may be reproduced, distributed, or transmitted in any form or by any means, including photocopying, recording, or other electronic or mechanical methods, without the prior written permission of the publisher, except in the case of brief quotations embodied in critical reviews and certain other non-commercial uses permitted by copyright law. Full credit must be given to the author and to the *Ontario Mathematics Gazette*. For permission requests, contact the Editor of the *Ontario Mathematics Gazette*.

The views expressed or implied in this publication, unless otherwise noted, should not be interpreted as official positions of OAME/AOEM.

▲ Table of Contents

Up Front

▲ Editor's Report
by Timothy Sibbald 2

▲ President's Message
by Sandra Jean Price 4

Regular Features

▲ The *Abacus* . . . . . 8-Page Insert

▲ Blast from the Past! John Charles Fields
by Jeff Irvine 5

▲ OAME/NCTM Report: Happy Fresh New School Year!
by Jacqueline Hill 7

▲ What's the Problem? Fun with Factorials!
by Shawn Godin 8

▲ Linking Literacy and Math: Using Ill-Defined Problems to Remove the Barrier Between School and Lived Experiences of Students
by David Costello 12

▲ In the Middle:
Sidewalk Starters
by Carly Ziniuk 14

▲ Dynamical View of Proportionality: Families of Ratios
by Jérôme Proulx 17

▲ Elementary Math Matters: The Anchor, the Deputy Prime Minister, and the Math Ed Professor
by Lynda Colgan 19

▲ OMCA News: December 2022
by Heather Theijsmeijer 22

▲ Coding in the Classroom: The Destreamed Grade 9 Mathematics Curriculum
by Iain Brodie and Hatice Beyza Sezer 23

▲ A Deep Dive into a Desmos Activity
by Nico Rowinsky and Dr. Xiaoheng Kitty Yan 30

▲ Mb4T (Mathematics by and for Teachers): Number Patterns
by Jennifer Holm 34

Special Features

▲ Book Review: How dynamic is Dynamic Math 9?
by Tim Sibbald 11

▲ OAME Financial Statement 48

Articles

▲ Family Math Nights: Engaging Students and Their Parents Outside of Class
by Dr. Jeff Irvine 25

▲ Solving Linear Equations: Metacognition and the Birthday Present Principle
by Dr. Jeff Irvine 36

▲ A Lesson in Culturally Relevant Curriculum: Focusing on Locally Developed Mathematics
by Taylor Murie and Ann Kajander 40

Notices

▲ OAME Sticky Graph Pads . . . . . ?

▲ OAME/AOEM Financial Statement 2021–2022 48

▲ OAME Board Members 2022-2023 . . . . . Inside Back Cover

Advertisements

▲ CCS 5

▲ Wintergreen . . . . . Inside Front Cover

▲ OAME 2023 Conference . . . . . Back Cover

▲ About the *Ontario Mathematics Gazette*

The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

Editor
- ▲ **Tim Sibbald**
  gazette@oame.on.ca

Associate Editor
- ▲ **Anne Yeager**
  anne.yeager@rogers.com
- ▲ **Peter Saarimaki**
  saarimaki@rogers.com

Associate Editor (French articles)
/Rédactrice adjointe (articles français)
- ▲ **Kyla Kadlec**
  kyla.kadlec@oame.on.ca

***Abacus* Co-Editors**
- ▲ **Marc Husband**
  marc.husbandl@oame.on.ca
- ▲ **Tina Rapke**
  tina.rapke@oame.on.ca

Copy Editor
- ▲ **Gitta Berg**
  gitta.berg@sympatico.ca

Design and Production
- ▲ **Penny Clemens,** Graphic Designer
  pennyydezign@gmail.com

Printing & Binding
- ▲ **Pole Printing,** Box 69, 1-89 King Street East
  Forest, ON N0N 1J0 (519) 786-5112
  poles@xcelco.on.ca

Advertising Manager
- ▲ **Robert Sherk**
  4366 Snider Rd., Verona, ON K0H 2W0
  Home (613) 374-1515
  robert.sherk@oame.on.ca

Submission of Advertisements
Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** by email or at the above address. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

Advertising Rates
Advertising rates are available by telephoning, emailing, or writing to the Advertising Manager.

▲ Editor's Report



TIM SIBBALD, OCT, PH.D.
Gazette@oame.on.ca

Tim Sibbald is a professor in the Schulich School of Education, Nipissing University, with a focus that includes mathematics education. He is a former President of OAME/AOEM and has held a variety of other OAME/AOEM posts over the years.

Do you remember being taught about *the pause*? Every teacher has had to learn this fundamental skill of recognizing that wait time needs to be measured to suit the students for whom you are waiting. Many teachers work at it and review it through their careers. It becomes second nature, perhaps, but nuanced as different students are in one's classes, and different classes show different social characters.

I was led to think about the pause recently. My youngest grandchild, at the age of less than a year, decided to teach me a thing or two (the best teachers are willing to learn from every opportunity children and students give them). We were out for a morning walk on the government dock when the child took an interest in some ducks in the water. We paused, but how long was enough? After the ducks, we came upon a couple of seagulls, and a slow-motion pause ensured they alighted. At the end of the dock, we had a pause to watch waves coming in on the breakwater rocks. Here the pause required wait time for waves to reach the rocks, and then a different wait time to ensure enough waves had broken for the toddler to realize it wasn't just on one rock.

Over the last year, I have engaged in research pertaining to Early Childhood Education, and while I was not directly researching students, there were implications for students. Questions of what "play-based" means and how it dovetails with learning had arisen in conversations. How, I wondered, does the pause work within play-based learning moments? While a grandchild gave insight, I am sure that many Kindergarten teachers and Early Childhood Educators (RECEs, as they are designated in Kindergarten classrooms) could expound in much more detail.

What is the take-away from all this? After years of teaching, we gain a level of comfort in some basic pedagogical tools such as the pause. Sure, we work at it and improve our handling over the years, but when do we make time to revisit such tools and think conceptually about their usage? There is an argument that it becomes like a reflex, and we do not need to think about it, but theoretically,

conversion to a routine simply makes the knowledge of the pause tacit or hidden. To think about it requires revealing it or making it explicit by paying attention to it. Then, and only then, can one examine it to understand how far you have developed it. That understanding is important if one is going to think about their usage conceptually.

There is effort involved, and you may wonder if the pause is something to be thought about conceptually—a response not unlike students wondering why you are asking them to do something. The reason for thinking about the pause and other fundamental instruments of pedagogy is because they hold such an important position in the pedagogical toolkit. What is more, the times have been changing, and who is to say whether a pause during coding is the same as a pause in other circumstances? Perhaps the pause has connections to social–emotional learning (SEL)? In destreamed courses, or courses that follow them, is the pause the same as in the differentiated prior courses? We do not need full formality to respond to these questions, but we do need to be alert to the possibilities and vigilant about evolving our own practices.

While the approach I have portrayed is reflective—the teacher examining their own practice—there are collaborative techniques for also learning about such aspects of teaching. In this case, having a trusted colleague observe a lesson can be good professional learning. A key detail, however, is to decide on the scope of the observation, such as watching how the teacher uses the pause, or tracking who gets asked questions on a seating plan of the class. These approaches make elements of teaching, such as the pause, explicit and open to professional conversation. It entails more than simply reflection, but often generates a rich discussion, where both teachers learn from each other.

While it was a toddler out for a walk who inspired my thinking about the pause, the destreamed Grade 9 course is the exclamation point on needing to think about the pause in classrooms. It is not an isolated instance, but it surely stands out as a key place, where the pause could hold lingering bias from the Academic and Applied courses, where teachers routinized their use of the pause.

Thank Yous

I have the pleasure to work with a remarkably talented team. A talent they have that is rarely acknowledged is the seamless way they work around variations in timing. For this issue, preparation was earlier than usual to accommodate my own planned travel. I was flattered and impressed that the editorial team and all the columnists opted to move to an earlier time, rather than arranging a guest editor. The level of professionalism to simply move the entire preparation process without any notable problems speaks of a remarkable group of individuals, editorial team and columnists, who make the *Gazette* happen. To all of them I say, "Thank you!" I sincerely appreciate your efforts to accommodate a disruption in my own schedule.

In This Issue…

Our President, **Sandra Jean Price**, brings a personal perspective to some of the challenges that gives an interesting perspective about SEL. **Jacqueline Hill** draws attention to change in her NCTM report. **Heather Theijsmeijer** speaks to what has been, and will be, going on with OCMA.

Among the columns, **Lynda Colgan** speaks up about the math politics that have been taking place in California that may be the canary in the proverbial coal mine. **Jérôme Proulx** digs into a proportion problem and shows a variety of ways students may think about it. **Iain Brodie** and **Beyza Sezer** give a brief of their interview of **Alice Aspinall**, who received a Prime Minister's Certificate, and, as usual, the full interview is available through OAME Talks. **Jeff Irvine** focuses on John Charles Field, who every math student in Canada should be aware of as a matter of national pride. **Carly Ziniuk** takes inspiration from Ron Lancaster's past columns (He has stopped his columns—at least for now.), using photos of sidewalk murals. **Shawn Godin** addresses problem solving related to factorials. **David Costello** begins a series of columns with a focus on thinking mathematically. Last, but certainly not least, **Jennifer Holm** looks at reasoning for patterns constructed with skip counting.

The articles in this issue address a good variety of topics so that all readers will find something of interest. **Taylor Murie** and **Ann Kajandar** give a detailed approach to addressing culturally relevant pedagogy in a locally developed classroom. **Nico Rowinsky** and **Xiaoheng Kitty Yan** give a thorough look at pedagogy around the use of a Desmos activity. As things have opened up from the pandemic, it is timely to have **Jeff Irvine**'s article promoting the benefits of family math nights. In a second article, **Jeff Irvine** looks at solving equations and what he calls the birthday present principle.

Missing from this issue is the Fields Report, which was due to the earlier than normal development of the issue. It is sure to be back in the next issue. ▲

If you go, let the Editor know… readers would appreciate hearing about it!



President's Message



SANDRA JEAN PRICE
sandra.jean.price@oame.on.ca

Sandra Jean has been on the Board of Directors of OAME/AOEM for almost 20 years. She is currently teaching in the Durham District School Board and has more than 25 years of teaching experience. She has had many roles, including classroom teacher, Math Coach, and Instructional Leader of online Math Additional Basic Qualifications (ABQs). Sandra Jean has contributed to a variety of Ministry projects and was a coordinator of the most recent OAME/AEMO writing projects. Sandra was the recipient of the 2008 OAME Award for Exceptional and Creative Teaching in Elementary Mathematics for her innovative instructional practices, integration of technology, and belief in success for all.

In my first "President's Message" in September, I focused on the opportunity we have as educators to begin the school year with a fresh start. I hinted at the numerous changes that we have experienced throughout the last two years. These changes have been the topic of various conversations in my school through the past few months. There have been many conversations around the continued implementation of the Revised Elementary Mathematics Curriculum that was released in 2020. Due to the upheaval during the pandemic, many teachers have yet to fully implement everything that is within the revised curriculum. Some of my colleagues are very anxious about teaching specific concepts that are either new to the curriculum (e.g., Mathematical Modelling, and Coding) or new within the grade level. Others are concerned about running out of time and not "covering" every expectation. While the additions and adjustments to the elementary mathematics curriculum were needed and are integral in our quest to help students learn and embrace mathematics, both the Revised Elementary Mathematics Curriculum and the Destreamed Grade 9 Mathematics Curriculum have another aspect to them that is not as focused on content knowledge. Perhaps this is the part we need to spend more time thinking about and discussing.

"You just can't do math." This one statement shattered my entire world when I was in Grade 7. Things had not been going so well in my personal life, but I was a very strong student, and that was my one constant that allowed me to keep some kind of control and success in my life. But for some reason, I began to struggle partway through Grade 7. The vice-principal was my math teacher and I did not understand the concepts being taught. I suppose I should have asked for help more often, but to be honest, I was afraid of my teacher. I was afraid that he or other students in the class would think I was stupid. Instead, I just sat in silence and tried to read the textbook to try to figure things out. I don't even remember what the concepts were. I just remember how I felt: lost, alone, and a complete and utter failure.

After one particularly unsuccessful attempt of a math test, my teacher pulled me aside and made that hard-hitting comment to me. In hindsight, I believe that maybe he was trying to be kind and give me a reason why I did poorly, despite wanting to understand. But what he actually did was confirm my worst fears. I was a failure and would never be successful in math. I stopped trying entirely… what was the point? And my marks in my other subjects began to slide as well. Thankfully, the year ended before I could give up completely in all my subjects.

Grade 8 began with my mind still set on the fact that I couldn't do math, but that changed very quickly after I met my new math teacher. One of the first things she said to the entire class was that her job was to help us understand… all of us, and that if we didn't understand, her job was to explain it and teach it in a different way until we did. In her math class, we used manipulatives and drawings (the technology at that time was limited to calculators). She would teach things in multiple ways, help us make connections, and encouraged us to ask questions. We were told repeatedly that making mistakes was, in fact, learning. She enjoyed it when we would come up with a different way of solving a problem than she had anticipated, and taught us to embrace alternative solutions. We had choice and we certainly had our voices heard. She always celebrated successes, no matter how small they seemed. She created a community of learners, who just happen to be learning math in her classroom. She was truly ahead of her time.

I did much better in that class. In fact, by Grade 9, I was taking the enriched math course offered at my secondary school. Again, I do not remember exactly what concepts I learned during that Grade 8 year, or which ones I enjoyed the most, but I do remember how I felt: successful, valued, and that I belonged.

The Social–Emotional Learning sections, the "Curriculum Context" in the elementary curriculum document, and the "Some Considerations for Program Planning" sections in both the elementary and Grade 9 documents, help provide a foundation upon which teachers can connect with students. Unfortunately, these are the sections that many of my colleagues do not spend much time examining. In an effort to adequately prepare students, they tend to focus immediately on the specific content expectations. Some are concerned

about running out of time and not finishing everything.

I encourage all teachers to read the beginning of the curriculum documents. There is a wealth of information in there that reminds us of why we teach, and what is important in order for students to learn. Allowing for choice allows students to explore mathematics through a lens that connects with them. Along with the use of high-yield instructional strategies, this will ultimately lead to better understanding. Providing opportunities for students to share, and regularly encouraging student voice, promotes a sense of belonging. By investing time doing these things and getting to know your students, the rest of the teaching/learning experience will likely be more successful. The idea of creating a learning community in which students feel safe enough to make mistakes and take risks is not an impossible task. It just needs the same attention that is given to the curriculum expectations within the curriculum documents that are primarily perceived as content driven.

As much as we want our students to remember the important concepts that we teach them, we should also want them to learn about themselves. We want them to love learning. We want them to connect to mathematics in a meaningful way and see how integral it is in their lives. We want them to feel like they are successful and that they belong. We want to help create that positive feeling that they will remember when looking back on their education.

"What we know matters, but who we are matters more." —Brene Brown ▲

CCS *for your Math supplies!*

Calculators & Accessories
Smartview Software, CBR2, Navigator

STEM & Coding
Texas Instruments Rover, Hub, Modular Pacs & Bread board

Bat'ry Gard™
prevents battery theft

All Calculator Batteries
Permanent Lithium, NiMH Rechargeable and one use

Calculator Repairs
We repair or replace your broken calculators

Cases for all Calculators
we sell Havilland cases & our own custom cases

SHARP TEXAS INSTRUMENTS CASIO

24 Rogate Pl, M1M 3C3, Toronto 1-877-CCS-EDUC
info@ccseducational.com • www.ccseducational.com

▲ Blast from the Past! John Charles Fields



JEFF IRVINE
dr.jeffrey.irvine@bell.net

Jeff Irvine, Ph.D., has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

Four years before Canada was born, John Charles Fields was born in the bustling town of Hamilton, population 19 000. After high school, he had to choose between the two universities in Ontario: University of Toronto, or Queen's University in Kingston. He chose the University of Toronto, graduating with a bachelor's degree in 1884. Interestingly, this was the same year that women were admitted to the university for the first time.



Image from Wikipedia

Since there were no doctoral (i.e., Ph.D.) programs in Mathematics in Canada, Fields went to Johns Hopkins University in the United States, graduating with a doctorate in 1887. This forced migration from Canada for graduate studies was the beginning of Fields' view of mathematics and scientific research as an international pursuit, not limited by nationality.

After graduation, Fields taught for a few years at American universities before moving to Europe in 1891. In Europe, he associated with many of the renowned mathematicians of his time, including Felix Klein, Karl Weierstrass, and Max Planck. Fields developed a close friendship with Gösta Mittag-Leffler, which would last throughout his life. Fields returned to Canada in 1902 and became a lecturer at the University of Toronto.

Fields' focus was on raising the profile of mathematics

and mathematics research in Canada. He actively solicited funds for research and helped establish the National Research Council of Canada. He served as president of the Royal Canadian Institute from 1919 to 1925 and was instrumental in bringing the International Congress of Mathematics to Toronto in 1924. Subsequently, Fields was in ill health for several years and died in Toronto in 1932. He left money from his estate to fund the prestigious Fields Medal.

The Fields Medal

The Fields Medal is sometimes referred to as the Nobel Prize in Mathematics. Up to four medals are awarded every four years to support up-and-coming researchers in mathematics. There is also a modest monetary award, currently $15 000. The design of the medal follows Fields' conception of mathematical research being worldwide.



Images from the International Math Union

The medal was conceived by a Canadian, designed by a Canadian (R. Tait McKenzie), and was struck by the Royal Canadian Mint. The medal features a bust of Archimedes (a conceptual image, since no images of Archimedes have survived). The obverse of the medal contains a Latin inscription, "Mathematicians from around the world honour noteworthy contributions to knowledge." The winner's name is inscribed around the edge of the medal, but no nationality is included.

The medal was first awarded in 1936, four years after Fields' death, and then every four years, beginning in 1950. The Fields medals were awarded earlier this year (see www.mathunion.org/imu-awards/fields-medal/fields-medals-2022).

The Fields Institute

Established in 1992, the Fields Institute for Research in Mathematical Sciences is located in a building on the campus of the University of Toronto. Its purpose is to bring together mathematicians from Canada and around the world, and to support collaborations between mathematicians and researchers in other areas including physics, biology, computer science, engineering, medicine, and economics. In 2002, the Fields Institute established lifetime appointments of Fields Institute Fellows to recognize and support Canadians who have made outstanding contributions to mathematics. Another example of activities supported by Fields is the Society Investigating Mathematical Mind-Expanding Recreations (SIMMER), described by Alan Chan in a 1997 *Gazette* article.

Of particular importance to practising teachers is the Fields Mathematics Education Forum—monthly meetings that bring together university researchers in mathematics, in mathematics education, and teachers of Kindergarten to Grade 12 mathematics, as well as community college mathematics instructors. The forum meetings feature speakers, discussion of issues in mathematics education, and presentations by members, including teachers. Reports on the activities at the Fields MathEd Forums are a regular feature of the *Gazette* (e.g., Craven, *OMG* 49(3), 2011; Mendaglio, *OMG* 60(2), 2021).

Beginning in 1997, in partnership with OAME/AOEM, the Fields Institute has been involved in mathematics curriculum development for the Ontario Ministry of Education. Prior to 1997, all new mathematics curricula were developed solely by the Ministry of Education, with limited involvement of teachers or other stakeholders in curriculum. The involvement of OAME/AOEM and the Fields Institute represented a significant shift in curriculum development, which has continued through more recent curriculum revisions.

References

Chan, A. (1997). The "fields" of dreams. *Ontario Mathematics Gazette, 36*(3), 29–30.

Craven, S. (2011). Fields Institute MathEd Forum notes. *Ontario Mathematics Gazette, 49*(3), 19–21.

Mendaglio, A. (2021). Events at the Fields. *Ontario Mathematics Gazette, 60*(2), 10–11.

Reim, E., & Hoffman, F. (2011). *Turbulent times in mathematics: The life of J.C. Fields and the history of the Fields medal.* American Mathematical Society and The Fields Institute.

University of Toronto. (n.d.). Fields Institute biography. www.fields.utoronto.ca/aboutus/jcfields/ ▲

The 2022–2023 OAME/AOEM Board of Directors and Chapter Reps!



▲ OAME/NCTM Report: Happy Fresh New School Year!



JACQUELINE HILL
jacqueline.hill@bell.net

Jacqueline Hill is a Grades 9–12 teacher of mathematics. She is a Past President of OAME and OMCA, as well as the recipient of the award for Exceptional Teaching in Secondary Mathematics. She also wrote the "Director's Dialogue" for the Gazette *for a number of years.*

CHANGE! There has been so much change in the past three years, for example, dropping off calculators for recycling, only to have students find them in their lockers that were unavailable when the school was closed due to the pandemic. Another example was mailing copies of EQAO reference sheets to students who didn't have access to a printer at home. There was also preparing evaluations that were individual to each student taught online. Wow! That was a lot of change. Change from my six pods of students (four, five, and sometimes six in a pod), teaching with a SMART Board®, whiteboards, vertical surfaces, and a full host of manipulatives at our fingertips, to an online-only presence. My students would often refer to their classmates as their "virtual friends."

So… how can NCTM help you navigate this change? Or in finding balance when the pandemic wanes?

C – Cultivate – Cultivate growth by going to school board-provided professional development, attending OAME/AOEM professional development, and working on your own professional growth. NCTM has a virtual conference March 29 – April 1, 2023, which can support cultivating a department, workroom, and classroom that is ready to grow!

H – Help – Help students to find online tools that engage them and help fill their learning gaps.

A – Appreciate – Appreciate that students are resilient. Use this as a tool to grow a community in your classroom. Perhaps visit the "Notice and Wonder" section under teacher resources on the NCTM website.

N – Nurture – Nurture your students into the routines that will benefit them, from lining up and walking in the halls in a safe manner, to how to set short- and long-term goals for their success. Remember… they haven't socialized in the "school way" in a long time.

G – Gratitude – As adults, we often find time to think about the things for which we are grateful. However, for students, it may be something that needs to be broken down so that they understand it. Once, with student permission, I spotlighted a unique feature in their agenda. Not only did it have a calendar, to-do list, and place for writing notes, but it also had a section each week for "things I am grateful for." This student was happy to share that they were grateful for family, friends, the ability to afford a cellphone. After a class discussion on gratitude, students added to that list by talking about being grateful for their clothes, and the opportunities they have. As teachers, it is often hard to see the forest for the trees. As we navigate the change from pandemic to post-pandemic activities, we should be grateful for the teams we work with and the post-pandemic opportunities we will have.

E – Explore – Explore NCTM resources that will help you. Perhaps you will need a few more online resources for days your school is in an online mode (or you need a supply plan quickly). The "year game" might help: see www.nctm.org/Classroom-Resources/Year-Game/The-Year-Game/.

In conclusion, CHANGE is upon us as educators! We need to embrace this change. Since you have read to the bottom of my column, I can tell that YOU are an agent of change! Change for growth!

NCTM offers virtual and in-person conferences.

Virtual: March 29–April 1, 2023

In-person: Washington, DC – October 25–28, 2023
Chicago, IL – September 25–28, 2024

Until next time!

Yours in mathematical fun, Jacqueline

▲

Gazette Articles Needed!

The *Gazette* has a queue of articles that are ready for print. Some of you may be aware that queue was relatively long last year. However, we now have four authors in that queue and potentially run out of articles after the next issue.

Consider sharing what has been working in your classroom. The professionalism of communicating what you have done will enrich your own understanding and lead you to new ideas! (I speak from experience.)

Your Editor is an email away, if you would like to talk about an idea or the process: gazette@oame.on.ca.

▲ What's the Problem? Fun with Factorials!



SHAWN GODIN
GodinMathStuff@gmail.com

Shawn is a former mathematics teacher, department head, and consultant. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: presenting workshops, writing articles, working on local projects, and helping create mathematics contests.

Welcome back, problem solvers. We are continuing the theme "(mathematical) problems from my life" that we started in the last issue. When I was in Grade 8, I wrote the Junior Mathematics Contest, sponsored by the University of Waterloo, and it changed my attitude toward mathematics. Many of the problems in the contest were unlike the ones I was used to seeing in my math classes. I had to formulate my own plan in my attempts to solve them. There were others that I just could not crack, and that made me want to try to figure them out for myself. Contest day became that one day a year, when I could really experience my love of mathematics.

I was still new to problem solving and, as a result, made some classic errors. One problem sticks out in my mind—question 24 from the 1981 Junior Mathematics Contest—which I wrote when I was in Grade 10 (I know I am dating myself here), your homework from last issue:

If $90! = 90 \times 89 \times 88 \times \ldots \times 2 \times 1$*, then what is the exponent of the highest power of 2 which will divide 90!?*

As it turns out, I got this problem wrong. I know I was wrong because shortly after the contest, I was revisiting the problems. I had never encountered the factorial before and thought the problem was interesting. In my rethinking about the problem, I discovered my error, and it taught me a valuable lesson: always reflect on your work.

At the time of the contest, we didn't have access to the tools we do today. Let's take advantage of an online tool, *WolframAlpha*, and begin our investigation by "cheating." We can quickly calculate 90! = 1 485 715 964 481 761 497 309 522 733 620 825 737 885 569 961 284 688 766 942 216 863 704 985 393 094 065 876 545 992 131 370 884 059 645 617 234 469 978 112 000 000 000 000 000 000 000 ≈ 1.5×10^{138}, which isn't very helpful. However, if we ask WolframAlpha to factor 90!, we get $2^{86} \times 3^{44} \times 5^{21} \times 7^{13} \times 11^{8} \times 13^{6} \times 17^{5} \times 19^{4} \times 23^{3} \times 29^{3} \times 31^{2} \times 37^{2} \times 41^{2} \times 43^{2} \times 47 \times 53 \times 59 \times 61 \times 67 \times 71 \times 73 \times 79 \times 83 \times 89$, which gives us the result we are looking for: 86.

What fun is that? The thing to keep in mind about a nice problem is that the "answer" isn't the real goal. The goal is to *understand* what is going on in the problem and to use our higher-order thinking skills. In this case, knowing what the answer is will help us verify if our strategy is correct.

Thinking back to my younger self, I probably thought: each even number will produce a factor of 2 in the product; there are 90 ÷ 2 = 45 even numbers, so it must be 45 (the question had originally been a multiple-choice question, and 45 was one of the choices). Knowing myself, seeing the answer on the list would probably have satisfied me and I would be on to the next problem. However, when it wasn't multiple-choice, or if my answer wasn't on the list, I probably would have thought about it a little longer.

Since our original strategy didn't work, let's modify it. We will look at the highest powers of 2 that divide even factorials for small numbers, and see if we can find a pattern.

n	*n*!	Highest power of 2	*n*	*n*!	Highest power of 2
2	2	2^1	10	3 628 800	2^8
4	24	2^3	12	479 001 600	2^{10}
6	720	2^4	14	87 178 291 200	2^{11}
8	40 320	2^7	16	20 922 789 888 000	2^{15}

At first, there doesn't seem to be a pattern in the exponents: 1, 3, 4, 7, 8, 10, 11, 15. However, if we look at the differences in this sequence, we get 2, 1, 3, 1, 2, 1, 4. It seems that every second number is a 1. The entries that are not 1 correspond to when we multiply by a multiple of 4. If we dig further, we will see that 3 occurs when we multiply by a multiple of 8, and larger values when we multiply by a multiple of 16, and so on. Doing some calculations, we find:

- 90 ÷ 2 = 45, so, there are 45 multiples of 2 between 1 and 90
- 90 ÷ 4 = 22.5, so, there are 22 multiples of 4 between 1 and 90
- 90 ÷ 8 = 11.25, so, there are 11 multiples of 8 between 1 and 90
- 90 ÷ 16 = 5.625, so, there are 5 multiples of 16 between 1 and 90
- 90 ÷ 32 = 2,8125, so, there are 2 multiples of 32 between 1 and 90
- 90 ÷ 64 = 1.40625, so, there is 1 multiple of 64 between 1 and 90

Each even number contributes (at least) one 2 to our product, all multiples of 4 contribute (at least) *one more* 2 to our product, and so on. Hence, the total number of 2's in the product is

$$45 + 22 + 11 + 5 + 2 + 1 = 86,$$

which agrees with our result from WolframAlpha.

There is a more compact way we can write this result. The *floor function* $\lfloor x \rfloor$, yields the greatest integer that is less than or equal to x, so $\lfloor 5.8 \rfloor = 5$ and $\lfloor -5.8 \rfloor = -6$. Hence, the number of multiples of 16 between 1 and 90 can be calculated as $\left\lfloor \frac{90}{16} \right\rfloor = 5$. If we can define the function

$$p(n) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor,$$

then it will provide the highest power of 2 that divides n!. For example, if we put in 90, we get

$$\begin{aligned} p(90) &= \left\lfloor \frac{90}{2} \right\rfloor + \left\lfloor \frac{90}{4} \right\rfloor + \left\lfloor \frac{90}{8} \right\rfloor + \left\lfloor \frac{90}{16} \right\rfloor + \left\lfloor \frac{90}{32} \right\rfloor + \left\lfloor \frac{90}{64} \right\rfloor + \left\lfloor \frac{90}{128} \right\rfloor + \cdots \\ &= 45 + 22 + 11 + 5 + 2 + 1 + 0 + \cdots \\ &= 86. \end{aligned}$$

Notice that for powers of 2 greater than 90, the floor is zero and the contribution to the sum is 0. We can use logarithms to calculate the largest power of 2 less than or equal to 90 as $\lfloor \log_2 90 \rfloor = 6$, so we could have stopped the sum at 6. In Figure 1 below, I have defined a refined version of $p(n)$ in *Desmos* that only sums as far as it needs to.

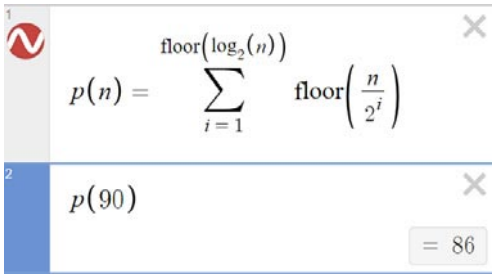


Figure 1: *A Desmos function to calculate the highest power of 2 dividing* n!

That was fun, so why stop here? As usual, let's look at our problem and see if we can extend it. The natural thing to do is to change the 90, but our technique will work in general, so we have already solved that problem. Similarly, if we change from powers of 2 to powers of any other prime, our method works. To make things interesting, we need to look at something different.

If we look at the computation for the value of 90! that we did using WolframAlpha, I noticed something that makes me wonder:

1. How many zeros does 90! (or any factorial) end with?
2. What is the first non-zero digit after all the zeros in 90! (or any factorial)?

You may want to take some time and try these for yourself. Don't worry, I will still be here when you get back.

For the first problem, it comes down to determining the greatest power of 10 that divides 90!. This looks like our original problem; however, when we use that method, we get

$$\sum_{i=1}^{\infty} \left\lfloor \frac{90}{10^i} \right\rfloor = \left\lfloor \frac{90}{10} \right\rfloor + \left\lfloor \frac{90}{100} \right\rfloor + \cdots = 9 + 0 + \cdots = 9.$$

When we check the output from WolframAlpha, we can see this is incorrect. What went wrong? Again, some computation might help us. Calculating,10! = 3 628 800, we see that 10! ends with two zeros, even though there is only one multiple of 10 in the numbers we are multiplying together. Doing some more calculations, you can determine that 5! = 120 is where the first zero shows up. If we rewrite the computation as $5! = 5 \times 4 \times 3 \times 2 \times 1 = 5 \times 2^2 \times 3 \times 2 = 2^3 \times 3 \times 5 = 2^2 \times 3 \times (2 \times 5) = 2^2 \times 3 \times 10$, we see that since $10 = 2 \times 5$ is *composite*, we will get 10's from combinations of the prime factors 2 and 5, not just multiples of 10. Thus, we need to calculate the largest power of 5 that divides 90!, which is

$$\sum_{i=1}^{\infty} \left\lfloor \frac{90}{5^i} \right\rfloor = \left\lfloor \frac{90}{5} \right\rfloor + \left\lfloor \frac{90}{25} \right\rfloor + \left\lfloor \frac{90}{125} \right\rfloor + \cdots = 18 + 3 + 0 + \cdots = 21.$$

Thus, we can conclude that $90! = 2^{86} \times 5^{21} \times (\textit{stuff}) = 10^{21} \times (\textit{other stuff})$.

Therefore, 90! must end with 21 zeros, which corresponds to our earlier calculation. It is worth noting that if we define

$$p_q(n) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{q^i} \right\rfloor$$

for a prime number q, it is the greatest power of q that divides n!. Notice that if q and r are primes, with $q < r$ then $p_q(n) \geq p_r(n)$ for all values of n (why?). Hence, $p_5(n)$ will tell us how many zeros n has, since $p_5(n) \leq p_2(n)$.

The second problem is more interesting. We can think of it as what is the units digit of

$$\frac{90!}{10^k},$$

where 10^k is the largest power of 10 that divides 90!. From our earlier "cheating" and the results of the last problem, we get that this should be $2^{65} \times 3^{44} \times 7^{13} \times 11^8 \times 13^6 \times 17^5 \times 19^4 \times 23^3 \times 29^3 \times 31^2 \times 37^2 \times 41^2 \times 43^2 \times 47 \times 53 \times 59 \times 61 \times 67 \times 71 \times 73 \times 79 \times 83 \times 89$.

When we multiply numbers together, only the units digits affect the units digit of the product. For example, the units digits of $52 \times 73 \times$, 82×33, and $46\,702 \times 94\,823$ are all $2 \times 3 = 6$. Thus, we can focus on just the units digit of the numbers involved. Because of that fact, powers of 11, 31, 41, 61, and 71 do not contribute to the product, since multiplying a number by 1 leaves it unchanged. Taking these things into

consideration, our desired number will be the same as the units digit of $2^{65} \times 3^{44} \times 7^{13} \times 13^6 \times 17^5 \times 19^4 \times 23^3 \times 29^3 \times 37^2 \times 43^2 \times 47 \times 53 \times 59 \times 67 \times 71 \times 73 \times 79 \times 83 \times 89$, or $2^{65} \times 3^{44} \times 7^{13} \times 3^6 \times 7^5 \times 9^4 \times 3^3 \times 9^3 \times 7^2 \times 3^2 \times 7 \times 3 \times 9 \times 7 \times 3 \times 9 \times 3 \times 9$, or just $2^{65} \times 3^{58} \times 7^{22} \times 9^{10}$.

Next, we do some patterning. The units digits of powers of 2 are: 2, 4, 8, 6, 2, 4, 8, 6, 2,…, a periodic sequence of period 4. We can then easily deduce that the units digit of 2^{65} is 2. Using similar logic, we can get the units digits of the other powers and calculate our result. I will leave the calculation and verification to the reader.

The mathematics contests from the University of Waterloo became an important part of my early education. I looked forward to the challenge of contest day each year. Mathematics contests were also a big part of my later career as I got my own students involved in writing contests. Additionally, contests became part of my professional growth through my involvement creating and marking the contests over the years, as well as attending problem-solving workshops held by the Centre for Education in Mathematics and Computing (CEMC).

You can gauge the desire for problem solving by the growth of the contests at the University of Waterloo. They started with two contests in the 1960's. The Junior math contest, for grades up to 11, ran until the early 1980's, when it was replaced by three contests: Pascal (Grade 9), Cayley (Grade 10), and Fermat (Grade 11). The Ontario Senior Mathematics Problem Competition, eventually renamed Descartes, ran until OAC disappeared from Ontario high schools in 2003. Along the way came the Euclid contest (Grade 12, 1975), the Gauss (Grades 7 and 8, 1977), Fryer (Grade 9, 2003), Galois (Grade 10, 2003), Hypatia (Grade 11, 2003), Canadian Intermediate Mathematics Contest (up to Grade 10, 2011), Canadian Senior Mathematics Contest (up to Grade 12, 2011), and the Canadian Team Mathematics Contest (up to Grade 12, 2012). Each year, students have many CEMC competitions, and other sources, to choose from, in which they can participate. The CEMC website contains many years of past contests for students to practise, or for teachers to use in their classroom. If you haven't already, I suggest you check out the resources on the CEMC website at www.cemc.uwaterloo.ca.

Over the years, many problems from the CEMC contests, and other math contests, have made their way into my columns. Below is a selection of problems from CEMC contests that I have discussed over the years. You can find all of these in the *Gazette* archives on the OAME website.

A student has two open-topped cylindrical containers. The larger container has a height of 20 cm, a radius of 6 cm, and contains water to a depth of 17 cm. The smaller container has a height of 18 cm, a radius of 5 cm, and is empty. The student slowly lowers the smaller container into the larger container. As the smaller container is lowered, the water first overflows out of the larger container and then eventually pours into the smaller container. Determine the depth of the water in the smaller container when the smaller container is resting on the bottom of the larger container. (Godin, 2013)

At the beginning of the winter, there were at least 66 students registered in a ski class. After the class started, 11 boys transferred into this class and 13 girls transferred out. The ratio of boys to girls in the class was then 1:1. Which of the following is not a possible ratio of boys to girls before the transfers?

(A) 4 : 7 (B) 1 : 2 (C) 9 : 13 (D) 5 : 11 (E) 3 : 5 (Godin, 2014).

If x and y are real numbers, determine the minimum possible value of the expression

$(x + 3)^2 + 2(y - 2)^2 + 4(x - 7)^2 + (y + 4)^2$ (Godin, 2016).

A (non-degenerate) triangle has side lengths 10, 17, and x, where x is an integer. For which values of x is the triangle obtuse? (Godin, 2018)

The numbers from 1 to 576 are written in a 24 × 24 grid so that the first row contains the numbers 1 to 24, the second row contains the numbers 25 to 48, and so on, as shown in the diagram below. Determine if it is possible to draw an 8 × 8 square around 64 of these numbers, such that the sum of the four corner numbers is 1646.

1	2	…	23	24
25	26	…	47	48
⋮	⋮	⋮	⋮	⋮
553	554	…	575	576

(Godin, 2020a)

Rectangle $ABCD$ has diagonal BD with endpoints $B(4,2)$ and $D(12,8)$. Diagonal AC lies on the line with equation $x + 2y - 18 = 0$. Determine the area of $ABCD$. (Godin, 2021)

I wrote about the original problem in this column in another of my columns (Godin, 2020b), "Problem-Solving Vignettes," which appears in the problems-solving journal *Crux Mathematicorum*. Interested readers can find some more factorial problems in that article.

And now for your homework. This month, we have a bonus… two problems!

1. *The number 24 678 051 is quite interesting because $2^8 + 4^8 + 6^8 + 7^8 + 0^8 + 5^8 + 1^8 = 24\,678\,051$. There are two other eight-digit numbers that are equal to the sum of the eighth powers of their digits. Find these numbers.*

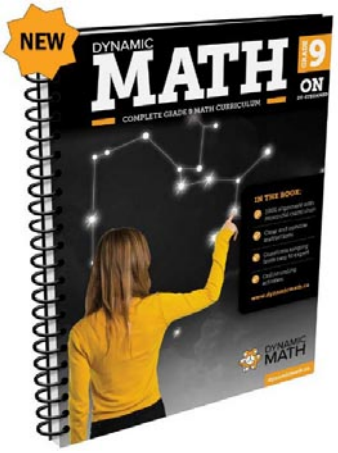
Continued on page 16

▲ Book Review: How dynamic is Dynamic Math 9?

BY TIM SIBBALD

Taylor, A.R., Kokoskin, B., & Huculak, T. (2022). *Dynamic math – Ontario grade 9 de-streamed: Complete grade 9 math curriculum*. Dynamic Classroom.

At the OAME-2022 Annual Conference, I moderated a session that presented this book. It caught my interest as a direct support for the destreamed Grade 9 course that was brought in during 2021. It is notable that in the conference description, it is described as a "resource," and the publisher's store calls it a "workbook"—it is not called a textbook.



The book contains culturally relevant content. For example, it opens with an activity that teaches about the medicine wheel, which is interesting (but also beyond my ability to review the cultural connection). There are mentions of famous mathematicians, both male and female. Some care may be needed, however, where as an example, it is stated that Pythagoras discovered the relationship after which his theorem is named. This is contrary to the spirit of the new curriculum, where it was known in China, Egypt, and Babylon, before Pythagoras either disseminated it concisely or proved it in full generality.

The presentation of content uses a high amount of scaffolding, with many details abbreviated, to rules with conceptual aspects portrayed only through patterns. Exercises are included with answers in the back, and they tend to emphasize text-based questions aligned with the knowledge category of achievement. This is consistent with the book being designed to "ensure students are well-prepared for the provincial assessment" (p. ii). Unfortunately, it means the resource is not suitable for supporting all aspects of the curriculum.

The book includes some coding, with examples using Scratch or mBlock programming. A strange detail is that the book encourages students to go to a publisher website for more activities; however, the terms and conditions of the website say it is age restricted and that, "Minors may access this Website and use its Service only under parental or adult supervision" (www.dynamicmath.ca/terms). This condition does not appear where students are encouraged, and either the publisher is ignoring their own terms and conditions, or placing an onus on teachers to be aware of details beyond the scope of the book itself.

Some aspects of the book represent choices. The book is a "consumable working text" (p. i), has "Do not photocopy" at the bottom of the pages, and requires a choice to use a department's budget for consumables for this resource. Similarly, a workbook is intended to be a support, which it can do, but requires the teacher to support the use of communication and perhaps augment with genuine hands-on applications or broader problem-solving approaches.

The book has some challenges. On page i, the NCTM Standards are described as a "national vision for mathematics in Canada," which is not true. While NCTM has looked at alignment, and it is considerable, there are differences, particularly with respect to financial mathematics.

Mathematically, on page 2, a rational number is defined as "numbers that can be written as fractions or decimal numbers." This is not accurate, though the definition of an irrational number and the examples that follow are accurate.

As far as I could tell, the book does not include the converse of the Pythagorean relationship, where three sides are specified and the triangle is classified as acute, right, or obtuse. The curriculum is not explicit about this detail, though it clearly fits with triangle properties. Similarly, the book does not appear to include reflex angles. This is an aspect of the curriculum that is also murky. Reflex angles are mentioned in the Grade 6 curriculum, and teachers will have to decide whether they should therefore continue to be supported in Grade 9 destreamed, or whether omission is focusing on the expectations of the grade.

The book does not have an index. So, finding specific content or checking if it is included (such as reflex angles) is awkward. However, the table of contents is sufficiently detailed to find most content and would suffice for students who have used the book.

Some features of the resource are good. The section regarding properties of circles is notably strong. The binding of the book allows it to lie open flat or folded so that only one page is showing. Students are likely to find this to their liking.

As I examined this book, I was led to wonder about the authors. The publisher does not include information on their website. However, a bit of sleuthing found that the authors are based in British Columbia and have many years experience. The reviewing team varies in experience, which potentially mixes different opinions around modern views

Continued on page 16

▲ Linking Literacy and Math: Using Ill-Defined Problems to Remove the Barrier Between School and Lived Experiences of Students



DAVID COSTELLO
david@costellomath.com

David is a principal, who taught in the Primary division before assuming roles of numeracy interventionist, numeracy coach, numeracy leader, and curriculum consultant for Prince Edward Island. David has also instructed university courses regarding curriculum, differentiation, mathematics, and language arts. He facilitates professional learning in the areas of mathematics and school development. Dr. David Costello recently published three books: Mathematizing Student Thinking: Connecting Problem Solving to Everyday Life and Building Capable and Confident Math Learners; Making Math Stick: Classroom Strategies That Support the Long-Term Retention of Math Concepts*, and* Using What Works: Strategies for Developing a Literacy-Rich Environment in Math.

Take a moment to consider your math classroom. If you don't have a classroom of your own, consider the classroom of an educator whom you support. How many times have you heard students communicate the following?

- Why are we learning about this?
- I don't want to do this.
- When are we ever going to use this?
- This doesn't make any sense.

If you are like me or the many teachers I have observed and/or talked with, the comments above are all too common. Regardless of the grade level or time of year, the comments above could have been made anywhere. Yet while we continue to hear these comments from year to year, we move forward in our instruction by continuing to emphasize, and provide students opportunities with, the mathematical concepts assigned to their grade levels.

In this column, I want to ask you whether we are only compounding the problem. If students are communicating the comments outlined above, we have a significant pedagogical and learning concern that needs to be addressed. We need to take stock of why we continue to hear those comments, and consider ways that we can move from students *doing* mathematics to students *thinking* mathematically.

And, why are students only *doing* mathematics? My guess is that we have mathematical experiences for students in the classroom that are not reflective of the experiences they have outside of school. There is a divide between what students are assigned as mathematics in school and how they use mathematics to make sense and solve problems they encounter in their lives outside of school. This needs to change. As educators, we need to do what we can to remove this divide so that students experience relevancy between math in school and in their lives beyond the school walls.

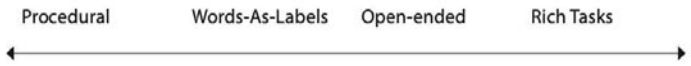
This is the first in a series of columns that will support educators in providing students with problems that are relevant, and with an approach that will assist in students becoming capable and confident math learners. Math learners are able to communicate their understanding, are creative and flexible in approaching the problem, and have autonomy in the decisions they make. We want students to be independent learners, so we must take time to critically examine whether our instructional practices are aligned with this goal.

The Problems We Assign

I have had the privilege to visit many mathematics classrooms across Canada and in the United States. I have witnessed great things. Educators are going above and beyond in identifying and providing students with tasks that reflect the grade-level curriculum. Regardless of which outcome is being explored, teachers take time to ensure that the assigned question and/or task aligns with the various aspects of the concept that is being explored.

These problems play a significant role in how students view themselves as mathematical learners and how they view mathematics in general. What I mean is that the problems we assign either position students in the role of passive dependent learners or as active independent learners. The problems that students have been typically assigned are called "defined problems." A defined problem has three characteristics. First, there is an initial state—this is the state students are presented at the onset of the problem. Second, there is the goal state—the state that is desired by the students. Third, there are obstacles—the stumbling blocks that students encounter between the initial state and the goal state.

I think it is fair to state that we have encountered many *defined problems* through our education. Within the scope of defined problems, I have observed a range of problem types (Costello, 2022):



Procedural problems are problems where the strategy is already identified and students must apply the steps accurately to find the solution. An example of a procedural problem is "Solve: 4.21–3.9."

Words-as-labels are problems framed in a context. Students must read through the problem and decide which strategy they need to apply to the situation. This action is similar to Pólya's "crafting a plan" stage in problem solving. Words-as-labels problems are commonly referred to as story problems or word problems. An example of a words-as-labels problem is as follows:

> *A group of educators needed to replenish the school library. The educators had $4648 already assigned for such purchases, but felt that they needed to reach a total of $10 000 to buy the necessary supplies. How much more did the group of educators need to raise to make the purchase?*

Open-ended problems provide students with opportunities for a greater degree of choice to select a strategy to apply. An example of an open-ended question is as follows:

> *When multiplying two fractions, the product is between one-half and nine-tenths. What could the two fractions be?*

Rich tasks are problems intended to be meaningful for students. When considering the familiar context within the problem, students have an opportunity to recall experiences or situations they have had in and out of school. Rich tasks are meant to be opportunities for engagement and interest. An example of a rich task would be as follows:

> *The community of Meadowvale was looking to build a park for children. When meeting with local students, the mayor of the community decided to set aside $5000. This money would be used to purchase items that would be of interest to the local students. What combinations of items and how many of each could be purchased by the town?*

Item	Price
Swings	$750
Slide	$1825
Soccer Nets	$975
Rock Wall	$2600

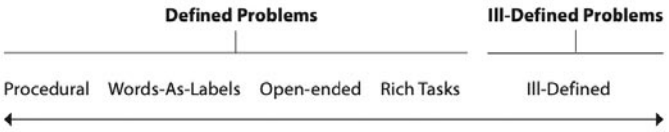
Each type of problem shares two commonalities. First, the starting point is mathematical. Educators have a mathematical concept in mind and then craft a problem to support student learning of the concept. Each problem may or may not involve a context. Second, each of the example problems has the three characteristics of a problem (initial state, goal state, and stumbling blocks). The difference within the problems is the level of engagement that students are afforded. The level of engagement could be in the forms of connections to lived experiences, opportunities to select from a range of strategies, and/or openness in possible solutions.

While the level of engagement varies, each of the *defined problem*-type examples provides students with all the information they need to solve the problem. This has tended to be how mathematical problems have been assigned to students. It is in this application of information that we run the risk of promoting passive learning. Students become reliant on being provided everything they need to solve the problem.

Remedying the Problem

Very rarely, outside of school, are we provided with all the necessary information required to solve a problem. Instead, we are given a problem and have to approach it in a way that is manageable. It is through the process of making the problem manageable that we take steps toward becoming independent learners. So, we need to move from providing students with everything they need to solve a problem to providing students with a context that requires them to take an active stance in their learning.

Consider the following diagram (Costello, 2022):



Ill-defined problems are missing one or more of the three characteristics of a problem (initial state, goal state, stumbling blocks). By missing one or more of these characteristics, ill-defined problems lack the structure of a defined problem. Ill-defined problems are messy, and, for this very reason, ill-defined problems are less common than defined problems.

An example of an ill-defined problem would be as follows:

> *The population of your community is growing. There are already limited housing options available to current residents. What can the community do?*

Within such problems, the starting point is not always perceived as mathematical, or is vague mathematically. Instead, the starting point is a context, which is what promotes relevancy for students. This is what makes it

meaningful. Assigning this type of problem addresses students wondering why they should care about answering the question. It moves students from *doing* math to *thinking* mathematically.

Solving the Problems That Matter

Once we have problems that matter to students, we now have to consider approaches they can use to reach a meaningful solution. When students are working with only defined problems, they can rely on Pólya's stages of problem solving (understanding the problem, crafting a plan, carrying out the plan and review). However, we have all seen students working through the stages, but doing so via the process instead of making meaning. As such, students are still only *doing* math.

We need to move beyond this. We need to move beyond *defined problems* to include some *ill-defined problems*. When working with ill-defined problems, Pólya's stages of problem solving are not enough because of contextual considerations. Instead, we need to move toward mathematical modelling. Through mathematical modelling, students take an independent stance in their learning and broaden their and our perspective of problem solving.

Final Thoughts

Let's consider the problems we are assigning students. While we may be offering different types of problems, are they all different versions of defined problems? If yes, we may be contributing to students being passive learners and to reinforcing the divide between the math students encounter in school and their lived experiences outside of school. When we reconsider the problems we assign, we then need to broaden our perspective of problem solving to include mathematical modelling. It is through this that students take agency in their learning and become less reliant on the teacher.

In my next column, I will delve into ill-defined problems and the mathematical modelling process. Through this exploration, I will share numerous examples of ill-defined problems, highlight opportunities for such problems in the classroom, and provide work samples that demonstrate the shift in student thinking.

I look forward to continuing the discussion of linking literacy and math via my website (www.costellomath.com), email (david@costellomath.com), or Twitter (@dr_costello).

Reference

Costello, D. (2022). *Mathematizing student thinking: Connecting problem solving to everyday life and building capable and confident math learners.* Pembroke Publishers Ltd. ▲

▲ In the Middle: Sidewalk Starters



CARLY ZINIUK
carlyziniuk@gmail.com

Carly Ziniuk teaches Grade 9 Mathematics, Grade 12 Data Management, and Advanced Placement Statistics at the Bishop Strachan School in Toronto, Ontario, Canada. She is very active in adopting real-life data to engage her students in solving problems.

In the June 2022 issue of the *Gazette*, Ron Lancaster's "Mathematical Snapshots" ascended to new heights, with his final column looking at staircases around the world. There, he inspired us to "step up," as he so often did, "to find where math lives." I accept Ron's challenge by instead taking you on a horizontal path, using sidewalk explorations as starters!

I consider a *classroom starter* to be a quick question or timely task that invites students to get moving as soon as they arrive in class. My starters endeavour to get students settled, ready to learn, and propelled forward. Photographs like those in Ron's "Mathematical Snapshots" column (available in the *Gazette* archives) are ideal starters. As Ron notes in his last column, a photograph showing where math lives, naturally asks students to "notice and wonder."

In August, the Midtown Yonge Business Improvement Area showcased five local artists with sidewalk murals that encourage the community to "come out and play" on the sidewalk. If you are interested in coming to play on these mathematical pathways, or bringing your students on a Math Trail, all five murals are within one kilometer between the Davisville and Eglinton subway stations. Mirna Chacin's beautiful photographs of these murals, some of which are reproduced in this article, come from the Mural Routes website, www.muralroutes.ca/sidewalk-play-murals/, if you cannot explore these in person.

I have chosen three of these murals as "sidewalk starters" for you to investigate with your students in the classroom. For each of these murals, I have created some questions and found some resources to connect and extend beyond these first steps into the world of mathematics. There are also suggestions for how to use chalk on a sidewalk near your school, in appropriate weather conditions, to step toward your own complementary

mathematical expedition.

***Tropical Snakes and Ladders*, by Andreacataro**
Yonge and Millwood (1953 Yonge St., Toronto)



This 6x6 square Snakes and Ladders board invites students to ascend the ladders or descend on the snakes. The traditional game is played with a die. This photograph could also be printed for quick play in a classroom as a tech-free alternative.

a) How many rolls on average do you think it would take to get from 1 to 36?

b) How would this change if Andreacataro made the board 7x7 or 8x8 or *n*x*n*?

c) If we created a Desmos table (like the one shown), based on the row number (*x*) and the leftmost value (*y*) on the Snakes and Ladders board, what would the graph look like?

x	y
1	1
2	12
3	13
4	24
5	25
6	36

d) How would the graph in c) compare if we changed the *y*-values to the numbers on the rightmost edge (6, 7, 18, 19, 30, 31)?

e) How do the graphical relationships created from leftmost or rightmost edges compare with the relationships that would be formed by using some other interior column (e.g., 2, 11, 14,… versus 3, 10, 15,…)?

f) How does this compare with having the *y*-values based on the numbers on the diagonal?

g) Andreacataro has used colour on the diagonal to create some symmetry. What colour are you most likely to land on when you play the game? What is the probability you land on that colour?

h) How would our tables and graphs from above change if Andreacataro made the board 7x7 or 8x8 or *n*x*n*?

***Animal Collison*, by Hemangi**
Yonge and Glebe (2008 Yonge St., Toronto)



This circular maze encourages students to select a local urban animal and traverse the animal's journey on the way to its home, while intersecting with other members of their ecosystem. In our mathematical classrooms, this maze can be an introduction to Graph Theory, especially extension topics such as Euler paths. Middle school-appropriate explanations of this topic can be found in the CEMC 7/8 Math Circles Graph Theory activities from March 9, 2022 and October 21, 2020:

- www.cemc.uwaterloo.ca/events/mathcircles/2021-22/Winter/Grade78_20220309_Graph_Theory.pdf
- www.cemc.uwaterloo.ca/events/mathcircles/2020-21/Fall/Grade78_20201021_GraphTheoryProofs.pdf

a) Which animal has the longest journey? How did you determine the lengths?

b) Does every animal intersect with every other animal the same number of times?

c) What is the measure of the angle between each pair of animals?

d) Draw six animals evenly spaced around a circle, and connect them to their habitats directly opposite, without intersecting with any other animal.

e) Does the number of points where any pair of paths intersect stay the same, or does it vary based on how you draw the paths? Explain your reasoning.

f) Imagine that we created a circular 2D "sidewalk" cake for our six animal friends. What is the maximum number of pieces of cake we could get if each animal got a straight cut?

***Colour Paths*, by Morgan Groombridge**
Yonge and Soudan (2149 Yonge St., Toronto)



In Groombridge's mural, the viewer can either perceive five characters emerging from a rainbow or converging toward it, depending on which direction you chose to wander along this sidewalk. Like Hemangi's pathways in the Animal Collison mural, Groombridge's intersecting paths encourage you to wander along the sidewalk, interacting with the other colourful wanderers.

a) How many closed regions are created by these overlapping paths?
b) Do any characters meet each other more than once? Which ones and how do you know?
c) What is the smallest number of intersections you can create with six curves? Is there a maximum (draw it)?
d) Can you meander along a path that includes colour pieces from each character on your way to the converging rainbow? If there is a path that is most efficient, describe it.
e) Imagine that we created a rectangular 2D "sidewalk" cake for our six characters. Would the maximum number of pieces of cake we could get, if each character were to get a straight cut, change if this sidewalk cake were rectangular instead of circular? Illustrate your thinking.

Related starters, for children as young as 7, are the classic Bridges of Konigsberg and the game Sprouts. The history of the Bridges of Konigsberg problem is explained at Plus Mathematics: Maths in a Minute at www.plus.maths.org/content/maths-minute-bridges-konigsberg and with examples at www.nrich.maths.org/2327. Sprouts was created by John Conway in 1967 while he was at the University of Cambridge. More information about Sprouts can be found at Cambridge's NRICH site, www.nrich.maths.org/1208. Euler Circuits, Cake Cutting, Bridges of Konigsberg, and Sprouts can all be started with paper and pencil, or with chalk and a sidewalk.



So many math teachers around the world have benefited from Ron's learning and teaching at conferences, in classrooms, and through his "Mathematical Snapshots" column in the *Gazette*. I can think of no better way to honour his unparalleled work than to continue to travel where math lives! Getting our students started on their mathematical journeys inside or outside starts with just a single step! I encourage you, echoing the incomparable Ron Lancaster, to step outside of your comfort zone to investigate the places math lives by generating questions, solutions, and murals! ▲

Continued from page 10

2. *The sides of a triangle have lengths 3, 7, and 8 cm, respectively. Show that the measures of the interior angles form an arithmetic sequence.*

Until next time, happy problem solving!

References

Godin, S. (2013). What's the problem? Cylinders within cylinders. *Ontario Mathematics Gazette, 51*(4), 8–9.
Godin, S. (2014). What's the problem? Ski school. *Ontario Mathematics Gazette, 53*(1), 7–10.
Godin, S. (2016). What's the problem? Expanding an ellipse. *Ontario Mathematics Gazette, 55*(2), 18–19.
Godin, S. (2018). What's the problem? Playing with Pythagoras. *Ontario Mathematics Gazette, 57*(1), 41–42.
Godin, S. (2020a). What's the problem? Scrutinizing square sums. *Ontario Mathematics Gazette, 58*(3), 20–22.
Godin, S. (2020b). Problem-solving vignettes: Fun with factorials. *Crux Mathematicorum, 46*(7), 291–293. www.cms.math.ca/publications/crux/issue/?volume=46&issue=7
Godin, S. (2021). What's the problem? Diagonals and area. *Ontario Mathematics Gazette, 60*(2), 11–12. ▲

Continued from page 11

regarding teaching. The team as a whole seems to be dominated by older voices.

Overall, I can imagine this book to be useful to some at-risk students. It is not a replacement for a quality textbook. The high level of scaffolding and short summaries will address the needs of some students, and, in my opinion, it should be used differentially as a support where such a support is necessary. ▲

▲ Dynamical View of Proportionality: Families of Ratios



JÉRÔME PROULX
proulx.jerome@uqam.ca

Jérôme Proulx is a professor of mathematics education (www.profmath.uqam.ca/~jproulx), and runs the Laboratoire Épistémologie et Activité Mathématique (www.leam.uqam.ca). His research focuses on epistemological issues related to cognition and/in mathematics. His current research program is dedicated to mental mathematics and school mathematics content development. (Photo by Nathalie St-Pierre of Université du Québec à Montréal)

For many, the introduction to proportionality marks a transition from additive to multiplicative thinking, when considering the relationship between the quantities. In many problems implying proportions, an additive entry (e.g., 3 to 6 is 3 more, 6 to 2 is 4 less) often leads to erroneous responses in contrast to the consideration of a multiplicative relationship (e.g., doubling, dividing in 3).

There are, however, some student strategies that appear to fit right in the middle of both multiplication and addition, and they are worth pondering. The strategies are often conceived as elementary, but a mathematical look at them through student work might tell a different story. In the next three columns, I will ponder and explore these additive strategies that students often come up with. In so doing, I will raise elements of mathematical significance. These elements, I argue, can in turn lead to an enriched understanding of proportions, through a dynamical view of proportionality itself. This first column addresses issues related to the interconnectedness existing between similar ratios, under the notion of family of ratios.

The Problem and the Strategy

Marco, an 11-year-old, was given the following task to solve (taken from Herron & Wheatley, 1978, which is outdated in terms of prices, but nevertheless brought out Marco's mathematical thinking):

If eggs cost 60 cents a dozen,
how much would five eggs cost?

Marco had to solve this task mentally, that is, without paper or pencil. To solve it, Marco first explained that since a dozen costs 60 cents, then it would cost half for 6 eggs, so 30 cents. He then added that 5 eggs represent one fewer egg than 6, so it would be 25 cents. He explained that there are 6 blocks of 5 in 30, so 6 times 5 cents gave 30 cents, and one of those blocks of 5 cents had to be subtracted from 30 cents. To summarize:

12 eggs → 60 cents
6 eggs → 30 cents
1 egg → 5 cents
5 eggs → 25 cents

What is this strategy all about?

Although one can recognize or infer multiplicative aspects in this strategy (halving, dividing by 5), additive elements appear central to its solution. Indeed, it does not call for 5/12 of a dozen to get to 5 eggs, but rather, subtracts the *one egg too many* out of 6 or a half dozen. One main difference, as well, is that whereas with "x 5/12" one gets the number searched for right away, this sort of strategy gives rise to numerous intermediary ratios: e.g., 6 eggs → 30 cents; 1 egg → 5 cents. These intermediary ratios, although not representing the required final, are still following or preserving the relation existing between eggs and price.

This sort of strategy focuses on the relation between the eggs and their price, that is, the ratio of eggs to price, through bringing forth numerous eggs–price couples that follow this relation (although not solution to the problem *per se*). In doing so, it points to an interesting feature of proportionality: there are numerous couples that follow or preserve the same ratio, that maintains that relationship here between eggs and price. Somehow, all these couples (12 eggs → 60 cents; 6 eggs → 30 cents; 1 egg → 5 cents; 5 eggs → 25 cents) are all part of the same *family of ratios*; they are all members of a family that preserves the same relationship. And this is what Marco's strategy gives rise to: numerous couples that are members of the same family of ratios.

There are other members to this family of ratios in addition to the four ones generated by Marco, even if some do not make much sense in the context of the current problem. Here are some examples:

10 eggs → 50 cents
60 eggs → 300 cents
1.2 eggs → 6 cents
2 eggs → 10 cents
0.5 egg → 2.5 cents
3 eggs → 15 cents
1.5 eggs → 7.5 cents

What these ratios bring forth is that there is more at stake than only an initial state (12 eggs → 60 cents) and a final answer (5 eggs → 25 cents). In this proportional situation, many other ratios exist and have relevance. Some of these ratios can simplify the problem (e.g., through the unit rate: 1 egg → 5 cents). Other ratios may be relatively simple to determine, but their utility is limited (e.g., dividing by 10 generates 1.2 eggs → 6 cents, which seems to be of limited value, particularly where fractions of an egg are not usually saleable).

Family of Ratios

The notion of family of ratios underlines the issue that any ratio in the family can lead to another ratio in the family. In other words, from any one ratio, you can get to another ratio that is "smaller" or "bigger." As such, there exist numerous paths to obtain the desired ratio when a problem is posed. For example, to get to 5 eggs and its associated value in cents, one can get to the value of one egg (1 egg → 5 cents) and multiply this value by five:

12 eggs → 60 cents
1 egg → 5 cents
5 eggs → 25 cents

Or, one can find that 2 eggs are worth 10 cents, then get the price for 10 eggs by multiplying it by five (10 eggs → 50 cents), and finally halving this to get to 5:

12 eggs → 60 cents
2 eggs → 10 cents
10 eggs → 50 cents
5 eggs → 25 cents

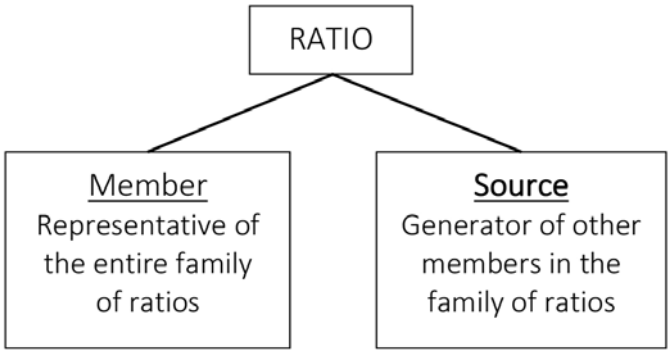
One can also get to 15 eggs and their price of 75 cents by seeing that there are 4 groups of 3 eggs in 12 eggs, and thus, 4 groups of 15 cents in 60 cents (12 eggs → 60 cents; 3 eggs → 15 cents). By adding one more of these groups, one gets to 15 eggs and their price (15 eggs → 75 cents). Then, dividing by 3, the value of this group of 15 eggs leads to 5 eggs for 25 cents.

12 eggs → 60 cents
3 eggs → 15 cents
15 eggs → 75 cents
5 eggs → 25 cents

And there are many other routes that can be taken, with 2 eggs, 3 eggs, etc. All these paths to get to "5 eggs → 25 cents," with choice and preference by the solver that might be guided according to some ratios being easier to obtain than others for *that* solver. Marco, interestingly, mentioned before giving his solution that for him, "dividing 60 by 12 was too hard," so he had to use another route to get to 5 eggs and their price.

Depending on the problem at hand, and on one's preferences or abilities, there are always ways to transform the initial ratio into other ratios that may be considered easier or more accessible. It could be "1 egg → 5 cents"; "6 eggs → 30 cents"; "2 eggs → 10 cents"; "0.1 egg → 0.5 cents," etc. All have the possibility to make solving of the problem more accessible, depending on the problem's context and the solver. In that sense, there are no "better" ratios in a family of ratios, since it is dependent on the context and the solver (see also Mochon, 1993, regarding this). The "1 egg → 5 cents" might be thought of as an *easier* ratio in some cases with the presence of a unit, whereas in other cases "2 eggs → 10 cents" could be said to be more *useful* because of the presence of a 10, or again, "120 eggs → 600 cents" could be seen as more *versatile*, with the flexible manipulation of 120 and 600 and their numerous factors and divisors.

As such, none of these family members represent the "ultimate" member of the family. There are no hierarchies, as they all have the capacity to be transformed into one another or to produce other ratios. Hence, any member of a family of ratios has a double role: it is both a *representative* of that family and a source to generate other ratios in the family. As a family member, each ratio is a representative of the family and indicates what this family is about, that is, the relation governing the quantities. As a source, each ratio is also a *generator* of other members of the family that follow this same relation. The following figure illustrates this.



Dynamic View of Proportions

This view of proportional ratios as members of the same family outlines the fluidity existing from one member ratio to the next. The search for the target ratio in a problem can, in that view, be seen as a "sorting-through" endeavour, using members of the same family for developing the right one. Searching for the desired ratio, the ratio couple that solves the problem becomes the same as searching through the family itself, going through its members, to find the one

needed. This view also underlines the flexibility existing within one family of ratios, where there is an infinity of possible paths to be taken to get to the desired ratio (e.g., from 12 eggs → 60 cents" to "6 eggs → 30 cents" to "2 eggs → 10 cents" to "1 egg → 5 cents" to "0.1 egg → 0.5 cents," and so forth.

Lesh et al. (1988) oppose a static view of proportions that assigns an equality between two things, to a dynamical view of proportions, more aligned with the establishment of a linkage between ratios and their similarity through a transformation from one to the other. For them, this dynamical view leads to essential components of proportional reasoning. They explain the following:

> The equation A/B=C/D can be thought of as a static (=) relationship between two simple mathematical systems which are described independently by the ratio relationships A/B and C/D. However, it can also be thought of as a dynamic transformation which maps one simple mathematical system (described by the ratio relationship A/B) into an "equivalent" system (described by the ratio relationship C/D). It seems to us that the recognition of structural similarity is an essential component of proportional reasoning. (p. 100)

The notion of family of ratios seems to tap right into Lesh et al.'s (1988) suggestion for conceiving ratios and proportions along a dynamical view. The notion of a family of ratios steps away from a focus mainly placed on initial ("given") and final ("solved") states when working on proportional situations, and opens the task to consideration of numerous intermediate ratios existing "in between." This is what Marco's strategy brings forth, giving rise to various members of the same family of ratios that are all possible routes to get to the needed ratio for solving the problem.

Above all, the notion of a family of ratios points to the fact that ratios exist in a plural space, where there is an infinity of them that follow the same relation. Going from one ratio to another, and back again, illustrates how fluid and flexible proportions can be. In short, it shows how proportionality is to be conceived in a dynamical fashion.

References

Herron, J.D., & Wheatley, G.H. (1978). A unit factor method for solving proportion problems. *Mathematics Teacher, 71*(1), 18–21. www.jstor.org/stable/27961131

Lesh, R., Post, T.R., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). National Council of Teachers of Mathematics.

Mochon, S. (1993). When can you meaningfully add rates, ratios and fractions? *For the Learning of Mathematics, 13*(3), 16–21. www.jstor.org/stable/40248090 ▲

▲ Elementary Math Matters: The Anchor, the Deputy Prime Minister, and the Math Ed Professor



LYNDA COLGAN
lynda.colgan@queensu.ca

Dr. Lynda Colgan is Professor Emerita, Faculty of Education, Queen's University, and Executive Director (Education and Development, Science Rendezvous). Lynda is currently working on a number of research, resource-creation, and teacher-education projects, funded by the National Sciences and Engineering Research Council of Canada (NSERC) and the Social Sciences & Humanities Research Council (SSHRC). She will carry on her long-established commitment to STEM outreach by continuing to seek out creative avenues to engage students, parents, and educators in mathematics at home and in the classroom.

My morning routine is, well, routine. After getting out of bed, letting Annie out into the backyard and Bella into her catio, I pick up my morning newspapers from the mailbox, then head back inside to put on the kettle and fill the French press to make the obligatory carafe of strong coffee that heralds the start to the day. Next is *Wordle*, followed by *Ponder* on my phone. My attention then turns to the newspapers. Satisfied that I am up-to-date with the goings-on of my city, province, country, and planet, I settle in to do the crosswords. Well sated, finally, I turn on the computer.

Same story with the laptop, for I am a creature of habit. First, I check Outlook and then Gmail. After deciding what needs an immediate response and what can wait, I turn to Twitter. There, I catch up on a decidedly curated selection of news, event announcements, and blissfully, non-sensical dog and cat posts. It is also on Twitter that I get my daily doses of inspiration from folks that I respect and trust, and stay *au courant* with the world of mathematics education.

It was on Twitter that I, like the rest of Canada, watched Lisa Laflamme describe being blindsided by a painful dismissal from CTV after 35 years as an anchor. It was on Twitter that I learned of the vicious verbal assault directed at Deputy Prime Minister Chrystia Freeland by a man as she walked into an elevator in Alberta. Ageist, sexist, misogynistic, abusive actions and words that clearly had traumatic impact on the individuals, and significant emotional impact on me as a woman reading and processing this

horrific content.

With these and other powerful women on my mind, my neurons made a connection. "Why," I asked myself, "haven't I seen anything from Jo Boaler lately on Twitter?"

As many *Gazette* readers will know, Stanford Professor, Jo Boaler, has made a huge and positive difference to mathematics education in Ontario. Many school districts ran book clubs as professional development opportunities—*Mathematical Mindsets: Unleashing Students' Potential through Creative Math, Inspiring Messages and Innovative Teaching* and *Limitless Mind: Learn, Lead and Live without Barriers* were *de rigueur* reading. Faculties of Education adopted these and other books, like *The Elephant in the Classroom: Helping Children Learn and Love Maths*, to be required or recommended readings in pre-service and in-service programs. It seemed that as a community, we shared the view of Carol Dweck (who made the concept of growth mindset famous), "Boaler is one of those rare and remarkable educators who not only know the secret of great teaching, but also know how to give that gift to others" (youcubed, 2022).

For the last two decades, Dr. Boaler has been a keynote speaker, not only at provincial mathematics education conferences in Canada, but around the world, spreading important messages, i.e., everyone can learn math—there is no such thing as a math gene or a math brain; believe in yourself; mistakes and struggle are important to our learning; and depth of thinking is more important than speed. At these events, many classroom teachers experienced epiphanies of sorts, buying into the passionate arguments Jo made for adopting richer, more open teaching methods and encouraging children to adopt a growth mindset.

So, why "crickets" from Jo Boaler on Twitter of late?

A quick Google search provided some information, but the story was not clear when I tried to piece the disjointed articles together. So, I turned to colleague, Sunil Singh, a close friend of Jo (she wrote the forward to his latest book, *Chasing Rabbits*) for an answer to my question. He directed me to his August 20, 2022 blog post: *Dehumanizing Math Educators: The Jo Boaler Story You Don't Know.* This is what Sunil wrote:

> *Death threats. Rape threats. PTSD. Never thought these horrific words would enter a math blog… But, there is no possible way to tell the complete story without them. This is what Jo Boaler is now dealing with. What kind of climate of discourse and understanding about mathematics education must exist to contribute to such unimaginable and unrelenting hostility?*

It turns out that there are firestorms in California other than those we see on the nightly news, wreaking a huge path of blazing destruction. These "other" conflagrations are not the natural phenomena we have witnessed via video, but are, in fact, manmade. The cause? The publication of *California Mathematics Framework*, written by a team led by Dr. Boaler herself.

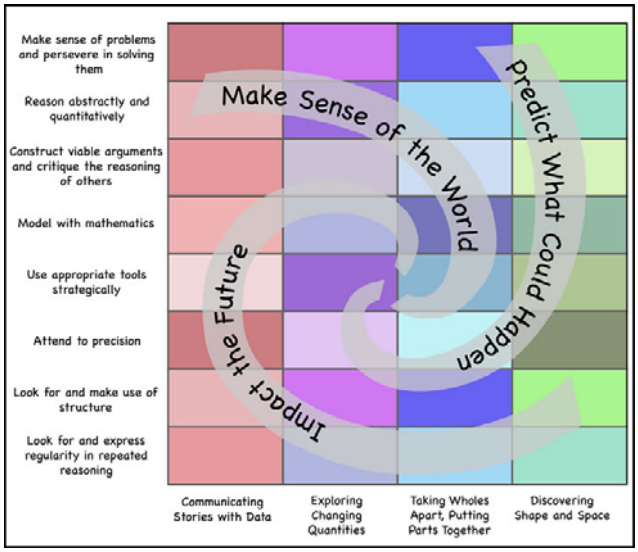


Figure 1: *According to the California Department of Education, this figure shows the Three Drivers of Investigation (DIs) that provide the "why" of learning mathematics: Making Sense of the World (Understand and Explain); Predicting What Could Happen (Predict); Impacting the Future (Affect); The DIs overlay and pair with four categories of Content Connections (CCs), which provide the "how and what" mathematics (CA-CCSSM) is to be learned in an activity: Communicating stories with data; Exploring changing quantities; Taking wholes apart, putting parts together; Discovering shape and space.*

The California Mathematics Framework (CMF) is a 900+ page document, written by a five-member team, including Jo, that reported to a 20-person oversight team. Work to revise the mathematics curriculum in California began in 2019 as part of a well-established eight-year revision cycle by the California Department of Education, with a goal of helping more students thrive in science, technology, engineering, and mathematics.

Condemnations quickly emerged when the first draft was released in late Spring 2022, with critics, principally college- or university-level mathematicians, Advanced Placement (AP) and International Baccalaureate (IB) instructors, loudly denouncing the elimination of Algebra in middle schools (pushing it to Grade 9), the de-emphasis on calculus, and the application of social justice principles to math lessons. Parents and some associations (such as the National Association for Gifted Children), too, were upset by what

they perceived to be inequitable treatment of gifted and talented students because the recommendations discourage streaming prior to secondary school and ask teachers to refrain from labelling students as "naturally talented" in math.

Four hundred mathematicians, who argue that their voices were not heard in the development of the CMF and that the completed document does not reflect an appropriate mathematics content pipeline to post-secondary courses, signed an open letter claiming that graduates of the program will be doomed to fail in fast-paced college or university STEM programs. (The open letter is titled *Replace the Proposed New California Math Curriculum Framework* and may be found at www.independent.org/news/article.asp ?id=13658.) They accuse the authors of the CMF of "offloading" the responsibilities of K–12 math onto them, in effect, forcing them to teach low-threshold remedial courses to equip students with the necessary skills and content that post-secondary instructors argue should have been mastered well before graduation from high school.

Some mathematics educators, too, were quick to express disapproval, saying that the CMF would hurt historically marginalized students the most by injecting too many social justice-related topics that distract from the math. They accuse Jo Boaler and her team of *de-mathematizing* math. Tom Loveless, a retired math education expert, who worked for the Brookings Institution, a national think tank, expressed the widespread sentiment like this: "The way you get social justice in mathematics is to teach the kids math. It's not by dressing up mathematics in social justice" (Hong, 2021a).

With controversy swirling and hundreds of individuals and groups submitting responses to the CMF oversight board, The State Board of Education has pushed back the adoption of the California Math Framework to sometime in 2023, and the writing team is back at the table.

Heated debate around, and strong reactions to, proposed changes to mathematics education is nothing new. Over the past 70 years in Ontario alone, we have lived through major changes to curriculum content and popular pedagogies, with each "new" release representing a triggering event, ranging from a sweeping education movement (like the publication of the NCTM *Principles and Standards*) to the election of a new government.

Every revised document brings with it change—something that is always threatening and difficult. Reactions to that change reflect individual reactions to the core beliefs held by various stakeholders. Challenges to those deeply rooted beliefs evoke many questions, the first of which is "why?" and the second, "who?"

The "why" question can usually be addressed quickly and satisfactorily as responses to external factors, such as, in recent times, the availability of powerful technological tools.

But the "who" question seems to raise folks' hackles. There is no simple answer to who "owns" mathematics curriculum. Mathematicians see it as their purview. Mathematics education researchers claim that it falls under their area of expertise. Epistemologists believe that they should play a role in curriculum development because it is this cross-section of cognition and philosophy that can explain how we build, define, and evaluate knowledge. Developmental psychologists see their work having a place in continua. Classroom teachers contend that their first-hand knowledge of children and learning must not be ignored. Parents and guardians, who are their children's first and most important educators and role models, want their needs to be addressed in the curriculum. The list goes on and on.

True, there have been very heated discussions over the decades (Irvine, 2021), and over time, some voices have become louder, while others have turned down the volume. At the end of the day, draft documents become policy, a consensus document (Sibbald, 2020, p. 1), and everyone gets on with the principal task of educating our children.

Everyone acknowledges, *silently*, the existence of the hierarchical "chain of blame" attributed to the lack of achievement in mathematics and children's (as well as society's) negative disposition to it. Parents/guardians do not prepare their children for school. Elementary teachers don't understand the importance of families of number facts. Secondary math teachers teach content, not students. Mathematicians only teach math because they must—they would much rather be doing research—so they do a poor job of it.

Boaler, her team, and other advocates of the framework responded to criticism of the CMF *vocally*, explaining why mathematics educators played leading roles in the development of the document. Jo justified the decision, saying:

> University professors—who work with older and more motivated students who elect to take higher-level math courses—might not have the insight needed to build a K–12 curriculum, especially for historically marginalized student groups. Mathematics educators understand education, and mathematicians have no experience studying education. Mathematicians sit on high and say this is what is happening in schools (Hong, 2021b).

And with that, the match was lit.

The most scathing indictment of the CMF came from

Stanford Professor, Brian Conrad (described as an "actual" mathematician), who accused Boaler and her team of doing sloppy science: misrepresenting studies and selectively citing research to make points they wanted to make in support of their recommendations.

The *Stanford Review* (the university's independent newspaper) focused more on the fact that Boaler had no right to charge $5000 per hour as a consultant to a low SES school district because she was "known to be advocating for lowering, not increasing educational expectations." In the same article, other allegations of financial abuse were hurled at Boaler, and it was erroneously reported that youcubed (the website she co-founded) had been suspended. The authors of the article encourage readers to contact their political leaders and others with influence to stop the CMF.

Based on Sunil's blog post, it is clear that many vitriolic, abusive, threatening responses did, in fact, come in, and were then shared widely on social media. Jo's own reaction has been to state that while the messages are egregious examples of maltreatment and misinformation, she promises to continue to stand up for equitable mathematics. With that, we come full circle. According to Sunil, Dr. Keith Devlin, a renowned Stanford mathematics professor, has gone on record saying that he more or less says the exact same things as Jo Boaler. Why then has there yet to have been one dart thrown in his direction? Because mathematics, like the media and politics, has a gender problem.

There still exists a pervasive belief that mathematics is a male domain. Mathematics is typically described as hard, complex, and based on thinking—not feeling—and these characteristics, being masculine traits, make it an unsuitable field for a woman. The notion that top mathematicians are born, not made, and that more males are born that way, still has strong credibility.

We are in need of many long overdue changes to reshape and reframe mathematics, as well as attitudes about mathematics and those who belong in it. Failing to act, according to Keith Devlin, "carries a risk of getting stuck in a tunnel while the world changes outside the tunnel walls." In speaking about the current controversies around the California Mathematics Framework, Devlin (2022) says,

> to avoid [getting stuck in a tunnel], we should continually strive to identify and assess the choices that have been made, who made them, and with what purpose in mind. If there is not a constant, heated debate about what math is and how we teach it, we're doing it wrong.

Heated debate, yes. Misogynistic and abusive attacks, no. We've been in that tunnel far too long.

Continued on page 24

▲ OMCA News: December 2022

HEATHER THEIJSMEIJER (OMCA PAST PRESIDENT)
OMCAmath@gmail.com

The Ontario Mathematics Coordinators Association (OMCA) is looking forward to another great year of mathematics learning with Math Coordinators, Consultants, and Coaches from across the province.

As an organization, we have grown over the past two years, with our virtual meetings allowing us to regularly meet with members from across Ontario. To maintain this level of accessibility, it was agreed that our monthly members' meetings would continue to be virtual. Our May meeting will be the exception: scheduled in conjunction with the OAME2023 Conference in Toronto, we will gather in person. At the time of writing, we are looking forward to an outstanding lineup of speakers over the first half of the school year, including Dr. Pamela Seda (October), Dr. Juan Carlos Chavez (November), David Poras (December), and Dr. Robert Q. Berry (January).

Curious about our meetings? Each meeting includes updates from stakeholder groups sharing similar interests as OMCA (such as the EQAO, the Math Knowledge Network, the Ontario Ministry of Education, and more), networking opportunities, resource showcasing, and a guest speaker. Non-members are welcome as guests for the guest speaker component of the meeting, for a fee of $30.

We are pleased to announce that we will be returning to an in-person format for our annual retreat! This will include a day and a half of networking, collaborating, and learning from guest speakers. Details are still being finalized for this February event. Keep an eye on our website for more information.

From Pikangikum to Peel, and from Walpole Island to Williamstown, new members are always welcome to join the OMCA. Contact us directly at OMCAmath@gmail.com, or visit our website at www.omca.website (note the different domain), to learn more about the organization, find out about upcoming events, and connect with us. ▲

The Editor finally caught up with our newest Associate Editor, Peter Saarimaki. Good chatting over tapas in Barcelona!



▲ Coding in the Classroom: The Destreamed Grade 9 Mathematics Curriculum



IAIN BRODIE
iainbro1.625@gmail.com

Iain was an elementary school educator for 28 years, and is now into a second career, teaching in-service and pre-service teachers. He teaches face-to-face at Ontario Tech University and virtually at Western University. He is the current editor of the Math+Code 'Zine, *www.researchideas.ca/mc/.*



HATICE BEYZA SEZER
hsezer2@uwo.ca

Beyza is a former teacher and long-term researcher. She is currently a research associate in education at Western University, and her research focus is the integration of computational thinking into mathematics education. She is an editor at Math+Code 'Zine*, www.researchideas.ca/mc/.*

Quotations may have been lightly edited to reflect the difference between spoken and written language.



Alice Aspinall is a high school teacher and recipient of the Prime Minister's Certificate of Achievement for Excellence in Teaching Science, Technology, Engineering, and Mathematics (STEM) (www.cbc.ca/news/canada/windsor/windsor-teacher-stem-math-prime-minister-1.6259770). We were fortunate to meet with Alice and ask her about her experience implementing the new Ontario destreamed Grade 9 mathematics curriculum, and what her impressions are about what coding in her classroom has added to her students' learning of mathematics.

Because of her studies in mathematics, where she also studied computer science, Alice has a well-informed view of what the coding in our curriculum is and is not. The coding in the curriculum is not what a computer scientist or programmer might think of it being—it is more of a way of thinking about a problem in order to break it down and solve it.

> If you look closely at the curriculum, the coding section, the way I interpret it is algorithmic thinking, right? That's how I view it. It's really about algorithms and thinking in a very simplistic step-by-step style. It's not writing code at all. We use Scratch to do some coding, but really, that's just a series of steps in writing an algorithm. It's not actual coding as I would have known it, right? When I first heard that coding was in the curriculum, I thought, oh, like, what language are we going to be using? This is going to be really difficult, but it's not like that at all. So students were surprised to see that when we say coding in this new curriculum, what we really mean is just algorithmic thinking, and that's what I had to explain to them. We have to break down what we do into the most miniscule steps. That is what we're going to be looking at, and so that brings about a different kind of thinking, right?

This algorithmic thinking is the ability to break down some math that we want to explore or to use in solving a problem in such a way that we can write out the steps so they can be used by a computer. It is similar, but much more complex and useful than the algorithms our students will have experienced in arithmetic, and it is important that we show our students the potential that algorithmic thinking brings to our abilities to engage and dive into the mathematics we are learning. A good analogy for this is how we teach our students to understand and use metaphor in their writing to enrich their literary lives. Teaching students to code can have a similar effect on their mathematical lives.

It is this enrichment that everyone who has included coding in their classrooms has noticed, and it has the most dramatic effect on some of those whose mathematical journeys may have gotten a bit off track. In our past columns, our interviewees have identified the positive effect coding has on those students who have learning difficulties, as it provides them with an environment that gives them instant feedback and the safety of being able to quickly fix bugs in their programs. Alice identified another set of students who already have an interest and experience in coding, and what sparked their interests.

> The biggest challenge for me [at the beginning] was I didn't really know where everybody was. Of course, we have some students who are gamers, and have taught themselves well beyond where I am in my coding journey, so they're real experts in it. I would give them something to do, where they would go above and beyond, and in some cases, these were the students who didn't have any interest in what we were doing on

other days, but they were experts in the coding realm. That made for such an interesting dynamic, and now, suddenly, students who never were really enjoying things in math class were the experts, and they're going around and showing other students how to do things. It really changes the dynamics.

These students were largely self-taught and had learned through trial and error, or as we have learned to call it, trial and improvement. This brought up the topic of gaming, which is seen by many as a distraction, but Alice pointed out that there are some good things that games can teach our students.

> So yes, there may be some bad parts to gaming, for sure. But I think on the flip side, there's a lot of really good learning that's happening, perseverance, things that our children are learning by making all those mistakes, and starting over and really persevering to want to beat a level or to get something in the game they're playing.

The area of gaming is something that Beyza and I have been interested in from our conversations over the last year with educators—not the playing of games, but the creation of games to demonstrate student learning. This is a genre that students are familiar with, so when we ask them to create a game to show their understanding of a mathematics concept—or science, geography, history, English, health and physical education—they know what is needed game-wise for the player, and can concentrate on the learning of the math in the game. This could also lead to students understanding concepts at a deeper level than they would otherwise.

Coding can also have a positive effect on those students who normally thrive in math class, rushing to solutions valuing correctness, and it comes through the way trial and error (improvement) plays a big role in learning to code.

> So much trial and error, and the students who really want to get it right the first time? We all have those students, right? They were struggling when we were doing coding because the first idea they had wouldn't work out. And you know, the best thing about writing code is that you know right away if it works or doesn't work. You immediately know if there's an error, it's very obvious and they [students] don't like that. They want to just get it right the first time, or they want to be told how to do it right the first time.

In marketing, there is the idea of a speed bump, something that slows a person down so that they are drawn to look at a product more deeply or more carefully. In classrooms, we use this idea in the form of curious puzzles or discrepant events that cause students to stop, "Wait! What?" Alice points out that in this case, we can also reach those quick-thinking students, who may not necessarily think deeply, as they are slowed down, to consider some of the depth of the mathematics they are doing as well.

Seymour Papert said that learning mathematics through coding was like learning French in France—the idea of immersion in a language, in a culture. In this column, we have highlighted Alice's thoughts and examples, put the flesh on the bones of Papert's idea as it applies directly to our math classrooms: what you will see and experience with your students when introducing coding into your math classroom; what it will bring to your students; and, most importantly, why you should implement it beyond it being a curriculum requirement. We talked about a lot more than just this with Alice, and she shared many more insights and ideas with us about her first year working with the new Grade 9 destreamed curriculum. You can download and listen to our full conversation at the OAME Talks podcast wherever you get your favourite podcasts. ▲



Continued from page 22

References

California Department of Education. (2021). *The California mathematics framework*. www.cde.ca.gov/ci/ma/cf/

Conrad, B. (2022) *Public comments on the CMF.* www.sites.google.com/view/publiccommentsonthecmf/#h.ns5n6hdqa4x8

Devlin, K. (2022, May 3). Mathematics and the real world. *Devlin's Angle.* www.mathvalues.org/masterblog/mathematics-and-the-real-world

Hong, J. (2021a, November 19). Understanding the debate behind California's new math framework. *Cal Matters.* www.calmatters.org/education/k-12-education/2021/11/california-math/

Hong, J. (2021b, November 21). Why revamping the way California teaches math is so contentious. *KQED.* www.kqed.org/news/11897105/why-revamping-the-way-california-teaches-math-is-so-contentious

Irvine, J. (2021). Blast from the past! The new math. Ontario *Mathematics Gazette, 59*(4).

Stanford Review. (2022, June 6). *Review investigation: Jo Boaler, Cathy Williams, and the woke math scam.* www.stanfordreview.org/review-investigation-jo-boaler-is-worse-than-we-thought/

Sibbald, T. (2020). Editor's report. *Ontario Mathematics Gazette, 59*(1), 2–3.

Singh, S. (2022, August 20). *Dehumanizing math educators: The Jo Boaler story you don't know.* https://sunilsingh-42118.medium.com/dehumanizing-math-educators-the-jo-boaler-story-you-dont-know-eb856d246571

youcubed. (2022). *Limitless mind.* www.youcubed.org/limitless-mind/ ▲

▲ Family Math Nights: Engaging Students and Their Parents Outside of Class



DR. JEFF IRVINE
dr.jeffrey.irvine@bell.net

Jeff Irvine, MA, MBA, PhD, CFP, OCT, has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

The recently released *Ontario Curriculum, Grades 1–8: Mathematics,* 2020 explicitly recognizes the value of family math nights as a parental outreach strategy that yields positive results (Ontario Ministry of Education [OME], 2020, p. 14). A simple Internet search for "family math night" yields 154 million results. In investigating a large number of the results, the overwhelming majority involve tips, resources, and ideas for elementary school math nights. This article outlines some considerations for why you should utilize family math nights in middle and secondary school, rather than exclusively in Kindergarten to Grade 5,

One very successful version of family math nights conveys messages that math is fun, active, hands-on, and worthwhile. Family math nights in this category include games, puzzles, movies involving math, and mathematical activities in other subject areas. Often, prizes are involved as additional motivation for participation. These family math nights have no explicit curriculum messaging, but do encourage whole-family participation in children's learning and enjoyment of mathematics.

Below are some descriptions of family math nights in which I have been involved. These are, of course, not the only possible formats, but are offered to provide suggestions for building your own family math nights. I have found that the responses to family math nights, while varying greatly in participation, have built rapport with parents (including guardians), and have served to dispel some of the concerns parents frequently express around "I didn't learn math this way," as well as generating interest and motivation for students.

Secondary School Family Math Nights

Most of my involvement in family math nights was at the secondary level. One class of family math nights was Math Sine-ama, where we showed movies and cartoons involving math. Popcorn and soft drinks were sold by the student council. The goal of these movie nights was to illustrate connections between mathematics and the real world. The movies had to be carefully chosen. Many commercial movies, such as *Good Will Hunting* and *A Beautiful Mind*, send the wrong message: math is for the elite, the very bright, the select few. This is also true of many television series, such as *The Big Bang Theory*. The message that we wanted to convey is that math is used by vast numbers of "regular" people, in a wide variety of occupations, and that math occurs in a myriad of situations in the real world. Many successful Math Sine-ama nights used movie clips found by our media arts students, illustrating math "misteaks," situations in which the mathematics in the movie clips was incorrect.

The majority of our family math nights were targeted to our Grade 9 students, both current and incoming from Grade 8. We ran two family math nights per year, one in the Spring for incoming students, and a second family math night early in the Fall, to help make our new Grade 9 students and their parents more comfortable with their new school and the structure of their new mathematics classes. These family math nights, especially the Spring night, were whole-school affairs, with participation from many departments other than math. The math department was a participant in all the nights, and other departments opted in or opted out, depending on the year. No explicit curriculum messages were included; the entire night consisted of parents and students moving to various locations within the school and being active participants in mathematics-related activities.

Prior to the family math night, especially the Spring night, we sent flyers and school maps to our feeder schools' Grade 8 students. The maps outlined where activities would be held, and the flyers gave brief descriptions of what would occur. We always included a challenge problem for the students and their parents. Examples (from Irvine, 2018, pp. 35–36) of these problems included: the twelve sticks, the painted cube, pizza cuts, triangular numbers, and others. Grade 8 families were invited to submit solutions, and winning entries won π-shaped cookies baked by our family studies department.

Here are some of the activities from the family math nights, first for other departments, and then for the math department:

Physical Education

Activities typically involved body measurements, such as arm span versus height, hand span versus foot size, and foot length versus height. One very popular activity involved a pressure plate that measured force when a student or parent jumped from it and was then compared to vertical jump height. Our current Grade 9 students utilized technology to construct scatterplots of the data, and generated lines of best fit.

Family Studies

Students and their parents were involved in baking π-shaped cookies that could be taken home at the end of the night; scaling up recipes to feed families of various sizes; cutting patterns for various clothing, which the students could then take home to sew; and scavenger hunts within the family studies rooms to find items on a list. All participants won π cookies.

Science

Science activities included building molecules from manipulatives; constructing electrical circuits, using computer software; cart racing in the school halls; online astronomy investigations; density experiments; and sometimes the science department even got ripple tanks out of storage and families explored wave patterns.

Technology

We had a very active computer art department, so many of their activities involved families designing artworks, using computer programs. Participants could save their work on USB drives and take it home. Other technical areas like the woodworking shop were also involved, with families designing projects to pursue at home.

Mathematics

While other departments participated on an occasional basis, unsurprisingly the math department was the constant for every family math night. Over the years, the department used a wide array of activities, with several inclusion criteria: the activity should be fun and involve activity of some kind; it should be inclusive so that both parents and students could be involved; it should require little or no prior mathematical knowledge, making the activity non-threatening, especially for parents who might have been out of school for extended periods of time and might lack a high level of formal education. Current Grade 9 students were available as coaches, and in some cases, to extend activities based on what they were already learning in Grade 9 mathematics classes. Prizes were awarded. Below is a list of activities that were used.

- Walk this way: This activity involves walking in front of a motion detector to match a distance–time graph given on the screen. From the *Grade 9 Course Profiles* (OME, 1999), this activity was really popular.
- Going around in circles: This is another motion detector activity, in which participants walk around a hula hoop placed on the floor and predict the shape of the resulting graph (Irvine, 2018, p. 36).
- Sometimes the hula hoops were used in a competition, counting the number of rotations of a hula hoop, and the time the hoop was maintained in the air. Scatterplots were created of number of revolutions versus time.
- Figurate number model construction using manipulatives: triangular numbers, square numbers, pentagonal numbers. Pattern finding.
- Learning styles inventory: Students and parents completed a learning styles inventory (e.g., OME, *Grade 9 Course Profiles,* 1999), and current Grade 9 students constructed circle graphs, using technology. Both students and parents were interested to learn more about their children's (and their own) learning modalities. Copies were printed to take home.
- Magic squares: Construction of magic squares and connections to history, like the Lo Shu square (Figure 1, reproduced from wikipedia.org/wiki/Lo_Shu_Square) and Albrect Dürer's Melancholia (Figure 2, reproduced from wikipedia.org/wiki/Melencolia_I), which contains a magic square, in which the two middle numbers in the bottom row give the date of the engraving (1514).
- Number fun: Abundant, deficient, perfect, and amicable numbers were investigated. Calculators were provided for checking results.
- Active pattern puzzles: Such as frogs, pizza cuts, crocodile river, marble mountain (Irvine, 2018). Manipulatives were provided.
- Logic problems, sometimes called grid problems: E.g., match five people with their occupations, given a number of clues.

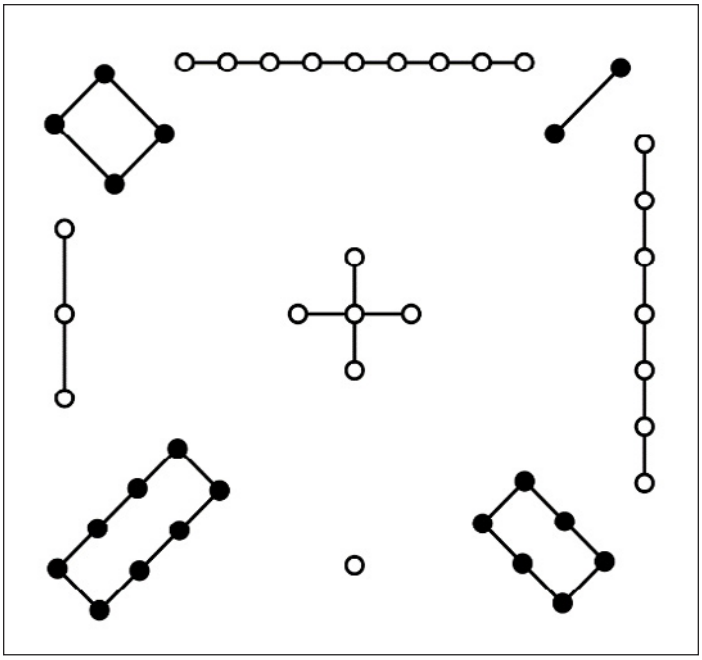


Figure 1: *The Lo Shu magic square, c. 650 BCE*



Figure 2: Melancholia*, engraving by Albrecht Dürer, 1514*

Each family math night lasted approximately two hours. The night was split into four half-hour rotations, but families could choose to stay at the same location if they were really involved in the activity. At the end of each family math night, families completed a brief exit survey, in which they indicated how they felt about the night, and posed questions that they might want answered by school staff.

There was no introductory messaging. Messages about what Grade 9 mathematics instruction looked like were embedded in the activities. Since the school was in a very diverse community, math instruction was often very different from what the parents and students had experienced in their countries of origin.

Family math nights were a non-threatening way to introduce Ontario pedagogy and also provide a fun night out for the families.

Below are brief descriptions of family math nights in which I was involved at the middle school and elementary school levels. They all had familiar themes: active, fun, interesting, non-threatening.

Middle School

For one family math night in a middle school, I used resources such as the short video *What does good math instruction look like?* (edugains.ca/newsite/math/school leader/resource_collection.html) to illustrate how math instruction has changed since many parents were in school. This was followed by students and their parents following a one-hour schedule with four rotations to classrooms, where teachers provided activities that illustrated what current math pedagogy involved. The activities used manipulatives, technology, and discussion as parents and students solved problems together.

At a different middle school, some family math nights were pure games nights. Some games nights used commercial board games, like Monopoly®, Payday®, and Yahtzee®. Math expectations were not explicitly mentioned, but students and their parents had fun, played together, and began to think of the school as a community hub. The principal provided pizza for all attendees. On other family math nights, more mathematical games were played, such as can be found on the mathies.ca website (Figure 3). Once again, the principal provided pizza.

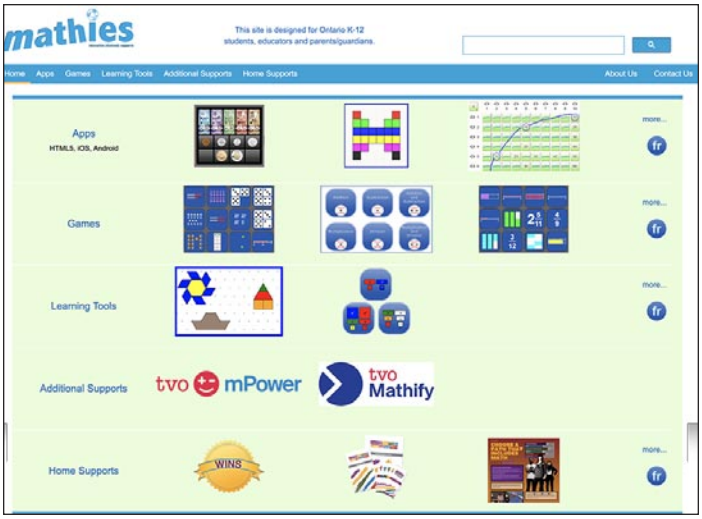


Figure 3: *mathies.ca*

For many students, middle school is a traumatic, uncomfortable time. Jacqueline Eccles has postulated stage-environment fit theory (Gutman & Eccles, 2007), which suggests that the move to middle school is about more than just changing buildings. Students must often adjust to much larger buildings; very different instructional techniques; increased use of teacher-centred pedagogy; loss of friendships as some students move to other schools; conflicting desires of students craving more personal freedom while entering what is often a more rigid environment; cliques and bullying; and, of course, coping with puberty and associated physical and psychological changes. Family math nights are one way to address some of these changes and increase student comfort levels in their new environment, while helping to maintain some parental involvement in their schooling.

Kindergarten to Grade 5

Family math nights can involve some explicit information around content, pedagogy, or assessment in addition to students and parents participating in activities that reinforce these messages. It is vital that the activities be active and hands-on, and involve both the students and their parents.

I was the invited speaker for a family math night, delivering the message that math instruction is now participatory, social, problem-oriented; the school was a math talk learning community, where student concerns, opinions, and alternate solutions were valued, and students were encouraged to take risks. I was followed by a wonderful presentation by four Kindergarten teachers, who demonstrated through activities how students transitioned from the play-based Kindergarten program to the Grade 1 math curriculum expectations. All the activities involved both students and their parents, were easy to understand, involved manipulatives, and allowed students to "become the teachers" for their parents.

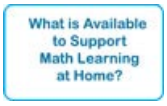
Another family math night in a different Kindergarten to Grade 5 school featured a very limited introduction, followed by over an hour of activities taken directly from teachers' instructional materials. Again, the activities were active, hands-on, and involved manipulatives to achieve mathematical expectations. This school had a very high ESL population and extremely high student turnover. Since many of the parents were uncomfortable with English, a series of five family math nights were held, presented in different languages: English, Punjabi, Urdu, Mandarin, and Tagalog. Community response was excellent; parents were comfortable with having information given in their first languages; and they learned a lot about how Ontario mathematics education differed from the education systems in their homelands. The learning was by example, through the activities in which they and their children participated.

At many of these family math nights, parents were given information on resources that were available, such as those found under "Home Support" on the mathies.ca website (Figure 4).

Lynda Colgan, professor at Queen's University, is a passionate advocate of family math nights, in addition to other outreach activities to engage whole-family participation in mathematics education. She has taught family math night courses for her pre-service and in-service teachers (Colgan, 2015); she has extensive training in family math and parent-engagement strategies (Colgan, 2008); she is a frequent speaker to parents on the topics of engagement, motivation, and math pathways (Colgan, 2016, 2018); and she has recently completed a two-year project, which developed a website for parent resources and ideas to engage families in mathematics (Colgan, 2019). A major focus of her work is to reduce or eliminate the parental influence on students resulting from "I could never do math" comments that are sometimes pervasive when parents discuss mathematics and mathematics education. To dispel this influence, Lynda sees family math nights as an important element. "The impact of well-organized family math programs is powerful" (Colgan, 2008, p. 14).

While the majority of family math nights continues to occur in elementary schools, secondary schools can reap abundant rewards through encouraging whole-family participation in activities that engage students in the study of mathematics. By reaching out to incoming Grade 8 students, and by encouraging participation in these events by current Grade 9 students, positive attitudes toward mathematics can be enhanced and supported, as well as build student confidence and self-concept.

Home Support



What is Available to Support Math Learning at Home?

---

mathies.ca hosts parent resources, as well as Ministry-developed digital mathematics learning tools, games, and activities. Postcard

---

Curriculum – Illustrated Guides

These illustrated guides highlight overall and specific expectations by grade level. They provide visuals of key mathematical concepts, tips about the math for parents, and sample activities that can support student learning at home. Currently, Grades 1 to 3 are available.

Activities – Working in Number Sense (WINS)

Working in Number Sense (WINS) activities are designed for parents/guardians/learning partners to engage with a child, using the learning tools and games by mathies.ca to deepen understanding and build fluency of mathematical concepts. Currently, counting, composing/decomposing, representing/comparing/ordering whole- number activities are available.

Doing Math with Your Child

This guide provides hands-on activities so that parents, guardians, caregivers, and other family members can help support the development of mathematics abilities in Kindergarten to Grade 6 learners. It includes tips as well as practical activities that can be used at home and in your local community. Enjoy learning together!

Doing Math with Your Child - variety of languages

Click < here > for instructions on ordering materials from Services Ontario.

Home Connections Activities, Grades 1–6

Activities located on mathies.ca are the activities in the Home Connections section of the *Guides to Effective Instruction in Mathematics,* with additional discussion prompts.

Inspiring Your Child to Learn and Love Math



This toolkit is for parents of children in the elementary grades in Ontario (Junior Kindergarten to Grade 8). The goal of this resource is to provide research-based information to help parents strengthen their own knowledge and understanding, as well as confidently support their child's learning in mathematics. It emphasizes the many ways in which parents play an important role in their child's education.

Encourager votre enfant à apprendre et à aimer les mathématiques.

Partnering with Your Teen, Grades 7–12



This online document provides information for parents, guardians, caregivers, and other family members on ways to support the development of their teen's mathematical abilities.

Figure 4: *Home supports available on the mathies.ca website*

References

Colgan, L. (2008). Hey, it's elementary! RSVP… you're invited to learn about family math and to come to Queen's. *Ontario Mathematics Gazette, 46(*4), 14–16.

Colgan, L. (2015). Hey, it's elementary: Parents as mathematics teaching partners: A toolkit for Ontario families. *Ontario Mathematics Gazette, 54*(2), 14–16.

Colgan, L. (2016). Hey, it's elementary: Choose a path that includes math: Bringing parents and teens together for planning mathematics pathways and pursuits. *Ontario Mathematics Gazette, 55*(2), 11–14.

Colgan, L. (2018). Hey, it's elementary: Share-worthy parent engagement materials for math… no fake news! *Ontario Mathematics Gazette, 56*(4), 37–40.

Colgan, L. (2019). Hey, it's elementary: Building parent engagement in math: A project to support the implementation of the renewed math strategy. *Ontario Mathematics Gazette, 57*(4), 15–18.

Gutman, L., & Eccles, J. (2007). Stage-environment fit during adolescence: Trajectories of family relations and adolescent outcomes. *Developmental Psychology, 43*(2), 522–537.

Irvine, J. (2018). A focus on student understanding: Georg Pólya's problem-solving heuristic. *Ontario Mathematics Gazette, 56*(4), 34–36.

Ontario Ministry of Education. (1999). *Course profiles, grade 9 academic.* Queen's Printer for Ontario.

Ontario Ministry of Education. (2020). *The Ontario curriculum, grades 1–8: Mathematics.* www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics/downloads ▲

▲ A Deep Dive into a Desmos Activity



NICO ROWINSKY
nico.rowinsky@gmail.com

Nico Rowinsky has been passionate about teaching math for over 20 years. With a focus on incorporating technology to enhance the classroom experience for his students, Nico has become an enthusiast of making and using Desmos activities. He is one of only a handful of Desmos Fellows from Ontario and teaches math at the middle school level at the Toronto District School Board.



DR. XIAOHENG KITTY YAN
xh.yan@utoronto.ca

Dr. Xiaoheng Kitty Yan was a former high school mathematics teacher. She is currently a post-doctoral fellow at OISE, University of Toronto. Her research focuses on proof and reasoning at the secondary and university levels. With much interest in digital technology, she also seeks new ways of using technology to support student learning.

As a teacher in the middle of the school year, I think it is fair to say that there is little opportunity to dive into rich academic papers that analyze and theorize pedagogical ideas. (To be honest, I'm lucky if I have matching socks.) Yet, over a casual conversation with Dr. Xiaoheng Kitty Yan about Desmos activities and the notion of intellectual needs, I found myself motivated to dive into a Grade 8 Desmos activity and see how it might measure up with respect to pedagogical theory.

So, if you are interested in Desmos, and would like to see how students respond to their activities, join us on this adventure. Let us start with the notion of *intellectual needs* and an overview of the Desmos activity.

Intellectual Needs

Each classroom has its unique culture, structure, and dynamic. However, one similarity of all classrooms is a wide range of students' needs. While addressing the social, cultural, psychological, and intellectual needs of all students is mathematically and pedagogically challenging, using meaningful and engaging mathematical tasks is an effective way to address a variety of learning styles (Day, 2015; Piggott, 2008) and meet students' intellectual needs in the classroom (Harel, 2008, 2010). *Intellectual need* is defined as a necessary condition for constructing an intended piece of knowledge (Harel, 2013). It often manifests an unsettling stage that learners experience in justifying how and why a particular piece of knowledge came into being. "Math shouldn't feel pointless. Math isn't pointless. It may not have a point in job [y] or [z], but math has a point in math" (Meyer, 2015). This very point in math referred to by Meyer is *intellectual need*.

There are five types of *intellectual needs*: the need for: 1) certainty, 2) causality, 3) computation, 4) communication, and 5) structure. In this paper, we focus on the following three needs:

- *The need for causality* (enlightenment) refers to the need to explain the causes of a phenomenon, to understand what makes a phenomenon the way it is.
- *The need for computation* is the need to represent a situation symbolically and manipulate the representing symbols in order to probe into the referential meanings for the symbols involved in the manipulation.
- *The need for structure* refers to the need to organize knowledge into a systematic, logical structure and the need to make connections.

The Desmos Activity

The Desmos activity, Turtle Time Trials (Desmos, 2021), involves interpreting, comparing, analyzing, and connecting multiple representations of linear relationships, including number lines, equations, graphs, tables of values, and cute animations, all of which fit nicely into the Grade 8 curriculum, and particularly the mathematical processes. Specific curriculum expectations in Strand C – Algebra, Patterns, and Relationships. C1.1, C1.2, and some of C4.

This particular activity consists of ten screens (see Figure 1). Situated in the context of turtle races, the story is set up unconventionally—while one turtle is at the starting line, the other starts almost halfway in (see Figure 2). Students are first asked to tell a story about the first race. On later screens, they analyze and manipulate races, using different mathematical representations. As new races are generated, the story unfolds.

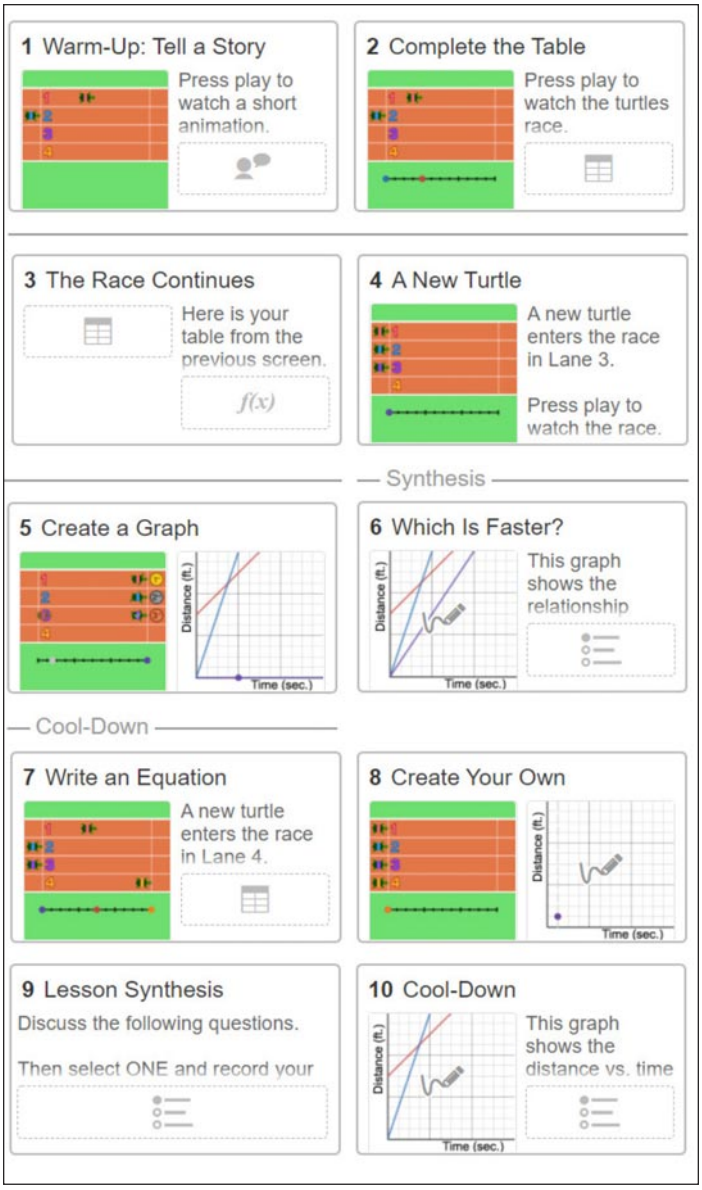


Figure 1: *The Turtle Time Trials activity*

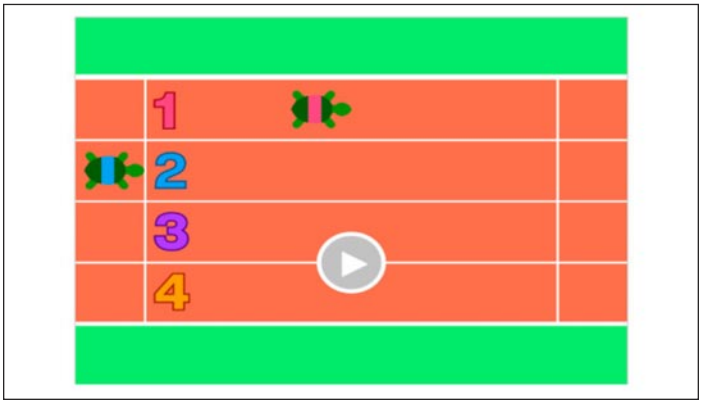


Figure 2: *Tell a story*

Among the nine screens in the Turtle Time Trials activity, we present Grade 8 students' responses in screens 1, 7, and 8. Names of well-known mathematicians, as pseudonyms, were assigned to each student, a unique feature embedded in Desmos activities.

Warm-up with a Story

Let's get into the lesson. It starts with a prompt to watch a short animation (as shown in Figure 1). The animation is of two turtles racing on a racetrack. A slow turtle in lane 1 starts the animation almost halfway down the track, while a faster turtle in lane 2 is at the starting line. Eight seconds later, the race is finished. The prompt? "Tell a story about what you see." As a start to a lesson, this task has pedagogical value, which involves easy entry, student voice, and prior knowledge. In particular, students are asked to use their own readily available language, their own narrative, to create and convey a series of events that take place in 8 seconds.

For example, Figure 3 shows Hewitt's story:

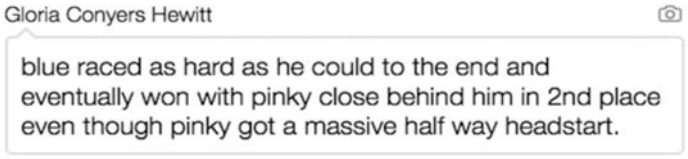


Figure 3: *A story told by Hewitt*

Hewitt gets right to the point, highlighting some of the main events of the 8 seconds. Other students wind a series of events together, crafting backstories and plot lines to explain what they see. For example, Banach (see Figure 4) added a moral touch to the story: cheaters never win. The warm-up allows students to explore with their voice and experiences, and understand a seemingly simple situation that we will eventually sculpt into a deeper understanding of mathematical representations.



Figure 4: *A story told by Banach*

The Need for Computation

On screen 7, students are asked to introduce a fourth turtle to a race. On the left, the race is ready to be run. On the right, each turtle's linear equation awaits analysis and comparison. Figure 5 shows the race at 3.1 seconds, with unit feet per second:

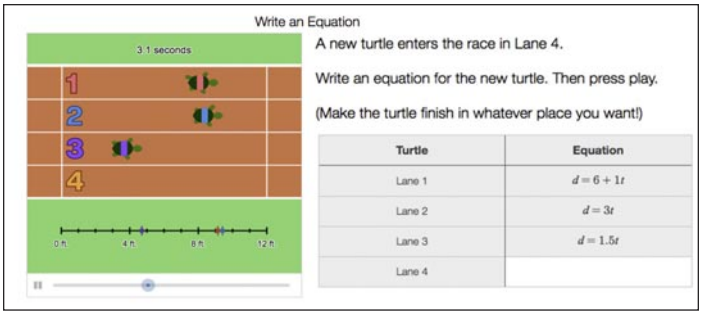


Figure 5: *Introducing a fourth turtle to a race*

Students are asked to write an equation for a new turtle and press play, with a small, but important caveat: "Make the turtle finish in whatever place you want!" No right or wrong answer, just write an equation and see what happens. As shown in Table 1, students indulged in turtles that travel faster than the speed of sound, even light ($d = 1\,000\,000\,000\,000t$), turtles that travel at the speed of π ($d = \pi t$), as well as turtles that travel backwards ($d = -10t$)—they move to the left and vanish off the screen. As students tried one equation after another, they compared and contrasted; they paused; they probed; they were constructing and looking for meaning in their symbols. As a result, the algebra came to life. This *need to compute*—not in the sense of number-crunching—comes from the multiple representations with which they were playing. Specifically, they engaged in representing a situation (the race) symbolically (with the equation), manipulating the representing symbols, and therefore interacting with the representations. Don't be surprised if you see some of these in your class too.

Table 1: Students' Responses to the Table

Turtle	Equation
Lane 4	$d = 3.01t$
Lane 4	$d = -10t$
Lane 4	$d = 800\,000t$
Lane 4	$d = 0.5t$
Lane 4	$d = t$
Lane 4	$d = 1000\ t$
Lane 4	$d = 1000\,000\,000\,000t$

We don't have a favourite here and find each as fascinating as the next (the students will too). Each example provides an opportunity for a class discussion.

The Need for Causality

As a final example of intellectual needs, screen 8 provides perhaps the most delightful mathematical moment in the lesson. Students were prompted with: "Create your own turtle race by sketching the graphs for up to four turtles." Figure 6 shows a race created by Taimina, at the 0.3-second mark. Can you spot where that is on the graph? Taimina has attempted a few exploratory moves here. Firstly, we saw some very steep slopes; Purple, followed by Red and Blue, getting steeper and steeper. And finally, Orange was introduced and had a slope going down. What will happen with the Orange turtle?

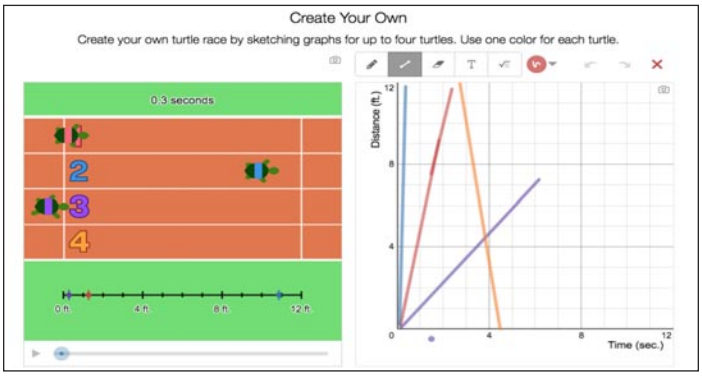


Figure 6: *A race created by Daina Taimina*

The student created this scenario just to see what would happen. In class, the combinations are infinite. Figure 7 shows Turing's work, some broken blue and red lines, some scattered orange dots, and a purple line with a short gap. The still is of the 2.5-second mark. Can you see when the Orange turtle will make a brief appearance? The game of drawing and seeing the results leads each student on their own path.

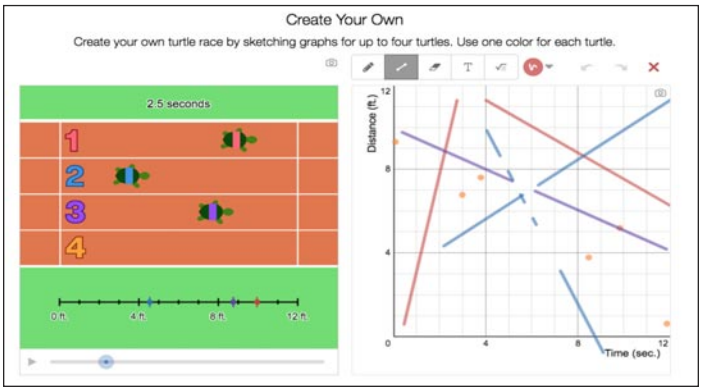


Figure 7: *A race created by Alan Turing*

In Figure 8, Rice took a more artistic approach to the graph. What will happen if I draw a turtle as my graph? Secondary teachers might see that the graph does not pass the vertical line test. Awesome! Other teachers might question, "Why exactly are there three turtles in lane 1?" Why not? This provides an opportunity for students to tackle how the graph leads to this phenomenon happening and triggers a desire to explain it, or for it to be explained.

The students were using the canvas as a playground to see what would happen. Several students asked why there were so many turtles in lanes 1 and 3. This desire to explain such phenomena is reminiscent of the headache to which Meyer (2015) was referring; in Harel's terms, the need for

causality (enlightenment).

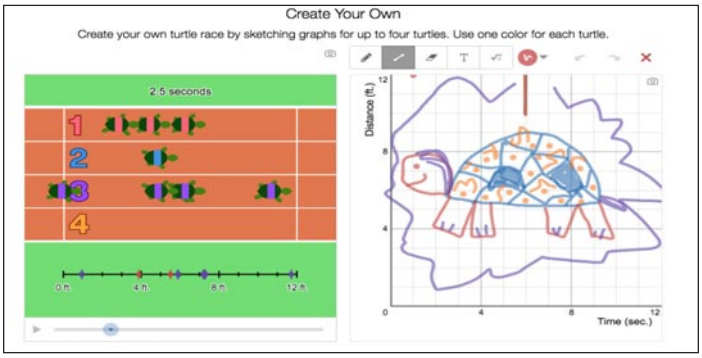


Figure 8: *A race created by Marjorie Rice*

As is common in Desmos activities, some Teacher Moves are suggested on this screen that capture the spirit of this task:

> This screen is intended for students to experiment and play with the turtle race context. Invite students to try various conventional and unconventional scenarios, using proportional and/or non-proportional relationships. We intend for this to be a creative experience for students. We encourage you to emphasize those virtues whenever you see them in your class.

The Need for Structure

Throughout the activity, the students have been exposed to and engaged in different representations that allow them to build and organize their knowledge through connections. As students are playing, drawing, and exploring, the fundamental question arises: "How can you tell which turtle is fastest?" The lesson synthesis of Turtle Time Trials[1] offers the students a choice of Table, Graph, or Equation (see Figure 9). They are now given choices to elaborate on the representation that works best for them.

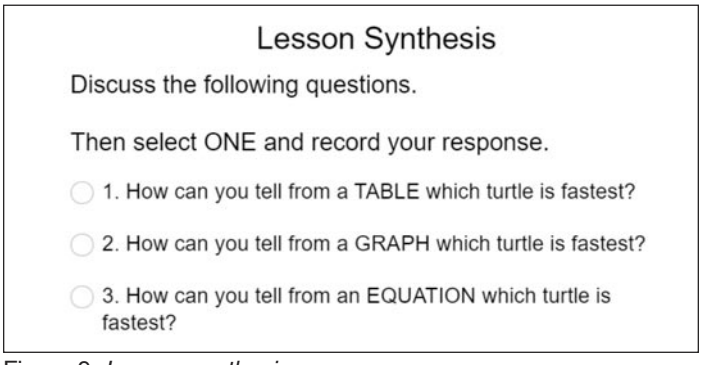


Figure 9: *Lesson synthesis*

This screen may not address the need for structure in a technical sense, but provides student agency to make connections and verbalize the underlying mathematical structure. It works as a culminating task of the activity, and also allows the teacher to showcase how these different representations highlight information in different ways. Since the connections among a Table, a Graph, and an Equation are a key component in understanding proportional and non-proportional relations, letting students choose their preferred method of representation provides an opportunity for divergent choices on converging ideas. Your "teacher moves" on this screen will come in handy. In what follows, we exemplify the synthesis of the lesson by the students.

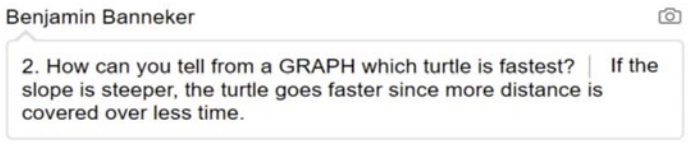


Figure 10: *A synthesis by Benjamin Banneker*



Figure 11: *A synthesis by Al-Khwarizmi*

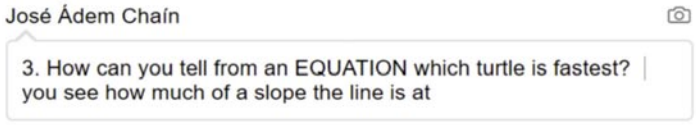


Figure 12: *A synthesis by José Adem Chain*

In the above examples (Figures 10–12), all students interpreted the same scenario, but made specific decisions on how to communicate their understanding. They drew on different representations to describe the concept of "fastest" in the turtle scenario. Benjamin's attention went to the slope in the GRAPH, Al-Khwarizmi mentioned feet per second in a TABLE, while José favoured the EQUATION to identify where the slope resides. Each choice of representation shows their understanding of the concept, while referring to the same structure.

Final Remarks

As teachers, we all seek effective and sensible ways of teaching (in-person as well as remote), and we find comfort in incorporating Desmos activities into our daily teaching routines to support learning that is inquiry based, technology enhanced, and playful at heart. These activities meet the curricular expectations and place emphasis on meeting students' intellectual needs for computation, causality, and structure. In our modest attempt to leverage the power of technology and interactive digital learning, we hope that the use of Desmos activities can enhance the mathematical experience of both students and teachers.

Continued on page 36

▲ MB4T (Mathematics by and for Teachers): Number Patterns



JENNIFER HOLM
jholm@wlu.ca

Jennifer Holm is an Assistant Professor at Wilfrid Laurier University and works with Primary/Junior and Junior/Intermediate pre-service teachers, as well as in the field supporting current mathematics teachers. She is interested in developing mathematics knowledge for teaching with both pre-service and in-service teachers. She focuses on the beliefs and opinions that pre-service teachers hold about mathematics and teaching and the connection they have to past experiences. She uses this research to support future teachers in developing beliefs and knowledge that will encourage and support effective teaching practices in mathematics.

One of the most wonderful aspects of mathematics for me is number patterns. A lot of mathematics can be represented and discussed in this way. Even simple number patterns provide an opportunity beyond asking students to determine the next few terms. Consider having students predict how to figure out a later stage in the pattern, without completing each step of the pattern. In a table of values, this helps develop an understanding of algebra. One easily defined pattern to start with is what happens when adding sequential odd numbers together (i.e., 1, 1 + 3, 1 + 3 + 5,…).

Odd Numbers	Total
1	1
1+3	4
1+3+5	9
1+3+5+7	16

Figure 1: *Table of values for consecutive odd numbers*

Although a straightforward way to figure out the pattern is to use a table of values or T-chart like in Figure 1, with students, this is not the best way to begin. Filling in the table of values one step after another encourages a recursive way of determining the pattern, since it implies a need to figure out the next stage by determining the one before it. What happens if we ask students to predict what they would get if they added the first ten consecutive odd numbers together? Although they could just add the numbers together or continue the table of values, providing square tiles or linking cubes presents a concrete and much more interesting way to visualize the pattern.

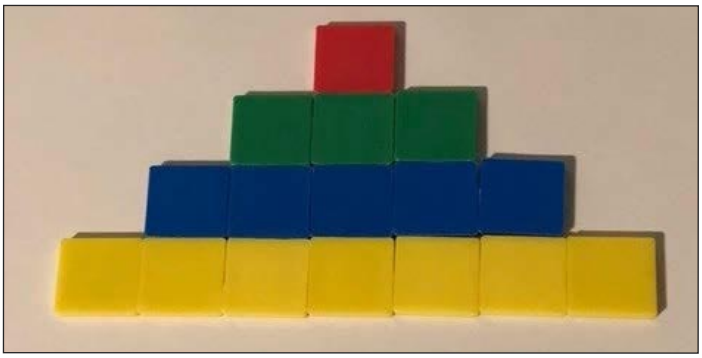


Figure 2: *Image of first four consecutive odd numbers*

In this instance, linking cubes are most helpful with students, since each odd number could be created as a single linked piece. This article uses square tiles to make the images clearer (see Figure 2 for the first four odd numbers). Building the numbers with concrete materials gives time for discussions about how you could continue to build the "stairs" and the interpretation that there is always a square added to each side of each new row. Discussion can be directed to questions about how many in the future rows, and how one knows.

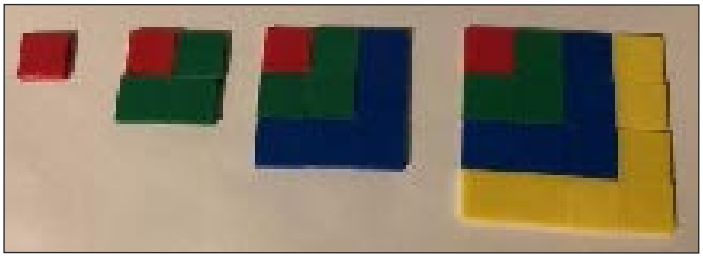


Figure 3: *Image showing what play discovered about adding consecutive odd numbers together*

Having students "play" with the different odd numbers being combined, and what happens when you add the next one, leads to an interesting observation: you can create squares (see Figure 3). As shown, the first stage has one square. Adding 1 + 3 gives you four, or a two-by-two square. Adding 1 + 3 + 5 gives you nine, or a three-by-three square. This pattern continues as you continue adding consecutive odd numbers together. Once you have determined this part of the pattern, then you can ask how this can be used to determine what happens when you add the first ten consecutive odd numbers. What about the first twenty? Looking at the image and the pattern facilitates recognition that the side length of the square is equal to the number of odd numbers added together, or the stage in the picture (i.e., term number). For example, the first four odd numbers added together gives 1 + 3 + 5 + 7 = 42 = 16. Using this logic, the first ten would be 102 = 100.

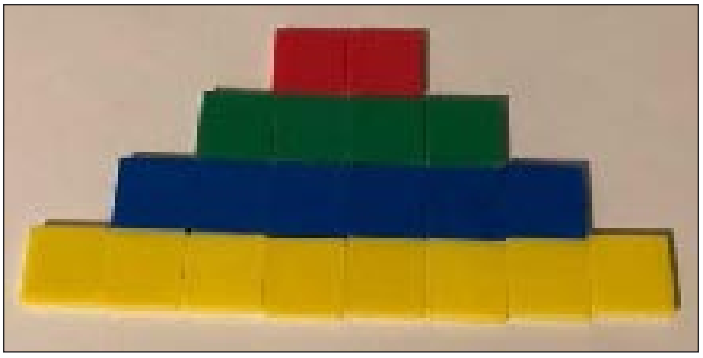


Figure *4: Image of consecutive even numbers*

I found that when working with teacher candidates, they were surprised that adding the consecutive odd numbers created perfect squares, since creating something "even" with odd numbers seemed incongruous. Posing the question "How could you predict what would happen if you added consecutive even numbers?" provides a way to compare and contrast the two problems. Once again, the use of linking cubes or square tiles provides a way to manipulate the numbers to determine how to build the pattern and figure out what the value would be. Figure 4 shows the first four consecutive even numbers. Using similar logic as the first problem with odd numbers, how could you determine the sum for the even numbers?

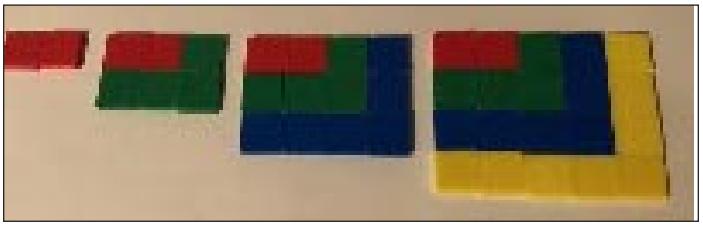


Figure 5: *Image of combining consecutive even numbers*

Play makes it clear that this time, you are not making square numbers, but instead, making rectangles (see Figure 5). Determining the side lengths of the rectangles is one way to determine the pattern so that one can figure out the tenth stage. In examining the rectangles, the height is the number of even numbers added together (i.e., the term number), and the base is one more than that value. For example, adding the first four even numbers corresponds to a rectangle that is $4 \times 5 = 20$.

I love asking why this happens. The fascinating part of number patterns presents itself when trying to figure out the "why?" If we look at Figures 3 and 5, what can we observe about the odd numbers making squares and the even numbers making rectangles? Before continuing to read, how might you answer the question about why this happens? What might your students say about it?

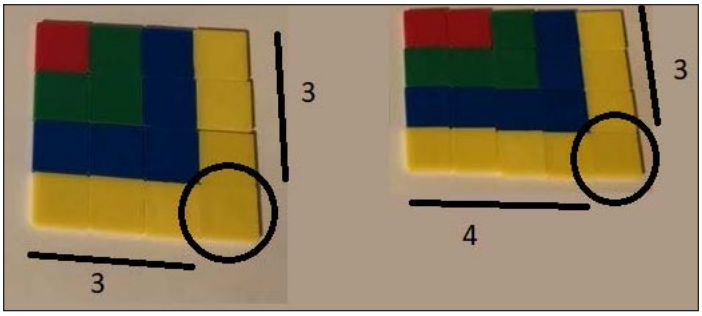


Figure 6: *Image showing adding first four odd numbers on the left and first four even numbers on the right*

The nature of the number of odd numbers or even numbers makes it clear why one makes squares and the other does not. To make the rectangular shape, you need a square in the corner (see Figure 6). When you add an odd number such as 7 in the yellow, you have an equal number on each side (i.e., $7 = 1 + 2 \times 3$) when you fold it over the existing pattern. This would be the same for all odd numbers, given that the initial one is a square (i.e., $n = 1 + 2 \times k$). With even numbers, you do not have the same pattern. The initial two squares will always form a rectangle, and when you fold the next even number around it, you would still be unable to create a square (i.e., $8 = 1 + (3 + 4)$). This would mean that with odd numbers, you could create a square, but with even numbers, the numbers added will not form the same pattern. Regardless of what you consider of why this happens when making the rectangular shapes, the pattern presents a place for conversations about mathematics.

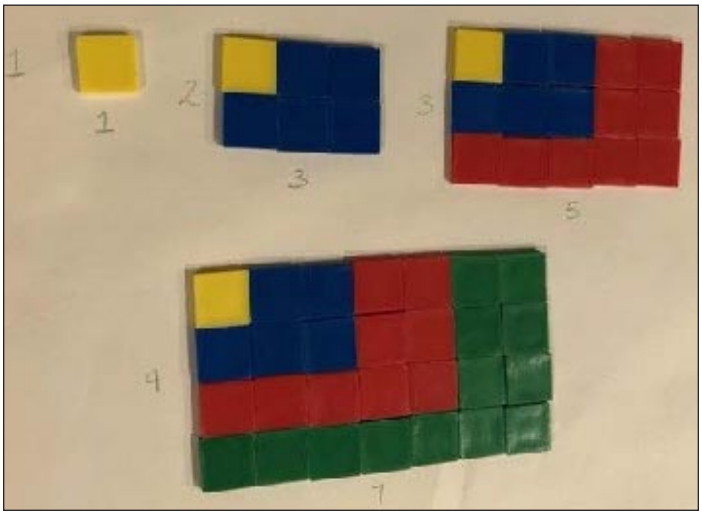


Figure 7: *Adding skip-counted odd numbers*

From here, the conversations with students can expand to considering their ideas. What other patterns hold interesting observations that can be made when we use manipulatives to create the numbers? What happens if we add odd numbers, but we skip-count among them and do not add them consecutively? For example, 1 + 5 + 9 + 13?

Is there a predictable pattern we can create here? If you try to create rectangles with the initial stages, another interesting pattern emerges. Figure 7 shows that the height is the number of odd numbers you have added together, and the base shows consecutive odd numbers! The height is the stage or term number (n), and the base is two times the base number minus one ($2n - 1$). Adding the first four, for example, the height would be four and the base is seven, so 1+ 5 + 9 + 13 equal to 28.

There are many other patterns that students can explore to see what they can learn. Not all patterns are going to give neat squares or rectangles, but encouraging students to discuss what they see and learn can be a fascinating exploration in mathematics. Other questions could include skip-counting by 7. Since this is a difficult multiplication sequence for students to memorize, is there a way to develop a pattern that helps to predict future numbers? What about predicting future sequences of the Fibonacci sequence? What about rows in Pascal's triangle? So many number patterns are available for exploration and learning! ▲

Continued from page 33

Note:

[1] This lesson synthesis appears in the Desmos curriculum version of Turtle Time Trials, and may differ from the free version available at desmos.com.

References

Day, L. (2015). Mathematically rich, investigative tasks for teaching algebra. *The Mathematics Teacher, 108*(7), 512–518. doi.org/10.5951/mathteacher.108.7.0512

Desmos. (2021). *Turtle time trials.* www.teacher.desmos.com/activitybuilder/custom/5da9e2174769ea65a6413c93?collections=featured-collections%2C5e72d58a20ae4e061b73b546

Harel, G. (2008). What is mathematics? A pedagogical answer to a philosophical question. In R.B. Gold & R.A. Simons (Eds.), *Proof and other dilemmas: Mathematics and philosophy* (pp. 265–290). Mathematical Association of America.

Harel, G. (2010). DNR-based instruction in mathematics as a conceptual framework. In B. Sriraman & L. English (Eds.), *Theories of mathematics education* (pp. 343–367). Springer.

Harel, G. (2013). Intellectual need. In K.R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 119–151). Springer.

Meyer, D. (2015, June 17). If math is the aspirin, then how do you create the headache? https://blog.mrmeyer.com/2015/if-math-is-the-aspirin-then-how-do-you-create-the-headache/

Piggott, J. (2008). Rich tasks and contexts. *NRICH*. www.nrich.maths.org/5662 ▲

▲ Solving Linear Equations: Metacognition and the Birthday Present Principle



DR. JEFF IRVINE
dr.jeffrey.irvine@bell.net

Jeff Irvine, PhD, OCT, has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

Solving linear equations is an important expectation in our math curriculum from Grade 1 through Grade 10. In Grade 1, a specific expectation from the Number strand is shown in Figure 1. The expectations begin with solving equations involving addition and subtraction, with recognition that equations are one of the possible representations of a situation, and numbers limited to less than 50. They then progress through equations involving larger numbers, and by Grade 4, the concept of verification is introduced as part of the expectations (Figure 2). By Grade 7, students are expected to solve equations involving integers (Figure 3), as well as equations involving decimals and multiple terms (Figure 4). By the Grade 9 Destreamed course, students are asked to create, solve, and verify equations in various contexts (Figure 5). This expectation is much less detailed than in the previous curriculum.

Missing from the new curriculum is any mention of the balance method for solving equations, which is fundamental to the concept of equality (Figure 6).

B2.4 use objects, diagrams, and equations to represent describe, and solve situations involving addition and subtraction of whole numbers that add up to no more than 50

Figure 1: *Grade 1 expectation, Number strand (2020)*

C2.2 solve equations that involve whole numbers up to 50 in various contexts, and verify solutions

Figure 2: *Grade 4 expectation, Algebra strand (2020)*

B2.4 use objects, diagrams, and equations to represent, describe, and solve situations involving addition and subtraction of integers

Figure 3: *Grade 7 expectation, Number strand (2020)*

C2.3 solve equations that involve multiple terms, whole numbers, and decimal numbers in various contexts, and verify solutions

Figure 4: *Grade 7 expectation, Algebra strand (2020)*

C1.5 create and solve equations for various contexts, and verify their solutions

Figure 5: *Grade 9 Destreamed expectation (2021)*

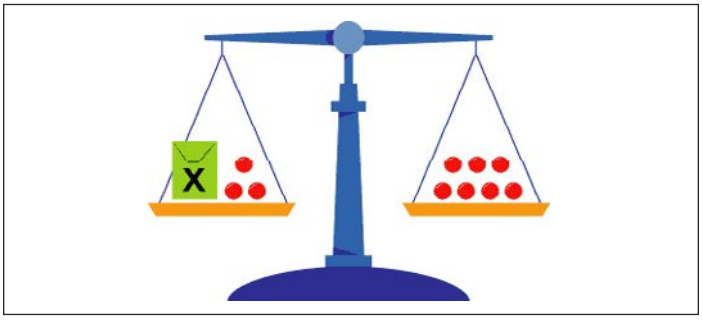
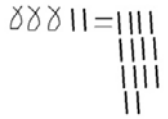


Figure 6: *The balance analogy*

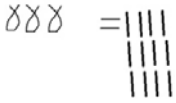
Since many students in middle school consolidate understanding through the use of concrete materials, teachers often provide students with manipulatives to develop equation-solving concepts. The *Abacus* contains several excellent articles on how this can be done; for example, the "Links to Manipulatives" columns in 57(2), 2018 or 54(1) and 54(2), both 2015. This current article supports the transition for our students from concrete materials to more abstract symbolic manipulations. It contains ideas that I have found, invented, or used during my career in the classroom.

Bags and Sticks

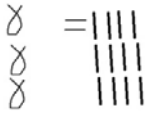
W.W. Sawyer, former professor emeritus at the University of Toronto, has published some marvellous ideas about helping students make the concrete-to-abstract transition. In his bestselling book, *Mathematician's Delight* (1943), Dr. Sawyer talks about "bags" (the unknown) and "sticks" (ones) in introducing equations. Figure 7 gives an example. The bags contain the unknown, hidden from view.



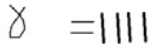
The = is read as "balances". So, 3 bags plus 2 sticks balances 14 sticks.



Remove 2 sticks from each side of the balance. So 3 bags balance 12 sticks.



Group the sticks on the right side into 3 equal piles.



So, 1 bag balances 4 sticks.

Figure 7: *Bags-and-sticks equation solution*



Dr. Sawyer goes on to point out that if you cut off the top of the bag, you get an "*x*," which is what most mathematicians use to represent the unknown.

As teachers, we sometimes suffer from "know too much" syndrome. For example, many of us have solved literally thousands of linear equations during our careers as students and teachers. So, when faced with an equation such as $3x + 7 = 22$, our solution might look like $3x + 7 = 22$, $3x = 15$, or even $3x + 7 = 22$, $x = 5$. However, when our students are first learning to solve equations, it becomes a somewhat more formidable problem, since they must make decisions about what to do, then do it (hopefully without mechanical issues), then decide on the next step, do it, and so on.

These decisions are part of student metacognition. Metacognition is what students need to do before actually solving a problem: goal setting, planning, strategy selection, evaluating alternatives; as well as during problem solving: monitoring clarity, monitoring accuracy, debugging; and after problem solving: evaluating the reasonableness of results, considering implications of a solution, performing "What if?" analysis (Marzano & Kendall, 2007; Meijer, Veenman, & van Hout-Wolters, 2006). This article focuses on strategies to assist students in developing metacognitive knowledge while learning about equation solving. It is important that students be allowed to develop their metacognitive knowledge through trial and error prior to the suggested strategies in this article. Once students have begun building their metacognitive strategies around equation solving, the teacher can provide strategies such as the ones in this article to help consolidate student metacognition and support those students who may need additional scaffolding in their metacognitive thinking. The balance analogy is fundamental to methods for equation solving, and is used extensively in

these strategies. It also suggests some strategies that address different student learning modalities: auditory, visual, and kinesthetic–tactile. All the strategies are classroom tested over a long number of years.

Think-Aloud

As teachers, we sometimes do not voice the steps that we go through to solve an equation. For students, this can be confusing. Since we know so much about the topic, we may skip the "thinking," "decision making," or "what's next" steps, which are vital to our students' understanding. Think-aloud is an instructional strategy that asks us to speak our "inner voice" aloud, as an aid to student understanding. Here is an example of a think-aloud when solving the equation $9x - 3 = 24$.

We want to know the value of $1x$.

So we need to isolate x on one side of the equals sign.

To get rid of the subtract 3, we can add +3 to both sides of the equation.

That gets us to $9x = 27$.

Now we need to eliminate the 9.

Since the 9 is multiplying x, we can eliminate it by dividing both sides of the equation by 9.

That gets us $x = 3$.

We're done!

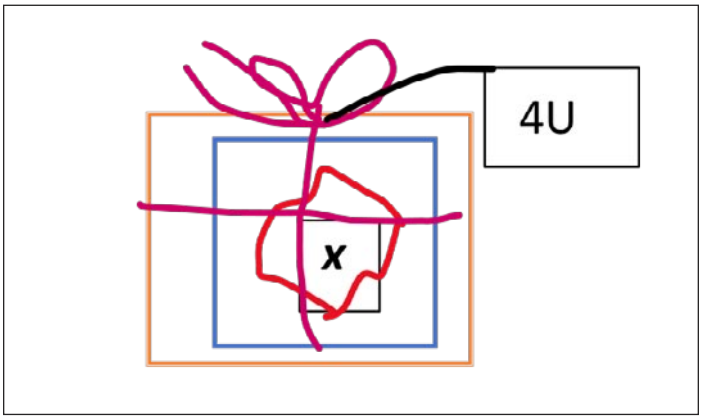
We should check to make sure that we're correct, by substituting into the original equation to make sure that it balances.

$9(3) - 3 = 27 - 3 = 24$

Think-aloud is particularly useful when students first encounter something new in their equations, such as variables on both sides, brackets, or fractions. As students gain facility with the new situation, think-aloud can move from the teacher to the student, having students voice their thinking and solution steps to a partner, using the Timed Retell[1] strategy. I have found that using think-aloud reduces student confusion and builds confidence. This strategy especially helps students whose primary learning modality is auditory, and when combined with visual elements illustrating the steps, it is also useful for visual learners.

The Birthday Present Principle

This is a fun analogy to help students make decisions about the order in which to "undo" steps in an equation. I have found that it helps students who are beginning to transition from concrete to abstract thinking. It is useful for all types of learners.



I'm going to give you a birthday present.

Since I'm a math teacher, my present to you will be an x.

First, I'm going to wrap the x in some nice tissue paper.

Then I'm going to put it into a gift box.

Then I'll wrap the box in some pretty paper.

I'll put some ribbon around the box.

I'll put a bow on the top.

Finally, I'll attach a card wishing you Happy Birthday.

Now, when I give you the present, what's the first thing you're going to do?

You would remove the card.

Then remove the bow.

Then take off the ribbon.

Take off the wrapping paper.

Open the box.

Remove the tissue paper.

Happy Birthday!

So, the moral of the story is that to unwrap the present, the first thing that you would undo is the last thing that I did. When combined with the mnemonic BEDMAS, it gives us an "unwrapping" strategy for solving equations (Figure 8). While this may seem algorithmic, thinking in this manner supports students in making metacognitive decisions on how to proceed.

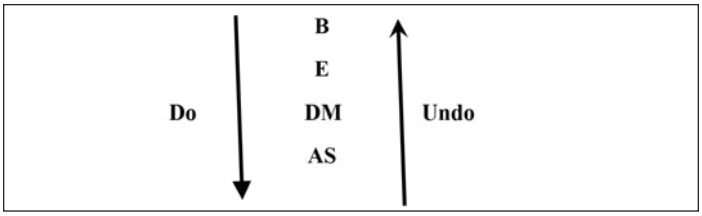


Figure 8: *Do (BEDMAS) and Undo*

For example, when the teacher built the equation $4x - 9 = 15$, the "do" steps would have been "multiply by 4 and then subtract 9." So, to solve this equation, the student "undo" steps would be "add 9 to both sides, then divide both sides by 4," resulting in $x = 6$.

When beginning to solve equations, or when additional complexity is introduced, students were encouraged to identify the sequence of steps that they would take, before completing the solution (Figure 9).

$$\begin{array}{c} \textit{2nd} \qquad \textit{1st} \\ -3(2x + 4) - 7 = 11 \\ \textit{4th} \quad \textit{3rd} \\ -3(2x + 4) = 18 \\ 2x + 4 = -6 \\ 2x = -10 \\ x = -5 \end{array}$$

Figure 9: *Identifying sequence of steps prior to solving*

As students become more proficient with abstract thinking, the explicit identification of the sequencing will become unnecessary. However, in the beginning, these sequencing steps will reduce the need for "what's next" thinking.

Broken Equations and How to Mend Them

I call equations "broken" when they contain uncollected like terms, i.e., like terms that have been broken into two or more pieces, either on one side or both sides of the equals sign. When teaching equation solving, my class identifies three classes of "broken" equations:

1. Uncollected like terms on one or both sides of the equals sign
2. Variables on both sides of the equals sign
3. Brackets on one or both sides of the equals sign

Solution Strategies developed by students:

For situation #1, collect like terms on each side.

For situation #2, use the "undo" operations to collect all variables on one side of the equals sign.

For situation #3, use the distributive property to remove the brackets, and then proceed as in situations #1 and #2. These strategies and use of the term "broken" support all three learning modalities.

For the equation in Figure 9, we did not remove the brackets before proceeding. However, as equations become more complex, removing the brackets often reduces the number of steps needed to solve the equation. Of course, many of the techniques described in this article are somewhat algorithmic. There are multiple ways to approach equation solving. However, I have found value in using the analogies, such as the birthday present principle, as providing students with a viable and visible method for approaching complex equations.

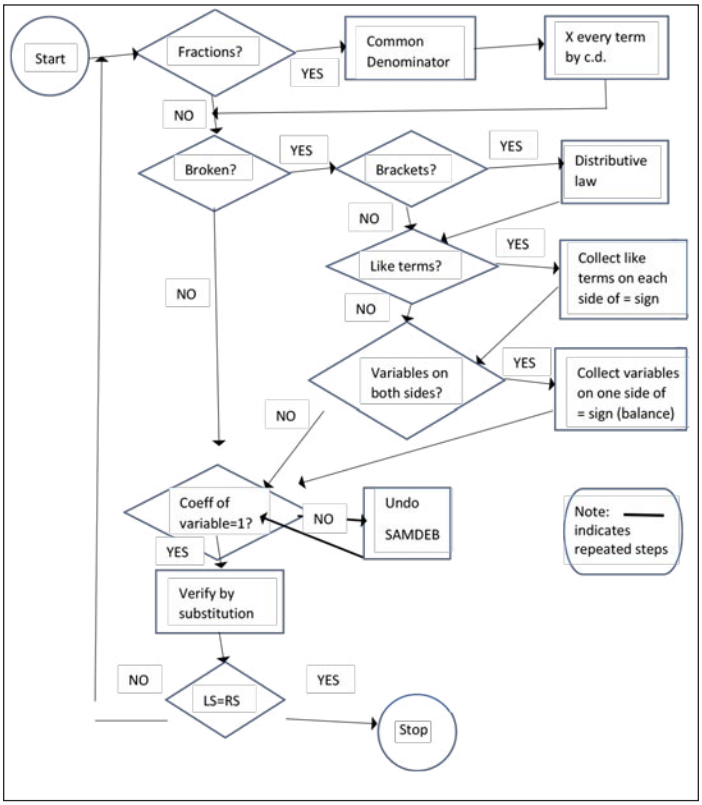


Figure 10: *Sample flow chart for Grade 9 Academic or Grade 10 Applied*

Putting It All Together Metacognitively

Once students have been exposed to all the varieties of equations necessary for the grade-level expectations, two strategies were employed to consolidate students' metacognitive knowledge of equation solving. In the first activity, students work in groups to sequence the steps required to solve a sample equation. Each group was given an envelope containing information on two colours of card stock, cut into strips. On the first colour were questions such as, "Are there fractions in the equation?"; "Is the equation 'broken'?"; "Is the coefficient of the variable =1?" On the other coloured strips were suggested operations required to reduce the equations' complexity, such as, "Find a common denominator"; "Multiply every term in the equation by the

common denominator"; "Collect like terms on both sides of the ="; "Use the distributive property to remove brackets"; "Undo operations, using SAMDEB." Each group's task was to sequence the questions and operations to build a logical plan for solving the equation. This activity addresses all three learning modalities: auditory through group discussion, visual through the visual elements on the card-stock; and kinesthetic–tactile through the physical sequencing of the card-stock strips.

The second activity involved a whole-class co-construction of a flow chart for solving equations. For this activity, the various flow-chart elements were written on large chart-paper pieces, and the class participated in sequencing the elements by sticking them to a whiteboard. The completed flow chart then remained available on a classroom wall for students to consult as needed. Figure 10 illustrates a possible version of the flow chart for Grade 9 Destreamed or Grade 10 Applied. This activity also supports all three learning modalities. Students were encouraged to consider alternative routes for equation solving, and these alternatives could be added to the flow chart for other students to consider.

By actively addressing student metacognitive needs, these activities reduced confusion, increased confidence, and improved solution accuracy in solving linear equations. Similar activities were employed in Grade 10 for solution of quadratic equations and for solving systems of equations; in Grade 11, for solution of trigonometric equations; and in Grade 12, for solution of exponential and logarithmic equations. The time required to implement these strategies was minimal, but the gains in student understanding were substantial.

Note [1] In Timed Retell, students work in pairs. The first student outlines their thinking in solving the equation. The second student then retells the thinking of the first student. The activity can be timed.

References

Kestell, M.L., & Kubota-Zirivnij, K. (2015). Links to manipulatives. *Abacus, 54*(1), 3–5.

Kestell, M.L., & Kubota-Zirivnij, K. (2015). Links to manipulatives. *Abacus, 54*(2), 3–5.

Kestell, M.L., & Kubota-Zirivnij, K. (2018). Links to manipulatives. *Abacus, 57*(2), 3–5.

Marzano, R., & Kendall, J. (2007). *The new taxonomy of educational objectives* (2nd ed.). Corwin Press.

Meijer, J., Veenman, V., & Van Hout-Wolters, B. (2006). Metacognitive activities in text-studying and problem-solving development of a taxonomy. *Educational Research and Evaluation, 12*(3), 209–237. doi.org/10.1080/13803610500479991

Ontario Ministry of Education. (2020). *The Ontario curriculum, grades 1–8: Mathematics.* www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics/downloads

Sawyer, W.W. (1943). *Mathematician's delight.* Penguin Books. ▲

▲ A Lesson in Culturally Relevant Curriculum: Focusing on Locally Developed Mathematics



TAYLOR MURIE
tamurie@lakeheadu.ca

Taylor is a recent graduate of the Master of Education for Change program at Lakehead University, with specializations in Indigenous Education and Social Justice Education. Her thesis was titled Re(righting) Mathematics Education: Changing Students' Attitudes Towards Mathematics Through Indigenous Knowledge and Worldview. *Most recently, she has been teaching Introductory Mathematics at the College level, and Curriculum and Instruction in Mathematics at the University level.*



ANN KAJANDER
akajande@lakeheadu.ca

Ann is an experienced high school mathematics teacher, who currently works with pre-service teachers at the elementary and secondary levels. Her research interests include mathematical knowledge for teaching, teacher education, and students at risk. Her most recent book, together with co-editors Jennifer Holm and Egan Chernoff, is Teaching and Learning Secondary School Mathematics: Canadian Perspectives in an International Context.

Locally developed mathematics classes can present special challenges for teachers, and our region of Northwestern Ontario is no exception (LDCC Project, 2005; Sedor, 2019). This article describes the design and field test of some of the materials developed during one phase of a recent study of Grades 9 and 10 locally developed mathematics classrooms in our region (Kajander et al., 2018). The concept of combining Indigenous Knowledge-(IK) focused and reform-based mathematics practices appeared to be a successful approach for many disengaged students in the field-test classroom (Murie, 2019). The driving force behind the creation of these resources was to engage chronically disengaged students, as well as providing a hands-on, accessible mathematics lesson that was culturally relevant to students. Although each locally

developed classroom is unique in its composition, location, and structure, we hope that sharing this lesson along with the noted successes will be of value to other educators. This article focuses on the resource-development aspect of a three-year study of Grades 9 and 10 locally developed mathematics classes (Kajander et al., 2018), provides a summary of the context and background, and shares a field-tested lesson based on this work.

Context and Goals

In our region of Northwestern Ontario, students may travel up to 800 km to come from remote regions, to live in a larger urban area to attend high school (Murie, 2019). In our region, many of these students disproportionally end up in locally developed mathematics classrooms (Kajander et al., 2018). Teachers in our region find that these classes have the widest range of student needs of any mathematics classrooms, and report a lack of suitable resources (Sedor, 2019). These challenges motivated our work.

IK-focused mathematics has become a topic of interest in the past few years; however, teachers reported that there is a lack of practical resources available for high school mathematics (Kajander et al., 2018). The goal of this particular aspect of the larger study was to explore how students might respond to an IK-highlighted lesson.

A problem-based (three-part) mathematics lesson with IK-highlighted content was developed and implemented in a Grade 9 locally developed mathematics classroom. The classroom where the lesson was presented was similar in its makeup to previously observed locally developed classes that were part of the larger project. The demographics of this class were also consistent with the literature on locally developed mathematics education. For example, there was one newcomer in the class, who spoke very little English, while there was a disproportionately higher number of Indigenous students compared to the Applied and Academic math courses, as may be typical (Galabuzi, 2014; Livingstone, 2014). What this means is that a larger portion of the locally developed classes identify as Indigenous in comparison to Academic or Applied classes, where a smaller proportion of the class would identify as Indigenous (Galabuzi, 2014; Livingstone, 2014). Research indicates that this may be for various factors, and in Northwestern Ontario, these factors include gaps in education due to lack of qualified teachers on reserve schools, intergenerational trauma from residential schools, less funding per student for federally funded schools, a history of various forms of abuse in families, etc. (Government of Canada; Indigenous and Northern Affairs Canada, 2017). Note that exact numbers of self-identified Indigenous students were unable to be obtained due to ethics; however, about 50 percent of the class used the Indigenous workroom frequently. The use of the workroom was interpreted as an informal sign of students' self-identification of Indigenous heritage.

We were interested to observe if there were any changes in student attitudes toward mathematics when this style of an IK-focused lesson was implemented. During the field test, informal comparisons were made to student responses in other locally developed mathematics classrooms, as had been observed over the previous two years, involving four iterations of the course with three different teachers (Kajander et al., 2018). While we recognize the situated nature of both the lesson itself and the classroom observations, it is hoped that the shared lesson and initial field-test data might serve as an example for adaptation to other regions and classrooms.

Background to Lesson

Students enrolled in Ontario locally developed-level mathematics have not been studied to any extent regarding standardized student achievement measures or attitudes. The potential benefit of locally developed classes is that teachers can tailor the courses to student needs; however, there is a concurrent problem with this openness, in that it also means that success in the course cannot be monitored at a provincial level. Given the flexibility of these courses, these students are not included in the provincial government's Education Quality and Accountability Office (EQAO) Grade 9 testing (EQAO, 2014). One of the goals of this testing is to "inform the improvement of instructional programs and strategies to help all children succeed" (EQAO, 2014, p. 3). Language such as this can be confusing, as in fact, students in the locally developed stream are not assessed by EQAO (Macaulay, 2015). This may also mean that these courses fall by the wayside in terms of board support or funding.

Streaming has also been found to increase the achievement gaps between groups of students, and may lead to inequitable outcomes for students in the lower streams (Applied and locally developed) (Macaulay, 2015). Indigenous students are also overrepresented in streams such as Applied and locally developed mathematics, and thus this streaming may extend the disadvantage gap (Hamlin & Cameron, 2015; King et al., 2009; Livingstone, 2014). Indeed, Ontario will be following a destreamed Grade 9 mathematics curriculum to replace the Academic and Applied courses, to begin September 2021. At the moment, however, the locally developed courses, offered for students whose needs "cannot be met" (LDCC Project, 2005, p. 2) by the other policy documents, are to remain.

In more remote regions, student needs may have unique aspects (Murie, 2019), with *decolonization* being one of them. In education, decolonizing is the process of recognizing colonialism, and making efforts to diminish and extinguish its power (Aikenhead, 2017, p. 78), while *indigenizing* is defined as incorporating IK, world views, and teachings in the classroom (Battiste & Henderson, 2009). Both decolonized and IK-highlighted content have been systematically excluded from schools and curriculum, eclipsed by Eurocentric knowledge systems (Battiste & Henderson, 2009; Godlewska et al., 2010). Decolonizing and incorporating IK-highlighted mathematics are considered processes to move toward reconciliation in relation with Indigenous peoples/community through education. Systemically, it may be difficult for non-Indigenous teachers to implement IK-focused teaching without guidance from local Indigenous educators, knowledge holders, and community when mainstream non-Indigenous schooling has been and continues to be settler focused and Eurocentric (Battiste, 2013; Belczewski, 2009; Munroe et al., 2013; Stavrou & Miller, 2017). All students, including both Indigenous and non-Indigenous, can benefit from decolonial and IK-highlighted pedagogies in STEM classrooms, including math (Aikenhead, 2017; Borden, 2013).

Reformed (problem-based) mathematics pedagogies that pay attention to place/land, student inquiry, and collaborative learning have the potential to better align with Indigenous world views (Russell & Chernoff, 2013). Traditionalists in mathematics endorse practices that include rote memorization and teaching by telling, whereas the reformists believe that mathematics pedagogies should be based on student-driven investigation or inquiry (Russell & Chernoff, 2013). Therefore, this lesson used a decolonial lens and IK-highlighted content, and was designed as a problem-based lesson.

Designing and Implementing the Lesson

Following extensive observations of locally developed classes over the previous two years, a new two-day lesson was designed and taught based on decolonizing principles, as described above (Kajander et al., 2018). A first-person perspective from the local First Nations community was used to bring forth authentic Indigenous voices relevant to the local geography.

To assess the effectiveness of the lesson, we interviewed the regular classroom teacher and support person, as well as one student participant. Unfortunately, only one student provided signed ethics forms, so only this student was interviewed. Students enrolled in locally developed-level classes often have external factors that impact their education, and many do not attend on a regular basis for many other issues than just academic (Kajander et al., 2018). This can make it difficult to retrieve signed ethics forms from students. The classroom teacher and student support person were interviewed for approximately 30 minutes each after the lesson took place. Before the implementation of the lesson in the locally developed classroom, one student volunteer, whom we will refer to as Harry, was interviewed about his experiences in mathematics, his favourite activities, and what he liked or did not like about his current and past mathematics classes. We also recorded classroom observations of the students, and made notes on their level of engagement, whether or not they seemed excited to do the work assigned by the teacher, and what attitudes they displayed toward mathematics. Harry was interviewed again after the lesson to determine his thoughts and clarify our field notes about engagement and student attitudes.

Lesson Field Test

It is critical to note that the first author, who taught the lesson, was a qualified high school teacher, who had been engaging in efforts to decolonize herself prior to teaching this lesson, and who attempted to deliver the lesson in a way that was grounded in Indigenous world views. First-person Indigenous voices local to our region were included in the lesson, and the students were welcomed to contribute their own knowledge throughout the two-day lesson. While the content of the lesson was important, the delivery and acknowledgement of the teacher's whiteness was equally important. She was known to the students, having spent approximately two months in the classroom prior to the lesson field test.

The teacher also positioned herself within the class at the beginning of the lesson. She identified herself as White, but explained that it was important to bring more First Nations perspectives and practices into the classroom. This would promote reconciliation, recognize the contributions of IK in mathematics, and seek social justice for marginalized groups (Tuck & Yang, 2012).

Before the lesson was delivered, the majority of the class was observed to be consistently and chronically disengaged. On any given day, only two to four students were actively engaged at various points in the lesson. Of the remaining five or six students typically attending, none were focused on the lesson, and often spent time with their heads on the desk, listening to music, or playing on electronic devices. These observations must be kept in mind when interpreting the response to the new lesson. The lesson used local

geography in consultations with First Nations members as to context and appropriateness. A description, lesson plan, and materials are provided in the Appendix.

Lesson Context

This lesson context is mapping and transportation in mathematics. This lesson was developed based on an available lesson created by the First Nations Education Steering Committee (FNESC), with a Grade 8 audience in mind; however, we felt it would fit the scope of the locally developed-level mathematics, as the students in this course can be up to four grade levels behind (FNESC, 2011; LDCC Project, 2005). Edits were made to the original lesson to make it more relevant to a Northwestern Ontario audience on the traditional territory of the Anishinabek and Métis. This lesson was selected as exemplifying a cross-curricular hands-on approach, which fits the definition of decolonized content. It also incorporates Indigenous content through the history of transportation in Anishinabek culture, which effectively embeds Indigenous knowledge into the lesson. Modern forms of transportation were also discussed throughout the lesson.

Effects on Student Attitude and Engagement

During the field test of the IK-focused lesson (see Appendix), students immediately responded to the "Minds On" activity. No students were on their phones, and I (first author and teacher of the lesson) and the other educators in the room noted that every single student was watching the video that was presented to the students. Most of the students who were chronically disengaged raised their hands to answer questions, and the quality of responses received in this section of the lesson noted that students were attentive while watching the video, as they were able to comprehend and answer questions on the content with which they were presented.

During the "Action" portion of the lesson, every student participated and completed building a model canoe. (Recall that on a regular day, most students would not even attempt the work for the day.) All students also tested their canoes to see how they travelled across the "river." The majority of the class also completed the worksheets that accompanied the lesson (see Appendix), and answered questions during the "Consolidation" portion. In terms of completion of a worksheet provided on a typical day in the classroom, it was rare to see students complete a worksheet, and only about five students would actively work on it. Almost all students made efforts to complete some of the worksheet, and about ten students finished it.

During follow-up interviews after the lesson took place, the regular classroom teacher and support person both underscored the unusual level of participation and engagement during the lesson, compared to a normal day to the first author. As well as engagement, the attendance record showed a significant increase from previous classes, and more students returning for the second day of this lesson than would normally be in class (Murie, 2019).

Conclusion

Overall, the observations suggest the potential of this type of lesson and context for engagement and learning in locally developed classes such as those typical in our region. Through incorporating an IK-highlighted lesson and a hands-on activity with relevance to real-world contexts, engagement and interest appeared to be heightened. Students appeared more interested, invested, and curious about mathematics through the Indigenous cultural content during the field test, compared with previous observations. This could have occurred due to interest in the Indigenous knowledge being presented or due to the problem-based style of the lesson; however, both go hand in hand. In classes in which attendance is often 50 percent at best, with students often off task even when present, the very fact that all the present students were working on the activities was encouraging. While the materials offered in the Appendix necessarily relate to our own context, it is hoped that other teachers will find value in them as a starting point for their own locally developed lessons.

References

Aikenhead, G.S. (2017). Enhancing school mathematics culturally: A path of reconciliation. *Canadian Journal of Science, Mathematics and Technology Education, 17*(2), 73–140. doi.org/10.1080/14926156.2017.1308043

Battiste, M. (2013). *Decolonizing education: Nourishing the learning spirit.* Purich Publishing.

Battiste, M., & Henderson, J. (Sa'ke'j) Y. (2009). Naturalizing Indigenous knowledge in Eurocentric education. *Canadian Journal of Native Education, 32*(1), 5–18,129–130. https://search.proquest.com/docview/755262421/abstract/645956F9D41483DPQ/4

Belczewski, A. (2009). Decolonizing science education and the science teacher: A white teacher's perspective. *Canadian Journal of Science, Mathematics and Technology Education, 9*(3), 191–202. doi.org/10.1080/14926150903118326

Borden, L.L. (2013). What's the word for…? Is there a word for …? How understanding Mi'kmaw language can help support Mi'kmaw learners in mathematics. *Mathematics Education Research Journal, 25*(1), 5–22. doi.org/10.1007/s13394-012-0042-7

Education Quality and Accountability Office. (2014). *Guide to EQAO assessments in secondary school.* Queen's Printer for Ontario.

First Nations Education Steering Committee. (2011). *Teaching mathematics in a First Peoples context, grades 8 and 9.* https://aned.public.sd61.bc.ca/wp-content/uploads/sites/

112/2019/02/FNESC_Teaching_Math_in_a_First_Peoples_Context_Grades_8_and_9.pdf
Galabuzi, G.-E. (2014, Winter). Race and the streaming of Ontario's Children and Youth. In D. Clandfield & G. Martell (Eds.), *Restacking the deck: Streaming by class, race and gender in Ontario schools.* Canadian Centre for Policy Alternatives: Our Schools/Our Selves, Special Issue in association with Everybody's Schools: An Education Policy Institute, 185–226. http://search.ebscohost.com/login.aspx?direct=true&db=eue&AN=96719998&site=ehost-live
Godlewska, A., Moore, J., & Bednasek, C.D. (2010). Cultivating ignorance of Aboriginal realities. *The Canadian Geographer, 54*(4), 417–440. doi.org/10.1111/j.1541-0064.2009.00297.x
Government of Canada; Indigenous and Northern Affairs Canada. (2017). *Let's talk on-reserve education: Survey report.* www.sac-isc.gc.ca/eng/1509019844067/1531399883352
Hamlin, D., & Cameron, D. (2015). *Applied or academic: High impact decisions for Ontario students.* People for Education. https://exchange.youthrex.com/report/applied-or-academic-high-impact-decisions-ontario-students
Kajander, A., Flessa, J., Lampo, A., & Sedor, K. (2018). Pitfalls of autonomy: The overlooked challenges of teaching locally developed mathematics in Ontario high schools. *Canadian Journal of Science, Mathematics and Technology Education, 18*(2), 164–176. doi.org/10.1007/s42330-018-0013-4
King, A.J.C., Warren, W.K., King, M.A., Brook, J.E., & Kocher, P.R. (2009). *Who doesn't go to post-secondary education? Final report of findings for Colleges Ontario collaborative research project.* Social Program Evaluation Group. https://educ.queensu.ca/sites/webpublish.queensu.ca.educwww/files/files/Research/SPEG/SPEG%20Who%20Doesn%27t%20Go%20To%20Post-Secondary%20Education.pdf
LDCC Project. (2005). *Locally developed compulsory credit courses, grades 9 and 10: Mathematics.* Peel District School Board. www.edugains.ca/resources/CurriculumDocuments/LDCC_Math_9_10.pdf
Livingstone, D.W. (2014, Winter). Class, race and gender differences in schooling. *Our Schools/Our Selves, 23*(2), 9–39. http://search.ebscohost.com/login.aspx?direct=true&db=eue&AN=96719994&site=ehost-live
Macaulay, A. (2015). *Effective practices in grade 9 applied mathematics* [Doctoral thesis, University of Toronto]. https://tspace.library.utoronto.ca/handle/1807/71461
Munroe, E.A., Lunney Borden, L., Murray Orr, A., Toney, D., & Meader, J. (2013). Decolonizing Aboriginal education in the 21st century. *McGill Journal of Education, 48*(2), 317–337. https://ezproxy.lakeheadu.ca/login?url=http://search.ebscohost.com/login.aspx?direct=true&db=edb&AN=91705865&site=eds-live&scope=site
Murie, T. (2019). *Re(righting) mathematics education: Changing students' attitudes towards mathematics through Indigenous knowledge and worldview* [Thesis, Lakehead University]. https://knowledgecommons.lakeheadu.ca/handle/2453/4514
Russell, G.L., & Chernoff, E.J. (2013). The marginalisation of Indigenous students within school mathematics and the math wars: Seeking resolutions within ethical spaces. *Mathematics Education Research Journal, 25*(1), 109–127. doi.org/10.1007/s13394-012-0064-1
Sedor, K. (2019). *Factors affecting the teaching of grade 9 locally developed mathematics* [Thesis, Lakehead University]. https://knowledgecommons.lakeheadu.ca/handle/2453/4504
Stavrou, S.G., & Miller, D. (2017). Miscalculations: Decolonizing and anti-oppressive discourses in Indigenous mathematics education. *Canadian Journal of Education, 40*(3), 92–122. https://ezproxy.lakeheadu.ca/login?url=http://search.ebscohost.com/login.aspx?direct=true&db=eue&AN=125457082&site=eds-live&scope=site, https://cje-rce.ca/wp-content/uploads/sites/2/2017/10/4.-2382-Stavrou-Miller-Sept-18.pdf
Tuck, E., & Yang, K.W. (2012). Decolonization is not a metaphor. *Decolonization: Indigeneity, Education & Society, 1*(1), 1–40. www.nwic.edu/wp-content/uploads/2016/08/Decolonization-Is-Not-A-Metaphor.pdf

Appendix

This appendix provides the lesson plan used in this field test, along with a description, handout templates, and student worksheet. It should be noted that the lesson as is needs to be modified to the geography of students in the region; in our case, the context is set in the Kaministiquia River in the traditional territory of the Fort William First Nation.

Title: Mapping & Transportation
Subject: Mathematics (Grade 9)
Time: 2 x 75 minutes
Strand: Developing and Consolidating Concepts in Measurement

Desired Results

Lesson Description

This lesson focuses on practical applications of measurement. Students will first learn about the traditional practices of the Indigenous peoples in the traditional territory of the Fort William First Nation regarding travel. Students will then complete an activity determining canoe travel across the Kaministiquia River, and they will use appropriate measuring tools and units to measure how far the canoe travelled (the hypotenuse of a triangle). In the consolidation phase, the length of the hypotenuse will be compared to the anticipated horizontal distance, and different impacts on the canoe's travel will be discussed (for example, current or wind), and current practices of travel will also be explored.

Ontario Curricular Overall Expectations

- DCMV.01 – estimate and measure length, capacity, and mass, in order to consolidate understanding of the metric system;
- DCMV.04 – communicate information about measurement concepts.

Ontario Curricular Specific Expectations

- DCM1.01 – investigate, discuss, and describe applications from everyday life and the workplace that would involve the measurement of length in commonly used metric units;

- DCM1.06 – demonstrate accuracy in measuring length, capacity, and mass in everyday applications, using teacher-selected tools, and record the measurements, using the correct abbreviations for metric units;
- DCM1.08 – estimate and use the measurements of length, capacity, and mass in everyday applications;
- DCM4.01 – organize measurement information, using a simple framework, draw conclusions from this data, and make decisions based on it;
- DCM4.02 – verbalize their observations and reflections regarding measurements, and ask questions to clarify their understanding;
- DCM4.04 – communicate, orally and in writing, the solutions to measurement problems and the results of investigation, using appropriate terminology, symbols, and form.

Lesson Goals (Clearly identify what students are expected to know and be able to do, as if asked the purpose by a principal.) I can...

- measure the distance a canoe travels, using appropriate tools and units
- describe some of the historic ways that the Indigenous peoples in this traditional territory travelled
- make connections between what factors impact the travel across a waterway

Success Criteria (Teachers will identify the criteria they will use to assess students' learning, as well as what evidence of learning students will provide to demonstrate their knowledge and skills, in student language.)

- Creating a labelled, complete diagram of the canoe's travel
- Demonstrating how to measure the canoe's travelled distance
- Completing the worksheet in full detail and in full sentences

Assessment

Assessment Mode: **oral/performance/written**

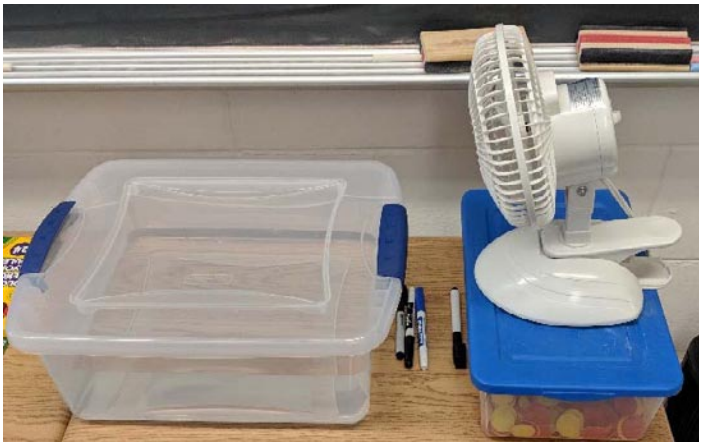
Assessment Strategy: formative, listening to students talking in groups, observing, looking at written work

Assessment Tool: completed worksheet, talk among groups

Materials

- Poster board
- Pencils
- Pens
- Large plastic bin
- Water
- Rulers
- Fan
- Erasable marker
- Canoe pattern
- Handout for students
- Glue or tape
- Marbles (or some form of weight for the canoes)

Prior to the lesson, the teacher should set up a bin of water with a fan at an appropriate distance away, similar to the image below.



Lesson Format: What Teachers Do/Say

Motivational Hook/Engagement/Introduction (Minds On)

- Explain that each community has unique stories of travel, and that the next two days will be focused on travel and how it relates to mathematics.
- Show Fort William First Nation video, *Pride, Confidence and Connection: Building More Than Just a Canoe* (www.northernontario.travel/paddling/fort-william-first-nation-birch-bark-canoe-building-near-thunder-bay-canada?s=1320).
- Ask students (historically) how people would travel in our area (either by land or water) and via what method? (canoe, walking, etc.)
- Ask students (historically) why people travelled (hunting, collecting food, visiting family, feasts, celebrations, warfare, sports, etc.).
- Bring forward the idea of water travel and how rivers flow with a current.
- Ask students to brainstorm what would happen to the canoe when travelling from one side of the river to another, and how we could find out how far the canoe travels.

During: Working on It: Action (Hands On): What the students are doing

- Introduce the Canoe Model Activity.
- Have students demonstrate the effect of current on a model canoe as it travels across a model of the Kaministiquia river (i.e., the bin of water with fan turned on and blowing across the water to mimic current).
- Student will construct their own canoes from poster board and tape and have them "travel" across the river.

Steps for students:

1. Cut out the canoe figure.
2. Assemble the card-stock canoe.
3. Measure the length and width of the container filled with water, and record it on the diagram on their worksheet.
4. Without the fan running, determine how much of a push will make the canoe drift from one side of the bin to the other. You may have to add some weight to your canoe with the supplies provided, to avoid tipping.
5. Run the fan and give the canoe a steady push across the river (container).
6. Mark with erasable marker where the canoe gets to the other shore, and measure the distance along the container from the starting end to where the canoe touched the other shore (side A).
7. Measure the distance across the river, the width of the container (side B).
8. Ask students how they could find how far the canoe actually travelled.
9. Have the students measure side C (the hypotenuse).

After: Consolidation: Reflect and Connect

1. Discuss how this is applicable in the real world when travelling across a river in a boat when a strong current is present.
2. Discuss students' findings and what they mean in terms of a real canoe or real boat.
3. Discuss implications of different factors for travellers when crossing waterways.
4. Discuss the relationship between the hypotenuse and the other sides of the triangle students created.
5. Discuss various scenarios where they want to end up at a specific point on the opposite river bank, and how they should approach their travel.
6. Connect back to the Indigenous peoples in this area—Why might they have been crossing the river? Why might we cross a river today?
7. Throughout the discussion, have students complete the worksheet based on these questions.

Extension Activities/Next Steps

- Discuss angles and other properties of triangles.

Special Education Notes: Differentiated Instruction Considerations/Accommodations/Assessment

- The instruction is differentiated for all learners, as all students are able to help build the model canoe and measure the dimensions needed for this activity.
- The lesson has a hands-on, practical approach, which helps engage more students in the lesson.

Cross-Curricular Links: Oral, Reading, Writing, Media, Social Studies, Science, Math, Visual Arts, Drama, Music, Health & Physical Education (circle)

- History
- Geography
- Science
- Health & Physical Education

Canoe Template

Different contexts may use different traditional canoe styles and adapt this lesson to their location. Templates can be created by drawing one side of a canoe and creating a mirror image, and then joining both in the centre. Sample canoe styles can be found on page 7 of *Northern Ojibway Style Birch Bark Canoe Building*, at www.occc.ca/library/Northern%20Ojibway%20Style%20Birch%20Bark%20Canoe%20Making.pdf.

Worksheet Package

Crossing the River

Ontario is laced with rivers. First Peoples often used canoes on rivers and lakes for travel. Crossing a river is tricky business. If you want to get to the opposite side across from where you start, you can't just head straight across. The river's current will be pushing you downstream while you are trying to get across. It's the same idea when you are crossing a lake: you have to consider the effect the wind will have on your path.

Materials

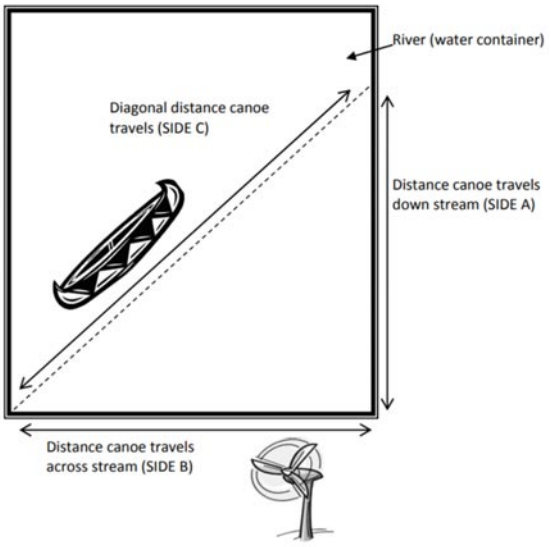
- Large plastic container (your river or lake) filled with water
- Model canoe that you made
- Erasable marker
- Fan (the current or wind)
- Metre stick or measuring tape

Directions

1. Set up the fan at one end of your water container (river) as shown in the figure on the next page.
2. Measure the distance of side B (the width of the river), and record it in the table on the next page. This will be the same value for each river crossing.
3. Give your canoe enough of a push so it can reach the other "shore" (other side of container).
4. Use the same amount of push and set up your fan on the lowest speed so that it will blow your canoe downstream, while still allowing your canoe to reach the other shore. With the fan blowing, push your canoe across the river.
5. Use the marker to put a line where the canoe hits the other shore. Measure alongside A (distance canoe travels downstream), and record this distance in the table.
6. Repeat steps 3-5 on medium speed.
7. Repeat steps 3-5 on high speed.

Reminder: the longest side of this triangle (that is opposite the 90° corner) is called the

_____.



Finding the distance travelled (hypotenuse)

Fan speed	Side A (cm)	Side B (cm)	Side C (cm)
Low			
High			

Question Worksheet

1. When might you see a situation like this (a boat crossing a river) in real life?
2. What things would you consider when crossing a waterway in a boat or canoe?
3. Compare the length of the hypotenuse (side C) with the length of the other sides. Is it longer or shorter? What might that mean for the boat operator?
4. If you wanted to reach a specific spot on the other side of a river and you knew there was a strong current, where would you start your journey? Label it on the diagram below.



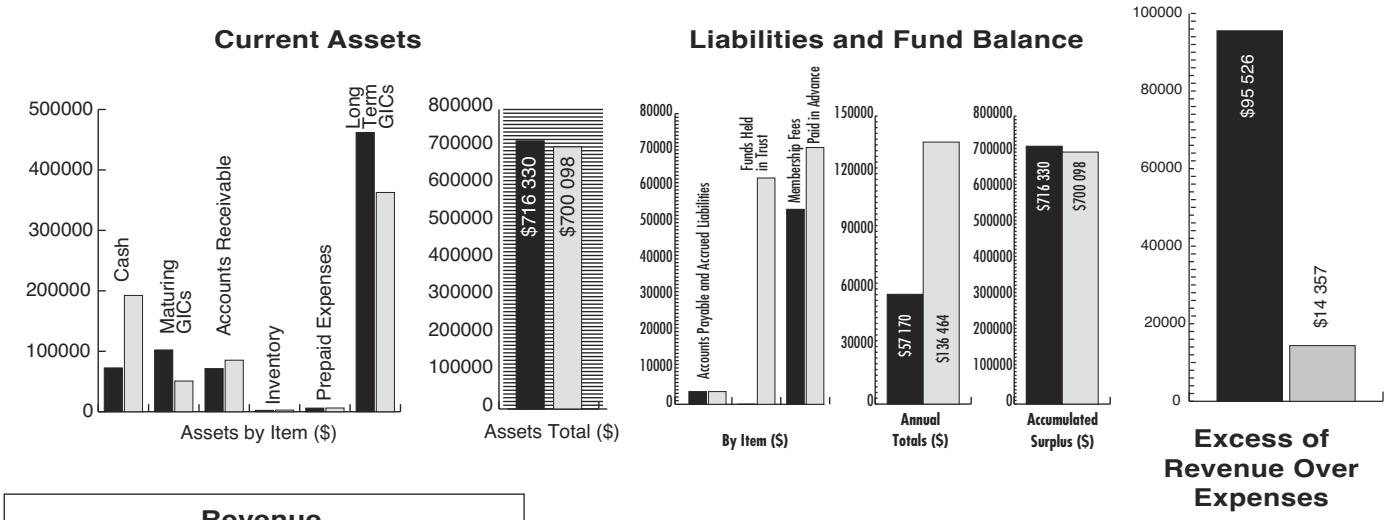
5. Historically, why might Indigenous peoples have wanted to cross the river?

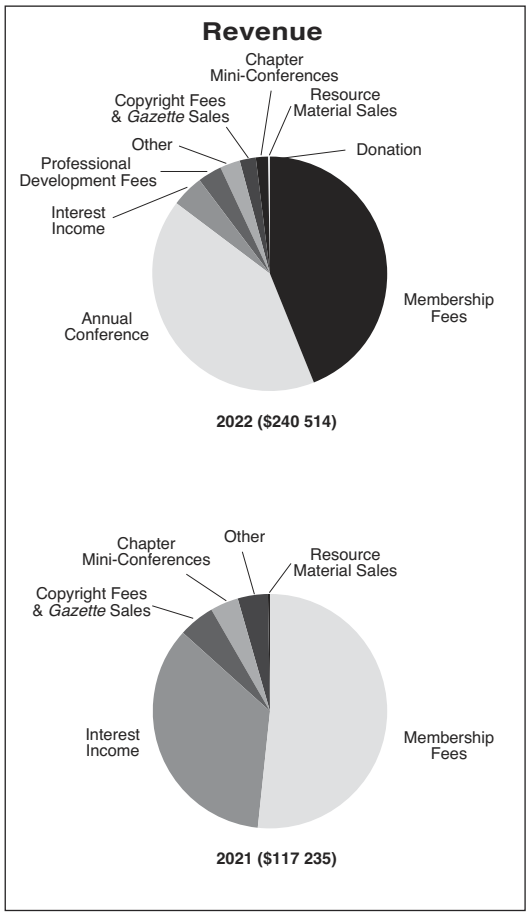
▲ OAME Financial Statement 2021-22

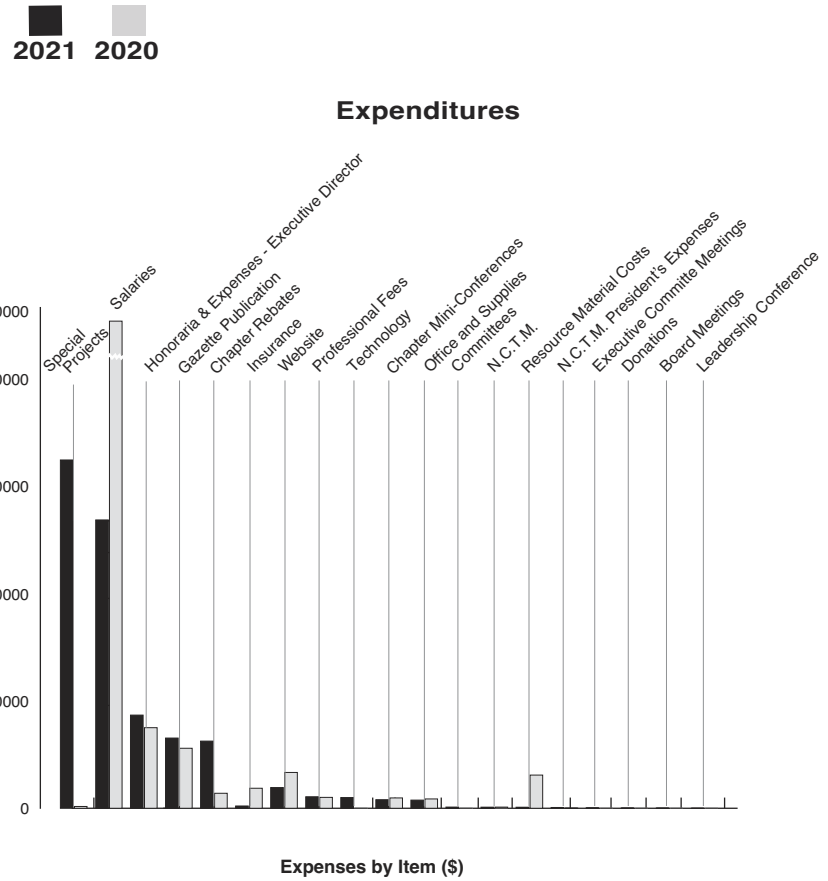
Notice to Readers

On the basis of information provided by Management, we have compiled the statement of financial position of Ontario Association for Mathematics Education, Association Ontarienne pour L'Enseignement des Mathematiques as at August 31, 2022, the statements of operations and changes in net assets for the year then ended, and Note 1, which describes the basis of accounting applied in the preparation of the compiled financial information ("financial information").

London, Ontario, October 7, 2022, GRANT THORNTON CHARTERED PROFESSIONAL ACCOUNTANTS







▲ 2022-2023 OAME/AOEM Board

OAME WEBSITE
WWW.OAME.ON.CA

DIRECT GENERAL INQUIRIES TO: EXECUTIVE DIRECTORS

Fred and Lynda Ferneyhough
4751 Mack Ave.
Plympton-Wyoming, ON N0N 1J6
Business Phone: 647-466-0674
eds@oame.on.ca

PRESIDENT

Sandra Jean Price
Oshawa, ON
sandra.jean.price@oame.on.ca

VICE-PRESIDENTS

Melissa Black
Kanata, ON
melissa.black@oame.on.ca

Bart Vanslack
Oakville, ON
bart.vanslack@oame.on.ca

PAST PRESIDENT

Judy Mendaglio
Mississauga, ON
judy.mendaglio@oame.on.ca

EMERITUS

Bill Otto
Stillwater Lake, NS
bill.otto@oame.on.ca

CHAPTERS & REPRESENTATIVES

CHAMP: TBD

GOLDEN: Mark Chubb
St. Catharines
mark.chubb@oame.on.ca

GVMA: Glen McMillan
Guelph, ON
glen.mcmillan@oame.on.ca

ISOMA: Cal Armstrong
Burlington, ON
cal.armstrong@oame.on.ca

MAC²: Stacey Collins
Barrie, ON
stacey.collins@oame.on.ca

NOMA: Heather Theijsmeijer
Kagawong, ON
heather.theijsmeijer@oame.on.ca

NWOAME: Chelsea Carlson
Fort Frances, ON
chelsea.carlson@oame.on.ca

$O_3$4ME: Chris Stewart
Chesterville, ON
chris.stewart@oame.on.ca

PRMA: Jennifer Clark
Bowmanville, ON
jennifer.clark@oame.on.ca

QSLMA: Siobhain Broekhoven
Kingston, ON
siobhain.broekhoven@oame.on.ca

SAME: Franca Genco
Scarborough, ON
franca.genco@oame.on.ca

SWOAME: Elizabeth Pearsall
Kingsville, ON
elizabeth.pearsall@oame.on.ca

TEAMS: Alvarine Aldridge
Toronto, ON
alvarine.aldridge@oame.on.ca

WOMA: Kristin Methot
Dorchester, ON
kristin.methot@oame.on.ca

Y⁴MA: Gina Micomonaco
Richmond Hill, ON
gina.micomonaco@oame.on.ca

DIRECTORS – TERM EXPIRING 2023

Elli Weisdorf
Toronto, ON
elli.weisdorf@oame.on.ca

Sam Garrison
Toronto, ON
sam.garrison@oame.on.ca

DIRECTORS – TERM EXPIRING 2024

Lori Yee
Pickering, ON
lori.yee@oame.on.ca

Jimmy Pai
Ottawa, ON
jimmy.pai@oame.on.ca

DIRECTORS – TERM EXPIRING 2025

Kit Luce
Barrie, ON
kit.luce@oame.on.ca

Jason To
Toronto, ON
jason.to@oame.on.ca

Kelly McKie
Ottawa, ON
kelly.mckie@oame.on.ca

Cameron Redsell-Montgomerie
Toronto, ON
cameron.redsell-montgomerie
@oame.on.ca

NCTM REPRESENTATIVE

Jacqueline Hill
Lindsay, ON
jacqueline.hill@oame.on.ca

WEBSITE COORDINATOR

Greg Clarke
Orillia, ON
web@oame.on.ca

GAZETTE EDITOR

Timothy Sibbald
North Bay, ON
gazette@oame.on.ca

ABACUS CO-EDITORS

Marc Husband
Toronto, ON
marc.husband@oame.on.ca

Tina Rapke
Toronto, ON
tina.rapke@oame.on.ca

50th Annual OAME Conference

@OAMELEARNS
OAME2023.CA

OAME2023

BEANFIELD CENTRE
EXHIBITION PLACE
TORONTO
APRIL 27-28, 2023

PRE-REGISTRATION
NOV 1, 2022
FULL REGISTRATION
FEB 1, 2023

EMPOWERING ME
TORONTO

KEYNOTE SPEAKERS:

ROCHELLE GUTIERREZ



https://education.illinois.edu/faculty/rochelle-gutierrez

DAN FINKEL



https://mathforlove.com/-dan-finkel

DEBORAH LOEWENBERG-BALL



https://soe.umich.edu/directory/faculty-staff/deborah-loewenberg-ball

FEATURED SPEAKERS:

OCTAVIA BECKLES
SHANTEL CLARK
ROBERT DUROCHER
LINDSEY GOLD
HEMA KHODAI
CATHY MARKS KRPAN
ANTHONY MELI & JASON TO
MARIAN SMALL
HEATHER WARK

HOSTED BY NOMA AND TEAMS

NOMA
NORTHERN ONTARIO MATHEMATICS ASSOCIATION

Toronto Educators Association for Mathematics
TEAMS

design by Drew Ferneyhough bedowynn@gmail.com