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Ontario Mathematics Gazette

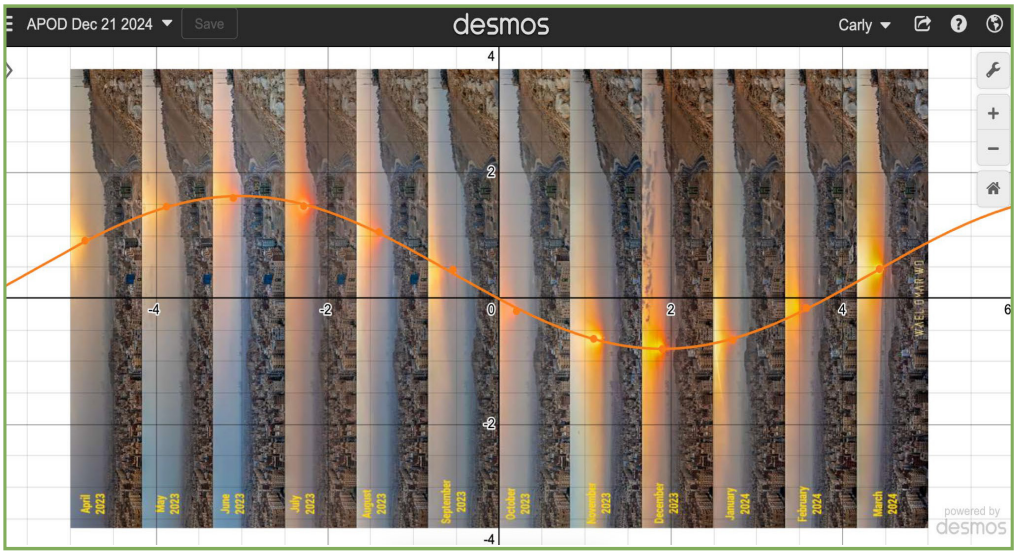
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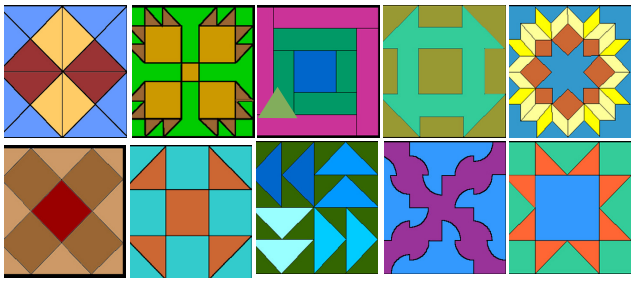
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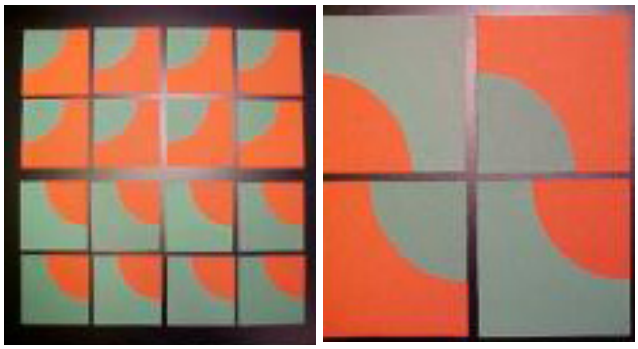
Vol. 63#4
June 2025

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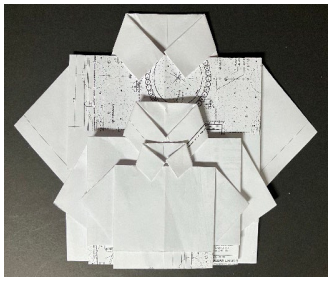




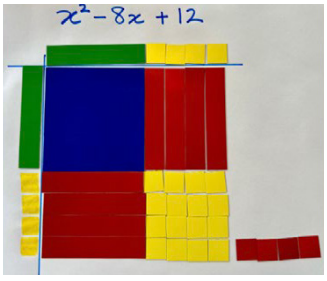
See Elementary Math Matters: Quilts as the Holders of Stories and Geometry

See Mb4T (Mathematics by and for Teachers): Fraction Talks

See In the Middle: All About Calendars



See The Math Teacher's Home Base: Origami in the Classroom



OAME 2026 Let's {Talk} Math

KEYNOTE SPEAKERS



Dr. Pamela Seda



Pamela Harris

FEATURED SPEAKERS

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Octavia Beckles
Dr. Alex Lawson
Chris Luzniak
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The *Ontario Mathematics Gazette* (*Gazette*) publishes news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *Gazette*:

- The *Gazette* is aligned with the OAME/AOEM Vision, Mission, and Strategic priorities. These can be found on the website www.oame.on.ca under "About OAME" and "Guiding Principles."
- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles are to be word–processed, MS Word is preferred, and prepared according to the *Publication Manual of the American Psychological Association*, Seventh Edition. However, please use single-line (not double) spacing and only one space after each period. Articles should not exceed eight numbered pages of text, though there is some flexibility. Figures, images, and photographs should be placed close to the relevant text and include captions. The photographer's permission is required, and for photos of students under the age of 18, permission of a parent or guardian is required.

Please submit your article in one blind file (i.e., identity of author is not evident), and include author names, contact information including email and mailing addresses, photos—head and shoulders, biographies—less than 100 words, and all content removed for blinding in a second file. Please email these two files to Tim Sibbald at gazette@oame.on.ca.

Upon review, you will be notified whether your article has been accepted for publication (as is, or pending minor or major revisions) or declined. The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME/AOEM.

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The views expressed or implied in this publication, unless otherwise noted, should not be interpreted as official positions of OAME/AOEM.

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About the *Ontario Mathematics Gazette*

The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

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Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** by email or at the above address. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

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Editor's Report



TIM *SIBBALD*, OCT, PH.D.
gazette@oame.on.ca

Tim Sibbald is a professor in the Schulich School of Education, Nipissing University, with a focus that includes mathematics education. He is a former President of OAME/AOEM and has held a variety of other OAME/AOEM posts over the years.

If you jump into your time machine and zoom back to the time when you contemplated going to Teachers College, what did your decision-making process look like? In hindsight, you likely see yourself having made the right decision. However, if your foray in time slips a little, and the policies of today moved to the time you made the decision, I wonder if you would make the same decision.

The times have changed. Teachers College moved from a one-year to two-year program in 2015. At the time, enrolments were cut in half, which has slowly led to the shortage of teachers today. You would also have to register with OCT (Ontario College of Teachers) to get a number so that you could then apply for a time slot to write the Math Proficiency Test (MPT). Perhaps as a reader of the *Gazette*, your interest in math would make you feel okay.

There are several catches. First, the MPT is not just a math test; it includes a section on pedagogy that is about broad policies from *Growing Success: Assessment, Evaluation, and Reporting in Ontario Schools* (2010) and *Learning for All: A Guide to Effective Assessment and Instruction for All Students, Kindergarten to Grade 12* (2013), which are not specific to math. Second, it is required that each section of the test is passed at the level of 70 percent, which constitutes a high-stakes test; that may not faze math teachers, but if your teachables are focused on the arts, you still have to pass the MPT to be certified to teach by OCT.

If this editorial is giving you a sense of déjà vu, it may be a faint remnant of my mentioning the MPT in my editorial in June 2021. Since the implementation during the pandemic, the MPT was taken to court by teacher candidates, and the Province lost. However, the government appealed and the ruling was overturned. It was reinstated last year and is now in full force.

A concern that drove the court case was the question of whether the test was equitable. However, before you think too much about equity, consider the following statement

made by the President of OTF (Ontario Teachers' Federation):

> In the case of the MPT, it quickly became clear that teacher candidates from equity-seeking and minority groups were disproportionately unsuccessful. As one illustrative example, individuals who wrote the English version of the test were 17% more successful than beginning teachers who wrote the French-language version (Durocher, 2023, p. 1).

The test remains available in English and French, though I am not aware of any statistics regarding results this time around.

The test environment… oh yes, the MPT must be completed online. You may read that it can be completed in person, but that means you can go to a site, and they will proctor the MPT while you use their computer. This addresses individuals with poor Internet connections or if someone's computer is not capable of handling the remote testing computer requirements, where the invigilators monitor test takers via the camera on their computer. (If that is not "1984" enough for you, the rules say you should not move your lips during the test. Evidently, they are worried about test takers conversing with people out of sight of the camera.)

The test-taking online environment includes numerous accommodations. It has a high contrast option, a capacity to read one line at a time, a zoom feature, an option where it will read the question, etc. This aspect of the testing, though one can blissfully ignore it, is informative about the range of accommodations and what they look like in practical terms.

There is also a calculator. If you want to use a calculator, it has to be theirs. The calculator is almost identical to one I got for Christmas in 1979. It is a rudimentary calculator with a square root button and a pi button. There is no square button, even though the Pythagorean theorem occurs in some questions. The issue with this calculator is not that it is rudimentary. The issue is that students may not think about the calculator in advance and end up writing the test without knowing. The fact that the calculator does not have brackets, does not display the calculation, and does not follow order of operations (it does sequential completion of operations) will be annoying to math-adept candidates, but will likely cause issues for students who are unaware. Teacher candidates who are concerned about the MPT ask why modern calculating tools are not permitted. While there is recognition that the test raises questions about order of operations and exponent laws, these could be done without a calculator, and then allow a modern calculator for the questions that entail more elaborate calculations.

The test begins with five mental-math questions that must be completed without a calculator; then 45 math questions, where you may use the supplied calculator; and, finally, 21 pedagogy questions. Let me show you a few. The questions are all multiple choice with four options.

Mental-math sample questions include 1 + (–2) – (–3) (Answer 2), $3^4 \div 3$ (Answer 27), $24 \div (2/3)$ (Answer 36), and, "What is the value of the 2 in 425 380?" (Answer 20 000). These questions reflect the Grades 3 to 9 range, around which the test is designed.

Calculator questions cover a range of topics and grade levels. Most are routine questions, with some being multi-step, but I will point out a few that are notable. Consider:

"What is the value of b in this equation: $\frac{b}{4} = b - 12$?" This particular question allows multiple approaches. If you are okay with fractions, you might isolate the variables on the left and solve using fraction subtraction or addition. Alternatively, you could multiply every term by 4 to eliminate the fractions and then isolate variables and solve as an integer question. A third approach is that, as a multiple-choice question, there are four available contenders for the answer, and substitution of 3, 6, 18, and 48 will rapidly identify the correct answer.

The math questions that allow calculators emphasize fractions in various ways, and graphing related to the linear concept. There is a range, however, such as a question that asks for the area of a shape composed of a rectangle with a semi-circle extruding from the long side and an isosceles triangle from a short side. In this question, a key detail is that the isosceles triangle is denoted by two hash marks showing equal length sides, and one has to notice a third hash mark at the other end of the rectangle that shows the triangular region is in fact equilateral. The questions range in form such as one giving the value of a variable and asking for which equation it is a solution. Consider: "Which option describes the sum of $\frac{11}{5} + \frac{1}{7}$?" Is it equal to 1, equal to 2, between 1 and 2, or between 2 and 3? Since 11/5 is more than 2 and it is addition, the answer must be between 2 and 3. Often this sort of test-taking logic can be used without having to determine an exact answer.

The pedagogy questions are oriented to knowing policies. Generally, they are not specific to teaching mathematics, despite the moniker of a "math test." Consider that a question asks which of "good," "very good," "satisfactory," and "needs improvement" is not listed in *Growing Success* for reporting on learning skills and work habits.

Which option is ***not*** *listed in* Learning for All *as an example of a formativ*e assessment tool or measure?

A. Provision of timely descriptive feedback to students
B. Use of assessment results to guide further instruction

C. Inventory of student interests to guide further instruction
D. Use of the moderate marking process to support ongoing assessment and monitoring of student learning

This highlights the wordiness that is common among pedagogy questions. The answer is C, which can be found on page 29 of *Learning for All*. Another sample question is:

According to Growing Success*, which student behaviour demonstrates the learning skill and work habit of self-regulation?*

A. The student follows instructions with minimal supervision.
B. The student seeks clarification and assistance when needed.
C. The student takes responsibility for and manages own behaviour.
D. The student devises and follows a plan and process for completing work and tasks.

In fairness, some math pedagogy is included, such as a question checking that candidates recognize that multiplying is not a math process (as per Strand A). However, this may be argued as simply showing knowledge of the language in the policies, as opposed to understanding the overall intent of the curriculum. Perhaps a more pungent example of the gap between memory and understanding is:

According to Growing Success*, which qualifier is used to describe student performance at Level 2?*

A. Some
B. Limited
C. High degree
D. Considerable

Granted this question aligns with candidates having seen the achievement chart. However, I doubt that it separates proficient from non-proficient teachers, and I also wonder about the implication that this is promoting the use of such vague language with students or parents—it lacks utility because it is the polar opposite of descriptive feedback. A question that leads to similar questions of its value for assessing proficiency asks how "Level 1" is related to "the provincial standard."

Overall, the test is not unreasonable. However, there are valid concerns with its implementation being strictly online and various hindrances within the online environment. Particular concern is earned by the use of an outmoded calculator that many candidates are unfamiliar with—I appreciate the goal of using it, but teacher candidates have asked why order of operations can't be checked in the mental-math section and then allow a square button as a minimum or a calculation display, or a scientific calculator, since these are routine tools in the modern world. Equally valid are questions about the requirement that the formula sheet is online and necessarily blocks the view of questions. To paraphrase a student, "It is as if the test has been designed for their ease of marking, rather than the ease of students completing the test." And that, students note, violates the principles of the very policies they are expected to adhere to. As a professor, I have found myself having to point out that the policies teacher candidates are learning apply to K–12 education, not to candidates at the entrance to a profession. But, as a requirement for admission to a profession, I am concerned that it is not modelling the behaviour and the policies expected of teacher candidates who become professional teachers. While I do not expect the same policies to apply, I worry that teacher candidates are being shown professional oversight as a machine, and that may have implications for how teachers perceive the roles they fill.

In the News…

Congratulations to **Lynda Colgan** as a recipient of a King Charles III Coronation Medal. The award acknowledges, "Dr. Colgan's tireless efforts continue to inspire the Kingston community, nurturing a sense of wonder and curiosity in even the youngest learners."

In case you missed it, OAME/AOEM Vice-President **David Petro** was on CBC's Pi Day coverage: www.cbc.ca/news/canada/windsor/windsor-pi-day-1.7484380.

In This Issue…

Kit Luce, OAME/AOEM President, brings wishes for the summer. **Jason To** brings news of activities at the OMCA. **Pamela Brittain** reports on many activities that have taken place at the Fields Institute for Mathematical Sciences during the Spring.

Jacqueline Hill arises twice in this issue! She has written her final "NCTM Report." She is also recognized by **Fred Ferneyhough**. During the editorial process, the latter led **Peter Saarimaki** to point out that Jaqueline is one of eight people to be President of both the OAME/AOEM and OMCA.

David Costello promotes teaching metacognitive skills. **Jérôme Proulx** considers how percents are added and cancelled. **Iain Brodie** and **Hattice Beyza Sezer** point out the benefits of bugs—coding missteps, that is. **Lynda Colgan** draws on quilting and history for culture-based mathematics. **Ron Lancaster** makes shirts and a heart for origami mathematics. **Carly Ziniuk** crosses cultures with several calendar considerations. **Sofia Ferreyro-Mazieres** considers the process of completing the square. **Jeff Irvine** stays current with flow theory. Last, but by no means least,

Shawn Godin focuses on ratio problems—he was concerned about the length of his column, but it is quite an interesting read that suits thinking about in the summer.

Three articles and a book review are included. **Vera Sarina** raises concerns about the glossary in the most recent curriculum documents. **Marc Husband**, **Tina Rapke**, and **Christina de Simone** bring a strategy for addressing circumstances where students bring in standard algorithms they have learned outside of class. **Jeff Irvine** considers what makes a teacher effective. The book review, by **Tim Sibbald**, reflects on a book about **Roger Penrose**.

Thank Yous

As we close this volume of the *Gazette* with issue four, I am reminded of the team of people we have, making our beloved double-blind peer-reviewed publication the ongoing success that it is. The Fab Four: **Anne Yeager**, **Peter Saarimaki**, **Gitta Berg**, and **Penny Clemens** are pillars of the *Gazette*, who have firmly supported all the authors and columnists for years. I could not ask for a better team. Thank you!

I also appreciate the long-term support of **Robert Sherk**, who is the advertising manager for the *Gazette*. Robert is a subtle and highly effective person who, from my perspective, simply makes the advertising happen seamlessly. It is also a benefit knowing that **Kyla Kadlec** is available to address any French articles that come to the *Gazette*. Thank you!

Over the last year, we have had reviews of possible articles completed by (alphabetical by first name): **Alexis Galvao**, **Ann Arden**, **Barb Seaton**, **Cathy Chaput**, **Chris Stewart**, **David Petro**, **Jacqueline Hill**, **Jennifer Holm**, **Jimmy Pai**, **John Rodger**, **Judy Mendaglio**, **Kameal Mina**, **Kiran Pain**, **Lynda Colgan**, **Mallory Andrews**, **Matthew Oldridge**, **Paul Alves**, **Ralph Connelly**, **Robert Cop**, **Sandra Jean Price**, **Sonia Ellison**. Thank you, and apologies if I have omitted anyone.

Lastly, but equally importantly, I would like to thank all the authors who have submitted articles, columns, letters, puzzles, cartoons,… everything that readers look forward to. It is the willingness of authors to professionally share their thinking that has seen our association persist and benefit so many professional teachers. Thank you to all of you!

Reference

Durocher, Y. (2023, Nov. 30). *The Ontario Teachers' Federation disappointed by Ontario Court of Appeal decision to uphold Math Proficiency Test for beginning teachers*. www.otffeo.on.ca/en/wp-content/uploads/sites/2/2023/11/The-Ontario-Teachers%E2%80%99-Federation-disappointed-by-Ontario-Court-of-Appeal-decision-to-uphold-Math-Proficiency-Test-for-beginning-teachers-1.pdf ▲

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▲ President's Message



KIT LUCE
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Kit has been an active member of MAC² for many years and joined the Board of Directors in 2022. She is a District Mathematics Facilitator for Grades K–9 with the Simcoe County District School Board. In her work, Kit creates math resources to support educators in their instruction and assessment practice, as well as developing and facilitating professional development. Kit is an Additional Qualification instructor and course writer, and she has written and reviewed work for a variety of publishers. She has been involved in Ministry writing projects and was a Content Lead for the recent OAME/AOEM writing projects. Kit received the 2024 OAME Award for Leadership in Mathematics Education.

The end of the school year brings with it the excitement of what is to come, with warmer weather, time to rejuvenate, time to learn, and time with friends and family. For many of us in different parts of Ontario, this has been a winter of intense weather, multiple bus cancellation days, and even days with school closures. There has been the stress of wondering how we can possibly teach all of the curriculum when many days have been lost to weather-related events. But we've made it to June, and we have time to reflect on the year, what went well, and what changes we want to make next year. This reflection often leads us to consider what we want to learn to better support our students. The summer break also allows us time to pursue further learning, through taking AQ courses, university courses, and engaging in learning through workshops and conferences.

OAME/AOEM's vision includes an equitable mathematics education community, where everyone experiences high-quality, research-informed, and engaging activities. Professional learning opportunities are central to realizing this vision by equipping educators with the latest research, effective strategies, and innovative tools to enhance their instruction.

One of OAME/AOEM's strategic priorities is to support all OAME/AOEM's members by providing value for membership, including quality professional learning, resources, and opportunities to network. This commitment underscores the importance OAME/AOEM places on continuous development for its members. This past year, there have been quite a few professional learning opportunities offered by OAME/AOEM's Board of Directors and by the local chapters. These have included the Rethinking Disability and Mathematics Leadership Conference with Dr. Rachel Lambert, and a book study about the same resource, facilitated by TEAMS (Toronto Educators Association for MathematicS). The OAME Talks series featured webinars with math educators throughout the year, and SAME (Scarborough Association for Mathematics Education) facilitated a book study to learn more about different types of problem structures with David Costello and his book, *Messing Around with Math*. Other chapters hosted math nights, meet-and-greets, social events, and virtual workshops to support their members by providing access to leading mathematics educators, both in person and virtually. These opportunities for professional learning culminated in the OAME Annual Conference in Durham with Making Math Come Alive in 2025, where educators from across the province and beyond came together to learn, discuss, debate, and play. We hope that you have been able to participate and engage in some of this learning. The financial constraints of many of our school boards have meant that professional development is often limited in our work, and OAME/AOEM can serve as a valuable resource to further our instructional practice.

Engaging in professional learning ensures that our teaching is grounded in current research-informed practices, ultimately benefiting student learning outcomes. By continually learning, we are better equipped to create inclusive environments and develop students' critical-thinking skills, enabling them to become independent and creative problem solvers. As you reflect on the past year this summer, I hope that you consider connecting with OAME/AOEM in the coming year and joining us at the local chapter level in person or virtually, to explore research-informed mathematics education practices you can apply in your classroom. For our Francophone members, who are also members of Association francophone pour l'enseignement des mathématiques en Ontario (AFEMO), we celebrate with you on the development of your new website (www.afemo.on.ca), and know that you join us in encouraging ongoing professional learning in mathematics education.

What do you want to explore and learn about in the coming year? What topics would you like to see presented in workshops? What books or resources would you enjoy investigating with colleagues? We'd love to hear from you. Reach out to your local chapter, or you can email members of the Executive Committee directly (you can find the email address for chapters and the Board of Directors list on the OAME/AOEM website and regularly appearing in the *Gazette*). Enjoy the articles in this issue of the *Gazette* as you enjoy your well-deserved summer, and we look forward to coming back together in the Fall to learn and grow together. ▲

▲ Recognizing Jacqueline Hill

BY FRED FERNEYHOUGH,
EXECUTIVE DIRECTOR

One of the joys of teaching is the incredible people whom you meet during your journey in education. There are great students you will remember for a long time, and hopefully, some of those will keep in touch with you. There are also colleagues who have touched you as a friend, a mentor, or an inspiration. Most likely, many of those will be other math teachers, but I can also count friends from other disciplines, who have touched me in some way. For those of us who are OAME/AOEM members and have attended one or more of the annual conferences, we have the added advantage of encountering amazing teachers from around the province, both speakers and participants.



Jacqueline Hill in Gazette, *42(1)*

During a conference many moons ago, I was advised to catch a session being presented by Jacqueline Hill, and that was great advice. This was in the time period when our curriculum was moving away from just having students attain skills. We were fortunate to have many people who had already transitioned to different methods, and Jacqueline was one of the leaders in that effort. Each time that I heard her speak, I took away something that I could use immediately.

Later, at the annual banquet, I introduced myself to tell her how much I had enjoyed her presentation. "Oh, you're the calculator guy! Pleased to meet you." I didn't know that she'd even heard of me. Then she did something that impressed me. She asked if I could write a program on the calculator to demonstrate a particular feature for her Grade 10 class. I learned that night that one of the attributes of great teachers is that they are never hesitant to admit that they don't know something and that you might know something that they can learn. During the speeches, I took out my calculator and wrote the program for her. You'd have thought that she had been given an early Christmas present. She was delighted and asked how she could get a copy of the program and get it into her calculators.

Later, I was asked by the *Gazette* Editor to write a column that demonstrated how to use calculators in our secondary curriculum. To the best of my knowledge, Jacqueline was the only person who not only tried out everything that I wrote, but sent me feedback on how it went in her classes. And, if something that I contributed was not pertinent to what she was teaching that year, she would try it out anyway, just to learn something new.

I have since realized that many teachers share that attribute of looking for new things to take back to their classrooms, schools, and/or districts. We are indeed fortunate in Ontario. Fast forward a few years and I joined the OAME/AOEM Board of Directors. Jacqueline was not only a great teacher, but also a coordinator of math in her district and very active in the Pine Ridge chapter of OAME/AOEM, as well as the OAME/AOEM Board of Directors. She was a driving force within the board and then took on the coordination of the annual conference in 2003. Not long after that, she moved into an Executive role of OAME/AOEM and served a year as President.

One unusual pattern with OAME/AOEM Presidents is that they serve as President-Elect for a year, do their term as Presidents, then a year as Past President, and then most of them leave the board, never to return. I find that pattern disappointing, as it is a huge loss of experience and wisdom. Jacqueline is one of three Past Presidents who are currently on our Board of Directors. She returned about ten years ago to serve as the NCTM Representative. Since then, she has chaired SPaRC—the Strategic Planning and Review Committee (a subcommittee of the Board of Directors). As always, she never sits on the sidelines when she can be actively contributing in some fashion.



Jacqueline Hill in Ye Olde Archives (on OAME website)

Jacqueline has announced that it's time for her to leave the Board of Directors. This will leave more time for her to enjoy her children and her quilting group. She leaves the board with our heartfelt appreciation for her contributions and our sense of loss because she has meant so much to us. ▲

▲ What's the Problem? Wrestling with Ratios



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Shawn is a former mathematics teacher, department head, and consultant. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: presenting workshops, writing articles, working on local projects, and helping create mathematics contests.

The problem we will explore in this column was originally from the "Brainteasers" column from the July/August 1998 issue of *Quantum* magazine, which has been discontinued. This is unfortunate, as there were wonderful articles and problems involving mathematics and physics, and it was a collaborative effort of many groups, including the National Council of Teachers of Mathematics (NCTM).

> ***B239 Goldbricking.*** *Three bars of gold alloys of different percentages of gold have masses 1 kg, 2 kg, and 3 kg. Can you find a way to divide these pieces in order to make another three bars with masses 1 kg, 2 kg, and 3 kg, but with equal percentages of gold? (Your method must work no matter what the original percentages of gold may be.)*

This is a beautiful little problem involving ratio, rate, percent, and fractions. Rather than just looking at the solution and discussing things that you can do with it in your classroom, we will consider several ratio questions. The problems and their solutions are not lengthy, but we will discuss several ways to approach and think about these questions, from concrete to abstract. We will also take some time, when appropriate, and go beyond "the answer" and think a little deeper about the concepts involved.

As a way to explore the problem, I wanted to come up with several ratio challenges that might scaffold the solution. I went to one of my favourite sources: math contests. Math contests provide wonderful problems you can use in your classroom because they tend to be different from questions in standard classroom resources, and you will find challenges that offer different paths of engagement. I went to the website of the Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo. I used their Problem Set Generator tool to pick questions from their Gauss (Grades 7 and 8), Pascal (Grade 9), Cayley (Grade 10), and Fermat (Grade 11) contests. I ended up with too many questions and as a result, I have created a video that shows you how to use the Problem Set Generator, as well as discusses several that didn't make this column. You can check out the video at https://youtu.be/rJsgiXYRxC0.

We will start with problem #6 from the 2011 Pascal contest for Grade 9 students.

> *Peyton puts 30 L of oil and 15 L of vinegar into a large empty can. He then adds 15 L of oil to create a new mixture. What percentage of the new mixture is oil?*
>
> *(A) 75 (B) 25 (C) 45 (D) 50 (E) 60*

We know that we have a total of 30 L + 15 L = 45 L of oil, and 45 L + 15 L = 60 L of the mixture, in total. Thus, the percentage that is oil is

$$\frac{45}{60} \times 100\% = 75\%,$$

which is answer *(A)*. Wonderful. Now on to problem….

However, let's suppose we are not writing a contest, and we may not have a class of Grade 9 students in front of us. There are a lot of important things going on in this question and techniques that we can consider. For example, I could use linking cubes, where each red cube represents 1 L of oil, and each blue cube represents 1 L of vinegar. Then we could represent the new mixture as in Figure 1.

A couple of subtle things are going on here. In the beginning, we are looking at litres of oil and vinegar in the ratio 30:15, where this means that for each 30 "units" of oil, there are 15 "units" of vinegar. Later, when more oil is added, the ratio becomes 45:15. In both these cases, the numbers in the ratio match the number of litres of each liquid present in the mixture, but that does not need to be the case. For example, if we noticed that all the numbers in the problem are divisible by 5, we could have grouped our linking cubes in groups of 5.

Figure 2: *Grouping the cubes by 5*

We can replace each line of five red cubes with one orange cube, and each line of five blue cubes with one purple cube, to get an equivalent ratio.



Figure 3: *An equivalent ratio*



Figure 1: *Representing the problem with linking cubes*

From there, we could go one step further to group the new cubes in groups of 3, or if we noticed that all numbers in the original problem were multiples of 15, we could end up with

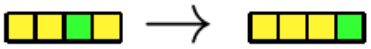


Figure 4: *Reducing the ratio to lowest terms*

At this point, we can either recognize that $\frac{3}{4}$ of the mixture is oil, which is 75%, or we could finish it off by going the other way and bringing in a bunch with the representation shown in the last rectangle, to get



Figure 5: *Representing as a percentage*

There is quite a bit that we can glean from this problem. First is the idea that when we see a ratio, we are comparing "equal things" in one sense or another. That is, in the original problem, we had 45:15, which represented litres of each of the fluids involved, in total. However, in our different models with blocks, we came up with 9:3, 3:1 and, finally, 75:25, where our "equal things" were 5 L collections of each liquid, 15 L collections of each liquid, and percentages of the whole mixture, respectively. This allows us to see that, as when working with fractions, we can multiply or divide each term of a ratio by the same value and get an *equivalent ratio,* that is, a ratio that has different numbers, but represents the same relationship between our quantities being compared.

Thus, we could have done the problem using strictly equivalent ratios. That is, the work we did with the cubes could be represented as

$$\begin{aligned} 45{:}5 &= (45 \div 5){:}(15 \div 5) = 9{:}3 \\ &= (9 \div 3){:}(3 \div 3) = 3{:}1 \\ &= (3 \times 25){:}(1 \times 25) = 75{:}25 \end{aligned}$$

Note that instead of replacing our "groups" with a new representative cube, as we did earlier, we could just rearrange the cubes to represent the relationship. Doing this, we can simultaneously see the ratios 45:15 = 15:5 = 9:3 = 3:1 in Figure 6 below. I leave it to the reader to discover these ratios and how we could have gotten from the start to this point.

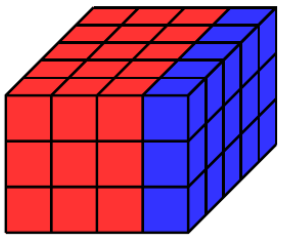


Figure 6: *Many ratios in one model*

The other thing we can deduce from the equivalent ratios is the common algebraic tool. That is, in our original, if we reduced the ratio to lowest terms as

$$O{:}V = 45{:}15 = 3{:}1,$$

where O and V represent the number of litres of oil and vinegar, respectively, then we can deduce that

$$\begin{aligned} O &= 3k \\ V &= k \end{aligned}$$

for some real number k (since we are dealing with litres of liquid, so we can have partial litres as well). Then, since we are dealing with percentages, the total would be 100, so

$$\begin{aligned} O + V &= 100 \\ 3k + k &= 100 \\ 4k &= 100 \\ k &= 25, \end{aligned}$$

from which we can deduce that 75% of the mixture is oil (and 25% vinegar).

Another important point when we are dealing with ratios is that in some cases, the things we are comparing are parts of some *whole*, as in this problem. We had a mixture of oil and water, and although the first part of the question was focusing on the parts of the mixture, the end was asking about the portion of the whole that is oil. This should get us thinking about fractions, which are also a powerful way of displaying ratios, but we must use caution. If we look at our problem, it started out as a mixture of oil and vinegar, where our *whole* was 45 L of a mixture that was $\frac{2}{3}$ oil and $\frac{1}{3}$ vinegar, to which we added more oil to make a new mixture, where the whole was 60 L that was $\frac{3}{4}$ oil and $\frac{1}{4}$ vinegar. Students may be tempted to try to write out an operation using fractions to obtain the result in one calculation. Although this can be done,

$$\frac{45\times\frac{2}{3}+15\times\frac{1}{1}}{45+15}\times 100\% = 75\%,$$

it is not obvious to most students (unless they know something about weighted means). Fractional form is useful in representing ratios. However, one must exercise caution because two fractions may not have the same *whole*. We will return to some problems where we "mix" things with different ratios later. First, let's look at a similar question.

The following was problem #17 from the 2012 Gauss contest for Grade 7 students.

> *The ratio of boys to girls at Gauss Public School is 8:5. If there are 128 boys at the school, then how many students are there at the school?*
>
> *(A) 218 (B) 253 (C) 208 (D) 133 (E) 198*

The information given in the ratio relates the number of boys to the number of girls, whereas the question is asking about the total number of students, so we should be careful.

One way or another, we need to determine the number of girls before we can work out the total (or do we?).

Students who are not comfortable with the algebraic technique presented in the last problem can still arrive at this result without the formal use of algebra. We could have written the question as

$$8:5 = 128:?,$$

where we are dealing with equivalent ratios. We know we can multiply or divide all the terms in a ratio by a constant, to obtain an equivalent ratio. Thus, as $128 \div 8 = 16$, we can conclude that

$$8:5 = (8 \times 16):(5 \times 16) = 128:80,$$

from which we get that there are $128 + 80 = 208$ students at the school. Answer is *(C).*

The numbers in the problem are probably a bit too large for us to solve the problem using manipulatives; however, we can think of equivalent ratios and generate the table in Figure 7.

Boys	Girls	Total
8	5	13
16	10	26
24	15	39
32	20	52
40	25	65
⋮	⋮	⋮

Figure 7: *Looking for a pattern*

From the table, we could either continue until we had our solution, or we could notice that $40 \times 3 = 120$ and $120 + 8 = 128$, so our total should be $65 \times 3 + 13 = 208$. We could also have written things out as

$$B{:}G{:}T = 8{:}5{:}13,$$

where T represents the total. Then we could have used *unit ratios* to solve the problem:

$$\begin{aligned} 8{:}5{:}13 &= (8 \div 8){:}(5 \div 8){:}(13 \div 8) \\ &= 1{:}0.625{:}1.625 \\ &= (1 \times 128){:}(0.625 \times 128){:}(1.625 \times 128) \\ &= 128{:}80{:}208 \end{aligned}$$

Reflecting on this result, we note that if we would have worked with

$$B{:}T{:}8{:}13,$$

we could have found our result without needing to determine the number of girls.

Let's use some of what we have seen so far and approach a problem as a *contest writer*. The following was problem #13 from the 2011 Fermat contest for Grade 11 students.

Xavier, Yolanda, and Zixuan have a total of $50. The ratio of the amount Xavier has to the total amount Yolanda and Zixuan have is 3:2. Yolanda has $4 more than Zixuan. How much does Zixuan have?

(A) $16 (B) $8 (C) $14 (D) $13 (E) $30

To begin with, we see that our pool of $50 is broken into the ratio 3:2, which has 5 "parts." Thus, we can conclude that Xavier has $30, while Yolanda and Zixuan have $20 *between them*. Right away we can eliminate choice *(E).*

Since Yolanda has *more* than Zixuan, we can conclude that Zixuan has *less than half* of the money they have between them, or less than $10, which leaves *only* answer *(B)*, the correct answer.

Pretty neat! Although my career focused on teaching secondary students, I strongly suspect there are many students in Grades 8 and 7 (and below) that can follow this argument, and many of them could do it themselves. Recall that the problem was from a contest for Grade 11 students. As such, when searching through contests for questions to use with your class, you need not restrict yourself to contests intended for the grade you are teaching. The problem need not be used verbatim; it can be adjusted to suit the needs of you and your students. For example, just changing $50 to $20 makes the last problem reasonable to solve with manipulatives, lowering the floor, while retaining the problem's richness. Hence, depending on your collection of students, you are likely to see a variety of different solutions, which is a great thing.

The downside of the "multiple-choice contest solution" is that, wanting only the answer doesn't leave much room to present a solution or communicate one's thinking. As such, this type of exercise could be used as a "Number Talk" (Humphreys & Parker, 2015), where students would discuss their strategies. For example, you could eliminate the choices, give students a short period of time, and ask them for a reasonable estimate ("a bit less than 10" would be a great response).

Interested readers may like solving the problem using more traditional tools.

The following was problem #22 from the 2005 Gauss contest for Grade 7 students.

In a bin at the Gauss Grocery, the ratio of the number of apples to the number of oranges is 1:4, and the ratio of the number of oranges to the number of lemons is 5:2. What is the ratio of the number of apples to the number of lemons?

(A) 1:2 (B) 4:5 (C) 5:8 (D) 20:8 (E) 2:1

An accessible approach, amenable to manipulatives, is to work with equivalent ratios. There are three things being compared, but we are looking at them two at a time. Suppose we group them all together, but leave the unknown one blank. That is, the first ratio given in the problem could

be written as

$$A{:}O{:}L = 1{:}4{:}?,$$

while the second could be written as (note the different question mark symbols represent different unknown quantities):

$$A{:}O{:}L = ¿{:}5{:}2$$

These two ratios are equivalent, so we can make them look the same. Since they have one term in common, let's work with it (note different symbols again represent different unknown quantities):

$$1{:}4{:}? = ¿{:}5{:}2$$

$$(1 \times 5){:}(4 \times 5){:}(? \times 5) = (¿{:}\times 4){:}(5 \times 4){:}(2 \times 4)$$

$$5{:}20{:}\square = \Delta\, 20{:}8$$

Since the two ratios are equivalent, and their middle term matches, all their terms must match, which means $A{:}O{:}L = 5{:}20{:}8$, from which we can get the answer *(C)*.

That was slick! What if we used manipulatives? We can use red, orange, and yellow linking cubes to represent apples, oranges, and lemons, respectively, in which case, the two ratios would be represented by the cubes in Figure 8.

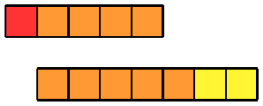


Figure 8: *Representing with cubes*

However, the units are not the same (otherwise, the same number of oranges would be in each construction). We could keep representing equivalent ratios for each until they match. For example, since there are more orange pieces in the bottom, we could make an equivalent ratio on the top to get

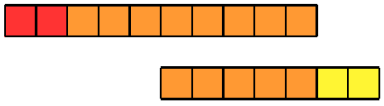


Figure 9: *Two ratios, different units*

Now there are more on the top, so we could go up to the next equivalent ratio on the bottom to get

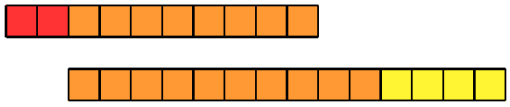


Figure 10: *Still not the same units*

leaving more orange on the bottom. Continuing in this manner, we will eventually get to the point shown below

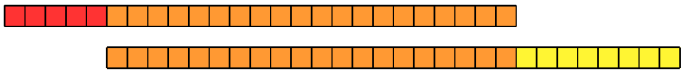


Figure 11: *Matching units in two ratios*

and we can get our answer.



Figure 12: *The answer*

However, we could do the problem in one step, if we keep in mind that the orange cubes in the two blocks represent the *same collection of oranges*. Thus, if we take the cubes that represent the original $A{:}O = 1{:}4$ and turn it vertically, each of the 4 orange cubes must be part of some $O{:}L = 5{:}2$ block. If we draw one so that they share one of the orange cubes, we get the situation in Figure 13.

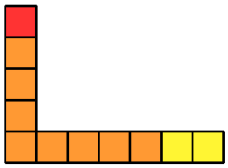


Figure 13: *Starting to merge the ratios*

We can consider this as a $O{:}L = 5{:}2$ block that grew off one of the oranges from the $A{:}O = 1{:}4$ or vice versa. Since all orange cubes are "shared" with both ratios, we continue this process until we get

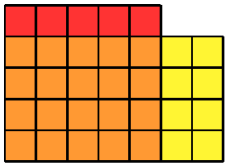


Figure 14: *The end result*

giving us our answer. This method will "work" in all cases, giving a valid ratio, although in some cases, it might not be in lowest terms (can you think of a case where this would happen?).

Let's continue with problem #13 from the 2010 Pascal contest for Grade 9 students.

> *There are 400 students at Cayley High School, where the ratio of boys to girls is 3:2. There are 600 students at Fermat Collegiate Institute, where the ratio of boys to girls is 2:3. When considering all the students from both schools, what is the ratio of boys to girls?*
>
> *(A) 2:3 (B) 12:13 (C) 1:1 (D) 6:5 (E) 3:2*

This is an interesting problem, as we are "mixing" two different "mixtures" made up of different "concentrations." In both cases, we are looking at groups of boys and girls. In Cayley High School, there are more boys than girls, and in Fermat Collegiate Institute, there are more girls than boys. The ratios of our "mixtures" are opposite of each other.

Since we are bringing these two groups together, it is going to be impossible for either of the original ratios to survive, because to do that, our ratios would have to be

equivalent, and they are not. As such, we can eliminate choices *(A)* and *(E)*.

If we continue logically, if we had *the same* number of students in each school, then we could assemble groups of 5 students from each school: 3 boys and 2 girls from Cayley High School, with 2 boys and 3 girls from Fermat Collegiate Institute, to get a big group of 5 boys and 5 girls, or a ratio of 1:1. However, since the schools do not have the same number of students, we can eliminate this possibility *(C)*.

Since our schools have different numbers of students, we can think of bringing the 400 students from Cayley High School and "mixing" them with 400 students from Fermat Collegiate Institute. As we saw above, this will result in the ratio of boys to girls being 1:1; in other words, we have the same number of boys as girls. However, we have 200 students left from the girls-rich Fermat Collegiate Institute. Throwing them in the mix will mean our final ratio will have more girls than boys. The only remaining choice that satisfies this is our answer *(B)*.

What would happen if we used a less abstract method, opting for a more concrete solution, and went back to linking cubes: blue for boys and pink for girls (yes, I know, I'm showing my age). Since the ratios for each school are reversed, in each smallest group, we will have the same number of students. If we think deeply about the situation, since the ratio of student populations is 400:600 = 2:3, then for every 2 students at Cayley High School. there are 3 students at Fermat Collegiate Institute. Furthermore, this means for every 2 equally sized *groups* of students at Cayley High School, there are 3 equally sized *groups* of students at Fermat Collegiate Institute. Using our cubes, the answer is staring us in the face.

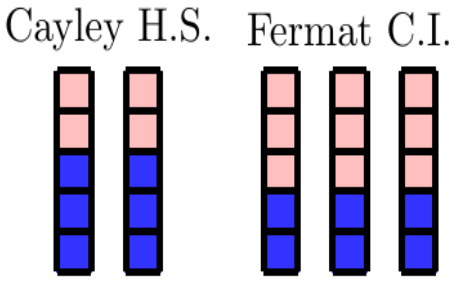


Figure 15: *The final ratio with cubes*

It's hard to deny the strength and beauty of a solution like that. As such, it is always a good idea to consider different methods and solutions.

We will consider one more problem before we go on to the original. The following was problem #11 from the 2011 Fermat contest for Grade 11 students.

Suppose m *and* n *are positive integers with* m < n*. The value of* $\frac{m+3}{n+3}$ *will be:*

(A) equal to 1
(B) equal to 3
(C) less than the value of $\frac{m}{n}$
(D) greater than the value of $\frac{m}{n}$
(E) equal to the value of $\frac{m}{n}$

Let's use what we have learned about mixing mixtures to solve this Grade 11 contest problem, using pre-high school skills. We can think of this as initially, we had a ratio of two things as m:n, where $m<n$, so we have a larger concentration of the second item than the first. Adding 3 to both the numerator and denominator of the fraction means we are mixing in a new mixture with ratio 3:3 = 1:1, which is richer in the first than we had originally. Thus, we expect the mixture to be between the two, less heavy in the second, but not quite 1:1. Interpreting this in terms of our original ratio, we could then write the new ratio as m´:n, where m´ > m. As $\frac{m+3}{n+3}=\frac{m'}{n}$, we can deduce that

$$\frac{m}{n}<\frac{m+3}{n+3}<1,$$

and we can see our answer is *(D)*.

There is another thing we can see from this line of reasoning. We (should) know that

$$\frac{a}{b}+\frac{c}{d}\neq\frac{a+c}{b+d},$$

except in very specific circumstances (when?). However, our discoveries about mixing mixtures allow us to see that if $\frac{a}{b}<\frac{c}{d}$, then

$$\frac{a}{b}<\frac{a+c}{b+d}<\frac{c}{d}.$$

There are many ways you can justify this. Using algebra, it falls out quite nicely, but you can also argue it using the definitions of fractions as "parts of a whole." I will leave it to the interested reader to investigate these demonstrations.

OK, let's (finally!) return to our original problem. How shall we approach it? More confident students, and teachers, might be tempted to begin algebraically. If that is your preference, make an attempt. I will, instead, approach the problem in a concrete way and represent things with linking cubes, as we have done in several places already. Since the gold bars are of different, unknown, percentages, I will indicate this by using different colours for the 3 bars. Also, since the bars are of different masses, I will represent the 1 kg bar as a line of blue cubes, the 2 kg mass as 2 lines of red cubes, and the 3 kg mass as 3 lines of green cubes, as shown in Figure 16, below.

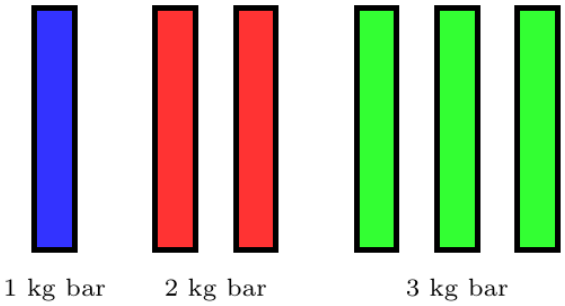


Figure 16: *The original three gold alloy bars of different concentrations*

We want our final bars to have the same percentages of gold, but how can we do that? What if we take our 6 *equal* bars and share them equally among the 6 new bars? We could do this by breaking each line of cubes up into 6 equal pieces (i.e., each line has 6 cubes in it). Then each line would contribute 1 cube to each "new" line, as shown below.

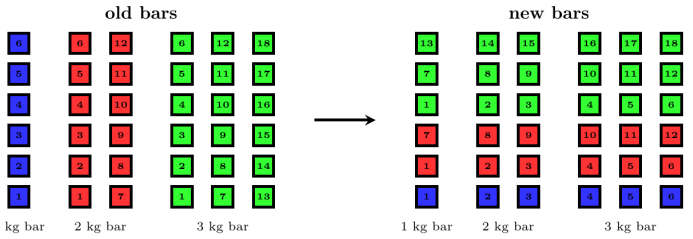


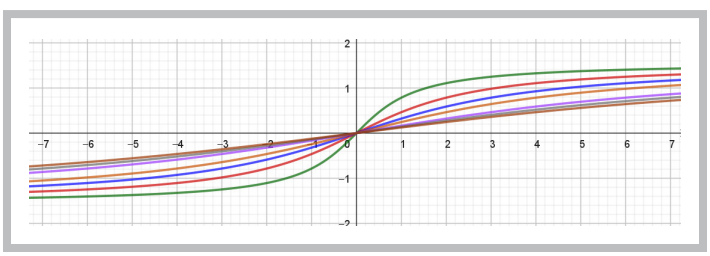
Figure 17: *The final arrangement*

This means that if we divide each of the original bars in the ratio of 1:2:3 and then put all the small pieces together, all the middle pieces together, and the large pieces together, you will be left with 1 kg, 2 kg, and 3 kg, which, when melted together, will be made up of the same percentage of gold. Interested readers may enjoy verifying this.

I hope you have enjoyed this deep dive into ratios. Check out the video I mentioned earlier for solutions to *more* ratio problems, as well as tips on how to find more. Have a great summer and happy problem solving!

Reference

Humphreys, C., & Parker, R. (2015). *Making number talks matter: Developing mathematical practices and deepening understanding, grades 4–10.* Stenhouse Publishers. ▲



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The Ontario Mathematics Coordinators Association (OMCA) supports the leadership of mathematics consultants, coaches, and coordinators from across the province. With almost 200 active members meeting monthly via Zoom, these leaders connect to share experiences, learn among peers, and receive updates from partners, including the Ontario Ministry of Education and the Education Quality and Accountability Office.

Monthly OMCA Meetings and Professional Learning

Over the past few months, OMCA has welcomed several insightful mathematics leaders to deepen the capacity of our members.

- November 2024: **Lisa Anne Floyd** shared various ways to "celebrate coding" for young and older students, including how pseudocode is a superb beginning for (but not a replacement for) coding in the math classroom, the importance of debugging activities to promote thinking, and cross-curricular opportunities with coding using Micro:Bits.
- December 2024: **Kyle Webb** deepened our understanding of Thinking Classrooms with some new developments, including new ways for students to write notes, as well as little moves to support teachers' implementation of Thinking Classroom practices.
- January 2025: OMCA welcomed **Doug Duff** and **Nicole Lesage**, whose presentation, *The Importance of Instructional Leadership and Its Power to Change Schools*, highlighted the critical use of mathematical models, such as the open number line, and part/whole models to drive rich and deep learning.
- March 2025: **Pam Harris** joined OMCA to share how problem strings are a terrific instructional approach to scaffold and support students as they develop conceptual understanding of number and algebra.

Continued on page 48

Useful Theories to Know About: Flow Theory



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Jeff Irvine, Ph.D., has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

Have you ever become so absorbed working on a math problem that everything else seems to fade into the background? Perhaps you lost track of time, and felt immersed in the problem?

In sports, this is sometimes referred to as "being in the zone" or "being in the groove." An athlete "in the zone" is ultra-focused and typically exhibits a high-performance level that is seemingly beyond the level of their peers. This usually happens with elite athletes faced with a highly challenging task. Outside of sports, this state of being is called "flow."

Kyra Kristensen-Irvine has described, to me, times during her university basketball career, when she seemed to be playing in a different time dimension than her opponents. "They all seemed to be playing in slow motion. I knew what they were going to do before they did it. I was so focused that I could make moves at a much higher speed than the other players."

Flow Theory

Mihaly Csikszentmihalyi (Chick-sent-me-hi-ee) was born in Hungary, where his formal education was interrupted by World War II. He emigrated to the United States at the age of 22. Despite not finishing middle school and never attending high school, he wrote and passed entrance exams to the University of Chicago, earned a doctorate, and became a distinguished professor of psychology. Csikszentmihalyi spent most of his professional life investigating flow. He identified flow as a psychological state characterized by energized focus, complete absorption, and enjoyment when performing an activity. Csikszentmihalyi (1975) identified a critical element of flow as the balance of skills and task challenge. He proposed the flow channel (Figure 1) to illustrate this concept. If one's skill level is low relative to the challenge of the task, the result is anxiety. Alternatively, if one's skill level is significantly higher than the challenge of the task, the result is boredom. However, when the task challenge matches the skill level, there is potential for achieving a state of flow.

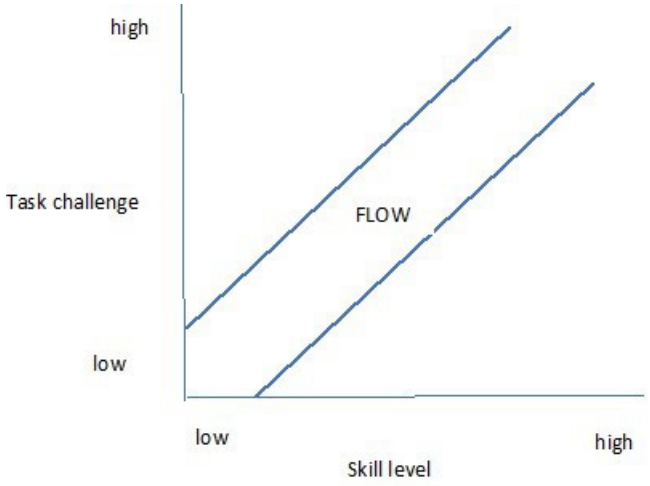
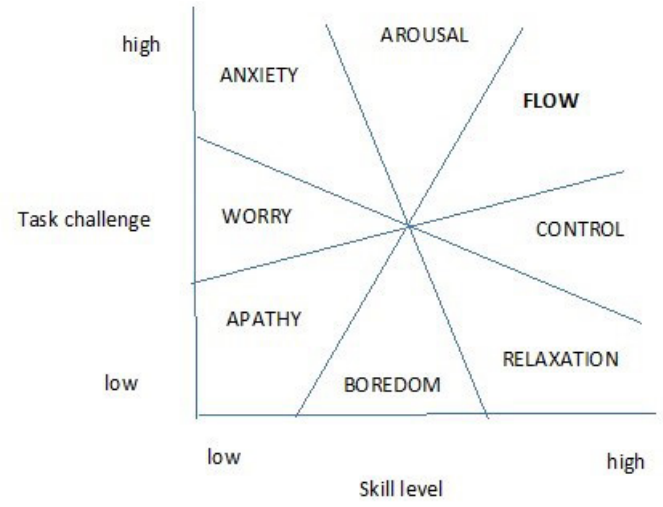


Figure 1: *Flow channel has skill level equal to task challenge.*

Csikszentmihalyi (1990, 1997, 2000) subsequently extended his consideration of flow by examining eight potential matchings of skill level and challenge (Figure 2). Consider first tasks with low levels of challenge. If the participant's skill level is also low, the task is achievable, but apathy will result. If the participant has a medium level of skills, the result is boredom, and if the participant has a high level of skill, the participant will easily accomplish the task and will experience relaxation. Next, consider tasks with a medium level of challenge. Participants with low levels of skill will experience worry; participants with high skill levels will feel in control of the situation. Finally, if the challenge level of the task is high, participants with low skill levels will be anxious, since they will be unable to successfully complete the task; participants with medium skill levels will feel arousal as they attempt to meet the high level of challenge. However, participants with high skill levels will experience a state of flow as they rise to the challenge of the task.



Figuer 2: *Flow and related psychological states*

In an interview with *Wired* magazine, Csíkszentmihalyi described flow as "being completely involved in an activity for its own sake. The ego falls away. Time flies. Every action, movement, and thought follows inevitably from the previous one, like playing jazz. Your whole being is involved, and you're using your skills to the utmost."

Why Should Math Teachers Care about This Theory?

As teachers, we want our students to realize their potential and engage with the work with high levels of intrinsic motivation. The flow state is the ultimate target, where the student is engrossed with their work and highly motivated to persist. Peter Liljedahl (Simon Fraser University) feels that the flow channel identifies maximum engagement for our students. Even though flow is a relatively rare state, we need to strive to move all our students to maximize engagement. Sometimes this will result in flow; but we should always teach in ways that engage and encourage excellence in all our students. There are conditions that we can create that maximize the potential for our students to have a flow experience in our classrooms. In an interview about applying flow theory to education (Beard, 2015), Csikszentmihalyi identified conditions that encourage or support flow experiences:

- Most importantly, the activities must be enjoyable. This supports engagement and generates intrinsic motivation.
- The experience must match the student's skill level, while still be challenging (Csikszentmihalyi's flow channel). Thus, we need to utilize problems that have low floors (all students have an entry point into the problem), high ceilings (every student can excel to their potential), and wide walls (problems allow for lateral as well as hierarchical thinking).
- The activity must have personal meaning for students, and impact the student's well-being in a positive way.
- The activity must have clear goals that are personally important to the student.

While these conditions will not guarantee flow, there will be no chance of flow experiences if these conditions are not present. At the very least, activities with these attributes will generate meaning and engagement for our students.

If You Want to Learn More

Bakker, A.B. (2005). Flow among music teachers and their students: The crossover of peak experience. *Journal of Vocational Behavior, 66*, 26–44. doi.org/10.1016/J.JVB.2003.11.001

About flow in the music classroom.

Bassi, M., & Delle Fave, A. (2012). Optimal experience among teachers: New insights into the work paradox. *The Journal of Psychology, 146*(5), 533–557. doi.org/10.1080/00223980.2012.656156

About achieving flow in your own teaching life.

Jackson, S.A., & Csikszentmihalyi, M. (1999). *Flow in sports: The keys to optimal experiences and performances.* Human Kinetics.

For those who want to learn more about flow in sports.

Spencer, J. (n.d.). *What is flow theory? What does this mean for our students?* [Video]. https://youtu.be/iUsOCR1KKms

This excellent video provides teachers with all the tools necessary for creating classrooms in which flow state can be achieved.

References (other than "read more")

Beard, K. (2015). Theoretically speaking: An interview with Mihaly Csikszentmihalyi on flow theory development and its usefulness in addressing contemporary challenges in education. *Educational Psychology Review, 27*, 353–364. doi.org/10.1007/s10648-014-9291-1

Csikszentmihalyi, M. (1975). *Beyond boredom and anxiety: Experiencing flow in work and play.* Jossey-Bass.

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Etymology of "Random"

In French, the word *randonnée* means to ride on horseback. This led to *random* coming to English around 1300, when it meant to ride a horse at full speed without much care. Shakespeare, 250 years later, used it to mean without a particular aim or purpose. It was in the 1650's, that the meaning became an occurrence without conscious choice.

Barber, K. (2006). *Six words you never knew had something to do with pigs.* Oxford University Press.

Musing on Percentages: Adding and Cancelling %



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Percent is everywhere—all around us in finance, shops, foods, news/media, surveys, and so on. Percentages may seem like a rather simple mathematical theme that has little depth. However, when one starts to pay attention to its usage, percentages can be seen as filled with surprising elements that are mathematically rich and conceptually fascinating, hence worth pondering. This is something that became quite salient for our research team, as we have been preparing experiments with students in classrooms about percentages in mental mathematical environments.

My columns presently endeavour to raise and address some of these peculiarities, which we came to call "curiosities," related to the concept of percentages. These musings can be seen to be as much about our usage of "%" in everyday life, in schools, or in mathematics in general. This fourth column addresses issues related to the addition and cancellation of %. It is my hope that these issues, and their underlying mathematical properties, will intrigue you.

(Wrongfully) Adding and Cancelling %

In one previous column (Proulx, 2024), issues of commutativity were raised where percents were discussed as being grounded in a multiplicative structure. At the same time, % is frequently seen as being added up in different situations. For example, anyone who has done some work with % and calculators will have more frequently used the ⊕ key than the ⊗ key. As well, it is quite frequent to see, although conceptually wrong, a list of % added to one another, for example, on companies' annual reports, where % of profits are added from one month to the other to offer the total annual profits. The same is to be said in salary raises, where the % gained each year over a number of years are added together to sum up the entire % raise over these years. I say conceptually wrong here because adding % is a risky business. This column explores this idea.

Let's start our quest with a recurrent situation:

> *The Blue Grocery Store has a 25% discount on their whole-wheat pasta. When ready to pay, the cashier offers you a ticket to scratch for an additional discount on your purchase. You scratch a 5% discount. After congratulating you, the cashier explains that this amounts to a total discount of 30% on your pasta and enters it. You raise a skeptical eyebrow, not convinced of this direct addition of %. Is the 30% discount equivalent to a 25% discount followed by a 5% discount?*

This situation is intriguing, since numerically, 25 + 5 = 30 makes it tempting to transfer it to a situation with % and fall into the trap with 25% + 5% = 30%. However, as hinted in previous columns, one idea residing at the core of % is the fact that percents are relative to a unit of reference. In this case, the unit of reference is different for the 25% and the 5%. Let's have a closer look.

In a nutshell, there is a difference between applying a 5% discount to a full amount and applying a 5% discount to an amount *already* reduced by 25%. In this second case, the amount on which the 5% is applied is smaller than the initial amount (by 25%), which leads the 5% discount to also be smaller (also by 25%). In the case of the 30% discount, *all* the 30% is applied to the initial amount: as if the 25% *and* the 5% were both applied to that same initial amount, which is not how such discounting is done in practice. Thus, having a 30% discount in "one go" is more advantageous than having a 25% discount followed by a 5% discount, since all of the 30% would be applied to the initial amount. In calculational terms: (0.75) × (0.95) = 0.7125 and not 0.70.

Let's now have a look at another connected situation:

> *The Green Grocery Store gave, by mistake, a 40% discount on their whole-wheat pasta. Looking to rectify the mistake, the following week, the manager raised the price by 40% to even things out. Explaining this to colleagues, they are not convinced that this procedure would do it. Will the increase of 40% cancel out the previous 40% discount?*

Building on the previous explanations, in relation to the % unit of reference, the colleagues' reactions seem the adept ones. The 40% increase in the second week is made on a price… already diminished by 40%, which makes the value of the 40% increase smaller (by 40%) than if it had been applied to the initial amount of the first week. Again, in calculational terms: (0.60) × (1.40) = 0.84 and not 1.00.

Here as well, at the heart of this situation lies the % unit of reference, where a % is constantly to be considered in relation to its unit of reference: 40% in one situation is not necessarily equivalent to 40% in another situation. Thus, both 40% events in this situation do not cancel each other out. In other words, adding A% and then subtracting A% does not amount to having done nothing, or do not sum up to 0%.

Analyzing the Cancelling of Percents

One might wonder what is really happening at the Green Grocery Store. There is a multiplicative issue at play, but how does it work? One way to make sense of it, beyond issues of relativity of the % and different units of reference, is to explore different scenarios in a systematic way, that is, by analyzing how small variations of adding and subtracting can lead to discrepancies. Table 1 illustrates a progression, with an initial amount of *A* of % from 10% to 90%, each one "compensated" by its complementary percentage.

Table 1: *Effects of Subtracting and Then Adding a Percent*

Percent %	Compensated A(1+%)(1-%)	First Difference	Second Difference
10	A × 1.10 × 0.9 = 0.99		
		-0.03	
20	A × 1.20 × 0.8 = 0.96		-0.02
		-0.05	
30	A × 1.30 × 0.7 = 0.91		-0.02
		-0.07	
40	A × 1.40 × 0.6 = 0.84		-0.02
		-0.09	
50	A × 1.50 × 0.5 = 0.75		-0.02
		-0.11	
60	A × 1.60 × 0.4 = 0.64		-0.02
		-0.13	
70	A × 1.70 × 0.3 = 0.51		-0.02
		-0.15	
80	A × 1.80 × 0.2 = 0.36		-0.02
		-0.17	
90	A × 1.90 × 0.1 = 0.19		

When considering the variations observed in Table 1, it is possible to see the impact on the residual value as the percentage goes from 10% to 90%: the residual value is becoming smaller and smaller, from 0.99 of the initial amount *A*, to 0.19 of the value. Moreover, the rate of shrinking increases, as the first difference shows; these residual values diminish from 0.03 ending with a diminution of 0.17. However, here, something mathematically significant appears: the difference between each diminution is constant, as shown by the second differences. Hence, throughout all these variations of % and values, something constant emerges. And these constant second differences are, usually, defining a second-degree relation. Let's consider this option.

At the calculational level, how can these differences and constancies be explained? In the table, additions and subtractions of % can be represented symbolically in the following way, with *P* as the numerical value of the compensated %: $A \times (1 + P) \times (1 - P)$. With some algebraic manipulations, this leads to:

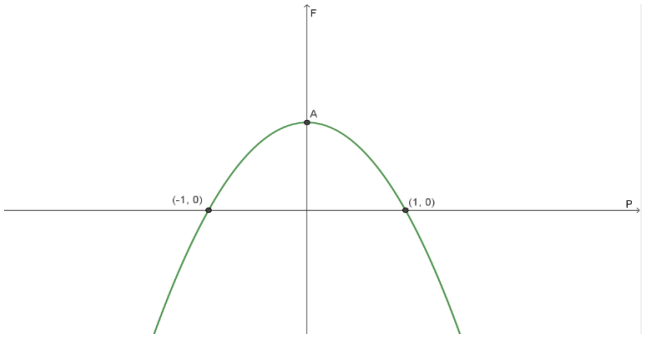
$$A \times (1 + P) \times (1 - P) = A \times (1 - P + P - P^2)$$

$$= A \times (1 - P^2)$$

And thus appears the second-degree relation, with P^2. Does this second-degree relation help us explain the growth observed in the table? It does. Informally, one could say that the larger the value of *P* becomes (that is, the larger the value of the adding/subtracting of the % becomes), the larger the value of P^2 becomes too, because it is squared. In the same way, the more the value of P^2 enlarges, more of what's left, the residual, becomes smaller, since it has been subtracted from 1. If the initial amount is *A* and the final amount is *F*, the equation that represents this second-degree relation is:

$$F = A \times (1 - P^2)$$

Here, again, the larger the value of *P* becomes, the larger the value of P^2 becomes, hence the residual obtained from $(1 - P^2)$ becomes smaller, as does the value of the final amount *F*. All this can also be graphed, with the value of *P* on the *x*-axis, and the final amount *F* on the *y*-axis:



What is quite interesting with this graphical representation is that it makes clear that the above table of additions and subtractions merely covers a portion of possibilities, that is, only the values in the 1st quadrant. What happens in the

other three quadrants? These other three quadrants concern situations in which the values are negative: negative amounts in the 4th quadrant, negative % in the 2nd quadrant, negative % and amounts in the 3rd quadrant. Making sense of these various "negative" situations is quite intriguing and represents a nice challenge, which I leave for the reader to have fun with!

Of particular interest, however, are some points of intersection in the graph, where some information about the situation can be drawn out. One of them, for example, is the intersection of the curve with the *x*-axis at (1, 0). This point expresses the fact that a decrease of 100%, followed by an increase also of 100%, gives 0% as a final amount. This may appear obvious, but is also made crystal clear with the graphical representation of the situation. Another intersection point worth paying attention to is the maximum (0, *A*). This point indicates that, whatever the situation with increases and decreases in %, the final amount *F* will never surpass the initial amount *A*. That is, there is no combination of adding and subtracting of % that can enable one to obtain an amount bigger than the initial value. It is simply impossible to obtain more than 100% of the initial value! This is an issue that the graphical representation underlines.

This sums up some intriguing aspects concerning the issue of adding and canceling %, and the relation that exists between both actions. But this is only the first attempt. Indeed, in the next column, which will be in the Fall, additional explorations of the nature of percent that cancel each other out will be presented, and again through situations, algebraic equations, and graphical representations. Until then, enjoy 100% of your summer, and don't hesitate to contact me if you have any issues with % that you want to share!

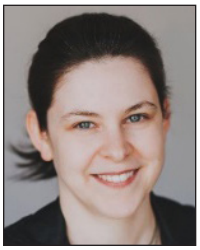
Reference

Proulx, J. (2024). Musing on percentage: Commutative %. *Ontario Mathematics Gazette, 63*(2), 37–38. ▲

"From a study of mathematics the student should learn to appreciate the nature of proof, logical structure, and deductive reasoning, and obtain training in the fundamentals of business transactions" (p. 172, Ontario Royal Commission on Education, 1950—known as the Hope Report).

"The typical secondary school of 1891… Practically all pupils studied… arithmetic; and algebra. …Two-thirds or more studied… geometry…. The percentage of students who took… trigonometry [was]… 4" (p. 18, Ontario Royal Commission on Education, 1950—known as the Hope Report).

▲ Events at the Fields: Research Day, Nurturing Creativity, and Examining Learning Resources



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Research Day at the MathEd Forum

The January MathEd Forum was focused on research and started off with a keynote talk by Jérôme Proulx from the Université du Québec à Montréal (and *Gazette* columnist). Jérôme's talk focused on his research involving mental mathematics and the strategies that children use when they solve mathematical questions. Their research found that students are quite successful at solving different tasks, and collectively use many different strategies to arrive at their answers. Many of these strategies were highly innovative, and even if the students didn't get the correct answer, they still came up with a strategy to attempt to solve the problem.

The second talk addressed research considering the potential for integrating modelling in CÉGEP to develop critical thinking. The talk by Shophika Vaithyanathasarma (Boston College) considered how critical thinking and mathematical modelling are related, and how the connection can address the concerns that people have about information provided to them. The link is not direct within the curriculum leading to the question of who is accountable for developing these skills; is it philosophy? French literature? maybe physics integrated into math courses? She also asked what an educator's relationship with mathematical modelling is, and how it affects their ability to teach it. The presentation explained the conceptual framework that revealed many teachers have misconceptions about what a model is (an application versus a process).

The next presentation, by Andrew Arizaga under the supervision of Francis Duah (both Toronto Metropolitan University (TMU)), was an evaluation of a mathematics

intervention, using propensity score analysis as an alternative to randomized control. Andrew explained that students have difficulty transitioning from high school to university-level math courses, and many universities have transition programs to deal with this. The focus of his research is a transition program called ACES program at TMU. While the program's goals are to improve understanding and grades in mathematics, Andrew's interest is the effectiveness of the course. Currently they use randomized control trials, but the use of propensity scores can potentially address some sampling constraints and issues. Andrew showed the model he used and explained some of the variables in relation to the ACES program. He used "R" to analyze the data and show the results, including how they were input and interpreted. He then spoke about the challenges of conducting a propensity score analysis.

The second keynote of the day was by Molade Osibodu (York University), who has examined coloniality in the Cambridge assessment and curriculum. Molade started by providing insights into her background and schooling in Africa, and her exposure to the Cambridge assessment and curriculum. Her main concern was the lack of African history and cultural details in the materials. Since the Cambridge curriculum is used in over 160 countries, it remains very colonial in the content, which drives the perception that the Cambridge curriculum is the benchmark of knowledge. She defined coloniality, gave a brief history on the Cambridge curriculum, and then considered three sections: power, knowledge, and being. Who has the power? What knowledge is being passed on? And how is education being used? Is it being used to dehumanize and impose inferiority? Is the curriculum perpetuating elitism values? Is local curriculum being used? Molade states that African voices are often not heard, and there is a question of who is worthy of discussion. Testimonials for the Cambridge curriculum come predominantly from global north institutions, and there's a lack of African representation in the curriculum and in the consultations. This begs the question of whose experiences are being seen and who is being consulted. Those creating the curriculum state that the Cambridge curriculum is neutral, and Molade argued that this is not the case—that it doesn't empower the learners, and reinforces the idea that Europe is the centre of power, which marginalizes African ways of knowing. She ended the talk with a vision for a transformed Cambridge curriculum that uses humility to shape perspectives and to acknowledge the inherently political nature of the content.

Another research snapshot was given by Awa Ndiaye (University of Ottawa). Awa's research is focused on ways to improve parent and student engagement, and to enhance teaching practices. She ran a pilot project involving her own teaching and some workshops with parents, where data was collected. From these workshops, it was found that there was a positive impact from parental involvement, and this will form the foundation for Awa's ongoing doctoral research.

The final research snapshot examined Number Talks in elementary classrooms. It was given by Marc Husband, Evan Throop-Robinson, and Allison Tucker (St. Francis Xavier University, Nova Scotia). A Number Talk is a short whole-class discussion focused on mentally solving an arithmetic question. The presenters stated that there is little research explaining how these talks work and the scope of educational benefits they address.

February MathEd Forum

Boris Koichu (Weizmann Institute of Science, Israel) spoke of Hamilton's use of four-dimensional numbers (i.e., quaternions) to do different types of mathematics. He spoke about how creativity isn't necessarily an everyday thing, and how it can come in a flash, one-time manifestation of mathematical creativity. He gave a detailed account of a Grade 10 student in Israel, who is not typically creative mathematically, but who had a moment of creative problem solving in response to an assignment. After the assignment was over, she returned to her usual problem-solving techniques. The talk ended by speaking about how teachers can create environments where students have more choices that might lead to more creative moments.

A panel discussion about creativity in mathematics education followed. The panel included Demetra Peter Pantazi (University of Cyprus), Mark Applebaum (author of 16 books and CEO of the Kangaroo Israel Contest), and Viktor Freeman (University of Moncton, New Brunswick). Each speaker started with a short presentation on mathematical creativity. Demetra spoke of providing students with multiple-outcome tasks and multiple-solution strategy tasks, including open-beginning and ending problems as well as math investigations, and modification of everyday problems by removing information to make them ill defined. Mark addressed preferences for different types of math challenges, and his research findings showing a preference for puzzles that can trigger creativity and encourage experimentation, reflection, and refinement of reasoning. Finally, Viktor presented problems from a past project called Kani, a virtual community of problem solving that operated from 2000 to 2016, and was a collaboration between the University of Moncton and a nearby school district—it has an archive of over 700 problems.

March MathEd Forum

The focus in March was textbooks. Miroslav Lovric (McMaster University, Hamilton) provided a history of university textbooks, and questioned whether some aspects hinder students' ability to develop their own solutions. Karen Lawrence and Kathryn Vrhovnik (Mohawk College, Hamilton) considered resources, and how textbooks fit into the broader ideas of course development. They spoke of alternatives such as digital materials, open educational resources, and other tools. Frosina Stojanovska-Pocuca (Mohawk College) addressed resources from K to 12, college, and university, and focused on the difficulties educators face, choosing from the large volume of resources available to them. Sean Fitzpatrick (University of Lethbridge, Alberta) presented about the "PreTeXt" format and its potential for the creation of online, interactive textbooks, and resources. PreTeXt allows for the creation of open textbooks, using XML format, which allows writers to enter metadata and convert it to other formats including print, HTML, EPUB, and even braille. Stefanie DiSimoni, Hillary Giles, and Richard White (Hamilton-Wentworth Catholic District School Board) spoke about their experiences with textbooks and resources in high school classrooms over the past 30 years. After a brief history, they explained the current use of online learning and learning management systems, which allow teachers to use multimedia such as videos and interactive tools, as well as resources from other teachers and websites, along with interactive worksheets for homework. The final presentation of the day, by Charles Anifowose (Vretta Inc., Toronto) and Mariya Boyko (Red Thread Innovations, Toronto), focused on the rapid changes in education, and how there's often a content lag in static resources like textbooks that is not seen in online resources. They spoke about the importance of using data to improve these systems, and discussed the differences between macro (aggregate) information and (micro) detailed student work, and related these to challenges in mathematics specifically. They noted that doing math in a digital space is not always natural, and it poses challenges for capturing student work and providing feedback. They ended the talk by providing some examples of tools for helping to capture detailed student work, including things like Graspable, Mathagon, and Lean.

The day ended with a panel discussion with representatives from educational publishing companies and an Ed tech product manager—Scott Brayne and Caitlin Smith (Cengage Learning Canada, Toronto), Greg Nosal (McGraw-Hill Canada, Toronto), Amie Plourde and Jeff Gulley (Pearson Canada, Toronto), and Mariya Boyko (Red Thread Innovations). The panelists discussed the role of printed textbooks and their evolution, where the panelists acknowledged the continued need for print, but a definite shift toward digital—how digital resources offered more flexibility, as well as ways Artificial Intelligence (AI) can contribute to content. The panelists then spoke about considering both teachers and students and their different needs when creating resources, and that accessibility is a major focus. The panelists were asked about AI and its role in textbook development, and how to balance high-quality peer-reviewed content with a demand for faster, more customizable digital materials. The panelists explained that AI has been used in educational platforms for some time in relation to adaptive learning, and it's being seen as a tool that can enhance efficiency and personalization, but that there is a responsibility to use AI effectively and ethically within teaching. They suggested that, while AI can be used to create personal learning experiences, it cannot replace the role of subject-matter experts, and that peer-reviewed content is essential.

Overall, the theme of the March MathEd Forum was an agreement that the role and the format of textbooks is rapidly evolving, and that there are now many considerations that educators need to pay attention to when developing course materials and resources. While the role of the textbook remains important, new digital tools, along with open access and free resources, need to be considered just as much.

Coming Up

The Fields Institute continues to run various programs involved in math education and is looking forward to future events, including an exhibit of artwork highlighting dyscalculia (more details to be released this summer), and continued programs for both students and teachers in mathematics. More information on these programs can be found at www.fieldsacademy.ca/eep/ or by emailing pbrittai@fieldsinstitute.ca.

We also invite you to check out our Math and Astronomy resources associated with author Brittany Luby's children's book, *When the Stars Came Home*. These resources can be found at www.fieldsacademy.ca/math-in-books/. ▲

Graduation and Degrees

Graduation is from the Latin *gradus*, meaning a step (and related to *gradual*). It symbolizes a step in the process of becoming educated and led to the usage of *graduate* in the 1400's. Related to this, *degree* comes from *degradus,* which is Latin for a step down. In the 1300's, it became the term for completion of a university step in education (universities having been around since roughly 1100).

Barber, K. (2006). *Six words you never knew had something to do with pigs.* Oxford University Press.

▲ Brock University's Department of Mathematics and Statistics Names Its Learning Centre after Professor Emeritus Eric Muller

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The naming of the Learning Centre is an honour that recognizes all of Eric Muller's outstanding contributions to mathematics teaching and learning at Brock University. This includes the development of the concurrent education programs, as well as his sustained dedication to the broader mathematics community, including OAME/AOEM, where he founded the Golden Section Chapter. We invite you to celebrate the contributions of this trailblazer of mathematics education in Ontario.

Dr. Eric Muller joined Brock University in 1967 after completing his Ph.D. in Theoretical Physics at the University of Sheffield (England). At that time, Brock University was only in its third year of existence, with 11 departments and about 700 students. Over his many decades at Brock, Eric wore a variety of hats. In addition to the significant role he played in the development and teaching of the undergraduate mathematics courses, his other contributions included serving on the University Senate, seven Presidential Committees, and the Board of Governors. Eric will always be respected and celebrated for his dedication to mathematics education.



Eric Muller at the renaming celebration of the Math Learning Centre (October 3, 2024, courtesy of Monique Muller)

In the 1970's, his work led to the integration of the St. Catharines' Teachers College into Brock University. In the early 1990's, he played a major role initiating and developing a B.Sc. concurrent program at the Junior/Intermediate (Grades 4–8) levels, dedicated to improving the teaching of mathematics and science in elementary schools. This success led to developing further concurrent programs at the Senior level (Grades 9–12), and in 2003, Eric, by then cross-appointed with the Department of Pre-Service Education, led the design of a B.Sc. Honours concurrent program, with a major in Mathematics, to strengthen the teaching of mathematics in secondary schools. In 2001, Eric was part of a team that developed the MICA (Mathematics Integrated with Computer Applications) program (see www.ctuniversitymath.ca/mica-then-now/). This program includes cutting-edge mathematics courses combining mathematics and programming to solve sophisticated real-world problems. Recently, a course was added that was specifically designed for future teachers to meet the new Ontario school mathematics curriculum expectations related to coding.

In the early 1970's, there weren't many professional learning opportunities available to mathematics teachers. Recognizing this need, Eric began running an annual get-together for regional secondary school mathematics teachers, featuring presentations by fellow mathematicians and teachers. This group eventually turned into the Golden Section Chapter of OAME/AOEM. Until his retirement in 2004, Eric Muller remained involved in hosting the Golden Section annual meetings at Brock University. Eric also led the organization of the provincial OAME Conference at Brock University in 1976, and was a key supporter when the provincial conference returned there in 2010.

Over his decades of involvement with OAME/AOEM and the Golden Section, Eric was always creating opportunities to make math accessible to all students. He recognized the challenge of educators, including himself, "to help our students learn mathematics and enable them to use it in ways that they can appreciate its value, [and] experience its role in society" (*Ontario Mathematics Gazette* [*Gazette*], 57(3), p. 7). When the National Gallery of Canada's exhibition of works by M.C. Escher came to the Rodman Hall Arts Centre in St. Catharines in 1997, Eric seized this occasion to organize a Celebration of Mathematics and Visual Art. Prior to the opening of the exhibit, Eric pulled together mathematics and visual arts teachers from local school boards to create a collection of mathematics activities based on the exhibit's featured artwork. The activities developed were meant for use with all mathematics and visual arts students, regardless of grade level or their

ultimate destination after high school. This booklet of activities was shared not only with school boards within the Golden Section; Eric and his writing team also travelled to other OAME/AOEM chapters and school boards across the province to share this resource. The booklet is presently being updated and will soon be available online.

Some of the other resources Eric developed, which have been used first by Golden Section teachers, and thereafter by other teachers from across the province include the following:

- *Brock Bugs:* A free two-player board game to discover some early probability concepts, which also turned and expanded into an online free computer game, E-Brock Bugs (www.brocku.ca/mathematics-science/mathematics/brock-bugs-computer-game/). A description can be found in the March 2019 *Gazette* (pp. 38–42).
- *Math Trails:* Trails were developed for Niagara Falls and Welland Canal, which facilitate students exploring mathematics in familiar surroundings through guiding questions. These resources are currently being updated before they become available online.

Eric always embraced technology, and he's seen a lot of it! Whether it was box calculators with manual cranks, electrical calculators, Minitab software, or computer algebra systems such as Maple, he was always looking for ways to improve student learning and understanding of mathematics. Eric's ability to always keep moving forward serves as an inspiration to us all.

OAME/AOEM has had a long-standing relationship with the Fields Institute for Mathematical Sciences, but did you know that it was Eric who established the Fields MathEd Forum (www.fields.utoronto.ca/activities/workshops/mathed-forum)? Because of his connections with local, provincial, national, and international associations, Eric was invited by the Fields Institute to initiate a conversation among mathematics educators at all levels. Consequently, teachers at elementary, middle, and high schools, as well as instructors and faculty at colleges and universities, gathered to focus on mathematics education. So, since 1996, the Fields MathEd Forum has been meeting one Saturday each month during the academic year.

Throughout his career, Eric was the recipient of numerous awards, both for his teaching at Brock University and his work advancing mathematics education. Among these awards are the Lifetime Achievement Award (OAME, 2005), Fellow (Fields Institute for Research In Mathematical Sciences, 2002), the Michael Smith Award for Science Promotion (NSERC, 2002), the Adrien Pouliot Award for Significant and Sustained Contributions to Mathematics Education (Canadian Mathematical Society, 1999), Award for Distinguished University Teaching of Mathematics (The Mathematical Association of America, Seaway Section, 1996), Award for Outstanding Contributions to University Teaching (Ontario Confederation of University Faculty Associations, 1994), and the Brock Alumni Award for Teaching Excellence (1990).

In the early 1970's, Eric ran Saturday morning mathematics workshops for Grades 6–8 students that gave them opportunities to dive into and explore mathematics topics outside of their classroom curriculum. The workshops developed into summer mathematics and science day camps for local students, and eventually into one-week camps, where students would stay overnight on campus. Over the years, Eric wrote many articles for the OAME journal, the *Ontario Mathematics Gazette*. In a recent one, he noted:

> The real challenge for many of our students who do not find it interesting to spend so much time learning the "rigid grammar" of mathematics, but would be more engaged if they could tell, read, and write stories. My view is that we are now in a much better position to balance our mathematics teaching between "grammar" and "stories." (*Gazette*, 57(3), p. 7)

Eric dedicated much of his time to making the math learning experience accessible and engaging for Brock students, future teachers, local youth, and the greater community. His extraordinary legacy will live on with the Eric Muller Math and Stats Learning Centre.

When speaking with teachers about the Celebration of Mathematics and Visual Art, Eric would often end with a thought that beautifully encapsulates our roles as mathematics educators: "Like an artist brings to life dead plants and other materials through composition, a mathematics educator should bring lifeless mathematics symbols and objects to life."

Reference

Muller, E. (2019). Promoting mathematics—Getting involved with OAME. *Ontario Mathematics Gazette, 57*(3), 7–8. www.tinyurl.com/3e757t5w ▲



▲ Meaning Making in the Classroom: Going Deeper When Teaching About Problem Solving



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David is a principal, who taught and had various board roles in Prince Edward Island. David has also instructed university courses regarding curriculum, differentiation, mathematics, and literacy. He facilitates professional learning in mathematics and school development. Dr. David Costello has published four books, with two of his most recent books being: Messing Around with Math: Ready-to-Use Problems That Engage Students in a Better Understanding of Key Math Concepts, *and* Mathematizing Student Thinking: Connecting Problem Solving to Everyday Life *and* Building Capable and Confident Math Learners.

In many conversations with math educators, I hear that the number one goal for students is that they can problem-solve. Educators want students to be able to work through a problem and reach a solution. With this instructional goal, educators dedicate much of their instruction and student learning experiences to problem solving. However, in these same conversations, I hear from educators that students are still struggling with problem solving. All too often, students either give up before reaching a solution, or they rely too heavily on the teacher to tell them hints to move forward in the problem-solving process.

While educators spend countless hours supporting students as problem solvers, I think we need to take a step back. We need to critically reflect on the approaches that we have taken, and how we can strengthen these approaches to better reach our goal of having students as independent mathematicians who can problem-solve.

What I want to emphasize in this column is the importance of problem solving, and the role that teachers play in supporting students as problem solvers. It is not enough that we provide students problems to engage with. Instead, we need to recognize that many students will require educators to teach them all that is inherent in problem solving. We need to provide instruction, whether it is through explicit instruction, teacher modelling, and/or "think-alouds." We need to model the problem-solving process—not just at a surface level, but from a comprehensive perspective. In short, we need to make the implicit of problem solving explicit.

In this column, I will provide a brief overview of problem solving. From there, I will share an important consideration that we must be cognizant of, and that we must include in our teaching repertoire—teaching about problem solving. It is during this process that I want to add a consideration that seems to be missing from much educational policy—metacognition. When we consciously address metacognition during our teaching about problem solving, we will see the result that we are seeking—students who are independent problem solvers.

Problem Solving

Problem solving is a multifaceted approach that can be described as a cognitive process where an immediate plan or solution is not obvious. The student must determine an appropriate plan based on the information provided within the task (Costello, 2022). Within this cognitive process, the student encounters productive struggle as they work throughout the task.

Within the problem-solving process, Pólya (1945) has identified four stages that have come to be common in many classrooms:

Stage	Description
Understanding the Problem	• Understand what the problem is asking. • Identify known information. • Identify unknown information. • Identify the question within the problem.
Crafting a Plan	• Consider possible strategies that can be used to solve the problem. • Identify a plan that best addresses the unknown information.
Carrying Out the Plan	• Implement the selected plan. • Check the work as the plan unfolds.
Reviewing Your Work	• Check to see if the response makes sense. • Check to see if the response answers the question. • Identify how you can use this experience moving forward.

While I have shared an overview of these four stages, I want to emphasize that the stages are not always treated in isolation. Often, students will be working throughout the process and may be working within two or more stages at any given time. Also, this problem-solving process can be quite messy in that the student may be going back and forth between stages before reaching a solution. It is not necessarily a linear, straightforward process.

Knowing how complex the problem-solving process can be, it is not a surprise that the Ontario Ministry of Education (2020) has identified the necessity that educators must

teach students about problem solving. In fact, the Ministry of Education has identified the teaching of problem solving as a high-impact instructional practice in mathematics.

Teaching about Problem Solving

Within the *High-Impact Instructional Practices in Mathematics* (Ontario Ministry of Education, 2020), it is stated that teaching students about problem solving is about making the thinking within problem solving explicit. It is about highlighting, for students, all the various thinking points that exist within problem solving. It goes above and beyond listing the four stages of problem solving. It focuses in on the thinking that the students must engage with from the first stage through to identifying the solution.

Within the teaching about problem solving, the teacher must make time to emphasize productive struggle. Students must understand that such struggle is inherent to problem solving. When the teacher makes time for this acknowledgement, students will not give up when they encounter such struggle. Instead, they will recognize that productive struggle is part of problem solving and can be a valuable part of the learning process.

However, when engaging in teaching about problem solving, educators will still ask why students are not independent when working through a problem. I believe that is because students are not engaged in metacognition when working through the problem. Students are not spending time thinking about their thinking, and questioning if their method is making sense. By making space and time for metacognition, in addition to teaching about problem solving, educators will set students up for success whereby students will not be depending on teachers to tell them if they are correct or not.

Adding Metacognition to the Solver's Toolbox

For far too long, students wait until the end of the problem-solving experience before they question the reasonableness of their response. In fact, in many classrooms, students are encouraged to double-check their work when they believe they have solved the question. This is far too late in the problem-solving process to wait to check on the reasonableness of their response.

Students must judge the reasonableness of their work throughout the entire problem-solving process. They must take time to consider their route to a solution from the instant they engage with the problem. When they are working on understanding the problem, they need to consider multiple ways of viewing the problem. When they select a plan, they need to take time to consider how the plan arises in different representations (hands-on, pictorial, and symbolic). When they carry out the plan, they need to consider corroboration between different representations. And, yes, one more time, they need to think about their thinking when they believe they have solved the problem. It is this thinking about their thinking, metacognition, that will support students as independent problem solvers.

Marzano (1998) referred to metacognition as the engine that drives thinking. In fact, Garofalo and Lester (1985) documented that mathematics instruction without metacognition is inadequate. Students need to be engaged in thinking about their thinking, so that they can self-monitor their problem solving. Think about your class or the classes you support. How many students rely on the teacher to tell them if they are correct or not. We should not be complicit with this. We want students to judge the reasonableness of their own work, throughout the problem-solving process.

When students engage in metacognition throughout the problem-solving process, they will persevere through struggle. They will persevere because their metacognition will assist them in determining when they are making progress toward a solution, or when they must make alternate plans in their approach (and will sometimes offer an alternative approach). One can say that metacognition serves as a compass for students, and that students will rely on this compass to reach a solution.

Putting It Together

We need to teach students about problem solving. However, when we do this while highlighting the role of metacognition, students will grow in their skills toward being independent problem solvers. Educators need to address the importance of metacognition and how this can be infused throughout the entire problem-solving process—from start to finish. It is not only when students engage in reviewing their work as a summary exercise. Instead, when metacognition is highlighted throughout the teaching of problem solving, students will endure through productive struggle and persevere to a meaningful solution.

Final Thoughts

By including metacognition throughout your teaching about problem solving, students will be given the necessary tools to become independent problem solvers. We do not want students to rely on the teacher to tell them if their work is correct or not. We want students to be judging the reasonableness of their work throughout the entire problem-solving process. As stated earlier, metacognition is the compass that will encourage students to persevere through stumbling blocks.

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▲ MB4T (MATHEMATICS BY AND FOR TEACHERS): FRACTION TALKS



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Recently I was attending a professional learning session with a group of secondary mathematics teachers. The facilitator, the school board's Mathematics Learning Partner, began her workshop on algebra tiles with a simple question: "How many of you are using algebra tiles in your Grade 9 classrooms?" Out of a room of nearly 50 educators, only a handful of hands went up. I will admit, I was surprised. Having seen algebra tiles woven into pre-service teacher training over the years, I assumed their use had become more commonplace in actual classrooms.

However, this moment made me realize something important: algebra tiles are still not widely used, and we need to keep talking about them. Despite their proven effectiveness in supporting conceptual understanding, algebra tiles remain underutilized in many secondary mathematics classrooms, but when they *are* used, intentionally and meaningfully, the impact is profound (Nunez-Lopez et al., 2024). This brought me right back to my own mathematics class, where I watched students go from politely nodding to suddenly getting it with algebra tiles. There is nothing quite like seeing this shift happen.

Yet, completing the square is widely recognized as a topic that can feel abstract, and for many students, arbitrary. But when we let them build the square, literally, with algebra tiles, suddenly it all clicks. Algebra tiles can be an incredibly effective way to make this topic more accessible and concrete for students. With a combination of manipulatives, visual reasoning, and symbolic notation, students begin to understand not just *how* the process works, but *why* it's called "completing the square." They move from concrete to representational to abstract algebraic thinking.

This progression reflects the Concrete–Representational–Abstract (CRA) approach, rooted in Bruner's theory of cognitive development: enactive (action based), iconic (image based), and symbolic (language based) (Bruner, 1966). In the CRA sequence, students typically begin by physically interacting with a concept in its most tangible form (concrete); then move to drawing or using digital tools to represent the idea visually (representational); and finally, transition to working with symbolic notation and algebraic generalizations (abstract). When teaching how to complete the square, the CRA model helps demystify the process, and makes the structure of quadratic expressions feel more intuitive and meaningful to students.

To begin our journey into completing the square, it is helpful to activate prior knowledge and foundational understandings. That is why, before diving into quadratics, you can begin with a review of the area model of multiplication. This visual model is familiar to many students from earlier grades, and reviewing it helps lay the conceptual groundwork for the algebraic thinking to come. Holm (2023), in her article *The Wonders of Area Model,* illustrates beautifully how we can use the area model from the very beginning, with single-digit numbers, and carry it all the way through to algebraic expressions. Just as it is illustrated, we might start with a problem such as 13 × 12, and work through it in different ways using decomposition strategies (Holm, 2023). This warm-up, using the area model with base-ten blocks and demonstrating partial products, lays the groundwork for decomposition, a skill that becomes crucial when we begin to complete squares, using visual tools such as algebra tiles.

From there, we are ready to move confidently into the realm of quadratics. Using our understanding of the area model and algebra tiles, we can now have our students expand perfect square binomials, for example, $(x + 2)^2 = x^2 + 4x + 4$ (as seen in Figure 1). Take out your algebra tiles, an area model mat, and practise expanding expressions of the form $(x + m)^2$ or $(x – m)^2$. Here are some you can try: $(x + 5)^2$, $(x – 3)^2$, and $(x + 7)^2$.

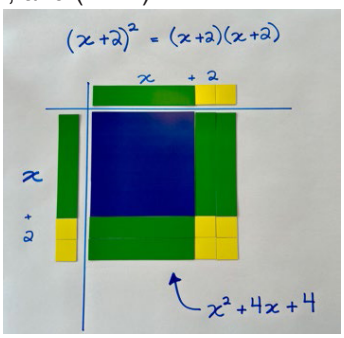


Figure 1: *Solving* (x + 2)² *using the area model*

One of my favourite games to play with students is what I call the *Mystery Square Match-Up* game (as seen in Figure 2). It is a quick and engaging way to get students thinking about the structure of a perfect square trinomial, and how it is obtained by multiplying two identical binomials. For this game, I prepare cards with an expression of the form $(x + m)^2$ or $(x – m)^2$ for each of my students. Using their algebra tiles, each student uses the area model to expand their expression. They then leave the algebra tiles on their desk, and the cards get handed back and redistributed to students. The students get out of their seats and go around the room, looking for their match. As students find their matching algebra tiles, they write down the trinomial expression and are prompted to discuss what clues they used and what patterns they recognize with an elbow partner.

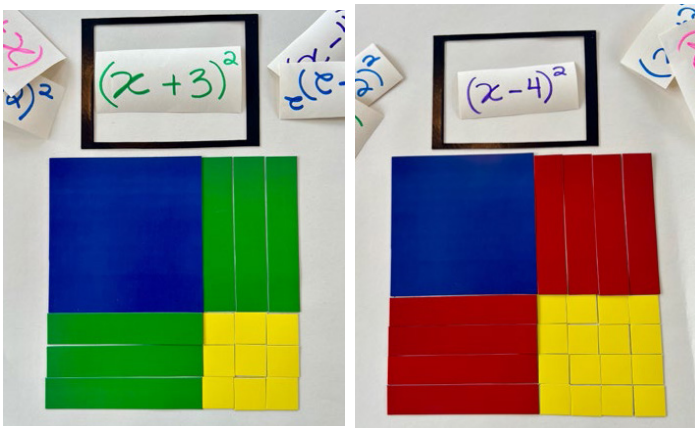


Figure 2: *Mystery Square Match-Up game*

Now to further deepen their understanding: I offer a list of quadratic expressions such as: $x^2 + 14x + 24$, $x^2 – 6x + 9$, $x^2 – 16x + 64$, $x^2 + 11x + 30$, $x^2 + 18x + 81$, and ask: "Which ones can be rewritten as $(x + m)^2$ or $(x – m)^2$?" Students work in groups and use the think-aloud strategy to share their thinking. Walking around the room, you can often hear comments such as: "Is the middle term double something? Is the last term that something squared?" After further discussion, we conclude that a perfect square of a binomial, such as $(a + b)^2$, results in a trinomial, $a^2 + 2ab + b^2$, where the first and last terms are perfect squares, and the middle term is twice the product of the square roots of the first and last terms.

After allowing ample time to work with algebra tiles, students are ready to explore the process of completing the square. Let's look at the expression $x^2 + 8x + 20$, using the algebra tiles provided (see Figure 3). Students typically begin by placing the x^2 tile in the top left corner and arrange the $8x$ tiles around it to begin forming a square. As they attempt to complete the square, they soon discover that, while they can form a full square with the addition of 16 unit tiles, the expression already contains 20 units, leaving 4 extras. This creates a natural and meaningful entry point into the concept—they can now see that $x^2 + 8x + 20$ is not a perfect square, but it comes close. Here students often notice that $x^2 + 8x + 20$ comes close to equalling $(x + 4)^2$, but there were 4 left over from 20, so $x^2 + 8x + 20 = (x + 4)^2 + 4$. The concrete representation, through the use of algebra tiles, reinforces the process of completing the square, and in this example, demonstrates clearly where the additional 4 units come from.

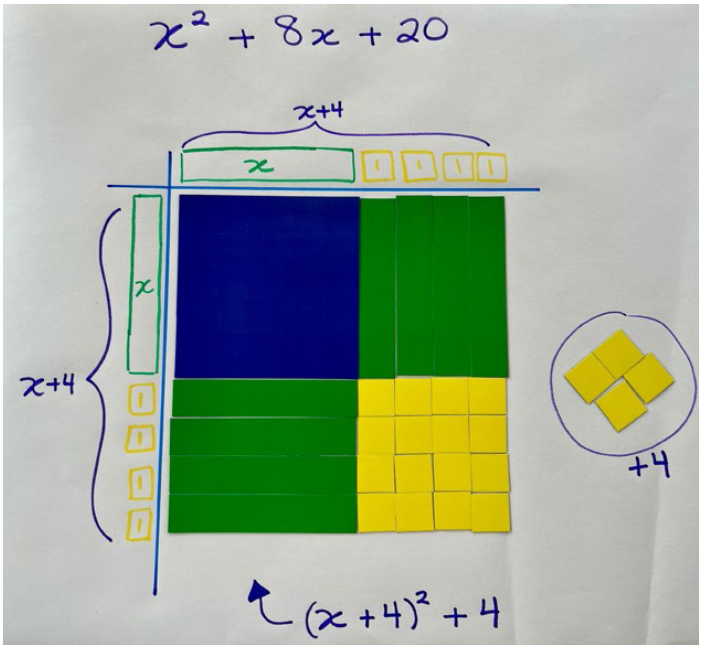


Figure 3: *Completing the square. Example 1:* $x^2 + 8x + 20$

Let's try another example: $x^2 + 6x + 3$. You may have started with placing the x^2 tile, followed by arranging the $6x$ tiles around it, to begin forming a square (as seen in Figure 4). As you are attempting to complete the square with the 3 unit tiles, you may realize that there is a gap—there are not enough unit tiles to form the final corner of the square. By exploring while using the algebra tiles, you may have determined that you will need 9 unit tiles to complete the square, but that the original expression includes only 3. Rather than simply add 6 extra unit tiles, we need to maintain the equality of the expression by adding zero in the form of 6 positive and 6 negative unit tiles. This subtle, but powerful, conceptual move, adding a net value of zero to preserve the expression, allows you to complete the square visually without altering the original value (as seen in Figure 4). You can now conclude that $x^2 + 6x + 3 = (x + 3)^2 – 6$.

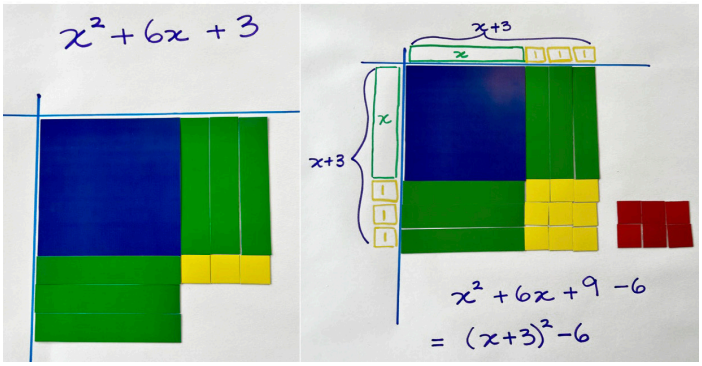


Figure 4: *Completing the square. Example 2:* $x^2 + 6x + 3$

Algebra tiles are a helpful tool to support the conceptual understanding of concepts such as completing the square. However, oftentimes, when we introduce the process of completing the square without the support of algebra tiles or visual models, we inadvertently lose the meaning behind the method. Students are left wondering, "Why is it called completing the square? Where is the square?" or, perhaps more concerning, they don't question it at all and simply proceed with rote memorization of the steps. Too often with expressions like $x^2 + 10x + 15$, it is taught as a series of steps to follow. For example, first you would begin by identifying the coefficient of the linear term, which in this case is 10. Completing the square transforms the quadratic expression into the vertex form $(x + p)^2 + q$ by constructing a perfect square trinomial, so in this example, we take half of the coefficient of x, which is 10/2 = 5, and then square it: $5^2 = 25$. This value, 25, is the number needed to complete the square. We then add and subtract 25 (i.e., a value of zero) within the expression so that we maintain its original value: $x^2 + 10x + 15 = x^2 + 10x + 25 + 15 - 25$. Next, we group the first three terms, which now form a perfect square trinomial, and combine the last two values into a single value. The original expression $x^2 + 10x + 15$ can thus be expressed in completed square form as $(x + 5)^2 - 10$. In many cases, if one only uses the written method, it results in students imitating mathematical procedures, rather than conceptually understanding them.

Rather than jumping straight to abstract thinking, students benefit from having the concrete (algebra tiles), the representational (drawings or using digital tools like Polypad), and the abstract (symbolic notation). Consider the expression $x^2 - 8x + 12$ (as seen in Figure 5). In the concrete stage, students use algebra tiles to model the expression, placing one x^2 tile and 8 negative x tiles evenly. To complete the square, they recognize that they will need 16 unit tiles to form a 4 by 4 square, but they only have 12. This reveals a missing area of 4 units, and by maintaining equality, they arrive at a concrete representation of $(x - 4)^2 - 4$. In the representational stage, students in this example used Polypad to manipulate tiles to visualize both the square and what is missing. They also annotate each step along the way, showing the progression from $x^2 - 8x + 12 = x^2 - 8x + 12 + 4 - 4 = x^2 - 8x + 16 - 4 = (x - 4)^2 - 4$. In the abstract stage, students take half of –8, square it to get 16, and adjust the expression by adding and subtracting 16, rewriting the expression as $(x - 4)^2 - 4$. Seeing all three phases simultaneously, students not only learn how to complete the square, but also understand why the steps work, connecting structure, meaning, and strategy.

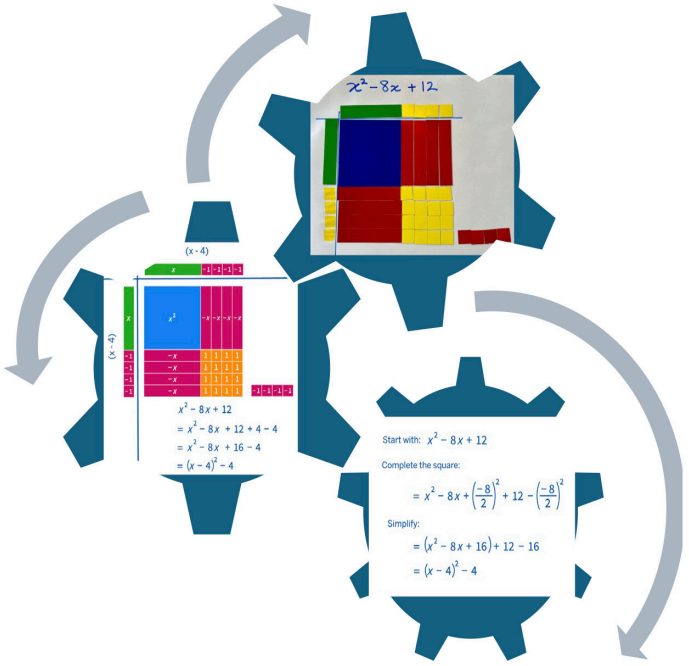


Figure 5: $x^2 - 8x + 12$ *using CRA (Concrete–Representational–Abstract) framework*

When used intentionally within the CRA (Concrete–Representational–Abstract) framework, algebra tiles become a powerful vehicle for bridging conceptual and procedural understanding, and guiding students from tactile exploration to symbolic fluency, in meaningful and lasting ways.

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In the Middle: All About Calendars



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Carly Ziniuk teaches Grade 12 Data Management and Advanced Placement Statistics at the Bishop Strachan School in Toronto. She is active in adopting real-life data to engage her students in solving problems. Among 30 global STEM leaders, Carly was part of the 2024 NASA Space Apps Collective. As a Science Communicator, she is working to connect astronomy and mathematics for learners of all ages across Canada.

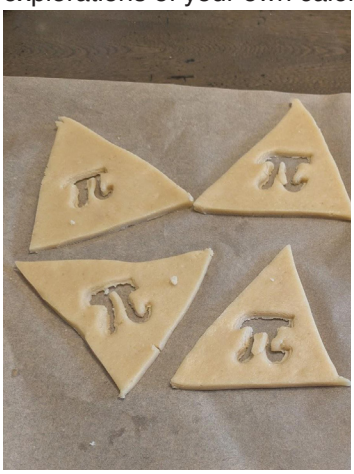
Author's Note: I have used the guidelines as presented by NASA's Style Guide *throughout this piece when referring to the Moon, the Sun, and Earth/earth. The* NASA Style Guide *capitalizes Earth, when it is used as the name of the planet. In this case, it is not preceded by "the," since one would not comparably say "the Mars." If referring to ground or soil, but not the planet, then "earth" is lower case. Capitalization of "Sun" and "Moon" refers specifically to our Sun and our Moon (but lower case, for example, with "Jupiter's moons" or the moon Titan). Use of "the" with Sun and Moon is appropriate for emphasis or clarity. The Canadian Space Agency (CSA) uses similar style conventions.*

Additionally, since this piece considers some historical, religious, and cultural references with respect to the calendar, I have consistently used BCE (Before the Common Era) and CE (Common Era). These abbreviations are secular alternatives to BC (Before Christ) and AD (Anno Domini, "in the year of the Lord").

I love my calendar. I think many teachers do. In many ways, a calendar is the structure by which we interact with each other: booking when we will see each other, agreeing when communal special days happen, and planning when an important event can occur. Calendars note the repetition of time, and the organization of a calendar is built on both modern and traditional cultural connections to observable astronomical phenomenon. Consider, for example, the seasons of the Sun or the cycles of the Moon along with their collective human activities, such as holidays or festivals.

This year, our beloved Pi Day (March 14, 2025) began in the Western Hemisphere with a lunar eclipse, an alignment that happens approximately once per year. The beautiful red colour of a lunar eclipse can be seen from anywhere on the night side of Earth because we are positioned between the Sun and the Moon, casting Earth's shadow on the Moon. I've written here before about the rarer solar eclipses, but it's lovely when the universe is celebrating right along with the Mathies.

This year, however, for the Jewish community around the world, March 14 was a calendrical Hat Trick, since the holiday of Purim began on the very same day. Purim is a holiday of joy, celebrated by giving treats to one another, especially triangular-shaped filled cookies called hamantaschen. This curious synchronicity had me learning a lot about calendars and our own mathematical, astronomical, historical, and cultural fascination with them. It also had me baking some novel hamantaschen (Figures 1 and 2), and musing about how often the calendar was showing off math just like the exposed filling of these cookies. This column can encourage you to do some similar explorations of your own calendrical favourites.





Figures 1 and 2: *Hamantaschen cookies made for Pi Day, 2025*

The Gregorian calendar is the most widely used across the world, for both commerce and science, and has existed since the Renaissance. It is often known as the civil calendar, even in cultures who use other arrangements for their religious observance. The Gregorian calendar is primarily solar, which means it represents the changing position of the Sun and thus the cyclic nature of Earth's seasons. NASA's Astronomical Picture of the Day from last year's Winter Solstice (December 21, 2024), A Year in Sunset (apod.nasa.gov/apod/ap241221.html) is actually a series of images recorded on the 21st of the month from the same location outside of Cairo. This visual representation of our Earth's relationship with the Sun, as represented by calendar date, is shown in Figure 3.



Figure 3: *A Year in Sunset, NASA's Astronomical Picture of the Day, December 21, 2024*

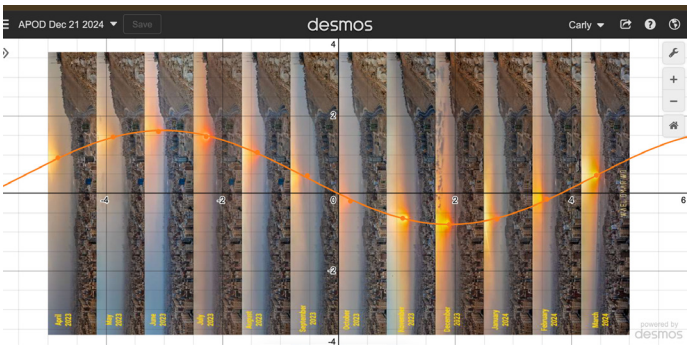


Figure 4: *Desmos sinusoidal regression on the sunset placement for NASA's 12/21/24 APOD*

Figure 4 illustrates what happens if one takes Figure 3, rotates the image, and uses a moveable point and the newest regression feature on Desmos (desmos.com/calculator/hhxki2zr1w). While this is beyond the scope of most middle school learners, it does demonstrate for these students, however, the cyclical and seasonal nature of sunsets, and makes a connection to a graph. I enjoy using images like these to have younger students think about how graphs can represent their own real-life experiences. You may be curious to note that the amplitude connects directly to the tilt of Earth.

The original Julian calendar, originating from the time of Julius Caesar (approximately 50 BCE) had 365 days and added one extra day every four years, with no exceptions. Almost 1500 years later, the Gregorian calendar modified the Julian calendar to space leap years differently, to make the average calendar year 365.2425 days long (Gregorian) instead of 365.25 (Julian). Earth's revolution around the Sun is now understood to take approximately 365.2422 days.

The Astronomical Applications Department of the United States Navy (aa.usno.navy.mil/faq/JD_formula) includes sample pseudocode for the Gregorian shift, summarized here for middle-level learners:

- Leap years are divisible by 4…
- except for when they are exactly divisible by 100…
- unless the year is exactly divisible by 400.

Let me start a list of questions that will continue through this column, and that you might ask about this calendrical shift:

1. What is the difference between these two calendars in seconds per year?
2. What is the percentage error for the Julian calendar compared to the Gregorian calendar?
3. Compare both calendars to the length of the annual revolution.
4. What is the most recent year that was a leap year? What is the next leap year?
5. What is the most recent year that was an exception to the "every four years" leap year rule? What is the next year that will also be an exception?
6. What is the most recent year that had both exceptions? What is the next year that fits with both exceptions?
7. Create a pattern and corresponding graph that compares the number of cumulative leap years to the calendar year.

Since many middle school classrooms prominently display calendars, they can be a ready source of questions. One example is the well-known Square Box Calendar problem that I adapted from University of Cambridge's Nrich version (nrich.maths.org/problems/calendar-capers), Calendar Capers.

Choose any 3x3 square on a calendar page (see Figure 5).

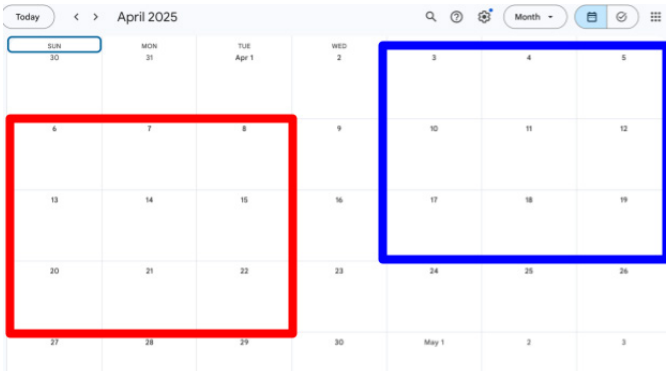


Figure 5: *Square Box visuals on the April 2025 calendar*

8. What do you notice about the sum of the corners of your squares? Is your pattern consistent?
9. For each of the rows, columns, and diagonals that **pass through the centre**, what do you notice about their sums?
10. For each of the rows and columns that **do not pass through the centre**, what do you notice about their sums?

Another Nrich extension is Calendar Sorting (nrich.maths.org/problems/calendar-sorting), which could also be used to support pre-algebraic thinking, and adapted into short coding exercises as well. NCTM's *Teaching Children Mathematics* (October 2002, doi.org/10.5951/TCM.9.2.0096) also has a series of Calendar Math activities, which includes questions that use a calendar and are oriented to probability, patterning, and date formats for learners from K–6. For example, one sample question, "*Today's date*," encourages students to choose the date on the calendar and find as many relationships as possible to express that numerical value, to include the month (i.e., October 24 is 10, 24), and challenges students to determine whether there are more ways to find 1024 or 2410.

11. Or consider including the year and asking, "How many ways can you make 17, using the numbers 1,0,2,4,2,5?"

Perhaps you have wondered why some holidays like Ramadan, Passover, the Lunar New Year, or Easter are not on the same numerical date on the Gregorian calendar every year, like Canada Day is. These seemingly "unpredictable dates" hint that those cultural and religious observances are responding to a different calendar arrangement.

The Hijri Islamic calendar is strictly lunar, which means their holidays occur at different solar seasons. It has 12 months, with each month beginning with the New Moon. Traditionally, the beginning of each month was based on actual observation of the Moon's crescent (known as the *hilal*), which meant that each month in the Hijri calendar depends on how Earth is positioned with respect to the Moon and thus has either 29 or 30 days. Although not all Muslims consider the crescent to be an Islamic symbol, the common use of the Moon symbol harkens back to a time when navigation occurred by night, but also this important component of their calendar. It is also, of course, a symbol of the International Red Cross and Crescent Society, the world's largest humanitarian organization.

12. Mathematically, why is the annual lunar Hijri calendar shorter than the civil (i.e., Gregorian) calendar?

What is curious to note is that astronomically, everyone on Earth sees exactly the same phases of the Moon throughout the lunar cycle. The consistency of the Moon would have been comforting for migratory or diasporic people. Obviously, this is not the same for the Sun! Commonly, in a lunar calendar, the odd-numbered months have 30 days, and the even-numbered months have 29 days. The repetition of the lunar cycle and how it connects to the Gregorian months is something many middle school students might not have noticed. You and your students can download and print a Moon Phases Calendar and Calculator (jpl.nasa.gov/edu/resources/project/make-a-moon-phases-calendar-and-calculator). In early January, NASA provides a new lunar insert for each new Gregorian calendar year.

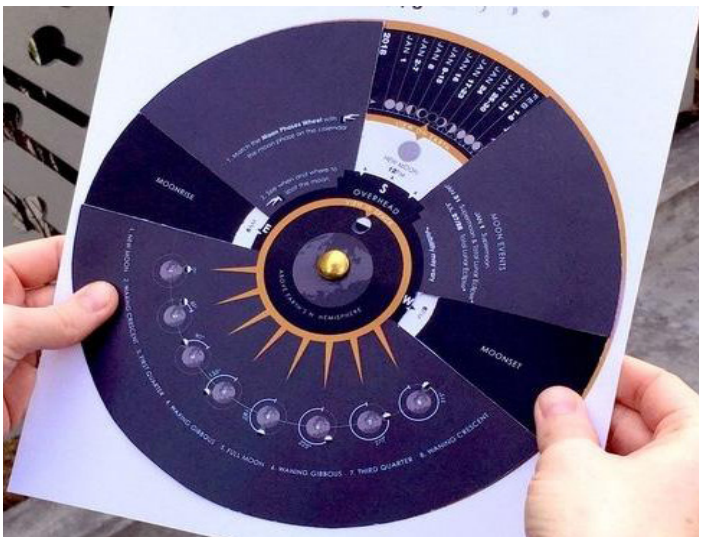


Figure 6: *JPL/NASA's printable Moon Phases Calendar*

However, "leap year" calculations also happen in the Hijri calendar, since it is either 354 or 355 days. The United States Navy (aa.usno.navy.mil/faq/islamic) explains that this calendar occurs in 30-year cycles, with 11 of the 30 years having an extra day added to the last month. This demonstrates a very practical use of some tricky fraction mathematics.

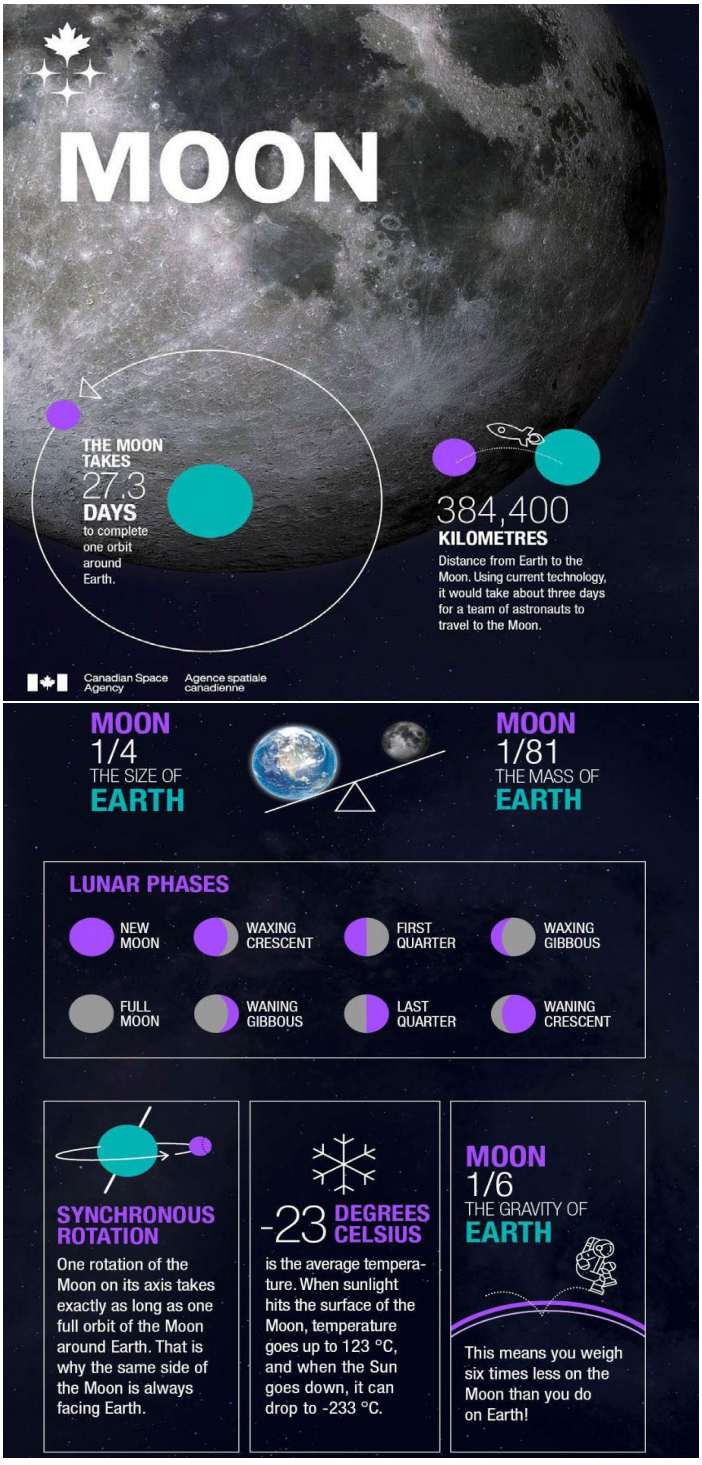


Figure 7: *The Moon in numbers – infographic, 2018 (asc-csa.gc.ca/eng/multimedia/search/image/8382)*

According to the Canadian Space Agency infographic in Figure 7, the Moon takes about 27.3 days to complete its evolution, but 29.5 days to change from New Moon to New Moon. As the Moon completes each 27.3-day orbit around Earth, both Earth and the Moon are moving around the Sun. Because of these changes in positions, sunlight appears to hit the Moon at a slightly different angle on day 27 than it does on day 0—even though the Moon itself has already travelled all the way around Earth. It takes a little more than two additional days for sunlight to hit the Moon in the same way it did on day zero. This is why the lunar cycle months have a length of between 29 and 30 days.



Figure 8: *Illustration of the rotation of Earth and the Moon relative to the Sun (NASA/JPL, 2024) (science.nasa.gov/moon/moon-phases)*

Ramadan is the ninth month of the Islamic calendar and lasts from the sighting of the crescent moon until that recurs again. For the observance of Ramadan, many adult Muslims fast from dawn to sunset, eating by waking before the sun, and staying up late. For example, in 2025, the sunrise–sunset fast for Ramadan occurs during the short winter days of February and March in Canada, yet a decade ago, the fast days here were very long, as the holiday overlapped the Summer Solstice in June and July.

13. Why does the length of the fast during the days of Ramadan for Muslims depend on where in the world they are fasting?
14. Where do you think the longest Ramadan fast day on Earth was for Muslims in 2025?

In considering the answer to the previous question, you may be curious to note that for polar areas with very extensive fasts, some Muslims follow the timing of Mecca, the holiest city in Islam. Similarly, Muslim astronauts schedule their fasting times (and other religious observances) based on the time zone of the last place where they stood on Earth.

From approximately 200 BCE, the traditional Chinese calendar was anchored to the transition of the Moon. This legacy is evident with the designation of the *Lunar* New Year, a holiday recognizing the annual festival celebrated in China, Japan, Korea, Vietnam, and many other Asian countries. For most of these countries, they use the Gregorian calendar for day-to-day affairs, but culturally, the Chinese calendar blends solar, lunar, and other cycles for the timing of traditions, such as transitions between the 12 animals of the zodiac.

The traditional Chinese calendar adjusts the lunar cycle

to maintain the Lunar New Year and mid-Autumn festivals with their solar seasonal anchoring. This calendar uses a 60-year (sexagenary) lunar cycle, resetting the count to align with the solar cycles, after a 60-year period. This cycle is based on 12 zodiac animals combined with 5 heavenly elements. They add an intercalary month, determined originally by the work of Chinese astronomy.

The Jewish calendar is another unique solution to the mathematical problem of the two astronomical phenomena of the Sun and the Moon. Celestially, the Jewish community have a lunisolar year, which means it has lunar months, reflecting its similarity to the Islamic calendar, and yet maintains the seasonal arrangements given to Earth by the Sun.

Just like our joy at finding Pi Day starting with a colourful lunar eclipse aligning with Purim, the Jewish calendar reflects community celebrations, and astronomical experiences. Called intercalation, the average 354-day-long lunar year is adjusted to the 365-day solar year by the periodic introduction of leap months, like the Chinese calendar, in order to assure that the major Jewish festivals fall in the proper season annually. Unlike Ramadan, which can rotate from summer to winter in a decade, Passover, for example, always must occur with the onset of Spring.

Originally, Jews also began each month based on the physical observation of a New Moon, with the decision to add the leap month based on observation of natural agriculture-related events. Historically, Rosh Chodesh, the start of each new month in the Jewish calendar, was determined by two different people declaring they've spotted the first sliver of the Moon. It is interesting to note the communal nature of this feature of the calendar in that it required someone actively looking for the Moon and not doing so alone.

Between the years 70 and 1178 CE, a set of explicit mathematical rules gradually replaced both Moon and Earth observations. For the 700 years following that, mathematicians, both formal and informal, began to create both algorithmic and algebraic models to calculate where the holidays would occur to maintain the lunar and solar connection. In 1800, Gauss created a two-page formula for computing the date of Easter (See *Gazette* 17(3), pp. 45–46), which has a religious connection to the Jewish holiday of Passover (Pesach in Hebrew), and thus is an algorithm for the entire Jewish calendar. It is now regularly referred to as the Gauss's Easter algorithm or Gauss's Pesach formula.

Astronomically, the Metonic cycle is a period of approximately 19 years. This is the time it takes for the lunar phases to repeat at approximately the same time in the solar year, with 235 lunar months.

15. Show that 19 solar years are approximately 235 lunar months.

Month length now follows a fixed schedule, which is adjusted based on the Molad interval, a mathematical approximation of the average time between New Moons. Jewish intercalation adds an additional 30-day "Leap Adar" month in 7 out of every 19 years to align the lunar and solar calendars.

For those 19 different valid patterns that the Hebrew calendar years may take, the year is distinguished uniquely by the length of the year and the day of the week on which Rosh Hashanah (the first day of the Jewish New Year) falls. Rosh Hashanah can only fall on Monday, Tuesday, Thursday, or Saturday, and the calendar adjusts appropriately. Thus, the pattern may repeat itself several times in a few years, and then not recur for a long time. In the discussions I have had with Jewish historians and astronomers, many have proposed that the extensive involvement of Jews in the history of mathematics and space exploration may be connected to the requirement to understand and maintain sophisticated mathematical and astronomical properties. Volvelles are spinning discs used to keep track of astronomical and mathematical information, and this includes documenting the details of the lunisolar Hebrew calendar. Jews created their own calendrical volvelles, documented patterns arising from interactions of prime and composite factors, and created algebraic equations so that the correct placement of the holidays was efficient and could be widespread throughout the diaspora.

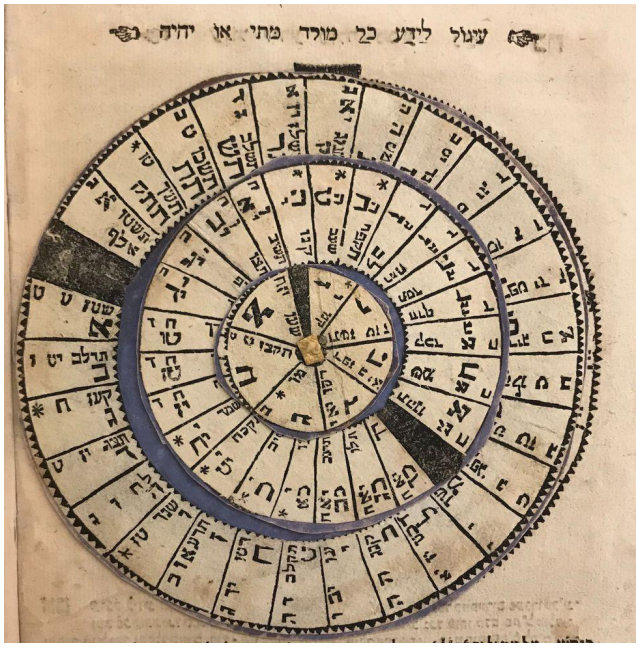


Figure 9: *Hebrew calendar volvelle, Sefer Evronot (Book of Intercalation) (blogs.loc.gov/international-collections/2019/08/stargazing-yiddish-theater-and-hebrew-calendars-come-together-in-the-library-of-congress/volvelle-from-offenbach-1722-talia)*

From as far back as the Middle Ages, Jews created and maintained Sefer Evronot, a genre of printed works showing calculated astronomical data, including predictions for calendars that could predict the Jewish holidays. These Sefer Evronot often included volvelles, like the one in Figure 9. This example from the United States Library of Congress originated in 17th-century Germany. Suffice it to say, there's a lot of math going on here for your students to explore.

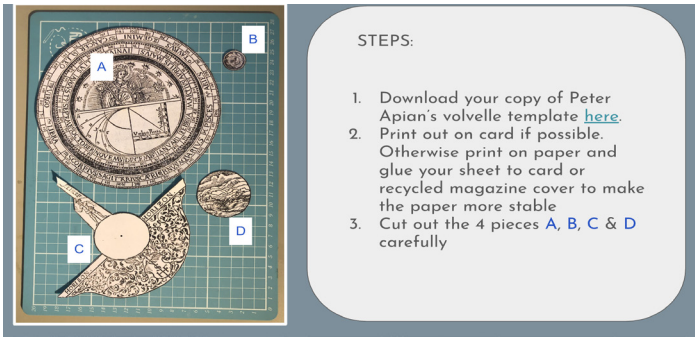


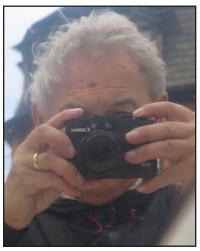
Figure 10: *Create your own volvelle? (Source: https://marshlibrary.ie/make-your-own-volvelle/)*

Volvelles were also used as slide charts to distribute information about the Gregorian calendar's connection to the changing Sun. Dublin's Marsh Library provides information (see Figure 10) to create a volvelle measuring the angle of elevation of the Sun, modelled on a volvelle from the 15th century. Details can be found at www.marshlibrary.ie/make-your-own-volvelle.

Ida Rhodes, a Ukrainian-born Jewish mathematician, designed the United States government's first computer language for Social Security and the Census, along with coining the word "software" in the 1950's (see mathshistory.st-andrews.ac.uk/Biographies/Rhodes). For over ten years, she worked mathematically to adapt a simplified version of Gauss's Easter/Pesach algorithm to calculate Hebrew dates. In 1977, Rhodes created the first Jewish computer calendar that later became Hebcal, originating the free Open Source software movement, a version of which is still the basis of iCal (Apple Calendar) and Google Calendar.

After all these activities and questions, we can see that calendars are a direct communication between our celestial understanding and our daily pursuits. You may want to offer some gratitude to Ida Rhodes, if you use a version of an online calendar regularly to organize your practical life. In this modern culture, at least for me, Rhodes' work continues to demonstrate how we establish routines, how we have agency to make plans, and how we do things together as a community. I hope you have also found it to be a motivating exploration of mathematical communication throughout different cultures and histories. ▲

▲ The Math Teacher's Home Base: Origami in the Classroom



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Ron is an Associate Professor Emeritus at the University of Toronto, where he taught mathematics courses for pre-service teachers for 17 years. Ron taught middle and high school mathematics for 21 years in coed public schools and an all-girls school. Ron's professional activities include conference presentations/consultations in North America, Asia, England, the Middle East, Africa, and India. Ron has authored/edited over 200 articles for NCTM and OAME. Ron received the 2015 Margaret Sinclair Memorial Award Recognizing Innovation and Excellence in Mathematics Education, awarded by the Fields Institute, and was the recipient of the 2022 OAME Lifetime Achievement Award.

Welcome to "The Math Teacher's Home Base," where we will explore the power of creating a resource-rich, inspiring space for all things related to teaching mathematics. This column is dedicated to helping you build a personal teaching environment that not only supports lesson planning, but also nurtures a lifelong passion for math—whether you're brainstorming lessons, experimenting with activities, or simply enjoying math for fun.

I want to focus on two simple, but powerful, origami models—the shirt and the heart—both of which I have used in my teaching for decades. These models are not just exercises in paper folding, but also powerful tools for illustrating mathematical concepts. As you'll see, there's a wealth of mathematics hidden in the creases and folds, and these models can serve as excellent starting points for exploring geometry, symmetry, and proportions.

Before reading further, I suggest you make a shirt and a heart by following the instructions provided at the following websites.

How to Make an Origami Shirt
www.origamiway.com/origami-shirt/

How to Make an Origami Heart
www.origamiway.com/origami-heart.shtml

The Shirt: A Perfect Model for Geometric Transformations and Proportions

As we fold the paper to create the shirt, we can introduce a discussion on geometric transformations—a key area of

geometry that helps us understand how shapes change and move in space. The folding process involves several transformations that can be explored and understood in a hands-on way.

Reflection (Symmetry):

When we fold the paper in half, we're reflecting one-half over the crease, creating symmetry. One side of the shirt mirrors the other, which is a perfect example of reflection—a transformation where each point on one side of the crease is mapped to a corresponding point on the other side.

Translation.

Folding the sleeves or shifting parts of the shirt often involves translation—moving a portion of the paper from one position to another along a straight line. While the piece of paper is translated, it doesn't change direction or orientation.

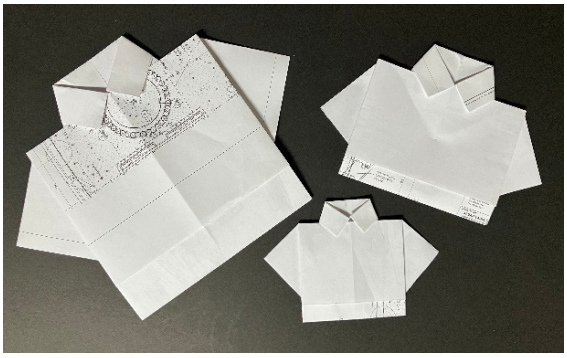
Rotation.

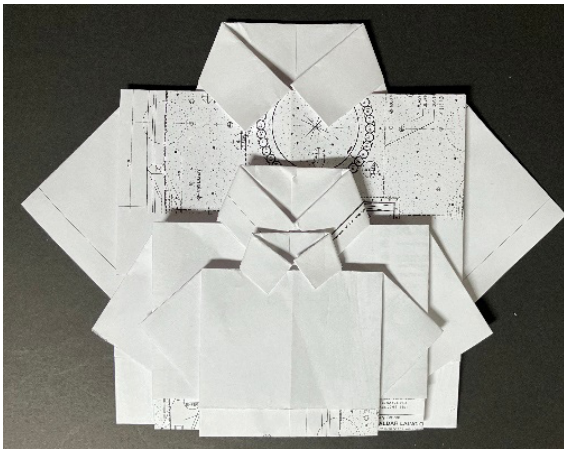
Some folds, such as forming the collar, require rotating a flap of paper by a specific angle. The flap is rotated around a point, often by 90 degrees, to create the desired feature of the shirt.

Dilation (Scaling).

By changing the size of the paper (Photographs 1 and 2), we observe a scaling effect, but not true dilation, since the paper sizes are not proportionate. If the paper sizes were proportionate, we would see dilation—a transformation in which the overall size of the shirt increases or decreases, while maintaining the same proportions for features like the collar or sleeves. However, because the paper sizes are different, the proportions of the features will change as the size of the paper changes. The shirts in these photographs were made from the following paper sizes: Tabloid, which measures 17 by 11 inches, and is also called Ledger when used in landscape orientation; Letter size, the standard paper size in the United States and Canada, which measures 11 by 8.5 inches; and Half Letter, also known as Statement size, which measures 8.5 by 5.5 inches.

To deepen their understanding, students can compare the resulting shirts by measuring and analyzing key features—such as the width of the collar, length of the sleeves, or overall shirt height—and calculating how those dimensions change across the different paper sizes. They might create ratio tables or graphs to visualize how the features scale in a non-uniform way. As an extension, students can be challenged to redesign the folding process to preserve specific proportions, or to experiment with cutting down larger sheets to create their own proportionate rectangles and test for true dilation.





Photograph 1 (Top): *Three shirts*
Photograph 2 (Bottom): *Three shirts*

Each of these transformations—reflection, translation, rotation, and dilation—becomes a tangible part of the folding process. As students engage with the shirt model, they're not just folding paper, they're experiencing geometric concepts in action.

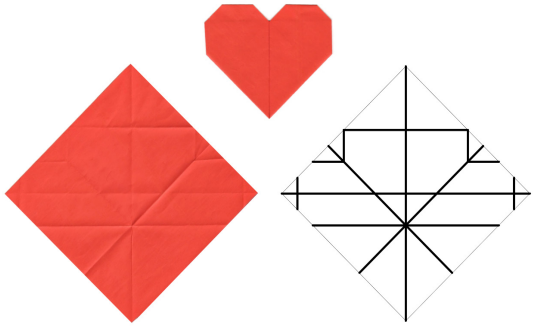
In the classroom, consider pairing students up to make a shirt. Have different pairs of students make shirts from the three sizes of paper. Then do the comparison by amalgamating groups that used different sizes of paper.

The Heart: Proportions, Symmetry, and Crease Analysis

The heart model is well suited to Valentine's Day, but beneficial year round. It offers a beautiful opportunity to explore symmetry and proportions, but one of its most powerful features is what happens when you unfold it (Photograph 3). Once opened, the heart reveals a fascinating network of creases—each line a record of the mathematical steps taken during the folding process. These crease patterns allow students to reverse-engineer the model and investigate the geometric relationships that underlie its construction.

Analyzing the creases gives insight into how folds divide the paper into fractions, how angles are formed, and how symmetry is preserved throughout the model. In upper-

grade classrooms, trigonometry can be introduced to calculate or estimate the angles formed by intersecting folds, creating a natural link between origami and geometry concepts. The crease patterns often reveal a variety of triangles—right, isosceles, scalene, equilateral—and offer opportunities to discuss their properties. In some cases, especially where right triangles appear, students can apply the Pythagorean theorem to explore relationships between the side lengths. Beyond triangles, the unfolding process frequently reveals other familiar shapes such as squares, quadrilaterals, trapezoids, and parallelograms. These geometric figures can lead to valuable conversations about their properties, classifications, and internal angle sums. This kind of analysis is not limited to the heart model—it's a universal feature of origami. Whether simple or complex, unfolding a finished piece invites rich mathematical discussions around line symmetry, congruence, proportional relationships, and geometric reasoning. This reflective practice helps students develop a deeper appreciation for how origami brings geometry to life—literally—in their hands.



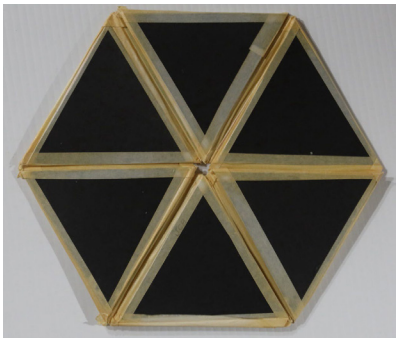
Photograph 3: *The heart*

Benefits of Using Origami to Teach Math

What's exciting about using origami in math instruction is that it makes abstract concepts tangible. Whether you're discussing symmetry, transformations, or proportions, origami helps students visualize and experience these concepts in a concrete, interactive way. The folding process itself requires focus, patience, and spatial reasoning—skills that are essential in mathematics. By creating something with their hands, students gain a deeper understanding of the relationships between shapes and their properties. Origami also promotes critical-thinking and problem-solving skills. As students follow step-by-step instructions, they must understand and anticipate the effects of each fold. Additionally, when mistakes are made, they can often be corrected by re-folding, which teaches resilience and the importance of process over perfection.

A Quick Note on Hexahexaflexagons

Although not strictly an origami model, hexahexaflexagons (Photograph 4) are another fascinating example of paper folding with mathematical significance. I first learned about these intriguing objects through one of Martin Gardner's books. You can find instructions for making them at www.auntannie.com/Geometric/HexaHexaFlexagon/ and www.instructables.com/Trihexaflexagons-and-Hexahexaflexagons. Hexahexaflexagons are made from a strip of 19 equilateral triangles (Photograph 5), and they are a type of flexagon that can be folded in multiple ways to reveal hidden faces. While they may focus more on geometric exploration and puzzle solving than traditional origami shapes, they also provide valuable opportunities for students to engage with concepts of symmetry and transformation. A particularly engaging activity for students is studying the patterns that emerge on the sides when the flexagon is flexed, offering a hands-on exploration of symmetry and cyclic structures. If you have time, I encourage you to explore some of the flexagon activities available through various mathematical resources—there's always something new to discover!



Photograph 4: *Hexahexaflexagon*



Photograph 5: *Net used to make a hexahexaflexagon*

Resources in My Home Office: A Shout-Out to Origami USA

Origami USA is an incredible resource that I've relied on since the late 1980's. The organization runs an annual conference and has a fantastic bookstore, where enthusiasts can purchase books and high-quality paper. However, the biggest benefit for me is their journal, *The Paper*, which is published three times a year. For a membership fee of $40 annually, members gain access to every single issue of *The Paper*, from Number 1 to the current issue (Number 146). *The Paper* contains a gold mine of material for mathematics teachers at all levels, with a particular focus on the

mathematical applications of origami. In addition to *The Paper*, Origami USA publishes another resource called *The Fold*, which features articles, diagrams, and much more.

Recent Articles from *The Paper*:

- *How Origami Boosts Spatial Skills and Math Confidence in Young Learners* (Winter 2004)
- *Six Pointed Torus* (Autumn 2024)
- *Origami as a General Education Math Course: Before, During, and After the Pandemic* (Spring 2024)
- *Getting Started with 3-D Origami* (Summer 2023)
- *How I Learned to Teach Origami to Blind Students* (Spring 2023)
- *How Much Folding Went into the James Webb Space Telescope* (Autumn 2022)

For more information, visit https://origamiusa.org/.

Recommended Books from My Home Library (Photographs 6 and 7):

- *A Plethora of Polyhedra in Origami* by John Montroll
- *Easy Money Origami* by Marc Kirschenbaum
- *Explore Math with Origami* by Marc Kirschenbaum and Daniel Scher
- *Flexagons Inside Out* by Les Pook
- *Marvelous Modular Origami* by Meenakshi Mukerji
- *Origami with Dollar Bills* by Duy Nguyen
- *Origami Omnibus* by Kunihiko Kasahara
- *Paper Capers* by Jack Botermans
- *Magic Books and Paper Toys* by Esther K. Smith
- *Perfectly Mindful Origami* by Mark Bolitho
- *Structural Packaging* by Paul Jackson
- *The Art and Science of Geometric Origami* by Jun Maekawa
- *The Complete Book of Origami Polyhedra* by Tomoko Fuse
- *The Guide to Hawaiian Style Money Folds* by Jodi Fukumoto
- *Tipping with Origami Money Folds* by Arnold Tubis
- *Toys and Tales with Everyday Materials* by Sudarshan Khanna and Gita Wolf
- *Unfolding Mathematics with Unit Origami* by Betsy Franco



Photograph 6: *What's on my shelf?*



Photograph 7: *Bills can be replaced by sheets of paper*

▲

▲ Elementary Math Matters: Quilts as the Holders of Stories and Geometry



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Lynda Colgan is Professor Emerita, Faculty of Education, Queen's University, and Executive Director of Science Rendezvous. Lynda collaborates with the Assessment and Instruction for Mathematics (AIM) Collective, which engages researchers and educators across Canada to support the development of tools for assessment and instruction in mathematics learning from Kindergarten through Grade 3. She continues to work on a number of funded research, resource-creation, and teacher-education projects. Lynda remains deeply committed to STEM outreach and can be found regularly at events that engage students, parents, and educators in science, technology, engineering, and mathematics at home and in the classroom.

Over the course of my career, I had many wonderful opportunities to learn with and from educators, parents/guardians, and students in villages, towns, and cities across Ontario. There are too many to list, but I have vivid memories of *Logo*[1] workshops in the town of Spanish, *Family Math* activity nights in Peterborough, an *Early Numeracy* conference in Windsor, *Math and Art* creativity in North Bay, and an evening of *Mathemagic Number Tricks* performed in London. However, one *Math and Quilting* workshop in Owen Sound was of particular significance to the community and me.

Why? Owen Sound, Ontario, has a long and well-documented Black History heritage. The city is known to have been the northern-most terminus of the Underground Railroad, the secret network of citizens who helped escaped slaves reach freedom in Canada. To commemorate the significant date when slavery was abolished throughout the British Commonwealth—August 1, 1834—Owen Sound founded an Emancipation Picnic in 1862. Today, Owen Sound has the distinction of holding the longest running Emancipation Picnic in North America; an amazing 163 years.

In 2004, a cairn (or stone memorial) commemorating the escaped slaves and their contributions to society was unveiled in Owen Sound's Harrison Park. The cairn is a living, reverential monument to the courage and hard work of those abducted from their native Africa and forced into slavery in the West Indies and the United States before escaping into Canada.

The cairn embodies cultural, historic, and artistic symbols. The windows are based on those in the first Black church in Owen Sound, *The Little Zion.* The colourful quilt patterns laid on the grass represent the "freedom" quilts that were purported to be embedded with a kind of pictorial code to be read by an enslaved person on the run, with warnings about the area's immediate dangers, or directions about next steps. The stones in the wall originate from locations in Canada, the United States, and Africa, each with a direct connection to slavery or the abolition movement. New stones having the appropriate provenance continue to be added.



Figure 1: *Photo of Owen Sound's cairn from the Grey County Black Heritage Society Facebook page. It was in 2005 that I was asked to do a Spatial Sense and Geometry workshop in Owen Sound by the district-wide organizers for the Bluewater School District. How better, I thought, to teach, reinforce, and extend curriculum expectations about 2D and transformational geometry than to focus on the quilts that were at the heart of the local cairn?*

To prepare, I turned to the book *Hidden in Plain View* (Tobin & Dobard, 1999), which had caused a stir among scholars and collectors when it related a story about a complicated secret code of quilt patterns, now commonly called the Charleston Code, being used as a means of communicating signals to American slaves. For example, the *Bow Tie* quilt (Figure 2 left) told those travelling the Underground Railroad to *dress in disguise to appear of a higher status,* while the *Bear Paw* design (Figure 2 middle) urged travellers to *follow an animal trail through the mountains to find water and food.* The *Log Cabin* pattern (Figure 2 right) provided relief: *Seek shelter now, the people here are safe to speak with.*

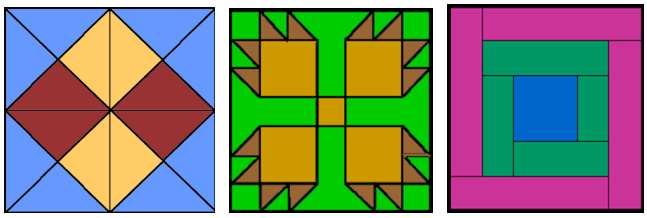


Figure 2: *From left to right, the* Bow Tie*,* Bear Paw*, and the* Log Cabin *quilt square*

Controversy still surrounds the quilt codes, largely because their original "story" was oral and shared by just one elderly African-American woman, Ozella Williams. Since oral histories leave no written record, there is no written proof that the codes in the quilt patterns actually existed. As a result, quilt and slave historians, as well other academics, challenge the idea of the codes, while others continue to celebrate them, including the artists, quiltmakers, and museum curators who in July 2024 hosted the Royal Ontario Museum At-Home exhibit, *The Secret Codes: African Nova Scotian Quilts.*

In spite of the debates about the veracity of the secret messages contained in the squares, quilting remains an excellent tool to teach a wide range of geometry-related vocabulary and concepts, and so the workshop in Owen Sound proceeded. The culminating task was for each table grouping to create a collaborative quilt by assembling ten square patterns: *North Star*, *Flying Geese*, *Shoo Fly*, *Bow Tie*, *Crossroads*, *Wagon Wheel*, *Monkey Wrench*, *Bear's Paw*, *Log Cabin*, and *Drunkard's Path*[2]. In a classroom, the culminating task would be to create a classroom quilt, with each student contributing a quilt square that could be printed onto a large white sheet using iron-on transfers and on a T-shirt to take home.

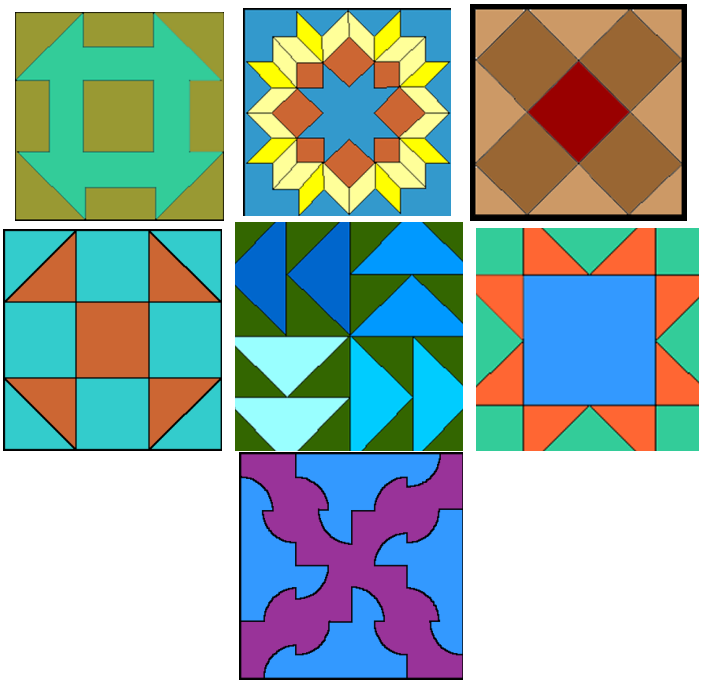


Figure 3: *Quilt squares. Top row from left to right:* Monkey Wrench*,* Wagon Wheel*,* Crossroads*. Middle row:* Shoofly*,* Flying Geese*,* North Star*. Bottom row:* Drunkard's Path.

The warm-up exercises about quilt squares were taken from books by the late Canadian curriculum developer, Jill Britton—*Teaching Tessellating Art* (Britton & Britton, 1991)

and *Symmetry and Tessellations* (Britton, 2000)—as well as *The Secrets of Interlocking Patterns* (1999) by mathematician and quilter, Jinny Beyer.

While it may not be immediately obvious, each of the quilt squares is what is known as a "four-patch" pattern, meaning that each completed square has four distinct quadrants, each of which in turn is a square. This is evident in the *Flying Geese* (Figure 4) square, where we observe the top left quadrant is rotated 90° right to create the pattern in the top right quadrant. There are many ways to describe the transformations that can create the bottom right and bottom left quadrants; for example, the top left quadrant is rotated 90° left to create the bottom left quadrant, while a glide reflection of the top left quadrant can generate the bottom right quadrant. What other transformations are possible?

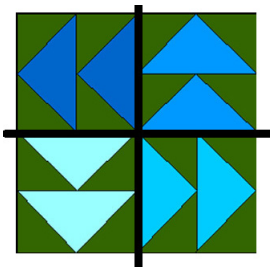


Figure 4: *The* Flying Geese *quilt square with horizontal and vertical lines of symmetry to emphasize the four patches (quadrants) that comprise the rows and columns in the completed square*

The workshop participants were asked to use visual analysis and recognition of repeating patterns to identify the shapes used within the quilt squares; describe their positions and orientations; and suggest transformations (translations, rotations, reflections) that could generate the final four-patch square. There were animated discussions and much cutting of brightly coloured construction paper into various squares and triangles; assembled and reassembled on grid paper as the participants explored geometry, using colours, patterns, and shapes.

I had learned through my own trial-and-error efforts to create the quilt blocks that the most challenging quilt square to deconstruct visually was the *Drunkard's Path* square. To avoid any frustration in recreating this geometrically intricate design, I led participants through the steps that would create this unique four-patch. The step-by step instructions are in Table 1.

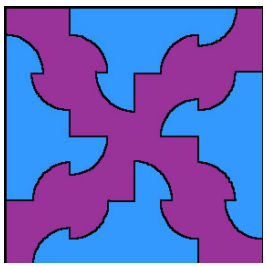
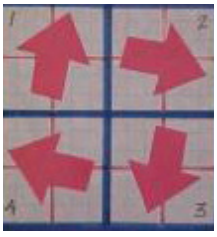


Figure 5: *The* Drunkard's Path *square. According to Ozella Williams, slave catchers from the south hunted for runaways with tracking dogs. Since drunkards walk in a crooked line, back and forth, never in a straight line, this pattern told the runaway slaves to zigzag on their journey to confuse the dogs.*

Table 1: The six-step process to create the Drunkard's Path *quilt block*

Step Number	Image of Step	Text Instructions
1		You will need two colours of construction or scrapbook paper. Cut 2 squares and 2 circles out of each colour. The diameter of the circles should be 2/3 the length of the sides of the squares.
2		Fold the squares and circles into fourths. Mark the fold lines lightly with a pencil.
3		Glue each circle onto the centre of a square of the opposite colour. Use the lines to position the circles.
4		Cut each square into fourths along the marked fold lines. You will now have 16 small squares.
5		Each large *Drunkard's Path* square block is made up of 4 smaller identical blocks. Assemble the first small block as shown.
6		Assemble 3 more identical small blocks. To form the *Drunkard's Path* block, the small blocks are rotated 90° right, starting in the upper left quadrant.

Step Number	Image of Step	Text Instructions
7		Your finished *Drunkard's Path* quilt block should look like this.

The participants were proud of their final quilt blocks—a seemingly complex design made possible by the repeated use of deceptively simple shapes that interlock perfectly. I'd like to think that after the experience, they would agree with Dr. Gerda de Vries, a University of Alberta mathematics professor and self-taught quilter, who believes that every quilt is a mathematical object, in addition to being beautiful and comforting. My hope would be that they would concur, based on the intricacy of their own *Drunkard's Square* that, as Professor de Vries says, "The patterns in quilts might appear to be random, but they're actually quite calculated—even though they don't need an actual calculator to make. By finding the patterns in these quilts, then, we are doing geometry..." (Black, 2016).

Notes

[1] Logo is a simple programming language devised by Seymour Papert, Cynthia Soloman, and Wally Feurzeig in 1967 for educational purposes. The name "Logo" is derived from the Greek logos, meaning "word" or "thought."

[2] For the purposes of the workshop, I took the names of the squares from *Hidden in Plain View*; however, I also explained that each pattern had multiple names, depending on geography and cultural origins. For example, the *Bear's Paw* is also known as *Duck's Feet* in the *Mud and Hand of Friendship*, while the *Flying Geese* pattern in also called the *Sawtooth Star*.

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Morse Code: – . – – / – – – / . . – // – – . / – – –/– // . . /–

▲ A Close Look at the Glossary of the New Ontario Mathematics Curriculum



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Vera Sarina has been a teacher for more than two decades. Her undergraduate degree was in mathematical linguistics, but after a few years working on research, she found her calling in teaching. She taught coding to young children; mathematics and other subjects in Primary, Intermediate, and Senior grades; wrote mathematical contests for city and provincial math Olympiads; and taught math Additional Qualification courses for teachers. Currently she teaches math courses to pre-service teachers at Western University, London, Ontario. Her love for history has led her to find many solutions to modern problems of teaching and learning mathematics in the global history of mathematics.

Like most teachers, I appreciate the new innovative ideas in the new curricula: the elementary Ontario mathematics curriculum (Ontario Ministry of Education [OME], 2020) and the destreamed Grade 9 mathematics curriculum (OME, 2021). The Grades 1–9 curriculum focuses on fundamental mathematics concepts and skills, rather than fragmented and very specific mathematical concepts and rules. The curriculum endeavours to compel students to value deep thinking and making connections, and to become capable and confident math learners. However, for the community of math educators, there is one area that is worth looking into more closely and, perhaps, more critically, that is, the definitions embedded in the curricula. This article is written as a suggestion for a wider math educational community, including the OME, to discuss the accuracy and relevancy of some of the definitions embedded in the Grades 1–9 curriculum. The suggested revised definitions are just examples of possible improvements to those in the curriculum. They must be viewed only as possible entry points to the discussion of the work that needs to be done with some of the existing curriculum definitions.

Until recently, I had never opened and read the pop-up definitions of the mathematical terms embedded in the most recent curricula's learning expectations. I did not need to read definitions of the concepts I taught, since I knew them. I assumed that the definitions in the curricula had been written by mathematicians and, without a doubt, were

correct. After all, if the revisions process for curricula is detailed and extensive, I relied on that without checking the definitions of concepts. The literature dedicated to the role of definitions in teaching and learning mathematics states unequivocally that mathematical definitions are essential in developing mathematical knowledge and hence central to the learning of mathematics (Gilboa et al., 2018). Definitions in mathematics "help [form] the concept image, but they very often have a crucial role in cognitive tasks. They have the potential of saving you from many traps which are set by the concept image" (Vinner, 1991, p. 69).

It is a known fact that definitions in any specialized area, such as law, medicine, physics, and philosophy, play a crucially important role. For instance, we can read on the Department of Justice Canada website (Government of Canada, 2024) that "definitions are powerful provisions because they control the meaning of terms used throughout a legislative text." Change the word "legislative" to "mathematical," and the statement remains true: "Mathematical definitions control the meaning of terms used throughout a mathematical text." Mathematicians and learners of mathematics also appreciate the notion that mathematical definitions are prescriptive. The definition must prescribe the exact and correct mathematical meaning of a word.

From this stance, can we accept the following Grade 7 definition of a cylinder (OME, 2020):

> A three-dimensional object with two congruent, parallel bases, joined by parallel lines called elements. Cylinders can have polygons, curved figures, or a combination of the two as bases. Prisms are a special kind of cylinder that have only polygons as faces?

Let's examine this definition.

First, a cylinder does not have any obvious lines on it. This inconsistency is quickly "corrected" by the definition given on the same page in the Teacher Supports/Key Concepts*: "Cylinders have two congruent and parallel bases*; straight lines of equal length **can be drawn** that join one base to the other." Second, the sketch in Figure 1 represents a three-dimensional object that clearly matches the definition, but the object is not normally called a cylinder.

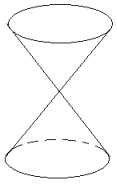


Figure 1: *Two cones connected at the vertices*

Note that if we look up the definition in the Glossary of the previous Ontario mathematics curriculum (OME 2005, p. 122), the ambiguity evaporates by one crucial addition: "Cylinder. A three-dimensional figure with two congruent, parallel, circular faces and **one curved surface**."

The third observation lies in the field of classifications of mathematical concepts. What do you define as a basic concept or term in some specific category? Let's return to the OME (2020) definition of a cylinder. "Cylinders can have polygons, curved figures, or a combination of the two as bases. Prisms are a special kind of cylinder that have only polygons as faces." According to this definition, a cylinder is a basic solid. Prisms are defined as being a special kind of cylinder. However, if we scroll down the page of the Grade 7 Spatial Sense Expectations, the formula for the volume of a prism is introduced through a rectangular prism, not a cylinder as shown in Figure 2.

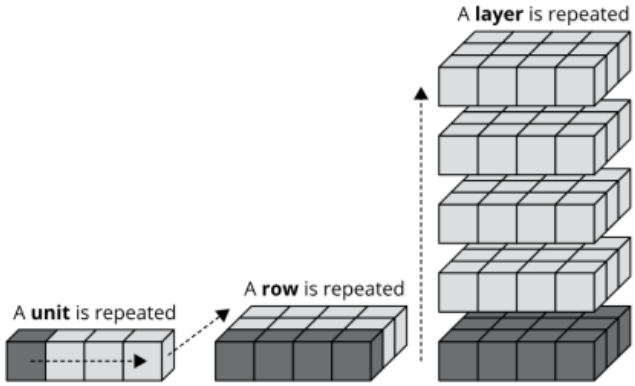


Figure 2: *Grade 7, Spatial Sense, Teacher Supports—Key Concepts*

Furthermore, below the diagram shown in Figure 2 is the following statement: "Therefore, the formula for finding the volume of a rectangular prism is area of the base × height. The same is true for any prism or cylinder." Therefore, the formula for the volume of a cylinder is a derivative of the formula for a rectangular prism, the same way as the formula for the area of a circle (*Area = pi x radius x radius*) is a derivative from the formula for the area of a rectangle. It stands to reason that "a prism" not "a cylinder" is a basic term through which other types of prisms are classified and defined.

Last, but not least, let's look at the definition of a cylinder again: "…Cylinders can have polygons, curved figures, or a combination of the two as bases. Prisms are a special kind of cylinder that have only polygons as faces." A cylinder is defined by two terms: bases and parallel lines. However, prisms, in the third sentence of the definition, are defined as having "polygons as faces." Faces? Since prisms are classified as cylinders, cylinders must have faces, which is

absent from their definition.

The above analysis shows that the suggested definition of a cylinder neither builds a useful, practical, and correct concept image of the concept known as "a cylinder," nor clarifies the broader concept of "a prism." However, this mishap can be easily mended. For instance, in the OME 2021 Glossary, the definition of "prism" is "a three-dimensional object with two parallel and congruent faces. A prism is named by the shape of its bases." Thus, we have a rectangle-based prism, a triangle-based prism, or a cylinder, which is a prism with a circular base. The definition does its job perfectly. Ontario Grade 9 students have an opportunity to fix and fine-tune their conceptual understanding of the geometrical solids named "prisms" and "cylinders."

We are obliged to ask ourselves a general question: What makes a definition a good definition? One of the most tried and trusted ways to build definitions that explain the meaning of a term is to follow the formal structure. The formal structure of a definition has two main parts, besides the term itself: the general group or category in which the term belongs, and the difference between the term and all other items in the same group or category. Assigning terms to the proper categories is the first indication of a good definition. However, careful reading of the definitions included in OME 2020 and 2021 curricula reveals that some definitions are placing terms in the wrong categories. Examples of such definitions are the definitions of "limit," "number system," and "geometric property."

Limit (OME 2021, Number, B1.3) is defined as "the long-term behaviour of a pattern or function, or the result as the number of terms increases." No, "limit" is not a behaviour. A limit is a value or number. "A limit is the value that a function (or sequence) approaches as the argument (or index) approaches some value" (Wikipedia.org). Yes, limits describe how a function (or sequence) behaves near a point, instead of at that point, but the category to which the term "limit" belongs is "value," not "procedure."

Number system (OME 2021, Number, B1.2) is defined as "a way in which number relationships are defined. For example, the base ten number system includes the digits 0 to 9, and the relationship of one place value to the next is a multiple of 10." This definition has many flaws and inconsistencies. However, we do not have to dig deep. The definition is not correct from the very beginning. The general category or group in which the term "number system" or "numeral system" belongs is ''any of various sets of symbols and the rules for using them to represent numbers," as *Encyclopedia Britannica* states, not "a way in which number relationships are defined." If we accept the OME (2020) definition, then the description of, for instance, the Greek number system found in *The Concise Oxford Dictionary of Mathematics* is incorrect: "The Greek number system used letters to stand for numbers. For example, α, β, γ, and δ represented 1, 2, 3, and 4, and ι, κ, λ, and μ represented 10, 20, 30, and 40." The description clearly describes the Greek number system, but it does not refer to or clarify the way in which number relationships are defined in the Greek number system. Also, note that the OME (2021) definition says "number relationships*"* in the first sentence, but describes "the relationship of one place value to the next" in the second sentence, which are different relationships.

Geometric property (OME 2020, Spatial Sense, Grades 1–8) is defined as "an attribute that remains the same for a class of objects or shapes. For example, an attribute for any parallelogram is that its opposite sides are of equal length." Red parallelograms have an attribute that remains the same for all of them: their colour. Is it a geometric property? Hardly so. Geometric property is not just any attribute, but a **geometric attribute**, such as shape, size, number of sides, or angles, that remains the same for a class of objects or shapes.

The second part of a formal definition of a term is "the difference between it and all other items in the same group." Some definitions from OME 2020 and OME 2021 do not do a proper job in defining the differences that distinguish a term from other terms in the same group or category. For example, the definition of a scatter plot in both OME 2020 and OME 2021 is:

A graph designed to show a relationship between corresponding numbers from two sets of data measurements associated with a single object or event: for example, a graph of data about marks and the corresponding amount of study time. Drawing a scatter plot involves plotting ordered pairs on a coordinate grid.

Let's compare two graphs shown in Figure 3: one is a line graph, and another is a scatter plot.

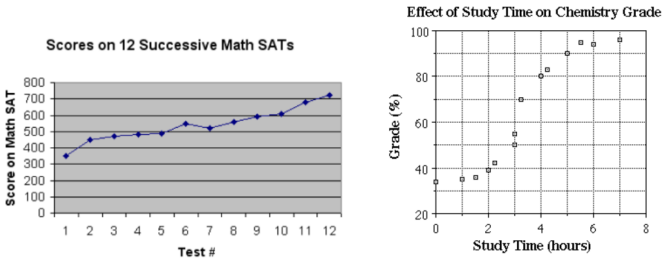


Figure 3: *Line graph versus scatter plot*

Both graphs fall under the curriculum definition of a scatter plot. The line graph on the left in Figure 3 (Note: SAT, or "Scholastic Aptitude Test," is a standardized test used for deciding admissions to many United States

colleges.) has every characteristic included in the OME definition of a scatter plot. It means that the OME definition does not describe the difference between a scatter plot and all other items in the same group called "graphs."

A reader of the article might have some doubts about the author's objectivity. Is she cherry-picking the examples of the curricular definitions? Perhaps only an insignificant portion of the definitions is not constructed methodically and with mathematical correctness, or reflective of the context of the grades for which they are written. As I was planning to write this article, I was asking myself the very same questions. I decided to check my objectivity by giving the pre-service mathematics teachers I teach at Western University the following assignment: "Find if you can identify at least seven incorrect, misleading, or contradictory definitions in the Grades 1–9 Ontario Mathematics curriculum. Explain why you think that each chosen definition is incorrect, misleading, or contradictory. Formulate a correct definition for the chosen concepts." Note that the students are aspiring Intermediate/Senior mathematics teachers. They are mathematically savvy, and at the same time, focused on teaching mathematics to school children. Below are some examples of their submissions. The curriculum definitions are bold-faced, students' reasons to rewrite the definition are italicized, and suggested revised definitions are in plain font.

Base (Shapes) (OME 2020): In three-dimensional objects, the face that is usually seen as the bottom.

This definition is wrong and misleading. If we take a triangular prism and place it with one of the rectangular faces on the bottom, by this definition, that rectangular face is the base. This is wrong.

New definition: The face of a polyhedron that is perpendicular to the direction that the height is measured. *~ Jacob*

Equivalent Fractions (OME 2020): Different fractional notation of the same part of a whole or group.

The definition is not wrong, but quite confusing. What does it mean for the same part? How are these notations different?

New definition: Equivalent fractions are the fractions that have different numerators and denominators, but are equal to the same value. *~ Vivian*

Percent (OME 2021): A ratio with a second term of 100. A percent is expressed using the % symbol.

The first sentence of this definition is very confusing. What does this mean? Let's say we are comparing apples to oranges and there are 52 apples and 100 oranges. Then the ratio of apples to oranges is 52:100. The second term is 100, but this is not a percent.

New definition: A number expressed as a fraction of 100, for example, 17/100 = 17%. *~ Jacob*

Symmetry (OME 2021): The geometric property of being balanced about a point, a line, or a plane.

What does "balanced" mean? Why not use the reflection term that is already defined in the glossary (and redefined above)?

New definition: The geometric property where a reflection of a shape through a point, line, or plane produces a mirror image of the shape. *~ Gerry*

Subset (OME 2021): A smaller set within a set.

This definition implies that the subset must be smaller than the original set, which is not necessarily true. A subset may contain all the elements of the original set, making it equal in size to the original set.

New definition: A subset is a set whose elements all belong to the original set. Any set is considered a subset of itself, and the subset may contain none (empty), some, or all elements of the original set. *~ Samantha*

Vertex (OME 2020): The common end point of the two-line segments or rays of an angle.

In three-dimensional objects, there can be up to three rays line segments.

New definition: The common end point of two or more line segments or rays. *~ Rolan*

***Y*-intercept (OME 2021): The value of *y* for a point (*x*, *y*) on the *y*-axis when *x* is zero.**

This contains no context for what a y*-intercept is and seems to imply that a* y*-intercept is simply the* y*-value of any point that lies on the* y*-axis.*

New definition: The value of *y* at the point (*x*, *y*), where a line intersects the *y*-axis, for example, the *y*-intercept of $y = 3x + 2$ is 2. *~ Andrew*

To be fair, there is a strange, but reassuring trend that one can observe in OME 2020 and OME 2021. In some cases, pop-up definitions and definitions from the Glossary differ. One example is the definition of "prism" that we have discussed earlier. The definition of "congruent" is yet another example of the trend. The pop-up definition is: "Two geometric figures are congruent if they are equal in all respects." However, in the Glossary, the definition of congruent is "*Having the same size and shape*," which is by any means a correct definition.

The latest Ontario mathematical curricula are not some

outliers with regards to choices of math definitions. The definitions of math terms in some recent North American curricula suffer from similar inattention. For instance, in the Glossary that is Chapter 14 of the *California Mathematics Framework* (California Department of Education, 2023), we can encounter the following definitions:

- "Quadratic expression. An expression involving a squared term, e.g., $x^2 + 1$, or a product term, e.g., $3xy - 2x + 1$." Thus, the cubic $3x^3 + x^2 + 1$ is a quadratic expression, since it involves a squared term.
- "Polynomial. An expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable(s)." Apparently, monomials and binomials are not considered polynomials in California, since they are expressions with one and two terms, respectively.
- "Number line. A linear representation of a set of numbers."—What does this mean? Do tally marks fit the definition?

The Ontario curriculum is prescriptive. The Ontario curriculum document states that "the curriculum for each subject or discipline is made up of a set of learning expectations that outline the knowledge and skills students are expected to learn and apply by the end of a grade or course." The OME (2020 & 2021) definitions of mathematical terms are a part of the knowledge that our students are expected to learn. However, as we have shown, a closer look at the definitions embedded in the most recent Ontario mathematics curricula reveals that a significant number of the definitions of crucial concepts are either misleading or contradictory, and in some cases, mathematically incorrect. As teachers of mathematics, we have a responsibility to teach correct and meaningful mathematics. What can be done to resolve the quandary with some of the definitions embedded in the relatively new Ontario Mathematics curricula? Perhaps with the curriculum posted online, a revision could be made by a working group specifically tasked to consider the Glossary. Teachers would need to be notified of such an important revision, but without a doubt, it would provide a resolution to the issue.

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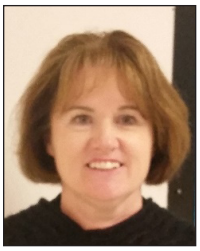
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Jacqueline Hill is a recently retired Grades 9–12 teacher of mathematics. She is a Past President of OAME and OMCA, as well as the recipient of the Award for Exceptional Teaching in Secondary Mathematics. She also wrote the "Director's Dialogue" for the Gazette *for a number of years.*

As I sit down to write this final edition of the NCTM math column, I am filled with a mixture of emotions—gratitude, nostalgia, and a deep sense of fulfillment. Over the past eight years, I've had the privilege of exploring the beauty and intricacies of mathematics from the NCTM lens with you, the readers, and it has been an incredibly rewarding journey.

Math has always been a subject of wonder, offering both the simplicity of numbers and the complexity of the universe. Through this column, I've sought to share that wonder with you—not just the dry equations or technical concepts, but the way mathematics shapes the world we live in, and the way mathematics shapes the children we teach. Whether we've tackled teaching pedagogy, explored NCTM history, or discussed real-world applications, I hope you've found moments of inspiration and insight.

There's something uniquely powerful about the language of mathematics. Math is the universal language. It connects us across time, space, and culture, and it helps us make sense of the world around us. From the smallest patterns in nature to the vastness of space, mathematics is a thread that ties everything together. In my columns, I've endeavoured to show that math isn't just for experts; it's for everyone. No matter your background, anyone can appreciate the beauty and logic that math provides.

While this may be my last column, my love for math and the curiosity that drives it will never fade. I encourage you to keep questioning, keep exploring, and keep finding joy in the patterns that surround you. Thank you for being a part of this journey. It's been an honour to share this space with you, and I hope the world of math continues to inspire you.

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▲ "Find *This* in *That*": How to Understand Algorithms



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Dr. Marc Husband is an assistant professor at St. Francis Xavier University in Antigonish, Nova Scotia. His research focuses on mental math and how making learners' processes visible and visual fosters understanding and confidence. Collaborating across educational contexts, his work explores using mathematical ideas as the foundation for meaningful teaching and learning.



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Dr. Tina Rapke is an associate professor at York University. Tina enjoys listening to students' mathematical ideas in classrooms and other community venues (such as public libraries). She is always looking for new ways to collaborate so that she can support all children in being confident mathematics learners.



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Cristina De Simone is a middle school mathematics teacher and is a Ph.D. candidate in Mathematics Education at York University. Cristina's current research explores parental experiences of engaging in mathematics with their children, endeavouring to bridge the gap between home and school learning.

The revised Ontario Mathematics curriculum emphasizes the importance of mental-math strategies to enhance student understanding and number sense. Despite this shift, traditional algorithms for basic arithmetic operations continue to find their way into classrooms, often introduced by well-intentioned parents or tutors. Previous researchers (e.g., Kamii & Dominick, 1998) have been concerned about the harmful impacts of introducing the algorithm prematurely prior to having a fuller understanding of why it works. This presents a unique challenge for teachers who want to support a deep understanding while being respectful of the person who introduced the algorithm (typically from well-meaning parents). The dilemma intensifies as teachers struggle with how to effectively use these algorithms without compromising student understanding and the curriculum goals. This article explores the prompt "find ***this*** in ***that***" for teachers to use, to help students make connections between traditional algorithms and their own strategies for adding, subtracting, multiplying, and dividing (Strand A, connection process, Ontario Ministry of Education [OME], 2020). To use it, you ask students to find things in the non-algorithm within the algorithm or vice versa. By employing the concrete prompt of having students "find ***this*** in ***that***," teachers support students to make connections between algorithms and mental-math strategies. This approach affords teachers a script they can use to turn the challenge of algorithms into valuable teaching opportunities, supporting the curriculum's objectives, while enhancing students' mathematical understanding.

Connection Making

Teachers are frequently provided with broad or superficial recommendations about connection making, such as encouraging their students to explore similarities and differences among "concepts, procedures, and representations" (NCTM, 2000). While looking at similarities and differences will offer students a way to examine two-solution strategies, this prompt may not make specific connections that will be useful in "understanding how mathematical ideas interconnect and build on one another to produce a coherent whole" (NCTM, 2000). In other words, asking students to find similarities and differences between different solution strategies may elicit surface-level connections from students like "this solution uses numbers" and "this one uses pictures."

In this article, we share what happened in a study with first-year teachers (who completed their B.Ed. program, but had yet to begin their teaching career) that focused on connection making because algorithms and understanding them took centre stage. Specifically, to foster connection making (and ultimately understanding of algorithms), teachers began to ask each other to "find ***this*** in ***that***" when examining two or more solution strategies. When considering "finding ***this*** in ***that***," it required the participants to look for a specific feature or mathematical property that appeared in both solution strategies. "Find ***this*** in ***that***" first appeared when the teachers generated several solution strategies for the subtraction question: In how many ways can you calculate 91–79? To support other teachers in envisioning how they themselves can use the strategy "find ***this*** in ***that***," we also share an experience we had in school when

we asked students to think about 738 ÷ 6.

"Finding *this* in *that*" with subtraction 91–79

When we asked first-year teachers to calculate 91–79, they were able to come up with several solution strategies. The variety of solution strategies (including the standard algorithm) become the landscape for asking, "Can you find ***this*** in ***that***?"

Solution Strategy 1: Use of Base-10 Blocks

Many teachers use base-10 blocks for subtraction because it allows students to visualize the concept of regrouping, making the abstract process concrete. By physically manipulating the blocks, students can see how 91 can be broken down into 80 (eight ten-blocks) plus 11 (eleven units). When subtracting 79 by removing seven tens and nine units, the students must exchange (aka borrow) a ten-block for 10 units, giving them 11 units. This method helps students grasp the idea of borrowing or regrouping more intuitively.

Solution Strategy 2: Traditional Algorithm

The traditional algorithm involves writing the numbers vertically, aligning the digits by place value, and systematically subtracting each column from right to left, "borrowing" from the next column on the left when necessary. While the traditional algorithm is efficient, it requires an understanding of place value and procedural fluency. Many students and adults alike struggle with understanding why the algorithm works.

Find **this** *in* **that** *for 91–79:* In the standard algorithm, when one is stumped with taking 9 (in 79) away from 1, one can "borrow" 10 from the 90 (in 91) and take it over to the 1 to make 11. To understand this process conceptually, one can "find ***this*** (the 11 from the traditional algorithm) in ***that*** (the base-10 blocks)." As one can imagine, the first-year teachers said, "Find ***this***," while pointing at the 11 in the traditional algorithm and said, "in ***that***," while pointing at the base-10 block representation.

The strategy of "find ***this*** in ***that***" came from first-year teachers thinking about supporting growth of understanding of the algorithm by connecting the solution derived using base-10 blocks to the traditional subtraction algorithm. The goal was to provide opportunities for elementary students to see the interplay between concrete manipulatives and abstract numbers. By physically grouping and regrouping the blocks, students can identify where the numbers 11 and 8 (which is actually an 80) come from, corresponding to the "borrowing" step in the traditional method. This visual and hands-on approach helps students understand that the "1" they borrow from the tens place in the algorithm is the same as the extra ten units added to the ones place when they use blocks. By finding these numbers in both the blocks and the algorithm, students can bridge the gap between concrete and abstract representations, reinforcing their understanding of the subtraction process and the importance of place value.

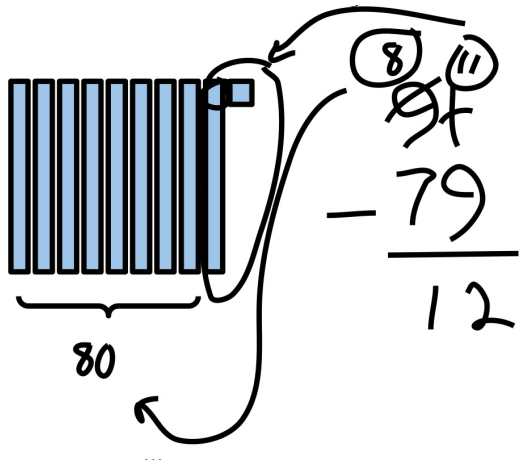


Figure 1: *Looking for the stacked algorithm in a model with base-10 blocks. In this model, the last ten-block, with one drawn in, shows the 9 from 79.*

"Finding *this* in *that*" with division 738 ÷ 6

Solution Strategy 1: Standard Algorithm

When we asked students in a Grades 5 and 6 classroom to share how they would solve 738 ÷ 6, students began by writing 738 (the dividend) under the division symbol and placing 6 (the divisor) outside (see Figure 2). First, they looked at the 7 (in the hundreds place) and asked, "How many times does 6 go into 7?" Since 6 goes into 7 once, they wrote 1 above the 7. They then multiplied 1 by 6 and wrote this below the 7, subtracted 6 from 7, and recorded the remainder (1) below the 6. Next, they brought down the 3, making the new number 13, and asked, "How many times does 6 go into 13?" The answer is 2, so they wrote 2 above the 3. They multiplied 2 by 6, subtracted 12 from 13, leaving 1, and brought down the 8 to make 18. Finally, they asked, "How many times does 6 go into 18?" The answer is 3, so they wrote 3 above the 8 and noted that the remainder is 0.

Solution Strategy 2: Multiplication and Estimation

We have found that some students prefer to do mental math and think of 738 ÷ 6 as a multiplication problem. They approach it similar to this:

First, I know 6 times 100 equals 600, so I started there. Then, I needed to get closer to 738, so I thought about what's left: 738 minus 600 is 138. I know that 6 times 20 equals 120, so I added that to get 720. Now I just need 18 more to reach 738, and I know that 6 times 3 is 18. So, I

added everything together: 100 plus 20 plus 3 equals 123. The answer is 123.

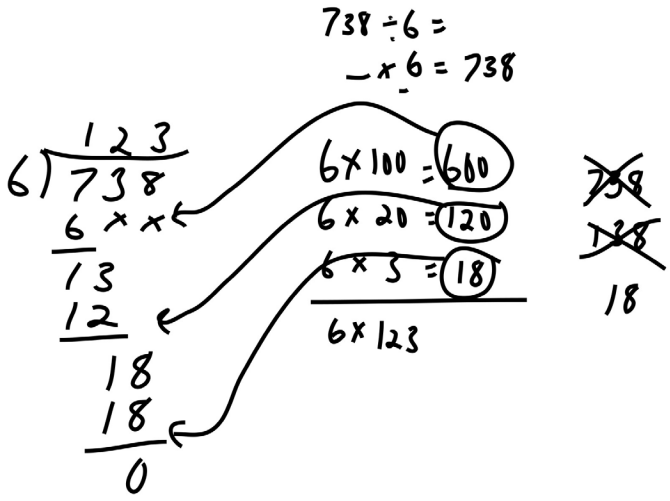


Figure 2: *Looking for multiplication in long division*

Finding *This* in *That*

In the traditional algorithm, 6 goes into 7 once, producing 1 in the hundreds place. In the multiplication/estimation strategy, this corresponds to the student using estimation in the hundreds place 100 to say that multiplying 6 × 100 = 600, and so there are 138 left unaccounted for.

After subtracting 600 from 738, the remainder in both strategies is the same: 138. In the traditional algorithm at this step, students may be focused on only the 13, and 6 goes into 13 (the next two digits) twice, producing 2 in the tens place. Similarly, in the multiplication strategy, the students do some estimation in the tens place by multiplying 6 × 20 = 120, subtracting 120 from 138.

This leaves 18 in both approaches. In the traditional algorithm, 6 goes into 18 three times, producing 3 in the ones place. The multiplication strategy mirrors this, with the student multiplying 6 × 3 = 18, subtracting 18 to leave no remainder.

In the traditional algorithm—6, 12, and 18 are the numbers of focus. In the estimation/multiplication strategy—600, 120, and 18 are the numbers of focus. Finding ***this*** (the 6 in the standard algorithm) in ***that*** (the estimation and multiplication strategy) can help students make connections and enunciate that the standard algorithm has place value stripped from it. In the event that there was a remainder, it would have provided an opportunity for discussion of how the remainder was dealt with in the standard algorithm versus the estimation/multiplication strategy. For example, was it left noted as a remainder, was it described as a decimal of a whole, and how are those two ideas connected?

Conclusion

We see great potential in teachers supporting growth of understanding of arithmetic algorithms by asking, "Find ***this*** in ***that***." This prompt can be used as a script by teachers with any algorithms they are relying on and the mathematical concepts involved. This approach prompts students to locate specific concepts, strategies, or relationships within broader procedures or representations. The Ontario Mathematics curriculum highlights that the connection-making process involves students "seeing the relationships among procedures and concepts" (OME, 2020). By prompting students to "find ***this*** in ***that***," connections that might only be loosely considered are intentionally identified and examined. The teacher's role is to clearly highlight particular connections between "concepts, procedures, and representations" (NCTM, 2000). Specifically, these connections may arise through students' ways of thinking about arithmetic or by linking a student's idea to a particular property or strategy the teacher tries to elicit. Connection making can play an important role in honouring what students bring into the classroom, particularly with algorithms.

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▲ Coding in the Classroom: Debugging—A Productive Struggle



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"In math education, the productive struggle is my favourite. If it's too easy, you're not learning." *~ Diane Tepylo*

Have you ever noticed how your students light up when they finally crack a tough math problem? That moment of triumph—that "aha!" feeling—is also present in coding when students tackle debugging. Debugging isn't just about fixing errors; it's about problem solving, perseverance, and deep learning. Instead of seeing errors as setbacks, we can help students view them as stepping stones to understanding. The best part? Most students enjoy debugging when they have the right mindset and tools!

Debugging: A Natural Fit for Learning

Think about how we approach problem solving in math. For example, we encourage students to break down problems, analyze patterns, and find solutions through trial and error. Debugging follows the same logic. It helps students think systematically, test hypotheses, and develop problem-solving skills that translate directly into mathematics. Coding affordances like abstraction, agency, and computational thinking all come into play, making debugging an essential learning experience, rather than just a technical fix.

A Simple Debugging Moment

Let us say a student's program is supposed to draw a square, but it only completes three sides. Instead of immediately pointing out the missing line of code, we can guide them through debugging:

- Ask them to explain their code step by step.
- Have them predict what the computer is doing versus what they expected.
- Suggest removing/adding one line at a time to see the effect.
- Encourage them to celebrate small successes along the way!

This process mirrors how we teach problem solving in math. When students work through their mistakes, they gain a deeper understanding and build confidence in their abilities. Debugging also fits right into the Ontario Mathematics curriculum's coding expectations. In Grades 1 through 8, students are encouraged to read and modify existing code and think about how their changes affect the outcome. When they debug, they are naturally doing both—analyzing what is happening, making adjustments, and observing the results. It is all about experimenting, learning from mistakes, and building stronger problem-solving skills that connect coding with mathematical reasoning.

Exploring Through Tinkering, Debugging, and Creating

Debugging is not an isolated task—it is deeply connected to how students explore and develop their ideas. When students tinker, they experiment freely, test boundaries, and play with possibilities. This unstructured exploration fosters creativity and curiosity, giving them the confidence to try new approaches. Creating builds on tinkering, as students begin to structure their ideas into concrete projects, designing, and building something meaningful. Debugging is what ties it all together—it is the process of refining, improving, and making their ideas work as intended.

Rather than seeing debugging as a frustrating roadblock, students can view it as a natural step in their creative journey. By integrating debugging into our math and coding lessons, we foster resilience and curiosity. So, the next time a student encounters an error, remind them that mistakes aren't the end of the road—they are just part of the journey toward discovery!

Debugging for Your Classroom

If you are looking for practical examples to bring debugging into your classroom, check out the *Remix My Code* resource (in the members-only resource files). It features engaging, real-world Scratch projects, where

students can practise troubleshooting and refining code, while reinforcing key math concepts. These projects encourage creative problem solving and align with the Ontario curriculum expectations. Creating your own authentic buggy code (i.e., code with errors or bugs in it) for students to fix, or debug (i.e., remove the bugs), can be difficult. *Remix My Code* was created by three teachers struggling with their own code. They had the idea to let their classroom students debug it for them and had great success. Their students felt less pressure when they were trying to debug someone else's code, and came up with a variety of creative solutions to fix their teachers' buggy code. Whenever you are personally having trouble with a coding project, save a copy to give to your students at some point, before trying to debug it yourself.

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Building OMCA's Network to Francophone School Boards

This year, OMCA has made a concerted effort to reach out to Francophone school boards and welcome their mathematics coaches, consultants, and coordinators to the OMCA family. Through our open house meeting in December, and additional connections to Francophone mathematics leaders, OMCA is working to ensure that our organization is supportive of all publicly funded institutions in Ontario.

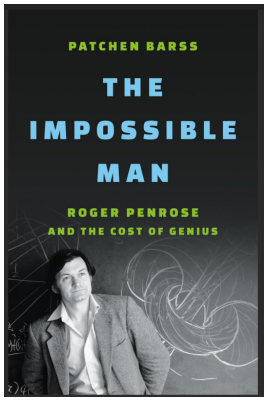
OMCA Retreat at Blue Mountain

Every February, OMCA organizes a retreat primarily for members to get together, network, share experiences, and learn together. This year, our keynote speaker was **Libo Valencia**, a very engaging mathematics teacher based in Chappaqua, New York, and author of *Math Play*. Libo shared his experiences centring joy and play in his mathematics teaching and in his personal life with his daughters, building their mathematics understanding through clever uses of toys and games. Libo also shared his love of, and expertise with, GeoGebra as a tool to make mathematics playful and fun.

OMCA is always open to new members who work in a dedicated role supporting educators with mathematics teaching and learning. Send us an email at OMCAmath@gmail.com to learn more about who we are and any upcoming events. ▲

▲ Book Review: The Cost of Genius, or the Price One Chooses to Pay?

Barss, P. (2024). *The impossible man: Roger Penrose and the cost of genius*. Basic Books.



REVIEWED BY TIM SIBBALD

When I came across this book, I was first impressed to find yet another Toronto writer, Patchen Barss, writing about people in the fields of mathematics and science. More surprising was to find that Ontario, Bayfield in particular, arises in the biography.

The book is well written, but in places, a difficult read. It is difficult because the "cost of genius" is steep, and the details are both eye opening and somewhat disturbing. I would argue, however, that the strength of the biography is that the reader is positioned to consider whether it was genuinely a cost or a choice to pay that particular price.

The book focuses on the way Penrose has lived his life. It is thin on mathematical and physics details, though there are some keywords, if one wishes to find more details. The account of his developing the kite and dart aperiodic tiling—a pair of tiles that tile the plane, but cannot make a repeating pattern—is woven from childhood through adulthood. The connection to Escher was a surprise, and that is very much a strength of the book.

Through the narrative, we meet various famous people, such as Stephen Hawking and John Conway. But the focus is balanced with details of family and relationships. It is a full-bodied account of the person who happens to also have become famous. Without saying too much, it is also interesting to see who is missing from the story.

While my initial reaction to the book was shock, I have subsequently begun to consider what it implies about giftedness, the challenge being that it is one experience, but the book very much reveals strategies that Penrose used to push himself ahead. That "cost" and some of its related strategies have possible implications for the way we work with gifted students. In that respect, there is a thought provocation beyond the biographical aspect. Overall, it is a thought-provoking read that *Gazette* readers are likely to find interesting. ▲

What Makes an Effective Teacher?



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Jeff Irvine, Ph.D., has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

In my last year as a vice-principal, we needed to hire a math teacher. There were 825 applicants for that one job. I know because I had to read 825 resumés. We interviewed six candidates for the job and hired a teacher straight out of a Teachers College. She was the best fit for the culture of the school and had the highest probability of positively impacting our math students. Shortly thereafter, the Ontario government implemented Regulation 274, which stated that any new hires must come from teachers with the top five seniority on the occasional teachers list. This regulation was based on the fallacy that teachers with more experience are better teachers.

So, principals were restricted to choosing from seniority, rather than merit, how well a new teacher might fit the school culture, or any other relevant factor. Some of you reading this article may have been impacted by this regulation.

In 2018, I wrote a paper with the results of investigating the relationship between teacher experience and teacher effectiveness. The short answer is that there is practically no relationship between the two. There is a small increase in effectiveness between a teacher's first year and their second year, mainly related to more efficient administrative tasks such as taking attendance and dealing with classroom management. But after that, extensive research shows no relationship whatsoever. So, a teacher with 20 years of experience is no more effective (or ineffective) than a teacher with two years experience. Teacher effectiveness is a consequence of what these teachers do to become more effective with their students.

We have all heard the George Bernard Shaw misquotation, "Those who can, do. Those who can't, teach." The actual quotation is from Aristotle: "Those who can, do. Those who understand, teach."

So, what makes a teacher effective? Here is what research tells us.

What Makes an Effective Teacher?

There has been an abundance of research into what makes a teacher effective. Among the characteristics identified are teachers caring about their students, and reflective practice, with teachers responding proactively to student questions and queries.

Teacher Care

In 2019, Sean Schat conducted an extensive investigation of students' perceptions of teachers caring about their students (Schat, 2020). He found that students who perceived their teachers as caring about student welfare performed better academically, as well as having higher self-esteem, self-confidence, and self-efficacy (the belief that one can accomplish challenging tasks). Schat also found that teachers who project an intentionally inviting stance (Purkey & Novak, 1995) foster in students a greater sense of belonging. This student sense of belonging has been shown to be a mediating variable in student achievement (Irvine & Kristensen-Irvine, 2025).

"People don't care how much you know until they know how much you care." ~ Theodore Roosevelt

Reflective Practice

Schön's (1984) research on reflective practice identified two dimensions of reflection: reflection-in-action, and reflection-on-action. Reflection-in-action refers to moment-to-moment decisions, such as those a teacher makes to adapt a lesson to student responses. Reflection-on-action, in turn, is the after-the-fact reflection on an event that allows for consideration of alternatives, identifies learning, and fosters potential future actions. Both types of reflection are important; reflection-in-action was found to be critical to effective teaching (Hattie, 2003). Reflection-in-action allows a teacher to be responsive to students' interests and curiosity, allowing students to guide and shape lessons. Reflection-on-action is useful to teachers who can then make modifications for future lessons on the same topic.

Edwards (2017) proposed two additional dimensions of reflection: reflection-before-action, and reflection-beyond-action. Reflection-before-action is an opportunity to integrate theory into practice and make decisions about the most appropriate instructional strategies before the lesson is taught. Reflection-beyond-action allows the teacher to

examine big-picture concepts of how the current lesson fits within an integrated curriculum framework.

Expert Teachers

John Hattie, who received his doctorate from OISE, has done extensive research into effective instructional strategies and effective teaching. Hattie (2003) identified five major dimensions in which effective teachers (whom Hattie calls "expert teachers") differ from teachers who are, in Hattie's words, "merely experienced." Expert teachers:

- can attend to affective attributes
- can identify essential representations of their subject
- can guide learning through classroom interactions
- can monitor learning and provide feedback
- can influence student outcomes

Some of these dimensions require further elaboration, and Hattie lists sixteen attributes that distinguish expert teachers within the five dimensions.

Expert Teachers Can Attend to Affective Attributes

Expert teachers have a high respect for their students, and work to understand their students' needs and wants. They care deeply about their students, and demonstrate that care in classroom interactions. Expert teachers accept responsibility for the learning outcomes of their students.

My own research on linkages among student motivation, attitudes, engagement, and achievement found that students who have positive attitudes toward mathematics, and who engage in their own learning, tend to have higher achievement levels (Irvine, 2020).

Expert teachers are also passionate about learning and about their subject; this passion for teaching and learning has been shown to increase student motivation, engagement, and self-efficacy (Mesa, 2012). Hattie (2009) found that teacher attitudes had a very large effect on student achievement (with a measured effect size of 1.04).

Expert Teachers Can Identify Essential Representations of Their Subject

Hattie found that virtually all teachers possess subject-matter knowledge; it is how this knowledge is used that differs between expert teachers and others. Expert teachers have a big-picture view of content, seeing it as an integrated whole, rather than a set of separate segments. This allows expert teachers to focus on the big ideas of both content and process; spontaneously relate classroom responses to big ideas; quickly reorganize classroom directions to be congruent with big ideas; and concentrate more on important aspects of learning. This allows expert teachers to be much more responsive to student input, as well as predict possible student errors (so they can think about how to respond effectively).

Expert teachers are very context bound and adopt a problem-solving stance, not in the sense of mathematical problems, but student focused, with a context of student learning and problem solving the pedagogical options for flexibility and pivoting of lesson directions based on student input. Flexibility and reflection-in-action are hallmarks of expert teachers.

Expert Teachers Can Guide Learning Through Classroom Interactions

Through their flexibility, expert teachers create an optimal learning environment, tailoring lessons based on student-directed choices. They let student responses and student questions guide the course of the lesson, reflecting-in-action to respond to their students' interests, inquiries, and thoughts on their learning.

In several states in the United States, math teachers are given scripted lessons, and are expected to follow those scripts without deviation. Hattie found that such practices are the antithesis of effective teaching. Effective teachers must have the flexibility to pivot their lessons to respond to their students' level of understanding and use student responses to shape the course of the lesson.

Expert Teachers Can Monitor Learning and Provide Feedback

Providing useful, actionable feedback is a mark of expert teaching. Hughes (2011, 2014) identifies three types of responses: "feed back," which discusses the current task; "feed forward," providing suggestions for next steps in the learning process; and "feed up," which relates current learning to the student's learning goals. Feedback must be a dialogue between the teacher and the student (Hattie & Timperley, 2007), not just a one-way teacher-driven discussion. Experts are more efficient at filtering relevant from irrelevant information, thus improving the quality of the feedback that they provide. Expert teachers are more adept at formulating and testing hypotheses about learning, identifying difficulties or alternative instructional strategies. For an example of using feedback in the mathematics classroom, see Irvine (2023).

Expert Teachers Can Influence Student Outcomes

Expert teachers engage their students in learning, and help their students develop their self-regulation, self-efficacy, and self-esteem as learners. They provide appropriately challenging tasks, focusing on mastery learning, rather than only achievement goals. Expert teachers endeavour to enhance both surface learning and deep learning. Hattie found that expert teachers focus 72 percent of classroom time on deep learning, with the

remaining 28 percent on surface learning. The non-expert teachers' ratio was exactly opposite: 72 percent on surface learning and only 28 percent on deep learning.

Conclusion

In summary, what does research tell us about the profile of an expert teacher? Expert teachers care deeply about their students as both students and as individuals; they use reflection-in-action to flexibly respond to student input; they have a big-picture view of the subject and the big ideas of the subject; they focus on student outcomes that include both cognitive and affective growth; and they have and exhibit a positive attitude toward their students, their subject, and the teaching–learning process.

The research identifies several aspects of effective teaching: perceived teacher care, subject-matter knowledge, informed and relevant instructional strategies, and demonstrably positive dispositions toward our students and our subject. The reason we all want to be effective teachers is to foster critical thinking and inquisitive natures in our students.

> "*In a completely rational society, the best of us would be teachers and the rest of us would have to settle for something less.*" ~ Lee Iacocca (former CEO, Ford Motor Company, father of the Mustang [and, sadly, the Pinto])

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I look forward to continuing the discussion of mathematical learning and instruction via my website (www.costellomath.com), email (david@costellomath.com), or X (@dr_costello).

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Dates to put on your calendars:

- **June 16–18, 2025** – NCTE–NCTM Joint Conference for Elementary Literacy and Mathematics Conference, Chicago, Illinois. See www.nctm.org/Conferences-and-Professional-Development/Joint-Conference/Joint-Conference-on-Elementary-Literacy-and-Mathematics/.
- **June 24, 2025** – NDCTM Summer Conference, Minot, North Dakota. See www.nctm.org/News-and-Calendar/Calendar-Events/Affiliates/NDCTM-Summer-Conference/.

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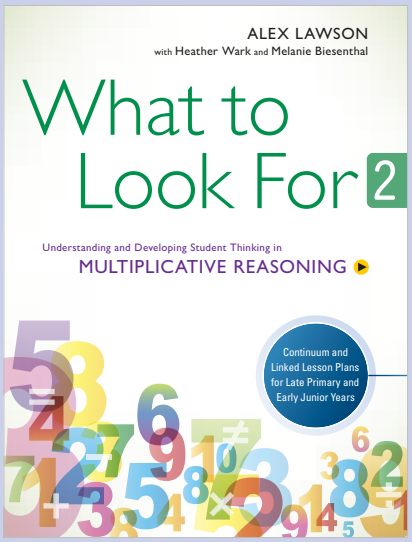
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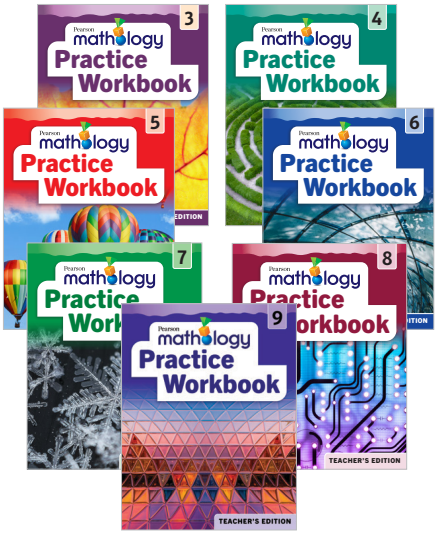
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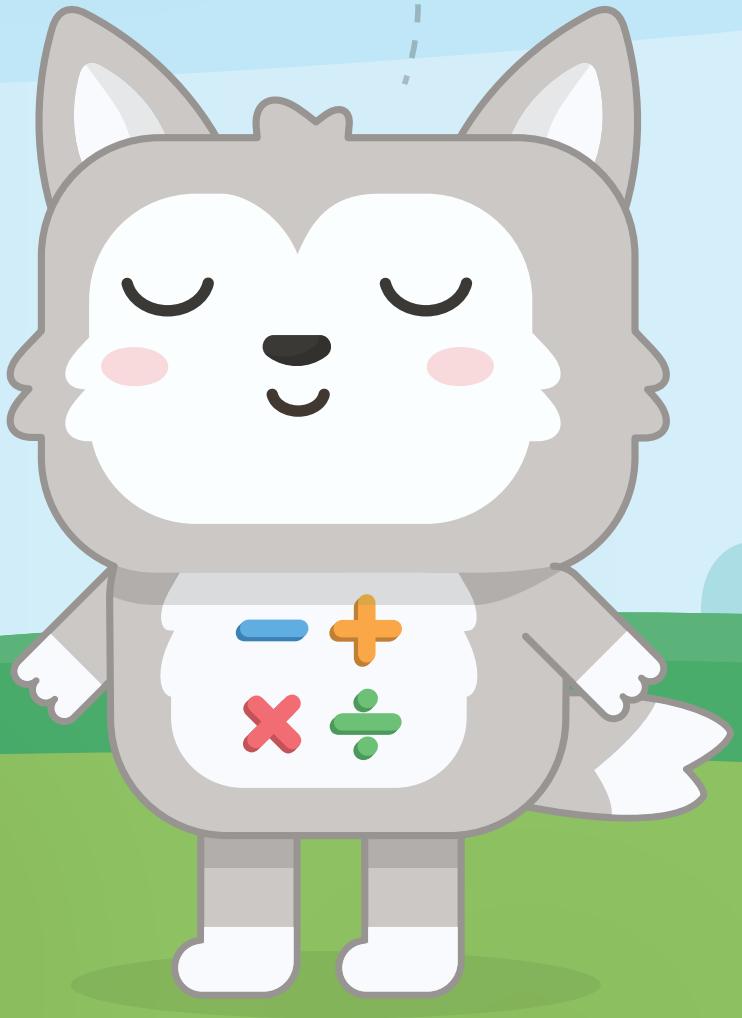
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