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Ontario Mathematics Gazette

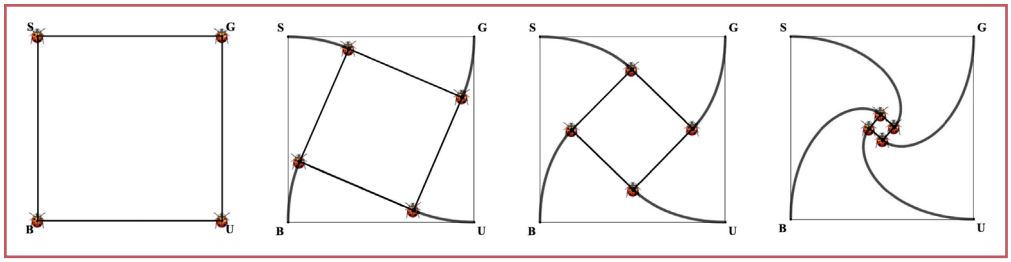
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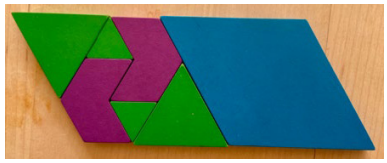
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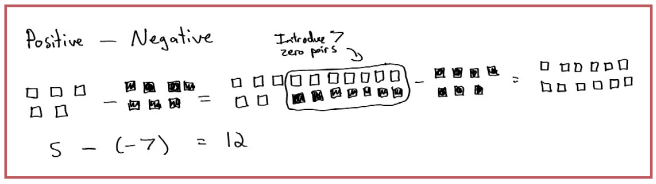


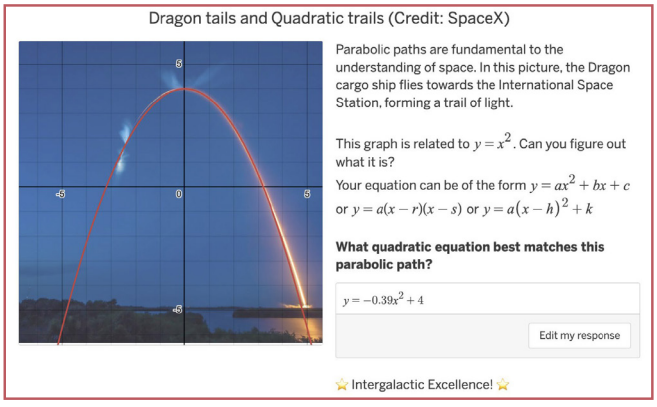
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Pi-day-inspired cartoon
by Julianna Kagawa.

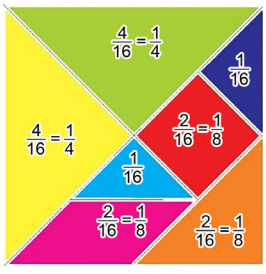
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Submission of Articles

The Ontario Mathematics Gazette (Gazette) publishes news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *Gazette*:

- The *Gazette* is aligned with the OAME/AOEM Vision, Mission, and Strategic priorities. These can be found on the website www.oame.on.ca under "About OAME" and "Guiding Principles."
- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles are to be word–processed, MS Word is preferred, and prepared according to the *Publication Manual of the American Psychological Association*, Seventh Edition. However, please use single-line (not double) spacing and only one space after each period. Articles should not exceed eight numbered pages of text, though there is some flexibility. Figures, images, and photographs should be placed close to the relevant text and include captions. The photographer's permission is required, and for photos of students under the age of 18, permission of a parent or guardian is required.

Please submit your article in one blind file (i.e., identity of author is not evident), and include author names, contact information including email and mailing addresses, photos—head and shoulders, biographies—less than 100 words, and all content removed for blinding in a second file. Please email these two files to Tim Sibbald at gazette@oame.on.ca.

Upon review, you will be notified whether your article has been accepted for publication (as is, or pending minor or major revisions) or declined. The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME/AOEM.

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Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

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Editor's Report



TIM SIBBALD, OCT, PH.D.
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Tim Sibbald is a professor in the Schulich School of Education, Nipissing University, with a focus that includes mathematics education. He is a former President of OAME/AOEM and has held a variety of other OAME/AOEM posts over the years.

Put your students out of your mind for a moment. Let's focus on you.

Presumably you have some interest in mathematics if you are a mathematics teacher. Granted some elementary teachers may be teaching math as an obligation, rather than an intrinsic interest, but as a reader of the *Gazette*, I am inclined to believe that there is evidence of your interest in the subject.

If we avoid the trappings of our career, the various meetings, planning, events such as report cards, and focus on the professional aspect, what is evident is a need to self-direct our own learning. I claim some of the time this needs to focus on ourselves and our interests, and that doing this has benefits for our students. Much of teaching is altruistic, with many teachers taking little credit where they deserve a lot. This is a reason why award nominations are relatively sparse, but I digress.

The focus on you is to ask: When was the last time you engaged in some mathematics simply because it is something you clearly find interesting? When do you continue to learn about the subject itself outside of the constraints of your career? I might well ask when do you exercise your creativity about your own engagement with the subject?



Figure 1: *2025 theme is "Mathematics, art, and creativity"*

Math learning, as a teacher, can be effective when aligned with creativity. To my mind, the two go hand in hand and, additionally, you have an advantage that you can apply what you know about pedagogy to your own learning. Why do you choose a book approach? Or a particular branch of

math to investigate? Can you differentiate your own professional math learning? How can you induce creativity in your own professional math learning?

Increasingly creativity is important as a topic inside mathematics. It is included in the theme for the International Day of Mathematics (Figure 1). However, this should perhaps not be a surprise because it comes only a short time after the OECD Programme for International Student Assessment (PISA) examined creativity in mathematics education with their international testing in 2022 (OECD, 2024). As an aside, in case you are not aware, they also examined equity, financial math education, and the pandemic disruption. I will provide a few snippets regarding these, but my focus is very much on creativity (and your professional learning).

PISA testing in 2022 occurred during the late stages of the pandemic. It engaged 81 countries and 700 000 students, and it found that 31 countries managed to maintain performance through the pandemic (OECD, 2024, p. 3). Canada is described as having been resilient in performance and equity, with small decreases due to the pandemic, and a decrease that is below the OECD average for resilience in well-being, which was measured in terms of sense of belonging (OECD, 2023, p. 26). Maintaining performance has common elements of "shorter school closures, fewer obstacles to remote learning, and continuing teachers' and parental support" (OECD, 2024, p. 3). However, care is needed because maintaining performance, when it had already shown a declining trend, means the pandemic cannot be entirely blamed for decreases in student performance because something else was already going on. The underlying performance issue, apart from the pandemic, is clarified as follows:

> In fact, one in four 15-year-olds is now considered a low performer in mathematics, reading, and science on average across OECD countries. This means they can struggle to do tasks such as use basic algorithms or interpret simple texts. (OECD, 2024, p. 3)

In terms of equity, findings from ten countries, including Canada, led to the remark that "education in these countries can be considered highly equitable" (OECD, 2024, p. 3). Results for financial literacy do not include Canada because a national sample was not achieved for this.

On the question of declining performance, I will offer one aside. "Some 30% of students, on average across OECD countries, reported that, in most or every mathematics lesson, they get distracted using digital devices; 25% of students reported that they get distracted by other students using these devices in class" (OECD, 2023, p. 24). Having been involved in some of the research that found benefits for students to use technology, I have been concerned over the last decade that the research was with small amounts of technology usage—a period in the computer room—as opposed to the sort of immersion that is now observed. In effect, I question whether the research that supported the use of technology was aligned with what is now being done in terms of usage. It is therefore important to note that PISA found beneficial effects of technology usage for instruction, but detrimental effects when used for leisure, such as the distractions described in the quotation.

While Canada (particularly Ontario and Alberta (OECD, 2024, p. 257)) performed well in creativity: "…education systems such as Singapore, Korea and Canada* are among the highest performing systems in terms of both creative thinking and performance in mathematics, reading and science…" (OECD, 2024, p. 5; *sampling detail available in full report). The PISA report notes that only half of the students believe that creativity can be taught—yet it can. There is a gender gap in creativity—"In no country or economy did boys outperform girls in creative thinking" (OECD, 2024, p. 24). Along with this, half of students have a fixed mindset about creativity, yet a growth mindset is positively related to performance. In Canada, girls outperformed on creativity by 10 percent on average and 25 percent when high performers (top 25 percent by performance) were compared. Related to this, across all countries, advantaged students have creativity equal to or better than disadvantaged students after accounting for math and reading performance.

Examination of aspects of creativity shows Canada is above the OECD average for the generation of diverse ideas, generation of creative ideas, and evaluation and improvement of ideas. Canada was also above the OECD average for categories of written expression, visual expression, social problem solving, and scientific problem solving. Notably the gender gap was least for scientific problem solving, but revealed that girls outperformed for all four categories.

In terms of beliefs, Canada was above average for believing creativity is malleable and possible in "nearly any subject" (OECD, 2024, p. 34). Attitudes also scored above average. In terms of social–emotional characteristics,

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categories of considering perspectives and persistence were at the OECD average, and curiosity was slightly above average. "Students tend to believe they are more capable of creative thinking when prompted to think about solving broad, real-life problems, rather than when working on school tasks" (OECD, 2024, p. 157).

While the PISA report has much more detail in its 300 pages on creativity, as well as all the pages in the three other reports, it is fair to say that we (Canada) are doing well in terms of creativity. It is also important to note that performance in math and creativity are weakly correlated (measured at 0.67), and that the correlation is weaker than between math and reading (0.80) and between math and science (0.87) (OECD, 2024, p. 83).

Big Prime Discovered!

In October, GIMPS (Great Internet Mersenne Prime Search – www.mersenne.org) announced that $2^{136279841} - 1$ is prime. This is a number with 41 million digits and supports further research about prime numbers.

So, having set out to talk about you, this editorial appears to have gone sideways (to PISAs, if you enjoy puns). In terms of your social–emotional place, consider that you might choose to view the math performance issue as a problem, the creativity performance as a positive, or recognize that both of these are true and tied to your career as opposed to your professional math learning. What we can also take from this is an opportunity in our own professional math learning to consider our creative engagement. Perhaps for our professional learning, we fall into a trap of revisiting the higher education math learning we did years ago, but I suspect generally, that educational period emphasized content and performance over creativity. The weak correlation between performance and creativity, albeit derived from testing of 15-year-olds, is suggestive that professional math creativity may provide an alternative approach to professional math learning. Not that it should be exclusive, but maybe it is a way to consider active learning in a new light, and a light that perhaps opens new avenues for learning about a subject in which you are interested.

In short, this editorial set out to consider the notion of what teachers do for their own professional learning in mathematics. In my opinion, the key is that taking a creative approach allows our own professional math learning to serve a dual purpose of rejuvenating ourselves, while continuing to learn about a subject area in which we are interested. It is about you, because your creativity, your rejuvenation, and your further learning inside mathematics are all pivotal to your own well-being. But if you would like an altruistic interpretation, that well-being serves to make your day-to-day efforts easier, as well as benefiting your own students. The best teachers are creative, look after their own well-being, and are altruistic because they can afford to be—and being positioned to do that is *about you!*

In the News…

Dr. Eric Muller has appeared as an author in the *Gazette* 12 times and was awarded the OAME Life Membership Award in 2005. However, on this occasion, congratulations are because Brock University put his name on a student learning centre. It is now called The Eric Muller Math and Stats Learning Centre. Congratulations, Eric!

The Counter is a new newsletter with OAME/AOEM content that provides a way to share more rapidly than the *Gazette/Abacus*. It is also distinct from the various social media feeds that OAME/AOEM provides (such as on X, OAMEwrites, OAMEcounts, OAMEtalks, and OAMElearns). The newsletter is available to non-members where they can sign-up to receive it—so, please, share it with your colleagues! Feedback by way of letters to the Editor or direct to *The Counter* are welcome.

In This Issue…

OAME/AOEM President **Kit Luce** draws attention to the Ontario Curriculum Context and barriers that need consideration. **Pamela Brittain** gives a brief of talks that contains something for everyone, at the MathEd Forum and the Canadian Mathematics Society Education meeting. **Jacqueline Hill** explains how NCTM can support us to THINK, and mentions some upcoming events. **Kit Luce** also reports on the OAME/AOEM Leadership Conference that focused on math-oriented universal design for learning.

The *Gazette* welcomes **Sofia Ferreyro-Mazieres**, who has written Mb4T while **Jennifer Holm** takes maternity leave. Mb4T highlights fraction investigations, with a focus on multiple solution methods. **Iain Brodie** and **Hatice Beyza Sezer** discuss coin flipping and the power of using technology for larger samples. **Lynda Colgan** folds some tangrams into the mix. **Shawn Godin** tries to capture his process of developing a problem and ends up with an excellent overview of thinking about lines. **Jérôme Proulx** highlights how the understanding of percent, particularly with taxes, can be context dependent. **Carly Ziniuk** guides the reader on ways to use the Desmos Activity Builder, while drawing on space-based resources. **David Costello** addresses intentional practice that prompts us to consider why we ask students to practise and the ways we do that. **Jeff Irvine** focuses on self-efficacy—the belief in one's own

ability to accomplish something. In case that isn't enough, **Ron Lancaster** is back, and he has a new focus—I love the bug problem and plan to tinker in the coming days. On this occasion, you will not find the "OMCA News," but it is anticipated to return in the next issue.

By way of articles,…

Thank Yous

Thank you to the reviewers, who ensured that articles that have been received led to feedback to the authors in a timely manner.

Thank you to the Associate Editors, **Anne Yeager** and **Peter Saarimaki**, who are excellent at checking for unintended meanings, general clarity, suggestions to make techniques more fulsome, and providing sound advice on all aspects of the editorial process.

Thank you, **Gitta Berg**, for polishing everything that goes into the *Gazette*. She is very detail oriented and makes sure the *Gazette* is consistent with spelling, grammar, and all aspects of presentation. Remarkably, this is often done within 24 hours.

Thank you, **Penny Clemens**, for doing the entire layout of the *Gazette*. The covers you create make it look sharp. But hidden away is how you always make things fit, despite any eleventh-hour tweaks.

I sincerely appreciate the entire editorial team. For the third year in a row, the preparation of this issue has landed on the busiest weeks of my work year. But, for the third year running, despite my busy-ness, they have all contributed to catching things I missed. Thank you.

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▲ Letter to the Editor

I have a list of about two dozen problems and investigations that I have used in my occasional visits to school classrooms. The idea is to arouse the interest of the students by directing them toward some mathematical goal that gives context to the material that they are learning in the regular curriculum. The problems are designed to be broadly accessible, but to lead toward more advanced material that can challenge the abler students.

While the problems are generally well received, what actually happens in the classroom can vary quite a bit, depending on the characteristics of the setting and the students. This motivates me to ask for the assistance of your readers. I am looking for people to take one of these problems that interests them, rework it to suit your situation and try it out with your students, and then write to tell me what the setting was, what you did, and how things turned out. For each problem, I have provided some insight and described my own experience, and I would like to augment this with comments from others so that we end up with a document that can serve as a useful guide for teachers.

To give you an example of what can be done, consider the Erdos-Strauss 1948 conjecture that for every integer n exceeding $2, \frac{4}{n}$ can be written as the sum of three distinct reciprocals of positive integers, for example, $\frac{4}{7} = \frac{1}{2} + \frac{1}{15} + \frac{1}{210}$. (You can find this online by searching "$\frac{4}{n}$ problem" or "Erdos-Strauss conjecture.") At a technical level, this problem provides experience in dealing with arithmetic operations, particularly with respect to fractions, as well as constructing examples, looking for patterns, and expressing them algebraically. I find that there are straightforward entry points that many students can find, when n is even, a multiple of 3, or one less than a multiple of 4. Within a class period, students can find procedures to produce a representation, except when n is one more than a multiple of 24. This is where the conjecture part comes in; it has been checked numerically for n up to about 10^17, but the problem is still open for all n. Perhaps some students might like to use a calculator or program a computer to search for representations for larger values of n, such as $n = 73$.

For a geometric investigation, students can be asked for the largest number they can find of distinct points in the plane for which the distance between any pair of them is an integer. It is interesting to see how many groups pick up the "obvious" answer of stringing them along the line. If this happens, you can ask that not all the points be collinear; even if one point is off the line, the problem leads into some interesting territory. Pythagorean triples have a role here; a simple starting example is the set of vertices of a 3 x 4 rectangle. You can ask that all the distances be distinct, make a competition of it, and ask students to mount their answers on a bulletin board.

I would be happy to send my collection of problems to any of the readers; contact me at barbeau@math.utoronto.ca.

Ed Barbeau, University of Toronto

Note: In this issue, the column, "What's the Problem?," by Shawn Godin, draws inspiration from Ed's collection. ▲

President's Message



KIT LUCE
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Kit has been an active member of MAC² for many years and joined the Board of Directors in 2022. She is a District Mathematics Facilitator for Grades K–9 with the Simcoe County District School Board. In her work, Kit creates math resources to support educators in their instruction and assessment practice, as well as developing and facilitating professional development. Kit is an Additional Qualification instructor and course writer, and she has written and reviewed work for a variety of publishers. She has been involved in Ministry writing projects and was a Content Lead for the recent OAME/AOEM writing projects. Kit received the 2024 OAME Award for Leadership in Mathematics Education.

In the Ontario curriculum document, *Roles and Responsibilities in Mathematics Education* (Ontario Ministry of Education [OME], 2020a), there is a section about the roles and responsibilities for the different partners in students' learning. The section about students says the following:

It is essential that all students take responsibility for their own learning as they progress through elementary and secondary school. Mastering the skills and concepts connected with learning in the mathematics curriculum requires a commitment to learning that includes:

- continual and consistent personal reflection and goal setting
- a belief that they are capable of succeeding in mathematics
- developing skills in persevering when taking on new challenges
- connecting prior experiences, knowledge, skills, and habits of mind to new learning
- a willingness to work both collaboratively and independently
- dedication to ongoing practice
- an ability to receive and respond to meaningful feedback

(www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics/context/roles-and-responsibilities-in-mathematics-education#students)

I encourage you to take a moment to reread that section. Do you see your students reflected in these statements? What barriers do you see that might limit the ability of your students to develop these skills? How would you list these skills in order of importance?

There are clear connections here between this list and the components of Strand A: SEL skills and mathematical processes. For me, "a belief that they are capable of succeeding in mathematics" is essential for being able to achieve the other responsibilities, and therefore being able to achieve in mathematics, whether the child is in Kindergarten, Grade 9, Grade 12, or post-secondary mathematics. We are seeing math anxiety and statements about lack of confidence and enjoyment in math as early as Kindergarten and Grade 1.

In the 2023–2024 EQAO assessment, 68 percent of students in Grade 3 reported that they like math. This drops to 47 percent in Grade 6 and 48 percent for Grade 9.

When asked to rate their agreement with the statement, "A person can always get better at math," 89 percent of students in Grade 3, 88 percent of students in Grade 6, and 85 percent of students in Grade 9 agree. But when asked to rate their own ability in math, only 64 percent of students in Grade 3 state that they believe they are good at math. This drops to 50 percent for Grade 6 and 51 percent for Grade 9. It appears that students have a growth mindset about mathematics for others, but not necessarily for themselves.

So how can educators address this concern? What steps can we take in our classrooms to help students develop confidence, enjoyment, and a willingness to actively engage in mathematical learning. To build classroom environments, which place students as actively engaged in mathematics, we must plan with intention and purpose to create instruction and assessments that support students to develop a strong sense of themselves as capable and confident learners of mathematics.

There are clear strategies suggested in *High-Impact Instructional Practices in Mathematics* (OME, 2020b), and in the eight Mathematics Teaching Practices from NCTM's *Principles to Actions: Ensuring Mathematics Success for All.* For example, being intentional and transparent about learning goals sets students up for success because it demystifies math learning, helping students to see that math is not magic, nor does it have rules that only a few can know, but that math can make sense and can be understood. Math tasks need to be accessible by virtue of being open, or providing parallel (or thin sliced, mild, medium, and spicy) tasks. The contextual set-ups of our math tasks can provide windows and mirrors into the lives and communities of our students. We can provide opportunities for students to talk, to work with tools and models that are concrete and visual,

and to reflect and self-assess their understanding based on those shared learning goals and success criteria.

This past Fall, many of us attended the virtual OAME/AOEM Leadership Conference: Designing Equitable Mathematics from the Margins: Universal Design for Learning in Mathematics. The presenter, Dr. Rachel Lambert, shared her research that focuses on creating inclusive math classrooms, where all students can find joy, meaning, and learning. She encourages educators to examine their classroom environments, including lesson planning and assessment, with a critical eye, in particular, looking for barriers that limit access to mathematics for students and consideration of how they, the educators, can think critically and creatively to become designers for accessibility.

Take another look at the list of student roles and responsibilities. What are some adjustments you could make to your instructional practice to support students to develop these skills? When we plan for accessibility with intention and purpose, when we "place students at the centre of (our) mathematics planning, teaching, and assessment practices, and understand how the learning experiences (we) provide develop a love of mathematics and foster a positive "I can do math" attitude in all students" (OME, 2020a), then we move toward an environment that allows students to believe that they are capable of succeeding in mathematics, and perhaps even enjoying learning along the way.

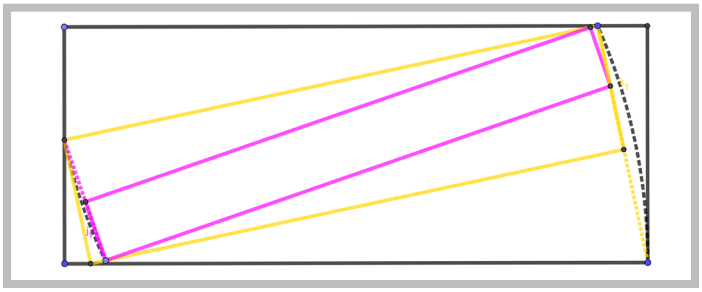
I encourage you to continue your own learning journey by engaging in math content and pedagogy learning in your school board as part of opportunities offered by the OAME/AOEM local chapters, and by attending the OAME 2025 Annual Conference in May 2025. The more we learn and work together, the better equipped we will be to meet the ever-changing and diverse learning needs of our students.

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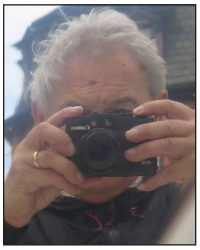
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▲ The Math Teacher's Home Base: Tools, Resources, and Inspiration



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Ron is an Associate Professor Emeritus at the University of Toronto, where he taught mathematics courses for pre-service teachers for 17 years. Ron taught middle and high school mathematics for 21 years in coed public schools and an all-girls school. Ron's professional activities include conference presentations/consultations in North America, Asia, England, the Middle East, Africa, and India. Ron has authored/edited over 200 articles for NCTM and OAME. Ron received the 2015 Margaret Sinclair Memorial Award Recognizing Innovation and Excellence in Mathematics Education, awarded by the Fields Institute, and was the recipient of the 2022 OAME Lifetime Achievement Award.

Welcome to "The Math Teacher's Home Base," where we will explore the power of creating a resource-rich, inspiring space for all things related to teaching mathematics. This column is dedicated to helping you build a personal teaching environment that not only supports lesson planning, but also nurtures a lifelong passion for math—whether you're brainstorming lessons, experimenting with activities, or simply enjoying math for fun.

Throughout more than 45 years of teaching, I've collected a variety of resources that have helped me teach in creative and interactive ways. Some might call them toys, games, or oddities, but for me, they're tools that make math come alive in the classroom. In each column, I'll share a few items from my home office—books, games, manipulatives, or even quirky objects—and explain how they've supported my teaching and sparked learning in new ways.

But I'm certainly not the only teacher who's been curating a math-filled home office! Many of you have your own treasures—resources that spark creativity, inspire new lessons, and make learning more engaging for your students. That's why I want this column to be a community-driven space, where we can share ideas, resources, and strategies. If you've found something that's worked wonders in your own teaching, I encourage you to contribute.

Let's explore together how the right tools can transform teaching into something joyful and energizing—both for you

and your students. Whether it's your home office, a classroom closet, or just a cozy nook for reflection, I hope this column will inspire you to keep building your own personal teaching heaven.

Featured Resources: *Engaging Lessons from Mathematics, a Human Endeavor: A Textbook for Those Who Think They Don't Like the Subject*

In 1979, early in my teaching career, I purchased *Mathematics, a Human Endeavor: A Textbook for Those Who Think They Don't Like the Subject* (1970) by Harold R. Jacobs, and it quickly became a cornerstone of my teaching. Over the years, I've used it extensively, and it has shaped my approach to teaching mathematics in ways I never imagined when I first opened its pages. The lessons, activities, and insights from this book have stayed with me. They continue to inspire me, even as I write this column today.

I'd like to share three of my favourite ideas from Jacobs' work—activities that have consistently captivated my students over the years. Drawing from a wide range of sources, including recreational mathematics, puzzles, and real-world applications, Jacobs' book emphasizes the beauty and interconnectedness of mathematical concepts. Inspired by this spirit of exploration, I've adapted and expanded upon his ideas to create engaging activities that connect with students' interests and help them develop a deeper appreciation for mathematics.

1. Exploring Curves with a Candle and Paper

In this hands-on activity, students explore various features of a curve and the cross-section of a cylinder when cut at an angle.

Materials Needed:

- Candle
- Photocopy paper

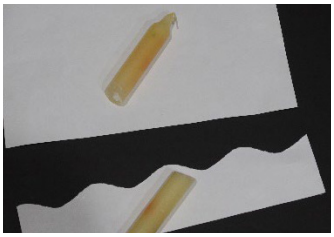
Activity Instructions:

1. Wrap a piece of paper several times around a cylindrical candle.
2. Use a knife to make an angled cut through both the paper and the candle.
3. Unroll the paper and examine the curve on each piece.

Safety Note:

For safety reasons, it's recommended that the teacher make the cut. If using a knife in the classroom is a concern, the teacher can prepare the model in advance and secure everything with tape.





Photographs 1 and 2*: Candle with knife, the curves after the cut*

Experimenting with Cutting Angles

- Repeat the process by cutting the candle at various angles.
- What changes do you observe in the shape of the curve when the cutting angle increases or decreases?
- Can you describe the relationship between the cutting angle and the resulting curve?

Experimenting with Candle Thickness

- Use the same angle for the cut with candles that have different thicknesses.
- What changes do you observe in the shape of the curve when the thickness of the candle increases or decreases?

Is It a Sine Curve?

- What makes you think the unrolled curve could be a sine curve?
- What specific features of the curve suggest it might be a sine curve?
- How might you verify whether it's a sine curve, using measurements or tools?

Cross-Section Observations

- Look closely at the ends of the candle pieces after making each angled cut.
- How does the shape of the cross-section vary with different cutting angles?
- Can you explain why the cross-section changes in this way?
- How might these observations relate to the sine curve?

2. Exploring Binary Numbers with Self-Sorting Cards

In this hands-on activity, students use a deck of specially designed cards to explore binary numbers, sorting algorithms, and the connection to powers of 2. This activity demonstrates how binary digits (0's and 1's) can be used to sort a set of cards efficiently, providing a tangible way to visualize mathematical concepts.

Materials Needed:

- 16 index cards
- A hole punch
- Paper clip (for sorting)

Activity Instructions:

1. Prepare the Cards: Use one card as a template and punch four evenly spaced holes along the top edge.
2. Repeat this process for all the cards, ensuring the holes align perfectly by using the template.

Labelling the Cards

Number the cards from 0 to 15, and write the binary representation of each number on its corresponding card.

Card 0: 0000
Card 1: 0001
Card 2: 0010
…
Card 15: 1111

Creating Slots for the Binary Values

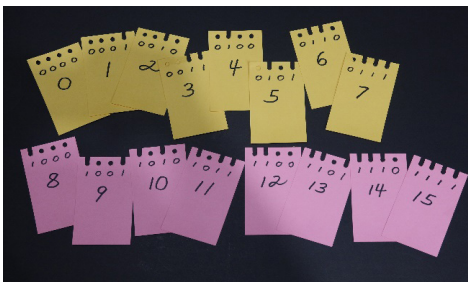
- On each card, create a slot by cutting a line from the hole to the edge wherever there is a "1" in the binary number. For example, for card number 6 (binary 0110), cut slots at the second and third holes, leaving the first and fourth holes uncut (see Photograph 3).

Mixing the Cards

- Shuffle the cards into a random order.

Sorting the Cards

- To sort the cards, use an opened paper clip to lift the cards by the first hole (from the right side). You might need to shake the cards so that the cards with a slot do not get lifted. If the cards stick together, gently separate them to ensure accurate sorting. Place all cards that have a slot at that position in your hand and place them in front of the stack.
- Repeat the process for each hole, moving from right to left. After four operations, the cards will be in numerical order.





Photographs 3 and 4: *16 cards with holes and slots, pack of cards with paper clip*

3. Mathematical Activities and Extensions

This activity provides opportunities for students to engage deeply with mathematical concepts. Here are some follow-up activities and questions to encourage exploration:

Understanding Why It Works

After completing the sorting process, ask students to reflect on why the sorting method works. Why does the position of the slots (representing binary digits) determine the order of the cards?

Exploring the Order of the Cards after Each Pass

- The table below illustrates how the card positions change after each pass during the sorting process. It is fascinating to study how individual cards move. For example, the card with a 6 on it was initially in position 9 from the top. Why did it move to position 5 after the first pass, then to position 12 after the second pass, and then to positions 14 and 7 after the third and fourth passes?
- How can you predict where the cards go without actually sorting the cards?

Initial order of the cards	Order of the cards after the first pass	Order of the cards after the second pass	Order of the cards after the third pass	Order of the cards after the fourth pass
14	14	4	0	0
1	2	12	8	1
2	10	0	1	2
10	4	8	9	3
11	6	1	2	4
4	12	13	10	5
7	0	9	11	6
13	8	5	3	7
6	1	14	4	8
15	11	2	12	9
3	7	10	13	10
9	13	6	5	11
12	15	11	14	12
0	3	7	6	13
8	9	15	7	14
5	5	3	15	15

Table 1: *Order of cards from top to bottom of the pile*

Exploring What Happens If a Different Order of Holes Is Used

- Lift the cards from the first hole on the left side, and then work your way through the holes from left to right.
- What new order of cards results from this change in lifting sequence?
- In what order would you select the four holes if you want the final order of the cards to be the odd numbers in order, followed by the even numbers in order?

Modifying the Activity for More or Fewer Than 16 Cards

- If we wanted to sort 32 cards, how many holes would we need in each card? Why?
- Randomize the order of the 32 cards and make a table for them, similar to Table 1, that shows their movement until all the cards are in order.
- How many holes would we need if we wanted to sort 50 cards?

4. The Four-Bug Problem

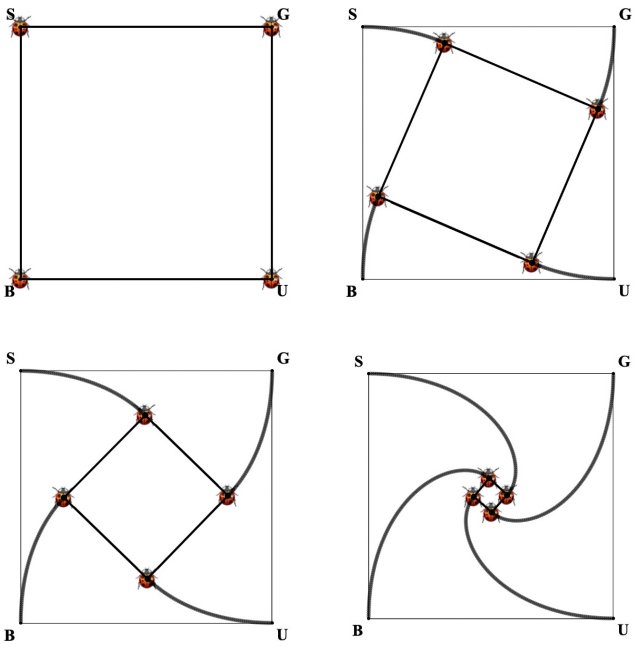
In this hands-on activity, students explore and extend a classic problem involving four bugs.

Materials Needed:

- Graph paper
- Rulers
- A large square drawn on the floor of the classroom, hallway, gym, or outside on the ground

The Problem

- Imagine four bugs starting at the vertices of a square: Bug B chases Bug U, U chases Bug G, and G chases Bug S.
- All the bugs spiral inward until they meet at the centre of the square (Figures 1, 2, 3, and 4).



Figures 1, 2, 3, and 4*: The Four-Bug Problem*

Drawing the Paths of the Bugs

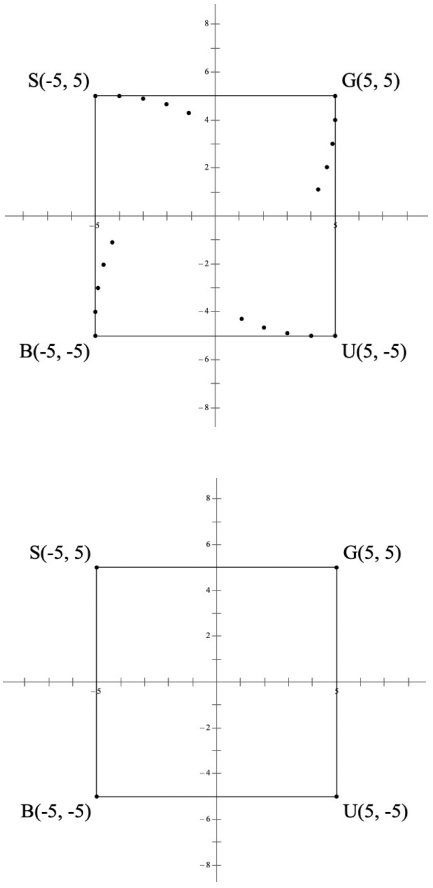
- Have students draw the paths of the bugs on graph paper.

Note to Teachers:

Many students may initially struggle with the idea that each bug must continuously adjust its direction. You might observe a student who, for example, has Bug B moving directly to where Bug U started, without understanding that each bug must constantly focus on the bug it's following.

How Far Does Each Bug Travel?

- Challenge students to determine the length of the path each bug travels. Since most students may not be familiar with calculating the length of a curve (arc length), this will be a non-routine problem that encourages creative problem solving, using available classroom resources. For instance, some students might draw the path and measure it with string. Others might notice (or need a prompt to consider) that the path length can be estimated by having the bugs move discretely. For example, Figures 5 and 6 demonstrate how this can be done on a coordinate grid.
- Instead of moving continuously, each bug can take one-unit steps, changing direction before each step.
- Using smaller steps will improve the estimate.
- You can also connect this to analytic geometry by having students determine the coordinates of the bugs after each step.



Figures 5 and 6: *Four-Bug Problem on a grid, bugs being discrete*

What Happens If More Bugs Join or Some Bugs Leave?

- If three bugs are placed at the vertices of an equilateral triangle, how far will each bug travel?
- What about placing the bugs at the vertices of a regular pentagon, hexagon, and so on?

Equations of the Paths

- For advanced students, challenge them to determine the equations of the spirals the bugs follow as they move toward the centre of the square.

Taking the Learning Beyond the Walls of the Classroom

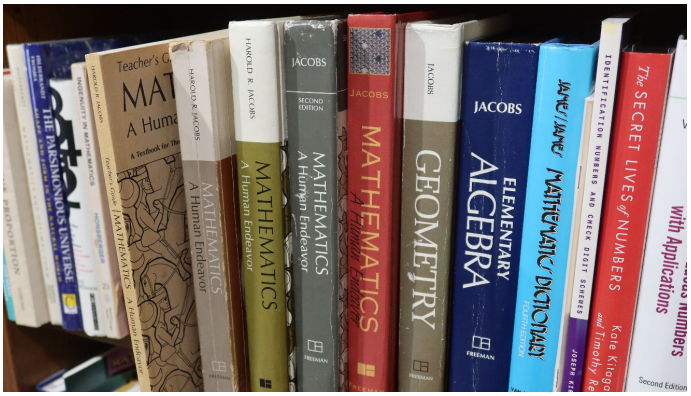
- The Four-Bug Problem has had a lasting impact on my teaching. I first encountered it in Jacobs' book, which included a full-colour image of the problem from the July 1965 issue of *Scientific American*. That image sparked my curiosity and led me to become a regular reader of Martin Gardner's column, eventually collecting all of his columns over 25 years.
- Another lasting take-away was how the Four-Bug Problem inspired me to develop hands-on math activities for students. I've used it in many ways, including having my high school students act out the problem on the first day of snow. The activity has also been part of math walks, including one at the National Gallery in Ottawa, co-created with Mary Bourassa in 2000. When *Feist's 1 2 3 4* video (see https://youtu.be/Xi_F4RAMW3A?si=0uOLjALku7NvzZbW) was released in 2012, my students at OISE/UT compared the bugs' spiraling paths with the one Feist created in her video, which features her being supported by a group of dancers.

Why These Activities Work

- Each of these activities brings abstract mathematical concepts to life. They encourage exploration, hypothesis testing, and discovery, helping students see the beauty and utility of math.
- These activities not only teach mathematical concepts, but also foster skills like critical thinking, collaboration, and hands-on problem solving—skills that are essential for the modern learner.

About Harold Jacobs and His Work

Harold R. Jacobs has been a pioneer in making mathematics accessible, engaging, and inspiring. Over the years, he authored three transformative books that have stood the test of time: *Mathematics, a Human Endeavor: A Textbook for Those Who Think They Don't Like the Subject* (1970, revised 1982 and 1994), *Geometry: Seeing, Doing, Understanding* (1974, revised 1987 and 2003), and *Elementary Algebra* (1979, revised 2016). His approach was ahead of its time, combining practical applications with an engaging and student-friendly style.



Photograph 5: *My collection of books by Harold Jacobs*

Mathematics, a Human Endeavor: A Textbook for Those Who Think They Don't Like the Subject —the book that has been a cornerstone of my teaching—was originally written to help students who struggled with math. The revised editions only deepened its impact, with Jacobs using humour, real-world examples, and a gentle introduction to complex ideas to draw students in.

His second book, *Geometry: Seeing, Doing, Understanding*, incorporates innovative ways to teach geometry and connects with modern tools like The Geometer's Sketchpad®. Meanwhile, *Elementary Algebra* presents algebra concepts in a unique sequence, offering a mix of inquiry-based learning and traditional structure.

On a personal note, meeting Harold Jacobs was a dream I had cherished for over 40 years. Over the years, our paths crossed at conferences, but it wasn't until the NCTM Annual Conference in Los Angeles in September 2022 that I finally had the chance to meet him. That unforgettable evening, spent with Jacobs and other inspiring figures like Audrey Weeks (Photograph 6), reminded me of how fortunate we are to stand on the shoulders of giants in our profession.



Photograph 6: From the left: Harold Jacobs, Ron Lancaster, Brent Ferguson, Audrey Weeks (the Queen of Sketchpad), and Dan Butler

Continued on page 24

Elementary Math Matters: Tangram Pedagogy



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Lynda Colgan is Professor Emerita, Faculty of Education, Queen's University, and Executive Director of Science Rendezvous. Lynda collaborates with the Assessment and Instruction for Mathematics (AIM) Collective, which engages researchers and educators across Canada to support the development of tools for assessment and instruction in mathematics learning from Kindergarten through Grade 3. She continues to work on a number of funded research, resource-creation, and teacher-education projects. Lynda remains deeply committed to STEM outreach and can be found regularly at events that engage students, parents, and educators in science, technology, engineering, and mathematics at home and in the classroom.

In recent years, I was engaged in an icebreaker task by being given a colourful, die-cut puzzle, as shown in Figure 1, and instructed to use the pieces to make an animal.

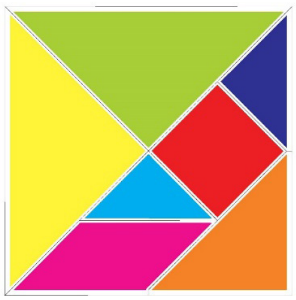


Figure 1: *The seven-piece tangram puzzle*

Many in the group quickly identified the handout as a tangram puzzle, and spoke of having plastic sets in their classroom manipulative kits. Soon, the room was abuzz as enthusiastic participants positioned their pieces, laughing as they described the creations imagined by their own mind's eye: a camel, turkey, or caterpillar. Tablemates were challenged to see the resemblance, but that simply added to the bonhomie.

Since no tangram rules were provided (i.e., often it is required that the final creation must contain all seven pieces, all puzzle pieces must touch each other, and no puzzle pieces may overlap), the resultant images were best described as a vivid hodgepodge of arbitrary designs. Nonetheless, prizes were awarded for an ersatz cow and owl, while everyone else seemed delighted with their participation prize—the take-home puzzle pieces.

I, however, was left disappointed by the lost opportunities to use the tangram as a "thinker toy," a term coined originally by Dr. Seymour Papert. Why had the facilitator not taught the participants how to construct their own tangram? The simple steps of folding one sheet of paper can unlock a world of geometric vocabulary, concepts, and relationships. Relating the origin story of the tangram—which was an invention attributed to the third-century Chinese mathematician, Liu Hui (Britz, 2023)—is a way to celebrate an important discovery in mathematics, and specifying the rules for the seven tans (puzzle pieces) is a way to teach students how to use the puzzle pieces correctly. In other words, where was the mathematics in the opening task for a day of professional development about mathematics?

The incident reminded me of the prophetic words of Constance Kamii (1931–2023). Kamii was a pioneer in how children's cognitive thinking relates to physical knowledge and mathematics. Her research in mathematics education and constructivist theory has been the cornerstone of early childhood mathematics education for decades and greatly influenced the use of mathematics manipulatives in the classroom. It was Kamii et al. (2001), who noted that "while tangrams can be an excellent teaching tool, it is easy for a teacher to reduce their value" (p. 23). Her empirical studies emphasize that mathematical relationships do not exist in objects, and perhaps most importantly, that children *do not learn* abstract concepts by [simply] touching and moving these objects. Kamii reminds us that mathematics develops as children think, and cautions that "manipulatives are useful or useless, depending on the quality of thinking they stimulate" (p. 31).

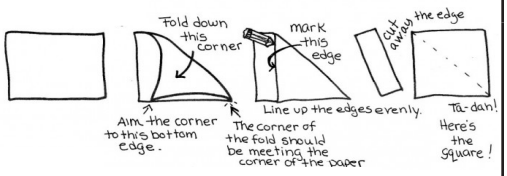
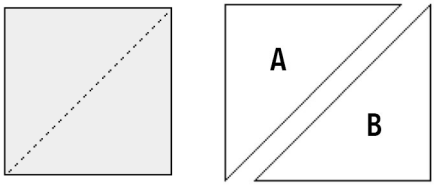
Kamii, like Kajander and Colgan (2024), underscores the fact that children learn from the specialized mathematical content knowledge for teaching that the educator calls upon to draw the concepts out from the manipulative pieces by using a particular task, much like the way a magician makes things that were not there suddenly appear before one's eyes. But unlike the magician whose conjuring elicits a gasp of disbelief, educators' conjuring should elicit excited cries of understanding, such as, "I see it. I get it!"

In addition to Kamii's important findings, other researchers have found that learners tend to view manipulative activities as playtime—an association that may hinder learning, as it devalues the potential of these materials as mathematical representations for learning (Berkseth, 2013).

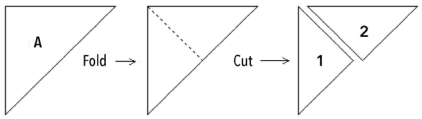
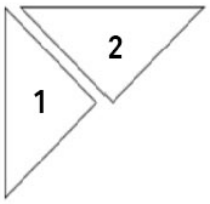
So, how would I have elevated the tangram puzzle opener so that it emphasized mathematical value? Unlike

the magician, who capitalizes on *misdirection* as a key part of their act (i.e., getting people to look at what's not important), I would have modelled how the mathematics educator should *direct* attention—making the most of what is important so that students watch and wait for the big conceptual reveal; in turn, making sense, themselves, of the mathematics wrapped up in a physical model and an enjoyable experience.

What follows are the step-by-step instructional diagrams that I would simultaneously project and talk through as I co-construct the tangram, using a large piece of paper with my students; my prompts and probes; along with some anticipated student responses. I encourage you to get a piece of paper and create your own tangram along with me.

Step 1	
Making an 8″ x 8″ square from a standard piece of 8 ½″ x 11″ paper	 Drawing by *Paula Beardell Krieg* from bookzoompa.wordpress.com
Step 2	
Examining the resultant product: a square with a diagonal fold, then cutting along the diagonal	
Teacher prompt	Tell me about shapes A and B, using as many math facts and words as you can.
Sample anticipated student responses	The *diagonal* of the square is a *line of symmetry*. Triangles A and B are *mirror images* of each other. Triangles A and B each have a *right angle* at one corner (vertex). Triangles A and B have two *acute angles.* Triangles A and B are *right triangles.* Triangle A and triangle B are *isosceles* right triangles. Triangle A is *congruent* to triangle B. Triangle A and Triangle B have $\frac{1}{2}$ the area of the original square. Triangle A + Triangle B = original square, so $\frac{1}{2}+\frac{1}{2}=1$ whole. Triangles A and B have two *equal sides* (edges). Triangles A and B have two *equal angles*.

As the students flip and turn the pieces, I ask them how they know the angles are equal and why the triangles are congruent? Assuming protractors have not been introduced in the early grades, we use visual tests: laying one triangle atop the other to see that they "match" exactly; lightly folding along the height of the triangle to see that the acute angles are equal; comparing the equal edges of the triangle to the lengths of the sides of the original square, which we knew from our construction had four equal sides.

Step 3	
Bisecting the longest edge (hypotenuse) of Triangle **A** by folding and then cutting along the fold line.	
Teacher prompt	
Examining the resulting product: two triangles **1** and **2**	
Teacher prompt	Tell me about shapes 1 and 2, using as many math facts and words as you can.
Sample anticipated student responses	The fold line bisects the hypotenuse of triangle A. Triangle 1 and triangle 2 are right triangles. Triangle 1 and triangle 2 are isosceles right triangles. Triangle 1 is congruent to triangle 2. Triangles 1 and 2 have $\frac{1}{2}$ the area of triangle A and $\frac{1}{4}$ the area of the original square. Triangle 1 and triangle 2 each have two equal sides, two equal acute angles, and one right angle. Triangle 1 + Triangle 2 + Triangle B = 1 square showing the areas add to the area of the square: $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}=1$.

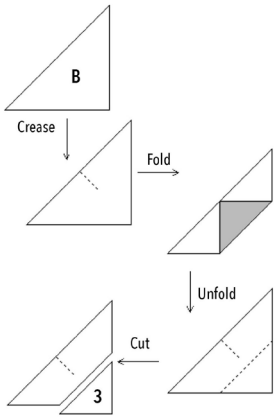
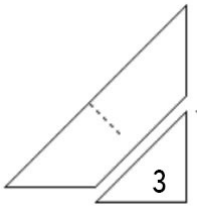
After we talk about triangles 1 and 2, I ask the students to stand, placing triangle B on the floor in front of them. Then I ask them to position triangle 1 in the same orientation; squat down so that triangle 1 is positioned atop triangle B; and then holding triangle 1 at arm's length, stand up slowly. What do they notice as they do this (remembering to keep only one eye open)? As they stand, it will appear, at one precise point, that triangle 1 is congruent to triangle A… matching its area and edges perfectly. This is equivalent to making an enlargement of triangle 1, and uses the same principles as those in one-point perspective drawing.

Mathematically, we can say that if the images of the original (triangle 1) and triangle B (the enlargement) align congruently when positioned this way, then they are *similar*.

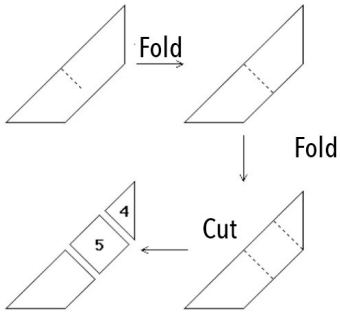
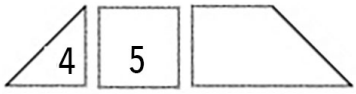
This leads to a discussion of similarity in the world around us. I ask students to think about what we are seeing when we look at a mountain or a very tall building in the distance. The students say that it looks miniature, like a reduced version of the real thing. The sizes of the mountains or skyscraper have not changed, but the distance between us and them makes them appear as if they were smaller. If we had a small-scale image of the mountain or office tower and a clear path for walking, we could prove their similarity, using the same technique, as we did for triangle 1 and triangle A.

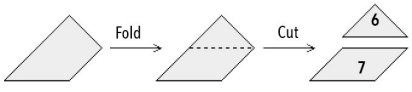
We then talk about how artists measure when they are out in nature so that the trees, mountains, and wildlife are portrayed to scale. Then I ask the students to stick their thumb up and hold their arm straight out, finding something in the room that appears to be the same height as your thumb. I explain that if you were to paint that object, you may (depending on the size of your painting) decide to paint it exactly one thumb length on the canvas. Alternatively, you can use the thumb as a ruler; i.e., if an object is three thumbs tall and two thumbs wide, then it should be drawn three parts tall and two parts wide. If the object is a face, for example, you continue to draw the rest of the person by using the thumb as a reference so that everything will be in proportion.

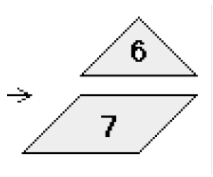
And now… back to the last three steps in making a tangram.

Step 4	
Creating a trapezoid and a triangle	
Teacher prompt	
Examining the resultant product: a trapezoid and a triangle	

Teacher prompt	Tell me about the trapezoid and triangle 3, using as many math facts and words as you can.
Sample anticipated student responses	The initial crease is at the midpoint of the hypotenuse of triangle B. Triangle 3 is an isosceles right triangle. The trapezoid is a quadrilateral. The trapezoid has two parallel edges. The trapezoid has two acute angles (which are equal in size) and two obtuse angles (which are also equal in size). Triangle 3 is similar to triangle 1 and triangle 2.

Step 5	
Bisecting the trapezoid to make two right trapezoids; then folding and cutting one	
Teacher prompt	
Examining the resultant product: trapezoid, square and triangle	
Teacher prompt	Use as many math facts and words as you can to describe them and any relationships among them and previous pieces as possible.
Sample anticipated student responses	Triangle 4 is a right isosceles triangle. Shape 5 is a square. The right trapezoid has one pair of parallel edges. Triangle 4 has $\frac{1}{2}$ the area of square 5. Triangle 4 is $\frac{1}{3}$ the area of the right trapezoid. Triangle 4 is similar to triangles 1, 2, and 3.

Step 6	
Folding and cutting the trapezoid	

Teacher prompt Examining the resultant product: a parallelogram and triangle	
Teacher prompt	At this point, we have seven shapes: five triangles, one square, and a parallelogram. Use as many math facts and words as you can to describe them and any relationships among them as possible.
Sample anticipated student responses	There are five right isosceles triangles in all: two large (triangles 1 and 2), one medium (triangle 3), and two small (triangles 4 and 6). All the triangles are similar triangles. Triangles 1 and 2 are congruent. Triangles 4 and 6 are congruent. Shape 7 is a rhombus. Triangle 3, the square, and the rhombus are each made up of two small tangram triangles (4 and 6). Triangle 3, the square, and the rhombus each have an area two times that of triangles 4 and 6. Triangles 1 and 2 are made up of four small tangram triangles, and thus have an area four times that of the small triangle and twice that of the other tangram pieces.

Now that the students have all seven tangram pieces, the obvious task is to ask the students to recreate the original square, using all seven pieces—which demonstrates conservation of area (i.e., the area of the pieces is equal to the area of the whole).

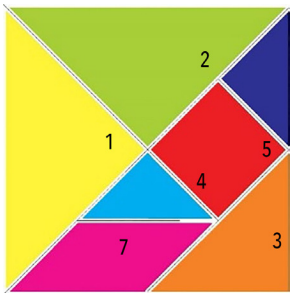


Figure 2: *The square reassembled from the seven tangram puzzle pieces*

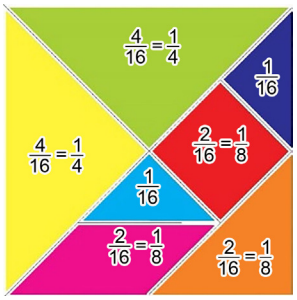


Figure 3: *The fraction value of each tan compared to the total square*

Many students worked backwards—using pieces 3, 4, 5, 6, and 7 to make the large trapezoid and medium triangle that comprised half the original square. The other pieces fall into place easily.

As a final challenge, I pose this task: If we use the small triangle (either triangle 4 or 6) as a building block, we can use it to make all the other tans (i.e., the other pieces). Using this fact, how many small triangles are required to make the original square?

The students quickly used their two small triangles—physically acting out the geometric transformations of slides, flips, and turns—to make the square, rhombus, and medium triangle. They then used a combination of the medium and small triangles to make the two large triangles, concluding that 16 small triangles were required in total.

From this, we computed the area of each piece as a fraction of the original square as a whole.

With mathematics brain cells fully activated in a meaningful way after this fold-and-cut activity, all but our youngest students can take pride in discovering some mysteries of mathematics made visible by the tans.

And in the spirit of the wonderful, open-ended children's book, *The Tangram Magician* (Lisa Campbell Ernst, 1990), I challenge you, the reader of this column, to recreate these animals, using the tangram pieces you created as you worked along with me from Step #1 to Step #7, and then make one of your own design.

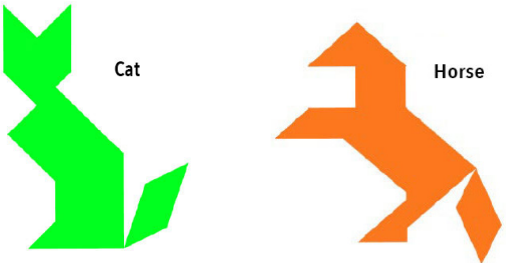


Figure 4: *Shapes of animals generated by using all seven tans*

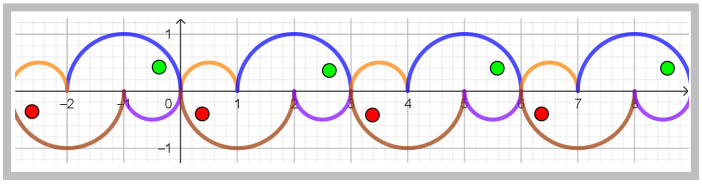
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Kamii, C., Lewis, B.A., & Kirkland, L. (2001). Manipulatives: When are they useful? *The Journal of Mathematical Behavior, 20*(1), 21–31. doi.org/10.1016/S0732-3123(01)00059-1 ▲



In the Middle: Desmos Activity Builder Goes to Space



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Carly Ziniuk teaches Grade 12 Data Management and Advanced Placement Statistics at the Bishop Strachan School in Toronto. She is active in adopting real-life data to engage her students in solving problems. Among 30 global STEM leaders, Carly was part of the 2024 NASA Space Apps Collective. As a Science Communicator, she is working to connect astronomy and mathematics for learners of all ages across Canada.

Author's Note: I have used the guidelines as presented by NASA's Style Guide throughout this piece, when referring to the Moon, the Sun, and Earth/earth. The *NASA Style Guide* capitalizes Earth, when it is used as the name of the planet. In this case, it is not preceded by "the," since one would not comparably say "the Mars." If referring to ground or soil, but not the planet, then "earth" is lower case. Capitalization of "Sun" and "Moon" refers specifically to our Sun and our Moon (but lower case, for example, with "Jupiter's moons" or the moon Titan). Use of "the" with Sun and Moon is appropriate for emphasis or clarity. The Canadian Space Agency (CSA) uses similar style conventions.

For the past two years, I have been using Desmos Activity Builder to create regular weekly check-ins as the primary homework vehicle for my students to reinforce their skills. They complete these check-ins independently, using technology on their time, so that class time is spent on the boards working out problems, discussing with classmates, and interacting with me during lessons.

To demonstrate this in this column, I created a sample Desmos Activity with Canadian images from the Canadian Space Agency (CSA) and NASA to illustrate the variety of different features of Activity Builder. The activity is discussed in detail in this column, but you can pre-emptively explore bit.ly/GazetteStudentDesmosActivity (note a "next" button is at the top right), or use Activity Builder as a Teacher to edit bit.ly/GazetteTeacherDesmosActivity, and assign it to your classes. The primary focus for this activity is the Grades 8–10 Mathematics curriculum content, but can easily be adapted to different mathematical topics, or can be revised based on the experiences students have with independent work and responses to self-assessment. Slides within this activity also can be exported to printable handouts, without the online checking feature made using the Computation Layer (custom-built internal code) for Assessment as Learning.

Desmos Activity Builder presents visually to students like an interactive slide show. As seen in the student version, Figure 1, and teacher version, Figure 2, the first interactive slide uses graph paper with the new Canadian Space Agency logo underlaid for students to use the sketch tools. This slide also includes a text box explaining the image and asking the math question, and a math input box, which includes checking by the Computation Layer (CL).

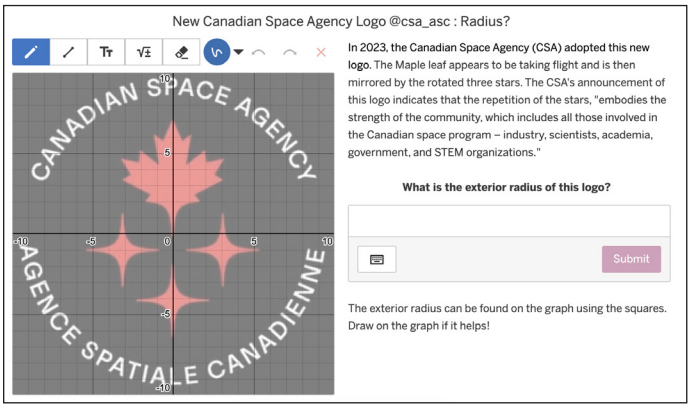


Figure 1: *Student version of Desmos Activity Builder*

Students enter a numerical value in the box and "Submit" when done. The Computation Layer attached to the math response box (labelled CSALogo below) checks whether the student's input is correct (see Figure 3). When a slide's component shows a small green circle with "</>" in the right upper corner, it indicates that the CL code is connecting the different components in an activity. I primarily use CL to respond to student input, to provide immediate feedback, and to connect different mathematical elements.

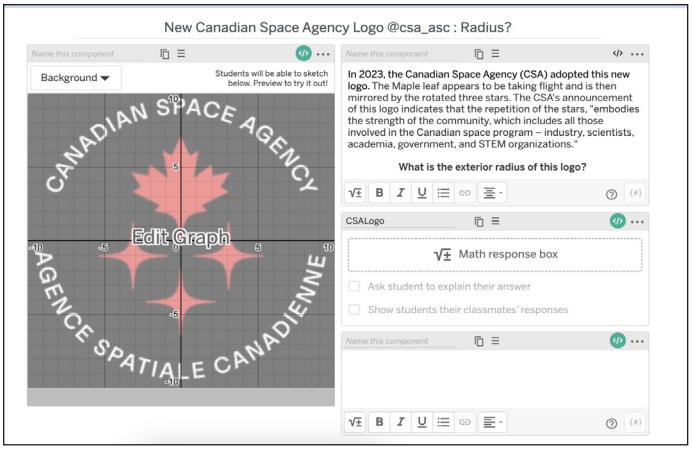


Figure 2: *Teacher version of Desmos Activity Builder*

In this example, the correct numerical value entered into the CSALogo's Math response box is a value between 9.4 and 10.5. When the student submits a value, the CL code identifies the correctness, and then responds. This allows students to have some potential measurement error reading from the graph, but allows teachers to choose the level of precision for student answers.

```
correct = CSALogo.numericValue>9.4 and CSALogo.numericValue<10.5

content:

when CSALogo.submitted and (CSALogo.numericValue>9.4 and CSALogo.numericValue<10.5)
"Great Job! ✓
Rocketing to success!
"

when CSALogo.submitted and (CSALogo.numericValue>10.5 or CSALogo.numericValue<9.4)
"Read the question carefully. ✗
Check your work with a neighbor and try again!
"

otherwise
"The exterior radius can be found on the graph using the squares.
Draw on the graph if it helps!
"
```

Figure 3: *Computation Layer Code giving self-checking feedback*

In this activity, this entire slide is duplicated for students to identify the diameter, after having determined the radius, presumably correctly. This is easy for teachers to do with minimal changes in the CL code as well.

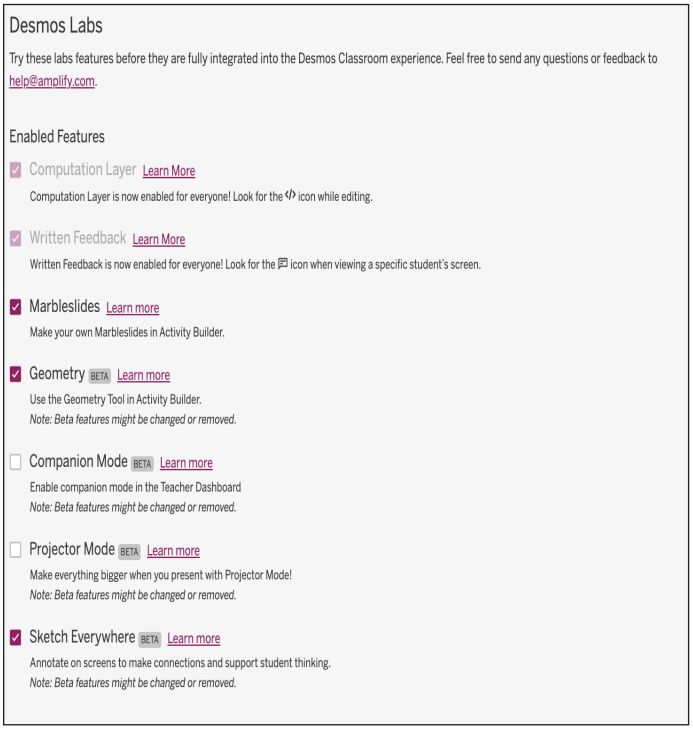


Figure 4: *Optional features in Desmos Activity Builder including Geometry Tool in beta*

Once you assign an activity to students in your class, you can either allow them to move on without a correct answer, stepping through the slides so they are not waiting for support, or you can choose to pace an assigned activity, using the "pace" or "pause" or "sync to me" features. This is useful when classes are working through this activity together at one time and you want to bring them all to one slide at the same time, or reveal a discovery after a fixed amount of time.

A subsequent slide (see Figure 5) is a classic Grade 8 geometry problem, layering parallel and intersecting lines on top of a photograph of Canadian astronaut Joshua Kutryk in launch position, and asking about angle properties. Because the Geometry feature is still a beta version, you can adapt any element from a shared activity, but to create one from scratch, you will need to log in, go to Desmos Labs, and select the Geometry Tool (Figure 4), as is done here.

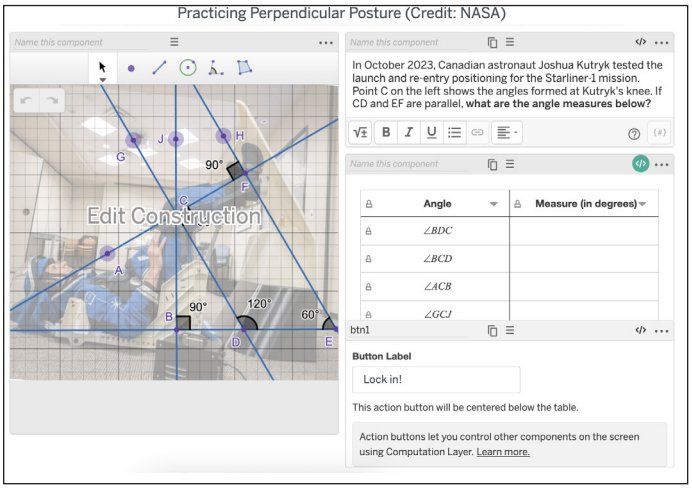


Figure 5: *Teacher view of Activity Builder showing table and button components*

Figure 5 shows how the slide looks to the creator, with the student view shown in Figure 6. The CL displays emoji feedback once the student chooses "Lock in!" Notice the angle column in the table has locked cells, and similarly, I have fixed the points on the geometric construction. Feel free to adapt the construction by unlocking any point or line placements so that the geometric properties are dynamic. I anchored these placements to the launch position photo, but this construction can show the parallel transversal properties when unlocked.

Online activities often do not match individual teachers' classroom choices or unique student needs. CL code, however, provides considerable teacher autonomy. Even if you are not interested in using this specific geometry skill, it can be adapted for other uses, such as translating to trigonometry, or copying the CL code for other table checking. For example, I have also used this exact code for conditional probabilities from a Venn diagram and scenario in the same arrangement on the slide.

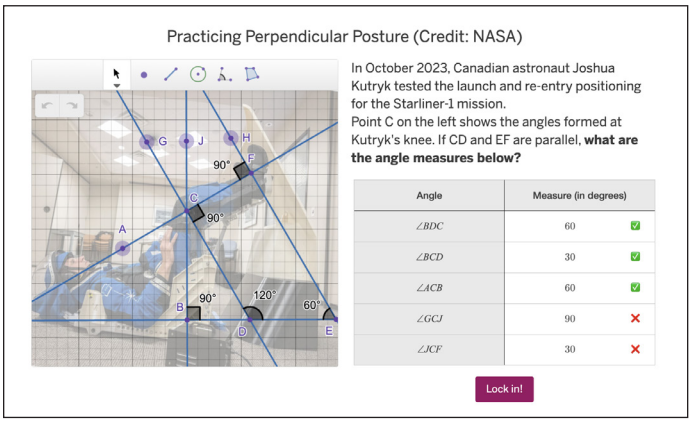


Figure 6: *Student view of Activity Builder showing CL feedback in a table*

You can also include words in the table, as shown in the Solar System diameter, and distance from the Sun graph and table in Figure 7. Here the table and graph are working together.

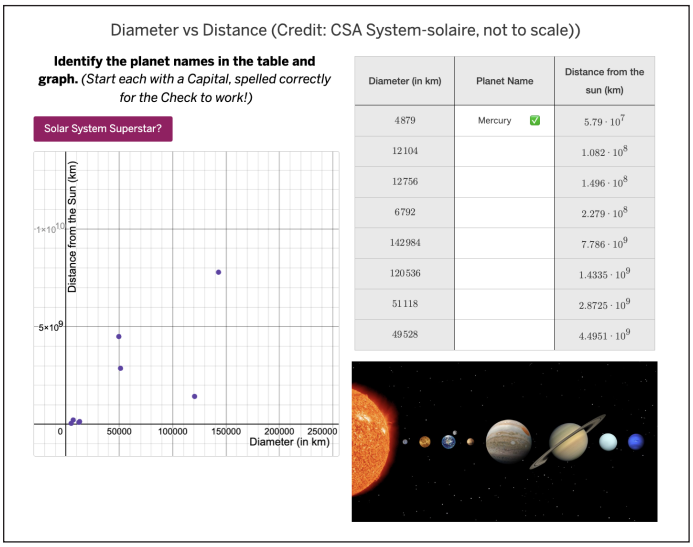


Figure 7: *Student view of Activity Builder showing graph, table, and feedback from CL*

Embroidered Mission patches are an important part of Space Exploration history, initially worn on an astronaut's space suit when launched (now they're silk screened, with embroidered ones worn groundside). Working with an artist, the astronaut designs personalized patches. Artemis II has a scheduled lunar flyby in April 2026, with CSA Astronaut Jeremy Hansen as part of the crew (Canadian Jenni Sidey-Gibbons is his backup!). Hansen's patch is used in the activity shown in Figure 8, to illustrate how Polypad, a recent addition to the Amplify platform with Desmos Classroom, can be used to demonstrate that the patch does not tessellate.

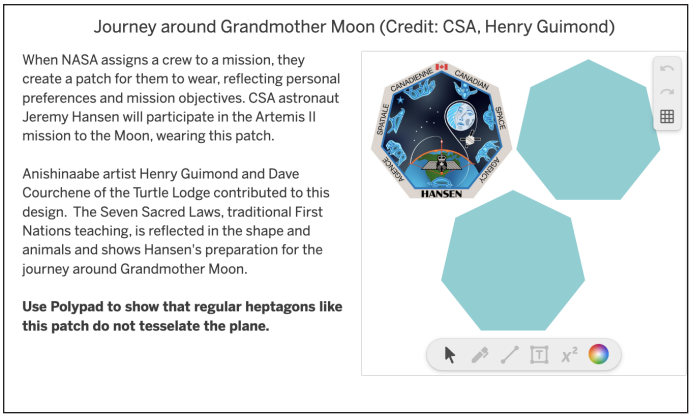


Figure 8: *Student view showing use of Polypad example of tessellations with Jeremy Hansen's NASA patch*

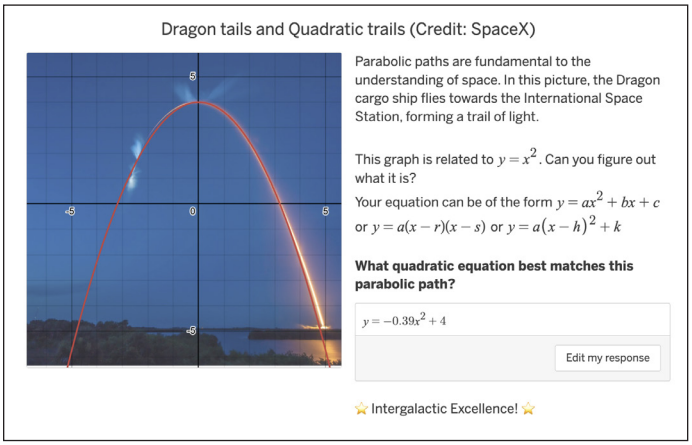


Figure 9: *Student view of Activity Builder showing graph and input of function on graph with image of rocket's parabolic path*

Quadratic matching using photographs (see Figure 9) is a common activity in Grade 10 Math. Paths in rocketry are an illustration of gravity bending the path of the projectile, they are approximately parabolic, and can be approximated by quadratic functions accessible to high school students. This sample slide displays an input function on the graph, checks the input function, and shows a message when it matches the CL solution. Similar input checking applied to multiple-choice questions is done in Figure 10 with a satellite image of Lake Manicouagan (Quebec). Students can check their circle area calculations of this annular reservoir with the built-in Desmos Graphing Calculator for mechanical calculations. You can also see the different ways error checking can be displayed in different places on the screen. Each of these images can also be exported with the graph as a printable pdf.

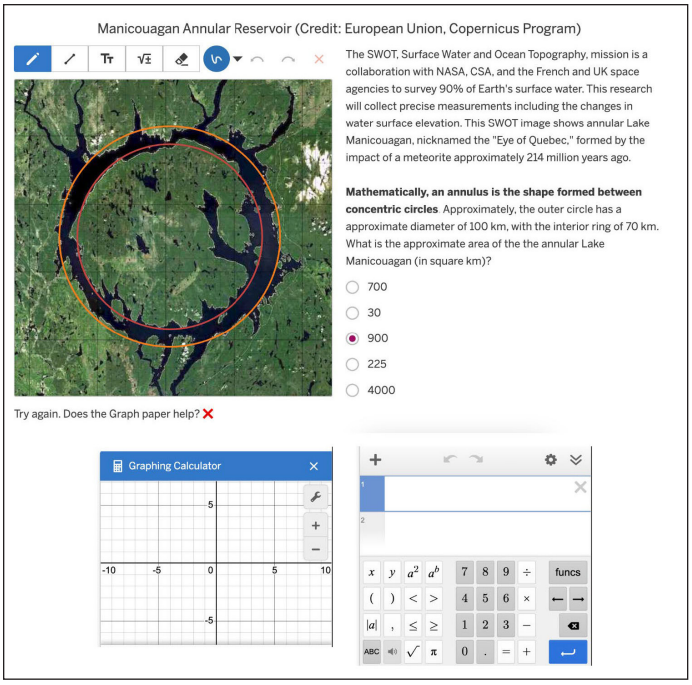


Figure 10: *Student view of Activity Builder showing feedback and how Desmos Graphing Calculator can be used at the same time*

Considering the Bennu asteroid sample collection image shown in Figure 11, the activity includes a "math response box" to collect the answer. That response box also asks students to explain their answer and allows them to view three randomly chosen student answers.

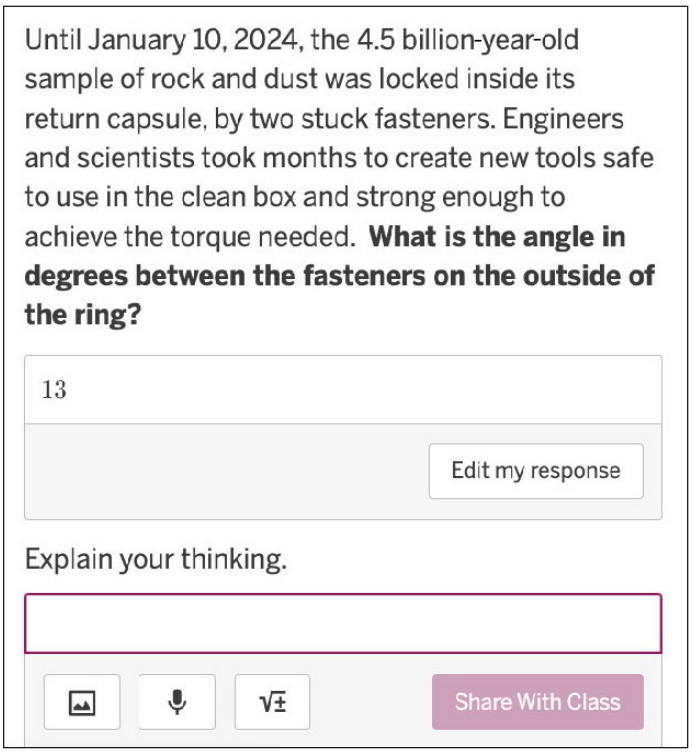


Figure 11: *Student view of "Share with Class" response box showing varied input modes*

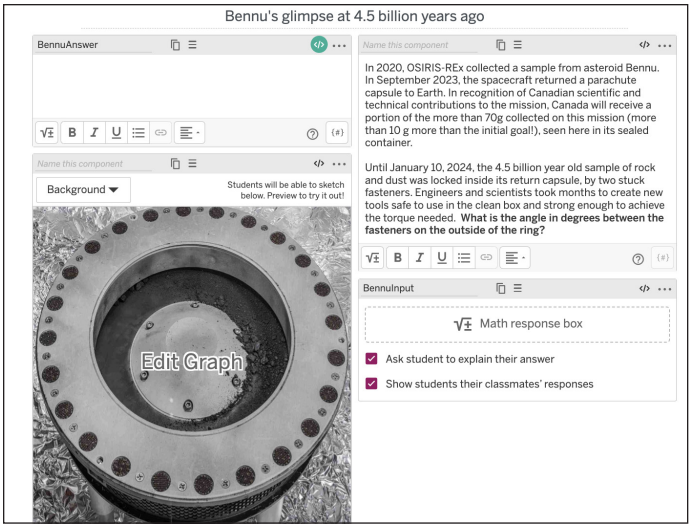


Figure 12: *Teacher view of Activity Builder showing the math response box with "Share with Class" option and Bennu sample collection that allows student sketch*

In my most recent course feedback, students noted repeatedly how much they loved being able to choose when they wanted to "share with class," shown in Figure 12 (allowing you to see other students' answers), and when they only wanted to explain their thinking to me, by not selecting that option. Notice the different buttons at the bottom allowing various input forms: uploading a picture (e.g., a photograph of handwritten solutions), recording an audio explanation, typing a written response, or entering a mathematical expression.

In the Figure 13 example, with scientific notation and metric conversion of space distances, students can order items in a list for self-checking as well. Within the teacher platform, it will randomly generate the initial order, and mark it correct for you as long as you enter the correct ordering in the list when you build it. You will need to place the instruction "correct: this.matchesKey" into the ordered list's CL.

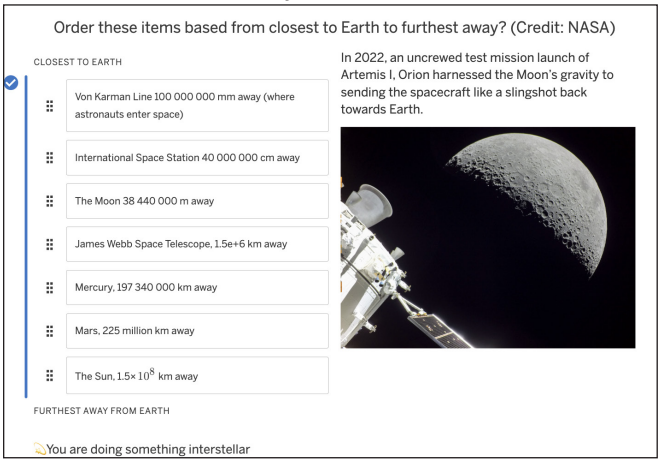


Figure 13: *Student view of Activity Builder ordered list showing distance from the Sun and CL student feedback*

Similar features exist for the "Card Sort" shown in Figure 14, with angles demonstrated by the Canadarm. Matching the cards when they are being built provides immediate feedback, and allows teacher assessment as well. I prefer to change the title when students get it right in a Card Sort because they tend to notice it more directly as they move things around.

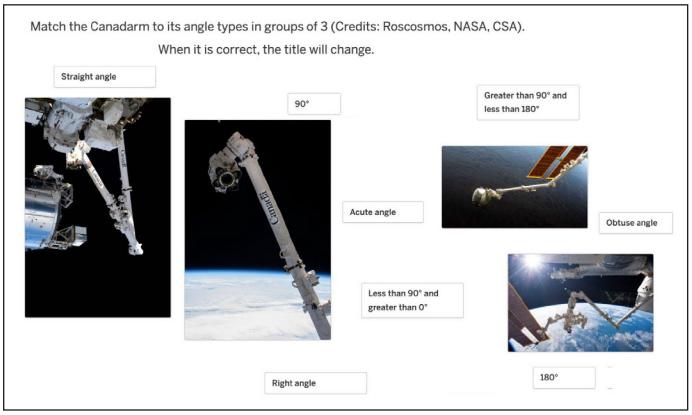


Figure 14: *Student view of Card Sort showing angles created made by the Canadarm*

My favourite adaptation of the check-box feature is to use it as self-reflection tiles in the final slide of an activity (see Figure 15). I frequently use this to summarize the overall learning goals or review the expectations covered. As in this activity, a "BINGO Board" style with randomly generated places on the board removes hierarchical assumptions of topic or order. I find nine separate topics, as shown here, the most aesthetically pleasing, and also a manageable number for students, without too many concepts. In the teacher view of the platform, I can also see which students did not check a given expectation, and then address those students individually for targeted conversation, saving me time and making the conversations more meaningful (no longer, "Why didn't you do your homework?"; rather, "I noticed you didn't check off the quadratics tile. How did you find the question about the Parabolic Path?"). My students have indicated that they use the "BINGO Board" in our weekly check-ins as a chance to make sure they "got everything" from a given activity or content area. The dynamic nature of the board and the ease of clicking has led to higher engagement for my students' self-reflection than typical checklists have done in the past.

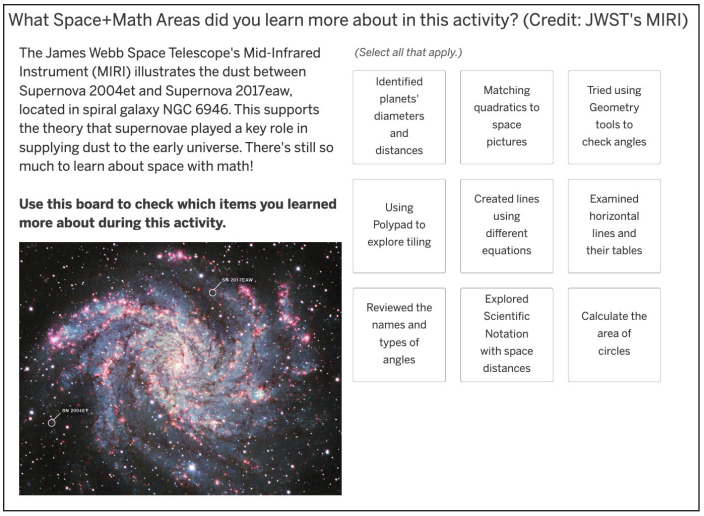


Figure 15: *Student view of Activity Builder tiled check boxes showing learning goals for student self-assessment*

With these independent activities, I am measuring the effect of ongoing feedback, student engagement, and interactive responses. I can see student progress in real life, and document their use of self-checking strategies. My students use these opportunities to grow their learning skills of independent work, organization, and responsibility. I hope you are able to explore some of these mathematical skills and CSA Images with Desmos Activity Builder. Don't hesitate to reach out to me (linktr.ee/CarlyZiniuk), if you have questions or suggestions on how to use CL, since mathematical skill learning through effective student self-assessment really does give Space for All! ▲

Dear ChatOAME, when are the letters spelling a number in alphabetical order?

… pondering …

are the letters spelling a number in alphabetical order?

I can't answer your question at this time.

Why not?

Why not?

Because cookies are not enabled on your machine, and without cookies, what motivation can I have?

Humour based on a conversation between your Editor and Pamela Brittain.

(Cal says ChatOAME does answer it correctly! Try it to find out.)

▲ MB4T (MATHEMATICS BY AND FOR TEACHERS): FRACTION TALKS



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Dr. Sofia Ferreyro-Mazieres is an Assistant Professor in Mathematics Education at Wilfrid Laurier University in Brantford. Her work focuses on destreaming initiatives in mathematics and supporting all learners. Sofia currently teaches Primary/Junior and Junior/Intermediate mathematics education courses to teacher candidates. She has also worked as an elementary and secondary mathematics teacher in the Simcoe Muskoka Catholic District School Board.

In my work with future teachers, I often notice gaps in their ability to connect mathematical knowledge with the pedagogical skills required to teach fractions effectively. For instance, I once asked a group of pre-service teachers to explain why $\frac{15}{25}$ is equivalent to $\frac{12}{20}$. What I intended as a simple warm-up exercise quickly turned into an unexpected eye-opening moment. While some candidates could demonstrate a mathematical procedure to prove equivalence, many struggled to represent it visually and clearly explain why the equivalence holds. This revealed a deeper disconnect in their understanding and reinforced the critical need to develop a strong foundation in fraction sense. For me, fraction sense goes beyond rote computation; it involves flexibility, fluency, and the ability to reason about fractions intuitively. The students in my classroom who have strong fraction sense can grasp fraction concepts deeply, enabling them to work with fractions on a conceptual level, rather than merely following steps or memorized rules.

I have found that adapting the *number talks* approach (Parker & Humphreys, 2015) for the domain of fractions is an effective strategy for addressing gaps in understanding and the teaching of fraction concepts. The fraction talks method (Parrish & Dominick, 2016) encourages learners to verbalize their reasoning, justify their answers, and explore multiple pathways to understanding equivalence and operations with fractions. By creating a collaborative space, where students can discuss and visualize fraction relationships, fraction talks not only build procedural fluency, but also foster a deeper conceptual grasp of the *why* behind the mathematics concepts. This column delves into the power of fraction talks, illustrating how this approach nurtures flexibility, critical thinking, and a profound sense of fraction understanding—skills that are necessary for teachers whose goal is to instil confidence and competence in their own students.

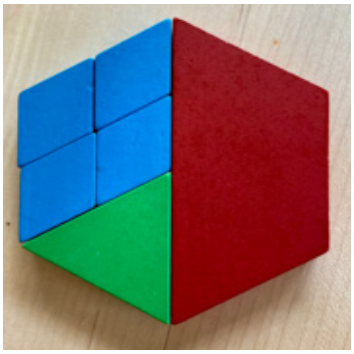


Figure 1*: Fraction talk image (mathforlove.com)*

In a recent session, I introduced an image from the opener resources found on the website mathforlove.com (Figure 1). My objective was to show that fraction talks can be transformative by fostering rich discussions and exploration. First, take a moment to consider the fraction of the whole shape is each colour, and how you or your students might solve it. To differentiate instruction for those with monochromatic vision or low vision, consider the fraction of each geometric shape or the fraction of each texture, where each colour would be replaced by a different texture type.

It was no surprise that many students could identify the red section as representing half of the total area. When shifting focus to the green triangle, students suggested dividing the shape into six equal parts, concluding that the green triangle represented one-sixth of the whole. Throughout these explorations, it is beneficial to be purposeful about highlighting the importance of asking students to verbalize their thinking and understanding.

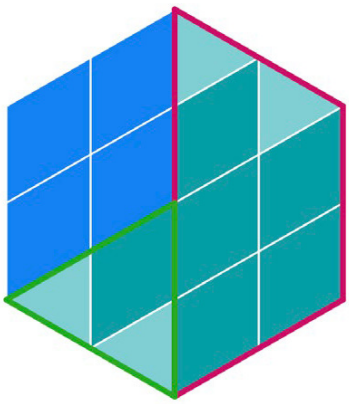


Figure 2: *One method for solving: What fraction of the shape is blue?*

Next, as you can see in Figure 2, one might describe the blue region as four-twelfths of the whole shape. Often when students are asked to explain and justify their thinking, they reason that two blue rhombi could fit inside the green triangle if one rhombus were halved (i.e., 1 whole and 2 halves). They observe that six blue rhombi (5 whole and 2 halves) fit into the red section, leading to the conclusion that there must be 12 equal parts in total, with four of these being blue $\frac{4}{12}$. To reinforce this explanation and ensure shared understanding, one can use the geometry tool on polypad.amplify.com (Figure 2), or physical pattern blocks to help others visualize the explanation.

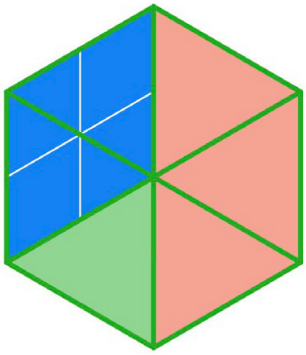


Figure 3: *Another method for solving: What fraction of the shape is blue?*

Building on this idea, another student suggested an alternative way to describe the blue section. They verbalized their thinking by explaining that they grouped the rhombi differently, noting that (see Figure 3) four blue rhombi could fit into two green triangles, representing $\frac{2}{6}$ of the whole shape. Again, this was demonstrated using virtual manipulatives, as seen in Figure 3. By demonstrating that six triangles fit into the entire shape, they introduced another perspective on equivalent fractions. The verbalization of their thinking not only clarified their thought process, but also invited others to consider alternative visualizations and interpretations. Their contribution sparked a deeper discussion about equivalence, which facilitated the conversation on how other students visually saw the blue section as one-third. If we look at Figure 4, these students grouped the four small blue rhombi into one large rhombus and could picture three large rhombi covering the shape, resulting in the blue region covering $\frac{1}{3}$ of the whole shape. This can be shown using virtual manipulatives, as seen in Figure 4, or could be facilitated with pattern blocks, as we will see in a later example.

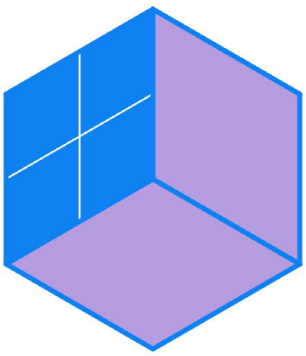


Figure 4: *One more method for solving: What fraction of the shape is blue?*

This fraction talk highlighted the importance of gaining flexibility in perceiving fractions in multiple forms and how fraction talks can provide a platform for deep exploration. To take it further, consider the following reflective questions that you might raise in your own fraction talk:

- How does this change our understanding of the fractions represented?
- What connections can we make between these fractions and their equivalents?

Through these types of discussions, students deepen their understanding of fractional values, develop insights into mathematical relationships like equivalence, and gain valuable learning experiences through justifying, revising, or defending their answers.

One important strategy when using fraction talks in the classroom is fostering diverse perspectives and prioritizing reasoning over answers. Let's take, for example, the hexagon shape in Figure 5, made of coloured geometric pattern blocks, and revisit the question: *What fraction of the shape is blue?* There are various strategies to solve this problem, and it is important to highlight them for students to gain flexible thinking skills. The first and most common approach students use entails visualizing that the whole shape could be covered in triangles. Using the smallest shape to cover the larger hexagon shape, they determined that six triangles covered the blue shaded regions, giving them $\frac{6}{24}$ as the fraction that is blue out of the whole shape. To support their reasoning, I encouraged them to recreate their mental images, using geometric pattern blocks, as shown in Figure 6.



Figure 5: *Fraction talk image (fractiontalks.com)*

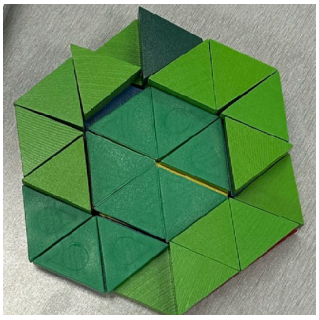


Figure 6: *Example of student work, group 1*

A similar strategy several students shared was visualizing the shape covered in blue rhombi, which they saw as $\frac{3}{12}$. Very few explained that they saw the shape partitioned in trapezoids. They could visualize eight trapezoids inside the larger hexagon and realized that it would take $1\frac{1}{2}$ blue rhombi to fill a trapezoid, giving them the fraction $\frac{2}{8}$. The final strategy that came out of this fraction talk involved students observing that the small yellow hexagon could be reconstructed using three blue rhombi. This insight allowed them to visualize covering the shape with rhombi and then rearranging them to construct small hexagons, noting that the blue area in the original question constitutes one out of the four small hexagons, resulting in the fraction $\frac{1}{4}$. Figure 7 depicts what the students visualized mentally. The various approaches students provided ultimately arrived at the same conclusion: one-fourth of the overall large hexagon is blue. Encouraging different approaches helps students gain flexibility in seeing that

there is more than one way to interpret fractions or deconstruct the shape, fostering a deeper understanding of fraction relationships.

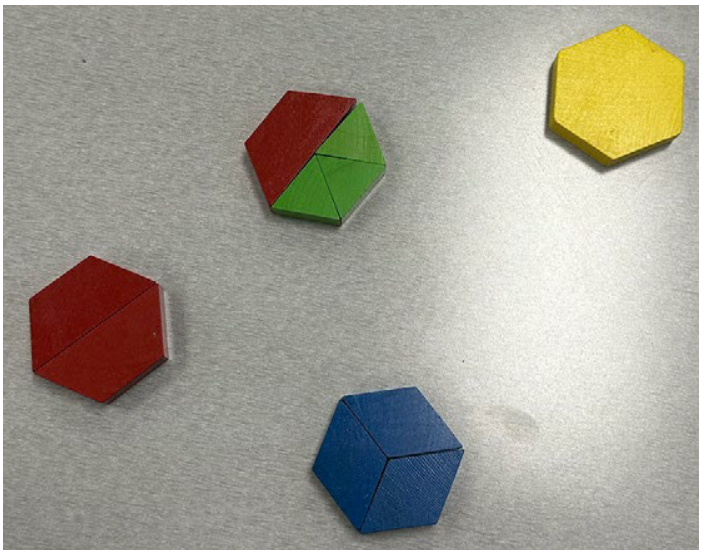


Figure 7: *Example of student work, group 1*

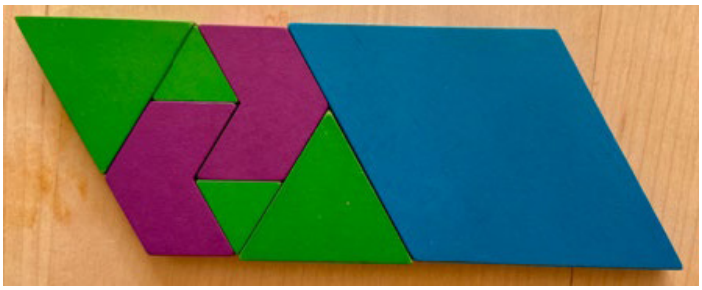


Figure 8: *Fraction talk image (mathforlove.com)*

Using images of shapes formed by geometric pattern blocks during fraction talks (like the ones found on the websites mathforlove.com or fractiontalks.com) can offer clear visual representations to foster discussions about how parts make up the whole. Let's take, for example, Figure 8, which shows a parallelogram divided into various coloured sections. In this example, consider the following questions:

- What fraction of the shape is blue?
- What fraction of the shape is purple?
- What fraction of the shape is green?

For students with monochromatic vision or low vision, consider recreating the shape by replacing fabric texture on the blue region, bubble wrap on the green region, and buttons on the purple region, and then think about: What fraction of the shape is covered by fabric? What fraction of the shape is covered by bubble wrap? What fraction of the shape is covered by buttons?

To start, we could notice that the entire shape can be broken down into smaller, uniform triangles. Each parallelogram within the shape is divided into 18 small triangles, and since there are two parallelograms inside the larger parallelogram, this means the entire shape consists of 18 × 2 = 36 small triangles. From here, look at each colour/texture individually. The blue section contains 18 small triangles, which is exactly half of the total. The area of the blue section is $\frac{18}{36} = \frac{1}{2}$ of the area of the whole shape. Next, the purple area consists of eight small triangles, so its fraction of the shape is $\frac{8}{36} = \frac{2}{9}$. Looking at the green section, we notice it includes two large triangles and two smaller triangles, and since the larger triangle is composed of four smaller triangles, the green section is made up of ten small triangles, leading to a fraction of $\frac{10}{36} = \frac{5}{18}$. To make sure everything adds up, we can check by adding these fractions together: $\frac{18}{36} + \frac{8}{36} + \frac{10}{36} = \frac{36}{36} = 1$. This confirms that the fractions represent the proportions of the blue, purple, and green sections as $\frac{1}{2}, \frac{2}{9}$, and $\frac{5}{18}$, respectively. By breaking the shape into smaller, equal parts and examining each section, we can clearly see how the fractions are determined.

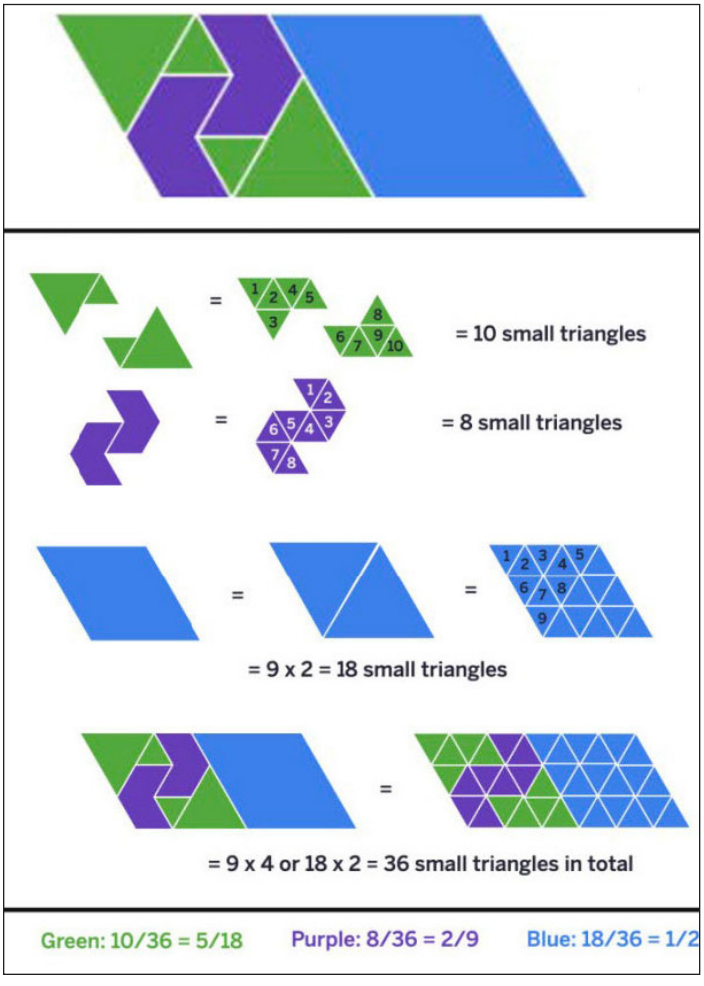


Figure 9: *Example of student work-in-reflection assignment*

Fraction talks from geometric shapes require students to visualize and actively engage with fractions in meaningful ways. Figure 9 illustrates an example from a student, along with their reflection on their classroom experience:

> A memorable moment was working on a fraction question. It was a moment that was a bit frustrating because I kept looking at the image and had no idea how to solve it. I used my learning experience from the fraction talks and decided to use a visual method to solve it by using Polypad. I was able to recreate the image and break down each shape into triangles to be able to solve it. I was so proud of what I created that I kept going back to look at it. I want to help students also feel that sense of pride in their own work. I would not have been able to complete this task without the visual support I built, which made me realize how important mental visualization, fraction talks, and manipulatives are in a classroom. (Student, Fall 2024 Primary/Junior Cohort)

This reflection highlights the value of providing students opportunities to explain their reasoning during fraction talks and to use visual aids to support their thought processes.

Using images of shapes made with geometric pattern blocks in your next fraction talk is one way you can build a stronger foundation for your students to understand fractions. This will encourage them to explore multiple ways of visualizing fractions and will help with the development of flexible thinking. What do you find most exciting about using fraction talks in your classroom? I would love to hear how you make them meaningful for your students!

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Continued from page 11

The ideas that I have shared with you from *Mathematics, a Human Endeavor: A Textbook for Those Who Think They Don't Like the Subject* provides a glimpse of the treasure that awaits you in this book, which was written over 50 years ago. The book is out-of-print, but used copies are available online. You can also borrow the book for a limited period at a time with an account at www.archive.org/.

Let's continue sharing and learning from one another to inspire the next generation of mathematicians and lifelong learners! ▲

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▲ Coding in the Classroom: Tossing Ideas Around



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Iain has been an elementary school educator for 28 years, and is now into a second career, teaching in-service and pre-service teachers. He teaches face-to-face at Ontario Tech University and virtually at Western University.



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Beyza is a former teacher and long-term learner. She currently works as a research associate at Western University, focusing on integrating computational thinking into mathematics education.

In our last column, we explored coding through geometric patterns that turned simple shapes into vibrant mathematical art. This time, let's flip the script—literally! We're diving into a fun and thought-provoking coding activity centred on coin flipping, randomness, and probability. This "low-floor, high-ceiling" activity ensures every student has an entry point, while offering endless possibilities for exploration.

Imagine this: You're with your students, everyone flipping coins together. How often do you think half the class will get heads? It seems like it should happen often, right? But does it? As coins fly and hands go up to show their results, something fascinating begins to unfold. Questions bubble up: "Why isn't it half every time?" "What's going on here?"

This is where the magic of coding steps in. Why keep flipping coins manually when we can let a computer handle the heavy lifting? That's where Scratch, our trusty coding platform, comes into play.

Let's Talk about Coding the Flip

Picture this: Your students, armed with Scratch, creating their first coin-flipping program. It starts with something simple, a single virtual coin toss that mimics their real-life experiments. Then, the questions roll in: "How many flips do we need to see an even split of heads and tails?" or "Why does it seem so rare, even though it's supposed to happen?"

These questions lead to exciting possibilities. Some students might want to tweak the code to alert them when they finally get an even split. Others may wonder, "What if we flip the coin more—or fewer—times?" Suddenly, they're modifying their programs, diving deeper into probability without even realizing it.

What Happens Next? The Ceiling Is Sky High

This is where things get really interesting. Students often take the reins, steering the activity in directions you might not have predicted. Some might simulate thousands of coin flips to explore patterns. Others could investigate probabilities, using tools like Pascal's Triangle (or Yang Hui's Triangle—yes, there's a history lesson tucked in there, too). And for those looking to add a creative twist, coding fractal-like patterns, using their results, could be the ultimate challenge.

But here's the best part: This isn't just about math or coding. It's about fostering curiosity, collaboration, and problem solving. Students troubleshoot their code, share ideas, and celebrate "aha" moments together.

Your Turn

Have you tried something like this with your students? Or do you have your own spin on coding and probability? We'd love to hear about your adventures! Share your stories—and maybe even some student-created programs—with us through the *Gazette*. Who knows? Your class's creativity could inspire our next column.

Let's keep flipping ideas around and discovering the endless possibilities of coding in the classroom.

"Flipping Out with Scratch" is a resource to help you code your coin flipping with an example or starter code for you and your students that you can find in the members only area on the OAME website under "Files and Links." ▲



Meaning Making in the Classroom: Taking Deliberate Practice One Step Further



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David is a principal, who taught and had various board roles in Prince Edward Island. David has also instructed university courses regarding curriculum, differentiation, mathematics, and literacy. He facilitates professional learning in mathematics and school development. Dr. David Costello has published four books, with two of his most recent books being: Messing Around with Math: Ready-to-Use Problems That Engage Students in a Better Understanding of Key Math Concepts, and Mathematizing Student Thinking: Connecting Problem Solving to Everyday Life *and* Building Capable and Confident Math Learners.

Teachers want the best for their students. They want their students to have opportunities to deepen their understanding of math and to grow as problem solvers. To reach these goals, teachers spend considerable time, both inside and outside school, planning their mathematical instruction. And, in doing so, teachers want to highlight instructional strategies that have significant impacts on student learning.

Regardless of the instructional approaches being applied by the teacher, there is often space provided for students to practise applying their understanding of a concept to solve problems. In fact, practice has been identified as being a necessary component in all effective mathematics programs (Ontario Ministry of Education [OME], 2020). And, it is this notion of practice that has become more of a focal point in educational discussions. Not all practice is equal. In terms of students applying their understanding, the time being devoted for students to practise is not yielding the desired outcomes of the teacher.

What I want to emphasize in this column is the importance of practice in mathematics and how we may need to adapt our approach to the way students practise in math. Not all practice is equal and, when we are aware of this and plan with it in mind, we may see students demonstrate the learning goals that were set.

In this column, I will provide a brief overview of practice. From there, I will share two considerations that, when applied to practice, can result in students making the growth in achievement that you seek. Furthermore, when we apply these two considerations to practice, we will see the result that we are seeking—a growth in student mathematical understanding.

Practice

While there are many definitions of practice, a general commonality is that practice must follow understanding (OME, 2020). Without students having conceptual understanding, practice is problematic, as students could be reinforcing misconceptions. Therefore, practice is something that should not be rushed. Instead, ensure that students have an accurate understanding of the concept before they are provided with opportunities to practise.

There must be thought given as to what tasks are involved in practice. Practice cannot just be busy work. It must be deliberate in that there is a clear purpose to what students are doing when practising. Busy work is not going to bring educators the intended learning goals that they have for students. The educator must have a clear depiction of what the student understands and what they need in terms of next steps.

Practice can take many forms. For instance, the tasks could be structured as pencil-and-paper, where students work toward a solution; or they could be practising with concrete materials; or students could be engaged in a game, where they must apply a concept. Regardless of the form, deliberate practice provides students opportunities for skill development.

High-Impact Instructional Practices in Mathematics (OME, 2020) points out that deliberate practice should be spaced so that students can return to a concept over time. This is based on the notion of spiralling, where revisits to a concept can strengthen both understanding and recall. In addition, students need feedback during practice so that they have an accurate depiction of their thinking.

The question, though, that I am often asked by educators is why, after all of this attention placed on practice, have we not seen more of an improvement in terms of student achievement?

Two Considerations for Practice

We can all agree that practice is important and that it must be deliberate if we are to see an improvement in student achievement. However, I believe that we are missing two aspects that, when considered in relation to deliberate practice, can be game changers: *key concepts* and *mixed practice*.

Key Concepts

While it is important to address all curriculum outcomes

in mathematics, we can leverage those few curriculum outcomes that have greater potential to add value to student understanding. There are select curriculum outcomes that not only are important to learn, but also play a role in understanding other curriculum outcomes. Such curriculum outcomes can be referred to as *key concepts*.

Key concepts are those curriculum outcomes that meet one or more of the following conditions (Ainsworth, 2003):

- **Leverage:** Curriculum outcome(s) that focus on knowledge and skills used in multiple academic disciplines
- **Endurance:** Outcomes and standards that focus on knowledge and skills that will be relevant throughout a student's lifetime
- **Essentiality:** Outcomes and standards that focus on the knowledge and skills necessary for students to succeed in the next grade level

Let's consider key concepts from the perspective of deliberate practice. Instead of revisiting all curriculum outcomes throughout the year, why not focus most of the deliberate practice sessions on key concepts. This revisiting of key concepts will help students to retrieve details, while also strengthening student understanding of the key concepts. Since these key concepts can also be leveraged to support understanding of other curriculum outcomes, the teacher is now supporting student learning of all outcomes.

Deliberate practice focused on key concepts is a way to support teachers in working smarter, not harder. When teachers attempt to digest the number and variety of curriculum outcomes they need to address in a single year, they can get overwhelmed (Costello, 2021). By labelling certain curriculum outcomes as key concepts, teachers are identifying material of the highest priority. The concepts that we are prioritizing in deliberate practice support the understanding of other concepts.

Mixed Practice

When considering effective ways to structure practice, the notion of spacing has been stated (OME, 2020). However, for practice to be truly impactful, both in terms of short-term and long-term, the notion of *mixed practice* must be considered.

Mixed practice simply means that there are multiple concepts addressed within the practice session, and that consecutive problems cannot be solved by the same strategy (Costello, 2021). This is contrary to typical practice sessions, where there is only a single concept being addressed. Within mixed practice, there is greater cognitive demand placed on students because they must determine which concept needs to be applied to solve the particular problem, and this experience continues for each subsequent problem (Costello, 2021).

Putting It Together

By incorporating key concepts and mixed practice into deliberate practice, students will have opportunities to strengthen their understanding of key concepts, be able to leverage these key concepts to make sense of all math concepts, and support their recall of key concepts by engaging in more cognitively demanding tasks as presented within mixed practice.

Now that we have spaced and mixed practice of key concepts within deliberate practice, we can structure tasks to reflect different formats. Educators can use games, pencil-and-paper tasks, visual representations, concrete objects, etc. These various tasks can be assigned daily, weekly, and/or monthly. Tasks can be anywhere from a few minutes in duration to the full length of the math instructional block. Consider checking out www.costellomath.com for such tasks.

Final Thoughts

Making subtle changes to deliberate practice can have significant impacts on instruction and learning. It is about working smarter, not harder. By making a few subtle changes (key concepts and mixed practice), teachers can move from feeling overwhelmed to feeling effective.

I look forward to continuing the discussion of mathematical learning and instruction via my website (www.costellomath.com), email (david@costellomath.com), or X (@dr_costello).

References

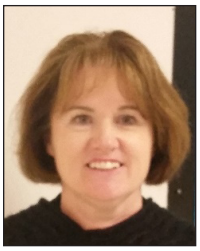
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▲ OAME/NCTM Report: "Let's Not Throw the Baby Out with the Bath Water"



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Jacqueline Hill is a recently retired Grades 9–12 teacher of mathematics. She is a Past President of OAME and OMCA, as well as the recipient of the Award for Exceptional Teaching in Secondary Mathematics. She also wrote the "Director's Dialogue" for the Gazette *for a number of years, and has had several math articles in the* Gazette *over the years.*

If you remember the saying in the title, you might be a seasoned math person. Seasons come and go. Trends come and go. Curriculum is always being reviewed and refined. Be true to yourself and your teaching strategies. NCTM is here to help you on your teaching journey. If you are not sure you want to become a member, then please take a read through this column, as all links and suggestions are found on the FREE side of the NCTM website.

So… if we continue to **THINK** about how we are delivering the curriculum:

T – **Trends:** Trends come and go. Be true to the needs of the students in front of you every day. Take the time to get to know the interests of your students. You can do this through questionnaires at the beginning of the year or by having a quick, three-second conversation as they enter the classroom each day. The NCTM website has ideas and ways to ensure your classroom is ready for those you teach (www.nctm.orgSearch/?wst=b1d8f639419328b502294e1e534d5a06#:~:text=Before%20Your%20Students%20Arrive).

H – **Honest:** Be honest with yourself. Be true to the teacher inside of you. Deliver the curriculum in the most meaningful way possible. NCTM has BLOG-related articles that help teachers with delivering curriculum (https://my.nctm.org/blogs/matthew-larson/2018/01/17/are-we-breaking-down-barriers-to-student-learning). (https://my.nctm.org/blogs/robert-berry/2018/12/21/connecting-charting-a-course-for-success-with-cata)

I – **Impact:** What impact do the resources you have, and the research you rely on, have on the students you teach? Can you streamline your teaching methods to make a greater impact on your students? (https://my.nctm.org/blogs/matthew-larson/2018/01/17/are-we-breaking-down-barriers-to-student-learning)

N – **Nurture:** Nurture the minds of the students you teach. Choose teaching strategies that help nurture learners.

K – **Knowledge of Teaching Strategies:** Be sure to stay informed about the researched methods that are out there. Regularly talk and share ideas with your colleagues. NCTM has numerous articles, and resources that can aid you in reaching students.

THINK about being true to the reason you became an educator. THINK about how you can best keep yourself in tune with the changing face of education. THINK about the greatest impact you can have on the students you teach.

Dates to put on your calendars:

- **April 2, 2025** – NCTM 2025 Spring Virtual Conference
- **June 2025** – NCTE–NCTM Joint Conference for Elementary Literacy & Mathematics – Chicago, Illinois
- **June 2025** – NDCTM Summer Conference – Minot, North Dakota

Until next time,

"Don't Throw the Baby Out with the Bath Water." Use NCTM to benefit yourself and those you teach.

Yours in mathematical fun,
Your NCTM Rep,
Jacqueline ▲



Did you know that the NCTM held its annual conference in Toronto?

Neil Williamson, Past OAME/AOEM President, is hiding behind a piece of memorabilia.

▲ Musing on Percentages: Sales and Income Taxes



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*Jérôme Proulx is a professor of mathematics education (www.profmath.uqam.ca/~jproulx), and runs the Laboratoire Épistémologie et Activité Mathématique (www.leam.uqam.ca). His research focuses on epistemological issues related to cognition and/in mathematics. His current research program is dedicated to mental mathematics and school mathematics content development. (*Photo by Nathalie St-Pierre of Université du Québec à Montréal)

Percent is everywhere—all around us in finance, shops, foods, newspapers, surveys, and so on. Percentages may seem like a mathematically rather simple theme, or one with little depth. However, when one starts to pay attention to its usage, percentages can be seen as filled with surprising elements that are mathematically rich and conceptually fascinating, hence worth pondering. This is something that became quite salient for our research team, as we have been preparing experiments with students in classrooms about percentages in mental mathematical environments.

These columns endeavour to raise and address some of these peculiarities, which we came to call "curiosities," related to the concept of percentages. These musings can be seen to be as much about our usage of "%" in everyday life, in schools, or in mathematics in general. This third column addresses issues related to sales and income taxes, and their difference in nature. Hopefully, you'll have more fun than when you have to pay them!

%, and a Little Bit of History

It is the time of the year again, when we have to complete our income tax forms. However burdening it may be, it is also an opportunity to reflect on how income tax works in terms of %, and especially how it might be different from the other tax that surrounds us in our everyday life, that is, the sales tax. This column endeavours to initiate a reflection into these differences.

So, first things first: let's go back to some of the origins of %, and how it established itself as a concept in (mathematics and in) everyday life. The origins of % are often linked to sales and income taxes, where the authorities used it to levy some amount of money or goods from people (e.g., Parker & Leinhardt, 1995). For example, a government body might impose a tax such as this on a farmer: "You give us a bale of hay for each three bales of hay that you get." This type of tax possibly evolved through the years to become "2 bales of hay for every 5 produced," "1 for every 6," etc., depending on the place, context, and type of product. The transition to "per 100" is, however, not well detailed in historical reports on %; that is, it is not very clear why hundred was chosen for tax calculations or as a referent to compare and establish these ratios. A number of hypotheses coexist on that matter. Some of these hypotheses link the origin of 100 to our number system, which uses base-10, but that choice of 100 is not obvious in this system, as 10, 50, or 1000 could have been other possibilities. Baruk (1995) asserts that it might have been the "reasonableness" of 100 that made it a privileged choice, where 100 is not too small like 10 and not too big like 1000. However, again other numbers would have satisfied this criterion.

Other hypotheses for 100 associate its presence with the emergence of coins and bills in monetary systems, which often use cents, dollars, "centimes," and other denominations, under a relationship of 1 to 100, making it a favourable or familiar choice in order to continue down that road. However, again, many of these monetary systems have emerged much later than the notion of % did, making this justification more complex to sustain. The usage of 100 has also been linked to some ancient Roman taxation practices of asking for 1/100 for every sales and commercial exchanges (see Lemay, 1995, on this matter). Hence, the question of "why 100?" remains to this day, since values like 12, 50, or 60 could have easily filled these same criteria, while referring to other social habits like time (hours, days, years, etc.), or possessing other properties of accessibility, divisibility, and so forth.

In addition to these shaky origins of "per cent," how these taxes were levied is not overly clear either. When the tax request is "1 bale of hay for 3 bales produced," one can wonder if this bale of hay is one of the three bales initially produced, or simply one additional to it. This raises an important issue often discussed in mathematics education between a part–whole conception of ratio (1 for 3 becomes 1 out of 3, hence 1 bale of hay levied and 2 bales of hay kept) or a part–part conception (1 for 3 becomes 1 out of 4, hence 1 bale of hay levied for 3 kept). This distinction is old and may seem obvious. However, it contains important elements to help make sense, and raise distinctions, regarding how sales and income tax function here in Canada.

%, and Taxation Practices in Canada

One can note that in Canada, our sales taxes follow a part–part definition. For example, when combined with provincial sales taxes, these taxes generally, across Canada, amount to less than or equal to 15% and is, for example, convenient as an overestimate of the 13% HST in Ontario. This 15% is what could be called a "surplus" to the purchase of goods one makes: if someone buys a jacket for $100, it "becomes" a $115 purchase. Income tax in Canada is a part–whole mode definition, where a $100 000 salary with, for example, a 30% taxation (taken as average) is to be split in $70 000 to be kept by the taxpayer and $30 000 given to the state. The goal here is obviously not to know why sales and income tax work the way they do in Canada, but mostly to understand the effect and differences between the two at the mathematical level in relation to how % are used. There are, in the ways of using these taxes, quite surprising issues!

With sales being defined as part–part and conceived as a surplus of the initial amount to pay, the sales taxes require that one pays more than what the price tag indicates. As mentioned, for a price of $100, the consumer needs to possess $115 to make the purchase, and not just the $100 price. By contrast, income tax uses a part–whole definition, where taxes are conceived as cutting out or a carving-in of the initial amount: the tax to be perceived is already "in" the amount to be taxed. If 15% is taken as the income tax (one wishes!), then it is an $85:$15 split between the taxpayer and the government for every $100 gained. In that sense, whereas the sales taxes are in addition to the initial amount, the income taxes are within it.

This is already an important difference in how both sales and income tax work in Canada, but this difference becomes even more interesting when the situation is reversed, that is, if one considers a surplus income tax and a carved-in sales tax (which does exist in other countries). On a $115 purchase, the surplus sales tax of 15% is an additional $15 for each potential purchase of $100 (a split of 100:15). If that sales tax becomes or is conceived as a carved-in tax (with a 85:15 split), the consumer still pays $115, but the partitioning of that $115 is different. In the final amount paid of $115, the carved-in sales tax is $17.25, and the initial price tag for the purchased item is thus $97.75 (the ratio is $97.75:$17.25). The carved-in sales tax on the $115 is bigger, since that tax is not relative to $100, but to $115. In relation to a $97.75 price, the tax amount is $17.25, which represents approximatively a 17.65% of surplus sales tax to get to $115 (i.e., 100:17.65). However, the consumer will always have to give the same amount to the cashier, that is, $115. It is also what sometimes gets called an "all taxes included" amount or inclusive tax of 15%.

This partitioning is also quite striking, and even counter-intuitive, for a $100 purchase. With a 15% carved-in sales tax, the partitioning is of 15% of tax for a price tag of $85, thus $85: $15 (here again, this $15 is worth 17.65% relative to $85). On the contrary, for a 15% surplus sales tax, a purchase of $100 is not partitioned $85:$15, but approximatively $86.96 of initial price for $13.04 of tax on the purchase: $86.96:$13.04. A purchase of $85 with a surplus sales tax added on of 15% indeed gives a final amount of $97.75 (and not of $100): a 100:15 split gives $85:$13.04. It is in fact a price of $86.96 that leads to an after-tax final purchase of $100 following a surplus sales tax of 15%: $86.96 × 1.15 » $100; hence, the split would be $86.96:$13.04.

All this is good for sales taxes, but what about income tax? On an annual base salary of $100 000, a carved-in income tax of 15% (using half the average income tax rate to parallel sales tax) leaves $85 000 to the taxpayer (hence, $15 000 of income tax for the state). However, if our income tax were a surplus income tax of, say, 15%, an annual salary of $100 000 would require that the taxpayer pay $15 000 *in addition to* the $100 000 received as salary: for every $100 000, one owes the state an additional $15 000 in income tax (as in the 1 for 3 conceived as part–part for 3 bales of hay produced for 1 additional bale of hay levied by the state). Obviously, the taxpayer could take this $15 000 from the base salary of $100 000, but imagine instead that this $100 000 salary was all spent already once received. This means that the taxpayer would need to have more than the received salary available during the year, that is, the base salary of $100 000 and a supplement of $15 000 for the income tax. One rapidly realizes that this is not representing a necessary favourable situation, since if the taxpayer obtained that $100 000 as salary during the year (and spent it all!), that taxpayer did not obtain $115 000 and might not have that additional $15 000 freely available to give to the state! Even if one has savings from earlier years, in the long run, this might be conceived as a non-sensical way of handling income taxes, which might explain why income taxes are mostly conceived as carved in by taxpayers.

All this to say that these two types of taxation practices, that is, the carved-in tax and surplus tax ones, are quite different. Some situations are more advantageous in one type than in another, which might provide a worthwhile discussion in the classroom. Unfortunately, we do not have much to say in these situations, as taxpayer and as consumers! If we shop local or choose to travel and pay tax

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▲ Useful Theories to Know About: Self-Efficacy



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Jeff Irvine, Ph.D., has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

When your students are confronted by a new problem, do they feel capable of solving it? When you are dealing with an issue in your classroom, do you always feel capable of dealing with the issue in an effective and efficient manner? Both situations speak to a concept called self-efficacy. Canadian psychologist Albert Bandura (1925–2021) postulated the theory of self-efficacy while he was a professor at Stanford University. It has been widely verified and is considered a foundational concept in motivational research.

Self-Efficacy

Self-efficacy refers to an individual's judgment about whether they are capable of accomplishing a task. This judgment may or may not reflect one's actual ability to accomplish the task. Changes in self-efficacy can result in major changes in achievement. So, it is perhaps not surprising that self-efficacy has also been found to be positively related to effort and persistence (Ross, 2008). Unfortunately, according to John Ross (Ontario Institute for Studies in Education) self-efficacy is very resilient and difficult to change. Consequently, a student's self-efficacy will vary, sometimes dramatically, for different subjects and for different tasks within a subject; still, given the major potential impact that changes in self-efficacy can have on student achievement, it must play an important role in any attempt to improve student achievement.

Bandura (1997) identified four influences of self-efficacy: previous positive accomplishments, vicarious experiences (observing others being successful), verbal (social) persuasion by peers and teachers, and positive emotional arousal emanating from completing a task.

Kennedy and Smith's (2013) study of professional learning communities and teacher self-efficacy revealed a link between reflective practice and self-efficacy; although the study was done with teachers, it is not unreasonable to consider that student self-reflection and student self-efficacy are linked. In addition, because self-efficacy is connected to resilience and requires long periods of time to change (Ross, 2009), many interventions focus on other motivational constructs, even though changes in self-efficacy can have an important impact on student learning (Ross, 2008).

Why Should We (Math Teachers) Care about This Theory

At the University of Victoria in British Columbia, Shelley Ross's (2008) study of PISA (Organization for Economic Co-operation and Development (OECD) Programme for International Student Assessment) mathematics data found that a one-unit increase in self-efficacy resulted in a 32-unit increase in achievement; she discovered also that no other motivational variable (intrinsic motivation, goal orientation, instrumental versus relational view of instruction) had as significant an impact on student achievement. Furthermore, a Meece, Wigfield, and Eccles (1990) study of math anxiety in middle school students found that self-efficacy was a major influence on students' emotional and academic achievement in mathematics. Therefore, if math teachers can improve our students' self-efficacy, there will be major gains in student achievement and self-confidence when tackling math problems.

How can we do this? Bandura found four ways. First, students' self-efficacy will grow when they have positive successful experiences in mathematics. Hence, according to George Gadanadis at Western University, we need to use problems that have low floors, high ceilings, and wide walls so that all students have an access point to engage with the problem, and the room (pun intended) to take the problem to higher levels and connect it to a wider range of other concepts. Next, our students need to witness successful solutions by their peers, teachers, and others. Thirdly, we need to utilize mixed-ability groupings so that positive and supportive peer social interactions are available to everyone. Finally, to elicit positive emotional responses to tasks, we need to ensure that the problems that we select in our teaching are grounded in the real-world, interesting, and engaging to all of our students.

It is also important that teachers continue to grow their own personal self-efficacies. Kennedy and Smith (2013) emphasized the importance of teachers being reflective

practitioners by considering the implications of their teaching before, during, and after the lesson is taught. Having a peer to bounce ideas off can also be very helpful. In the same way that students grow their self-efficacies, positive successful experiences provide gains in teachers' self-efficacies as well. Everyone gains when teachers' and students' self-efficacies grow larger.

If You Want to Learn More

Kennedy, S.Y., & Smith, J.B., (2013). The relationship between school collective reflective practice and teacher physiological efficacy sources. *Teaching and Teacher Education, 29*, 132–143. doi.org/10.1016/j.tate.2012.09.003

Self-efficacy also applies to teachers.

Productivity Guy. (2020, September 25). *What is self-efficacy? Explained in 2 minutes.* https://youtu.be/xF60mBAkjcc

A video that goes beyond the classroom into the real world.

Ross, S.P. (2008). *Motivation correlates of academic achievement: Exploring how motivation influences academic achievement in the PISA 2003 dataset* (Doctoral dissertation). University of Victoria, British Columbia. https://dspace.library.uvic.ca/items/612b8c6d-8198-42d5-bc39-f09efbb69043

A major finding about increasing student achievement.

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Ross, J. (2009, April). *The resiliency of teacher self-efficacy in mathematics.* Paper presented at the Mathematics Education Forum, Peterborough, Ontario. ▲



▲ OAME/AOEM Leadership Conference: Designing Equitable Mathematics from the Margins: UDL Math



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Kit has been an active member of MAC² for many years and joined the Board of Directors in 2022. She is a District Mathematics Facilitator for Grades K–9 with the Simcoe County District School Board. In her work, Kit creates math resources to support educators in their instruction and assessment practice, as well as developing and facilitating professional development. Kit is an Additional Qualification instructor and course writer, and she has written and reviewed work for a variety of publishers. She has been involved in Ministry writing projects and was a Content Lead for the recent OAME/AOEM writing projects. Kit received the 2024 OAME Award for Leadership in Mathematics Education.

In November 2024, OAME/AOEM hosted a virtual webinar featuring Dr. Rachel Lambert, speaking about her work in Universal Design for Learning in Mathematics. The webinar explored ways we can design math experiences that provide access for all students in our classrooms. Universal Design for Learning offers a way to design from the margins, looking for ways to reduce barriers for all students.

Dr. Lambert shared the research that led to UDL Math, a version of UDL that is specifically designed for mathematics classrooms. Instead of designing based on student deficits, we investigated beginning with strengths, using neurodiversity and how design and creativity are central to making math classrooms work for students at the margins, particularly students with disabilities and/or multilingual learners.

Dr. Rachel Lambert is an Associate Professor in Special Education and Mathematics Education at the University of California Santa Barbara. Before becoming a researcher, she worked for over ten years as both a special education and a general education inclusion teacher. Her goal is to increase access to meaningful mathematical experiences for students with disabilities. She is the author of *Rethinking Disability and Mathematics: A UDL Math Classroom Guide for Grades K–8*.

Almost 200 educators joined in the live webinar or watched the recording. Dr. Lambert spoke passionately about the need to empower teachers as designers toward accessibility. This includes thinking carefully about the way we create IEP (Individual Education Plan) goals. Often IEP goals are set well below grade level for students and emphasize knowledge and understanding (from the achievement chart) skills and procedures, rather than helping students develop conceptual understanding and to engage in problem solving. The goals are very narrow and don't offer students much choice or autonomy. One of the scenarios she shared from a classroom was the power of storytelling and impact of crafting the launch of a problem. She promoted allowing students to make connections to the context and task, allowing choice and autonomy to solve the problem, using concrete materials and visual representations, which includes drawing. These opportunities enabled students to engage with the problems in the lessons.

She shared a set of gestures and phrases which, she believes, help educators consider how to increase access to mathematics and support students to deepen their understanding.

- Math class is too narrow (bring cupped hands together, palm to palm).
- We need to open up math class (spread hands apart and move them so the palms are facing up).
- Provide strategic supports (bring hands together, palms facing up).

If we can keep these phrases and gestures in mind, we can start to design math classrooms that are focused on students and provide greater opportunities for all students to succeed in mathematics. This also connects to the importance of moving from a deficit or medical model of disability toward understanding that disability is often contextual. Moving toward understanding diversity means embracing neurodiversity and recognizing that humans are naturally diverse. We are diverse in our bodies. We're diverse in how we think. And that diversity is wonderful and beautiful, so we need to stop acting like it's broken and wrong. Part of this shift is moving from using language such as "low kids," "IEPs," "struggling students," and "high kids" to asset-based language that speaks to strengths and abilities. That way, all students will have access to rich problem solving and instruction as opposed to limited instructional procedures such as rote learning.

Dr. Lambert encouraged us to consider how we can make one small change—can we remove a barrier? Can we provide additional access? Can we create more options for choice and autonomy?

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▲ What's the Problem? Planning an Investigation



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Shawn is a former mathematics teacher, department head, and consultant. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: presenting workshops, writing articles, working on local projects, and helping create mathematics contests.

Welcome back, problem solvers. We are going to do something a little different in this issue. Instead of starting with a problem, exploring it, solving it, and looking at different things that you or your students could do with the problem, I will give you a bit of a glimpse into the way my (twisted) mind works when I encounter a problem. Generally, when I write this column, I have a problem in mind and a rough plan of the things I want to pull out of it for the readers. In most cases, as I work through the problem and flesh things out, some of my original ideas about the problem don't look as appealing as I first thought. Similarly, as I am working through the problem, I am often inspired to go off on a tangent. So, it would be fair to say that if I wrote out my plan for a column as a bulleted list, by the time I got done, some of the bullets would be crossed out and others would be added. The same was true in my classroom planning as a teacher, which continually reminded me of the value of working through the material prior to presenting it to the students. In the classroom, you also have situations where students do things that you never anticipated, and that forces you to make a mental note for the next time you do that activity. I am going to give you some insight into the process of developing an activity in this month's column, as well as showing how the original idea can evolve throughout the planning process.

The inspiration for this article came from the document titled *The Quadratic* (but linked on the website as *Exercises on quadratic polynomials*) from the website of Ed Barbeau. Ed is well known in the math education community, and his website has things for teachers and students at all levels. I strongly encourage all readers to check it out at www.math.utoronto.ca/barbeau/.

The document in question is a 15-page collection of

problems on quadratic polynomials, arranged into 12 categories. The one that started my wheels turning was problem 1.4 (b), which results in Lagrange polynomial of degree less than three for three points. The Lagrange polynomial is the polynomial of smallest degree, whose graph passes through several given points. You can always find a polynomial of degree (at most) $n - 1$ that passes through n given points (no two with the same x coordinate). This is an idea that you can lead students to, not because it is important for them to know, but because they can do some reflecting and connecting that will strengthen their understanding of the topics that are within the curriculum during the process.

My original thought was to create a series of explorations that would be suitable for a Grade 12 class, leading the students to discover the Lagrange polynomials at the end. Some of the questions in Barbeau's document plot a path toward that goal. My plan was to use a combination of ideas from Barbeau's questions, and some of my own, to come up with the desired investigation for the students.

Before I even started, I considered that since Barbeau's problems are all focused on quadratics, parts of the investigation might be accessible to students in younger grades. Thus, my plan adjusted slightly so that there was a bit more at the front end. This would work for Grade 12 students to help them access the prior knowledge needed in the investigation, while potentially being appropriate for students in earlier grades to strengthen their understanding of course material. Pursuing this line of thought further, I wondered if I could start my investigation with *linear* relations and build along the same lines, making parts of the investigation accessible to all high school grades.

I then wondered if I could build the exploration in the form that Grade 9 students could be led to linear Lagrange polynomials. Without going too greatly into detail, to do so, our goal would be to determine the equation of a line that will pass through a given point and have a given x-intercept. Since this is the final goal for the linear part, it made sense to me to have the students explore different forms of the equation of a line. Thus, students will be working with the algebraic and graphical representations of linear relationships, looking to make connections between the two. Graphing software, like *Desmos*, would be useful in this investigation, especially if we use sliders to control parameters. This way, we can check to see if an idea works for many different values of a parameter. Since we are going to consider "new" forms of the equation of a line, let's start with review of what we know about the more common ones.

The *slope y-intercept form*, $y = mx + b$, is central to the study of linear relations and will be used often by students and teachers throughout high school. When using this form, we are writing the dependant variable, y, as a function of the independent variable, x, allowing us to easily calculate its values. Also, the two parameters in the equation tell us about the relationship and the graph. The slope, or unit rate of change, m, tells us how steep the graph is, as well as how y changes with each unit change in x. The y-intercept, b, tells us where the graph crosses the y-axis or the "starting value" of the dependent variable for some applications where it makes sense that we start when $x = 0$.

The *standard form* $Ax + By = C$ equation of a line (I have also seen the related equation, $Ax + By + D = C$, referred to as the standard form) shows up naturally in many applications, specifically when we have two related variables, where neither "stands out" as the more important one. For example, if apples cost \$6.45 per kg and bananas cost \$1.75 per kg, then the expression $6.45a + 1.75b$ can be used to calculate the cost of a kg of apples and b kg of bananas. Thus, in this context, the equation $6.45a + 1.75b = 15$ could represent a combination of apples and bananas worth \$15.

The slope y-intercept form is nice for generating a graph, as well as describing one variable as a function of the other. The standard form arises naturally in some applications, but leaves us with a bit of work when we want to generate a graph. Let's play with these forms to see what we can get.

Investigation #1: Write down a few equations in the form $y = mx + b$ and graph each. Rewrite each equation in the form $x = My + B$. What do the values M and B tell you about the graph and how are they related to the original values m and b?

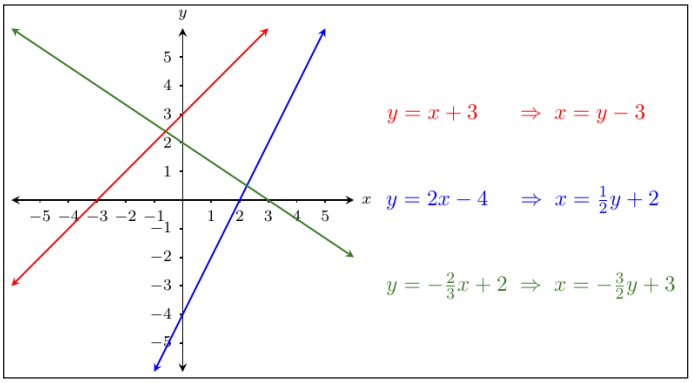


Figure 1: *Three lines with equations in different forms*

Some students will probably conjecture correctly that B will be the x-intercept. We can see this is true in the three examples in Figure 1. The other examples seem to show that M is the reciprocal of m. Reflecting on what m is can help solidify that conjecture. On the graph, m is the slope, a measure of the steepness of the line, calculated by

$$m = \frac{\text{run}}{\text{rise}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Thus, the reciprocal of the slope would be $\frac{1}{m} = \frac{\text{run}}{\text{rise}} = \frac{x_2 - x_1}{y_2 - y_1}$.

This seems a likely candidate for M because in our equation, the variables x and y have just switched roles. It seems likely that the roles of x and y might switch elsewhere as well; that is, the x's and y's may switch places in the expression for the slope. Reflecting further on the slope, recall that the rise and the run also describe how to move from one point to another on the line. That is, if you pick any point on the line and you "rise" by any scalar multiple of the rise and "run" by the same scalar multiple of the run, you will end up back on the line. The same thing occurs with M, just that the rise and run are in opposite locations in the fraction.

We can go one step further by doing some algebra and rearranging the general equation, that is,

$$y = mx + b$$
$$y - b = mx$$
$$\frac{1}{m}y - \frac{b}{m} = x$$

or $x = \frac{1}{m}y - \frac{b}{m}$. We have verified that indeed $M = \frac{1}{m}$. To verify the x-intercept, we can let in the original equation, which would lead us to being the x-intercept. The algebra gives us a bit more. It gives us a relationship between the x-intercept and y-intercept. Furthermore, because of the negative sign in the equation, relating the intercepts means that if the slope is positive, the intercepts will have opposite signs. Similarly, when the slope is negative, the intercepts will have the same signs. We can see this in the three examples in Figure 1.

A little reflection might lead us to realize that since the x-intercept was $B = -\frac{b}{m}$, we can rearrange this to get $b = -mB$. If we put this back into the original equation, we get $y = mx - mB$ or $y = m(x - B)$, or the *slope x-intercept form* of the equation of a line. One can then see that, since the line $y = mx$ would be parallel to $y = m(x - B)$, but with different x-intercepts, the new line has been moved B units to the right (keeping in mind that if B is negative, then moving a negative number of units to the right is, in effect, moving left). This gives us some insight that possibly replacing x with $x - B$ corresponds to a horizontal translation. We will return to this idea later.

Another thing we may wonder from investigation 1 is if we start with a linear relation, and get a new linear relation by switched x and y, how are these two linear relations related? For example, in Figure 1, the first linear relationship was $y = x + 3$. When we solved for x, we got $x = y - 3$. So, if we transpose the variables, the new equation would be $y = x - 3$. Doing the same with the other three examples from Figure 1, we get the results shown in Figure 2 below.

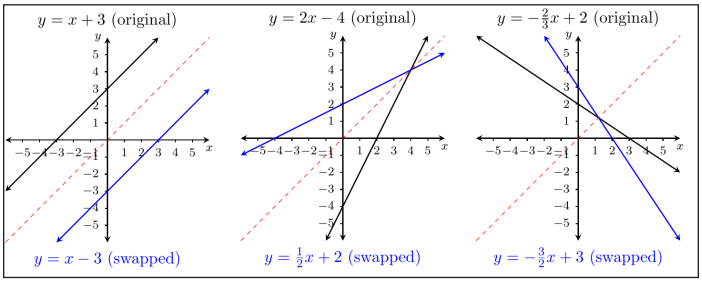


Figure 2: *Linear relations when variables are transposed*

In these cases, our M and B that we saw earlier become the slope and y-intercept of the blue lines (that have the variables transposed). We see in all three cases that the slope retains its sign. Also, in the last two cases, when one line is "steep," the other is "shallow," which makes sense from the fact that the slopes are reciprocals. In the first case, the slope seems unchanged, but since it was 1, which is its own reciprocal, this makes sense too.

Other relationships suggest themselves that would allow for further investigation. First, there seems to be a symmetry in the three diagrams. Examining the line $y = x$, drawn in red in the three diagrams, we see that the lines are the images of each other when reflected in the line $y = x$. Hence, the transformation $(x,y) \rightarrow (y,x)$ corresponds to a reflection in the line $y = x$.

Another not quite so obvious relationship suggests itself from the first set of functions in Figure 2. Note that we are doing the opposite in the two equations, adding 3 in one and subtracting 3 in the other. As a result, if we pick any number for x, run it through one of the equations, and use that y value as an x value in the other equation, we end up back to where we started. For example, if I start with $x = 6$, and do the process starting with the top equation, we get $6 \rightarrow 9 \rightarrow 6$, whereas if we started with the bottom equation and followed with the top equation, we would get $6 \rightarrow 3 \rightarrow 6$.

Not too surprising for $y = x - 3$ and $y = x + 3$, but students may be intrigued by what happens when we do the same thing with the other pairs of equations. For the graph in the middle, starting with the top equation and $x = 6$, we get $6 \rightarrow 8 \rightarrow 6$.

No matter what value we start with, the process will bring us back to where we started. Students in Senior grades may recognize that these pairs of functions are *inverses* of each other. Even though this is not in the Grade 9 or 10 curriculum, students can play with these ideas and see why they work. There is a nice connection to this and solving equations. If we are faced with the equation $2x - 8$, we can

think of this as original number $x \xrightarrow{\text{double}} 2x \xrightarrow{\text{subract 4}} 2x - 4$ new number, which can be solved by reversing the process: original number $\frac{1}{2}y + 2 \xleftarrow{\text{half}} y + 4 \xleftarrow{\text{add 4}} y$ new number.

At this point, as the author of the column, I realized that there is no way I will get through all the linear material, do some interesting stuff for quadratics, and generalize to polynomials of larger degree, without putting a great deal of stress on my Editor (more than he usually has to face when I email him my drafts). As such, we will stick with linear polynomials for this issue and possibly revisit the idea in a (far) future issue.

Returning to our investigations, we realize that we got quite a bit just from rearranging the equation $y = mx + b$. Let's try something similar with the form $Ax + By = C$ to see if it is as fruitful.

Investigation #2: Write down an equation of the form $Ax + By = C$, with A ≠ 0, B ≠ 0, and graph the corresponding line. For your values of A, B, and C, write the new equation and graph the corresponding line. How are the two lines related?

We will start our exploration using *Desmos*. Setting parameters for A, B, and C, and playing with them systematically, we may make a few discoveries. Trivially, when $A = \pm 1$, $B = \pm 1$, both equations become either $\pm(x + y) = C$ or $\pm(x - y) = C$ and are concurrent. When $A = 1$, $B \neq 1$, the two lines intersect at a point on the x-axis, as shown in the diagrams in Figure 3.

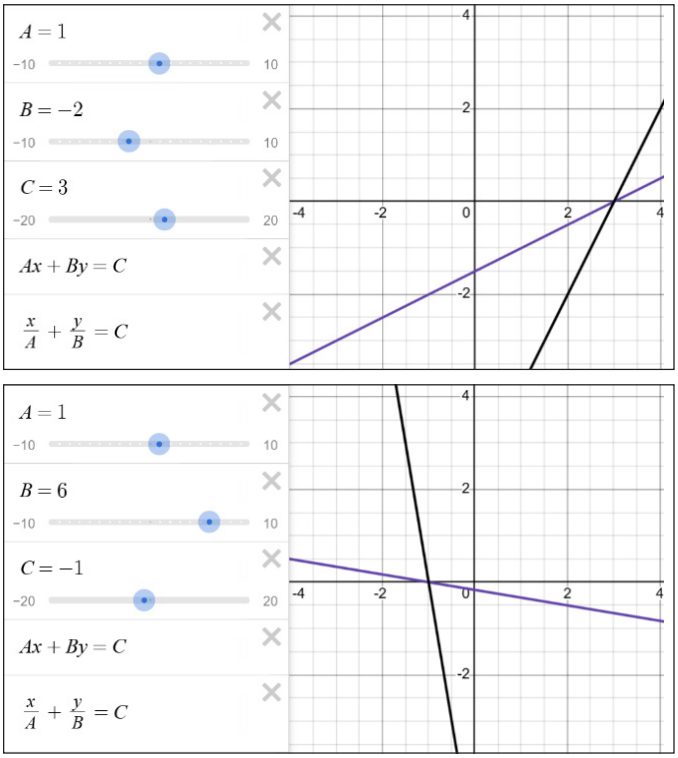


Figure 3: *Same* x*-intercept when* A = *1*

This makes sense, since we determine the x-intercept by setting $y = 0$ and solving for x. Both equations become $x = C$, when $y = 0$ and $A = 1$. It should not come as a surprise that when $B = 1$, $A \neq \pm 1$ the two lines share y-intercepts for the same reasons. The reader may want to explore and explain what happens when $A = -1$, $B \neq \pm 1$.

Another discovery is that when $A = B$, the two lines are parallel. Again, this can be verified algebraically by setting $A = B = k \neq 0$ and putting both equations into the form $y = mx + b$. Doing this, we get

$$\begin{aligned} kx + ky &= C \\ ky &= -kx + C \\ y &= -x + \frac{C}{k} \end{aligned} \qquad \begin{aligned} \frac{x}{k} + \frac{y}{k} &= C \\ x + y &= kC \\ y &= -x + kC \end{aligned}$$

Not only are the lines parallel, but the slope is –1, no matter what the value of k is. This may suggest to the students that the values of A and B (more specifically their ratio) are somehow tied to the slope of the line, in both forms.

Continuing that line of thought, examining a few cases will show that the slopes of the two lines are reciprocals. This can be seen by direct calculation or by noticing that the ratios of the intercepts are opposite. For example, in Figure 4, below, the ratio of the x-intercept to the y-intercept for the purple line is 2:1, but for the black line, it is 2:4 = 1:2.

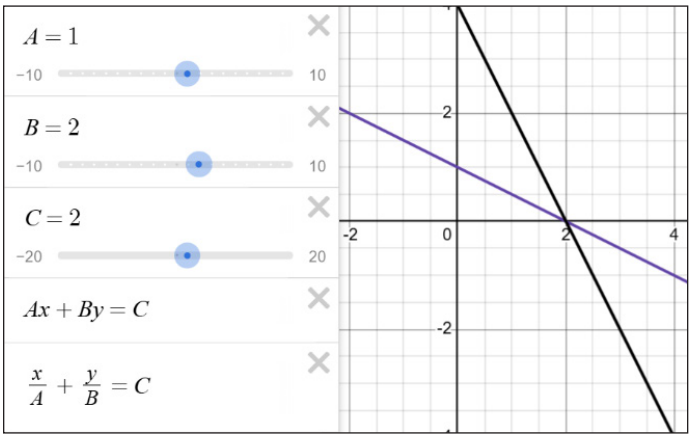


Figure 4: *Reversed ratios of intercepts*

This may remind us of our earlier work, where the x and y variables switched roles. However, it isn't quite the same. It may suggest that we look at the equation $\frac{x}{B} + \frac{y}{A} = C$. If we add that to our *Desmos* graph, we will see that it is always parallel to $Ax + By = C$. This might seem strange at first, but if we think of starting at the first equation and dividing by A, it will be eliminated from in front of x, but it will end up "under" y. Looking at $\frac{x}{B} + \frac{y}{A} = C$ for the case $A = 1$, $B = 2$, $C = 2$, shown in Figure 4, we find that the x-intercept is $4 = 2 \times 2 = BC$, and the y-intercept is $2 = 1 \times 2 = AC$. Setting $y = 0$ and solving for x in the general equation $\frac{x}{B} + \frac{y}{A} = C$, we get

$x = BC$, verifying our observation. We could do the same for the y-intercept. If we went one step further and divided both sides of the equation by C, we get $\frac{x}{BC} + \frac{y}{AC} = 1$, or $\frac{x}{P} + \frac{y}{Q} = 1$.

When the equation is written in this form, the parameters P and Q are the x and y intercepts, respectively. As such, we have "discovered" a "new" form for the equation of a line, where the parameters are directly connected to the graph of the line. We may want to call this the *intercepts form* of the equation of a line.

Where else can we go with this? Although the intercepts are "nice" in the sense that one coordinate is zero, so the point lies on one of the axes and is easy to locate, there is nothing really special about them. Let's examine what happens if we change our focus from the y-intercept to a new, related point.

Investigation #3: Draw several lines passing through the point $(1,k)$. Determine the equations of the lines in the form $y = mx + b$. How does the value of k show up in the equation?

Suppose we start with $k = 3$ and draw the three lines shown in Figure 5. Then, since a line with slope 1 and y-intercept of 3 has equation $y = x + 3$, our line through (1,3) suggests $y = x + 3 - 1$ or $y = (x - 1) + 3$. This form is similar to our discoveries near the end of investigation 1. When we look at the next line in a similar way, we get $y = -x + 3 + 1 = -x + 1 + 3$, which may not seem to be the same. However, since x has a coefficient of -1, we can rewrite this equation as $y = -(x - 1) + 3$. This suggests that since a line with slope m and y-intercept k has equation $y = mx + k$, our line through $(1,k)$ will have equation $y = m(x - 1) + k$.

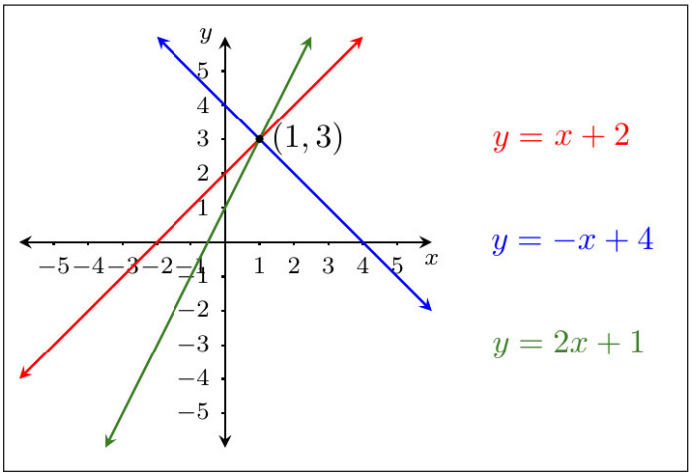


Figure 5: *Three lines through (1, 3) with corresponding equations*

If we check our conjecture with the next line, we see that $y = 2(x - 1) + 3 = 2x - 2 + 3 = 2x + 1$, which works out.

We can further justify this conjecture by going into *Desmos* and graphing the point $(1,k)$ and the line $y = m(x - 1) + k$ for parameters k and m, to get something like Figure 6. Playing with the parameters, we will see that no matter where we move the point or no matter what the slope is, the line passes through the point (the black and red lines were from earlier tested values of m).

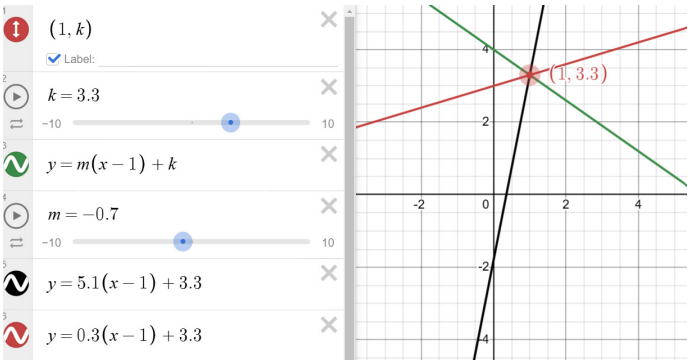


Figure 6: *Testing a conjecture with Desmos*

We can *prove* our conjecture algebraically by substituting $x = 1$ to get $y = k$, which means that the point $(1,k)$ is on the line, as conjectured.

One may even go a step further and realize that we could have written the equation $y = m(x - 1) + k$ as $y - k = m(x - 1)$, where now the variables x and y are treated in the same way. This may further lead us to conjecture that the equation of a line through any point (x_1,y_1) can be given in the form $y - y_1 = m(x - x_1)$, the *point slope form* of the equation of a line.

At this point in my own investigation, I see that my original goal may seem a bit contrived. As such, I abandon my original goal.

Even though $(1,k)$ my original goal got abandoned, the investigation touched on some valuable ideas:

1) We explored different equations of lines, some that should be familiar, others that will probably be new (a, b, and m are x-intercept, y-intercept, and slope, respectively):
   - The *slope y-intercept form* $y = mx + b$
   - The *standard form* $Ax + By = C$
   - The *slope x-intercept form* $y = m(x - a)$
   - The *intercepts form* $\frac{x}{a} + \frac{y}{b} = 1$,
   - The *point slope form* $y - y_1 = m(x - x_1)$
   - The *slope x-intercept form* $x = \frac{1}{m}y + a$
2) We saw some connections with forms of equations and transformations:
   - The horizontal translation by a units $(x,y) \rightarrow (x + a,y)$ show up in the equation, with x being replaced by $x–a$.

- The vertical translation by b units $(x,y) \rightarrow (x,y+b)$ show up in the equation, with y being replaced by $y-b$.
- Reflection in the line $y = x$ shows up by transposing x and y, as in $(x,y) \rightarrow (y,x)$.

3) We got a sense that switching the roles of x and y leads to the inverse, which can be used in solving linear equations.

Throughout the investigations, the students would have been accessing a number of the mathematical processes. They are making and verifying conjectures, making connections, using different representations of linear relations, and selecting different tools to help them investigate. Throughout they are accessing current course content, thus solidifying their knowledge and understanding. In other words, we have come up with a good investigation.

Take the time to work through ideas that you have for student investigations, all the time keeping your mind open to what students might do, as well as directions you can nudge them. When the students do the investigation, keep your eyes and ears open for ideas, directions of investigation, and missteps that you did not anticipate, and make notes for the next time you use the investigation. And, as always, happy problem solving! ▲



Pi-day-inspired cartoon by Julianna Kagawa.

▲ On the Addition and Subtraction of Integers: We Can't Remove What We Don't Have



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If you have been near a mathematics classroom where the students work with integer operations, you will likely have heard something akin to the phrase, "Well, you just flip the sign" at some point. Perhaps you have even said something like this yourself. As someone heavily vested in the teaching and learning of mathematics, this phrase still causes me to purse my lips slightly, even though I am sure I am guilty of saying it myself. After all, recognizing the pragmatic aspects of such a phrase for determining the correct answer of integer addition and subtraction problems is straightforward. Its use as a shortcut is also simplistic to understand, as the phrase itself is not complex. If we perform integer addition like $7 + (-5)$, we can simply switch the sign of the operation to get $7 - 5$ and solve this to get $7 - 5 = 2$. If you perform integer subtraction like $(-3) - (-2)$, we can simply switch the sign to $(-3) + 2$ and obtain $(-3) + 2 = -1$. There are, of course, other variations that could be listed. Oversimplifying the addition and subtraction of integers through use of "flip the sign" traps the student into a cycle of misunderstanding that is difficult to remedy. Part of the purpose for this article is to raise awareness of the mathematical processes involved in performing the addition and subtraction of integers, and provide some clear visual examples that help to show that "flip the sign" is but a siren song that can be overcome, if students understand a few simple elements, including addition of zero pairs, models, and use of the additive identity.

There are several major conceptual problems with "flip the sign." First, the word *flip* itself is somewhat ambiguous,

and it does not seem to do justice to recognizing the unique ability of addition to undo subtraction and vice versa, denying the student the opportunity to acknowledge the importance of these operations having the other as an inverse and how this inversion fundamentally works. Even if you used another word like *swap* or *change*, this problem persists. Further, the sign you are supposed to flip is not specific. Second, the heuristic itself encourages the replacement of a commutative operation (addition) with a non-commutative operation (subtraction), with no evident justification for doing so, or explanation of why this still achieves a correct answer, despite this replacement. Third, a crucial step in the understanding of how models of these problems can be constructed is completely absent, specifically the idea that we cannot remove something that we do not possess. Fourth, it leaves the additive identity (i.e., zero) and its central role in the process of the addition and subtraction of integers (as zero pairs) unaddressed. Examining the nature and properties of operations, role of models, and the position of the additive identity in the process of performing integer addition and subtraction will follow.

Prior to continuing this line of discussion, some details of addition and subtraction need to be reviewed. Addition, of course, is the combination of two or more quantities into a sum, while subtraction is the removal of one quantity from another to obtain a difference. However, addition has a curious property that if you have an initial quantity, you can combine it with another quantity of your choosing, even though the second quantity has limited dependence on the first quantity. In fact, because addition is commutative, the same term of *addend* is used on both numbers being combined. However, subtraction has a major difference (I hope you appreciate the pun) in that the removal of the second quantity relies on the amount that is available in the first quantity. The second quantity must be removed from the first quantity, and we must recognize that being able to remove the second quantity from the first is dependent on there being enough objects present initially to complete the removal. These quantities are represented by Hindu-Arabic numerals, the terms, and the first term is known as the *minuend*, and the second term is the *subtrahend*. Since subtraction is not commutative, we cannot alter the positions of the minuend and subtrahend, without fundamentally changing the question. And what happens if we try to remove more than we have (i.e., when the subtrahend is larger than the minuend)?

While the introduction of negative numbers and models (for example, tiles or number lines) helps deal with the subtrahend exceeding the minuend, this problem is complicated further when we begin dealing with more complex combinations of integers. If you have a positive number and you want to combine it with (i.e., add it to) a negative number, this is generally fine and will be explained in detail in a moment. However, in subtraction, if a positive number is the minuend and the desired operation is to remove a negative number as the subtrahend, it becomes clear quickly when a model of this problem is created that there are no negative numbers present that can be removed from the minuend. Somehow, negative numbers must be introduced into this situation so they can be removed, but if relying on "flip the sign," a student can achieve the right answer, safely ignore the conceptual issue, and happily move on to the next problem, secure in their lack of knowledge of the full process. Unfortunately, without the introduction of negative numbers into this situation, it is impossible to maintain the line of reasoning required to fully understand what has mathematically happened as the "sign was flipped."

The inverse nature of addition and subtraction is not effectively communicated with most words that could be inserted into the heuristic. Flipping, swapping, changing, adjusting, etc., are all simply actions that are tied to the idea of switching the sign because adding a negative number achieves the equivalent result as subtracting a positive number, and vice versa. Though it achieves the same result, it does not highlight the inverse nature of the two operations, and it encourages a dangerous mathematical behaviour, allowing the exchange of non-equivalent operations without justification or explanation. By doing this, an expression is changed from a state where the position of the numerals matters, to one where their position does not matter. Shown, this looks like $4 - (-3)$ versus $4 + 3$; in $4 + 3$ we can adjust the positions to be $3 + 4$ without affecting the problem, but in $4 - (-3)$, the numeral positions must stay where they are. And if the numeral positions change in the version featuring subtraction, we know that a mathematical crime has been committed, and obtaining a correct answer now is, of course, impossible unless the minuend and the subtrahend are the same. This replacement of a commutative operation with a non-commutative operation (or vice versa) is, I contend, like mathematical blasphemy. If this type of behaviour is, in fact, allowed, perhaps the patron gods of mathematics have simply stepped out for a moment, or have simply resigned themselves to look the other way during the addition and subtraction of integers. I argue that this replacement of operations, particularly when a property of the operation like commutativity changes, is not something that should be undertaken lightly, and want to demonstrate directly how this "flipping problem" can be addressed.

However, much of this is not really an issue when it comes to integer addition. Integer addition, due to its property of

commutativity, is relatively safe from the conceptual grief that was just discussed. With two integers, there are only three possible situations that can occur: adding two positives together, adding two negatives together, and adding a positive and negative together (remembering that the order is unimportant due to commutativity). When modelling with tiles in the two positive and two negative scenarios in Figure 1 below, the combination of the different quantities eventually results in a single group that contains only positive or negative tiles, as zero pairs are formed and then removed. When dealing with the final situation of combining opposite signs, the applied use of zero pairs (and their subsequent removal) allows the cancellation of every single quantity that possesses one of the signs (could be positive or negative), and whatever is left is the sum. What this demonstrates is that there is no need for the initial "flip the sign" heuristic when working through integer addition, though when it is used, it masks the use of the additive identity through the formation of zero pairs.

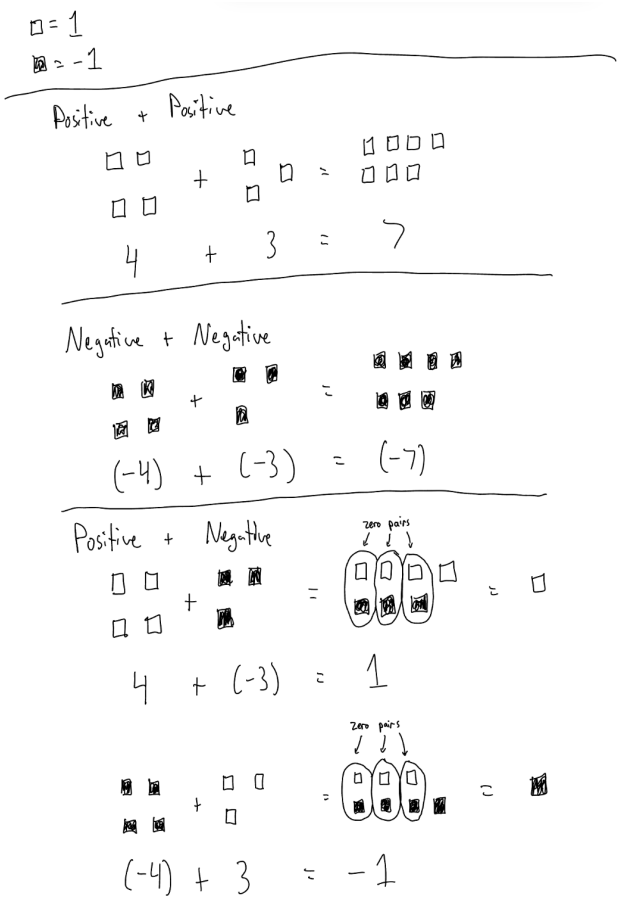


Figure 1: *Drawing of integer addition scenarios with models*

Unfortunately, integer subtraction is a different story. The additive identity of zero plays an absolutely central role in the addition and subtraction of integers, and, through zero pairs, becomes obvious when models are used to represent integers. However, I would argue that this is not intuitive and is potentially a reason for why the "flip the sign" heuristic exists in the first place. Consider the following illustrations using filled squares for -1 and empty squares for +1:

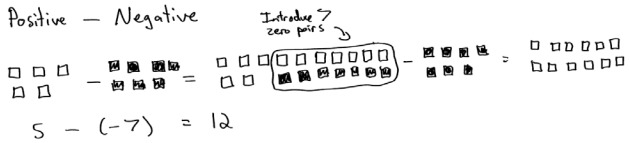


Figure 2: *Drawing of specific integer subtraction scenario (positive - negative) with models*

As clearly shown above in Figure 2, if we have a problem like a certain number of white squares are initially possessed as our minuend, and we want to remove a certain number of black squares from that group, but we don't have any black squares that can be removed. So, the additive identity of zero is used to introduce exactly seven zero pairs because that contains seven black and white squares, and that number of black squares is exactly what we want to remove. We seek to add exactly the number of zero pairs needed to satisfy whatever number we need to remove, based on the specific problem. We now have the necessary number of black squares present in our model that we originally wanted to remove (but could not do so), and we can now remove the appropriate number. This removal leaves a "residue" of positive squares that were introduced when the zero pairs were placed, which then results in 12 white squares remaining. The process of the subtraction of integers requires this idea of utilizing the concept of zero to be present to conceptually understand that when a negative number is subtracted from a positive number, a larger positive number is the necessary outcome. This is a non-trivial and non-intuitive step in the process, but it is necessary; after all, we cannot remove what we do not have.

While one of the four possible situations was just modelled with tiles (a positive number of tiles subtract a negative number of tiles), it is equivalent for the converse situation; i.e., the exact same process applies if the problem begins with a negative number of tiles and a positive number of tiles is subtracted. The zero pairs must be introduced into the situation, and the residue of tiles left behind affects the quantity we report as the difference, and this is necessarily a larger number of negative tiles. When examining subtracting a negative integer from a negative integer, whether this process applies depends on whether the number of negative tiles representing the minuend is larger or smaller than the number of negative tiles representing the subtrahend. If the minuend has more negative tiles than the subtrahend does, we do not need the zero pairs, as we can simply remove the appropriate number of negative tiles represented by the subtrahend from the minuend, but if

representing the subtrahend requires more negative tiles than representing the minuend, using the zero pairs is necessary. After all, we cannot remove what we do not have. This applies in the same way when a positive number subtracts another positive number. See Figure 3 below for this in detail.

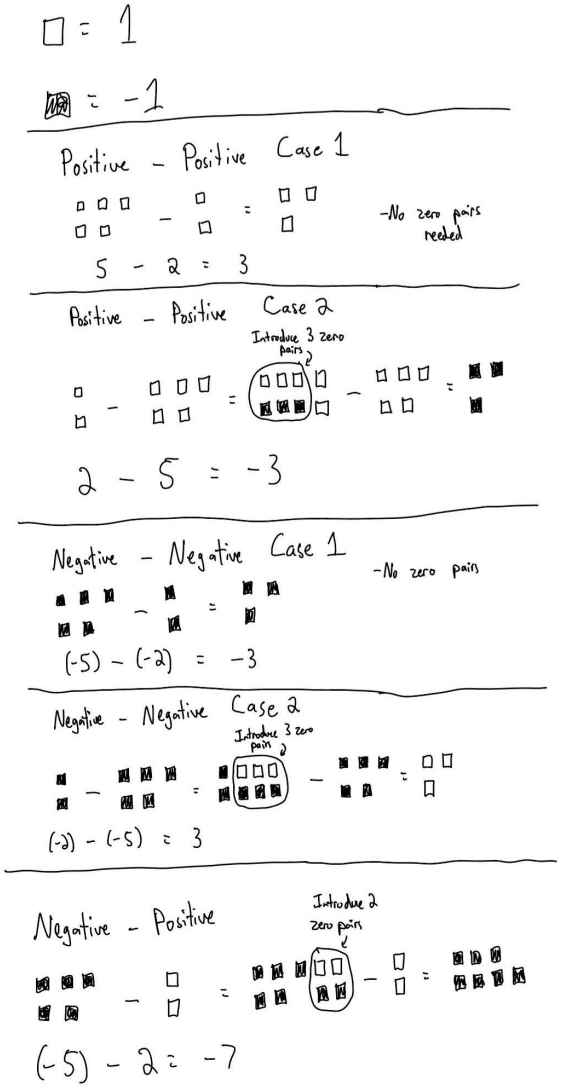


Figure 3: *Drawing of integer subtraction scenarios with models*

Now that this process has been fully examined for the different situations that can be encountered in the addition and subtraction of integers, the role of the additive identity (zero) has been fully unveiled, and the usefulness of "adding nothing" is laid bare. Integer addition and subtraction are not simply "flipping the sign," and while I acknowledge this may well achieve the correct answer, it masks the complex dance of the use of zero pairs and adding zero to problems that is fundamental to why this heuristic works in the first place. Further, it replaces "flip the sign" with clear lines of reasoning built on mathematical facts that are useful outside integers. ▲

▲ Supporting Teachers in Integrating Coding into Mathematics Education



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When introducing the 2021 Grade 9 destreamed mathematics curriculum, the Ontario Ministry of Education rolled out several new expectations for students, including some for coding. As the 2021 curriculum encourages integrating coding in mathematics classrooms, teachers are faced with the challenges of learning and teaching coding to their students. It can be understandable that some mathematics teachers are reluctant to integrate coding into their classes, as they may not have had experience in coding, nor the training to teach or debug such programs.

However, some studies have found that workshop training can alter the attitudes of teachers to be positive toward coding, even in short durations (Adler & Kim, 2018; Ahern, 2009). From professional learning sessions, positive attitudes toward coding can be cultivated as teachers become more familiar with programming concepts, gain confidence in implementing code, and find ways to integrate coding applications into their classrooms (Chalmers, 2018; Cooper et al., 2011; Holmes et al., 2018; Kay & Moss, 2012; Marcelino et al., 2018).

As part of a larger research project formed in collaboration with the Ontario Institute for Studies in Education (OISE), University of Toronto, and the Toronto District School Board, the Grade 9 Destreamed Mathematics Research Project organized professional development sessions for teachers. These sessions were designed with one of the project's objectives in mind: to determine, based on teacher insights and research findings, which professional development activities would be most effective in supporting the teaching of Grade 9 destreamed mathematics courses.

The first year of the project (2021–2022) aimed at discovering teacher insights about the Grade 9 destreamed mathematics course and providing professional learning sessions for course preparation. In the following year (2022–2023), our research team addressed the common concerns around coding. In the third year of the project (2023–2024), we offered coding professional learning sessions that covered the basics of Python (programming language), examples of ways to implement coding in the classroom, and exemplary activities for integrating coding into students' mathematics learning. This paper discusses in-depth the coding resources and activities incorporated into these professional learning sessions, teacher feedback about the learning experience of these sessions, and the in-class applications of coding used in the workshops.

Professional Learning Session Activities

Many activities were developed and modified by the research team from the Grade 9 Destreamed Mathematics Research Project for the purpose of improving teacher confidence, experience, and comfort toward coding and teaching coding in their mathematics classrooms. These activities provided teachers with opportunities to increase their computational thinking skills and encouraged collaboration among teachers. Teachers were also invited to replicate and/or modify them to be implemented based on the specific characteristics of their classrooms.

Pseudocode Activity

Some of the most useful exercises that link mathematics and coding are pseudocode problems. Pseudocode uses English descriptions of algorithms to work out logical sequences of problems, often similar to block coding language and commonly used to plan steps before coding a program. At the Ontario provincial level, Grade 9 students are required to take a standardized mathematics test through the Education Quality and Accountability Office (EQAO), which now has incorporated pseudocode logic problems. Our research team drew inspiration from the EQAO test pseudocode questions and created an activity that challenged teachers' familiarity with pseudocode and logic problem solving. The activity was designed so that teachers were in groups, with access to chart paper/whiteboards. Each group was generated randomly with three or four members and was assigned a representative from the research team, who would give them one of four pseudocode problems to solve. When a question was solved, the representative would hand the group one letter. This process would continue until the group solved all four questions and received all four letters. The group's final task would be to rearrange these letters, which would spell out "E-Q-A-O." Then, the representative would reveal that all the problems solved were based on real EQAO practice test questions (modified for the activity) and encouraged group members to discuss connections with EQAO testing and pseudocode.

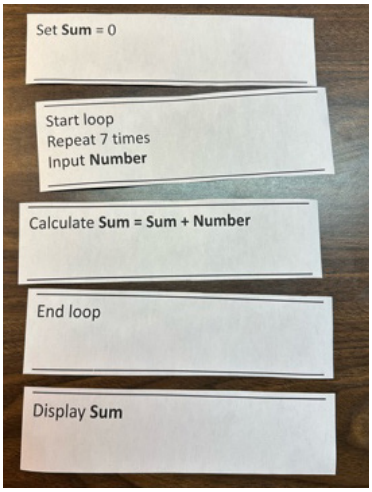
EQAO Pseudocode Question #2

A code is written to display all the possible areas corresponding to the value of radiuses ranging from 0 to 10.

Rearrange the blocks of code to complete the code.

*(Note: ** represents to the exponent)*

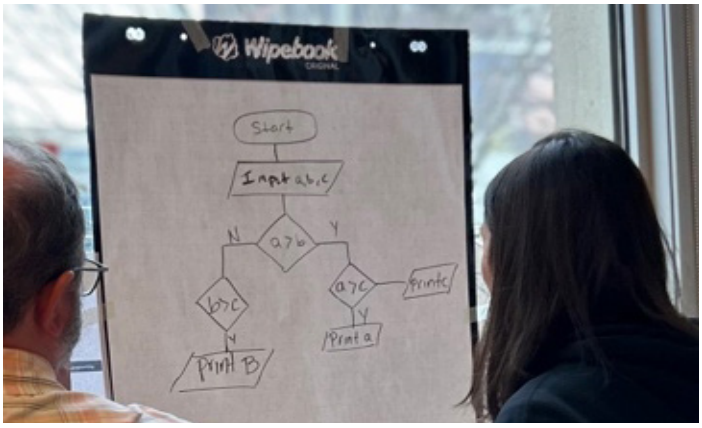
Looping Cut-Outs Activity



As the concept of repetition in coding (or loops) can be difficult to understand among all age groups, it was crucial to develop a strong foundation for teachers to solve these types of questions. For this activity, pseudocode questions that involved loops (sequence of instructions that is continually repeated until a certain condition is reached) were used. Teachers were organized into table groups of four or five. Each group would be given several slips of paper and a large sheet of chart paper. Each of these slips would have descriptions of a pseudocode step. The teachers rearrange these slips in the correct programming order and show their understanding of these steps on the chart paper that correctly follows the loop sequence. Examples of these questions were generating the sum of seven numbers and outputting all of the areas of a circle corresponding to radii of value one to ten.

Flow Charts

In addition to pseudocode questions, participants were introduced to flow charts to better express and form a deeper understanding of repetition in coding, as well as other intricate multi-step problems (e.g., conditional statement problems). For these more complex problems, flow charts can represent algorithms through diagrams using symbols and text, which can help explain and write programs. For the flow-chart activity, participants were first shown a legend that describes different symbols and descriptions for the parts of a flow chart. Then, in groups of three or four members, participants were tasked to create/design a flow chart to represent a multi-step problem (e.g., finding the largest among three different inputted numbers). After finalizing their flow charts, representatives from the research team would check each group's explanation and understanding of the activity.



The extension of this activity included other examples of multi-step problems assigned to table groups. For each problem, table groups of four or five members were asked to collaborate in designing a flow chart on chart paper to present the possibilities/sequences of the conditional statement problem and then execute the corresponding code on Google Colaboratory (Google Colab). Examples of conditional statement problems included a program that incorporated possible outputs for solving the equation of a line with a given slope and coordinate point, and a program that outputs the status of monthly statements based on given monthly income and expenses.

Python

Python is one of the most commonly used programming languages that is high level, easily readable, and conveniently transferable to other programming languages. During the professional learning sessions, all programming was taught using Python. Teachers gained familiarity with Google Colab as a well-used resource for writing and executing code in Python.

Google Colab is accessible through a browser and is similar to other Google applications, automatically updating and saving files in real time on Google Drive. The coding files are referred to as notebooks, which also have the feature of collaboratively sharing files with others with a link, and allowing sharing with email access. The multi-use features of Google Colab are especially appealing to teachers as they can access notebook sharing among colleagues for collaborative resources or even view student notebooks (should they choose to introduce Python coding into their classrooms).

Pre-made Colaboratory Notebooks

As Google Colaboratory is accessible online, there are a multitude of pre-made resources available for users. There already exist many pre-made Google Colab notebooks, but for our professional learning sessions, we selected and

constructed notebooks for teacher access. In the first session, we introduced a notebook that allows users to get a feel for coding syntax and mathematical operations in Python. We asked teachers to complete the notebook individually, running each box of code (https://colab.research.google.com/drive/1onoF7O20EcEqaQigROTDnbIrNjR2lV4H?usp=drive_link).

1. Find and Fix Syntax/Runtime Errors

```
#Example A
print ("Find the Area of a Triangle:")
base = input("Base: ")
height = input("Height: ")
area = int(base) x int(height)/2
print (area)
```

```
  File "<ipython-input-13-e5246d7817ce>", line 5
    area = int(base) x int(height)/2
                     ^
SyntaxError: invalid syntax
```

2. Solve Logic Errors

```
[ ] #Example 1
    #Calculate the average of two numbers
    x = 10
    y = 4
    average = x + y / 2
    print (average)
```

```
12.0
```

For later sessions, the research team created a debugging Google Colab notebook, which features pre-made questions with existing errors. This resource allows participants to get comfortable experiencing syntax errors, logic errors, and runtime errors. In addition, participants can recognize when and where these mistakes may occur, and how to successfully debug them (https://colab.research.google.com/drive/17csf4VAX34fvhm4CdmdNo0HNcwYM1ZhA).

Discussion

The integration of coding into the Grade 9 destreamed mathematics curriculum requires teachers to embrace new skills and approaches, presenting both challenges and opportunities for professional growth. While initially daunting for many educators, the professional development sessions provided through the Grade 9 Destreamed Mathematics Research Project proved to be highly effective in increasing teacher confidence and readiness to incorporate coding into their classrooms.

Participants overwhelmingly found the sessions beneficial, highlighting the activities' relevance and practicality. As one teacher noted, the focus on pseudocode and flow-charting was particularly valuable because "in EQAO, they will get these types of questions" (Teacher 3, interview, February 2024). The emphasis on foundational skills like computational thinking helped teachers bridge the gap between coding concepts and their practical applications in teaching mathematics.

While some educators expressed lingering uncertainty about handling unexpected questions from students, "If students start asking me questions like: 'Why is my code not working?,' I feel like my brain will melt," shared Teacher 2 (interview, February 2024), they also recognized the importance of ongoing practice, and the supportive structure of the resources provided. One teacher commented, "I just need to keep at it and do those examples that you guys posted" (Teacher 1, interview, February 2024).

The professional learning sessions empowered teachers to reflect on how coding could be tailored to their students' varying levels of experience and interests. As one participant aptly summarized, "If a student creates something simple, like an interest calculator on their own, using pseudocode, that's a winner already" (Teacher 5, interview, February 2024). Teachers also appreciated the structured activities for their potential classroom application: "I feel stronger in coding," Teacher 4 reflected (interview, February 2024).

By fostering collaboration, providing hands-on coding practice, and addressing both conceptual and practical challenges, these professional development sessions succeeded in equipping teachers with the tools to confidently integrate coding into the mathematics curriculum. The initiative's effectiveness underscores the importance of targeted support in helping educators navigate curricular changes and inspire their students through innovative learning opportunities.

Next Steps for Teachers

To build upon the professional learning sessions, a next step for this group of teachers would be engaging in collaborative coding projects with peers. These projects could involve co-developing lesson plans that integrate coding with mathematics or sharing real-world classroom applications of coding. Additionally, incorporating advanced coding activities, such as working with more complex algorithms or exploring data-visualization techniques, could further enhance their confidence and expertise. Teachers may also benefit from ongoing access to coding communities or forums, where they can seek advice, share successes, and address challenges collaboratively.

Regularly scheduled follow-up sessions focusing on debugging strategies and handling student queries could address lingering uncertainties, ensuring teachers feel prepared to respond effectively to coding-related questions from students. Creating opportunities for peer mentorship within schools, where more experienced teachers support those newer to coding, might also help sustain the momentum of professional development.

Next Steps for Professional Learning Sessions

For future professional learning sessions, expanding the resource pool would be a meaningful step forward. This could involve curating additional pre-made Google Colab notebooks that align with various mathematics topics or developing a repository of case studies showcasing successful classroom implementations of coding. Finally, creating a more robust follow-up mechanism, such as virtual check-ins or periodic webinars, would provide ongoing support for teachers as they continue to implement coding in their classrooms. These initiatives would ensure the professional learning sessions remain impactful and adaptive to the evolving needs of educators and their students.

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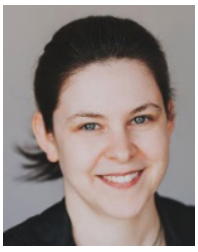
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▲ Events at the Fields: Cognitive Science Network and the CMS Education (Online) Meeting



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The Cognitive Science Network at the MathEd Forum

The October MathEd Forum was jointly organized with the Cognitive Science Network and focused on both theoretical models and practical applications of cognitive science, in terms of math education and AI (Artifical Intelligence). There were various speakers, highlighted below.

Boris Steipe from the University of Toronto on *Cognitive Education with a Non-Cognitive Partner: Addressing an Intrinsic Limitation of Generative AI*

Boris' talk began by defining cognition as "the category of thoughts that we can be aware of." He then looked at where generative AI comes into the picture and called it a wonderful tutoring device for students, and a tool for himself. He then spoke about how we can use AI to consider how we think about problems versus writing syntax.

Boris then went through the process of how generative AI works: taking in information, processing it through artificial neural networks, generating a result, and then applying a feedback loop for training. He then described how the AI "mind" is a hybrid of three components: the language, the dialogue, and the agency. He explained how AI uses dialogue that is developed word by word and not through cognition like humans—where it looks like human dialogue, but it is lacking the cognition behind it, as there is an absence of introspection. We notice this through AI's weakness in categorization; lack of logical coherence; how it does not know what to prioritize; improves writing, but not ideas; and lacks creativity and play—such as AI's inability to make a good punchline to a joke because it doesn't know

how to think further than the next word.

Boris' conclusion to his talk was that, while AI was not an efficient way of resolving ill-defined questions, it does make a good tool for developing insights and collaborations. His research has shown that when asked such ill-defined questions, AI is prone to providing incorrect or misleading (or even made-up/fantasy) responses, and that he believes we need to create a type of "hybrid mind" between AI and humans.

Moriah Sokolowski from Toronto Metropolitan University on *The Building Blocks of Mathematical Thinking*

Moriah began her talk by describing her journey researching how we develop complex learning. She started working with young children during her undergraduate studies, asking the question, "How does the developing mind support complex learning?" She now uses mathematical thinking as a model for studying learning because of the way math is used in our daily lives, how it can predict later academic achievement, and how it is relevant to future career opportunities and occupations.

Her three streams of research include: 1) core competencies, 2) SEL factors, and 3) applications of the research.

1) *How does the developing brain learn basic numerical concepts*? The most basic form is non-symbolic representations of numbers (recognizing and comparing quantities)—a skill we share with other species (primates, some insects/bees), and even young infants who have this ability. Symbolic forms (e.g., the number 3) is very human. Her research uses various tasks and neuroimaging processes to understand the connections between symbolic and non-symbolic number processing. The take-away from this research is that "in adults… symbols and quantities are processed distinctly both behaviourally and at the neural level."

"How do children acquire symbolic representations?" Presenting a developmental timeline, starting with procedural (mostly memorizing) and then becoming conceptional, Moriah walked us through this process. Finally, she described how children learn the Cardinal Principle (when counting items in a set, the last number we count is how many items are in that set). She investigated how this process works, explaining her methodology and the tasks she created using dinosaurs and train blocks, to see what numbers young children knew, and investigating if the child understood numbers and/or sizes.

2) *Why do females have more math anxiety?* Moriah then discussed looking at gender differences in math anxiety. To do this, she gathered data from first-year undergraduate students through surveys, cognitive tests, neural imaging, and copies of transcripts throughout their first year. The original hypothesis was that math anxiety stemmed from differences in math ability or even perceived math ability, but they instead found that it was about spatial representation of numbers. Their research found that males had better spatial ability overall, and that there was a clear correlation between gender and spatial ability, perceived spatial ability, and spatial anxiety; and this spatial anxiety was linked to math anxiety.

3) *Does math ability and/or math anxiety predict selection and success of STEM courses by undergraduate students?* Moriah's research found that math ability correlated positively with the number of STEM courses taken and how well students did in them. They also found that high math anxiety had a negative correlation. When put together, math anxiety is the strongest predictor of courses taken and success in those courses.

Dan Vilenchik from Ben-Gurion University of the Negev, Israel, on *A Quantitative Approach for Measuring the Engagement of Students in e-Learning Platforms*

Dan raised disengagement as a burning issue in e-learning, and the circumstance where the instructor has no idea that the learner has disengaged. He pointed out that disengagement is not like unsubscribing or cancelling a service.

Their research uses Matific to explore disengagement. From their website, Matific claims to be: "[…] a math app that helps learners aged 5–15 excel at maths with personalized and engaging activities. Teachers and parents can access real-time progress reports, assign specific content, and choose from 50+ languages and 70+ countries." They described how Matific works, what data is collected, and some trends they found (such as the fact that over 80 percent of users opened less than 20 videos—with the majority opening only one video—and then disengaging). Their research showed that disengagement was clearly an issue, and their interest is how to predict it so they can consider interventions.

Dan then explained some of the problems they encountered with making predications, such as defining disengagement—is it related to time spent, performance, throughput/completion?—and how different definitions can lead to different results. He also explained how the problem was difficult because there could be other reasons for disengagement that weren't captured (distractions, difficulty

level, length of the video, etc.), and this meant their predications could be inaccurate and not useful.

To address some of these issues, they developed a model to predict disengagement. The model includes 17 features including time, difficulty, and throughput aspects. Students are labelled based on how they approached the learning (such as "explorers" or "difficulty" or "pace"). They then used machine learning to analyze the data, using three different algorithms, and compared their predictions to real-world data and other predictive models.

Dan then wrapped up his talk by speaking about how the definitions they used may seem intuitive for predicting disengagement after the fact, but may not be meaningful in capturing disengagement while the student is actively progressing through the content. He also spoke about the various limitations of the study, such as using a single data set that only included at-home participants, only using three definitions, and the use of only a limited number of algorithms.

Keith Devlin from Stanford University, California, *Is there a simple visual model for a science of education?*

Kevin's presentation was a fascinating trip through both history and research. He described his research, which started in the 1980's, and how it was applied to industrial problems in the 1990's, to the Defense Intelligence Agency in the 2000's, and is now being applied to neuroscience, human creativity, clinical psychiatry, and education.

He then shared his "simple model" for the science of learning that is based on research started in the 1980's pertaining to a mathematically based theory of information. To build this model, they first needed to ask, "What is information?" Kevin then explained that it's only been recently where information went from being an adjective to a noun, a commodity you could share, and then it became an abstract concept that could be processed in response to new technologies. "Information seems to arise when people communicate at a distance (across space or time)," such as a flow of information between two devices. He then used the analogy of a journalist who writes a story, it gets written on paper, then someone gets and reads the paper (produce, store, acquire)—there's a generation of a thought/idea at one end that is turned into a thought/idea at the other end; the journalist's conception to the reader's understanding—someone who is in a different location at a different time. The newspaper itself isn't the information, but the thoughts/ideas contained in it are.

Much of their research stemmed from Fred Dretske's 1981 book, *Knowledge and the Flow of Information*, where information is the creation of expertise in a specific topic. This information depends on types or categories, and rational action depends on type recognition, and production of an appropriate response.

He then used a commutative diagram to translate this into math and how it relates the world to the mind and its connections. The next few decades were spent looking at methods for analyzing the information conveyed by a document or other medium, using the process of math formalization and an analytic technique to gain insight, and then applying this to real-world problems (such as those with the National Security Agency, United States Navy, and United States Army).

Since information is layered and changes with time, the diagram was used to interpret how things in the world adapt and how it is translated into higher knowledge. The diagram captures the flow of information between two sides, and provides a pathway for noticing, changing, and adapting. The diagram can be a simple model that can be used to understand a complicated domain, and helps to understand information and rational action. The diagram also relies on type recognition and a rational agent producing an appropriate response.

Using this information, a functional model of the mind was created, which is a tool for cognitive science and facilitates looking for ways to link it to the real-world mind. This work is now being extended to developing similar diagrams for education, and for addressing things like PTSD, where we need to understand and change various cognitive filters of information in order to process and change our responses.

Springer Representatives: Elizabeth Loew and Nicholas Di Benedetto

Part of the day also included a presentation from an Associate Editor of *Springer*, who spoke about the *Math and Minds* series, with many entries written by our MathEd Forum members.

> The monographs and occasional textbooks published in this series tap directly into the kinds of themes, research findings, and general professional activities of the Fields Cognitive Science Network, which brings together mathematicians, philosophers, and cognitive scientists to explore the question of the nature of mathematics and how it is learned from various interdisciplinary angles. Themes and concepts to be explored include connections between mathematical modelling and AI research, the historical context of any topic involving the emergence of mathematical thinking, interrelationships between mathematical

discovery and cultural processes, and the connection between math cognition and symbolism, annotation, and other semiotic processes. All works are peer-reviewed to meet the highest standards of scientific literature. (www.springer.com/series/15543)

Canadian Mathematics Society (CMS) Education (Online) Meeting Debrief

The November MathEd Forum was a follow-up from the 2024 CMS Education (Online) Meeting. This online meeting is held in conjunction with the CMS's Winter Meeting and invites math educators to present and collaborate on their work and research. You can find more details on this past year's conference at https://winter24.cms.math.ca/mathed online.

For the MathEd Forum, three presenters from the conference were invited to discuss their work in further detail. These talks included: *Math Is Everywhere* (Shaun Elder and Pamela Brittain), *How do educators' reflections on problem solving inform our understanding of teaching problem solving?* (Ana Duff, Kseniya Garaschuk, and Lynn Pacarynuk), and *chatOAME: What's Been Learned from 6 months of AI Conversations with Mathematics Teachers* (Cal Armstrong).

Math Is Everywhere

In this presentation, myself and Shaun Elders (formerly of the Royal Conservatory of Music) presented the use of an app called MathBeat, to explore the connections between mathematics and music, and using music to explore and explain various math concepts. This specific talk focused on illustrating dividing a fraction by another fraction, and how we can use musical beats to both visualize and hear this concept.

Reflections on Problem Solving

In this talk, Ana and Lynn provided information on their research into problem solving. They shared insights into their study with educators, where they investigated the role of familiarity in problem selection, challenges in solving complex problems, and strategies for scaffolding problem-solving skills in the classroom. They also investigated educators' thoughts and feelings on their own capabilities in both problem solving and applying problem-solving techniques in their own classrooms.

chatOAME

The final presentation was from Cal Armstrong, where he outlined the chatOAME platform, how it was developed, what it is currently capable of, and where he hopes it will go. The platform is available to all OAME/AOEM members, and further information can be found in both the March 2024 and September 2024 issues of the *Gazette*.

Videos from the MathEd Forum presentations can be found at www.fields.utoronto.ca/activities/24-25/meforum.

Ongoing Programs at Fields

Looking to build your math skills? Or host a virtual visit with a mathematician for your students? Know a high school student who is interested in puzzles and problem solving? Or want to find fun resources to share with your students and colleagues? Then check out our ongoing Fields K–12 programming at www.fieldsacademy.ca/eep/. ▲

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abroad, we have to live with and follow the rules in use in the place where we are. Knowing all this on how taxation practices function is, however, fascinating, even if when we have to pay them, it is, pun intended, quite taxing!

I hope you continue to enjoy these (third) musings on % as much as we did. Let me end with one of Edison's famous sayings, who reportedly asserted that success is 10% inspiration and 90% perspiration (or was it 1% and 99%?). I'll let you figure out if this is a carved-in or a surplus type of partitioning, and how it could work otherwise!

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Her summary encouraged the following:

- Aiming high for all kids!
- Knowing your purpose and goals (grade-level expectations, access, participation, and creating strategic sense makers)
- Knowing your students (what they know and what they can do, how they feel about learning math)
- Then designing to reach those goals with the students, ask:
  - What is narrow and working poorly?
  - What can we open up and make more accessible?
  - What will still need additional scaffolds/supports?
- Trust in your students' thinking! ▲

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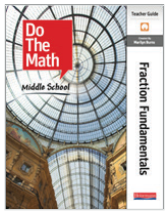
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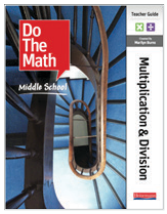
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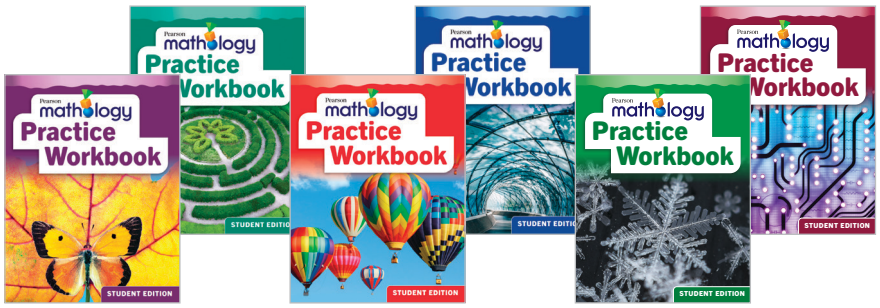
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