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Ontario Mathematics Gazette

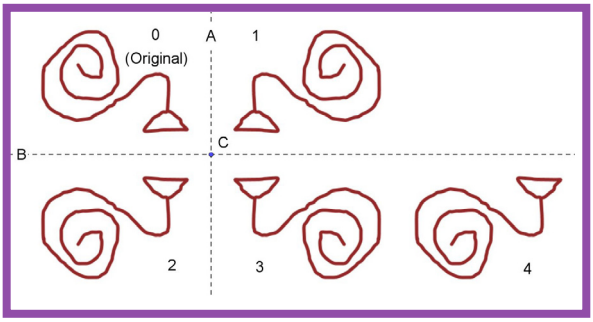
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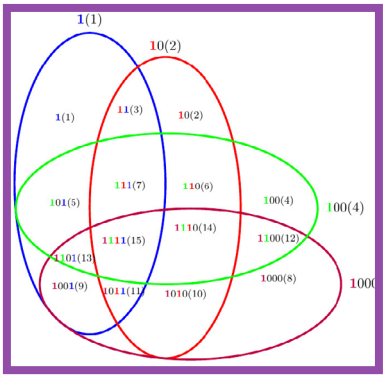
Vol. 62#3
MARCH 2024

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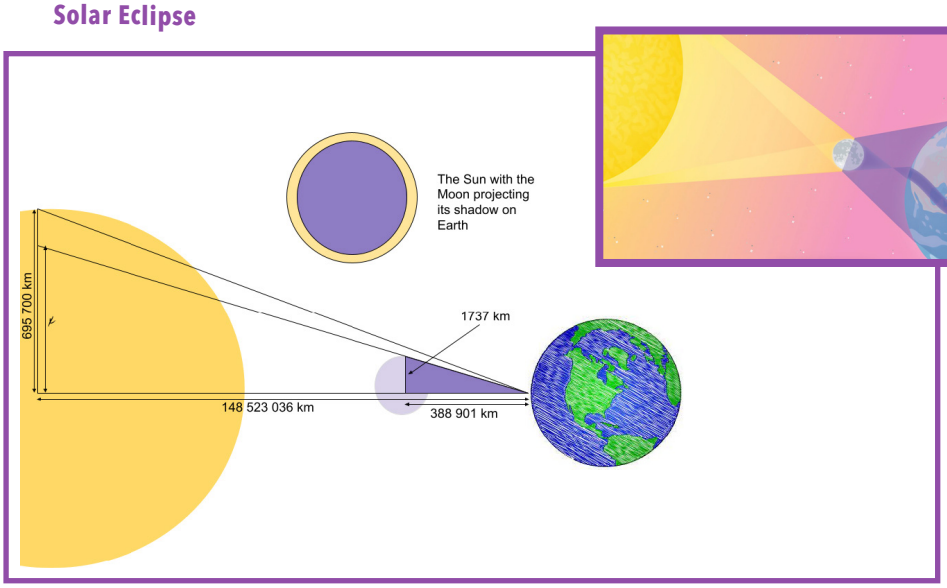


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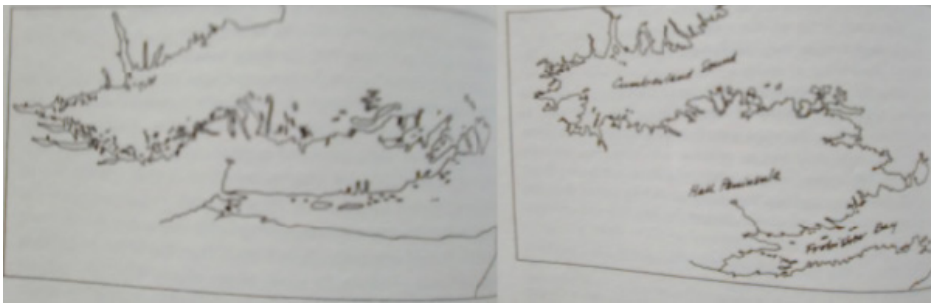
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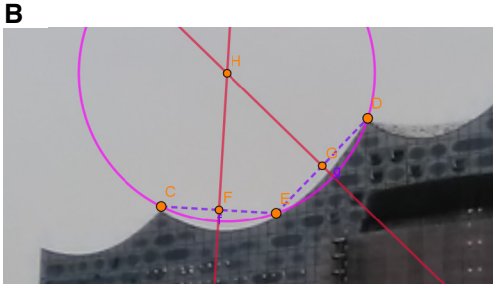
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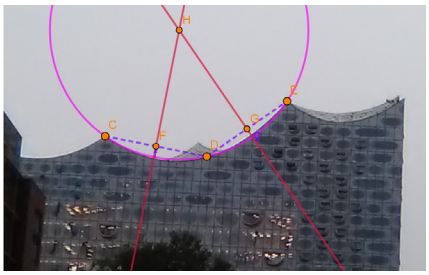


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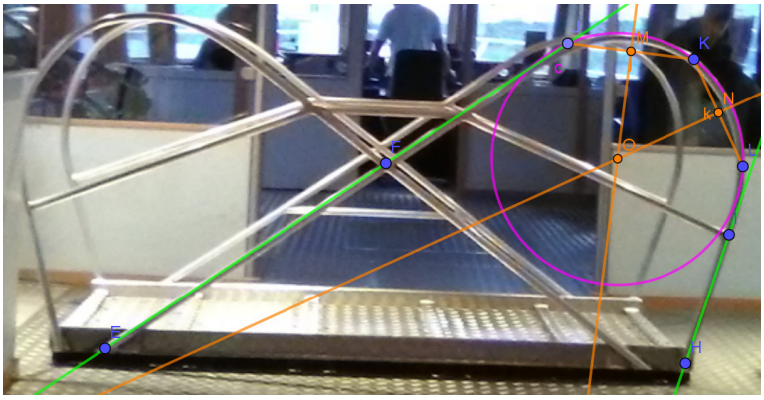


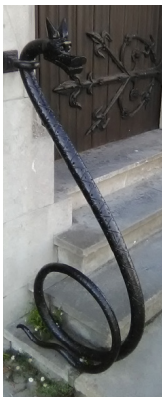
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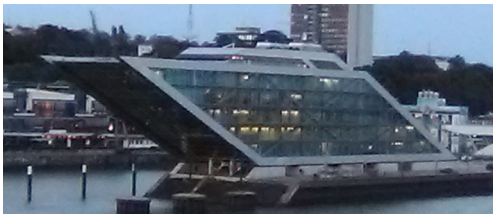


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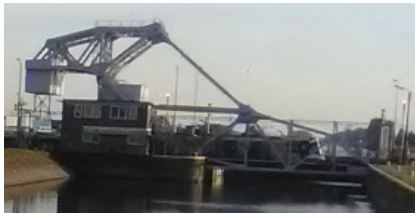




D



E



F



G



H



A) Indigenous Mathematics!
Here is a map drawn by an Inuit from memory (Left) and the modern mapping equivalent (Right). The maps are from *Acrtic Dreams* by Barry Lopez (The Harvill Press, 1986).

B) Hamburg, Germany, Skyline
If the roofline appears circular without any special geometric set-up by the photographer, is it likely to actually be circular? Could it be another shape and just appear circular?

Does the view on the right, from a different angle, change your thinking?

C) Costing a Ramp?
A ramp for a ferry in Hamburg, Germany, has metal tubing for handholds. The tubing is straight before bending in a circular arc and then straight again. Estimating the height of a railing for a ramp, determine the length of tubing required to build a ramp.

What length of material do you estimate would be needed to make this railing?

D) Usable Space?
Suppose space in a building is usable if a person five feet tall will not hit their head on the ceiling. So, a rectangular building has 100% usable space. Assuming floors are ten feet apart, what percentage of this parallelopiped building (in Hamburg, Germany) is usable space by this definition?

E) Levers?
I am not saying I am old, but we used to teach about levers in math classes—so we could talk about the bascule bridge in Bruges, Belgium, shown. Whatever happened to that? (Did it go over to physics?)

F) Cobblestone Patterns?
What pattern do you see first when you look at this picture of cobblestones?

What is the second pattern you saw? (Does one pattern necessitate the other pattern?)

G) Play-based Opportunity?
A sculpture made of shipping containers, located in Le Havre, France, is inspiration for play-based learning using carboard boxes. In later grades, students could fit parabolas; in younger grades use tape, string, and boxes to build a rendition.

H) Border in a Church
A border from Église Saint-Léonard, Honfleur, France, suggests modelling using ribbon and glue.

Submission of Articles

The *Ontario Mathematics Gazette* (*OMG*) is looking for news items, articles, and good ideas that are useful to mathematics teachers and mathematics education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *Gazette*:

- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles are to be word–processed, MS Word is preferred, and prepared according to the *Publication Manual of the American Psychological Association*, Seventh Edition. However, please use single-line spacing (not double). Articles should not exceed five numbered pages of text, and figures, images, and photographs should be placed in the text close to where they belong, with captions. The photographer's permission is required, and for photos of students under the age of 18, the written permission of a parent or guardian is required.

Please submit your article in one blind file (i.e., identity of author is not evident), and include author names, contact information including email and mailing addresses, photos—head and shoulders, biographies—less than 100 words, and all content removed for blinding in a second file. Please email these two files to Tim Sibbald at gazette@oame.on.ca.

Upon review, you will be notified whether your article has been accepted for publication (as is, or pending minor or major revisions) or declined. The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME/AOEM.

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The views expressed or implied in this publication, unless otherwise noted, should not be interpreted as official positions of OAME/AOEM.

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Submission of Advertisements
Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** by email or at the above address. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

Advertising Rates
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▲ Editor's Report



TIM SIBBALD, OCT, PH.D.
gazette@oame.on.ca

Tim Sibbald is a professor in the Schulich School of Education, Nipissing University, with a focus that includes mathematics education. He is a former President of OAME/AOEM and has held a variety of other OAME/AOEM posts over the years.

What is time?

Having been the Editor for a few issues (since September 2016), I find myself revisiting ideas. This is not a bad thing, if there is a new angle to explore, and I know I have previously considered ways teachers can be more efficient to realize some more time in the day. More recently I have been thinking about differences between actual time taken and perceptions of time. It is an odd topic that I noticed somewhat as a teacher—especially when class seemed to whizz by because there was a high level of engagement—but it was primarily as an academic and having a sabbatical that it really stood out. I have written about perception of time during a sabbatical elsewhere (Sibbald, 2022) and take a different approach here.

Increasingly, I notice how attitude toward time seems to be indicative of larger changes in attitude that either are signalling fundamental changes in society, or a topic that needs to be addressed. My interest is that I do not know which of these it is, but I am keenly observing and wondering about the phenomenon that I will describe.

Recently, a colleague sent me an opinion piece that suggested that declines in standardized test scores correspond to certain cellphone features becoming ubiquitous. In particular, the rise of data and the smartphone seemed, according to the opinion, to fit the timeline. The concept of time may be an important way to think about such an opinion. It is not the occurrence of cellphones or data that seem to be relevant. It is the behavioural change because online communities vie for student attention. Sometimes this is described in a derogatory way as cellphone users needing instant gratification. I think it is more authentic to suggest that it is the timeline that is important; cellphones allow active engagement. The cellphone gives access to online communities, since data (in the 4G sense) truly became available, and such communities engage people, particularly teenagers.

Consider, for a moment, the converse—any experience where you purposefully reduced the level of engagement—

what was that like? In my case, as a teen, I recall a disastrous dip into the world of correspondence chess. Instead of the player opposite you responding in seconds or minutes, there was a week-long wait for the post to contain a next move or two (an "if you do this, then" move). It was very challenging to play a game of chess on the scale of a month or two. But if you consider growing up playing chess this way, and then playing face-to-face were invented, what an extraordinary change that would seem to be.

For educators, a parallel universe has arisen. There are online communities that offer quick jolts of thought about educational issues. There are interesting and thought-provoking ideas circulating amidst the wild west of communications. As an example, a year or two ago, there was a blog claim (no longer available) that a pie graph in French is called a "Camembert," with a clear linguistic connection to the circular cheese. Now, I find dialogues dispelling this claim, shock being expressed that names for things vary in different languages, and a sprinkle of swearing because that is the sort of forum where the discussion is taking place. These communities are a quick click away and offer ready answers, or a plethora of opinions to posed questions, but, *do professionals risk having their professional acumen decline if that is exclusively their engagement in the larger conversations of education?*

I like to think of the *Ontario Mathematics Gazette* as a calm meditative realm that brings new ideas, but also presents ideas that have been thought through. At times, I wonder if this makes me old-fashioned and, frankly, I do not see any evidence of that. What I do find is that a degree of slowness, with time taken for due consideration of details, has a definitive place in professional communities.

Furthermore, I find the longevity of the archives to have no equivalent feature in the modern world of social media interactions—a thought that leads me to wonder if the lack of a historical record in social media sets people up to repeat the same conversations (assuming they rely solely on social media).

The point is that when time scales are perceived to be markedly different, we should consider the larger implications. Something triggers one to notice and, as Marshall McLuhan said, "The medium is the message." It is in consideration of this that I find the *Gazette* to have a position of importance for many of its features—peer review, editorial oversight, and a historical record, to name a few.

News

The International Day of Mathematics (IDM) is, of course, March 14 (i.e., Pi Day). This year, the theme is "Playing with Math." See www.idm314.org for details. Statistics that have been gathered have shown that Ontario has had little endorsement of the IDM. Perhaps this is because it often falls on the March Break, or, perhaps, it is that teachers engage in an unofficial way that does not appear in the statistics gathered by the International Math Union (who have led the development of the IDM). In light of two distinct possibilities, I would like to invite emails from the readership by teachers who do something for or related to Pi Day. Please let me know with an email to gazette@oame.on.ca.

Thank Yous

Updates have taken place in the archives. During last summer, various issues were rescanned, and thanks to the website team, are now posted in the *Gazette* archives. There are very few double-width pages, so reading many of the early issues is much easier. If you find any that requires adjustment, let me know.

Thank you to our *Gazette* reviewers. Over the last while: *Andrew Dmytriw, Andrew Morrison, Barb Seaton, Beverly Farahani, Carly Ziniuk, Christopher Stewart, Jacqueline Hill, Jennifer Holm, John Rodger, Judy Mendaglio, Kameal Mina, Linda Boucher St-Arnaud, Mary Bourassa, Niloofar Mani, Ralph Connelly, Robert Sherk,* and *Sonia Ellison*

Thank you also to a wonderful *Gazette* team. A quiet behind-the-scenes person on the team is **Robert Sherk**, who is the advertising manager. Please help Robert by mentioning the *Gazette* as a place to get the word out to many active math teachers across the province.

In addition, the team for this issue has included the usual suspects: **Gitta Berg**, **Penny Clemens**, **John Rodger**, **Peter Saarimaki**, and (because alphabetical order works that way) **Anne Yeager**. Notably, John picked up a few pieces with virtually no forewarning, which is a wonderful support—thank you! And two other members of the team were travelling and picked up as soon as they returned—thank you!

In This Issue…

Our President, **Sandra Jean Price**, brings a perspective on the role of Artificial Intelligence, but so also does **Cal Armstrong**. They were both inspired to write by a project that is underway, and they bring two perspectives, which

makes interesting reading. Also on a new front, **Shawn Godin** brings news that there is an effort to get a new version of the Ontario Mathematics Olympics going. Along with good news, there is some sad news. **Melissa Black** and **Cathy Hall** have written a tribute to **Pat Kehoe**, who was a Director of OAME and a Chapter Representative.

Pamela Brittain reports on **Lynda Colgan** receiving the Margaret Sinclair Memorial Award, a presentation about destreaming, and a Canadian Math Society event. **Jacqueline Hill** considers **Ron Lancaster**'s involvement with the NCTM, while **Heather Theijsmeijer** brings news from the OMCA.

David Costello explores the idea of a spectrum of equivalent problems and the implications for well-rounded instruction. **Shawn Godin** considers a curious problem to remove items from stacks, and continues to use his very informative process of including explicit links to the math processes. *At the end of his column, Shawn asks readers to send problems that show reasoning and proving—I encourage you to surprise him with the volume of responses*. We also have **Jennifer Holm** back, with a focus on instructional practices around measurement. **Iain Brodie** and **Hatice Beyza Sezer** write about liminal spaces and draw together ideas from several past interviews. **Lynda Colgan** questions whether pre-service programs are sufficient for teacher candidates. **Carly Ziniuk** considers geometry and measurement related to the eclipses. **Jeff Irvine** gives a brief synopsis of the Pirie-Kieren dynamic model of mathematical understanding.

Missing from this issue, **Jérôme Proulx** is taking a hiatus and will, hopefully, return in the Fall.

Reference

Sibbald, T. (2022). (Re)Discovering academia through a sabbatical. In T.M. Sibbald & V. Handford (Eds.), *The academic sabbatical: A voyage of discovery* (pp. 89–101). University of Ottawa Press. ▲

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▲ President's Message



SANDRA JEAN PRICE
sandra.jean.price@oame.on.ca

Sandra Jean has been on the Board of Directors of OAME/AOEM for almost 20 years. She is currently teaching in the Durham District School Board and has more than 25 years of teaching experience. She has had many roles, including classroom teacher, Math Coach, and Instructional Leader of online Math Additional Basic Qualifications (ABQs). Sandra Jean has contributed to a variety of Ministry projects and was a coordinator of the most recent OAME/AEMO writing projects. Sandra was the recipient of the 2008 OAME Award for Exceptional and Creative Teaching in Elementary Mathematics for her innovative instructional practices, integration of technology, and belief in success for all.

At the beginning of 2024, technology once again claimed some attention in my household. My son, who is in Grade 9, was taking English during the first semester. It was clear that his teacher was trying to deal with the reality of artificial intelligence (AI) applications (e.g., ChatGPT) and how students are trying to lighten their workload by using them. They had students complete multiple drafts, some even on paper, some on computer, with all versions saved and submitted. They also had to submit their final version to an online plagiarism and originality checker. Several students were required to rewrite their assignments. Although my child has not taken a math course in the first semester (the second semester begins after the writing of this message), I started considering how the use of AI may impact math classes. This has led me to reflect on how I use and have used technology.

When I was a student, calculators were only rarely allowed in most Grades 7 and 8 math classes. In high school, calculator use was much more accepted. I did, however, have a teacher who taught us how to use a slide rule. Graphing calculators were not a thing, and the early computer labs were for business students only. Years later, as I began teaching Grades 7 and 8, I taught with some teachers who believed that if students were allowed to use calculators, they would never learn how to do basic math calculations. Being a new teacher, I followed that lead until I began questioning assessments in which students clearly understood a concept, but made calculation errors. They were adamant that students needed to use their own brains, since "they will not always have a calculator with them."

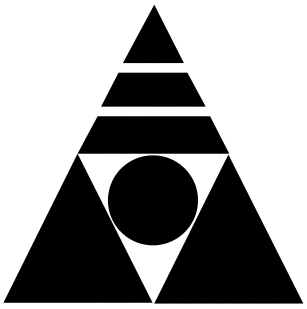
Fast forward to today. Most adults and many students do have a calculator with them in the form of an application on a phone. Students are taught how to use calculators in school. To clarify, basic computational skills are still taught, practised, and assessed, but calculators are used when needed as an appropriate tool when solving problems in many areas of mathematics when computation is just one small part of the goal of the activity. In fact, they are necessary tools, especially at the secondary level, since teachers no longer carry around books of trigonometry and logarithm tables. Students still use their number and operational sense to ensure that the answer the calculator produces is reasonable. After all, errors can still occur when entering the data.

But the conversation has now shifted to the use of AI in the math classroom. AIs can now solve equations. Does this mean that we need to stop teaching students how to solve equations? No, just as we did not stop teaching students how to do basic computations. But perhaps teachers need to rethink the types of questions that are being posed. If questions rarely reach beyond rote computation, then we need to dig deeper. Rote computations rarely lead to deep understanding of concepts and ability to solve complex problems using modelling and reasoning. If, as math educators, we can leverage AI instead of fighting it, it may be possible for students to eventually create complex mathematical models that were once out of reach. The challenge then becomes how to ensure that students are still "using their brains," while taking advantage of what the technology can do for them (that perhaps doesn't really "use their brains" terribly much).

All educators do need to work together to adapt and evolve. Some ways to begin to explore this technology, that is really in its infancy, include, but are not limited to: reading research on the topic, participating in a research study if one is available, contacting school board resource people or post-secondary educators, sharing ideas across panels, and trying out some of the AI applications. When teaching students about AI, educators need to ensure that students understand that AIs are not infallible and do make errors. They need to teach students how to use AI appropriately and ethically. After all, it is here to stay, and it has already proven to be a useful tool in many fields, including education.

Lately, I have been exploring a few different AIs in the creation of three-part math lessons. Some of the resulting lessons were very well explained and planned, while others were not. But there were some very interesting and effective activities within the lessons that I had never tried previously. I was able to input a grade-level text that the AI simplified to a lower reading level. This opens up a world where learning can be personalized and targeted, depending on student need. When trying things, however, there was also a lot of information that was not specifically based on the Ontario curriculum or assessment practices. To that end, OAME/AOEM has been exploring the creation of an Ontario-focused chatbot in the area of mathematics to support teachers with instructional pedagogy. While it is still in its early beta form, it is hoped that it will eventually be shared with our members.

Technology is experiencing exponential growth. It is up to us, as educators, to learn the new technology so that we can better prepare our students for jobs that may not even exist yet. This is a very lofty goal, but throughout the years, we have proven that we are ready to learn and can adapt to change. And that, in our world, is the first thing we need to teach. ▲



OAME/AOEM is starting the process of looking for a new Executive Director(s). The position would begin Sept. 1, 2025. The position includes, but is not limited to:

- managing all correspondence related to OAME/AOEM
- maintaining the membership records: reminding when expiry dates are looming, adding new members or information, sending receipts
- managing finances and budget of the organization from year to year and working with the accounting firm to prepare annual statements
- arranging for the printing and mailing of the *Ontario Mathematics Gazette* quarterly
- organizing and helping to plan all meetings through the course of the year
- being a liaison with OAME/AOEM's local chapters and board committees
- assisting with visioning and forward planning of the organization
- attending annual conferences and staffing a table in the sponsors area

The successful candidate(s) will be provided with training during a transition period.

Anyone interested in being considered for the position should contact Eds@oame.on.ca.

A Tribute to Pat Kehoe

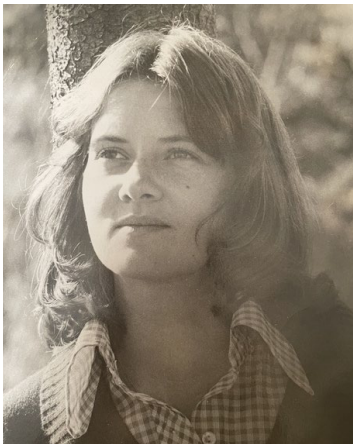
BY MELISSA BLACK AND CATHY HALL

It is with heartfelt sadness that we, on behalf of OAME/AOEM, bid a final farewell to a remarkable soul—our treasured friend and colleague, Patricia Kehoe. On November 16, 2023, Pat left us after a lengthy and valiant battle with esophageal cancer. We remember her fondly through this tribute as we celebrate her life… and her profound impact on mathematics education.

The obituary posted by her family, while sad, also brought a smile to our hearts. It was so warm, heartfelt, and real… just like Pat herself!



Sliding into third with her tea in one hand and the adoration of her six grandchildren in the other, Patricia Ann Kehoe succumbed to being too fabulous for this world. She will be ached for by her children, Shannon (Tyson Charlebois), Jim (Emma Kehoe), and Mikey… as well as her family—John Swain (Kim), Mary Swain, Peter Swain (Gisele), Nora Swain (Chis), Jane Proctor (Bernie)… and her many wonderful friends.



And Pat had many dear friends, both in her personal life and within her professional sphere. When she was first diagnosed with cancer and sidelined from all her usual pursuits, we at COMA organized a collection of items for a series of gift baskets. These contained, among other things, a beautiful handmade quilt and a selection of carefully chosen items meant to provide Pat with both physical and emotional comfort. We wanted her to feel the love and support of her dear friends within the Ottawa math-education community. Throughout her illness, during the good times and the bad, Pat maintained her positive attitude and thoroughly enjoyed heartwarming visits with former students, friends, and colleagues. She even embarked on some domestic and international travel while her cancer was in remission. Ever optimistic, she was determined to grab all the gusto she could!

Pat truly took to heart the inspirational words inscribed on a card that was tucked into one of her gift baskets. The message was a quotation by Maya Angelou, world-renowned poet, author, activist, and life coach:

"My mission in life is not merely to survive, but to thrive—and to do so with some passion, some compassion, some humour, and some style."

And that is what she did, not just during her illness, but throughout her remarkable life and teaching career!

Pat's impact on the lives of others is mirrored in another pertinent quotation from Maya Angelou:

"A great soul serves everyone all the time. A great soul never dies. It brings us together again and again."

This is what Pat did over and over again within her professional circle. She was an active and well-respected member of OAME/AOEM for more than 25 years. During that time, she served a three-year term as the COMA Chapter Representative to the Board of Directors and two three-year terms as an OAME/AOEM Director. Both of these positions represented a significant amount of volunteer time, travelling to Toronto for Board of Director meetings, and doing follow-up "homework" for various subcommittees and working groups. During this time, Pat also helped the Board of Directors host the provincial OAME/AOEM Conference in 2016 and again in 2018, when she served as Co-Chair of Promotions. She also willingly shared her math expertise as a workshop presenter at two provincial conferences in the early 2000's, when Ontario's newly revised math curriculum was rolled out and facilitators were needed to support its implementation.

One of Pat's most significant contributions to OAME/AOEM was her work on SPaRC (Strategic Planning and Renewal Committee), especially as it pertained to the restructuring of the Board of Directors. This involved a lengthy process of retooling the board's Constitution—not an easy or popular job—but Pat willingly embraced the challenge. Throughout this demanding initiative, she exhibited a knowledgeable and powerful presence within the working committee, as well as fostering a spirit of collegiality and co-operation among all board members when it came time to roll out the necessary amendments and proposals. No matter how tumultuous the discussions and negotiations, Pat showed herself to be a compelling, yet calming, voice of reason!

During her years as a member of our COMA Chapter (now rebranded as $O_3$4ME), Pat held many positions. She served as President, Vice-President, Secretary, Ottawa







Catholic School Board Representative, and Member-at-Large. She helped organize all of our chapter events and could always be found working on-site at those same events. She willingly shared her math knowledge and expertise by facilitating workshops at COMA mini-conferences and forums. She also served on the organizing committee for COMO (Carleton-Ottawa Math Olympics) and made herself available to offer a helping hand at the Saturday competitions. Her networking skills, welcoming manner, and quick wit made her a perfect fit as Co-Chair of the Hospitality/Social Events Committee whenever our chapter hosted the OAME Annual Conference in Ottawa. Most importantly, Pat was always "our touchstone" and "emeritus knowledge keeper" in our local OAME/AOEM chapter. She was passionate about helping.

Her dedication to the OAME/AOEM vision of "excellence in mathematics education" shone through in all that she did! An outstanding leader, her enthusiasm was infectious, and people are proud to share that they have been members of her team—whether within the OAME/AOEM Board of Directors, the $O_3$4ME chapter, the Ottawa Catholic School Board, or further afield in various aspects of provincial curriculum delivery and professional development. As a recipient of our local chapter's Lifetime Membership Award in 2020, she was a positive and powerful role model for those of us who strive to make a difference by promoting our $O_3$4ME Vision: "*Making teaching and learning mathematics accessible for all!*"

Our last words come from the ancient Greek leader, Pericles:

"What you leave behind is not what is engraved in stone monuments, but what is woven into the lives of others."

What Pat Kehoe has left behind is a colourful tapestry made up of milestones, moments, and memories interwoven with the lives of all those lucky enough to enter into her sphere of influence. May her legacy continue to serve as an inspiration for us all!

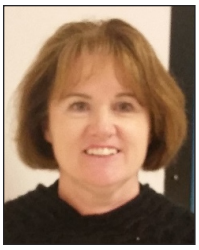
Final Thoughts from Melissa Black…

On a personal note, this sad news hit me hard. I loved Pat. She was such a dear friend and mentor to me all these years. She was the one who made me love algebra, after teaching me in Grade 11 at St. Paul High School. She made me want to go into teaching and to eventually teach high school mathematics at the Ottawa-Carleton District School Board for the past 23 years. I am going to miss her infectious laugh and easygoing nature!

She pushed me to get involved with COMA at the local level… and later supported my decision to work with OAME/AOEM at the provincial level. We had some good laughs, shared some wine, and roomed together for a few years during the OAME/AOEM Board of Directors meetings in Toronto. Whenever I needed some advice about my career or how to navigate through all the ups and downs, I knew I could turn to Pat for her supportive and helpful words of wisdom. I will miss this the most.

An Irish Blessing for you, dear friend…
"May the road rise up to meet you.
May the wind be always at your back.
May the sun shine warm upon your face,
The rains fall soft upon your fields,
And, until we meet again,
May God hold you in the palm of His hand."
Author Unknown ▲

▲ OAME/NCTM Report: Canadians Give True Contributions to NCTM!



JACQUELINE HILL
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Jacqueline Hill is a Grades 9–12 teacher of mathematics. She is a Past President of OAME and OMCA, as well as the recipient of the Award for Exceptional Teaching in Secondary Mathematics. She also wrote the "Director's Dialogue" for the Gazette *for several years.*

Sometimes I feel, as an educator and member of OAME/AOEM, our organization is just a wee raindrop in the ocean of what is NCTM. In the last column, I dug deep to find NCTM by the numbers. It was the "T" in NCTM, the "T" in team, that led several Canadian contributors to NCTM to step out of the shadows. In this column, I would like to highlight one such Canadian!

Ron Lancaster!



Ron, who is one of our OAME columnists, is also a prolific author for *The Mathematics Teacher* (with about 150 columns published by NCTM).

In 1994, Ron created "Media Clips" for *The Mathematics Teacher*. Ron co-edited/co-authored the column for three years. "Media Clips" ran for a remarkable 20 years under numerous editors. Ron created "The Mathematical Lens" for *The Mathematics Teacher* in 2004. He co-edited/co-authored the column with Brigitte Bentele (The Trinity School, New York) all the way through to 2019, when the column was retired. Ron and Brigitte helped design a series of full-size posters based on "The Mathematical Lens" columns for classroom walls to showcase the presence of mathematics in our lives.

Ron has presented workshops, three-hour mini-courses, and talks at 70 NCTM Annual and Regional Conferences. His 71st NCTM Conference will be the 2024 Annual Conference in Chicago, where he will be providing all attendees with a math walk in Millennium Park, which starts at the Bean.

For those new to OAME, or new to reading this column, it may seem like a lifetime of work that Ron has shared with both OAME and NCTM. I asked him a few questions that may help others along their journey.

(1) How did you become inspired to begin this work?

Near the start of my teaching career, a colleague gave me a copy of *Smart Soap Bubbles Can Do Calculus* by Dale T. Hoffman, published in the May 1979 issue of *The Mathematics Teacher*. That column resonated with me because of the hands-on activities and physical models. Hoffman showed me the importance of volunteer work, in this case, writing articles for educational journals. That led me along the path to becoming an author for the NCTM and the OAME.

(2) Did you have a mentor to help and encourage you?

In every school I have worked at, I had the good fortune of being surrounded by caring and talented colleagues and administrators, who made me a better teacher. This was especially true at Oakville Trafalgar High School. Because of the mentoring I received from Bob Loree and Gord Ofield, the Heads of the Pure and Applied Sciences Department, I was honoured with two National Awards of Great Merit from the Hilroy Fellowship Program (1983 and 1989). Those awards allowed me to apply for grants from the Sharing Teaching Excellence Program Travel Grants, which I used to speak at conferences in seven provinces. These early experiences of working with teachers motivated me to do more and led to a lifetime of travel and work around the world.

(3) What advice would you give to those starting out to share their ideas?

Anyone who wants to write articles and speak at conferences should do so for the love of helping teachers, learning new things, and building a professional learning community beyond the school where they teach. It is possible that this work will lead to opportunities where you will be paid, but this cannot be the reason for sharing ideas. This may seem obvious, but you will also need to have something to share that goes beyond standard material that teachers can easily find. You will need to constantly push yourself to come up with new ideas and strategies to teach concepts and to engage students. While everyone else is locking themselves into packaging everything they do within the latest trend, embrace these new ideas, but do so by being different, thinking out of the box, and taking time to veer off on tangents. Get off the road that everyone is travelling on and come up with your own ideas.

Again, a big thank you to Ron for taking the time to be highlighted in our NCTM deep dive… from North of the Border.

Continued on page 16

▲ Useful Theories to Know About: The Pirie-Kieren Dynamic Model of Mathematical Understanding



JEFF IRVINE
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Jeff Irvine, Ph.D., has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

In 1988, Susan Pirie (professor emerita, University of British Columbia) and Tom Kieren (professor emeritus, York University) proposed a paradigm-shifting theory of how students learn mathematics. This theory proposed that, rather than learning mathematics in a linear fashion, students learn in a non-linear, recursive, and iterative fashion, repeatedly returning to previous levels of understanding to consolidate and confirm knowledge. This was a revolutionary change in thinking about how students learn, and the theory has major implications for teachers of mathematics. There is abundant research evidence to support this theory, and it is surprising that more mathematics teachers are not aware of the theory and its implications.

The Pirie-Kieren Dynamic Model of the growth of mathematical understanding is shown in Figure 1. The concentric circles indicate that understanding builds from the student's beginning understanding (called primitive knowledge) and becomes increasingly sophisticated as they increase their understanding, while retaining everything that they have learned previously. In the figure, student learning is modelled using the line that traverses the circles. It begins at primitive knowing and follows a path that primarily migrates both outward (for example, during learning), but has periods where it moves inward (for example, during consolidation).

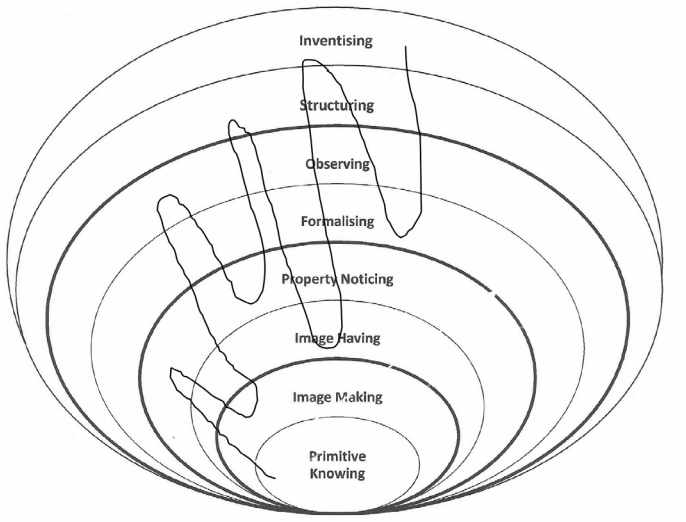


Figure 1: *The Pirie-Kieren Model of the dynamical growth of mathematical understanding*

Primitive Knowing represents the learner's initial knowledge about the topic. The use of the word "primitive" does not imply rudimentary or low-level knowledge; simply, this is the learner's knowledge state prior to engaging. In the next level, *Image Making*, "image" means any representation, including mental, visual, pictorial, and so on. *Image Having* represents a level where actual image making is no longer necessary; the learner can use the image without resorting to explicit image making or specific examples.

At the *Property Noticing* stage, the learner is able to construct context-specific properties by combining attributes of images. Formalizing occurs when the learner abstracts a common attribute or method from the properties noticed in the previous stage. *Observing* allows the learner to express their new knowledge as theorems. At the *Structuring* level, the learner collects appropriate theorems to form a coherent theory. Finally, *Inventizing* involves generating new questions based on a full, rich understanding of the topic, by breaking away from the preconceptions that enabled learners to reach this outer level. It is important to recognize that the levels of the Pirie-Kieren Model are not targets to be met. Rather, they represent stages in the students' understanding. Some students' understanding will progress to the outer levels of the model, while others will grow their understanding, but that growth will reach lower levels of the model.

Two important concepts distinguish Pirie-Kieren from other theories of learning. The first is *folding back*, which identifies learning as non-linear and recursive. Learners fold back to previous levels to solidify understanding and correct misconceptions. This is often represented on the diagram

by a "wandering" line that revisits previous levels of understanding. When they fold back, students retain all the new knowledge that they have gained, which Pirie and Kieren called *thickening*. So, students look at their previous learning through a new and thicker lens, which aids in developing their understanding.

Why Should We (Math Teachers) Care About This Theory?

The most important feature of the Pirie-Kieren theory is the folding back. If students learn in a non-linear, recursive, and iterative manner, teachers need to recognize that fact and adapt their teaching to support their learners. One way to do that is to provide differentiated levels of support, depending on the level at which a student is currently learning. Irvine (2017) provides one model by which teachers can do this. Differentiated instruction is not new. However, differentiation to support a non-linear model of learning is something that teachers may not be familiar with, and for which they may need support. Many math teachers assume a linear learning model by which a new topic is dealt with each day, and yesterday's topic is presumed to have been understood and mastered completely. The Pirie-Kieren model says that this is not the case, and that for real understanding, students need to revisit prior learning, to consolidate and to address misconceptions and false starts. The Pirie-Kieren model is a constructivist theory of learning. This means that students construct their own learning by adding onto or modifying learning that had previously been addressed. Through folding back, the Pirie-Kieren model illustrates how student understanding grows and becomes richer through repeated visits to prior learning.

If You Want to Learn More

A model of teaching to support the Pirie-Kieren model of learning is found in:

Irvine, J. (2017). From description to prescription: A proposed theory of teaching coherent with the Pirie-Kieren model for the dynamical growth of mathematical understanding. *Journal of Instructional Pedagogies, 17*, Article 5. www.tinyurl.com/yd5pyr25

Folding back is the major paradigm shift, where details can be found in:

Irvine, J. (2023). The Pirie-Kieren model: The critical concept of folding back. *Journal of Instructional Pedagogies, 29*, Article 13, 1–18. www.researchgate.net/publication/375071645_The_Pirie_Kieren_dynamic_model_of_the_growth_of_mathematical_understanding_The_critical_concept_of_folding_back

The original paper presenting the theory is:

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For the Learning of Mathematics, 9*(3), 7–11. www.flm-journal.org/Articles/339B51168E57EF2E1E42175EA8147D.pdf ▲

▲ Events at the Fields: Margaret Sinclair, Destreaming, and the Canadian Mathematical Society Education Meeting



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Pamela Brittain (Ph.D.) is an educator, author, illustrator, presenter, and math lover, who currently works for the Fields Institute. She has been involved in STEM education for over 20 years and strives to make math accessible and enjoyable for everyone.

It's been a busy few months at Fields, with some exciting presentations taking place at the MathEd Forum, including the Margaret Sinclair Memorial Award, talks about implementing destreaming in Grade 9 math, and the inaugural Online Math Education Meeting, hosted by the Canadian Mathematical Society (CMS).

September MathEd Forum—Margaret Sinclair Memorial Award

September brought us the 2023 Margaret Sinclair Memorial Award presentation by Dr. Lynda Colgan, Professor Emerita in the Faculty of Education, Queen's University, and long-standing *Gazette* columnist. Lynda's talk was entitled "Full Circle: My Journey Through Mathematics Education and the Unstoppable Path Back to Kindergarten."

The presentation began with an introduction from Margaret's family, an announcement that a donation has been made to continue the award in perpetuity, and that the criteria have been changed to allow opportunities for early year educators to be acknowledged. Lynda was then presented with the award by the Deputy Director of Fields, Dr. Deirdre Haskell (see Figure 1).

Lynda's talk focused on her journey with mathematics, including her connections to Margaret Sinclair herself. Lynda spoke about how her own personal passion for mathematics started in Kindergarten, where students explored how straight lines on a paper, when rolled, can form spiral diagonals—and she wanted to know why. This journey continued with opportunities to ride a square-wheeled bicycle, appearing on *Reach for the Top* (hosted by none other than the late, great Alex Trebek), and how her time as a student in the teacher education program changed her perspectives on mathematics and how to teach it.



Figure 1: *Deputy Director Deirdre Haskell presents Dr. Lynda Colgan with the 2023 Margaret Sinclair Memorial Award (Photo by permission of the Fields Institute for Research in Mathematical Sciences)*

One of the main lessons of Lynda's talk was that mathematics can be made accessible to all, and that younger students can grasp complex mathematical ideas, as long as they are presented effectively.

Lynda joined the faculty at Queen's in order to inspire the next generation of teachers, and she has made many other amazing contributions to the world of mathematics education, including the creation of an optional enrichment course for teacher candidates (complete with Sierpinski kites); her contribution of a math column for a Kingston newspaper; the creation of online resources; the writing of the book *Mathemagic! Number Tricks*; contributions to the TVO show, *The Prime Radicals*; creating math kits and story prompts for parents; participating in math podcasts and radio interviews; and much, much more—including being the current Executive Director (Education and Development) for Science Rendezvous.

Finally, we learned of her "full circle" moment when she went from an elementary student, passionate about mathematics, to a secondary teacher, to an elementary teacher, and finally to being involved with the Early Math program, bringing her back to what she believes is the hardest and most important job in mathematics, teaching Kindergarten.

Lynda's presentation was both inspiring and informative and brought to life Lynda's dedication to mathematics education. It showcased why she deserved to be named the 2023 Margaret Sinclair Memorial Award winner.

The September Forum was rounded out by three additional presentations. The first by Julie Houle (Lester B. Pearson School Board, Quebec) and Helena Osana (Concordia University) on "NyKK-5: Educator-Researcher Collaboration to Support Teachers' Early Numeracy Practice in Kindergarten" (where "NyKK-5" stands for "Numeracy Kit for Kindergarten 5-year-olds"), who spoke on why young children's mathematics is far more sophisticated than we think; is independent of where you are born, language, race, sex/gender, etc.—it universally contributes to our growth, starting from birth. The second by Heather Douglas (Carleton University) and Tracie Anthony (Grande Prairie Public School Division, Alberta) on "The Development and Implementation of the Early Math Assessment at School" looked at the question of how to evaluate and assess early learners' (K–8) approach to mathematics. What are those critical skills? What does the research tell us? And how predictive are these early skills into later math achievement? Finally, the third presentation was by Dr. Jo-Anne LeFevre (Carleton University) on "Connecting to Advance Assessment and Instruction in Math (AIM) education: The AIM Collective," where we learned a bit more about the AIM Collective and the importance of building collaborations among researchers and practitioners in math education.

These additional talks rounded out Lynda's presentation quite nicely and featured many of her colleagues and collaborators, who spoke on issues affecting the teaching of mathematics to early learners.

Recordings of these talks can be found at www.fields.utoronto.ca/activities/23-24/meforum-Sep.

October MathEd Forum—Destreaming

Dr. Doug McDougall, leading a team of researchers from the Ontario Institute in Studies for Education and the Toronto District School Board, has been looking at the destreamed Grade 9 math curriculum. The goal of the project is to understand how best to implement the 2021 Ontario Grade 9 mathematics curriculum and involved teachers from both Ontario and Alberta. The Alberta members, from Calgary, were included because Alberta has had a destreamed Grade 9 for about 15 years, given the structure of their middle school, which includes Grades 7–9.

This group created a series of online resources and then used case studies (semi-structured interviews, surveys, and document reviews) to review and evaluate them. The group used ten dimensions to evaluate the resources, including everything from the learning environment, communication with parents, assessment, and teacher attitudes and comfort with mathematics. The survey showed a lack of awareness by parents that the curriculum had changed and, for those who knew of the changes, concerns about what would happen to high-achieving and struggling students when placed in the same class. There were also concerns by teachers about how they could effectively implement the

resources that were provided.

Based on the survey, Dr. McDougall and his team created teacher workshops to explain methods of using the resources (for example, spiralling and how to implement it), along with some coding workshops (at the request of the teachers).

The day ended with a panel discussion of "stories from the ground" of how destreaming in Grade 9 has been experienced by the teachers teaching it, and an open discussion on destreaming and its multiple effects and considerations.

Recordings of these talks can be found at www.fields.utoronto.ca/activities/23-24/meforum-Oct.

November MathEd Forum – CMS Online Math Education Sessions

November was unique because MathEd Forum participants were invited to participate in the inaugural Online Math Education Meeting, hosted by the Canadian Mathematics Society. This two-day event brought together math educators and researchers from across Canada for an online conference. Topics presented included resources for teaching mathematics, puzzles and games, exploring the historical importance of where many of our mathematical concepts come from, and much more. Participant feedback on the event was positive, and we look forward to joining with Canadian Mathematics Society again next year. ▲

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▲ Meaning Making in the Classroom: The Importance of Having a Toolbox of Problems



DAVID COSTELLO
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David is a principal, who taught and had various board roles in Prince Edward Island. David has also instructed university courses regarding curriculum, differentiation, mathematics, and literacy. He facilitates professional learning in mathematics and school development. Dr. David Costello recently published four books: Messing Around with Math: Ready-to-Use Problems That Engage Students in a Better Understanding of Key Math Concepts, Mathematizing Student Thinking: Connecting Problem Solving to Everyday Life and Building Capable and Confident Math Learners, Making Math Stick: Classroom Strategies That Support the Long-Term Retention of Math Concepts, *and* Using What Works: Strategies for Developing a Literacy-Rich Environment in Math.

Teachers spend much time considering the mathematical idea(s) they are working with in the classroom. They examine the foundational aspects of the concept(s) so that they have an understanding of how to approach them in the classroom and what to look for in terms of student understanding. Teachers also take time to consider common stumbling blocks that students may encounter when working with the idea(s). To know the concept well is to be prepared in supporting student understanding.

What I am proposing, in this column, is the notion that not only do we, as teachers, have to consider the foundational aspects and common stumbling blocks associated with concept(s) being explored, but we must consider the types of problems we are assigning as a means to support student conceptual understanding. When reflecting on the questions you assign, consider the following (Costello, 2024):

- Where do you find the problem sets?
- Do the problems have similar structures?
- Why do you select these particular problems?
- Do the problems move learning forward, or are they busy work?
- Are the problems helping students meet learning goals?

The problems assigned in class influence the experience of students and their perspective of math (Costello, 2024). First, how open or closed a problem is will dictate how students can enter the problem-solving process and come to a conclusion. It will also impact the ability of the teacher to meet the various learning needs of students in the classroom. Second, the level of student engagement with the concept(s) will be influenced by the choice of problem assigned to students. Third, the type will be an opportunity for students to either problem-solve or practise. We must be cognizant that if students habitually encounter the same problem type, there is a likelihood that the experience becomes more like practice than problem solving. Fourth, the greater the variety of problems being assigned to students, the greater the likelihood that students broaden their understanding of the math concept(s).

Regardless of the mathematical concept that you are exploring, you can assign different problem types. By doing this, students will encounter the concept from differing perspectives, thereby strengthening both their understanding of the concept and their problem-solving abilities.

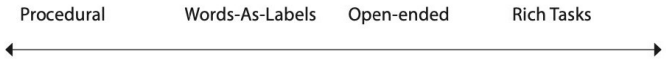
In this column, I discuss a selected mathematical concept from each of the following grade-level ranges: Kindergarten, Primary (Grades 1–3), and Elementary (Grades 4–6).

Kindergarten	Represent numbers in a variety of ways.
Primary (Grades 1-3)	Identify and describe patterns consisting of numbers to 1000.
Elementary (Grades 4-6)	Find the perimeter, area, and volume.

For each concept, I will provide an example of different problem types so that the reader can see that changing the problem gives students different encounters with the concept.

Toolbox of Problems

Consider the following diagram (Costello, 2022):



Procedural problems are problems where the strategy has already been identified. Students are required to apply a series of steps accurately to find the solution. *Words-as-Labels* problems are word problems, or story problems, whereby there is a context provided to students to work within. *Open-ended* problems are problems that have multiple entry points and multiple possible solutions so that students have choice in which strategy to apply to arrive at one of the many possible solutions. *Rich tasks* are similar to open-ended problems, with the only difference being that rich tasks are presented within a realistic context for students.

For the purpose of my work with various problem types, I reference this collection as a toolbox of problems. Similar to a carpenter or mechanic with a variety of tools in their toolbox, it is important for teachers to have access to a variety of problems, to address the targeted mathematical concept(s).

Kindergarten

Represent numbers in a variety of ways.	
Procedural Problems	Draw a picture of seven balloons.
Words-as-Labels Problems	There are three empty parking spots in the school bus lot. Draw a picture of a school bus for each empty parking spot.
Open-ended Problems	There are fewer empty spaces than filled spaces on a ten-frame. What is the number?
Rich Tasks	The world we live in can help us make sense of math. Think about things in our community. Where do we see numbers? What are things that we count? Make a book about numbers in our community.

Primary (Grades 1–3)

Identify and describe patterns consisting of numbers to 1000.	
Procedural Problems	Identify the error in the following pattern: 87, 85, 83, 81, 79, 78
Words-as-Labels Problems	There were 21 players in the line. The coach asked every third player to raise their hand. Represent this repeating pattern, using letters.
Open-ended Problems	Numbers in a pattern increase by a small amount. What could the first six numbers in the pattern be?
Rich Tasks	Sammy started fundraising with little money. Sammy received the same donation from each business she visited. Write a number sequence for how much Sammy had after each visit. What is the pattern rule?

Elementary (Grades 4–6)

Find the perimeter, area, and volume.	
Procedural Problems	Find the volume of a cube with a side length of 7 cm.
Words-as-Labels Problems	The community maintenance person needed to buy fencing for a playground. The dimensions of the rectangular playground are 26 m by 20 m. What is the perimeter of the playground? How much fencing is needed to fence the entire playground?
Open-ended Problems	The area of a rectangular stage is 96 m^2. What could the length and width of the stage be?
Rich Tasks	A student told his partner that there are four possible arrays for an area of 32 cm^2. Is the student correct? How do you know?

Take-Away

As can be seen within the three tables above, each of the four problem types addresses the mathematical concept. One is not better than the other. What is important is that students have opportunities with all problem types so that they are presented situations that are not always the same. And, as a teacher, by offering different problem types, we are provided with opportunities to assess student understanding of the mathematical concept(s) from a variety of perspectives, thereby increasing the likelihood that instruction will periodically match student learning needs.

I am not advocating for one problem type more than another. There is no magic formula as to the number of problems that should be assigned for each problem type. What I am advocating for, however, is that teachers assign a variety of problem types so that conceptual understanding can be strengthened.

Final Thoughts

The intent of this column was to highlight the importance of offering different problem types to students. When we, as teachers, rely on the same type of problem, we limit the opportunities for students to engage with the mathematical concept(s) from different perspectives, and we reduce their experiences as problem solvers. The four types of problems that I offered for each concept will provide students with opportunities to engage concepts from different perspectives, whether that is the openness of the problem or the relatability of the problem. By offering these four types of problems, we stand a better chance to address the range of learning needs in our classrooms and we, as teachers, are provided with a greater likelihood to gain a more detailed understanding of student learning.

I look forward to continuing the discussion of mathematical learning and instruction via my website (www.costellomath.com), email (david@costellomath.com), or X (@dr_costello).

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▲ MB4T (Mathematics by and for Teachers): A Discussion of Measurement



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Jennifer Holm is an Associate Professor at Wilfrid Laurier University and works with Primary/Junior and Junior/Intermediate pre-service teachers, as well as in the field supporting current mathematics teachers. She is interested in developing mathematics knowledge for teaching with both pre-service and in-service teachers. She focuses on the beliefs and opinions that pre-service teachers hold about mathematics and teaching and the connection they have to past experiences. She uses this research to support future teachers in developing beliefs and knowledge that will encourage and support effective teaching practices in mathematics.

As a teacher, I always enjoyed the measurement unit of the curriculum because I found so many wonderful hands-on ways to teach the concepts. Students were engaged because they were learning something practical that they could envision being used in real life, and the lessons were engaging due to the tactile explorations of concepts. However, I struggled with finding a way to effectively teach about conversion between units and why this was important to learn. Having taught school in the United States early in my career, I must admit that being able to use the metric system is much simpler, but there are still some challenges with ensuring understanding as opposed to memorization of how the different prefixes relate to each other.

A favourite problem I like giving to pre-service teachers, to help them think about conversion between units and how to teach it, is as follows: How many 200 cm^2 tiles fit in an area of 0.6 m^2? The lead of the courses at Lakehead University, Dr. Ann Kajander, first introduced me to a similar problem to think about metric conversions and the extent of its possibilities.

Take a moment to consider the problem and how you or your students might solve it. Would it surprise you to learn the most common answer I receive is zero? Take a moment to consider why students might answer this way, and think about ways you might help a student visualize this problem instead of memorizing conversions. How might you, or your students, use concrete materials to solve this problem and ensure that there are no misconceptions?

Laying the foundation for measurement units should include consideration of how we treat the units. I like to emphasize the fallacy of the idea that when working with units, you just remove them from the problem and add them in at the end. I find this notion problematic for so many reasons, but the first is the potential missed opportunities to lay a foundation for algebra. Consider the rectangle in Figure 1. To calculate the perimeter, students could just use the formula base on adding the sides: 2 + 2 + 4 + 4 = 12. Since this is a measurement, what does the sum mean? What size is the rectangle really? Working with children, they often ask how they know what the unit is at the end, or will just add the same letters as they see in the problem, without really thinking about it. Consider instead if they did the following: 2 *cm* + 2 *cm* + 4 *cm* + 4 *cm* = 12 *cm*. Now you can discuss how adding different centimetre measurements together would generate an end result that is also in centimetres. Although it seems simple, this can help set up a foundation for finding the area and being able to determine the proper unit: 2 *cm* × 4 *cm* = 8 cm^2. Since the units are being multiplied together, then centimetres squared would be correct. If teaching young students and understanding cm × cm = cm^2, using manipulatives would demonstrate that the area is made up of many smaller squares; thus, cm^2 describes this.

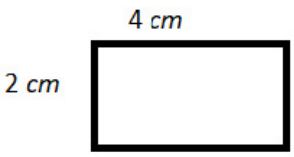


Figure 1: *Rectangle in centimetres*

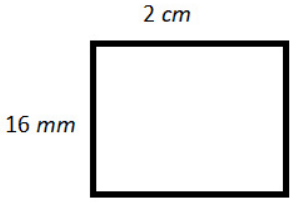


Figure 2: *Rectangle in centimetres and millimetres*

If we consider a rectangle like Figure 2, we can have a different conversation. Students who are not asked about using the units may do something like the following (where they erroneously interpret 16 mm as 160 cm): 2 + 2 + 160 + 160 = 324. Then, when it comes to the units, will either leave it off or be confused about what to do to complete the problem. A student who thinks about the units may set up the question as such: 2 *cm* + 2 *cm* + 160 *mm* + 160 *mm* = 2 cm + 320 *mm*. Although not a complete answer, it is mathematically correct, and it lays the foundation for algebra and thinking about combining like terms. If we think about it in algebraic terms as $2x + 2x + 160y + 160y = 4x + 320y$, then it becomes clear why the solution is not $324xy$. If, in measurement, we talk about how we need the units to be the same in order to combine them, then we can help reinforce the idea of combining like terms in the Intermediate grades.

Now to return to our original problem: consider how units may have led students to assume the answer is zero, and how we might use conceptual understanding to support a different outcome for the problem. The overwhelming majority of future teachers who answer this way give the same reasoning: If there are 100 *cm* in 1 *m*, then there are 60 cm^2 in 0.6 m^2. Inevitably, what I learn is that they have memorized the relationship between the prefixes, and have not considered that changing to squared measurements would alter the relationship. Since my students have seen what happens with the centi- and milli- (e.g., 100 milli*grams* is equal to 1 centi*gram*, just as if it were *metres*), there is an erroneous assumption it will always apply. Here is where keeping the units during the calculations can help with the understanding: having seen that cm^2 is created by multiplying a centimetre measurement by another centimetre measurement, it is evident that the metre squared part is not just appended to a centi- prefix. To now solve the problem, we could reinforce the memorization of the different relationships on how many cm^2 in a m^2 (i.e., 1 m^2 = 10 000 cm^2), or we could help students reason through the problem, using pictorial methods.

A good first pictorial method is to determine the conversion relationship if students only remember there are 100 *cm* in 1 *m*. Figure 3 shows an image that could help in determining this relationship instead of memorizing it. If we consider a m^2 as being one metre by one metre, then the same square would be 100 centimetres by 100 centimetres. By this logic, then 1 m^2 = 100 *cm* × 100 *cm* = 10 000 cm^2. Using this conversion, the problem can be solved.

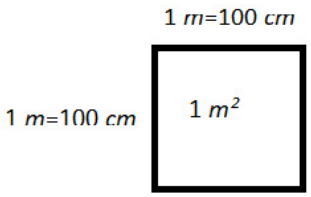


Figure 3: *Determining how many centimetres squared in a metre squared*

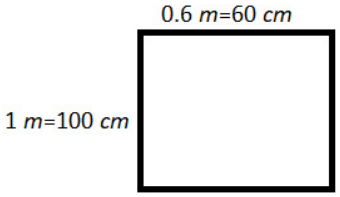


Figure 4: *Converting 0.6 m^2 to cm^2*

A second method that may be used is to try to draw out the problem and then solve for the number of tiles. If you consider that a 0.6 m^2 space could have the dimensions of 1 *m* by 0.6 *m*, then an image like Figure 4 could assist with solving the problem. By first setting up the side lengths in metres, you can then convert them to centimetres. This could tell you the entire area by multiplying the two sides together: 0.6 m^2 = 100 *cm* × 60 *cm* = 6000 cm^2. If you then divide by the size of the tile (200 cm^2), the answer would be 30 tiles.

Students may also take the pictorial representation a step further. Starting with the image in Figure 4, they may draw something like Figure 5 with a hypothetical tile size. If students propose the side lengths of the tile as 100 *cm* by 2 *cm*, you can draw the tiles on the image to determine the solution of (60cm/2cm-per-tile = 30 tiles) 30 tiles.

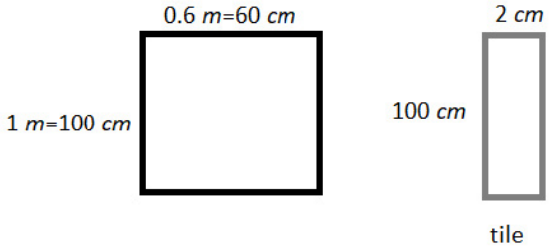


Figure 5: *Solving the tile problem with an image*

In the end, making sure that students do not omit the units until the final stage and stick them on the answer can lead to a deeper understanding of measurement concepts and help them to reason through different problems. This can also help by not having students memorize more conversions because area conversions can be derived. It is still helpful for students to see the difference in size of 1 m^2 compared to 1 cm^2 so that it becomes clear that there are 100 cm^2 in the shape. This could also extend to the third dimension, helping students see the difference in cubic measurements as well (1 m^3 = (100 cm)^3 = 1 000 000 cm^3). Taking the time to lay the foundation and allowing for students to use pictorial representations and reasoning to solve problems supports a stronger development of mathematical concepts. ▲

Continued from page 8

NCTM offers virtual and in-person conferences.

- **NCTM Virtual Conference:** April 10–13, 2024
- **National Council of Teachers of English (NCTE)** and **NCTM Joint Conference** on K–5 elementary literacy and mathematics in New Orleans, Louisiana, June 17–19, 2024
- **NCTM Annual Conference** in Chicago, Illinois, September 25–28, 2024
- **NCTM Regional Conference**, Kansas City, Missouri, February 5–7, 2025

Until next time!
Yours in mathematical fun,
Jacqueline▲

▲ What's the Problem? Playing with Processes: Part 3



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Shawn is a former mathematics teacher, department head, and consultant. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: presenting workshops, writing articles, working on local projects, and helping create mathematics contests.

Welcome back, problem solvers. In the last two issues (Godin, 2023a, 2023b), we started a four-part series focusing on the mathematical processes. We continue in this issue with the following problem from your homework:

> *You play a game where you have 15 piles containing 1, 2, 3, …, 15 counters, respectively. On each move, you may choose any subset of piles and take the same number of counters from each pile. For example, on your first move, you could remove 5 counters from each of the piles with 6, 9, 11, and 14 counters (leaving 1, 2, 3, 4, 5, 1, 7, 8, 4, 10, 6, 12, 13, 9, 15 counters). What is the fewest number of moves required to remove all the counters?*

This problem was problem 34 from *The Inquisitive Problem Solver* (Vaderlind et al., 2002). This is a lovely problem book. There are 256 problems, some of which have hints, and all of them have solutions. Many of the solutions are augmented with further queries for the reader to consider. These queries are also answered and, in some cases, the authors encourage the reader into further avenues of exploration. As one who adores "playing" with problems, this book is close to my heart.

As in the two last issues, we will keep in mind George Pólya's four steps in problem solving from his book *How to Solve It* (Pólya, 1957). Here they are as a reference, with some of his guiding questions, which are reproduced below in italics. As in the last issue, relevant mathematical processes are noted in **bold** within parentheses throughout the column.

1. UNDERSTAND THE PROBLEM. It is impossible to solve a problem if you don't understand what is being asked, the notation, or terms in the problem. This involves, in some cases, accessing prior knowledge. (**Reflecting**)

2. DEVISE A PLAN. (**Selecting Tools and Computational Strategies, Representing, Reasoning and Proving**) *When have you seen it before? Think about a related problem.* (**Reflecting, Connecting**)
3. CARRY OUT THE PLAN. As you work through each step of your plan: *Describe how you know that step is correct?* (**Reflecting, Communicating**) *How can you prove that it is correct?* (**Reasoning and Proving**)
4. LOOKING BACK. *How can you check the result?* (**Reflecting**) *Can you show this result differently?* (**Reasoning and Proving**) *How could you use the result, or method, for some other problem?* (**Connecting**)

In this issue, I want to focus on **Connecting**. Connecting is about seeing how things tie together. This tends to manifest itself in two ways: how certain problems are tied together, or related, and how mathematical concepts are intertwined (many times in a mysterious way). For example, in analytic geometry, there is a link between algebra and geometry, two seemingly unrelated fields. These two categories are highlighted in the rubric for Connecting from the Ministry of Education (Ontario Ministry of Education, 2008), shown in Figure 1.

We are ready to tackle our problem, using Pólya's four steps.

1. Understand the problem. We are given the starting conditions, the rules for each move, and we are searching for the quickest solution. Nothing too deep and not much prior knowledge, other than how to tackle a problem, is required.
2. Devise a plan. This is one of those problems where we have either seen something like it or not. If we have seen something similar, we can explore the other problems to see if anything in its solution can be useful (**Reflecting, Connecting**). If this is a new situation, it might be best to play with the problem and see what we can find. We will do this by looking at easier problems first to see if that leads us to a pattern (**Selecting Tools and Computational Strategies**).
3. Carry out the plan. Let's start with five piles containing 1, 2, 3, 4, 5 counters, respectively. This is a great problem to use some manipulatives to help us out (**Representing**). Counters or linking cubes would work well. It should be apparent that we can remove all the piles in five moves (**Reasoning and Proving**). This can be done trivially by removing 1 from the first pile, then 2 from the second, and so on, in any order.

However, if we continue to play, we will see there are some other, more interesting ways to remove all the counters in five moves. Suppose at each step we removed 1 counter from each pile, then we would get

$$1,2,3,4,5 \rightarrow 1,2,3,4 \rightarrow 1,2,3 \rightarrow 1,2 \rightarrow 1 \rightarrow \text{done}.$$

At each step, a pile is either removed—when it has 1 counter in it—or it is reduced to a pile of a smaller size. What happens if we removed 1 counter from just the piles with 2, 4, and 5 counters, respectively? We would get

$$1,2,3,4,5 \rightarrow 1,1,3,3,4.$$

Aha! Now, if we remove 1 counter from the first two piles and the last pile, we get

$$1,1,3,3,4 \rightarrow 3,3,3.$$

We can then finish by removing 3 counters from each of the piles, meaning we did the whole process in three moves, an improvement from five.

Where did our improvement come from? Notice that, in this case, when we did our moves, we ended up with multiple piles with the same number of counters in them. This is an advantage because we can then remove several piles in one move. Can we use this to improve our results? Let's think about it logically.

We started with five piles with unequal numbers. In the first step, we can remove one of the piles, which reduces us to four piles. However, if we are lucky or careful, the other

Connecting				
Criteria	Level 1	Level 2	Level 3	Level 4
Makes connections among mathematical concepts and procedures	Makes weak connections among mathematical concepts and procedures	Makes simple connections among mathematical concepts and procedures	Makes appropriate connections among mathematical concepts and procedures	Makes strong connections among mathematical concepts and procedures
Relates mathematical ideas to situations drawn from other contexts	Transfers ideas to other contexts and makes limited connections	Transfers ideas to other contexts and makes simple connections	Transfers ideas to other contexts and makes appropriate connections	Transfers ideas to other contexts and makes unique, original, or insightful connections

Figure 1: *Rubric for Connecting*

four piles might be reduced to piles of two different sizes. For example, if we started by taking 1 counter from piles 1, 3, 5, we would get 1,2,3,4,5 → 2,2,4,4, and then we can finish in two more moves by either removing the 2's then the 4's (or vice versa), or by removing 2 counters from the piles with 4 (or all the piles!), and then removing 2 from all the remaining piles. It seems at each move, the optimum strategy either cuts the number of different numbers in half, or removes one or more groups with the same number, and then cuts the number of different numbers in half. Notice if we had started by removing 3 counters from piles 3, 4, 5, we would get 1,2,3,4,5 → 1,2,1,2, which we can again finish off in three moves total (**Reasoning and Proving**).

Let's try a larger number (**Reflecting**, **Connecting**). If we start with ten piles and remove 1 counter from each of the piles with an odd number of counters, we get

1,2,3,4,5,6,7,8,9,10 →2,2,4,4,6,6,8,8,10.

Next, if we remove 2 counters from the piles with 2, 5, and 10 counters, we get

2,2,4,4,6,6,8,8,10→4,4,4,4,8,8,8,

which can be completed with two more moves, four in total.

At this point, I hope your "spidey sense" is tingling. In the last example, we removed 1 counter from some piles, then 2 counters from some piles, and we were left with piles of 4 and 8. Each of these numbers is a power of 2 (**Reflecting, Connecting**). Let's see what happens when we unleash the powers of two on the original problem. First, we remove 1 counter from the piles with the odd numbers (the first, then every second one), yielding

1,2,3,4,5,6,7,8,9,10,11,12,13,14,15
→ 2,2,4,4,6,6,8,8,10,10,12,12,14,14.

Since we will treat piles with the same number of counters the same way, we will eliminate redundancy and only record numbers to represent we have at least one pile with that many counters (**Representing, Communicating**). Thus, removing 1 from every second pile, then 2, then 4, and so on, leads to

1,2,3,4,5,6,7,8,9,10,11,12,13,14,15
→ 2,4,6,8,10,12,14 → 4,8,12 → 8 → done.

This means that we can solve our original problem in four moves, the same as when there were 10 counters. The reader might like to verify that we could also do the process in four steps if we first took 8 counters from all piles with 8 or more, then 4 from all piles with 4 or more, and so on.

5. LOOKING BACK. Let's take a deeper look at our solution and see its connection to other areas of mathematics (**Reflecting**). It turns out our solution depends on the binary representation of the number of counters in each pile (**Connecting**). Binary, or base-2, is used quite often in computer science due to the physical way a computer represents numbers, with "switches" being turned on or off to represent the binary digits 1 or 0, respectively. In binary, we represent numbers as sums of multiples of powers of 2, just like in decimal, where we use powers of 10. However, while in decimal, we use the ten digits—0, 1, 2, 3, 4, 5, 6, 7, 8, 9—as our multipliers of powers of 10; in binary, our digits are just 0 or 1. Hence, the decimal number 123 would be written as 1111011 in binary, since

$$123 = 1 \times 10^2 + 2 \times 10^1 + 3 \times 10^0$$
$$= 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 21 + 1 \times 2^0.$$

Let's see what happens to the numbers in our piles when we write the numbers in binary. In the table, the number in brackets is the decimal representation.

Start	Remove 1 (1)	Remove 10 (2)	Remove 100 (4)	Remove 1000 (8)
1 (1)	0 (0)	0 (0)	0 (0)	0 (0)
10 (2)	10 (2)	0 (0)	0 (0)	0 (0)
11 (3)	10 (2)	0 (0)	0 (0)	0 (0)
100 (4)	100 (4)	100 (4)	0 (0)	0 (0)
101 (5)	100 (4)	100 (4)	0 (0)	0 (0)
110 (6)	110 (6)	100 (4)	0 (0)	0 (0)
111 (7)	110 (6)	100 (4)	0 (0)	0 (0)
1000 (8)	1000 (8)	1000 (8)	1000 (8)	0 (0)
1001 (9)	1000 (8)	1000 (8)	1000 (8)	0 (0)
1010 (10)	1010 (10)	1000 (8)	1000 (8)	0 (0)
1011 (11)	1010 (10)	1000 (8)	1000 (8)	0 (0)
1100 (12)	1100 (12)	1100 (12)	1000 (8)	0 (0)
1101 (13)	1100 (12)	1100 (12)	1000 (8)	0 (0)
1110 (14)	1110 (14)	1100 (12)	1000 (8)	0 (0)
1111 (15)	1110 (14)	1100 (12)	1000 (8)	0 (0)

Thus, we need one step to remove each of the powers of 2, including 2^0. Hence, if our problem started with piles from 1 to n, inclusively, then $[\log_2 n] = k$ represents the largest power of 2 in the binary representation of n. That is $2^k \le n < 2^{k+1}$, and hence the binary representation of n has $k + 1$ digits (which are multipliers of $2^k, 2^{k-1}, \cdots, 2^2, 2^1, 2^0$). Therefore, we would need $[\log_2 n] = +1$ moves to complete the problem (**Reasoning and Proving**). In Figure 2, there is a Venn diagram of the numbers 1 through 15, organized into groups corresponding to powers of 2 (**Representing**). You can get a sense of how the numbers will be affected by each move (**Connecting**).

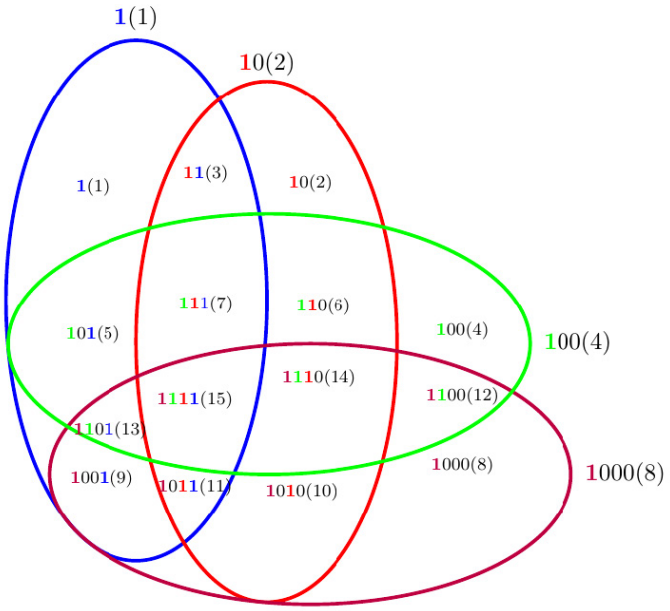


Figure 2: *Venn diagram of numbers containing powers of 2*

We can also make connections to other problems. A classic that shares the binary tie-in with our problem is the following. You have five cards with numbers written on them as follows:

Card 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31

Card 2: 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, 26, 27, 30, 31

Card 3: 4, 5, 6, 7, 12, 13, 14, 15, 20, 21, 22, 23, 18, 29, 30, 31

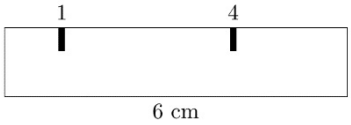
Card 4: 8, 9, 10, 11, 12, 13, 14, 15, 24, 25, 24, 27, 28, 29, 30, 31

Card 5: 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31

To make things interesting, the cards are not numbered, and the numbers are not arranged in numerical order on the cards. You give the cards to someone, and they look for their birthday on the cards. They hand back the cards that have their birthdate and you immediately state the birthday. For example, my birthday is on the 29th, so I would give cards 1, 3, 4, and 5. The question is how can I state the birthday immediately, without having to do a process of elimination? Also, how are the cards constructed (**Reasoning and Proving, Connecting**)? Could you modify the idea to another situation? Have fun with this.

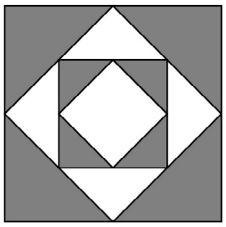
As in the last two columns, here are some problems from previous columns that involve some sort of connection between concepts or problems. Feel free to explore them for yourself and revisit the columns in the OAME archives (in the members-only area).

1. An unmarked ruler is known to be exactly 6 cm in length. It is possible to exactly measure all integer lengths from 1 cm to 6 cm, using only two marks, as shown in the diagram, at 1 cm and 4 cm, since $2 = 6 - 4$, $3 = 4 - 1$ and $5 = 6 - 1$.



Suppose that an unmarked ruler is known to be exactly 30 cm in length. Find a way of placing nine or fewer marks on the ruler to be able to exactly measure all integer lengths from 1 cm to 30 cm. (Godin, 2010)

2. Tell me everything you can about the diagram below. (Godin, 2013)



3. The point $A(0, 0)$ is the vertex of a square $ABCD$, and the point $M(5, 0)$ is the midpoint of side BC. Determine the coordinates of vertices B, C, and D. (Godin, 2015)

Now time for your homework… We are going to do something a bit different in this issue. In the next issue, we will do the last part of our series focusing on the Mathematical Processes. In that column, we will look at Reasoning and Proving. I would like readers to find a problem or two that they feel helps them see evidence of Reasoning and Proving in their students. I would love to hear about your problems. Send me an email with the problem, the source (if you know it), and your thoughts about why it is a good problem to demonstrate Reasoning and Proving. Be as brief or as elaborate as you wish—I'm retired, what else have I to do?

Until next time, happy problem solving!

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OMCA News: March 2024



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A belated Happy New Year from the Ontario Mathematics Coordinators' Association (OMCA)! Our organization continues to offer exciting opportunities for mathematics coordinators, consultants, and coaches throughout the winter and spring.

Shortly into the new school year, OMCA President Jackie Decker announced that she was moving into a new role as Education Officer with the Ontario Ministry of Education. This transition left a vacancy in the OMCA executive. In a motion made by Jackie, it was accepted that Past President Jason To would resume the role of President for the remainder of the school year, and Heather Theijsmeijer would transition back into the role of Past President for the same length of time. OMCA members collectively wish Jackie all the best in her new position!

Our members' meetings continue on a monthly basis throughout the school year. The January and March meetings wrapped up our series by Jennifer Hlavka from The Instructional Coaching Group. Over the course of five sessions, we explored different aspects of instructional coaching in the mathematics classroom, including educator partnership principles and goal setting. With the new provincial math funding model, our membership now includes many in-school instructional coaches, making this series particularly timely.

In January, OMCA worked with an outreach team from EQAO to offer an armchair session on the newly released Grades 3, 6, and 9 mathematics content. Results from some of the latest research into EQAO data were also shared. Members learned how to leverage the data to inform school and board improvement planning, as well as how to use released items as part of ongoing student assessments.

OMCA's annual in-person retreat took place on February 12–13, 2024 at Hockley Valley. Titled "Sparking Math Conversations," the event saw over 150 delegates come together to learn more about how engaging students in mathematics conversations can activate prior knowledge, generate deep discussions, provoke questions, reveal understanding, and help students engage in deep mathematical thinking. It was a full day and a half of learning

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In the Middle: The Geometric Perfection of a Solar Eclipse



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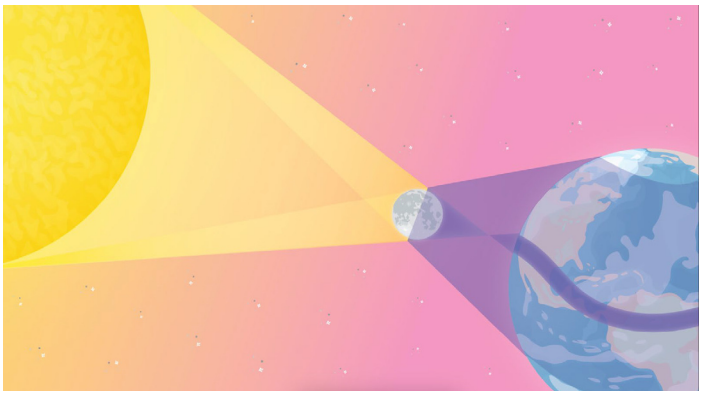
Carly Ziniuk teaches Grade 9 Mathematics, Grade 12 Data Management and Advanced Placement Statistics at the Bishop Strachan School in Toronto. She is very active in adopting real-life data to engage her students in solving problems. Among 30 global STEM leaders, Carly is excited to have been selected for the NASA Space Apps Collective for 2023–2024. As a Canadian educator, she will be contributing toward the pilot program's chosen theme: Building an Inclusive Global Space Community.

Author's Note: I have used the guidelines as presented by NASA's Technical Documentation Style Guide throughout this piece when referring to the Moon, the Sun, and Earth/earth. The NASA *Technical Documentation Style Guide* capitalizes Earth when it is used as the name of the planet. In this case, it is not preceded by "the," since one would not comparably say "the Mars." If referring to ground or soil, but not the planet, then "earth" is lower case. Capitalization of "Sun" and "Moon" refers specifically to our Sun and our Moon (but lower case, for example, with "Jupiter's moons" or the moon Titan). Use of "the" with Sun and Moon is appropriate for emphasis or clarity.

On April 8 of this year, some parts of Ontario (as well as Quebec and Atlantic Canada) will experience a total solar eclipse, with other Ontarians experiencing a partial solar eclipse. For most of our students, this will be the first time they are below the shadow of the Moon. When the Moon passes directly between Earth and the Sun, it blocks the Sun from view and creates a total solar eclipse. The precise timing and geometric alignment must be exactly perfect to give rise to this phenomenon. Thus, it provides you and your students with many rich mathematical opportunities.

As part of my role on the NASA Space Apps Collective this year, I have explored many areas of mathematics that are new to me, like the upcoming eclipse. In this role, I have had the good fortune to meet many curious STEM thinkers, like the astronomy educators at Discover the Universe. The Dunlap Institute of Astronomy and Astrophysics at the

University of Toronto, with their Astroindigenous portal and their partner, Discover the Universe (www.discoverthe universe.ca/eclipse), have been incredibly helpful by preparing Canadian educational resources related to this year's eclipse. The Canadian Space Agency has also provided quality learning materials for youth and educators in the Solar and Lunar Eclipse Toolkit (www.asc-csa.gc.ca/eng/youth-educators/toolkits/solar-and-lunar-eclipses/), which also includes some animated videos. Additionally, as explained in more detail in my December 2023 column, NASA resources can be readily adapted to our local context. I hope that the resources below will bring the geometric magic of the eclipse inside our classrooms well beyond April 8.



Every year for Pi Day, NASA and Jet Propulsion Laboratory (JPL) provide lessons, activities, backgrounds (like that above), and posters connecting pi to space learning with their *Pi in the Sky* annual problem sets. The problem "Epic Eclipse," from the NASA/JPL 2017 *Pi in the Sky*, finds the approximate surface area of Earth that will be covered by the disc of the Moon's shadow at any one time during the eclipse. More recently, "Eclipse Enigma" uses similar triangles to determine the percentage of the Sun obscured by the Moon.

Adapted from the NASA/JPL 2023 Pi in the Sky activity, "Eclipsing Enigma"

On April 8, a solar eclipse will be visible across North and South America. The Sun, with a radius of 695 700 km, will be 148 523 036 km from Earth. The Moon, with a radius of 1737 km, will be 388 901 km from Earth.

What percentage of the Sun's disk area will be obscured by the Moon? *Drawing not to scale, with the Earth sketch from RawPixel*

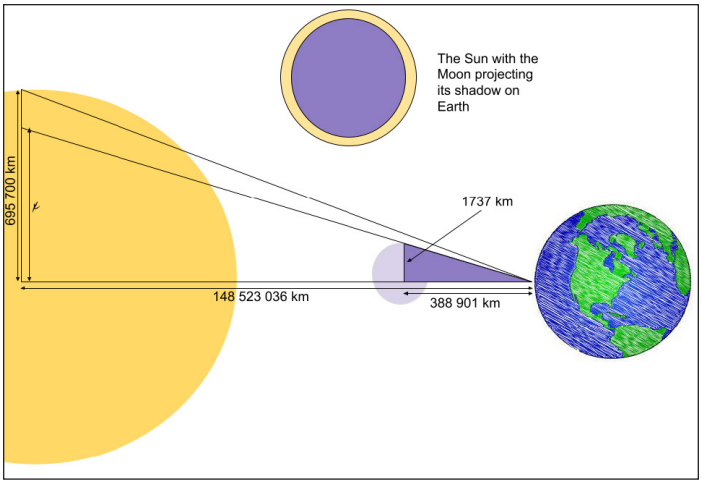


Figure 1: *Geometry of an eclipse*

With the diagram in Figure 1 above, find the radius of the Sun's disk area (purple) that is obscured by the Moon, using similar triangles.

$$\frac{1737\ km}{388\ 901\ km} = \frac{x}{148\ 523\ 036\ km}$$

$$x \approx 663\ 400\ km$$

Using the areas of the two circles (purple, formed by the Moon, and yellow, formed by the Sun), calculate the ratio of the Moon's obscured area to the Sun's total disk.

$$\frac{A_{Moon}}{A_{Sun}} = \frac{\pi(r_{Moon})^2}{\pi(r_{Sun})^2}$$

$$\frac{A_{Moon}}{A_{Sun}} = \frac{\pi(x)^2}{\pi(695\ 700)^2}$$

$$\frac{A_{Moon}}{A_{Sun}} = \frac{\pi(663\ 400)^2}{\pi(695\ 700)^2} \text{ or } 91\%.$$

As mentioned in the December 2023 issue, Space Math @ NASA (spacemath.gsfc.nasa.gov) contains longer multi-problem activities involving different mathematical concepts connected by a common astronomical theme. One example of this is 2012's "Solar Math," which contains several mathematical applications about the Sun, including calculating the size of sunspots from photographs, using proportional reasoning and scientific notation. Eclipse activities include Problem 516: "Hinode Observes a Solar Eclipse from Space" (Proportions, Calendar arithmetic), Problem 38: "Solar Eclipses and Satellite Power" (Geometry, Decimal operations), and Problem 143: "The Last Total Solar Eclipse Ever."

This latter activity involves the concept of angular size. When you look at anything, including the Sun and the Moon, you measure its "angular size": the amount of space it takes up in your field of view. In middle school, we measure angular size in degrees, and can discuss greater precision by measuring minutes (1/60 of 1 degree), and seconds (1/60 of 1 minute or 1/3600 of 1 degree). In senior high school

math courses, we might discuss angular size as measured in radians.

Adapted from the 2012 Solar Math activity, "The Last Total Solar Eclipse Ever"

Total solar eclipses occur when the Moon's angular size is almost the same as the Sun's, despite their vastly different sizes and distances.

What may be surprising to you is that the Moon is actually moving away from Earth! The rising tides in the oceans cause drag that slow Earth's spin rate, which means the Moon's gravity affects Earth's rate of rotation. Since Earth rotates on its axis faster than the Moon rotates about the Earth, our day–night axis is faster than the lunar cycle. At the same time, the Moon's gravity affects Earth, like the tides. The result is that Earth is slowly pulling the Moon to revolve faster to match the speed of Earth's rotation. The tidal forces pull the Moon to revolve faster at the same time the Moon is slowing down Earth's rotation. As the Moon revolves faster, it moves farther away, with an increasing orbit distance.

One consequence of the Moon moving away is that eventually the angular sizes will no longer match and total solar eclipses will not happen again. These questions help to determine when this will happen.

1. The *perigee* is the point in the orbit of the Moon nearest to Earth (the point furthest from the Moon is called the *apogee*). From the surface of Earth, the perigee's distance is 356 400 km, with its angular size 0.559º. If the Moon were double its distance from Earth, what would its new angular size be from the surface of Earth?

   If you double the distance, the Moon will appear half its former size, or 0.279º across.

2. If the Moon's distance increased by 50 000 km, how would the angular size change?

   The total distance is now 406 400 km,

   $$\frac{406400}{356400} = 1.14 \text{ } or \text{ } 114\%.$$

   The angular size would reduce by the same factor,

   $$\frac{0.559}{1.114} = 0.49^{\circ}.$$

3. The Sun's smallest angular size occurs near the summer solstice at a distance of 152 million km, with its angular diameter of 0.525º. How far away does the Moon have to be to match the Sun's apparent diameter?

   $$\frac{0.559}{1.114} = 0.49^{\circ}.$$

   The Moon's distance would have to be 1.06 times further away from Earth than it is on the summer solstice.

   356 400 × 1.06 = 377 800 *km*.

1. How much further away from Earth will the Moon be at that time?

   377 800 – 356 400 = 21 400 km.

2. If the Moon is moving away from Earth at a rate of 3.8 cm/year, how long will it take to move the distance where a total solar eclipse cannot be seen on Earth (your answer to question 4)?

   3.8 cm/year is the same as 0.000038 km/year

   $$\frac{21400}{0.000038} \approx 563 \text{ million years from now.}$$

The eclipse in 2017 travelled across most of the continental United States, and NASA provided a series of Science and Math Challenges (eclipse2017.nasa.gov/math-challenges) for classrooms across the United States. Many of these Math Challenges can be adapted for the 2024 eclipse.

NASA Eclipse2017's Math Challenges 2, 3, and 4, labelled "X Marks the Spot," use data as well as linear and quadratic equations to identify the historical and future paths of total solar eclipses. The other challenges include angular measure, conical calculations, and speed–time graphs.

Adapted from the Eclipse2017 Math Challenge 4: "X Marks the Spot with Quadratic Equations"

For our middle school students, this graph can allow them to explore quadratic equations graphically, without requiring equations.

According to GIS.Stackexchange.com, the extent of a given geographical area is the most extreme latitudes and longitudes for a given area on a map. Ontario's extent, or the rectangular area that would form the bounding rectangle of Ontario, are 49° 00' N, 114° 04' W to 60° 00' N, 139° 03' W.

Where does that extent fall on the map below?

How many total solar eclipses have occurred in Ontario?

By researching your school's latitude and longitude, where does it fall in the extent rectangle? How rare were total solar eclipses in your community (or will be!)?

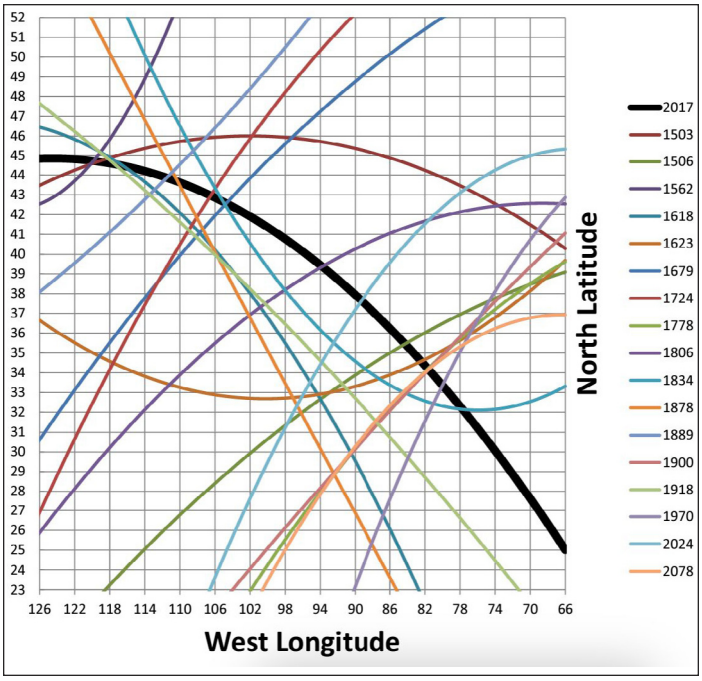


Figure 2: *Map from the "X Marks the Spot with Quadratic Equations" activity*

Further examination of this activity can identify the curves' equations and where they intersect in pairs. This translates practically into which locations were in the path of totality for two different eclipses. Middle school students can do the same, only graphically.

Adapted from the Nrich Math 4: "Perfect Eclipse"

University of Cambridge's Nrich Math's activity, "Perfect Eclipse," also provides an open-ended challenge for ages 14–16. It suggests using "data cards," like the one shown in Figure 3. First provide the data to small groups, asking students what information they need to model the situation. The diagram included in Figure 3 is given as a hint. What information can they discover, given the information in their card? Another teaching strategy might be to separate the data and give each student a single piece of information—what can they can come up with when they share their information and the data is collected together?

Additionally, provide data (in a handy .csv file) for other moons so that students can also investigate how common (or how rare!) an eclipse is for other moons in our solar system. As with most Nrich Challenge activities, they provide related Warm-up introductions and Think Higher extensions from their archives, in this case, problems involving inscribed triangles and gravitational paths, respectively.

Radius of the Sun: 695 500 km
Radius of the Moon: 1737 km
Radius of Earth: 6371 km
Smallest distance between Earth and the Sun: 147 098 074 km
Largest distance between Earth and the Sun: 152 097 701 km
Smallest distance between Earth and the Moon: 356 375 km
Largest distance between Earth and the Moon: 406 720 km
This image (not to scale) represents the positions of the Earth, Moon, and Sun in a solar eclipse.

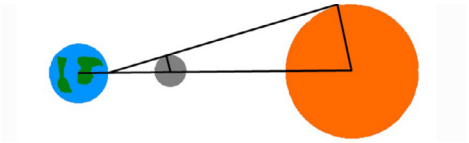


Figure 3: *"Perfect Eclipse" example*

If you are looking to find additional resources to support your students' understanding about the upcoming eclipse, especially for other disciplines like science and geography, use the JPL/NASA's educators' page (www.jpl.nasa.gov/edu/teach) with the tag "eclipse." Physics educators have modelled total solar eclipses in 2D and 3D, using GeoGebra and Desmos, as well as many different physical models that students can create. Discover the Universe's Astroindigenous portal, and especially Dr. Laurie Rousseau-Nepton's video, *Indigenous Perspectives on Solar Eclipses* (www.youtu.be/474HOSOcM6M), designed for the 2017 eclipse, which provides additional perspectives on this natural phenomenon.

A final note beyond the academic learnings of solar eclipses, eye safety is of primary importance during an eclipse and even a partial eclipse. Although the Moon blocks some sunlight during a solar eclipse, the light that is not blocked can damage eyes. During the process of the eclipse, as many people will want to stare at the phenomenon, Professor Ralph Chou (from the Optometry School at the University of Waterloo, presented by the Dunlap Institute) has a student-friendly eye safety video during a solar eclipse, updated for the upcoming eclipse (youtu.be/jvCOgdHzNIM). It bears repeating that during an eclipse, even though some sunlight is blocked by the Moon, the remaining light can still damage unprotected eyes. Proper eclipse glasses, correctly worn, allow you to safely look directly at all partial phases of the eclipse. Even if a tiny fraction of the Sun is visible, you must wear solar filters to prevent damage. The wonderful folks at Discover the Universe can help you and your school plan a safe "watching event." ▲

▲ Elementary Math Matters: Is it time to admit that math teacher prep is failing our Primary/Junior teachers and their students? I say it is.



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Lynda Colgan is Professor Emerita, Faculty of Education, Queen's University. Lynda is currently working on a number of research, resource-creation, and teacher-education projects, funded by the National Sciences and Engineering Research Council of Canada (NSERC), the Social Sciences & Humanities Research Council (SSHRC), and the Mathematics Knowledge Network (MKN). She will carry on her long-established commitment to STEM outreach by continuing to seek out creative avenues to engage students, parents, and educators in mathematics at home and in the classroom.

> *"Teachers must know the subject they teach. Indeed, there may be nothing more foundational to teacher competency."* (Ball et al., 2008, p. 404)

Having been an instructor for the Primary/Junior Mathematics Curriculum course at Queen's for almost 25 years, I inevitably received a phone call or email in August from an incoming teacher candidate with the same worried tone. The gist was this: "I'm going to be in your course starting in September. We're just going to learn how to teach math, right? We're not going to have to *do* math, are we?" My response was always the same: "You are going to *do* math every day." The responses from the other end of the phone ranged from silence to sobs, with many reactions in between—some of which were decidedly not suitable for the delicate sensitivities of this readership.

The question always astounded me. How could it possibly come as a surprise to an applicant to *any* Primary/Junior education program in this province that upon graduation, one would be responsible for teaching math for the grade level at which they were hired?

It is common knowledge that in K–Grade 6 schools, almost all classroom teachers are responsible for implementing the entire elementary curriculum. Here's the list:

1. The Kindergarten Program
2. The Arts
3. Health and Physical Education
4. Language
5. Mathematics
6. Science and Technology
7. Social Studies, History, and Geography

It is a full and demanding curricular gamut, with Ontario Ministry of Education policy mandates requiring one hour of mathematics and one hour of language daily. Yet, Primary/Junior teacher candidates, for the most part, do not begin their post-secondary education with an accurate end goal in mind, i.e., the reality of being a math teacher each and every day in a K–Grade 6 classroom.

The minimum admission criterion for a Primary/Junior program at most faculties of education in Ontario is a completed undergraduate degree. In a recent York University study (Gallagher-Mackay et al., 2023) based on longitudinal (2006–2015) cohort data collected from 156 580 students by the Toronto District School Board (TDSB), 99.8 percent of university-bound students completed English at the "University" level in Grade 12. Students are not *required* to take Grade 12 Math at either College or University levels, and the report found that only 70.8 percent of TDSB students completed Grade 12 Math, yet went on to post-secondary studies, including general Arts and Science programs at university.

But is a three-year baccalaureate degree an adequate criterion for admission to a typical consecutive teacher education program? According to Ginsburg et al. (2008) and others, the answer is a resounding, "No."

> Of course, educational credentials are only a proxy for relevant knowledge and skills acquired in the colleges and universities. The real issue is whether an undergraduate degree—especially an undergraduate degree in early childhood education—provides teachers with knowledge and skills useful for teaching early childhood mathematics. The answer is discouraging. The undergraduate degree—even with a major in early childhood education—is not a good predictor of classroom quality and children's academic outcomes (Early et al., 2005).

If a bachelor's degree—even one specializing in early childhood education—is an inadequate prerequisite for the classroom, then we must assume that the full development of the enormous body of knowledge and skills required for teaching in the Primary/Junior division resides solely at faculties of education.

Unlike most other schools/faculties that train professionals (such as Pharmacy or Optometry & Vision Science), there is no common denominator on how best to prepare future elementary teachers in mathematics. Wide variations in mathematics-related course content and requirements at

faculties of education exist because they are departments within post-secondary institutions, and thus governed by the rules and rights of the academy, including academic freedom. As a result, each faculty of education in Ontario structures its own program to provide what it considers to be the basis for its graduates to be certified by the Ontario College of Teachers. While consecutive teacher education in Ontario is, on paper, a two-year program, there is enormous variation in program design and duration, even with Ontario College of Teachers accreditation oversight of programs.

As a result, each faculty of education finds its own solution to balance "teaching methods" courses in all Primary/Junior subjects, in addition to addressing topics in educational philosophy, psychology, and law, as well as dealing with issues such as equity, Truth and Reconciliation, student mental health, and inclusion.

The culture of mathematics education courses in many pre-service programs is to focus on a brief introduction to the "strands" and "processes" of the math policy document; teaching math using children's literature; using games to support math in the classroom and at home; coding using *Dash* or *Bee-Bot* robots; using math manipulatives; and socio-emotional learning skills in math.

The culture of most mathematics education courses is *not* to concentrate on math content—numbers and operations (including place value and fractions), algebra, geometry, and data and probability. This situation is not unique to Ontario. In fact, a 2016 study by the National Council on Teacher Quality (an American body) found

> …that just 13 percent of the 860 undergraduate elementary teacher preparation programs reviewed, covered critical math topics, including numbers and operations, algebra, geometry, and data and probability. And in 2018, a similar review of graduate programs found that just *1 percent* of 201 programs covered these topics (Durrance, 2019).

If mathematics education classes for pre-service teachers focus on topics such as classroom management for math manipulatives, organizing activity centres, planning lessons, and using rubrics to evaluate general student understanding, when do prospective and novice teachers learn about the mathematical content and mathematical trajectories of the lessons taught with those concrete materials during those activities and lessons? When do they learn how to formulate the questions that will prompt mathematizing of the content? When do they learn to provide mathematical explanations using multiple representations to counter misconceptions and simultaneously promote understanding? When do they learn how to identify specific and well-documented misconstructions and remediate accordingly?

If Primary/Junior mathematics education courses do not make content proficiency a priority, how can we be sure that new teachers are equipped to provide the type of instruction required by learners to understand the math? The simple answer is that we can't. But we can design a program that could equip every teacher candidate with the knowledge, skills, dispositions, and resources to be able to do all of this and more, to the benefit of children in their classrooms, and to make the professional education school one of high repute.

It is time to eliminate the post-degree, "two-year" consecutive education program in Ontario (or worse yet, the one-year program Liberal leader Bonnie Crombie promises to institute if her party wins the next provincial election). We need applicants to Primary/Junior professional teacher preparation programs, who know from the onset of their post-secondary education that their ambition is to be a certified teacher, and that translates to taking and passing mathematics content courses that are specifically designed to meet the needs of the elementary teacher, and more importantly, the needs of the elementary learner.

It is time for *all* teacher education programs to become professional three- or four-year dual-track programs. In such programs, *each year*, all Primary/Junior teacher candidates would need to successfully complete a full-year mathematics[1] content course designed specifically for future teachers. In mathematics, that content "track" would focus on K–Grade 6 curriculum topics—numbers and operations, algebra, geometry and measurement, as well as data analysis and probability—and be taught *only* by mathematics education professors who can connect the math content for teaching and deconstruct it for learning. The parallel program will be comprised of education and pedagogical courses that cover subject-specific instructional strategies and resources, and a full complement of courses that fall under the umbrella of "the science of learning" (e.g., developmental psychology and cognition courses, diagnostic assessment, and remediation), taught by specialists in the domain, who have classroom connections.

Before you ask, "And where are we going to find faculty with those credentials?," I argue that such expert pedagogues (a term coined by David Berliner in 1986) exist. In mathematics departments, we once had luminaries like Dr. David Poole (Trent University) and Dr. Leo Jonker (Queen's University), who pioneered engaging and mathematics-rich courses for elementary teachers that had dedicated classroom components. Dr. Ann Kajander (Lakehead University) and Dr. Cathy Bruce (Trent University) are continuing the tradition with content-infused courses and mathematics department courses that are cross-listed as meeting both "education" and "mathematics" requirements.

The "science of learning math" stream is rich with faculty like Dr. Jo-Anne Lefevre and team from Carleton University, Dr. Zachary Hawes (Ontario Institute for Studies in Education/University of Toronto), and Dr. Helena Osana and team from Concordia University.

An important consideration within a dedicated concurrent program must be options for specialization, and one of the priority areas for a reformed approach to teacher education must be early mathematics—teaching Kindergarten to Grade 3 students. Effective early instruction in mathematics is the key to children's later academic success, and effective early instruction is predicated on a specialist teacher who can move young children beyond their informal, intuitive mathematics understanding.

The world of language instruction has been upended in positive and fruitful ways by the Ontario Human Rights Commission's *The Right to Read* inquiry report and the subsequent implementations of science-of-reading strategies provided by experts in developmental psychology, educational psychology, cognitive science, and cognitive neuroscience. The basic principle of the science of reading is to provide direct instruction in Kindergarten and the early years so that children acquire the building blocks of words and reading. Most current classroom educators are doing their best to learn the strategies required by turning to online courses, recent books, and support from professional organizations such as local chapters of The International Dyslexia Association and their school district educational coaches.

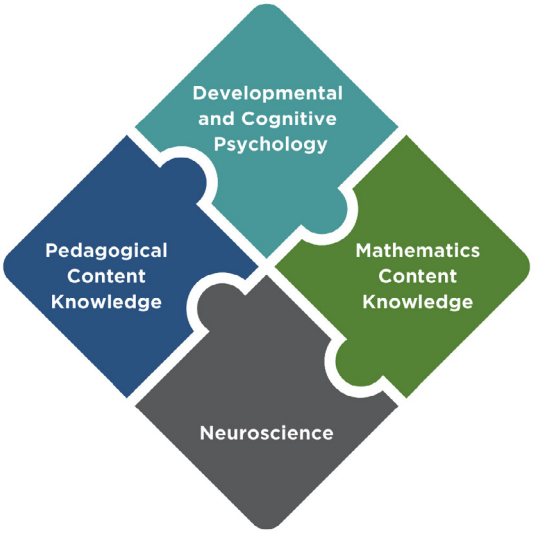
The time is right and ripe for faculties of education to adopt a similar research-based framework grounded in what cognitive science says are the most effective ways to teach and learn foundational skills in mathematics; supporting novice teachers to learn how to assess and diagnose gaps in learning, followed by how to teach those skills to a wide variety of students. Teachers cannot be expected to teach explicitly if they do not know what explicitly needs to be taught. Our Primary/Junior teachers (many of whom are intimidated by math) can only benefit if they are provided opportunities and time to learn—or relearn—skills like regrouping numbers alongside how to teach those skills to students.

In early December, in response to the release of the Programme of International Student Assessment (PISA) scores by students on a 2022 test of 700 000 secondary school students around the world (including 23 000 from Canada), administered by the Organisation for Economic Co-operation and Development (OECD), national newspaper headlines screamed "Canadian students' math scores continue steep slide, raising concerns about how subject is taught" (Alphonso, Dec. 1, 2023). At the same time, social media sounded the "sky is falling" alarm, and parents, students, educators, and the public chimed in with anecdotes and antidotes for a "failing system." Predictably, there was the call for "back to basics"; the elimination of calculators and cellphones in classrooms; an end to "inquiry" and "discovery" learning; and changes to teaching. Where were the voices demanding changes to teacher preparation?

Teaching and teacher preparation are inseparable. It has been long established that the fallback position is for teachers to teach the way they were taught themselves. or perhaps the way they were taught to teach at a faculty of education. In 1975, Dan Lortie called this the "apprenticeship of observation," noting that teacher candidates arrive at faculties of education having spent thousands of hours observing and evaluating teaching professionals in action—something that is not true in other professions, and a reality that leads to deeply rooted preconceptions and often misconceptions.

If education is to be seen as a profession "with teeth," whether one is on the inside looking out, or the outside looking in, we need to "up" the standards on teacher preparation. The rigour of the parallel structure requiring substantive content coursework and comprehensive pedagogical preparation before the move to the other side of the desk, would not only deter applicants who were not serious about becoming an elementary teacher, but would require faculties of education to rethink hiring mathematics educators with dissertations describing tangential issues about math, but containing no mathematics; mathematics departments to join other disciplines as proactive science communicators; and Psychology, Cognition, and Neuroscience faculties to claim their rightful (and too long denied) place in mathematics education as an integrated field.

In my 2023 Margaret Sinclair Memorial Lecture, *Full Circle: My Journey Through Mathematics Education and the Unstoppable Path Back to Kindergarten*, I spoke about how The Early Math Assessment at School (EMA@School) and The Assessment and Intervention in Mathematics (AIM) collective is the concretization of the science of math, as shown in the diagram below. It has taken over 50 years to integrate the complex knowledge about young children's learning of mathematics into a unified whole. Half a century of dedicated efforts by deeply committed researchers has finally emerged in a practical format that acknowledges the fact that Primary/Junior math is rigorous and multifaceted—worth much more than a distilled version of transmission in one class as part of the chaotic bricolage of current programs. It is time to honour 50 years of discovery by recognizing that elementary teacher preparation must be intensive, extensive, and long term, requiring specialized faculty who value, respect, and know mathematics, education, cognition, and classrooms.



Note [1] As well as language, science, history, geography, physical education, and art, but I will leave these for other subject associations to argue.

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▲ Coding in the Classroom: Coding is in a Liminal Space



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Liminal spaces are fascinating. They are the spaces in between—the transitions—the places of possibility and opportunity. Liminal spaces can be physical, like a doorway. On one side of the door, you are outside; on the other, you are inside. The liminal space is where you are right in between being outside and inside. You are no longer out, but you are not yet in. This is where we find coding in the classrooms of Ontario right now.

Coding is not on the outside anymore. It is in the mathematics curriculum for Grades 1–9 and the science curriculum for Grades 1–8, with it likely being added to more curricula in the future. Despite there being little in the way of provincial support for teachers to learn how to code and then learn how to teach through coding, teachers are once again stepping up and learning by and for themselves. School boards, coaches, and outside personnel are all busy trying to reach as many people as possible—a huge job—in order to get coding started in Ontario classrooms.

Nor is coding fully inside, yet! We are in that transitional space, not an unknown quantity, but not yet an accepted and fully realized part of a teacher's everyday practice. But that may actually be a good thing; let me explain liminal space with a metaphor with which we are more familiar.

Consider learning something. At one point, you have not learned it; at another, you have. A liminal learning space is that moment right in between the possibly confused "Huh?"

and the excited light bulb turning-on moment of “Aha!” We have all had one of those; hopefully, many of them. Think about that feeling you have just before everything becomes clear. You see something ahead that is full of possibility, potential, and opportunity. It is wide open before you focus in, finding that learning now in a specific place with an “Aha!”

We have talked with some fascinating people about what their coding practice is like, how they envision it as part of their students’ learning, and its potential to improve learning outcomes. Think of the possibilities. We have had Pekka Reinio (Sept., 2021) and Nicole Myers (March, 2023) use game creation to motivate and teach their students. Game creation allows for so many mathematical and scientific ideas to be learned and demonstrated. Any game is a model of something, be it physical or fantasy. To program it, you must create a computational model for it that convinces the user that they are in that place doing that thing. Even coding a simple game of Asteroids (see https://en.wikipedia.org/wiki/Asteroids_(video_game)) requires a working knowledge of Newtonian laws and a spaceship load of algebra. Really good game making also requires good storytelling, allowing teachers to cover even more curriculum with one learning object. Here is an opportunity to integrate language into math and science. What might the possibilities be?

Amanda Deneau (June, 2022) showed us that coding can allow us to spread our learning from the classroom outward by allowing our students to invent and innovate. Let them tackle problems and come up with computational solutions that help us all reach sustainable development goals. Should we expect that tinkering with problems will have students coming up with viable solutions? No, but giving our students the opportunity to grapple with big problems that automation and computation allow for, makes it more likely for them to potentially find one later when they continue in school.

Diane Tepylo (Dec., 2021) and Lisa Anne Floyd (Dec., 2023) pointed out the potential for learning, which computation affords our students, to tackle bigger problems than they would normally be capable of. Productive struggle is effective, but it is a tightrope requiring some balance. Allowing our students to harness the power of a computer to do repetitive computations enables them to learn at a different level than previously. Students no longer need to spend major portions of their time calculating, as Lupolph van Ceulen did, calculating pi to 20 and then 35 decimal places. Today students can hand off these mundane tasks to the computer by coding a program, thus allowing them to dig into and learn from the results more quickly and efficiently.

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▲ Rebranding the OMO



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Shawn is a former mathematics teacher, department head, and consultant. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: presenting workshops, writing articles, working on local projects, and helping create mathematics contests.

The Ontario Mathematics Olympics (OMO) was a celebrated annual OAME/AOEM event. The Olympics allowed teams of students at the Grades 7 and 8 levels to compete in a fun mathematical event that encouraged problem solving, communication, and, most importantly, collaboration. The first OMO was held in 1995, and the event ran annually until it was discontinued after the 2018 competition. The OMO was reluctantly discontinued due to increased complications and costs associated with holding such a large provincial event in person. Currently, the OAME/AOEM is hoping to rebrand and resurrect the competition in a different form. The new competition should allow for even more students to get involved, while retaining the charm of the original.

The working model for the new competition (which we will refer to from now on as nOMO, as in “new OMO”) will be that of a team mathematics league. Its structure was inspired by several existing mathematics competitions, for example, The Math League (www.mathleague.com) and the Newfoundland and Labrador Senior and Junior High School Math Leagues (www.math.mun.ca/~mleague/index.html). These contests consist of multiple parts that occur several times during the school year, as will the nOMO. However, unlike the two aforementioned competitions, the type of competition for nOMO will be different for each of the parts.

At this point, a very small committee is at work, trying to bring the nOMO to life. We foresee the separate parts of the competition as being very short (about 30 minutes), with some flexibility in timing to allow teachers to get the greatest amount of student participation. Students will work in groups as before, with some of the previous restrictions on the OMO groups dropped. Collaboration will still be at the heart of the competitions, and some competitions will have incentives for collaboration built into them.

At this point, the committee is looking to the 2025–2026 school year as the reboot year. However, the committee needs more members to iron out the details for the competitions and create all the contests. Ultimately, the competition will benefit from corporate knowledge, so committee members who are willing to stay on for several years are specifically being sought. If you are interested in creating fun, engaging activities for students (and finding a better name than nOMO), please consider joining the committee. The nOMO cannot move forward without you!

Inquires about the committee can be sent to godinmathstuff@gmail.com. ▲

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and discussions, facilitated by mathematics educator Chris Luzniak. Thank you to our sponsors for helping us make the retreat such an engaging experience!

While most of our monthly meetings remain virtual, our second in-person opportunity to gather this school year will be on Wednesday, May 1, ahead of the OAME Annual Conference in Kingston.

From Smooth Rock Falls to Niagara Falls, new members are always welcome to join OMCA. Contact us directly at OMCAmath@gmail.com, or visit our website at www.omca.website (note the different domain) to learn more about the organization, find out about upcoming events, and connect with us. ▲

Contiuned from page 28

It is this type of thinking that Alice Aspinall explained to us. Having students think about how to code a program to do similar tasks is a type of algorithmic thinking that transcends the mundane chore of algorithmic doing. Imagine how carefully a student must think about creating an algorithm to complete their calculations. Imagine also how well these same students need to understand the mathematical processes involved. Using coding as a tool like this does not replace any learning that some back-to-basics folk think is going missing in student learning. It is enhancing it and using it to accomplish more complex tasks or to try to solve really big, intricate problems using mathematics.

Coding is in a liminal space, but a liminal space is an exciting place to be. Possibilities and opportunities lie ahead. Some will be fruitful, some won't, but it will be interesting to find out which is which. ▲

▲ chatOAME: AI from an Ontario Perspective



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Cal is a veteran math teacher, who explores new ways of using technology in his classroom, and from his experiences at the Park City Mathematics Institute, he believes that self-reflection is a key skill for teachers to improve their practice and learn from their experiences. When on a sabbatical, he created the OneNote ClassNotebook, a tool inspired by Black & William's research on formative assessment and feedback. He challenges his own technology use by comparing and contrasting his practice against practical evidence and academic research as he strives to enhance his students' learning outcomes, without falling prey to a bandwagon.

Recall back to your B.Ed. classes when Lev Vygotsky, a psychologist looking at student learning, provoked the idea of the Zone of Proximal Development, that area where a student sits and can move to understanding that next bit of knowledge, if they are interacting with an activity alongside a "more knowledgeable other" who can guide them. That "other" is usually us, the classroom teacher. But who sits next to us and pushes us to the next stage of our professional understanding?

When I proposed to ISOMA (Independent Schools of Ontario Mathematics Association) and, subsequently, the OAME/AOEM Board of Directors, about creating a project to develop an Artificial Intelligence (AI), with a specific lens on the OAME's/AOEM's Mission and Vision, it was that "more knowledgeable other" of which I was thinking. So many math educators have questions, wonder about something, want to experiment with activities or assignments, or want to discuss thoughts or ideas. As busy people, we can't always take the time of our colleagues, and many teachers work on their professional growth at different hours due to other demands on their time. Twitter, now known as X, used to be a great space for collaboration around ideas, but it is no longer, and the multifurcation of social media has not helped. Too often, online discussion is pushed by corporate or political intents that aren't based on fundamentally strong tenets of excellence in mathematics education. By creating a special version of ChatGPT, I proposed that we create that "knowledgeable other" for OAME/AOEM members that reflects excellence: one that was trained on OAME/AOEM content, driven by

OAME/AOEM ideals, and driven philosophically with the best of Ontario mathematics education.

The lens I chose to take is constructed on four pillars: first, the content developed over the past 50 years in the OAME *Ontario Mathematics Gazette* and *Abacus* and, more recently, podcasts, presentations and position statements created by or for OAME/AOEM. The vast majority of this content is available only to OAME/AOEM members and would not be included in any other AI's knowledge base, making this chatbot's perspective unique, and uniquely Ontarian. The second pillar is Ontario Ministry of Education documentation, starting with *Growing Success* (2010), the more recent *High-Impact Instructional Practices in Mathematics* (2020), the Ontario Mathematics Practices, the Ontario Achievement Chart and, of course, all the Ministry curricula. The third pillar was Canadian education content that was deemed to align with OAME's/AOEM's Mission and Vision. This pillar is one that continues to grow and challenge. What are the writings from those outside of OAME/AOEM that are critical to "good Ontario mathematics"? Social–emotional considerations, Indigenous perspectives, human development, technology use—what are the ideas that need inclusion and emphasis in this lens on Ontario mathematics education? And the last pillar, educational philosophy, from which all of the former evolve: I included original documents from Dewey, Piaget, Papert, the aforementioned Vygotsky, among others, to ground responses in the foundational ideas of education. Pushing the AI responses through a lens based on those four pillars should ensure we get responses that are meaningful, and that push us to the next best stage of our understanding.

The project is still in its initial stages and, like many technology projects, has been overtaken by events. What started in the early Fall of 2023 as a tedious hard coding of content was catapulted into the future by developments in AI creation. By the end of December, a functioning model was being tested by a small group of teachers, who were using it for themselves. The discussions with chatOAME ran from the quotidian creation of multiple-choice questions for exit tickets, writing success criteria, to conversation assessment of Calculus units, rubrics for various tasks and then, and what I really appreciated seeing, a teacher discussing with chatOAME the assessment of student work as seen through the *Growing Success* lens from multiple perspectives. In the conversation, the teacher and chatOAME discussed the nature of evaluation in a simple example; chatOAME never disagreed with the teacher's approach, but rather, offered that the situation could be viewed differently, depending on your goals, objectives, and viewpoint.

As we all know, there is no one correct answer to how a particular question should be evaluated or assessed, but thinking deeply about simple details like this can enhance the teacher's understanding of what it means to assess, and how to do so, which prepares them better for dealing with student work in the future. It was this example that shows the value of having the more "knowledgeable other" in the form of a conversational chatbot: "Active, persistent, and careful consideration of any belief or supposed form of knowledge, in the light of the grounds that support it, and the further conclusions to which it tends, constitute reflective thought" (Dewey, 1933, p. 9). Through the use of a universally available chatOAME, Ontario math educators will be able to discuss their ideas, their thoughts, their questions with an AI that's been trained with the best of mathematics education content that OAME can offer.

Where does this project go from here? As of this writing, it is still being tested, but the collection of users continues to grow. A discussion of the project is ongoing with the Board of Directors. It should also be expected that the technology on which it is built will grow by leaps and bounds. The next few months will accelerate its capabilities considerably. Your feedback, as OAME members, is greatly valued, and you're encouraged to offer feedback to oame.ai@oame.on.ca or through a letter to the Editor.

The following is the transcript of an OAME member engaging with an early version of chatOAME, without any edits and reprinted with permission.

USER

This question is marked out of 3. I have the following solution from a student. Can you please provide a mark for this person and give your reasoning, please? I'll provide a response in the next prompt.

ASSISTANT

Absolutely, please provide the student's solution to the expression, and I will evaluate it based on the criteria typically used in Ontario mathematics assessments. The evaluation will consider the following dimensions from the Ontario Achievement Chart:

- **Knowledge and Understanding:** Correctness of the mathematical concepts and procedures.
- **Thinking:** Quality of the student's thought process in solving the problem.
- **Communication:** Clarity and effectiveness in communicating the steps taken to solve the problem.

For a question marked out of 3, we would likely allocate one mark for each of the above dimensions. Please share the student's solution so I can assess it accordingly.

USER

7.2 - (0.4*3)/0.2+0.6^2 = 7.2-1.2/0.2+0.6^2 = 7.2 - 0.06+0.6^2 = 7.2 - 1.26 = 5.94

ASSISTANT

Let's evaluate the student's solution step-by-step and assign marks according to the dimensions of the Ontario Achievement Chart.

Step 1: 7.2 - (0.4*3)/0.2+0.6^2 = 7.2 - 1.2/0.2 + 0.6^2

- The student correctly identifies the need to perform the multiplication within the parentheses first (0.4*3).
- **Knowledge and Understanding:** 1/1

Step 2: 7.2 - 1.2/0.2 + 0.6^2

- The student attempts to move on to the division (1.2/0.2), but incorrectly calculates it as 0.06 instead of the correct value of 6.
- **Thinking:** 0/1 - The student made a computational error in the division step.

Step 3: 7.2 - 0.06 + 0.6^2

- The student has an incorrect value here due to the previous error, but they correctly recognize the need to compute the square of 0.6. Unfortunately, they do not carry out this calculation, leaving the expression as 0.6^2. This shows a lack of completeness in the thinking process.

Thinking: 0/1 - The student did not complete the calculation for the exponent.

Step 4: 7.2 - 1.26

- The student incorrectly subtracts 1.26 from 7.2, which seems to be a combination of the incorrect division result (0.06) and an incorrect application of the square of 0.6 (presumably 1.2 instead of the correct 0.36).
- **Communication:** 0/1 - The student's communication of their process is not clear or correct. They did not communicate the correct application of the exponent, and their steps did not lead to a correct final answer.

Final Step: 5.94

- The final answer provided by the student is incorrect. The correct final answer should be 1.56.

Final Marks:

- Knowledge and Understanding: 1/1
- Thinking: 0/1
- Communication: 0/1

Total: 1/3

Reasoning: The student demonstrated understanding of the order of operations initially, but made a significant error in division and did not compute the exponentiation. They failed to communicate a correct and clear method to reach a valid solution. As a result, the student would receive one mark for the initial correct step, but lose marks for "Thinking" and "Communication" due to computational errors and incomplete steps.

USER

Is there a reason why you did not include an "Application" mark in this question?

ASSISTANT

In the context of the Ontario Achievement Chart, the "Application" category assesses the student's ability to apply knowledge and skills in familiar and new contexts. For the mathematical expression given, the focus was primarily on the correct execution of mathematical procedures (Knowledge and Understanding) and the ability to follow a logical thought process to arrive at an answer (Thinking), as well as clearly communicating those steps (Communication).

The expression in question did not require the student to apply mathematical knowledge to solve a real-world problem, make connections to other areas of mathematics, or extend the concept to a new context, which are typical requirements for assessing "Application."

However, if the question had been embedded within a real-world context or applied scenario, where the student needed to use the order of operations to solve a practical problem, then the "Application" dimension would be more relevant and should be assessed. In the case of this isolated procedural calculation, the "Application" category is not directly engaged.

To clarify, in the context of a straightforward computation like the one given, we typically look for the following:

- **Knowledge and Understanding:** Correct use of mathematical conventions, terms, and rules of operations.
- **Thinking:** Use of planning and processing skills to devise a strategy to solve the problem, which includes applying the order of operations correctly.
- **Communication:** Clear expression of the mathematical thought process, including the correct and systematic documentation of steps taken to reach the solution.

Since the question did not explicitly ask the student to demonstrate the use of the given calculation in a practical application, it was not assessed under the "Application" category. However, it is important to have a balance of all four categories (Knowledge and Understanding, Thinking, Communication, Application) across the overall assessment plan, to ensure a comprehensive evaluation of student learning.

USER

I'm going to provide you with my thought of this solution and how I would mark this question. Can you provide feedback on my thought and include your reasoning, please? I will provide an explanation of how I graded this question in the next prompt.

ASSISTANT

Certainly, I will be ready to review your grading approach and provide feedback based on the guidelines established by the Ontario curriculum, pedagogical best practices, and the principles outlined in the *Growing Success* assessment document. Please go ahead and share your evaluation process and rationale for the question, and I will offer my insights.

USER

Instead of having the process broken down and evaluate their solution line by line, like what you did, I looked at the solution all together and recognized that this student has considerable knowledge in the order of operation. He has successfully identified the correct order to complete multiplication in a bracket, then division and then exponent. The only mistake he made in the process was in the last line, where instead of going from left to right, he performed addition first.

ASSISTANT

Your holistic approach to evaluating the student's solution is valid and reflects an understanding of the student's overall grasp of the order of operations. It's commendable that you acknowledge the student's ability to identify the correct order for most of the operations. This perspective aligns with the *Growing Success* document's emphasis on considering the student's overall understanding and the process they demonstrate, rather than just the final answer [growsuccess.pdf].

However, I would like to point out a few considerations based on the student's solution provided:

1. **Multiplication within brackets and division:** While the student correctly identified the need to perform multiplication and then division, they made a computational error during the division step (1.2 / 0.2), which they reported as 0.06 instead of 6. This is not just a procedural error, but also a fundamental misunderstanding of division.
2. **Exponentiation:** The student correctly noted that exponentiation should be performed before addition and subtraction, but did not actually compute the square of 0.6, leaving it as 0.6^2.
3. **Final computation:** The student's final step of performing addition before subtraction was not the only error; there was also a computational error, as they subtracted 1.26 (which is not present in the original expression, nor a result of previous steps) from 7.2.

In your evaluation, it is important to recognize the student's correct understanding of the order of operations up to a point, but also to identify the areas where misunderstandings and computational errors have occurred. The "Thinking" and "Communication" categories of the Achievement Chart can help capture these aspects by assessing the student's reasoning process and the clarity with which they express their mathematical thinking.

When providing feedback to the student, it's beneficial to highlight their initial correct approach and also guide them through the errors to ensure they understand where and why they went wrong. This feedback can be an invaluable learning opportunity, helping the student to correct their misconceptions and improve their computational skills.

To summarize, while your evaluation is positive and acknowledges the student's strengths, it is equally important to address the specific errors made to support the student's learning and development. It's essential to balance encouragement with constructive feedback to help the student grow in their mathematical understanding and proficiency.

USER

Would you change your marking based on my analysis that this student has considerable knowledge in the order of operations, showed correct format by working vertically downwards and completing each calculation according to the order of operation; main area of improvement would be he could have correctly multiplied and divided decimals.

ASSISTANT

Given your analysis, it's clear that you acknowledge the student's strengths in understanding the order of operations and their organized approach to solving the problem. This is an important aspect of the "Communication" category, as it reflects the student's ability to present their work in a clear and systematic way.

However, as per the "Knowledge and Understanding" category, it's crucial to accurately perform calculations, including multiplication and division of decimals. This is a significant component of the student's mathematical understanding that needs to be addressed.

Your focus on the student's process is aligned with the Ontario curriculum's emphasis on mathematical processes, which includes problem-solving, reasoning, and proving, reflecting, selecting tools and computational strategies, and connecting [High-Impact Instructional Practices in Mathematics.pdf].

Assuming a three-mark question divided equally among "Knowledge and Understanding," "Thinking," and "Communication," and based on your analysis, here's how the marking might be adjusted:

- **Knowledge and Understanding:** While the student understood the order of operations, the decimal computation was incorrect. However, given your emphasis on their overall understanding, you might choose to award 0.5/1 for recognizing the order of operations, but missing the accuracy in calculations.
- **Thinking:** The student demonstrated the ability to follow the order of operations until the final steps, where they made an error. This indicates a lapse in their thinking process. You might award 0.5/1 for the correct set-up, but deduct for the calculation errors.
- **Communication:** The student communicated their steps clearly by working vertically and showing each step of the calculation. This aspect of their work could be fully credited, so 1/1 for communication.

This would result in a total score of 2/3, with the understanding that the student has shown considerable knowledge and clear communication, but needs improvement in executing decimal operations accurately.

It's important to provide feedback that guides the student on where the errors were made and how they can improve their decimal computations. This could involve practice with decimal division and multiplication, as well as careful attention to the details of their calculations.

In conclusion, it is appropriate to award marks for what the student has done correctly, while also providing specific feedback on areas for improvement, in line with the principles of assessment for learning and the Ontario curriculum expectations.

Reference

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. D.C. Heath & Company. ▲

▲ Transformations: The Frieze as a Stepping Stone to Tessellations



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Dr. Sibbald is a professor with the Schulich School of Education at Nipissing University. He teaches pre-service mathematics courses for Junior, Intermediate, and Senior divisions. He also teaches in the graduate program, most recently a course focused on mathematics education issues.

When teacher candidates examine the curriculum, it is not uncommon that transformations are raised as something they are not altogether sure about. The word "tessellation" catches some off guard, and they question their backgrounds in the topic and are unsure of practical applications. Often there is an underlying issue or perception that this is a two-dimensional topic (i.e., covering the plane) and little awareness of how it can be thought of as a one-dimensional problem. This article challenges the issue by showing how a "strip tessellation" is a rich topic, and clarifies the utility for teacher candidates, teachers, and students in daily life.

Transformations in the Curriculum

All tessellations have a foundation that relies on transformations. In a "strip tessellation," the transformations act like wallpaper border patterns—they are constrained in one direction, or dimension, and are repetitive without constraint in another direction or dimension. It is the constrained scope in one direction that distinguishes this from two-dimensional tessellations, and serves as a beneficial instructional stepping stone to developing a strong conceptual understanding.

Transformations are introduced in Primary grades as a source of patterns. Even in Kindergarten, play using jigsaw puzzles includes transformations. They are made explicit in Junior grades as translations, reflections, and rotations. In Grades 6 and 7, combinations of the individual transformations appear in the expectations. In Grade 8, a spatial sense expectation (E1.1) specifies: "identify geometric properties of tessellating shapes, and identify the transformations that occur in the tessellations" (Ontario Ministry of Education, 2020).

If we take the expectations apart, the Grade 8

expectation relates to tessellations. Due to transformation skills often being taught in two dimensions (i.e., the plane), tessellations in Grade 8 are often interpreted as two-dimensional. The implication is that in Grades 6 and 7, instruction is setting students up to understand combinations of transformations in the plane or two dimensions. A natural stepping stone, however, is to consider transformations along a strip or border, which has implications for the earlier grades. It is within this context that this article was born.



Figure 1: *Example of a frieze pattern along the roof line of the First Presbyterian Church in Chatham, Ontario*

Strip Transformations

Many older buildings include a horizontal decorative motif near the roof. This is called a *frieze* and serves structural purposes, while including an artistic element. In Figure 1, there is a frieze pattern just below the roof that appears as a sequence of small rectangles hanging downward.

The decorative aspect of friezes is also found in wallpaper borders. The strips, which may be 6–8 inches wide, are found on walls at the ceiling (perhaps as an alternative to crown moulding) or, sometimes, as a chair rail that allows a transition in decorating features midway up a wall. In both cases, there is an aesthetic aspect, though there can also be practical purposes behind their usage. For example, finishing the edges around a ceiling was difficult with plaster and, to a lesser extent, with drywall, leading to the use of a border or crown moulding as a way to address the challenges. In architecture, I have observed where a frieze is placed over the ends of structural features, such as nuts on threaded bars, and it protects structural components from weather.

Mathematical Models

Friezes and wallpaper borders have a fixed width and repeating patterns along the length. The repeating pattern is specifically a translation, perhaps so the pattern can be printed using an inked wheel. All repeating patterns have a smallest atomic component that is distinct. The atomic component is unique because it cannot be split into two or more congruent pieces. The smallest component is called a *motif*, which is not a term used in the curriculum, but it is used in the mathematics literature and will be used here. If, for example, you consider a border that is "… ABCABCABC…," the motif is "ABC" because it repeats by translation, and "ABC" cannot be subdivided into congruent pieces.

The transformations that apply to a motif are the ones described in the curriculum. However, there is a small twist, where the motif is copied by translation, so the motif cannot contain a translation in the same direction. For example, in the frieze "…YYYYYYY…," the motif is "Y" and not "YY" because this is not the smallest component. So, the challenge is to consider all possible ways that transformations can be used to generate motifs that can be translated to make friezes. This is a classic and important mathematical concept for developing an understanding of two-dimensional tessellations.

Consider a motif to be like a stencil shaped like a jigsaw puzzle piece. A frieze, modelled in Figure 2, is generated when the motif is translated and the stencil is copied. The question then arises of how many different types of motifs can be made, that is, how many mathematically unique motifs. Artistically the answer would be endless, but mathematically there is a limited number of transformations within the motif.



Figure 2: *Creating a frieze by translating copies of a motif*

The transformations in the curriculum, with the exception of translation, are based on symmetry, which leads to the consideration of how symmetry can inform the construction of motifs. For example, if the letter "A" is used as a motif to generate the frieze "… AAA…," it is different from whether "E" generates "…EEE…" because "A" has a vertical line of symmetry, and "E" has a horizontal line of symmetry. These motifs are different and lead to distinct symmetry properties of the friezes—notably the A-frieze has an infinite collection of vertical lines of symmetry, whereas the E-frieze has a single horizontal line of symmetry. (An astute observer will notice that the friezes were generated with horizontal translation, but the distinction remains, even if one were to use vertical translation.)

Motif Symmetries

Let's assume that a translation will always be in a horizontal right/left direction. This simplification for communication does not impact the result because a frieze can be rotated to align the translation direction with the horizontal. This also facilitates discussion of symmetry within a motif, using language suited to Intermediate students.

The symmetries that one might use within a motif are:

1. reflection through a horizontal line,
2. reflection through a vertical line,
3. reflection through a point,
4. a half rotation, or
5. a horizontal glide reflection, which is a reflection followed by a translation.

Note that the term "glide" does not appear in the curriculum, but is used in mathematics to express the composition of a reflection and a translation. The glide is also called "transflection" and is considered a symmetry; however, that symmetry is not entirely evident until it is used in a frieze pattern.

There are a couple of technical details to be aware of. Firstly, each reflection inside a motif is described as horizontal or vertical, but only because the translation direction was set as horizontal. Generally, the reflection inside a motif is either aligned with or perpendicular to the direction of translation used to generate the frieze pattern. Second, and this is a crucial clarification from the defining mathematical characteristics, the allowable symmetries must be repeatable; that is, if an action occurs inside the motif (say a half rotation), then repeating (i.e., two half rotations) must still define part of the motif. The requirement for repeatability is a defining requirement of a mathematical group. What is particularly important about this is that it disallows some possibilities. For example, translation by 45°, which would resemble "A^A," does not work because if it were done repeatedly, more and more copies would appear, and that is contrary to having a constrained direction (i.e., maximum height). (It may not show typographically, but the "A" and translated "A" that appears as the exponent are supposed to be the same size. Repeating would add a third exponent "A" that again would be the same size, etc.).

Some aspects of repeatability need consideration by students. Reflecting "E" through a horizontal line produces "E," which is the same, and further repeating produces the same result. Reflecting "A" through a horizontal line produces an upside down "∀," but then, if it is reflected again, it returns to "A"—this means the motif is constrained in the vertical direction, and this reflection is repeatable. Lastly, a glide reflection (a reflection followed by a translation), as shown in Figure 3, is allowed. In this case, repeating the glide reflection with the upside-down "A" does not restore the original "A" because the glide includes translation to the right, but in terms of the overall frieze pattern, it will overlap the next copy of the motif.

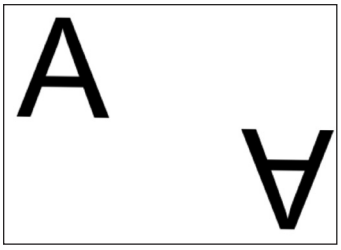


Figure 3: *A glide motif*

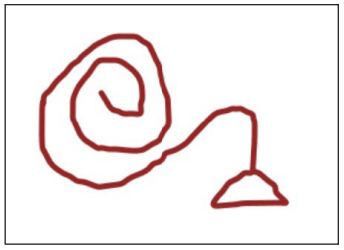


Figure 4: A squiggle

Imagine starting to build a motif with a squiggle, such as Figure 4, that has no symmetry. In Figure 5, the squiggle from Figure 4 is shown at the top left, labelled "0" and noting it is the original squiggle. The use of symmetry generates other versions of the squiggle. Squiggle 1 is a reflection through line A, 2 is a reflection through line B, 3 is a reflection through point C, and 4 is a glide that is a reflection combined with a translation. These are the only transformations that can generate a distinct motif.

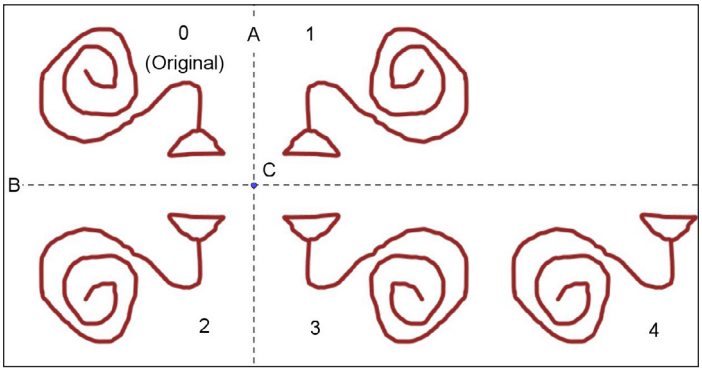


Figure 5: *Transformations of an original drawing*

Using Motifs to Generate Frieze Patterns

Each motif must contain the original pattern and may contain other transformed versions of the original pattern. This leads to the question of whether any combinations of transformations lead to the same frieze pattern.

The simplest case is to use the original squiggle "0" and generate the frieze from that—each motif will have no symmetry properties. The 0-frieze is shown in Figure 6, with the motif marked by a rectangle.



Figure 6: *The 0-frieze with a rectangle around the motif*

We can pair any transformation with the original to make a motif with a unique symmetry property. Combining "0" and "1," and writing "0–1," generates a motif with a vertical line of symmetry. The 0–1 frieze is shown in Figure 7, with the motif marked by a rectangle.



Figure 7: *The 0–1 frieze with a rectangle around the motif*

The other combinations—0–2 (Figure 8), 0–3 (Figure 9), and 0–4 (Figure 10)—all generate motifs with unique symmetry properties.

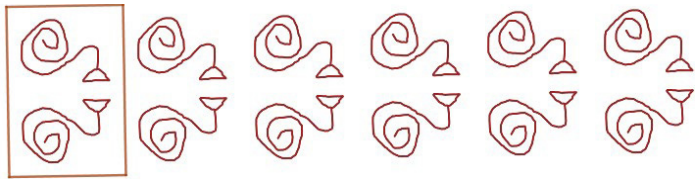


Figure 8: *The 0–2 frieze with a rectangle around the motif*

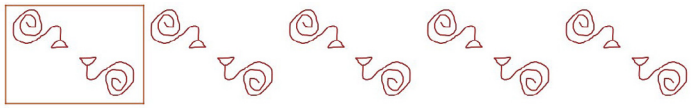


Figure 9: *The 0–3 frieze with a rectangle around the motif*



Figure 10: *The 0–4 frieze with a rectangle around the motif*

There are two more unique motifs that are based on combining more than one symmetry property. The "0–1–2–3–4" motif leads to the frieze shown in Figure 11. As a collection of five squiggles, it has both vertical and horizontal lines of symmetry, and that makes it distinct from any of the motifs based on pairs of transformations. Notice that the glide component, the lower right squiggle in the motif, could be positioned to overlap the lower left squiggle in the next copy of the motif—that is why 0–1–2–3 is not required.



Figure 11: *The 0–1–2–3–4 frieze with a rectangle around the motif*



Figure 12: *The 0–2–4 frieze with a rectangle around the motif*

Similarly, the "0–2–4" motif is a collection of three squiggles shown in Figure 12. The left pair has a horizontal line of symmetry, but the glide also gives this an alternating pattern when the frieze is constructed.

Seven Frieze Patterns

The collections of transformations: 0, 0–1, 0–2, 0–3, 0–4, 0–1–2–3–4, and 0–2–4 (Figures 6–12 examples, respectively) are the only collections of transformations that generate frieze patterns. Those seven options are the only possible ways to make friezes! The approach taken requires some consideration of other possible combinations of patterns. However, the requirement of repeatability—that transformations within the motif can be repeated within the border constraints—eliminates some contenders. The result that there are exactly the seven possibilities has been proven mathematically (see Martin, 1982).

If readers look at other resources, they will find the representations vary and often use lines of symmetry internal to the "squiggle." An example is the letter "N," which has symmetry as a reflection through a point placed in the middle of the letter. The approach taken here has the lines and points of symmetry outside of the squiggle to highlight the concept. I did find, however, that Table 1 was teacher friendly, unlike many other resources. In the table, F_1 is the 0-frieze, F_2 is 0–3, F_3 is 0–2 (the example could also be 0–2–4 with overlapping motifs), F_4 is 0–1, F_5 is 0–1–2–3–4 (with overlapping motifs), F_6 is 0–4, and F_7 is 0–2–4.

Frieze Patterns	Type of Symmetry			
	Point	Center Line	Perpendicular Line	Glide Reflection
F_1 LLLLLLL				
F_2 NNNNNN	x			
F_3 DDDDDDD		x		x
F_4 VVVVVVV			x	
F_5 HHHHHHH	x	x	x	x
F_6 ΛV ΛV ΛV	x		x	x
F_7 ΓL ΓL ΓL				x

Table 1: *An alternative description from www.cut-the-knot.org/triangle/Frieze.shtml*

For teaching tessellations, the frieze patterns clarify transformations. For two-dimensional transformations, many patterns are based on a rectangular, rhombus, or parallelogram grid. A single strip of this sort of grid can be thought of as a frieze. That, I claim, is a stepping stone to having a conceptual understanding of two-dimensional tessellations as a combination of transformations. It is not a complete story, since tessellations based on triangles or hexagons do not align with frieze patterns, but as stepping stones go, it can help the students who struggle, and solidify the transformation concepts for all students.

Reference

Martin, G.E. (1982). *Transformation geometry: An introduction to symmetry.* Springer.

Ontario Ministry of Education. (2020). *Elementary mathematics.* King's Printer for Ontario. www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics ▲

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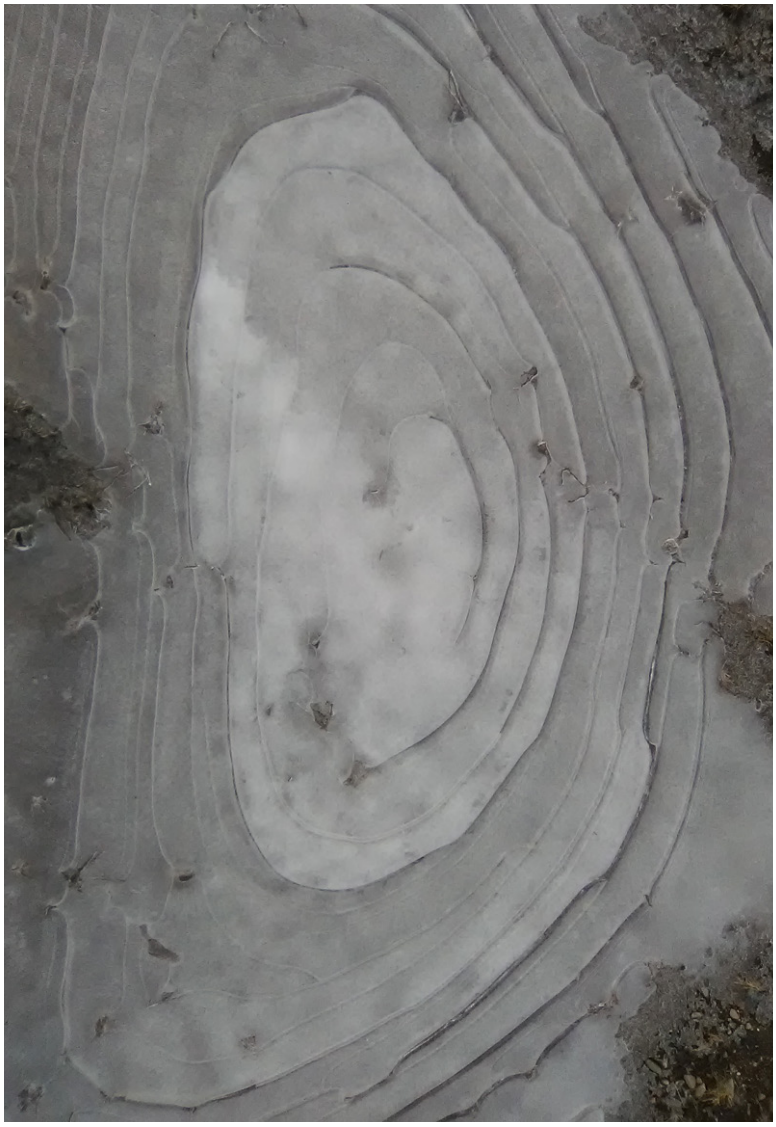
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I



J



I) Buoys, Oh Boy!
These can inspire various hands-on construction activities—Will it sink or float? How much is below the surface?

J) Contour Lines (Perth, Ontario)
How might students describe these contour lines? What reason might explain them not being round?

K) How Many Cannonballs?
Here is a pile of cannonballs in Puerto Rico. How many are there? What assumptions have you made?

L) Spot the Pentagons?
Using triangles to build a sphere generates many hexagons and a few pentagons. Can you spot any pentagons in this sculpture from Valette, Malta?

M) Coral Patterns?
Describe any pattern you see in this coral on display in Cartagena, Columbia. Does the pattern continue indefinitely, or does it break down after a while? Since coral is a living organism, do the changes in patterning look like choices? Can you tell?

L



K



M

