8-PAGE INSERT ELEMENTARY MATH
ABACUS

Ontario Mathematics Gazette

OAME – ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

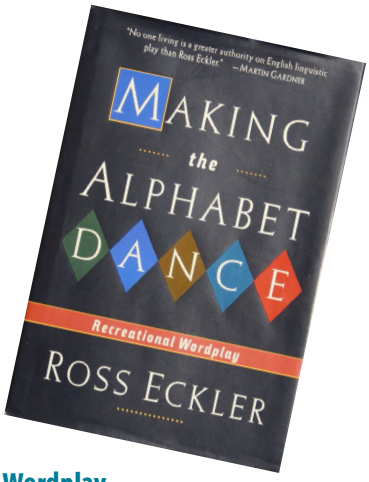
AOEM – ASSOCIATION ONTARIENNE POUR L'ENSEIGNEMENT DES MATHÉMATIQUES

Vol. 64#3
MARCH 2026

IN THIS ISSUE

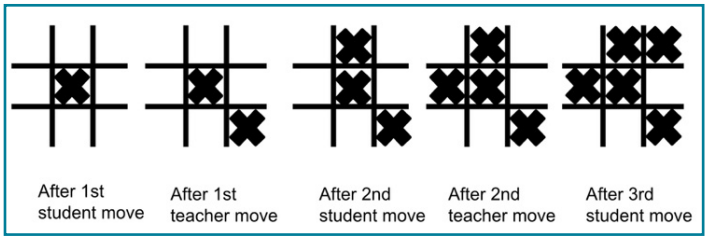
- Editor's Report
- President's Message
- OMCA News
- Events at the Fields: Problem Posing and Mathematics in Early Years
- OAME/NCTM Report: Spring 2026 NCTM Highlights for Ontario Math Educators
- Elementary Math Matters: Five- and Ten-Frames… Some Lessons from Six-Year-Old Xander
- Mb4T (Mathematics by and for Teachers): Partial Quotients: A Bridge to Understanding Division
- Play, Math, and Early Learning: Sorting
- Meaning Making in the Classroom: Understanding How Question Focus and Format Influences Student Understanding
- In the Middle: Making More of Financial Literacy
- What's the Problem? Amazing Arithmetic
- Useful Theories to Know About: HOTS and LOTS
- Musing on Percentages: % as a (Curious) Number
- The Math Teacher's Home Base: The Alphabet Dance: Symmetries, Solids, and Wordplay
- Engaging Tasks for Teaching Trigonometry
- *Notakto*–My Favourite Mathematical Game for K-12 Classes





See The Math Teacher's Home Base: The Alphabet Dance: Symmetries, Solids, and Wordplay

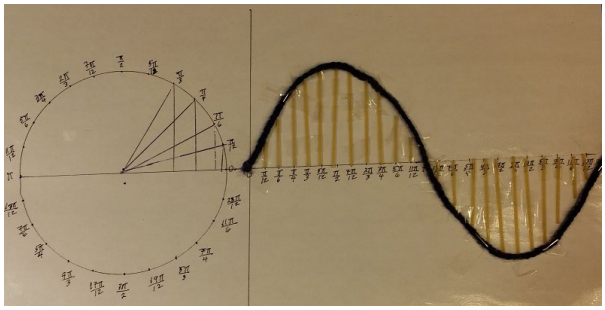
See *Notakto*–My Favourite Mathematical Game for K-12 Classes





See Play, Math, and Early Learning: Sorting

See Engaging Tasks for Teaching Trigonometry



OAME 2026 Let's {Talk} Math

KEYNOTE SPEAKERS



Dr. Pamela Seda



Pamela Harris

FEATURED SPEAKERS

Daniel Ansari
Octavia Beckles
Dr. Alex Lawson
Chris Luzniak
Cathy Marks Krpan
Marian Small
Ian VanderBurgh

WWW.OAME2026.CA

APRIL 30-MAY 1, 2026

FANSHAWE
London, Ontario

OAME WOMA TEAMS

@oamelearns

Submission of Articles

The *Ontario Mathematics Gazette* (*Gazette*) publishes news items, articles, and good ideas that are useful to mathematics teachers and mathematics teacher education. We are seeking submissions, preferably from mathematics teachers K–12 and other mathematics education professionals, that describe innovative and creative approaches to mathematics teaching.

Please keep in mind the following criteria when making submissions to the *Gazette*:

- The *Gazette* is aligned with the OAME/AOEM Vision, Mission, and Strategic priorities. These can be found on the website www.oame.on.ca under "About OAME" and "Guiding Principles."
- The ideas/activities must be of interest to the readership.
- The ideas/activities must be fresh and innovative.
- The mathematics content must be appropriate for the readership.
- The mathematics content must be accurate.
- The article must be well written and easily understood.
- The article and its ideas must be free of sexual, ethnic, racial, or other bias.
- The article must not have been previously published, nor should it be out for review by other publications.
- The article must be original.

Articles are to be word–processed, MS Word is preferred, and prepared according to the *Publication Manual of the American Psychological Association*, Seventh Edition. However, please use single-line (not double) spacing and only one space after each period. Articles should not exceed eight numbered pages of text, though there is some flexibility. Figures, images, and photographs should be placed close to the relevant text and include captions. The photographer's permission is required, and for photos of students under the age of 18, permission of a parent or guardian is required.

Please submit your article in one blind file (i.e., identity of author is not evident), and include author names, contact information including email and mailing addresses, photos—head and shoulders, biographies—less than 100 words, and all content removed for blinding in a second file. Please email these two files to Tim Sibbald at gazette@oame.on.ca.

Upon review, you will be notified whether your article has been accepted for publication (as is, or pending minor or major revisions) or declined. The Editor reserves the right to edit manuscripts prior to publication. Once an article is published, it becomes the property of OAME/AOEM.

PERMISSION TO REPRINT: No part of this publication may be reproduced, distributed, or transmitted in any form or by any means, including photocopying, recording, or other electronic or mechanical methods, without the prior written permission of the publisher, except in the case of brief quotations embodied in critical reviews and certain other non-commercial uses permitted by copyright law. Full credit must be given to the author and to the *Ontario Mathematics Gazette*. For permission requests, contact the Editor of the *Ontario Mathematics Gazette*.

The views expressed or implied in this publication, unless otherwise noted, should not be interpreted as official positions of OAME/AOEM.

▲ Table of Contents

Up Front

▲ Editor's Report *by Timothy Sibbald* 2
▲ President's Message *by Kit Luce* 4

Regular Features

▲ The *Abacus* . . . . . 8-Page Insert
▲ OMCA News
by Gerard Lewis 6
▲ Events at the Fields: Problem Posing and Mathematics in Early Years
by Pamela Brittain 7
▲ OAME/NCTM Report: Spring 2026 NCTM Highlights for Ontario Math Educators
by Gerard Lewis 10
▲ Elementary Math Matters: Five- and Ten-Frames... Some Lessons from Six-Year-Old Xander
by Lynda Colgan 11
▲ Mb4T (Mathematics by and for Teachers): Partial Quotients: A Bridge to Understanding Division
by Sofia Ferreyro-Mazieres 15
▲ Play, Math, and Early Learning: Sorting
by Jennifer Holm 18
▲ Meaning Making in the Classroom: Understanding How Question Focus and Format Influences Student Understanding
by David Costello 20
▲ In the Middle: Making More of Financial Literacy
by Carly Ziniuk 22
▲ What's the Problem? Amazing Arithmetic
by Shawn Godin 26
▲ Useful Theories to Know About: HOTS and LOTS
by Jeff Irvine 29
▲ Musing on Percentages: % as a (Curious) Number
by Jérôme Proulx 30
▲ The Math Teacher's Home Base: The Alphabet Dance: Symmetries, Solids, and Wordplay
by Ron Lancaster 33

Articles

▲ Engaging Tasks for Teaching Trigonometry
by Jeff Irvine 39
▲ *Notakto*—My Favourite Mathematical Game for K–12 Classes
by Peter Gustainis 44

Notices

▲ OAME/AOEM Board 2025–2026 48

Advertisements

▲ OAME 2025 Conference . . . . . Inside Front Cover
▲ Neufeld 14
▲ Pearson Education . . . . . Inside Back Cover
▲ Vretta . . . . . Back Cover

About the *Ontario Mathematics Gazette*

The *Ontario Mathematics Gazette*—ISSN 0030-3011—is indexed in the Canadian Education Index and is published four times per year. Its Canadian Publication Mail Product Sales Agreement Number is 40051074.

Editor

▲ **Tim Sibbald**
gazette@oame.on.ca

Associate Editors

▲ **Anne Yeager**
anne.yeager@rogers.com

▲ **Peter Saarimaki**
saarimaki@rogers.com

Assistant Editors

▲ **Steven Khan**
skhan6@brocku.ca

Abacus Co-Editors

▲ **Marc Husband**
marc.husbandl@oame.on.ca

▲ **Tina Rapke**
tina.rapke@oame.on.ca

Copy Editor

▲ **Gitta Berg**
gitta.berg@sympatico.ca

Design and Production

▲ **Penny Clemens**, Graphic Designer
pennyydezign@gmail.com

Printing & Binding

▲ **Pole Printing**, Box 69, 1-89 King Street East Forest, ON N0N 1J0 (519) 786-5112 poles@xcelco.on.ca

Advertising Manager

▲ **Robert Sherk**
4366 Snider Rd., Verona, ON K0H 2W0
Home (613) 374-1515
robert.sherk@oame.on.ca

Submission of Advertisements

Advertisements for publication in the *Ontario Mathematics Gazette* should be sent to **Robert Sherk** by email or at the above address. Deadlines for advertisements are January 23 for the March issue, April 1 for the June issue, July 1 for the September issue, and October 1 for the December issue.

Full-page advertisements are to be on 8.5" by 11" paper with a minimum of 0.5" margins and single sided. Each advertisement should be print ready, and colour advertisements should have no bleeds.

Advertising Rates

Advertising rates are available by telephoning, emailing, or writing to the Advertising Manager.

Editor's Report



TIM SIBBALD, OCT, PH.D.
gazette@oame.on.ca

Tim Sibbald is a professor in the Schulich School of Education, Nipissing University, with a focus that includes mathematics education. He is a former President of OAME/AOEM and has held a variety of other OAME/AOEM posts over the years.

Few teachers get to see what goes on in higher education—it is a statistical fact because there are not that many professors focused on training teachers and specifically mathematics education. In conversation with teachers, I often have a *Wizard of Oz* moment, where I feel I am revealing what is behind the curtain. In essence, it is not commonly understood what tenured professors do—a situation that has been muddied by increases in the number of contract faculty, as well as some professors being designated as being in a "teaching stream." Let me explain a couple of these before getting to the main idea of this editorial.

Faculty who are "contract" typically teach part-time and are often local educators, who teach a few pre-service classes. If they persist, they may gain rights to continue teaching specific courses, and they can receive some, typically limited, benefits. I did this while teaching high school to gain experience with post-secondary. By contrast, professors who are hired into a traditional tenured role (which I have been for over a decade) have permanence when they have earned tenure, provided they maintain standards. The traditional tenured role is 40 percent teaching, 20 percent service, and 40 percent research and scholarship. Professors hired in a "teaching stream" also earn tenure and have the same general role percentages, except that instead of research, they have 40 percent instructional scholarship.

Perhaps you have picked up on the use of "research" and "scholarship" and wondered about the difference. Suppose an academic would like to examine how students interact with a manipulative during a classroom lesson—this is research and requires approval by their Research Ethics Board, as well as approval by the students' school board. If the same academic considers the curriculum scope for using the same manipulative, and conducts a research synthesis (a formal collation of existing research) to interpret the relative benefits of the manipulative—this is an example

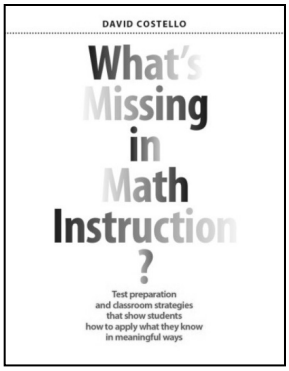
of scholarship. Academics often consider how they will approach an academic endeavour because of trade-offs between different methods. Instructional scholarship provides a means for examining best practices that are typically aligned with the type of position. For example, a professor who teaches first-year mathematics might conduct instructional scholarship with the goal of improving student outcomes from first-year university math courses.

There is a sticking point, in my experience, talking to leaders in education. Often leaders come to conversations looking for sharing of research. The implication is they want results of specific studies that one has conducted. The parlance tends to wain if one offers scholarship or points out that they are well informed about research on particular topics. While I have been aware of the language distinction, leaders often begin to show fading interest when they feel they are not getting "research" directly from the source. Yet, scholarship is research informed.

Against this backdrop, Canada has a neighbour with roughly ten times our population. The proportion of population corresponds to them having roughly ten times the number of academics. A strange detail is that leaders in education in Ontario tend to support their choices of presenters based on new books, which is scholarship as opposed to research. To clarify, in conversation, I get asked for research, but subsequently find presenters selected because they have published books, which is not research. This about-face is particularly evident when one hears of particular mathematical ideas being aligned with certain authors or academics, even though the focal idea has been around much longer.

For research, a key challenge is access. It is challenging to complete an application to a Research Ethics Board and to school boards when there is little coherence in the process between school boards. Timing for consideration of research projects in school boards varies considerably, in my experience. Supportive teachers can be a benefit, but unfortunately, there is a bureaucratic hindrance to the conducting of research in schools in Ontario. Too often leaders seem to want research, but are not clearly willing to support it in their own schools. Added to this challenge is that there is little upside to school boards because funding is scarce in Ontario, so coveted release time and professional learning are difficult to coordinate.

I could go on and on, but the point here is that support for the home team is not as simple as asking for current research. It is more effective to look for academics who are willing to present on topics where they have current knowledge. That is what the publishing of books illustrates, and it is evidently a proxy for presenter choice. However, all academics are busy working away at something (that might not include writing books), and it may be necessary to have a conversation to determine if there is common ground that could fit a presentation. In addition, most academics in education are well versed in supporting teacher development, and sometimes that is a matter of revisiting ideas that are known, but in a way that makes coherent sense of them.



In the News…

Congratulations to **Dr. David Costello**, a *Gazette* columnist, who has had another book published! If you would like a free copy, contact the Editor for details of how book reviewers are reimbursed. For details of the book, see www.pembrokepublishers.com/book.cgi?isbn=9781551383774.

The International Math Union has reported that the Web of Science (older readers may know this as the citation index) has dropped the field of mathematics. The concern is that "the field of Mathematics is more vulnerable to strategies to optimize status and rewards through publication and citation manipulation" (Pendlebury, 2023).

In this Issue…

Kit Luce reflects on teaching and events over the course of the year. **Pamela Brittain** draws on two MathEd Forums, with foci on mathematics in the Primary division and problem posing across all grades, which are both enticing because videos of the events are available. **Gerard Lewis** brings word of activities and events with OMCA. Gerard also draws attention to NCTM and has prizes left for the scavenger hunt, so he has extended it!

Sofia Ferreyro-Mazieres explains the benefits of using partial quotients in an article that has implications for many grades (including long division in Grade 12 Advanced Functions). **David Costello** considers a framework that facilitates thinking about questions in classrooms. On a similar theme, **Jeff Irvine** considers orders of thinking, with Bloom's and Marzano's taxonomies. **Ron Lancaster** draws out word play that has mathematical concepts, and mentions some resource gems along the way. **Carly Ziniuk** explores tipping and related financial math applications that highlight some of the many nuances of something that seems simple. **Jennifer Holm** considers the importance of noticing when play and math intersect as young children sort and organize

toys and other items. **Lynda Colgan** discusses the five- and ten-frames, and a pedagogy discussion she had with a sharp six-year-old. **Jérôme Proulx** considers a refined notion that a percent sign might be equivalent to a numerical value, and the consequences if one accepts that idea. **Shawn Godin** considers a problem involving selecting an arrangement of digits to achieve a purpose.

Jeff Irvine explains various approaches to teaching trigonometry with an eye to student engagement. **Peter Gustains** brings the game of Notakto to life with considerations of tactics and strategies that make one want to see what students will do with it.

Thank you…

Thank you, **Anne Yeager**, **Peter Saarimaki**, and **Steven Khan**. You have demonstrated that more hands make an improved balance of work. That has allowed us to dance more effectively to accommodate travel, holidays, and getting our task done. The professionalism you bring is second to none.

Thank you, **Gitta Berg** and **Penny Clemens**. Both of you are invaluable and continue to amaze with the specialized expertise you bring to the process. Your professionalism is also second to none.

Reference

Pendlebury, D. (2023, November 15). Highly cited research in 2023: The evolution of our evaluation and selection policy to support a robust scholarly landscape. Clarivate. www.clarivate.com/academia-government/blog/highly-cited-researchers-2023-the-evolution-of-our-evaluation-and-selection-policy-to-support-a-robust-scholarly-landscape/

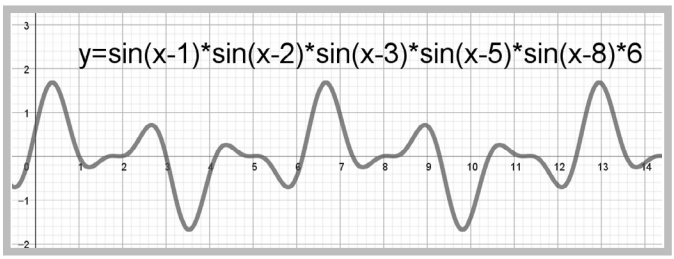
Letter to the Editor

Dear Tim,

First, I would like to thank you for your wonderful editing of Paul's article. I did see some of the material that you were given, and you definitely captured the most meaningful parts.

Please thank all those involved in making this tribute to Paul happen. It has meant a great deal to my family.

Sincerely,
Judy Lessard ▲



▲ President's Message



KIT LUCE
kit@oame.on.ca

Kit has been an active member of MAC² for many years and joined the Board of Directors in 2022. She is a District Mathematics Facilitator for Grades K–9 with the Simcoe County District School Board. In her work, Kit creates math resources to support educators in their instruction and assessment practice, as well as developing and facilitating professional development. Kit is an Additional Qualification instructor and course writer, and she has written and reviewed work for a variety of publishers. She has been involved in Ministry writing projects and was a Content Lead for the recent OAME/AOEM writing projects. Kit received the 2024 OAME Award for Leadership in Mathematics Education.

I write this message during a time when there has been a great deal of snowfall and challenging weather conditions where I live. I'm already beginning to yearn for sun and warmth. As we move into Spring, I've been reflecting on how our mission and vision come to life—not just in statements, but in the work we do together. OAME's/AOEM's commitment to promoting excellence and equity in mathematics education is evident in the initiatives of our chapters, the conversations sparked by OAME Talks, and the professional learning that connects research to classroom practice.

We see this vision in action through the incredible work of our chapters across the province, each creating opportunities for collaboration and growth. TEAMS (Toronto Educators Association for Mathematics) hosted a mini-conference focused on sharing best practices, creating spaces for collegial learning and equity in action. MAC² offered *Rethinking Fractions* with Shelley Yearley and Tara Flynn, a session that deepened content knowledge and strengthened instructional moves—an example of research-informed professional learning that benefits both teachers and students. NWOAME (Northwestern Ontario Association of Mathematics Educators) hosted a math night, exploring the evaluation of problem solving in high school mathematics, and GVMA (Grand Valley Mathematics Association) planned their first in-person gathering in several years.

One of the gifts of teaching is the rhythm of reflection—moments throughout the year when we pause, adjust, and refocus on what matters most for our students. We begin

the school year in September with excitement and anticipation, thinking about the new students we will meet and the math challenges we face together. In January, we come back from winter break in elementary or begin to think about semester 2 in secondary schools, and we can take the time to assess what has been working well, and what changes we want to make to support our students in their learning of mathematics. After March break, we reflect and adjust our plans to ensure that key concepts are addressed, and that students deepen their understanding of math at their grade level. And at the end of June, our thoughts turn to the successes, to the "aha" moments experienced in our classrooms, as well as reflection regarding the whole year and modifications for the next or coming years.

Looking ahead, I'm excited for our annual conference, *Let's Talk Math*, taking place April 30 and May 1 at Fanshawe College in London. This brings a chance to learn, share, and celebrate the work we do together. The theme centres around student voice, dialogue, and inclusion. The keynote speakers reflect these priorities: Dr. Pamela Seda, co-author of *Choosing to See* and creator of the ICUCARE® Equity Framework, will challenge us to design systems where every learner's brilliance is recognized. Pamela Weber Harris, author of *Building Powerful Numeracy*, will share strategies for reasoning-based mathematics and vertical coherence. Our featured speakers will explore research-informed classroom practices, bringing insight into how students develop numerical understanding. These voices remind us that excellence grows where equity, evidence, and engagement intersect.

Between conferences, OAME Talks continues sparking meaningful conversations. Jamie Mitchell invited us to reimagine assessments so they're more meaningful, inclusive, and resilient in the age of Artificial Intelligence. Chad Harris's *Wait, What Was I Learning?* sheds light on neurodivergent experiences in the classroom; Stacey Jacobs's *Empowering Math Learners* offered executive-function strategies to support student success; and *Etuaptmumk—Two-Eyed Seeing* invited us to braid Indigenous ways of knowing with western mathematics through partnerships, cultural learning, and community engagement. Each of these conversations reminds us that equity and excellence are not separate goals—they thrive together when we listen, learn, and act.

In November, our Leadership Series featured Dr. Elham Kazemi, who explored how storytelling can transform mathematics learning and foster belonging. She reminded us that we can support students when we *"ask kids what's making sense to them about a strategy, but then also compare their strategy with another one, and by listening to them, you see what questions they have, what they feel solid about, and what they're unsure about."*

Her message—that rich mathematical conversations flourish when classrooms are rooted in reciprocity—echoes our vision for inclusive, engaging learning environments.

The learning opportunities for mathematics educators offered by OAME/AOEM continue to be offered both in person and virtually. OAME2026 will take place in person in London, Ontario. Looking ahead to OAME2027, the conference will be offered in a solely virtual format. This structure will hopefully allow for greater accessibility to educators from throughout Ontario's regions and school boards. The conference theme for OAME2027 is *Mathematics Through Stories: Cultivating Courage, Curiosity, and Connection*.

As we are preparing to gather in London this Spring, I invite you to bring your voice, your questions, and your passion for high-quality mathematics education. Together we'll continue to promote, support, and advocate for the kind of learning every student deserves—rooted in equity, research, and vibrant mathematical thinking.

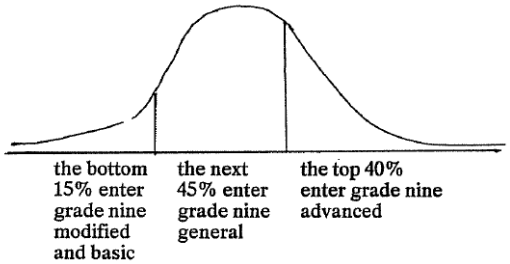
Thank you for the energy and ideas you bring to this work. Together we'll keep developing classrooms where every student sees themself as a capable math learner. I look forward to learning with you in London and continuing this journey together.

References

All references come from OAME/AOEM sites, www.oame.on.ca and www.oame.on.ca/mcis/index.php, except for the quotation from Elham Kazemi, which came from www.hep.gse.harvard.edu/blog/2025/08/25/teachers-change-lives-recording-and-recap/. ▲

From the Archives …

(*Gazette* and *Abacus* archives are available in the members-only section of the website.)



"Now the streaming is far from perfect, but it seems to me that it is also far from imperfect" (Henderson, 1979, p. 28).

Henderson, D. (1979). One of the basics, a humane approach to student evaluation. *Ontario Mathematics Gazette, 18*(2), 26–31.

OMCA News



GERARD LEWIS
(OMCA PAST PRESIDENT)
OMCAmath@gmail.com

The Ontario Mathematics Coordinators Association (OMCA) supports the leadership of mathematics consultants, coaches, and coordinators from across the province. With over a hundred active members meeting monthly via Zoom, these leaders connect to share experiences, learn among peers, and receive updates from partners including the Ontario Ministry of Education, Education Quality and Accountability Office (EQAO), Ontario Association for Mathematics Education (OAME), Francophone Association for Mathematics Education in Ontario (AFEMO), National Council of Supervisors of Mathematics (NCSM), and other educational delegates.

Monthly OMCA Meetings and Professional Learning

For the Fall and winter seasons, we welcomed speakers for two mini-series of professional learning. In September, October, and November, our learning centred around facilitating fraction learning for educators. This focus on strengthening content knowledge helped to support Priority Action 2 of the Ministry's "Taking Action in Mathematics" *look-fors* document[1]. Shelley Yearley and Tara Flynn led our members in the eight core concepts as part of the fraction-learning pathway, where the professional learning focused on the use of strong representations and unit fractions. They also focused on the effect of teacher efficacy on the learning of fractions by students, where teachers with high efficacy are more likely to try out new teaching ideas and strategies that are difficult, involve risks, and require that control be shared with learners. Additionally, teachers with high efficacy persist and attend more closely to the needs of struggling learners.

Our second mini-series in December, January, and March welcomed Diane Sweeney to explore the seven core practices for student-centred coaching with our members. The objectives of these seven practices included impacting student learning through assessments, evolving instruction to support good practices such as reasoning and problem solving, and fostering partnerships with the co-teachers and school leaders. Over the course of three sessions, we learned about the structure of a coaching cycle, which included setting goals, unpacking into learning targets, co-planning using student evidence, co-teaching with effective instructional practices, and measuring impact on student and teacher learning.

Ad-Hoc Committees

This year, OMCA wanted to extend focused learning and working-group opportunities to members who share common interests and responsibilities in their roles. The communications strategy committee is looking at the process of maintaining our new website and streamlining our virtual filing system. The digital tools and VASP[2] committee is advocating for the review of how digital tools in mathematics are permitted, and restricted, by a variety of school boards. The special education committee is collaborating on a framework to compare models of support provided to students with an Individual Education Plan (IEP) in mathematics. And the Kindergarten curriculum committee is developing strategies to support coaches and coordinators with the release of the new Kindergarten curriculum.

OMCA Retreat at Hidden Valley

Every February, OMCA organizes a retreat, primarily for members to get together, network, share experiences, and learn together. This year, our keynote speaker was Dan Finkel, an esteemed curriculum writer and founder of Math for Love, a Seattle-based organization devoted to transforming how math is taught and learned. The retreat theme of "Math Connections" was divided into three sections: connections to teacher and educator learning, with the lens of how we might support our teacher colleagues in our role(s); leadership connections, especially implementing strategies at the school and district levels; and family and community connections.

OMCA is always open to new members who work in a dedicated role supporting educators with mathematics teaching and learning. Visit our new website at www.sites.google.com/view/omcamath/home, or contact us at OMCAmath@gmail.com, to learn more about the organization, find out about upcoming events, and connect with us.

We look forward to seeing you at our upcoming Spring meetings on the following dates:

- March 9, 2026 – Virtual
- April 13, 2026 – Virtual
- April 29, 2026 – In person at Fanshawe College

Continued on page 9

▲ Events at the Fields: Problem Posing and Mathematics in Early Years



PAMELA BRITTAIN
pbrittai@fieldsinstitute.ca

Pamela Brittain (Ph.D.) is an educator, author, illustrator, presenter, and math lover, who currently works for the Fields Institute as their K–12 Academic Coordinator. She has been involved in STEM education for over 20 years and strives to make math accessible and enjoyable for everyone.

The October and November MathEd Forums focused on the formulation of problems by students, and mathematics teaching in the Primary division.

October MathEd Forum: Problem Posing

The October MathEd Forum was all about problem posing in mathematics, and included talks by Zack Wolske and Kitty Yan on *What is problem posing by students, and why is it important?,* Liisa Suurtamm (District School Board of Niagara) on *Creating a Culture of Problem Posing in Mathematics Classrooms,* and Allysa Lumley (York University) on *Encouraging Mathematical Curiosity Through Open-Ended, Flexible Assessments.*

Zack Wolske considered what it means to pose a problem. He asked questions like: What does it mean for students to be posing problems? What does it look like? How do we evaluate it? What types of problems are we interested in and why is it important? He mentioned how the Ontario curriculum, along with many others, refers to how problem solving builds agency in students and allows them to see themselves as competent in mathematics. However, Zack argues that problem solving alone doesn't cover the full spectrum of mathematical practices, and this is where problem posing comes in. Most in-class problem posing tends to focus on providing facts or a pattern or a picture to students, and then explicitly asking them to create a problem that is connected to the situation. Students then explain their problems, how they came up with them, and provide a solution. Instead, Zack suggests having students reformulate problems, or think about how their smaller problems can relate to a larger context, and the research shows that students are quite capable of posing interesting math problems. From a practical standpoint, Zack suggests providing students with examples of good problems, and then having students build on those to generate their own, though this requires the teacher to have built their own problem-posing skills as well. One method to developing these problems is to use "I notice" and "I wonder" statements as a starting point, and work toward turning those statements into mathematical problems.

Next, Kitty Yan talked about some of the theoretical underpinnings connecting problem solving and problem posing. She shared two recent papers on the subject and provided some background on the evolution of the field into a pedagogical practice that is being studied. She spoke about recent research evolving from looking at the products of problem posing, such as the actual problems generated by the students and their solutions, toward the process of problem solving. She does admit that the latter is much more difficult to evaluate than the former. She shared insight about how a student's mathematical content knowledge influenced their ability to create solvable or suitable questions, as did the problem's connection to a real-life context. She then broke down the four stages of the general problem-posing process model:

1) Orientation: Engage with and understand the initial situation or context.
2) Connection: Draw on prior knowledge and identify relationships or structures.
3) Generation: Construct and articulate new math problems based on these relationships or modifications.
4) Reflection: Evaluate, refine, and interpret the problems posed for mathematical soundness and creativity.

Kitty describes this process as providing a framework for creating problems, but also for guiding evaluation.

Liisa Suurtamm spoke about the place of problem posing in the curriculum, where it is explicitly stated (e.g., the Grade 9 expectations), and where we see it implicitly throughout the curriculum (e.g., the mathematical modelling expectations). She emphasized the importance of including both teacher-posed and student-posed problems. Students often struggle with the task of posing good mathematical problems, which creates deep thinking on the part of the students. With reference to Zack's talk, she noted that "I notice" and "I wonder" statements formed by students were often not initially mathematical, but served as good starting points by getting students engaged, which eventually led to creating mathematical problems. She then shared some images and asked participants to share what they noticed and wondered, while guiding the group toward making mathematical problems out of them. She noted that coding often leads students to pose problems, ask questions, and to think about what happens if we extend underlying concepts. She gave the example of using technology, like graphing calculators or Desmos, to inspire students to ask

questions such as, “I wonder if I could create a graph to write my name? What would that look like?”

Finally, Allysa Lumley spoke about using open-ended, flexible assessments at the University level, to encourage mathematical curiosity. She contrasted first-year engineering students with third-year mathematics students, and how each approached problem posing differently, based on their level of mathematical knowledge. The assessment problems she used were posed within a framework and within a specific context that left students feeling that there was a degree of creativity available to them. The specific question she posed to her students involved taking materials learned in class (drawing a two-dimensional curve, rotating it around an axis, and then writing an integral to describe the volume) and creating a physical three-dimensional model of the rotated curve. Students were provided with guidelines and a framework (it had to lie flat in 2D space; it had to have a hinge to open and close; it had to represent a curve; students had to be able to calculate the integral at the end; and bonus marks were awarded if the shape was unique—not just a cylinder or sphere). Allysa found there was a little bit of competition that took place among the students as well, as they wanted to create something their classmates wouldn’t think of or try.

Finally, students were invited to participate in a problem-solving session with the presenters, to implement and explore many of the topics discussed throughout the day. There was also a panel discussion with presenters, where students could ask questions and gain further insights into the methods and research that had been presented.

November MathEd Forum: First Principles: Mathematical Learning Pathways from K to 3

The November Forum focus was on our youngest learners and the unique challenges and opportunities of teaching mathematics to students in early years. The speakers included:

- Ashley Kozak (K–12 Math Program Lead with the Upper Grand District School Board), Brad Digweed (Student Achievement Leader with the District School Board of Niagara), and Matthew Oldridge (Mathematics Educator, Ontario College of Teachers), who spoke about *Framing Talks: Key Ideas and Central Questions*, along with
- Dr. Sandy Youmans (Queen’s University)—*Bringing It Home: Guided Play and Tangible Resources*
- Dr. Akihiko Takahashi (DePaul University, Chicago)—*Using Lesson Study to See and Support Mathematical Thinking*.

The day also included a collaborative session, led by Sandy Youmans, where participants brainstormed ideas related to the issues educators face when introducing mathematics to young learners, and potential solutions to those issues.

The first set of talks by Ashley, Brad, and Matthew were short insights from the K–3 area of mathematics education.

Ashley Kozak spoke of how important collaborations are between educators and researchers because researchers can often provide information to teachers that teachers may not have the time to engage in themselves. She explained the use of a K–3 screener to identify students in need of additional assistance, and the types of lessons and structures (including learning goals, big ideas, minds-on activities, explicit instruction, guided work, consolidation, and exit slips) to support student learning, as well as professional development to help support the teachers.

Brad Digweed spoke on the importance of spatial sense in foundational K–3 mathematics development, before moving on to more complex topics, for example, understanding the properties of the number 9, before advancing on to bigger numbers like 54 501, and that students need to have a strong understanding of what these numbers look like. What do 9 people look like versus 54 401 people, and how does that affect student understanding of numbers in general? He described how his school board began focusing on professional development for teachers intent on developing spatial sense in mathematics, including specific Additional Qualification courses.

Matthew Oldridge rounded out the short talks with discussion of the connections between practitioners and researchers. He discussed the idea of capturing what he calls “loose knowledge” that is more freely available now (like Pi Day jokes or the concept of infinity, which he views as knowable even to a Kindergarten student), and capitalizing on that to build lessons. He then spoke about ways in which young children approach math topics such as counting, and that studying developmental benchmarks in young students can help to address issues in later math courses, which linked to Ashley’s earlier presentation. Matthew’s overarching belief is that there are certain patterns and approaches that are common to how young children tackle mathematics, and that studying these approaches can help to address and deal with issues that may arise in later mathematics studies. He also believes that teachers build a kind of innate knowledge or sense of these approaches, and asked the participants to consider their own understanding of how young students learn and approach math.

Sandy Youmans discussed guided play, and provided some tangible resources for the classroom, along with research that shows the importance of early numeracy skills

as a predictor of later math proficiency. The way to build those foundational skills is what she refers to as *structured numeracy*—a combination of direct instruction and structured inquiry. Like Ashley's talk, she promoted the use of screeners, as they can help to show where early learners have gaps, and provide guidance to teachers to address them effectively. Sandy then provided some examples involving understanding the meaning of digits in a number, in terms of quantity, and being able to answer questions comparing the size of quantities. She also made an interesting reference to how the English language doesn't support place value like other languages do (for example, 12 in Chinese is ten two, whereas in English, it's twelve, which can be confusing to students). Sandy also compared the literacy skills found in the *Right to Read* report (by the Ontario Human Rights Commission) to foundational concepts in mathematics, and discussed the belief she and other educators have that a similar report needs to be created for mathematics. Sandy also walked through some resources from www.earlymath.ca, youngmathematicians.edc.org, the Erikson Institute Early Math Collaborative (www.erikson.edu/early-math-collaborative/), and DREME Family Math out of Stanford (www.familymath.stanford.edu). Finally, she spoke about the importance of using evidence-based, high-quality resources, and the need for collaboration, and how her involvement in the AIM (Assessment and Instruction for Mathematics) Collective had been invaluable to her.

The last speaker of the day was Akihiko Takahashi, who spoke about his experiences working with schools outside Canada, including the United States, Cambodia, and Denmark. He explained his research, which examined the percentages of students struggling in early mathematics, and how different levels of teaching ability from the teachers affected student learning, for example, the difference between a Level 1 teacher, who is able to show a step-by-step explanation, and a Level 2 teacher, who is able to explain the "why," and often uses diagrams or concrete representations to illustrate a concept. The key idea in the research is teacher-facilitated discussions and the use of problem solving to allow students to discover and explore new mathematical concepts. Akihiko reiterated the importance of incorporating investigations into lessons, but then formalizing learning through explicit instructions, applying it to similar situations, and developing student mastery and fluency. Akihiko discussed how their research studied various methods for effective professional development, to help teachers build skills and develop the expertise needed to become a Level 2 teacher. He ended his talk promoting the Teacher Network for Teaching Through Problem-Solving (TTP Network, ttp-network.org), where the research and many other resources (including a book club) can be found.

One common theme among all the presenters was the barriers and issues many of them face when trying to do research, and assisting the youngest students in their pursuit of mathematics learning. This was the brainstorming session's topic at the end of the day, where a lively discussion took place with participants, guided by Sandy Youmans, on what these issues are, and ways we, as educators, can address them.

As always, videos of the presentations can be found at fields.utoronto.ca/activities/25-26/meforum-Oct and fields.utoronto.ca/activities/25-26/meforum-Nov.

Ongoing Programs at Fields

The *Math Circles* program investigates mathematics challenges and puzzles in a collaborative and guided setting, where students can explore skills like looking for patterns, making conjectures, finding counter-examples, and constructing proofs. These sessions are open to students in Grades 7–12, and to anyone who wants to explore math in more detail. Participants don't have to be "traditionally strong" in mathematics to join; they just need to be curious about exploring mathematical puzzles. Math Circles runs on Saturdays from 2 – 4 p.m., in person, at the Fields Institute (222 College Street, Toronto), and is free to join. You can learn more, and have students register, at www.fieldsacademy.ca/eep/.

Our newest initiative is the *Math in Literacy* program, where we work with authors and illustrators of children's picture books, from communities typically underrepresented in mathematics, and provide math and science activities intended for K–6 teachers. Resources and information can be found at www.fieldsacademy.ca/math-in-books/. Our thanks go to House of Anansi and their Groundwood department for supporting this initiative.

Finally, our *Ask a Mathematician* program brings mathematicians virtually into Grades 4–10 classrooms to talk about what it's like to be a mathematician, and about unique and interesting fields of mathematics. Learn more at www.fieldsacademy.ca/aam/. ▲

Continued from page 6

Notes

1 The original document no longer appears to be available on the Ontario Ministry of Education website, but can be found at https://assets-us-01.kc-usercontent.com/fbd574c4-da36-0066-a0c5-849ffb2de96e/f2753f22-d275-4287-97d3-983ce10677df/Appendix%20B%20-%20Taking%20Action%20in%20Mathemathics%20June%202023.pdf.

2 VASP stands for "Vetting of Application Security & Privacy" in reference to the service that school boards can use to determine how risky certain digital tools are for student and teacher use. The service is provided by the Educational Collaborative Network of Ontario (ECNO), with details available at www.ecno.org/services-programs/vasp/. ▲

▲ OAME/NCTM Report: Spring 2026 NCTM Highlights for Ontario Math Educators



GERARD LEWIS
gerard.lewis@oame.on.ca

Gerry is an Elementary Mathematics Facilitator, who assists teachers of students with IEPs (Individual Education Plans) within the Toronto Catholic District School Board. Gerry has presented over 20 times across Canada and the United States, with a focus on equity, culturally responsive pedagogy, mathematical modelling, and integration of technology in the classroom. In 2024, he completed his Master of Education degree in mathematics education at York University. He is currently the Past President of the Ontario Mathematics Coordinators Association (OMCA), and for his work with mathematics teachers through the pandemic, he was awarded the Prime Minister's Award for Teaching Excellence in 2022.

As we head into Spring 2026, the National Council of Teachers of Mathematics (NCTM) continues to offer valuable opportunities for online and in-person professional learning and engagement that align with Ontario's current focus on deep mathematical understanding, equity, and problem solving.

Upcoming Dates and Events

- NCTM Spring Conferences—Indianapolis, Indiana (February) and New Orleans, Louisiana (March): These gatherings include sessions on early numeracy, mathematical discourse, and equitable pedagogy that resonate with Ontario's focus on concept development, beauty of mathematics, and student reasoning. The Indianapolis conference theme is "Navigating Crossroads Together: Innovating, Supporting, and Connecting," and the New Orleans conference theme is "Rhythm, Roots, and Reasoning: Cultivating Joie de Vivre in Mathematics."
- Webinars: NCTM's online offerings continue into 2026, including "Deliberate Practice in Action: Strengthening Instruction Through AI-Empowered Simulation" on March 4; "Spring 2026 Mathematics Education Trust Grants Webinar" on March 23, to learn more about how to submit a grant proposal; and "April 2026 MTLT Teacher Talk" on April 7, which will discuss the *Mathematics Teacher: Learning & Teaching PK–12* (MTLT) journal article, "Corvettes, Curve Fitting, and Calculus." All these sessions are open to members and non-members, and interested participants can find more information at www.nctm.org/online-learning/webinars/list.

Position Statement Highlight

NCTM's Position Statement on *Open Educational Resources (OER)* (June 2025) articulates the importance of selecting coherent and mathematically sound instructional materials, a priority that supports Ontario educators navigating both provincial resources and supplementary resources to ensure curriculum fidelity. In my coaching role, I support teachers who are inundated with board-supported and open resources, so this position statement helps to find the professional balance between seeking quality and collecting quantity of support. This position statement includes the following declarations:

- There is a place for Open Educational Resources.
- High-quality resources demonstrate coherence.
- Teachers must have agency, authority, and power.
- Teachers are responsible for their selection.

To learn more about these events or documents, and if you would like to connect, please reach out to me at gerard.lewis@oame.on.ca. Have a great Spring!

Scavenger Hunt

Thank you to those who participated in the NCTM Scavenger Hunt. Because there are many NCTM publications still available to give away, the NCTM Scavenger Hunt will remain open until March 13, 2026, to allow time for more entries. You can find the instructions and the questions at www.docs.google.com/presentation/d/1NegcIfPWSZWJbXf6NZereSfgOjtRwnnmGKWLqEBwfAM/edit?usp=sharing. ▲

Banana Patterns?

What patterns do you see with this bunch of bananas?

Can you find similar or contrasting patterns for the banana plant?

This photo was taken in Madeira, Portugal. Where else are bananas grown?



▲ Elementary Math Matters: Five- and Ten-Frames… Some Lessons from Six-Year-Old Xander



LYNDA COLGAN
lynda.colgan@queensu.ca

Lynda Colgan is Professor Emerita, Faculty of Education, Queen's University, and Founder and Faculty Lead for Science Rendezvous Kingston. In May 2025, she was awarded the King Charles III Coronation Medal by the Chancellery of Honours at Rideau Hall for her lifelong contributions to education. Her latest project is contributing author for an upcoming Springer publication, Teaching and Learning Elementary Mathematics. *Lynda remains deeply committed to her two loves: STEM outreach/public education and her English Cream Retriever, Annie.*

A common graphic used in elementary math is the five-frame as a single row of five squares, usually oriented horizontally, as shown in Figure 1.



Figure 1: *A five-frame is a grid with five empty spaces. It is shown here in a standard horizontal configuration.*

As a first step in instruction, children physically place items in each box sequentially from left to right to allow for accurate counting, while developing the concept of cardinality (the last number said represents the total quantity).



Figure 2: *A five-frame showing the standard way in which counters are placed sequentially from left to right to show the numbers from one to five. Positioning and touching each counter while saying the number aloud provides an important tactile experience that enhances cognitive development.*

This hand–brain coordination activity helps children to develop consistent visual patterns for number sense representations—an important developmental step in subitizing (quickly recognizing the total number of objects, without counting each one). It is also a key component of future mathematics concepts including decomposition and fluency about number relationships within the "anchor" construct of five, such as understanding that "four" is one less than "five" or "five" is one more than "four."

The next step is to introduce the ten-frame: a rectangle divided into two rows of five squares each, creating a rectangular grid made up of a total of ten congruent boxes. Each square in the ten-frame represents one unit.

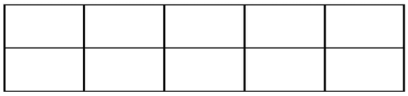


Figure 3: *A ten-frame is used to represent numbers up to ten. Since we have a base-10 number system, it is important for children to develop a strong understanding of this important anchor number to support mental math by clarifying familiar numerical relationships.*

There is consensus that ten-frames are powerful tools for building the critical developmental benchmark of the "anchor" number 10. Disparity emerges around the ways in which students are instructed to use and interpret the physical structure of the ten-frame. As with every mathematics manipulative, the goal is to have students capitalize on the representational power of the tool; this does not happen if students are counting each counter (dot) on a ten-frame. For example, in Figure 4 below, the goal would be for children to "see" multiple representations based on the anchor numbers; i.e., "7" is three less than 10 and two more than 5.

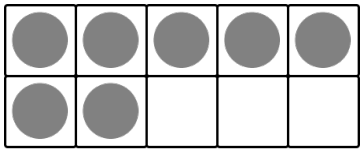


Figure 4: *The filled spaces visually represent the quantity, i.e., 7 spaces filled = 7.*

However, the procedure for filling the ten-frame is not standardized—some teaching resources suggest filling the top row from left to right and then the bottom row from left to right, while other resources recommend reversing the "fill" sequence, i.e., filling the bottom row first. Still other resources suggest that rules about the "fill" procedure are irrelevant—students should understand that if one row is filled—either the top or bottom—that row contains 5 and does not need to be counted. Regardless of the "fill" procedure taught (and evaluated), ten-frames are usually shown in a horizontal orientation.

Beyond mere counting and number relationships involving the anchor numbers, five- and ten-frames also provide opportunities to employ both perceptual and conceptual subitizing skills. For example, in Figure 4, recognizing "4" dots and "3" dots without counting allows children to see that the number "7" can be constructed in multiple ways, not exclusively through reliance on the anchor number "10." For example, in Figure 5, we see the

following deconstructed representations on the five-frame: 3 + 2 = 5, 2 + 3 = 5, to show addition. If on the five-frame, the two black counters are removed, we see that five subtract two is three (5 – 2 = 3), or conversely, if three grey counters are removed, we see that five subtract three is two (5 – 3 = 2). On the ten-frame, we see that 10 can be "made" by combining 7 and 3 or by combining 4 and 6. If we remove, rather than place, the grey counters, then we see that 10 – 3 = 7. If we remove the black counters, then we see that 10 – 7 = 3. Similar actions generate the number sentences 10 – 6 = 4 or 10 – 6 = 4.

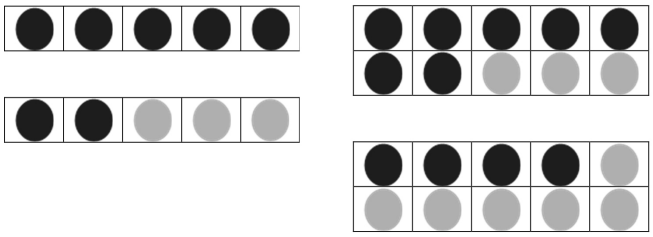


Figure 5: *On the left, to show different decompositions of the number 5, using the five-frame, the colour of the counters is changed. On the right, different decompositions of the number 10 are represented: 7 + 3 = 10 and 4 + 6 = 10.*

While these examples may be as familiar to *Gazette* readers as they were to me, I recently experienced an "Aha" moment when I was tutoring a child from my neighbourhood, Xander (not his real name)—an afternoon that brought my "researcher" hat to the fore. This reminded me (once again) of the wisdom of Dr. Seymour Papert, who believed that children must be empowered to take charge of their own thinking and learning process, rather than be forced to become passive receptors for top-down instruction.

After several counting sessions together, Xander and I arrived at "teen numbers," where a full ten-frame represents 10, and the items in the second row show the remaining "ones" (e.g., a full frame and three items in the second row represents 13).

I proceeded to lay out two ten-frames, as shown in Figure 6, to represent the number 13, using what I thought to be a standard configuration, and one that would be familiar to Xander.

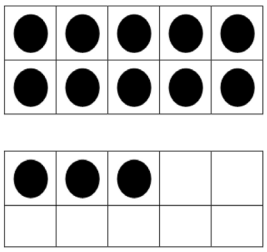


Figure 6: *Two ten-frames to represent the teen number "13" as the sum of 10 and 3*

Head on hands, Xander watched me closely and then stopped me in my tracks by sitting up straight, staring me in the eye, and insisting that I was wrong. I looked at him with a puzzled expression, genuinely not knowing what he would answer, and asked, "Xander, why is it wrong?"

Without hesitation, he proceeded to reorient the ten-frames vertically, as shown in Figure 7.

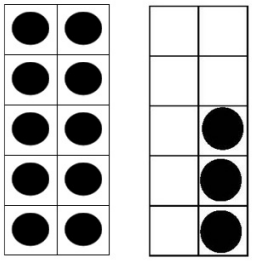


Figure 7: *Xander positioned the ten-frames vertically to represent the number "13."*

"Xander, tell me the difference. Why are your ten-frames correct, but mine are not?" In typical six-year-old fashion, after an exaggerated facial expression that said, "Okay, if I *have to*," Xander began (using his best teacher voice, I might add), "Lynda, please write the number '13' for me on this paper." I complied and wrote the number in the standard way.

13

Xander then asked me to look carefully at my number. "You see, in '13,' the '1' isn't really a '1.' It means 10 because it's on the left side. For a filled-up ten-frame, if you put the ten-frame the right way, it goes up and down the same way you write '1.' The '3' mean three ones. To make '13,' first you take one ten-frame and fill it up with 10 counters to make one 'ten.' Then you move it over (he motioned as he moved it to the left), and then you get another ten-frame and put in the three ones. When you put them together, you have '13'." To make his point more fully, he then wrote "1" and "3," as shown in Figure 8, and said, stretching out the syllables, "there it is, th… ir… teen."

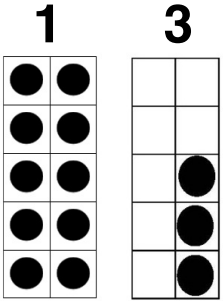


Figure 8: *Xander's representation of 13*

"Aha, now I get it, Xander, very good thinking." To confirm his understanding, I then asked him to show me the number "27," using his system. "Well, you need two full ten-frames to show '20,' and then the ten-frame in the 'ones' place needs to have seven counters." He positioned the ten-frames and counters, as shown in Figure 9.

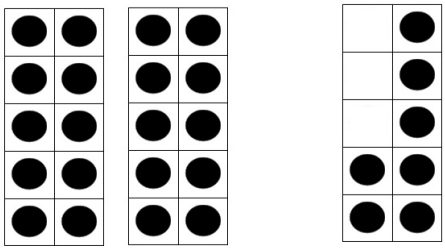


Figure 9: *To represent "27" using ten-frames, Xander purposefully positioned two full ten-frames to the left (because in his words, "that's where 10's go"), and then positioned the ten-frame showing "7" in what he called the "ones" place.*

I told Xander that his representation of 27 was correct, but I still asked him this question: "I noticed that you used the ten-frame to make '7' like this. (Please see Figure 10, left side.) Why didn't you show '7' this way? (Please see Figure 10, right side.) Isn't it still '7'? Xander looked at my ten-frame and then at me. "Yes, there are seven counters on yours, but I had to add them to make '7' after I counted four at the top and three at the bottom. With mine, I didn't need to count because I saw the filled part and knew it was five right away and I just added two. A ten-frame's just two five-frames side by side."

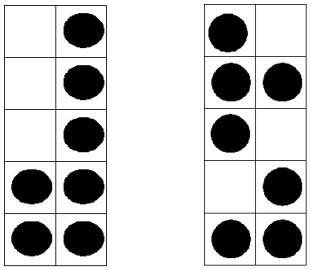


Figure 10: *On the left, Xander's representation of "7" on the ten-frame, and on the right, mine.*

Since Xander was able to represent other multi-digit numbers using the same strategy—filled ten-frames to the left and single digits in the ten-frame on the right (always based on his idea that a ten-frame is made up on two abutting five-frames), I pushed a bit further and introduced addition using ten-frames. The first task was to find the sum of 8 + 7, so I asked him to show me each digit on a separate ten-frame. His representations are on the left in Figure 11. When I asked him what actions he would take to show the addition, without hesitation, Xander said, "Well, you can make one of the ten-frames full like this (and he moved the three counters in the right ten-frame). You know the full ten-frame means one 'ten' and then there are five 'ones,' so you have 15 in total. Eight plus seven equals 15."

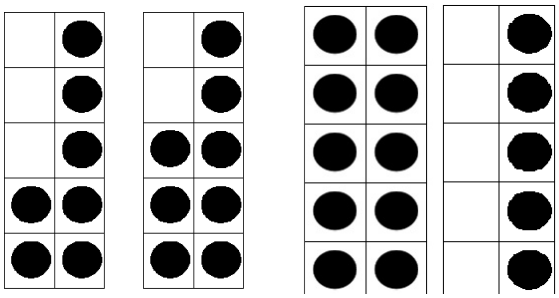


Figure 11: *To add 7 + 8, Xander rearranged the counters on the ten-frames, as shown on the left, to create the configuration on the right.*

When I said, "Xander, I have another question," I knew that our session was coming to an end. Patience now frayed, Xander looked at me and grumbled, "*Now* what don't you understand?" I promised him that there would be no more "asks" from me, if he could just explain one thing for me, and that was his reasoning for not filling the ten-frame on the right. I showed him what I meant. "Don't my two ten-frames mean the same as your two ten-frames? I just moved two counters to make 10." Please see Figure 12.

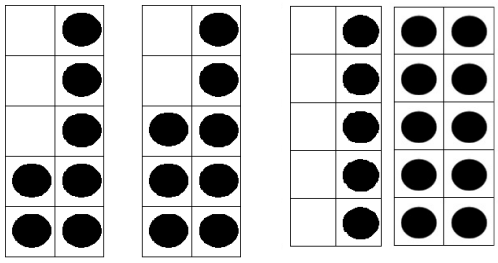


Figure 12: *I asked Xander why he did not rearrange two counters representing "8" and "7" in the configuration shown on the right by simply moving two counters.*

Xander looked at me in disbelief and said, "I already told you that's the wrong way to do it." As I waited for him to continue, I forced myself to remain silent while moving my finger back and forth between Xander's representation and mine and shaking my head as if I did not see the difference.

After a small huff, with eyes open as wide as possible, and a facial expression that screamed, "If I've told you once, I've told you a thousand times," Xander began his explanation. "Look here." Pointing to his configuration, he wrote "1" and "5" above the ten-frames, as shown in Figure 13. "You have to have the full ten-frame in the right place, so you know you can replace it with the '1' that stands for one '10.' Then you add the counters from the other ten-frame to get the total," said Xander, continuing, "but if you look at yours, it doesn't

help you get '15' because the ten-frames aren't in the right order. I *could* count by 5's to get the answer if I wanted to, but that's not the same as showing how you make the number. If I do write the numbers on yours, they're not in the right place. Get it?"

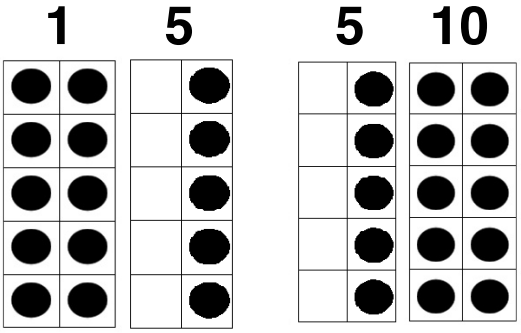


Figure 13: *Xander wrote the number "15," using its correct place-value convention, using the ten-frames on the left, and realized that the rule for representing two-digit numbers did not apply, if the ten-frames were filled in a way that did not reflect that convention.*

And with that, it was time for a snack—crispy, tart apples, freshly picked from the farm—and to call it a day for our mathematics lesson and reinforcement tasks. Xander understood many important concepts, and I was delighted that he did not default to the misguided (and oft-included in unvetted Internet sources) rule of "filling the one-frame that has the lowest number of empty spots," and chose to "make 10," using a system that reflected not only the physical appearance of the "1" in the tens place (i.e., his insistence on using vertical ten-frames), while also representing conventional place-value notation.

Textbooks and resources may suggest that there are hard-and-fast rules and standards for using ten-frames and may even go so far as to impose rigid strategies. Xander made short work of those artificial conventions—some of which were deeply rooted in my own practice as a mathematics educator. As my perspective on this ubiquitous manipulative changed by seeing it through Xander's eyes, it became clear that there is no "right way" to use this tool, and in fact, the typical horizontal configurations of ten-frames may indeed conflict with other models down the road, for example, base-10 blocks that use place-value mats that are labelled from left to right as 1000, 100, 10, and 1. Vertical stacks of units are used to make rods that represent 10. Vertically aligned rods are positioned in columns to make flats representing 100, and vertically piled flats are used to make cubes that represent 1000. Grid models used for other purposes, including standard bar graphs, also came to mind.

Xander found his own way to group the quantities, and reasoned out his own correct and logical strategies—ones that I came to respect and accept because they validated his uniquely constructed visual understanding—making mathematics make sense to him. And all I had to do was watch and learn. ▲

CODE N PLAY within MATH Curriculum

Coding will change how we teach and learn mathematics

Introduce CODING into some MATH lessons.
Try ready-to-use lessons within *FIRST STEPS IN CODING* ebooks.

Gr 1 to 4 Lessons
- Concrete stage (Code the KaiBot)
- Virtual stage with Free Block Coding



Gr 5 to 8 Lessons
- Advanced Block Coding with free coding tools

Grade 1 Sequential Events
Grade 2 Concurrent Events
Grade 3 Repeating Events
Grade 4 Nested Events

Grade 5 Conditional Statements
Grade 6 Solve Math with Code
Grade 7 Solve Math with Sub-Programs
Grade 8 Change Code affect efficiency/outcomes

Presented By:
Rudy Neufeld
Senior Author/CEO

NLS MATH AND CODING SINCE 1999

www.umathx.com

Visit http://tiny.cc/Examples-CodingInMath. Work with 3 Excellent Lessons in action.

Find Lessons & Supplies in our Shop: https://umathx.com/shop

Book a Free Thursday Webinar: https://forms.gle/3sP25Fieww2awKmi6

▲ Mb4T (Mathematics by and for Teachers): Partial Quotients: A Bridge to Understanding Division



SOFIA FERREYRO-MAZIERES
sferreyromazieres@wlu.ca

Dr. Sofia Ferreyro-Mazieres is an Assistant Professor in Mathematics Education at Wilfrid Laurier University in Brantford. Her work focuses on destreaming initiatives in mathematics, and supporting all learners. Sofia currently teaches Primary/Junior and Junior/Intermediate mathematics education courses to teacher candidates. She has also worked as an elementary and secondary mathematics teacher in the Simcoe Muskoka Catholic District School Board.

Students' first encounters with dividing multi-digit numbers provide some of the most informative moments for understanding mathematical thinking in the Junior classroom—not the neat, practised kind you see on worksheets, but the kind that arrives as a contextual math question: There are $168 to share equally among three friends. How much does each person get? In that moment, students rarely reach for a procedure. They focus on meaning. They begin to estimate, then share and group quantities, often using manipulatives or place-value reasoning to adjust and check their thinking. Some distribute tens first, others proceed in smaller increments of fives, and a few jump straight into a bold, efficient multiple because they trust their multiplication facts.

Consider the following example that uses partial quotients, which will be explained afterwards. *Mason, Kaer, and Maria made $168 shovelling snow. They share the money equally. How much does each person get?* Many students will begin by distributing the total in manageable amounts. Here is a "think-aloud" from one group who solved this question:

> *Let's give each person $40. That uses $120, which means we have $48 left. That works, so now we can give each person $10 more. That uses $30, so now we have $18 left. Okay so now let's give each person $5. That uses $15, so now we have $3 left. We can give each person $1 more. That uses $3 and now we have $0 left. This means that each person gets $56—hooray!*

As teachers, these early decisions are worth paying attention to. The size of the initial multiple (in this example, $40) that a student chooses often tells us more than the final answer ever could. Are they thinking in tens or hundreds? Are they cautious and incremental, or confident enough to take a larger step? These choices give us a window into students' comfort with place value, multiplication, and estimation long before they arrive to the correct answer.

This is precisely why partial quotients deserve a central place in how we teach multi-digit division. Partial quotients is a division strategy that offers an alternative to the traditional long division algorithm, by having students repeatedly remove convenient multiples of the divisor from the dividend, keeping track of how many groups have been accounted for until no further whole groups can be formed. Partial quotients are not a workaround for students who struggle with traditional long division. They are a way of making division visible. It honours what students already know about sharing, grouping, multiplication, and place value, and it provides a structure that helps those ideas scale up to larger numbers.

At its heart, partial quotients return to one central question: How many equal groups of the divisor can be removed at a time? Students work with convenient multiples of the divisor, keep track of how many groups they removed, and then add the partial quotients to find the total quotient. The work looks different from student to student, and that variation is not a problem. In fact, it is an instructional asset. When students choose different multiples, we can ask them to justify those choices. Those explanations often reveal how students are coordinating multiplication and subtraction, and whether they understand what each number in their work represents.

The sequence in the example above is powerful because it shows students monitoring the situation as they go. They are not following a script (as they would in the traditional long-division algorithm). They are responding to what remains. When we record this thinking using partial quotients, the structure captures that reasoning clearly:

> *Take 3 groups of $40 (3×40 = 120). Remainder 48. Take 3 groups of $10 (3×10 = 30). Remainder 18. Take 3 groups of $5 (3×5 = 15). Remainder 3. Take 3 groups of $1 (3×1 = 3). Remainder 0. Adding the partial quotients (40 + 10 + 5 + 1) gives 56, which clarifies that 3×$56 = $168.*

At this stage, what matters most is not efficiency. What matters is that students can explain why particular values were chosen and what it represents in the context of the question. Teachers can assess student understanding by asking questions such as, "Why did you switch from tens to

fives?" or "How did you know it was time to adjust the size of the groups you were using?" These questions help students reflect on their decision making. These conversations also help teachers assess whether students truly understand division as fair sharing, rather than as memorized procedures.

After students have experienced partial quotients in a familiar context, many are ready to see how the same reasoning can be recorded in a structured written form. This next example uses a step-by-step partial-quotients layout that makes the structure of the calculation clear, and allows students to track what has been accounted for and what remains. Consider the problem: *A bakery has 236 cookies that need to be packed into boxes of 4 cookies each. How many full boxes can be made?* This situation can be represented as 236 ÷ 4, and the students' first step would be to write the equation in a partial-quotients format, with 236 as the dividend and 4 as the divisor (see Figure 1).

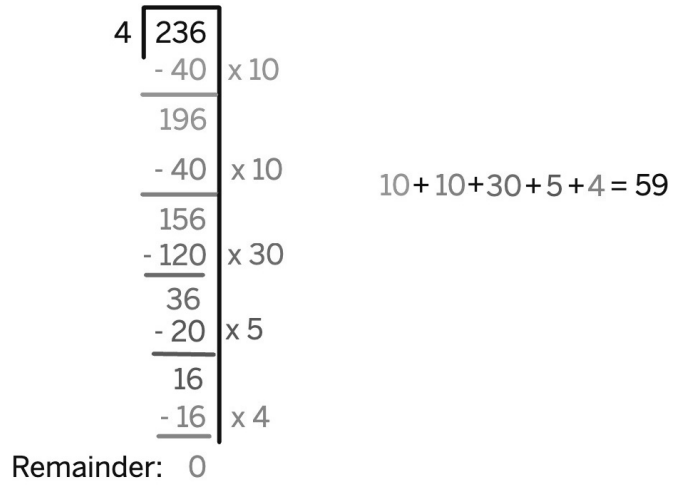


Figure 1: *Partial-quotients strategy, student 1 example*

Now it is time to start figuring out how many boxes of 4 will fit into 236. We want to use numbers that are easy to work with, so we begin with friendly numbers such as 10. We know that at least 10 groups of 4 will fit into 236, so we start there. We write ×10 on the right side to show that we are taking out 10 groups. Since 10 groups of 4 make 40, we subtract 40 from 236. This leaves us with 196. Now we need to figure out how many more groups of 4 we can make from 196. Again, we know that 10 groups of 4 will fit. We write ×10 to represent another 10 groups, and since 10 groups of 4 make 40, we subtract 40 from 196, leaving us with 156. At this point, we realize that we can make larger groups, so we try 30 groups of 4. We write ×30 to show that we are taking out 30 groups. Since 30 groups of 4 make 120, we subtract 120 from 156. This leaves us with 36. Now we need to figure out how many more groups of 4 can be made from 36. We know that 5 groups of 4 make 20. We write ×5, subtract 20 from 36, and are left with 16. Finally, we look at how many groups of 4 can be made from 16. We know that 4 groups of 4 make 16. We write ×4, subtract 16, and are left with 0. At this point, nothing remains, which tells us that all the cookies have been accounted for, using full boxes. If a non-zero amount had remained that was smaller than the divisor, it would indicate that no additional full boxes could be made and that the leftover cookies would form a remainder. To find the quotient, we add the partial quotients: 10 + 10 + 30 + 5 + 4 = 59. So, 236 ÷ 4 = 59, meaning that 59 full boxes of cookies can be packed.

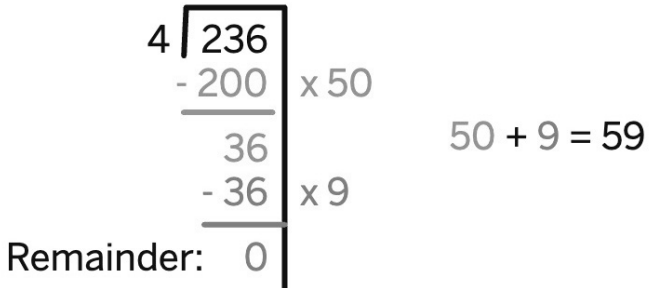


Figure 2: *Partial-quotients strategy, student 2 example*

Another student solving the same problem may be ready to work with larger values from the outset. They begin with writing the equation in a partial-quotients format (see Figure 2). They start by figuring out how many groups of 4 will fit into 236 and select a larger, convenient multiple. They choose 50, since they know that at least 50 groups of 4 will fit into 236. They write ×50 to show that they are taking out 50 groups. Since 50 groups of 4 make 200, they subtract 200 from 236, which leaves 36. Now they figure out how many more groups of 4 can be made from 36. They know that 9 groups of 4 make 36. They write out ×9, subtract 36, and are left with 0. To find the quotient, they add the partial quotients: 50 + 9 = 59. It is important to note here that what distinguishes this approach from the traditional long division algorithm is the choice and transparency of the thinking of each step. Rather than following a fixed sequence of digits and placements, the student is making deliberate decisions about how many groups can be taken out and why.

For many students, when transitioning from concrete models to written strategies, the *area model* provides an accessible entry point. Conceptually, it is doing the same work as partial quotients. Students remove friendly multiples off the divisor until nothing remains. The difference is that the thinking is organized spatially, which can make the structure easier to see. Let's consider the following situation: *A trail is 486 kilometres long and is divided equally into 3 sections for a relay event. How long is each section?* This situation can be represented mathematically as 486 ÷ 3.

Students begin by drawing a box. The dividend, 486, is written inside the box, and the divisor, 3, is written along the side (see Figure 3). The student might begin with a large, friendly number. Taking 100 groups of 3 uses 300, with 186 remaining. The remainder is carried into a new box, and it continues. Taking 50 groups of 3 uses 150, with 36 remaining. The 36 is carried into a third box (see Figure 4). Taking 12 groups of 3 uses 36, with 0 remaining. The values written on top of the boxes are then added: 100 + 50 + 12 = 162. Therefore, 486 km can be arranged into three sections of 162 km each.

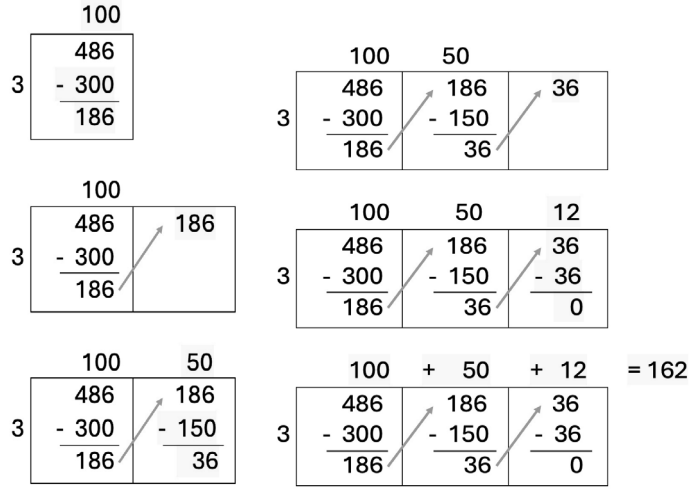


Figure 3: *Partial quotients, using the area model*

Figure 4: *Partial quotients, using the area model (cont'd.)*

The power of this representation lies in how it supports communication and explanation. Each box represents what remains to be shared. Each partial quotient can be justified with a multiplication fact. Teachers can ask, "Why did 100 make sense to start with?" or "How did you know 50 would still fit?" These questions keep the focus on reasoning, rather than layout.

This method is equally powerful when remainders appear because it helps students identify the point at which no further equal groups can be formed. Consider this: *A community garden has harvested 587 tomatoes. The tomatoes are packed into crates that hold 4 tomatoes each. How many full crates can be packed?* This situation can be represented mathematically as 587 ÷ 4. Using the same visual method as before, taking 100 groups of 4 uses 400, with a remainder of 187 (see Figure 5). Then taking 40 groups of 4 uses 160, with a remainder of 27. Then taking 6 groups of 4 uses 24, remainder 3. At this point, students decide to stop. The remainder 3 is smaller than the divisor, so no more whole groups can be formed. Adding the partial quotients gives 100 + 40 + 6 = 146, with a remainder of 3. This moment invites rich discussion. Asking, "Why do we stop here?" or "What does 3 represent in this situation?" or "What do we do with the leftover tomatoes?," helps students understand remainders as meaningful quantities, rather than leftover numbers to be ignored.

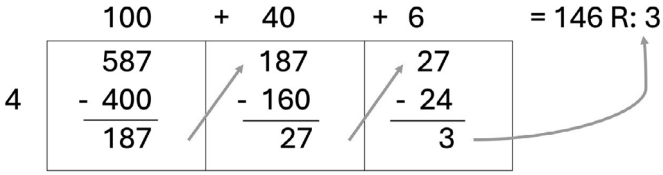


Figure 5: *One more example (with a remainder)*

With practice, partial quotients can support mental computation. Rather than relying on the traditional long-division algorithm that uses a fixed written layout to keep track of steps, students learn to organize their work through reasoning about multiples, remainders, and what has already been accounted for. The partial-quotients strategy supports flexible thinking, while preserving meaning, even as the numbers grow larger. Consider the division 824 ÷ 6. Here students continue to picture a situation in which 824 items are being grouped into sets of 6. Partial quotients allow students to hold on to that meaning while working efficiently. They may represent it as follows: 100 groups of 6 (100 × 6 = 600), remainder 224. 224 is 30 groups of 6 (30 × 6 = 180), remainder 44. 44 is 7 groups of 6 (7 × 6 = 42), remainder 2. At this point, the remainder, 2, is smaller than the divisor, so no more whole groups can be formed. The partial quotients are added: 100 + 30 + 7 = 137, remainder 2. What is important here is that each partial quotient represents a deliberate decision about how many groups to remove. The remainder is not an afterthought, and students can reason by explaining that "the remainder shows 2 items that cannot be shared equally" or that "2 are left over after making as many groups of 6 as possible." This example shows how using partial quotients can help students keep the logic of sharing and grouping intact as they work toward efficiency and abstract.

Whether students begin in a sharing context or move to recording their work through partial quotients, the underlying mathematics is the same. They are reasoning about how many groups fit, how much has been accounted for, and what remains. Partial quotients do not compete with the traditional algorithm—they explain it.

The goal is not fewer steps. The goal is intentional steps. When students can explain why a particular multiple was chosen, what each number represents, and why the process ended where it did, multi-digit division becomes more than a procedure. It becomes a way of thinking. In Junior grades, that way of thinking is what allows formal algorithms to take root with understanding, rather than memorization. ▲

▲ Play, Math, and Early Learning: Sorting



JENNIFER HOLM
jholm@wlu.ca

Jennifer Holm is an Associate Professor at Wilfrid Laurier University and works with pre-service teachers, as well as in the field supporting current mathematics teachers. She is interested in developing mathematics knowledge for teaching; focusing on the beliefs and opinions that pre-service teachers hold about mathematics and teaching and the connection they have to past experiences. She uses this research to support future teachers in developing beliefs and knowledge that will encourage and support effective teaching practices in mathematics. She is also the proud mom of two young learners, who are exploring the world of mathematics.

The focus of this column is noticing and encouraging sorting activities in early learners. Early learning frameworks include sorting as one of the skills for young learners (e.g., Ontario Ministry of Education [OME], 2007). While engaging in play, young learners will often organize items that have been presented to them. As a mathematics teacher, watching how the child organizes their toys can provide a look into their understanding of sorting and noticing attributes. We will examine how sorting provides a precursor for more complicated mathematics skills, and how we can both notice and encourage sorting behaviours with young learners.

Sorting is important for young learners because it provides a basis for more complicated mathematical understandings. Being able to determine attributes used for sorting allows children to use those attributes for patterning, and to begin to develop an understanding of how shapes fit into categories. In the new Kindergarten curriculum, the verb "sort" is in the Data Literacy and Probability strand: "sort objects or data using one attribute," and used within the Geometric and Spatial Reasoning strand with shapes (OME, 2026, A8.1 and A9.1). In the previous Kindergarten curriculum (OME, 2016), sorting is part of "investigation" in that children will "gather, compare, sort, classify, order, interpret, describe observable characteristics and properties, notice patterns, and draw conclusions, using a variety of simple tools and materials" (p. 23), as well as part of the mathematics behaviours of Kindergarten children (p. 75). In the elementary school curriculum, in Grades 1 to 3, sorting is specifically part of the Geometric Reasoning and Data strand, but an ability to describe attributes is also essential for stating pattern rules (OME, 2020). Observing and encouraging young learners to pay attention to attributes that they can use to organize or sort items provides a foundational understanding for these later mathematics concepts.

For preschool learners, encouraging learning through play is emphasized (OME, 2014). So, the ideal way to "teach" about sorting is to engage them where they currently are in their understandings. For young learners, when they are engaged in play, they are using sorting to experiment with which objects belong together in groupings, and this language can be helpful to encourage children to think about putting them into groups (or sorting). Allowing children time to experiment with their toys and play items can give them opportunities to determine attributes that they see in the world around them. These attributes, as we will discuss in this column, may not always be the ones that adults would see or encourage.



Figure 1: *Learning Resources sorting animals*

There are many early learning activities. For example, Learning Resources has created an "All Ready for Toddler Time" kit (Learning Resources, n.d.) that includes farm animals that children can sort (see Figure 1). There are five different animals and five different colours—but the pairs of the same animal are different colours. The instructions encourage toddlers to sort them by colour (where the two animals are different), or to sort them by animal (where the two colours would be different). Although not a "mathematical" noticing, working with items like this can also help identify if a child may be colour blind, by seeing if they are unable to differentiate between the different colours. The activity kit provides a helpful mat, with circles in the five colours (although the purple is a different shade), to encourage toddlers to match the colour with the circle. Watching this activity of "matching" could provide evidence of an inability to differentiate between colours, not just point to an inability to sort through the matching of colours. The idea is that these objects constrain the number of attributes (animal and colour) for the toddler to focus on when they are playing.

If you watch children with toys, though, it becomes clear that these are not necessarily the attributes that they will even notice or attend to when they sort. Before my son could name all the animals, he liked to group them by their ears (a favourite body part to name), and which had ones that stuck out from the heads of the animals. My two-year-old quite often lines up his toys on counters or shelves. I have watched and noticed that the lineup of the toys is identical whenever he pulls out the same toys. He also becomes quite frustrated if I try to move one of the toys to a new spot in the line, or if I help him put the toys on the line and do not do it in his correct order. He has a schema that he is using to determine how the toys need to be organized. If one looks carefully, he is clearly grouping the toys by an attribute that he has noticed when he examines his toys.

Consider the picture of dinosaurs in Figure 2, which is a favourite collection of his to sort: how might they be sorted?



Figure 2: *Dinosaur toys*

The first considerations might be by colour, size, or dinosaur type. When you give the dinosaurs to a toddler, you may see them sort in this way, but you also may see something different. The dinosaurs could be grouped according to which have two legs on the ground, and the ones with four legs. Some of the dinosaurs make sounds when you close the mouth of the dinosaur, so they could be grouped by the type of roar, and then the ones who do not make a sound at all. Texture could also play a role in sorting the dinosaurs because they clearly have different levels of "bumpiness" that could be used to determine an order, and which ones belong together.

It is important to let children explore and lead by their own understanding of which attributes can help them decide how the toys or items belong together. If we impose our own schemas onto children, we are missing out on learning about their understandings and could be discouraging their own creativity with the play. This parallels the importance of not imposing a framework on how elementary students view patterns or push a schema for determining algebra rules (see Kajander, 2014, 2015 for alternative considerations). If we are going to impose a framework or way for children to see the world, we might as well just teach them rules and formulas (see this discussed more in Holm, 2018) instead of how to think.

Children intuitively have a mathematical understanding of the world, and it is our role as teachers to acknowledge it and build from where the children are. With early learners, it is especially important that we learn to recognize moments of mathematical significance and consider how to move them forward. For example, as a child is putting dinosaurs in a row, and they seem to have a way of organizing them, you may be noticing sorting or at least the beginnings of determining an organizational structure by the child. In my son's case, the original order was one that I just could not completely figure out, but he was clearly using something in his head because he would move the dinosaurs to organize the newly presented dinosaur into a specific place. As near I could tell, colour played a part, but so did the "type" of dinosaur (e.g., the red and black one is from *Marvel's Spidey and His Amazing Friends*, so the other ones from the show would be one "type" of dinosaur). Once he started communicating clearly, he used words like "roar" or "bumpy" to describe the dinosaurs, and his organization of the dinosaurs had evolved (although the "Spiderman" dinosaurs are always their own special group).

Noticing is the start. Next, we need to gently ask questions and engage with the play to see the mathematical power in their activity. You could ask them to tell you about the dinosaurs and then listen to how they describe them: Are you seeing them use language about features of the dinosaurs (regardless of how sophisticated the terms)? Are they using the same structure each time they put the toys on the shelf and seem to be reaching for a particular toy, not just the one that is closest? If you hand them a toy, do they replace it with the one that "fits" next in their organization? One note is that, as a child ages, how they organize their toys may become more sophisticated (as I have observed with my son), or they may start to assess that certain toys belong in a different group, even when there are no other matching items. Another line of conversation you may have is to introduce another toy and see what the child does with it. If they have a line of dinosaurs and you hand them a dinosaur they have not seen before, what do they do? If they simply stick it at the end, they may not be sorting or using an organization, but if they change around the order to put the new one in a specific spot, they are probably using a schema to determine where it goes. To prompt

thinking about it, you might ask, "Where does this dinosaur belong in your line?" Since not all preschoolers are verbal, watching and observing their play is the best indication of what they are doing mathematically. Think about asking questions that are action based, rather than asking them to verbalize their thoughts. Show me, rather than tell me.

Although the tendency may be to always provide items, like the picture in Figure 1, that have a minimal number of attributes, you may be missing out on a wider understanding of how the child sees the world. This may be an interesting starting point, but I would recommend using different items that allow for more options, moving past it as the child develops, or even with the toys in Figure 1 changing the task by not providing the colour mat, which obliges a particular grouping. Using the items in Figure 1 may provide an idea that items can be grouped, but try to encourage different types of items. Even with the toys in Figure 1, there are many other ways to sort, not just by the type of animal or colour, if you have some imagination (as my son did with noticing the "ears"). Putting them in the hands of a toddler and asking them to show you how they might decide which ones belong together can give you an insight into what they are seeing.

In the end, early learners have an exciting view of the world, and it may be enlightening what they notice about the things around them. These early understandings hold mathematical potential, if we are carefully observing, and asking questions, or encouraging thinking. Now, it is important to note that children will not always sort or organize their items: sometimes the line is just a way of seeing their toys, or the purpose of putting this together could be for a different reason, such as being able to knock items over to hear the sound or to cause destruction. It is also important to note that not all children will sort or organize items; they may just be interested in imaginative or some other type of play. This could be where bringing in something like the items from Figure 1 and the colour mat may be helpful: you are helping children see that items can be grouped. Or engage in activities with items in their play area to see "where does this belong" (e.g., the spoons go in the kitchen play area, but the dinosaurs do not). This will give some starting points to talk about sorting as putting things where they belong, and also provide language as you are playing, that allows for noticing attributes (such as, "this item is really soft, but this one is rough"). Sorting is one of the many ways children can show their early understandings of mathematical concepts: we just need to be open to the way they see the world around them.

Continued on page 22

▲ Meaning Making in the Classroom: Understanding How Question Focus and Format Influences Student Understanding



DAVID COSTELLO
david@costellomath.com

*David is a principal, who taught and had various board roles in Prince Edward Island. David has also instructed university courses regarding curriculum, differentiation, mathematics, and literacy. He facilitates professional learning in mathematics and school development. Dr. David Costello has published four books, with two of his most recent books being: W*hat's Missing In Math Instruction: Test Preparation and Classroom Strategies That Show Students How to Apply What They Know in Meaningful Ways, *and* Messing Around with Math: Ready-to-Use Problems That Engage Students in a Better Understanding of Key Math Concepts.

The questions assigned to students in mathematics play a significant role in their learning. The more variety of questions students are given, the more likely they are to develop a broader understanding of the mathematical concepts.

Teachers should be aware of both the focus and format of questions assigned in the classroom (Costello, 2026). Focus is the category of thinking attached to the question. There are three categories considered in regard to question focus: Knowledge and Understanding, Application, and Thinking (EQAO, 2025; Ontario Ministry of Education, 2010, 2020). Students should experience a range of question focus so that they can deepen their conceptual understanding as they approach it from differing perspectives.

Question format refers to the structure of the question. Formats can range between closed-ended to open-ended. If students are continuously assigned the same format, their capacity to answer questions of differing formats may be hampered. Students need to be assigned various question formats so that they can better understand what is entailed in specific question formats, and, in return, can strengthen their ability to answer different question formats.

Questions are the entry point for students to apply their mathematical insight. When students are consistently assigned questions with similar focus and formats, they

may struggle with showing their mathematical understanding when presented with novel question foci and formats later in the course or in subsequent courses with other teachers. The more students engage with varied questions that cover all the foci and have a variety of formats, the more they strengthen their learning and their ability to demonstrate what they know.

In this column, I will share an approach that all math teachers can apply when assigning questions. When applied to instruction, teachers will see how students strengthen their grasp of math concepts and can become independent when engaging with a variety of questions that cover various foci.

Question Focus

The three cognitive levels used to categorize question focus are Knowledge and Understanding, Application, and Thinking questions. These three levels are rooted in the cognitive domain categories of Bloom's Taxonomy (Adams, 2015). Knowledge and Understanding questions are based on the lower two levels of Bloom's Taxonomy: remember and understand. Application questions relate to the apply level of Bloom's Taxonomy. Thinking questions correspond to the three highest levels within Bloom's Taxonomy: analyze, evaluate, and create. Students deepen their understanding of the concept and their ability to navigate different situations when they engage with questions representative of different focus areas.

The following table highlights a brief overview of each question focus (Costello, 2026).

Question Focus	Overview
Knowledge and Understanding	• Knowledge and Understanding questions target the student's ability to recall information, comprehend concepts, and apply such knowledge in direct situations. These questions require students to recall, recognize, solve, order, and measure. • These questions tend to be closed-ended.
Application	• Application questions target the student's ability to apply mathematical concepts to real-world situations, and require students to represent, model, determine, and implement. • These questions tend to be across the spectrum of the closed-ended to open-ended continuum.
Thinking	• Thinking questions target the student's critical-thinking and problem-solving skills, and require students to analyze, synthesize, draw conclusions, evaluate, and justify. • These questions tend to frequently use problem solving, entailing choice of processes or open-ended.

Question Format

The structure of the question can be referred to as question format. In addition to the cognitive demand of questions, the teacher must be aware of the format of questions they assign. Relying too heavily on one format may not be advantageous for student learning. When students encounter the same question format over time, their understanding of the concept may be limited as they are engaging it from similar perspectives and stances (Costello, 2026). Altering the format of questions will broaden student mathematical understanding by having them engage with concepts from different perspectives.

The following table highlights a brief overview of five question formats that can be used to broaden student understanding and increasing the problem-solving experiences of students (Costello, 2026).

Question Format	Overview
True or False	• Students are provided a statement to evaluate as true or false.
Multiple Choice	• Students are provided a stem with alternative responses. One of these responses is correct, with the remaining being incorrect.
Procedural	• Students are provided a prompt that requires them to perform an operation or calculation.
Word Problem	• Students are provided a contextual problem that requires them to understand the context, and then select and apply a concept to reach the solution.
Problem Solving	• Students are provided a novel situation that requires them to understand the situation and develop an approach to reach a possible solution.

Final Thoughts

The questions that students encounter are the gateway to mathematical understanding. It is what triggers curiosity, it is what motivates them to persevere through stumbling blocks, and it is a window into how concepts can be applied to reach solutions. Teachers must be cognizant of the importance of questions being assigned to students.

When considered in unison, the question focus and format allows teachers to address mathematical concepts from multiple perspectives (Costello, 2026). Adjusting the focus supports teachers in adding layers of meaning to the concept as students engage it from multiple perspectives. Question format allows teachers to assign different questions, ones that are representative of the spectrum of closed- to open-ended, that connect to the intended focus. By changing the format of questions they assign to students, students are provided a range of opportunities to engage

the concept.

I look forward to continuing the discussion of mathematical learning and instruction via my website (www.costellomath.com), email (david@costellomath.com), or X (@dr_costello).

References

Adams, N.E. (2015). Bloom's taxonomy of cognitive learning objectives. *Journal of the Medical Library Association*, *103*(3), 152. doi.org/10.3163/1536-5050.103.3.010

Costello, D. (2026). *What's missing in math instruction: Test preparation and classroom strategies that show students how to apply what they know in meaningful ways.* Pembroke Publishers Ltd.

Education Quality and Accountability Office (EQAO). (2025). *Released questions with provincial data: Mathematics.* Author. King's Printer for Ontario. www.eqao.com/wp-content/uploads/2025/11/math-resource-released-questions-g9-25.pdf

Ontario Ministry of Education. (2010). *Growing success: Assessment, evaluation, and reporting in Ontario schools (1st ed., covering Grades 1 to 12).* www.edu.gov.on.ca/eng/policyfunding/growSuccess.pdf

Ontario Ministry of Education. (2020). *Grades 1–8 mathematics curriculum.* Queen's Printer for Ontario. ▲

Continued from page 20

References

Holm, J. (2018). Mb4T (Mathematics by and for teachers): Examining the use of manipulatives and models for understanding. *Ontario Mathematics Gazette, 57*(1), 31–35.

Kajander, A. (2014). Mb4T (Mathematics by and for teachers): Visual patterns—an opportunity for growth in algebraic understanding. *Ontario Mathematics Gazette, 53*(2), 21–24.

Kajander, A. (2015). Mb4T (Mathematics by and for teachers): The importance of connecting patterns to multiple algebraic descriptions. *Ontario Mathematics Gazette, 53*(3), 20–23.

Learning Resources. (n.d.). *All ready for toddler time.* Author.

Ontario Ministry of Education. (2007). Early learning for every child today: A framework for Ontario early childhood settings. Best Start Expert Panel on Early Learning. www.betterbeginningssudbury.ca/wp-content/uploads/2015/09/continuum.pdf

Ontario Ministry of Education. (2014). How does learning happen? Ontario's Pedagogy for the early years. Queen's Printer for Ontario. https://files.ontario.ca/edu-how-does-learning-happen-en-2021-03-23.pdf

Ontario Ministry of Education. (2016). The Kindergarten program. Queen's Printer for Ontario. https://files.ontario.ca/books/edu_the_kindergarten_program_english_aoda_web_oct7.pdf

Ontario Ministry of Education. (2020). Mathematics. King's Printer for Ontario. www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics

Ontario Ministry of Education. (2026). Kindergarten curriculum. King's Printer for Ontario. www.dcp.edu.gov.on.ca/en/curriculum/kindergarten/grades/jk-sk/home ▲

▲ In the Middle: Making More of Financial Literacy



CARLY ZINIUK
carlyziniuk@gmail.com

Carly Ziniuk teaches Grade 12 Data Management and Advanced Placement Statistics at the Bishop Strachan School in Toronto. She is active in adopting real-life data to engage her students in solving problems. Among 30 global STEM leaders, Carly was part of 2024 NASA Space Apps Collective. As a Science Communicator, she is working to connect astronomy and mathematics for learners of all ages across Canada.

In the December 2025 issue, I highlighted some financial literacy resources for Grades 6–10. This column continues that effort with further adaptations of activities from the CEMC (The Centre for Education in Mathematics and Computing) workshop that I led with Pauline Martin this summer.

Pauline and I encouraged the teachers to consider ways to address financial literacy expectations as an application throughout the classroom learning, rather than in a single block of content. The examples provided allow you to explore how you might integrate these concepts into your classes now.

The curricular expectations centre decision making and perspective taking throughout (a checklist summary of the Ontario Financial Literacy, Grades 1–9 curriculum, can be reviewed at bit.ly/4oDc49W). Considering the impact of tips from different perspectives adds a small expansion to a classic percentage application. I prompt my students in trios to take on the different roles of patron, server, and employer with respect to tipping.

1. Discuss how tipping differs for the people involved:

	Advantages of Tipping	Disadvantages of Tipping
Patron		
Server		
Employer		

Table 1: *Perspective taking on tipping*

It is important to note that tipping, like many components of financial discussions, is a personal decision; being aware, sensitive, and flexible to different cultural practices and decisions is recommended. My students have been keen to discuss how online services like Uber and Door Dash collect and distribute tips. Social media is a source of many rants about the lack of tip transparency on many service apps, and how awareness of mathematical, media, and technical literacy can save you a significant amount of money. For example, Laura Brehaut's *National Post*, June 26, 2025, Toronto lifestyle piece, "*These groceries cost 45% more from Uber Eats than in-store,"* was inspired by Marie-Eve Fournier's June 19, 2025 *La Presse* reporting in Montreal, "*The grocery bill increases from $38 to $82."* Both articles indicate the author's specific orders and the overall cost increases with service fees and tip, along with taxes, beyond what is charged for in-person purchases.

2. Complete the following table, explaining the procedure you would use to calculate a tip at each of the rates. If the total bill you are considering is D dollars, explain the reasoning you would use to calculate the tip at each of the rates, and provide an algebraic expression.

	10%	15%	20%
Procedure for Calculating the Tip	Divide by ten		
Algebraic Expression for the Tip	$\frac{D}{10}$		
Algebraic Expression for the Total Bill	$D+\frac{D}{10}=\frac{11D}{10}$		

Table 2: *Expressions for different tip values*

Application problems that can be completed using code and algebra, with or without technology, can also help students expand their financial literacy. This is a financial literacy expectation that is repeated throughout middle school. Having students create and adapt algorithmic procedures can address this, while also addressing algebraic expectations.

3. Review the following comments for certain tipping rates for a bill of $41.40. Determine the approximate rate for which the patron is choosing to tip.

If the bill comes to $41.40, I divide it by 10 to get $4.14 and then add half of $4.14 or $2.07 to get a total tip of $6.21.	I just make the tip one dollar for every $10. So, for a $41.40 bill, I tip $4.
I take about one-sixth of the bill. I round $41.40 up to $42 and take 1/6th of that, which is $7.00.	I double the bill and divide by 10. So, for $41.40, first I get $82.80 and then divide by ten, getting $8.28.

Table 3: *Exploring different strategies for calculating tip*

Being able to estimate the tip and total can help students plan appropriately for their expenditures. Here are some problems that can be tackled with algebraic approaches and code.

4. After deciding to tip 15% at a restaurant on a $24.75 (after-tax) bill, the customer leaves three tens (i.e., $10 bills). Is this a reasonable amount? Is it more or less than 15%? What percentage tip is it?

Another option is to present a completed Scratch or Python code program that calculates tip or tax (see Figure 1), and ask students to explain the coded steps, without actually creating code. Andrew Ruston, a colleague, with whom I worked extensively when implementing the destreamed Grade 9 math for the first time in 2020, suggested seeing code as an alternate representation embedded in algorithmic expectations.

After modelling a 15% tip, we asked students to follow and explain the algorithms represented in the code of the programs we provided, and then in small groups, discuss how the same code would generate a 15% tip. In our classes, students first complete these tip calculations without coding, then add the coding option to highlight this as an algorithmic representation. I have found it efficient in a classroom to have students work on paper before ever touching their device.

Andrew and I subsequently asked students to adjust the program in the following ways.

5. Change the program so that it gives two outputs: the value of the tip and the total to be paid (tip + bill).
6. Change the program to ask the user what tip percentage to use, then calculate the tip and total.
7. Change the program so that it asks the user to rate the quality of the restaurant's service from 1–5 (1 being low, 5 being high), and then it recommends a tip based on that rating and calculates it.

I recommend discussing commutative mathematics with students about these problems. Commutativity is a property math students understand. Jérôme Proulx has been discussing Commutative Percentages in his December 2024 column, and I encourage you to check out his entire

"Musing on Percentages" series, starting in September 2024. As Proulx discusses in his June 2025 column, both income and sales taxes practices in Canada require students to understand percentages as part of future budget planning.

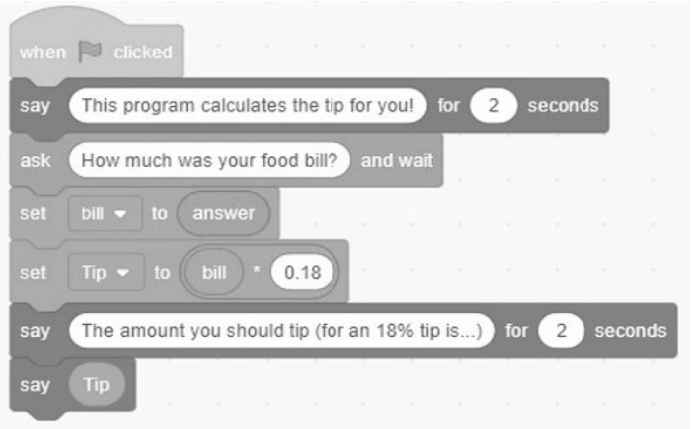


Figure 1: *Example of Scratch code for calculating a tip*

The two problems here originated with similar problems I used as oral starters in my classroom, using Building Thinking Classrooms, and then created as slides for the CEMC Conference. These examples show students how calculating tax, tip, and discount ahead of time can be part of budgeting for significant purchases.

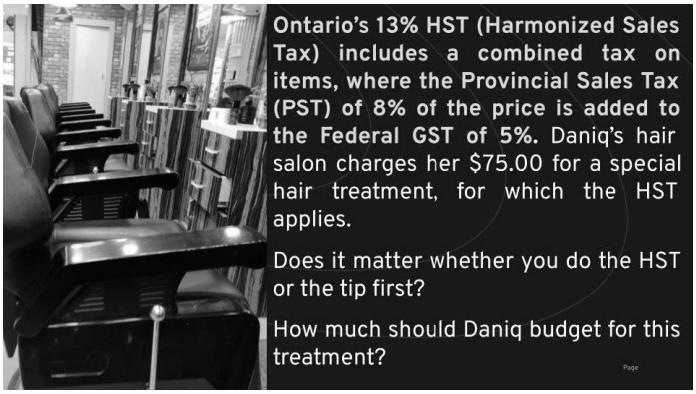


Figure 2: *Daniq's hair treatment, a sample financial starter problem for commutativity and percentages*

Other questions your students might be curious to ask are:

8. What is the total tax Daniq will need to pay?
9. Assuming Daniq is happy with her salon's service, what would be the total bill, given tax and tip? How much more would it cost than if she just doubled the HST fee as the tip?
10. Break down Daniq's overall bill into the following:
    - How much of the final bill goes to the salon?
    - How much goes to the Provincial government?
    - How much goes to the Federal government?
    - Does that change based on whether you calculate the tip on the original price or the total bill after taxes?
11. For her sister's birthday, Daniq decides to include the purchase of a gift card for the hair salon when she pays her overall bill. Why should Daniq be careful not to include the tax or tip on the gift card cost, and instead, only on her hair treatment?

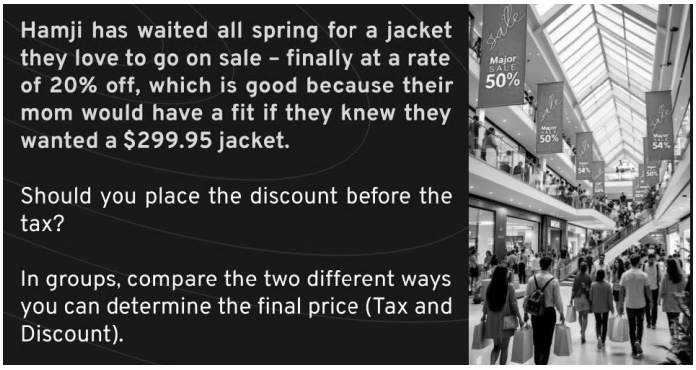


Figure 3: *Hamji's jacket, an additional problem expanding upon the percent and commutativity learning*

12. Does Hamji save money by having the discount applied before the tax? Explain your answer.
13. How much does Hamji save in the end?
14. Break down Hamji's overall bill into the following:
    - How much of Hamji's final bill goes to the store?
    - How much goes to the Provincial government?
    - How much goes to the Federal government?
    - Does the breakdown change based on whether you calculate the discount on the original price or the total bill after taxes?

One strategy I have used to incorporate financial literacy throughout the curriculum is to introduce financial vocabulary, including words like *appreciation*, *depreciation*, and *inflation*, while exploring mathematical topics such as slope. In my December 2025 column, I demonstrated how to use some of the content available from the Royal Canadian Mint to introduce these terms. The Bank of Canada Museum also provides a series of curriculum-linked resources to teach financial literacy from elementary to high school. Their activity to explain inflation, "If I had a million dollars… I'd be reasonably well off," was the inspiration, along with the 1990 Barenaked Ladies song, for an activity I made for CEMC, using Amplify's (formerly Desmos) Activity Builder.

The data here comes from that activity. In case you are unfamiliar with this Canadian classic, the song is written tongue-in-cheek about the different ways the songwriter would spend one million dollars in order to garner love. The

2020 Bank of Canada activity and my 2025 adaptation using Activity Builder do include a level of innuendo, and you may want to consider whether this activity is appropriate as written for your student. You can find the Student version at bit.ly/CEMCFinanceDesmosStudent, and the Teacher version, so you could edit or remove elements of it to fit your classroom, at bit.ly/CEMCFinanceDesmosTeacher.

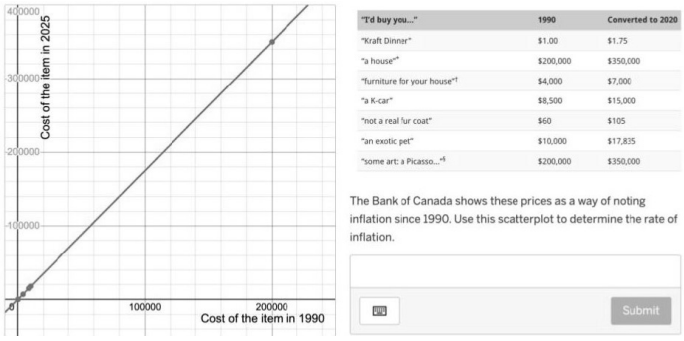


Figure 4: *Costs, comparing 1990 and 2020, of the items in "If I had a million dollars," using Amplify Activity Builder*

Students can explore the data and the scatter plot together, and the connection to the rate of inflation for the items in the song from 1990 to 2020 (Figure 4). Using different graphical representations in all areas of the curriculum has helped my students become more skilled at appropriately interpreting visuals for meaning, including financial literacy.

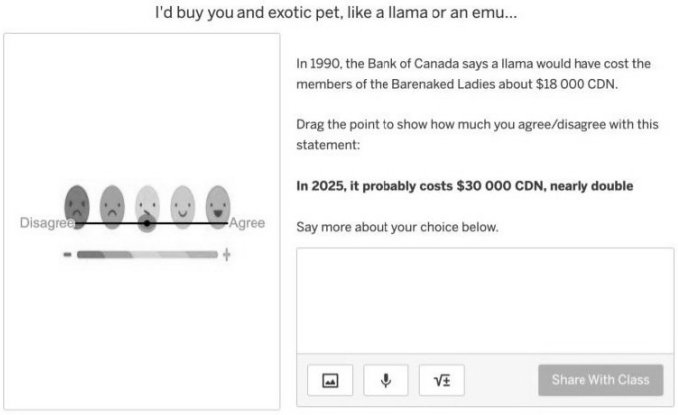


Figure 5: *Activity Builder slide showing "Share with Class" option to compare conclusion about inflation rate*

Amplify's "Share with Class" option, as shown in Figure 5, can be used for peer reflection on their slope calculation and using this moveable Likert scale. Using another scatter plot, illustrating changing house prices in Canada from 1990 to 2025 (Figure 6), students can explore slope in yet another way to explain inflation. This allows teachers to begin to discuss overall expectations around "Identify a past or current financial situation and explain how it can inform financial decisions, by applying an understanding of the context of the situation and related mathematical knowledge."

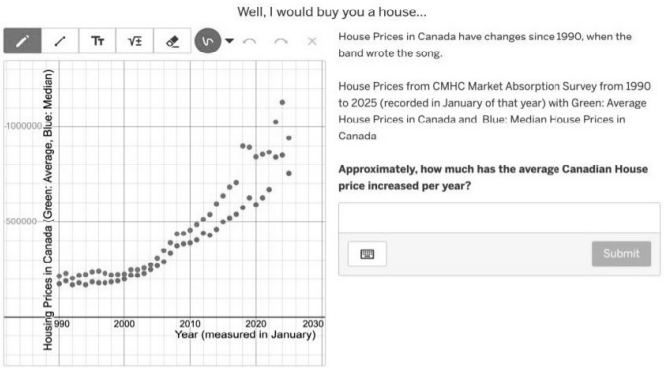


Figure 6: *Canadian house prices from 1990 to 2020 through Amplify Activity Builder with auto-checking, using Computation Layer*

An additional area which can slide into various other instructional topics is the expectation to "describe some advantages and disadvantages of various methods of payment that can be used when dealing with multiple currencies and exchange rates." This green dress problem, again referring to an iconic part of the song, has students explore whether a dress purchased in a local store in Canadian dollars with tax is really more expensive than the same dress purchased from the United States online site, given shipping and handling, international surcharges, and exchange rates (the one given here is from August 2025).

Notice how this problem addresses tax and exchange, while also having the students employ mathematical decision making. My students found this relatable, since they recognize the challenge that comes from finding something you want online, yet needing to determine carefully how much it will actually cost in the end. Using the Amplify Activity Builder activity can have auto-checking for this problem as well.

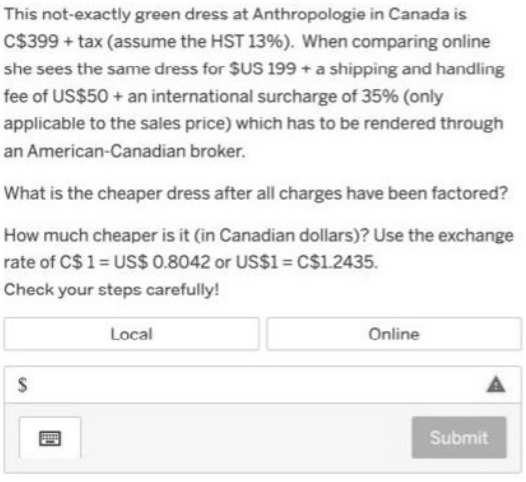


Figure 7: *Exchange example using auto-checking through Computation Layer*

If you have some time to explore that Activity Builder file, it will preview some of the other features and financial literacy topics I will be exploring in upcoming pieces. Preparing students for the Financial Literacy graduation requirement test will be more easily done contextually, and I am finding students responding positively to integrating this content throughout all areas. Spiralling throughout the middle school math content will help students approach these areas of budgeting, planning with mathematical thinking, and problem solving together, beyond just preparing for a single test. I hope these allow you to explore and experiment, both with the tool and the financial literacy content in 2026.

Resources

Brehaut, L. (2025, June 26). These groceries cost 45% mo re from Uber Eats than in-store. *National Post.* www.nationalpost.com/life/food/uber-eats-grocery-order-cost

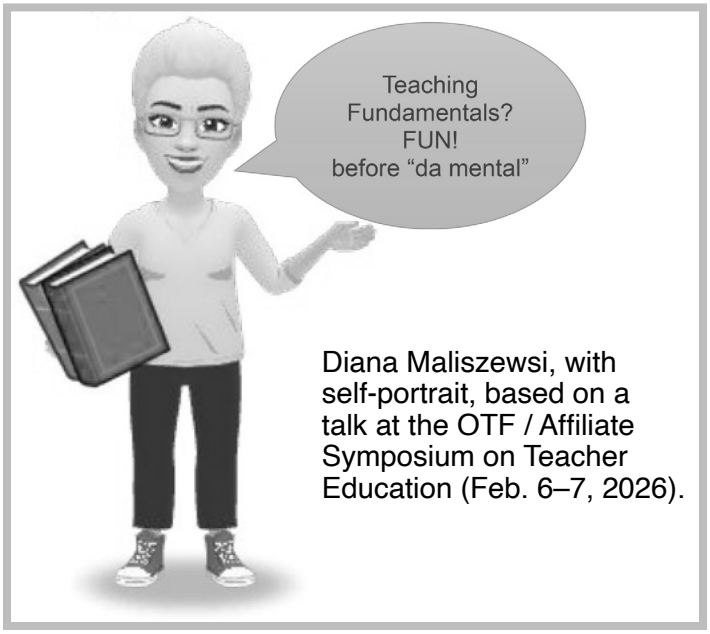
Canadian Mortgage and Housing Corporation (CMHC). (2025). *Market absorption statistics by Canada, provinces, CMAs, and large CAs.* Author. www.cmhc-schl.gc.ca/professionals/housing-markets-data-and-research/housing-data/data-tables/housing-market-data/market-absorption-statistics-canada-provinces-cmas-large-cas

Fournier, M. (2025, June 19). The grocery bill increases from $38 to $82. *La Press.* www.lapresse.ca/affaires/chroniques/ 2025-06-19/uber-eats/la-facture-d-epicerie-passe-de-38-a-82.php

Iddon, G. (2020, October 5). If I had a million dollars… I'd be reasonably well off. *Bank of Canada Museum*. www.bankofcanadamuseum.ca/2020/10/if-i-had-a-million-dollarsid-be-upper-middle-class/

Ontario Ministry of Education. (2020). *Mathematics (elementary)*. Ontario Ministry of Education's Designated Curriculum and Program (DCP). King's Printer for Ontario. www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics

Ontario Ministry of Education. (2025). *Mathematics (secondary)*. Ontario Ministry of Education's Designated Curriculum and Program (DCP). King's Printer for Ontario. www.dcp.edu.gov.on.ca/en/curriculum/secondary-mathematics ▲



Diana Maliszewsi, with self-portrait, based on a talk at the OTF / Affiliate Symposium on Teacher Education (Feb. 6–7, 2026).

▲ What's the Problem? Amazing Arithmetic



SHAWN GODIN
GodinMathStuff@gmail.com

Shawn is a former mathematics teacher, department head, and consultant. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: presenting workshops, writing articles, working on local projects, and helping create mathematics contests.

Welcome back, problem solvers. Some of the best problems are the easiest ones to state. Seemingly simple on the surface, there is a sea of richness to be discovered for those who take the time to dig deeper. The problem that we will examine in this issue falls into this category. This problem, and a large collection of others, can be found in an article by Ed Barbeau of the University of Toronto—on Ed's website, www.math.toronto.edu/barbeau/home.html, under "Talks, articles and compilations"—titled "A collection of problems and investigations."

Problem 1. *Using each of the eight digits* 1, 2, 3, 4, 5, 6, 7, 8 *exactly once, create two numbers whose difference is as small as possible.*

The first thing that comes to mind is semantics. Calculating 5678 – 1234 = 4444, while 1234 – 5678 = -4444, which is smaller. Hence, we obviously want to make the answer negative and "as large" as possible. I suspect this is not the intention of the problem. However, rather than discouraging it, I suggest you leave it and deal with this issue if any students bring it up. Generally, when people talk about the difference between two numbers, they are talking about the *absolute* difference between them, that is, the positive difference obtained by subtracting them in any order and taking the absolute value of the result. In this sense, the difference between 3 and 10 and the difference between 10 and 3 are both 7. If the negative case comes up, it is worth the time discussing the terminology, and then directing students to the intended problem.

This is the type of problem that can be used at many different grade and ability levels, with different expectations of the students. You could allow students to use calculators to help them in their search. You could disallow them calculators, compelling them to think about some properties

of numbers and operations to aid them in their solution. You may decide that this is a good place to give students a chance to practise coding.

Let's start at the lowest possible level: calculating without (at first) thinking. We will also look at a simpler problem, that is, using the four digits 1, 2, 3, 4, instead of the original list. This is a good problem-solving technique for students: investigate an easier, related problem, and use any insights gained from its solution to tackle the original problem.

In the simplified problem, we could either be dealing with a three-digit number paired with a single-digit number or with two two-digit numbers. We can solve this simpler problem by working out every possibility.

Trial and error, although not the ideal method of solving a problem, is still a valuable tool. In some cases, starting with trial and error can lead to insights that point in the direction of an improved strategy. However, to do this effectively, one must be systematic in one's search; otherwise, cases, or patterns, may be missed.

The numbers for this case are small enough that students could, conceivably, do an *exhaustive search*, checking every possible three-digit/one-digit pair ($4! = 4 \times 3 \times 2 \times 1 = 24$ cases) and every two-digit/two-digit pair ($4! \div 2 = 12$ cases). Exhaustive search or not, one thing should jump out early in the trial-and-error process. *All* three-digit/one-digit pairs have differences that are larger than *all* the two-digit/two-digit pairs. If we think about what subtraction is doing, it is finding out how far apart two numbers are. *All* two-digit numbers are between three-digit numbers and one-digit numbers. As such, the two-digit/two-digit pairs are going to be where our solution lies. This tells us that for the original problem, we should be looking at pairs of four-digit numbers.

We can picture this idea visually by looking at a number line. In Figure 1, we see that if we pick any three-digit/one-digit pair, their difference will be larger than the difference between the furthest separated two-digit/two-digit pair.

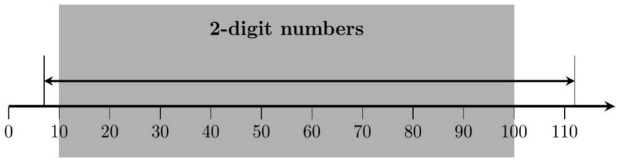


Figure 1: *Visualizing our problem on a number line*

Another thing we should discover is that the results are better when the tens digits of our two numbers are consecutive numbers. This also makes sense, as we are dealing with two-digit numbers, and the tens digits will contribute the most to the result. Since we want the difference as small as possible, we want the difference between the tens digits as small as possible, which can be obtained if they are consecutive numbers.

Using this insight, we can cut our exhaustive list down to six possibilities, shown below. We could do these computations by hand or visualize them on a number line.

$$42 - 31 = 11 \quad 41 - 32 = 9$$
$$34 - 21 = 13 \quad 31 - 24 = 7$$
$$24 - 13 = 11 \quad 23 - 14 = 9$$

Our insights cut our work down quite a bit. For the original problem, our two numbers could have been: seven-digit/one-digit, six-digit/two-digit, five-digit/three-digit, or four-digit/four-digit. The first three cases have $8! \div 2 = 40\,320$ possibilities, while the last has $8! \div 2 = 20\,160$. So, our first insight from the simplified problem, that we only need to check four-digit/four-digit pairs, reduces our systematic checking in the original problem from 141 120 cases to 20 160. The second insight, that the digits in the largest place values must be consecutive, reduces it further. There are only seven possible consecutive pairs for the first digits, which means that there are a mere $7 \times 6! = 5040$ cases to check. That is, we *only* need to check one twenty-eighth of the cases, woo-hoo! However, rather than rushing off to try our 5040 cases, let's keep exploring.

We'll move up from our first attempt at simplifying the problem to something a bit more complex, using the digits 1, 2, 3, 4, 5, 6. Even though we are almost back to the original problem, in this case, we would "only" have $5 \times 4! = 120$ cases to check. However, instead of checking cases, let's try something else.

Imagine we have any pair of three-digit numbers made using the digits 1, 2, 3, 4, 5, 6 each once, with consecutive hundreds digits, and we found the difference which isn't minimal. Could we swap two digits to get a better solution? Pick two random three-digit numbers, with consecutive hundreds digits, and give it a try before you read on. What can you discover?

Imagine we start with 562 and 413, with a difference of 149. Keeping the hundreds digits and swapping the tens and units digits in the first number yields $526 - 413 = 113$, an improvement. Similarly, if we had swapped the tens and units digits in the second number, we get $562 - 431 = 131$, better than the original, but not as good as our first swap. However, if we do both swaps, we get $526 - 431 = 95$, our best result yet.

What insights can we get from these results? It seems that when we made the larger number smaller, we improved our result; that is, the numbers are closer together. Similarly, when we made the smaller number larger, we again improved our results. As we are trying to make the difference as small as possible, we want to make the numbers as close

together as possible. Hence, since we are using 4 and 5 as our hundreds digits, we want to make the number 5XX as small as possible and the number 4XX as large as possible. Since 1, 2, 3, and 6 are left over, it suggests that our best result, with 4 and 5 as our hundreds digits, would be 512 – 463 = 49, far better than the results in the previous paragraph.

How can we use these insights to find our desired minimum for all such numbers? The largest and smallest *two-digit* numbers that we can make using the digits 1, 2, 3, 4, 5, 6 at most once are 65 and 12, respectively. How can we use this in our problem? The only unused digits are 3 and 4, which must be our hundreds digits of the smaller and larger number, respectively. These numbers are indeed consecutive, which gives us confidence. This suggests that our best result would be 412 – 365 = 47. The only other way to make the two numbers closer would be if they shared hundreds digits (which might suggest an extension).

Checking our first result with this logic indicates that if we have the digits 1, 2, 3, 4, the largest one-digit number is 4 and the smallest is 1, leaving 2 and 3 for the tens digits. Putting them together yields 31 – 24 = 7, as we discovered earlier. If we use it in the original problem, we get the largest and smallest three-digit numbers 876 and 123, so our smallest difference must be 5123 – 4876 = 247.

There are various ways you can approach this problem with a class. As mentioned above, you could start from an easier problem and work toward a more difficult one. You could even give various problems to different groups of students and then have the groups share with the class, or compare notes with each other, to work toward the solution of the original problem. For example, we could assign groups sets of four numbers: {0, 1, 2, 3}, {1, 2, 3, 4}, {2, 3, 4, 5}, …, {6, 7, 8, 9}, each group getting a different set of numbers. Although the groups will be solving "different" problems, they will get the same results, and it might help them see the structure of the problem before moving on to more digits.

The problem also submits itself readily to many variations. We have already looked at using fewer digits, and we could extend it to more digits with 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Interesting things will happen if we use an odd number of digits. We have already hinted earlier that interesting things might happen if we have multiple copies of digits. For example, if we use our method above with 1, 2, 2, 3, 4, 4, 4, 4, 5, 5, 5, 6, 7, 8, we will not get the smallest difference. Our method yields 4 876 555 – 4 122 344 = 754 211; however, you can create a pair of numbers with a difference of 26. That is quite an improvement! Can you find these numbers?

Of course, we could flip the problem over and ask for the maximum difference. This is not as interesting as the original problem, but it gets students to think about numbers and subtraction.

There is nothing that ties us to subtraction. We could easily have asked for the minimal or maximal sum. However, like the maximal difference problem, these are not as rich as the subtraction problem. As such, these other problems could act as a good warm-up to the difference problem. Sometimes, when designing multi-part questions for mathematics contests, this idea is used. The very interesting, very tough problem that you want to ask might be too much of a leap for most students. As such, a problem that allows students to brush up against an important idea can help them toward a solution. We can do similar things in our classrooms by providing scaffolding for students, or by having some of these "middle steps" in the back of our mind, so that we can prompt students who are struggling toward a more attainable goal, which will help them reach the main goal as well.

In a second email from Ed, he mentioned that someone he was corresponding with had asked students to produce two numbers, using the digits 1, 3, 5, 7, 9 once each, such that the product was maximal. Ed said, "I do not see a really neat systematic way of tackling problems like this when you take other sets of digits (allowing repetitions, if you wish), and this may be worth a more detailed look." What was I to do with this? I did what any self-respecting problem addict would do, I went down into my windowless basement office, closed the door, and got to work!

I came up with an algorithm that I am confident with, although I am not sure it would hold up to all possible generalizations (although I suspect I will continue trying to prove it). My first observation was that if I looked at my two numbers, the digits in each number would have to be in descending order. Otherwise, we could increase one of the numbers being multiplied, and that would increase the product. For example, if I thought 395 × 17 was the maximum, I could easily prove that wrong because 395 × 17 < 395 × 71 < 953 x 71.

Then, I thought of building the numbers one digit at a time. Thus, to maximize things, I would start with 9 × 7, the two largest numbers. I would need to use the 5. Now I could either just try 95 × 7 and 9 × 75 to see which is larger and proceed that way and finish it off quickly. However, I was thinking about generalization; that is, if I had two multi-digit numbers, A and B, and I wanted to attach the digit d to the end of one of them so that the product was maximized, to which would I append it? To do this, I wanted to assume that attaching the digit to the end of the first number was the best, and see what this implied. How do I attach a new

number to the end of an existing number? Looking at the examples above, I am interested in how I went from 9 to 95 or from 7 to 75. We can see that:

$$9 \times 10 + 5 = 95$$
$$7 \times 10 + 5 = 75$$

Therefore, $10A + d$ would attach the digit d to the end of the number A, which makes sense from the base-ten expansion of A. We can see that this works with several examples:

$$385 \times 10 + 1 = 3851$$
$$1792 \times 10 + 4 = 17\,925$$
$$98\,765 \times 10 + 4 = 987\,654$$

Thus, if adding the digit to the end of A is optimal, we must have $(10A + d)B > A(10B + d)$.

Expanding and subtracting the common term $10AB$ from both sides yields $Bd > Ad$.

Since d is a single-digit number, we must have $d \geq 0$.

If $d = 0$, this doesn't hold (this is not part of the original problem, but I am thinking in terms of a generalization where we could start with any collection of digits). However, if we think about it, if we start with two numbers A and B, adding a zero to the end of *either* of them is the same as multiplying that number by 10, so $(10 \times A + 0)B = A(10 \times B + 0) = 10 \times AB$, no matter where the zero is placed. However, if the digit is not zero, then we must have $B > A$.

In other words, it is best to append the digit to the end of A if it is the smaller of the two numbers. Using this logic, we would proceed from 9 and 7, to 9 and 75, to 93 and 75, to 93 and 751.

Let's try to check our result. Since we established earlier that, to maximize the product, the digits of our two numbers must be in descending order, we greatly cut down the number of cases we have to check. In essence, we have changed it to a problem where order doesn't matter, since the order—descending digits—is specified. Hence, to make a three-digit/two-digit pair, rather than needing to check $5! = 120$ cases, we just need to check $\binom{5}{3} = 10$ cases, which we can readily do to show that we have indeed found our desired maximum.

$975 \times 31 = 30\,225$	$931 \times 75 = 69\,825$
$973 \times 51 = 49\,623$	$753 \times 91 = 68\,523$
$971 \times 53 = 51\,463$	**$751 \times 93 = 69\,843$**
$953 \times 71 = 67\,663$	$731 \times 95 = 69\,445$
$951 \times 73 = 69\,423$	$531 \times 97 = 51\,507$

We can interpret, when $d > 0$, $(10A + d)B > A(10B + d) \Leftrightarrow B > A$ in another way. Suppose we take any two numbers, multiply them to get the product. If, when you "remove" the units digit from the larger number—making it one digit shorter—if it is still larger than the other number, then if the

Continued on page 32

Useful Theories to Know About: HOTS and LOTS



JEFF IRVINE
dr.jeffrey.irvine@bell.net

Jeff Irvine, Ph.D., has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

Most teachers are aware of the need to engage students' Higher-Order Thinking Skills (HOTS). However, John Hattie has found that only 28 percent of instructional time focuses on HOTS, while the other 72 percent of time focuses on Lower-Order Thinking Skills (LOTS). The most effective teachers do the exact opposite, spending 72 percent of their time on HOTS and only 28 percent of time on LOTS. So, what exactly are HOTS, how do we recognize them, and what teaching strategies will help our students address them?

HOTS and LOTS

HOTS may be identified using a taxonomy of learning, such as the Revised Bloom's taxonomy, shown in Figure 1. Alternatively, it may be identified using Marzano's New Taxonomy (MNT), shown in Figure 2.

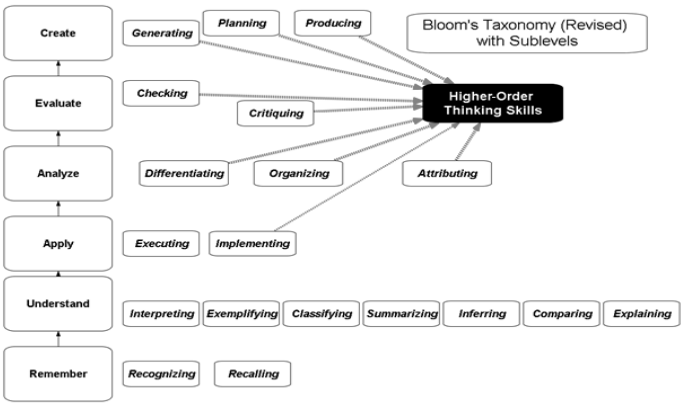


Figure 1: *HOTS on the revised Bloom's taxonomy*

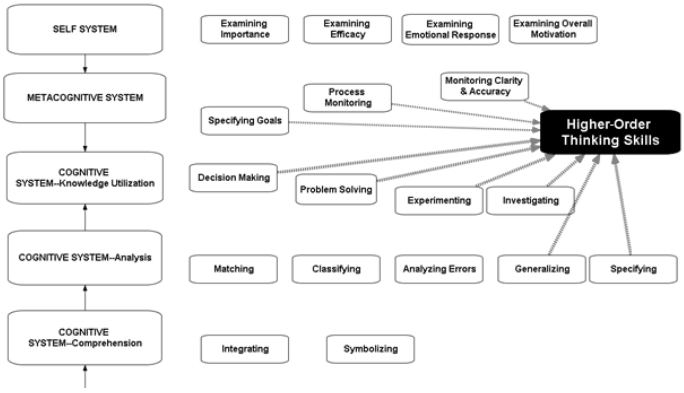


Figure 2: *HOTS on Marzano's new taxonomy*

HOTS include tasks such as formulating and testing hypotheses, problem solving, problem posing, posing "What If?" questions, applying strategies to new situations, explaining and justifying solutions, and identifying constraints. HOTS do not include applying a known strategy to a known situation. So, for example, constructing and solving a system of equations for a word problem about mixtures would not be identified as requiring HOTS, if the class had already previously done similar problems; however, if the mixtures was a new situation, then doing the same steps of constructing and solving a system of equations would be considered requiring HOTS.

HOTS would also be needed in situations of *far transfer*, where a student is confronted with a problem or situation significantly different from those already encountered in class. They would have to: analyze the problem, formulate a possible solution strategy, carry out the strategy, and review their solution to determine whether it was a feasible solution. Most of what we do in mathematics classrooms is called *near transfer*. The problems students face are generally similar to problems that they have already solved, using a familiar strategy. Research has shown that for students to acquire a deep understanding, some experiences with far transfer situations are necessary.

Why Should We (Math Teachers) Care About This Theory?

As teachers who want our students to realize their potential, we need to ask our students to go beyond LOTS and experience the need for HOTS. Peter Liljedahl's *Building Thinking Classrooms in Mathematics* provides some excellent ideas on how we can use HOTS to encourage our students to think and reason at high levels. We need to identify problems, situations, and investigations that encourage HOTS. George Gadanadis (Western University) says we need to utilize problems that have low floors (all students have an entry point into the problem),

Continued on page 32

▲ Musing on Percentages: % as a (Curious) Number



JÉRÔME PROULX
proulx.jerome@uqam.ca

Jérôme Proulx is professor of didactique des mathématiques in the mathematics department at the Université du Québec à Montréal. His research focuses on epistemological issues related to cognition and/in mathematics. His current research program is dedicated to mental mathematics and school mathematics content development. From January 2026 onwards, he is Reader of Mathematics Education in the School of Education, Communication & Society at King's College London, United Kingdom. (Photo by Nathalie St-Pierre of Université du Québec à Montréal)

Percent is everywhere—all around us in finance, shops, foods, media, surveys, and so on. Percentages may seem like a rather simple mathematical theme that has little depth. However, when one starts to pay attention to its usage, percentages reveal surprising elements that are mathematically rich and conceptually fascinating, hence worth pondering. This is something that became quite salient for our research team, as we have been preparing experiments with students in classrooms about percentages in mental-mathematical environments.

These columns endeavour to raise and address some of these peculiarities, which we came to call "curiosities," related to the concept of percentages. These musings can be seen to be as much about our usage of "%" in everyday life, in schools, or in mathematics in general. This seventh column addresses a particular way of conceptualizing the % sign as a number, which is powerful, intriguing, and puzzling.

Contextualizing the Issue

The concept of % challenges students in terms of understanding what it is about, interpreting it, and undertaking calculations with it. These challenges are also experienced on a daily basis by mathematics teachers, a situation which has led, from the 1930's onwards, to numerous recommendations in the professional mathematics education literature about how to approach the teaching of %. Ideas, activities, and advice have included using bar graphs, unit squares, rulers and rods, to mnemonics tricks, verbal explanations, and visual methods, all the way through various sorts of games.

Amidst all of these suggestions, one notion in particular is worthy of consideration. This curious idea was proposed in 1983 by Donald Byrkit, a professor from West Florida University. In a nutshell, Byrkit's idea is to conceptualize the % sign as being a number that is equal to 1/100. Let's unpack this idea.

Conceptualizing the % Sign as a Number

There exist many ways and suggestions to work with % and make calculations more accessible. Various school textbooks propose direct conversions of percentages and other numerical representations, for example, 35% and 0.35 or 35/100, and then operating with these numbers and fractions. It is also frequent to see the % sign talked about in terms of a "request" for multiplying by 1/100, where 35% becomes 35 × 1/100 = 35/100, or 35% of 12 is obtained by (35 × 1/100) × 12. And the same thing can be calculated with a multiplication by 0.01 (i.e., 35 × 0.01 × 12), or a division by 100 (i.e., 35/100 × 12).

Starting from the same idea, Byrkit suggested an additional step. He recommended affording a numerical value to the % sign. Yes, you read right: Byrkit suggested that the % sign be conceived as equal to the number 1/100. As mentioned, the presence of the % sign has frequently led some textbook authors to propose that one needs to multiply by 1/100, but this is not what Byrkit suggests. He asserts that the % sign is literally worth 1/100, leading to the following equality: "% = 1/100". To this, he adds that since the % sign is always used with another number, those two "numbers" are multiplicatively linked, as is usually done in algebra. Indeed, in algebra, when one writes 25*t*, this means that 25 and *t* are multiplicatively linked: 25*t* means (25 × *t*), and so 25% becomes (25 × %) and then (25 × 1/100).

Although intriguing, Byrkit's way of conceiving the % sign enables him to address a variety of % tasks with great ease. In particular, this helps explain the mathematical property that A% of B = B% of A. If % = 1/100, then A% of B becomes an operation with three separate numbers, namely A, % and B, as in (A)(%)(B), which can be transformed in A × 1/100 × B. And, through commutativity of multiplication in context of % (see also Proulx, 2024), B and A can be permuted. This gives:

A% of B
= (A)(%)(B)
= A × 1/100 × B
= B × 1/100 × A
= (B)(%)(A)
= B% of A

As such, an example like 85% of 20 gives:

85% of 20
= (85)(%)(20)
= 85 × 1/100 × 20
= 20 × 1/100 × 85
= (20)(%)(85)
= 20% of 85

Stumbling upon Some Caveats

Powerful and surprising as it may be, this way of considering the % sign as a number, that is, by attributing it a precise numerical value of 1/100, is not without caveats. As a matter of fact, these diverse permutations through commutativity of multiplication can lead one into awkward situations. For example, in 55% of 8, one can end up with:

55% of 8
= (55)(%)(8)
= 55 × 1/100 × 8
= 1/100 × 55 × 8
= (%)(55)(8)
= % 440

Ending up with an expression that "starts" with a % sign, to say the least, is quite unusual for us in English. (This said, I was recently informed that in some other languages, the % sign precedes the number; hence, our usual way of writing might be the one surprising to them!) Of course, one could resort back to translating it to "1/100 × 440," leading to 4.40 as a definite value, but it is not easy to verbalize what %440 means; "percentage four hundred and forty"? In fact, I have never encountered this way of writing in mathematics, nor of speaking, and it is my belief that this is not a generally accepted convention.

But this is not all. If % = 1/100, situations with successive % (as previously discussed in Proulx, 2025a and 2025b) can lead to unforeseen consequences that are even more puzzling. For example, consider a price increase situation: *If the price of a pack of pasta has increased by 10% last week and 5% this week, what is the total increase to a pack of pasta if its price was initially $3.50?* To find this amount, it is necessary to establish 10% of $3.50 and then find 5% of this new price (already augmented by 10%). With Byrkit's suggestion that % = 1/100, this situation could be represented with the following steps:

Step 1. Find 10% of $3.50, which gives the increase in the price of a pack of pasta for the first week. This can be written as (10)(%)($3.50).

Step 2. The pack of pasta is now worth 110% of 3.50.

Step 3. Find the 5% of this new price. This can be found by asking the value of 5% of [110% of 3.50], that is, (5)(%)[110% of 3.50] = (5)(%)[(110)(%)(3.50)] = (5)(%)(110)(%)(3.50).

Step 4. Add both increases : (10)(%)(3.50) + (5)(%)(110)(%)(3.50)

This result, namely (10)(%)(3.50) + (5)(%)(110)(%)

(3.50), can be calculated if the % = 1/100 is used. However, with permutations, it is possible to end up with % signs at different and surprising places. For example:

$$(10)(\%)(3.50) + (5)(\%)(110)(\%)(3.50)$$
$$= (\%)(10)(3.50) + (\%)(5)(110)(3.50)(\%)$$
$$= (\%)(10)(3.50) + (\%)(\%)(110)(5)(3.50)$$

Some of these are quite unreadable, except as symbol manipulation, and even more surprising when some are factored out:

$$(\%)(10)(3.50) + (\%)(5)(110)(3.50)(\%)$$
$$= (\%)[(10)(3.50) + (110)(5)(3.50)(\%)]$$

And if the % sign is rendered with an exponent, this achieves a "quadratic percent." Again, how fascinating!

$$(\%)(10)(3.50) + (\%)^2(110)(5)(3.50)$$

These oddities could go on for a while! Okay, maybe I am pushing it a little here, and one could say that in most cases, we do not need such oddities, and the situation would work perfectly fine. I concur. At the same time, pushing the idea is a nice way to show the limits of the suggestion. You might think of other puzzling situations that could be created with this "% = 1/100" idea. (Editor: 1 cm = 1 %m = 0.1 %%km.)

Concluding Remarks – Beyond Calculations!

These various examples and mathematical explorations aimed at underlining how Byrkit's suggestion possesses obvious calculational strengths. However, and this is important for school mathematics at least, his suggestion has significant limits, conceptually speaking, concerning what a % really is.

The challenges with mathematical concepts are seldom calculational: when one knows a mechanical process and can apply it systemically, all is well and done. It is the case with algebra, with conversions of %, with arithmetical operations, algorithms, and so on. Some students even do these with their eyes closed, as the saying goes. However, when it comes to making sense and understanding a problem, or when one is confronted with a situation requiring more than the simple application of rules and procedures, significant challenges may emerge. In mathematics, some would say, it is impossible to set aside meaning! Doing mathematics is hardly synonymous with the robotic application of rules and procedures… however powerful these may be.

Thus, what is there to think about in relation to Byrkit's surprising suggestion of % = 1/100? It has (calculational) strengths, and it has (conceptual) limits. One thing for sure is that whatever one thinks, Byrkit's suggestion has the merit of catching our attention, and to force us to really reason and reflect on the meaning of the % sign. This, one must concede, is a very good thing!

Let me know of other "tricks" you have. I'll give it my %110 for exploring them in depth and making sure I have a lot of fun!

References

Byrkit, D.R. (1983). Teaching percent. *The Mathematics Teacher, 76*(1), 8. www.jstor.org/stable/i27963288

Proulx, J. (2024). Commutative %. *Ontario Mathematics Gazette, 63*(2), 37–38.

Proulx, J. (2025a). Adding and cancelling %—Part 1. *Ontario Mathematics Gazette, 63*(4), 16–18.

Proulx, J. (2025b). Adding and cancelling %—Part 2. *Ontario Mathematics Gazette, 64*(1), 6–7. ▲

Continued from page 29

digit is appended to the end of the smaller number, becoming its new units digit, and the new numbers are multiplied, the product will be larger than the original.

That was a mouthful! Let's look at an example. Starting with 5391 and 8, if we ignore the units digit of the larger number, it is still larger, that is, 539 > 8. Hence, 539 and 81 should produce a larger product than 5391 and 8, which can be easily verified. However, as 53 < 81, moving the units digit 9 to 81, creating 53 and 819, does not produce a larger product than the last step.

As usual, while solving a problem, it is a good idea to keep your mind open to your internal questions to guide your investigation beyond "the answer." Some of these questions may not yield anything worthwhile, but that is OK. From time to time, you will stumble upon something where your efforts are rewarded.

Until next time, happy problem solving! ▲

Continued from page 30

high ceilings (every student can excel to their potential), and wide walls (problems allow for lateral as well as hierarchical thinking). We need to strive to move all our students to maximize engagement; but we should always teach in ways that engage and encourage excellence in all our students.

If You Want to Learn More

Here are some videos to consider as starting points for further consideration:

Aflatoun International. (2021). *Bloom's taxonomy and higher order thinking* [Video]. YouTube. www.youtube.com/watch?v=7zmV2fQ9tP4 Bloom's Taxonomy and HOTS.

Challenge Innovate Grow Teacher & Learner Centre. (2023). *What is higher order thinking?* [Video]. YouTube. www.youtube.com/watch?v=W9t2sPFZVJQ A brief overview of HOTS.

Hyman, D. (2015). *What happens when classrooms meet higher order thinking* [Video]. YouTube. TEDxAmsterdamED. www.youtube.com/watch?v=x2X0Yuks2TE Excerpt from a TED Talk. ▲

▲ The Math Teacher's Home Base: The Alphabet Dance: Symmetries, Solids, and Wordplay



RON LANCASTER
ron2718@nas.net

Ron is an Associate Professor Emeritus at the University of Toronto, where he taught mathematics courses for pre-service teachers for 17 years. Ron taught middle and high school mathematics for 21 years in coed public schools and an all-girls school. Ron's professional activities include conference presentations/ consultations in North America, Asia, England, the Middle East, Africa, and India. Ron has authored/edited over 200 articles for NCTM and OAME. Ron received the 2015 Margaret Sinclair Memorial Award Recognizing Innovation and Excellence in Mathematics Education, awarded by the Fields Institute, and was the recipient of the 2022 OAME Lifetime Achievement Award.

Welcome back to "The Math Teacher's Home Base." In our last column, we explored the intersection of poetry and combinatorics. In this issue, we are shifting from the rhythm of the stanza to the geometry of the letter. I've pulled several books from my shelf that treat the alphabet as a collection of shapes, symmetries, and sets to be manipulated. For those interested in further exploration of these concepts, I recommend checking out the *Gazette* articles "Make your own font?" (62(1), p. 6) and "Fun with Fonts" (62(1), p. 31).

A note on professional curiosity: I occasionally hear from colleagues who wonder, "What is the point of all this?" or "How is this used in the real world?" These are fair questions, but when we ask them, we are doing the exact thing we find so frustrating in our students when they ask, "When am I ever going to use this?" As educators, we must model a love for learning that isn't always tied to immediate utility. We should embrace the joy of discovery for its own sake. When we play with the alphabet, we are engaging in logic, structure, and pattern recognition—the very habits of mind that make us mathematicians.

It is also worth noting that understanding these geometric properties is relevant to modern technology; scalable fonts are the standard today and are defined using these very structural principles.

Section 1: The Geometry of Wordplay

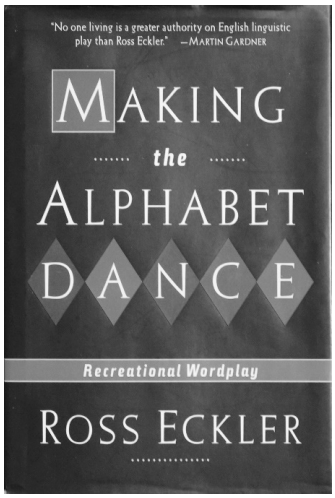


Figure 1: Making the Alphabet Dance *by Ross Eckler*

One of the most intriguing concepts in *Making the Alphabet Dance*, by Ross Eckler (see Figure 1), is the association of word groups with the five Platonic solids. He suggests that we can use these three-dimensional shapes as "containers" for sets of letters.

Various word groups can be associated with the five Platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. For example, one can label the vertices of a Platonic solid with different letters of the alphabet so that words can be formed out of the letters at the vertices bounding each face (Eckler, 1996, p. 213).

Using a cube, choose eight unique letters to label the vertices.

The Task:

Find a set of eight unique letters that, when assigned to vertices of a cube, have the property that, for every face, the four letters can be arranged to spell a word.

The Mathematical Difficulty: In a cube, every vertex is shared by exactly three faces. This means that if you place a letter on a vertex, that letter must be a functional part of three different four-letter words at the same time. This shared vertex constraint makes the fifth and sixth faces difficult to complete without patience, perseverance, and perhaps a dictionary. The task invites students to wonder if there is an easier way of organizing the letters on the faces. There is! Students might find it easier to place the letters on a 2D net (Figure 2) and to visualize how each letter is connected to the other two faces that they are part of.

Photograph 1 shows a solution to this task entered on a net. Also appearing in the photograph is a cube with the letters.

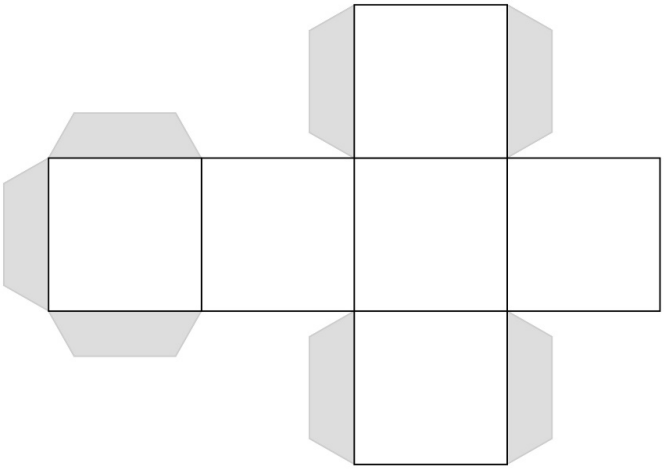
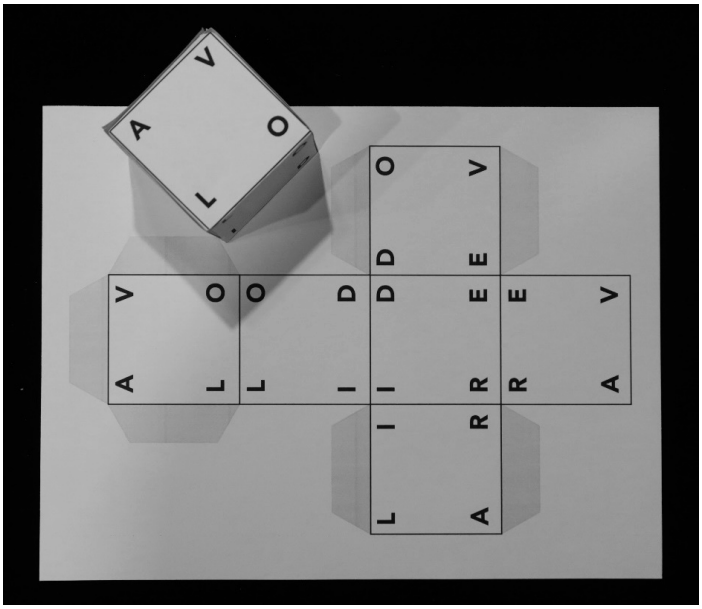


Figure 2: *Net for making a cube (senteacher.org)*



Photograph 1: *Mapping the eight-letter solution onto a 2D net and 3D cube. Created and photographed by Ron Lancaster*

The Four-Letter Shuffle

One of the most remarkable sections of Ross Eckler's work involves identifying 24 different words that represent every single way to arrange four specific letters.

The Poster Challenge (Phase 1): Teachers can display these 24 words on a poster to spark an initial investigation:

PArtiaL	triaL	tiarA
MaritAL	CUrtaiN	traiL
Stair	VEritaBLE	PartiCLE
MatriX	MILitarY	ratiNG
FLirtaTION	NitraTE	SatirE
LAriat	airtIGHT	DiatrIBE
tiraDE	Trait	SEMiartICULATE
WaitrESS	StariNG	ADMiratION

Question 1:

What do you notice about the lower-case letters in this collection of words?

Question 2:

Are there any cases missing in the collection? Are there any duplicates? How do you know?

The Three-Letter Quest:

The goal is to find a collection of words, where each word contains three chosen lower-case letters in a unique order. Once students have discovered the pattern in the four-letter set, teachers can introduce this creative task. Presenting the task here makes it clear where the investigation is going.

The Math Challenge:

How many words do you need to find to have a complete set for three letters?

The Mathematical "Why": The search for exactly 24 words is rooted in how we count arrangements. For any set of four unique items, the number of ways to arrange them is 4×3×2×1=24. For the students' three-letter quest, the target is 3×2×1=6 words.

The Word Triangle Challenge

In this final task for Ekler's book, we explore a sequence of words, where each new word is formed by adding exactly one letter to the previous set and rearranging the result.

The Poster Challenge: Teachers can display this full 17-line word triangle to spark student interest.

A
AT
TAN
RANT
TRAIN
RETAIN
CERTAIN
REACTION
MANTICORE
REMICATION
CREATIONISM
CREMATIONIST
REMASTICATION
RECLAMATIONIST
RECITATIONALISM
NONMATERIALISTIC
ANTICEREMONIALIST

Question 1:

What do you notice about the words and the number of letters per line as you move from the top down?

Question 2:

How many letters are used in this entire triangle?

Question 3:

Create a pattern like this, starting with a letter of your own choice. See how far you can go!

The Mathematical "Why": This linguistic exercise is a beautiful representation of triangular numbers. In this pattern, the *n*-th line contains exactly *n* letters. To find the total number of letters used in a triangle with *n* lines, students can apply the formula for the sum of the first *n* integers: $n(n+1)/2$. For this 17-line triangle, the total is 153 letters.

Section 2: The Palindrome Puzzle

We now move into the world of palindromes—words, phrases, or numbers that read the same forwards and backwards. This concept is a linguistic version of symmetry.

Real-World Symmetry: Teachers can start by showing everyday examples of palindromic branding and titles:



Photograph 2: *ABUBA—A famous steakhouse in Jakarta. Photographed by Ron Lancaster*



Photograph 3: ZOONOOZ—*The long-running magazine from the San Diego Zoo*

It is important to clarify that ABUBA (Photograph 2) possesses reflection symmetry as well as being a palindrome. In contrast, ZOONOOZ (Photograph 3) is a palindrome and has rotational symmetry, but it does not have reflection symmetry.

The Reverse-and-Add Challenge

Palindromes are a fascinating part of number theory. A popular investigation for students is the "reverse-and-add" sequence.

The Forward Process (Using 59):

1. Pick a number: 59
2. Reverse the digits: 95
3. Add them together: 59+95=154
4. Reverse 154 (451) and add again: 154+451=605
5. Repeat: 605+506=1111. Success—we have reached a palindrome!

The Challenge Questions:

Question 1:

Start with the number 75. Carry out the "reverse-and-add" process and determine how many steps it takes before you reach a palindrome.

Question 2:

Suppose you performed exactly one "reverse-and-add" step and ended up with the number 55. Identify the number(s) you could have started with.

Question 3:

Attempt the "Backward Path" for the number 89. Determine if there is any number that, when added to its reverse, equals 89. Explain your findings.

Question 4:

Starting with 89, it takes an incredible 24 steps to reach a palindrome. Find another number that also takes exactly 24 steps to reach its palindromic destination.

Teacher's Insight: Reverse-and-Add Mechanics

For Question 1 (75):

Students should find that 75+57=132, and 132+231=363. It takes two steps.

For Question 2 (55):

To find the starting numbers, students should look for pairs of digits that sum to 5 (without carrying). The possible numbers are 14, 23, 32, and 41.

For Question 3 (The 89 Mystery):

This is a "dead end" for two primary reasons.

The Multiples-of-11 Rule: When you add a two-digit number $10x+y$ to its reverse $10y+x$, the result is $11x + 11y$, or $11(x + y)$. This means any two-digit sum must be a multiple of 11. Since 89 is not divisible by 11, it cannot be the sum of a two-digit number and its reverse.

The Three-Digit Gap: The smallest three-digit number that is a sum of its reverse is 101+101=202. Since 89 is less

than 101, there are no three-digit (or larger) candidates either.

For Question 4 (The 24-Step Partner):

Students should discover 98. Like 89, it requires an extensive 24-step journey to reach the palindrome 8 813 200 023 188.

The Mystery of the Unreachable Palindrome

The "reverse-and-add" process typically leads to a palindrome quite quickly, but it also introduces students to one of the most famous unsolved problems in recreational mathematics: Lychrel numbers. While many numbers, like 89, eventually reach a palindrome after a long journey of 24 steps, there are others that seem to defy this logic entirely.

The number 196 is the most notorious example. Despite being carried out for millions of steps by computer algorithms, it has never reached a palindromic destination. This "196 Problem" serves as a perfect classroom bridge between simple arithmetic and the frontier of mathematical proof, showing students that even a process as straightforward as addition can lead to deep, unanswered questions.

Note for the Classroom: This sequence provides a natural invitation for students to move beyond pencil and paper. It is an excellent opportunity for rapid iterations or to write a simple program to see how far they can push the "196" experiment before the digits overflow the screen. Furthermore, coding allows students to explore the relationship between the starting number and the number of steps required to reach a palindrome. By generating a scatter plot of "Starting Number" versus "Number of Steps," students can look for clusters, outliers, or hidden patterns in the data, turning a linguistic number game into a robust statistical investigation.

The Search for the Palindromic Point

In coordinate geometry, a palindromic point is a pair of coordinates, where the x and y values are identical, such as (2, 2) or (–5, –5). These points all lie on the line $y = x$ and they can be found by graphing, or algebraically, through equation solving.

The idea of palindromic points can be combined with palindromic coefficients for a parabola. Consider this specific family of parabolas, where the coefficients are a, 1, and a—this palindromic parabola is:

$$y = ax^2 + 1x + a$$

An example case: Look at the graph in Figure 3 for $y = 8x^2 + 1x + 8$.

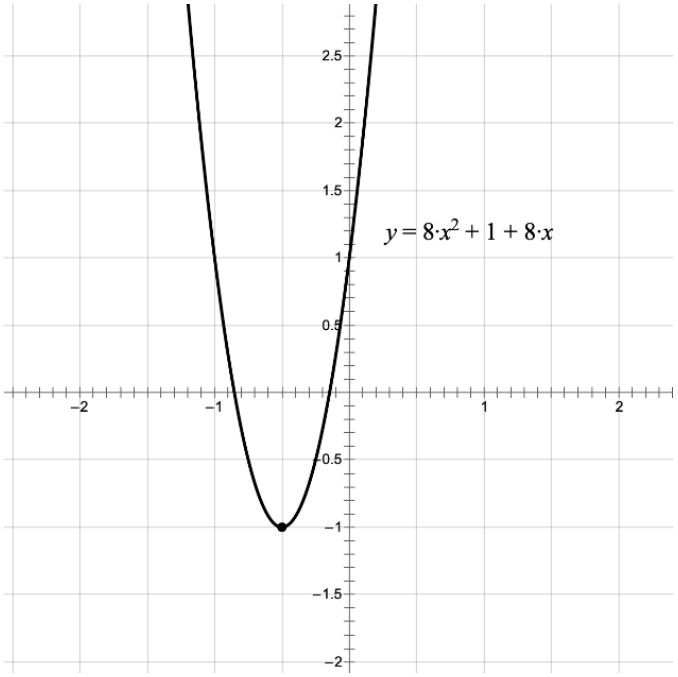


Figure 3: *Quadratic example, where* a *= 8*

The Investigation:

The Vertex Hunt: Use the graph or algebraic methods to find the coordinates of the vertex for the example above, where $a = 8$. Determine whether or not this is a palindromic point.

The Slider Challenge: Using graphing software, create a slider for the variable a. As you change the value of a, watch the vertex move. Suggestion: Add the line $y = x$, and add the points of intersection with the quadratic. Then, as the slider is moved, the palindromic points will be revealed and move. Determine if there is any value for a that lands the vertex exactly on the line $y = x$, and verify your answer using algebra.

Teacher's Insight: The Algebra of the Impossible Point

When students attempt to verify their slider observations using algebra, they are trying to find a value for a, where the x-coordinate and y-coordinate of the vertex are equal. They will not be able to find an example, and they can verify that it is impossible by finding the coordinates of the vertex as a function of a.

By completing the square for $y = ax^2 + 1x + a$, they will find that the vertex is located at

$$\left(\frac{-1}{2a}, a - \frac{1}{4a}\right).$$

By making the coordinates equal to each other, students will find that $a^2 = -0.25$, and as a result, it is impossible for the vertex to be a palindromic point.



Figure 4: *Palindrome gallery*

A Collection of Palindromic Books

***Was It a Cat I Saw?* by Tara Bontie** (Figure 4, top left): A beautifully illustrated book that uses the classic palindrome to explore visual and linguistic reflection.

***Otto: A Palindrama* by Jon Agee** (Figure 4, top right): A clever graphic novel, where every line of dialogue is a palindrome, telling a complete story through mirrored speech.

***Palindromania!* by Jon Agee** (Figure 4, bottom left): A vibrant collection of palindromic phrases and wordplay that showcases the author's mastery of the "forward-and-back" technique.

***2002: A Palindrome Story in 2002 Words* by Nick Montfort and William Gillespie** (Figure 4, bottom right): A unique literary experiment and technical feat—the entire story is one massive palindrome spanning exactly 2002 words.

Note for Educators: While this is a brilliant mathematical and linguistic achievement, this particular volume contains adult themes. It is intended for a mature audience and should be used as a personal reference, rather than a book brought into the classroom.

Classroom Activity: Although writing a 2002-word palindrome is a monumental task, this book serves as a fantastic prompt for students. Teachers can challenge their students to write their own "Palindromic Story"—perhaps starting with a goal of just 20 or 50 words—to experience the logic and perseverance required to maintain the pattern.

Section 3: The Art of the Flip

While a palindrome is a mirror in language (reflectional symmetry), an Ambigram is a rotation in design. Most ambigrams feature 180-degree rotational symmetry.

This design by Scott Kim (Figure 5) is a perfect example: the word "Upside" becomes "Down" when rotated, and "Down" becomes "Upside." In the world of ambigrams, the orientation of the page changes, but the meaning remains clear.



Figure 5: *Upside Down by Scott Kim (scottkim.com)*

Simple Rotational Symmetry:

Before diving into complex designs, we can find rotational symmetry in everyday words and numbers.

NOON: Write this word in capital letters. Notice how the "N" rotates into another "N" and the "OO" remains unchanged.

1961: This year is a numerical ambigram. The "1"s remain "1"s, and the "6" and "9" rotate into each other.

The "Super-Symmetry" Mystery: Mirror-Rotational Words

There is a rare group of words that respond to both reflection (mirrors) and rotation (flipping) in a way that seems to "cancel out" the transformation. These words possess horizontal reflectional symmetry.

The "Flip-and-Mirror" List: The following words look completely normal if you turn them upside down and view them in a mirror simultaneously:

BED
CHOICE
CHECKBOOK
COOKBOOK
DECIDED

The Rotational Challenge:

Question 1:

Write the word SWIMS in all capital letters. Rotate the paper 180° and describe the transformation of the word and its individual letters.

Question 2:

Examine the capital alphabet. Identify every letter that possesses point symmetry (looks identical after a 180° rotation). Explain the geometric difference between letters like “N” and “Z” versus letters like “E” or “F.”

Question 3:

Using the point-symmetry letters you identified, construct a new word that functions as a natural ambigram.

Question 4 (The Mirror-Flip Investigation):

Perform the “Combo” test on the word CHOICE (turn it upside down and hold it to a mirror). Describe the result. Find another word not listed above that shares this property, and explain the symmetry requirements needed for it to work.

Question 5:

Write the word MATH in all capital letters and perform the “flip-and-mirror” test. Explain why this word fails the test, while a word like BED succeeds.

Teacher's Insight: The Math of the Flip

Question 2: The point-symmetry letters are H, I, N, O, S, X, Z.

Questions 4 and 5: To pass the “mirror-flip” test, a word must have a horizontal axis of symmetry. A 180° rotation followed by a vertical mirror reflection is mathematically equivalent to a single reflection across the horizontal *x*-axis. Since B, C, D, E, H, I, K, O, X have horizontal symmetry, they work. MATH fails because its letters (A, M, T) have vertical symmetry (*y*-axis), which cannot “fix” an upside-down rotation.



Figure 6: *The ambigram library*

***Inversions* by Scott Kim** (Figure 6, top right): A master class in treating letters as geometric puzzles, showing how characters can be stretched and skewed to maintain symmetry.

***Wordplay: The Philosophy, Art, and Science of Ambigrams* by John Langdon** (Figure 6, top left): An exploration of the philosophy and art of the “flip,” including the famous ambigrams used in *Angels & Demons*.

***Topsys & Turvys* by Peter Newell** (Figure 6, bottom left): Classic illustrations that tell a completely different story when rotated 180°.

***Round Trip* by Ann Jonas** (Figure 6, bottom right): A brilliant use of negative space, where a black-and-white landscape journey transforms into a return trip when the book is turned upside down.

Concluding Thoughts: The Alphabet Lab

From the rigid vertex constraints of a cube to the fluid rotational “magic” of an ambigram, we have seen that the alphabet is far more than a tool for literacy—it is a playground for geometric and combinatorial thinking. By engaging with these tasks, students aren’t just playing with words; they are practising the perseverance, pattern recognition, and structural analysis that define mathematical inquiry.

Many of the foundational texts mentioned here, such as Ross Eckler’s *Making the Alphabet Dance* or Scott Kim’s *Inversions*, are true classics of recreational wordplay. While some may be out of continuous print, they are widely available as used copies, and deserve a permanent spot on

your classroom shelf. Adding these "visual math" books to your library ensures that the next time a student asks, "When will I ever use this?," you can simply hand them a mirror, a cube, or a copy of *Round Trip* and let the discovery begin.

Expanding Your Collection

If you and your students are looking for even more ways to manipulate the alphabet, I highly recommend adding these two titles to your professional library. Both contain a wealth of additional puzzles and linguistic curiosities that mirror the geometric and logical challenges we have explored in this column:

Words at Play (Figure 7, left): A brilliant collection that treats language as a playground, offering a wide variety of puzzles, anagrams, and structural word games that encourage mathematical thinking.

The Wordplay Almanac (Figure 7, right): Organized with a "discovery-a-day" feel, this book provides a constant stream of patterns and linguistic oddities that can be easily adapted for classroom warm-ups or logic challenges.

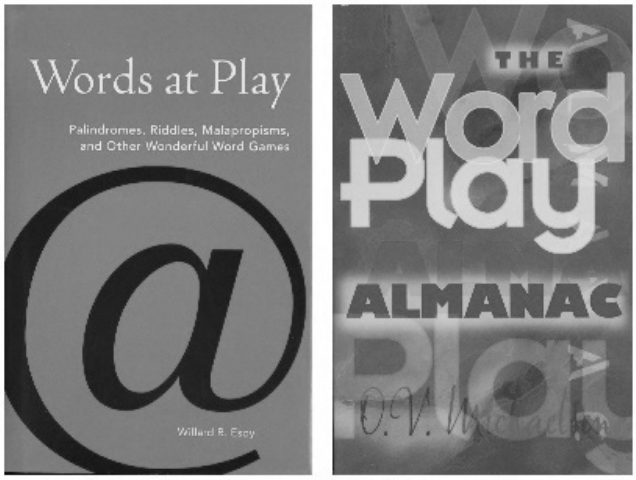


Figure 7: *Expanding your collection?*

▲



Fence, St. John's, Newfoundland (For more fences see March 2023 in the archives.)

▲ Engaging Tasks for Teaching Trigonometry



JEFF IRVINE
dr.jeffrey.irvine@bell.net

Jeff Irvine, Ph.D., has been a secondary mathematics teacher, department head, and vice-principal. He has taught at three faculties of education and at a community college. For several years, he was an Education Officer in the Curriculum and Assessment Policy Branch of the Ontario Ministry of Education, where his portfolio was Grades 7 to 12 mathematics for the Province of Ontario. Jeff is co-author or contributing author for 11 high school mathematics textbooks. With over 45 years in education, Jeff is particularly interested in the interplay of student motivation and mathematics achievement.

Student engagement has been shown to be a key factor in student achievement (Scherrer & Preckel, 2019). This is problematic in mathematics, since our subject has been found to be the least engaging of any subject in school (Shernoff et al., 2003). If we can get our students more engaged and motivated, the result will be greater participation, deeper understanding, and greater intrinsic motivation (the desire to learn for the sake of learning, rather than external motivations such as grades).

There are four components to engagement: behavioural, emotional, cognitive, and agentic (Ouweneel et al., 2013; Reeve & Lee, 2014). We would like our students to exhibit all four of these dimensions, but particularly cognitive engagement, thinking deeply about the concepts we want them to learn, flexible problem solving, self-regulation, the use of metacognitive strategies, and coping with failure (Conner & Pope, 2013).

Classroom environment and teacher behaviours play a large role in student motivation and engagement. Teachers play a critical role in maintaining and increasing student engagement (Schussler, 2009). Mathematics teachers are frequently criticized for failing to engage student interest and motivation (Archambault et al., 2012). Instructional strategies significantly influence in student engagement; strategies that are active involve students working in groups, employ problem-based learning, and ask students to explain their thinking, involve manipulatives, hands-on activities; and real-world problems were found to have long-lasting effects on student engagement (Moyer et al., 2018; Smith & Star, 2007).

Trigonometry is an ancient, but really interesting, branch of mathematics. Its usefulness stems from its wide applications to many real-life situations. Its interest for students arises because it is novel when introduced and different from the mathematics they had previously studied, as well as having an abundance of information about the uses of trigonometry. Trigonometry is a rich area to increase student engagement and motivation, if our instructional strategies fit the description above.

This article presents some student-centred activities that can be used to build student understanding of trigonometry. These activities allow for student agency, differentiated instruction, and intrinsic motivation, thereby engaging students in their own learning and deepening their understanding of mathematical concepts.

Trigonometric Ratios for a 60° Angle

This activity was originally developed by Texas Instruments for a graphing calculator, but transfers easily to using a spreadsheet such as Excel (Irvine, 2019). It should be done in groups.

Materials needed: protractors, rulers, lined paper (such as foolscap or looseleaf notebook paper; a set of numbered pieces of paper—I suggest numbers 3 to about 10, but each value must be less than the width of the paper.

Each group draws a slip of paper; this is the length of line that they are to draw.

1. Starting at the left margin of the paper, draw a line with length given by the slip of paper, in centimetres perpendicular to the margin.
2. Construct a right angle. The margin of the paper is useful for this.
3. At the right end of the line segment, construct a 60° angle.
4. Complete a triangle by joining the right end of the angle vertex to the vertical line formed by the margin.
6. Measure all sides of the triangle.
7. Record each group's results on a whiteboard. This is a chance to introduce the terms *opposite*, *adjacent*, and *hypotenuse* (although some students may propose the term *beside side* for adjacent).
8. Students then enter the class data into the first three columns of an Excel spreadsheet, with appropriate column headings. Set the spreadsheet to three decimal-place accuracy.
9. In the next three columns, compute the values of opposite/hypotenuse, adjacent/hypotenuse, and opposite/adjacent.
10. Students will quickly notice that the values on each of the last three columns are identical, no matter what the measurements of the triangle were.

This fun activity introduces and reinforces understanding of the primary trig ratios.

Extensions:

- Add columns for the secondary trigonometric ratios and label with correct names.
- Add a column to compute sin260° + cos260° to discover a special case of this identity.
- Recognize that the other angle in the triangle is 30°; compute the trigonometric ratios for 30°; possibly conclude that the ratios of complementary angles are equal.
- This activity could be linked to coding and presents a natural way that coding can be included in the curriculum. Coding could, as one example, be used to compute the tables on a computer screen.

Student-Constructed Clinometers

Almost every math teacher I know does this group activity. It is included here because it is really engaging for students. This is a clear demonstration of the usefulness of trigonometry. Instructions can be found on the Internet to construct a clinometer shown in Figure 1. However, some of the Internet instructions may be confusing, especially since they may omit information that an angle of 85° on the clinometer is actually giving a 5° angle of elevation (consider that sighting through a horizontal straw corresponds to the string showing 90° on the protractor). In assigning measurement tasks, be sure to include some that involve angle of depression as well (looking up = angle of elevation; looking down = angle of depression). One semester, I taught three Grade 10 classes. Thinking to save time, my first class built the clinometers for their investigations, and my other two classes used those clinometers to measure various objects. The two classes that had not built clinometers for themselves were far less engaged in the activity. Apparently, feeling ownership of the measurement tool was important to student agency.

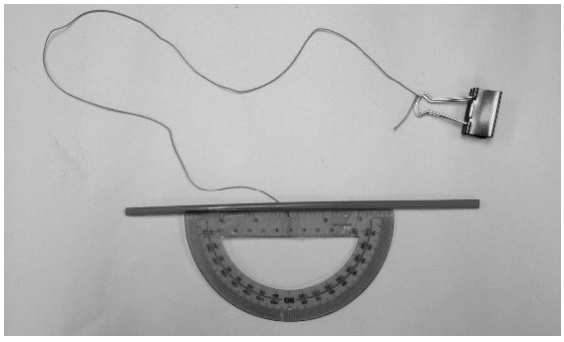


Figure 1: Student-constructed clinometer

Research Report on an Application of Trigonometry

Reports can be written, verbal presentations, or poster presentations, with the format chosen by the student.

How is trigonometry used in the following?:

- Architecture
- Astronomy
- Civil engineering
- Computer music
- Electronics
- Electrical engineering
- Mechanical engineering
- Navigation
- Oceanography
- Phonetics
- Seismology
- Surveying
- Autonomous robots

Alternatively, students can choose to investigate a specific instrument that uses trigonometry, such as a sextant or astrolabe. This activity allows for differentiated instruction by providing choice in both the topic and presentation format, and increases student agency and engagement in the task.

History of Trigonometry

Trigonometry is one of the oldest branches of mathematics. This project allows students to investigate a heritage or legacy in mathematics (Irvine, 2020). Students may choose to investigate a specific problem, such as:

- How did Hipparchus measure the distance to the sun?
- How was trigonometry used in building the pyramids of Egypt? (Figure 2)
- Who were the Harpendotae of ancient Egypt and what did they do?
- Who had trigonometry first: Egypt, Persia, India, Greece, another culture? How did Aristarchus of Samos (310 BC to 230 BC) calculate the distance to the sun?
- Investigate a specific development in trigonometry, such as the sextant for navigation.
- What was trigonometry's first use in surveying?

Students should choose both their topic and their reporting format. This allows for differentiated instruction and increases student engagement.



Figure 1: Student-constructed clinometer

Trigonometry RAFT

Role, Audience, Format, Topic—RAFT for short—is an excellent differentiated-instruction strategy. Because trigonometry has so many applications, it is a great subject for using RAFT. Students choose one row of a RAFT table (Table 1).

Table 1: RAFT for trig tasks

Role	Audience	Format	Topic
Airplane navigator	Pilot and crew	Verbal report	Position of airplane
Surveyor	Landowner	Written report	Property boundaries
Architect	Development company	Building blueprints	Design of building
Artillery spotter	Artillery crew	Radio communication	Range and elevation
Sound engineer	Music composer	Sound mix instructions	Mixing tracks
Medical technician	Doctor and patient	Written report	Results of body scan
Oceanographer	General public	Online news report	Depth of underwater reef
Seismologist	General public	TV news report	Strength and epicentre of earthquake

Trigonometry at the Movies

Students choose an application of trigonometry that appears in a commercial video. The application may be correct or incorrect, and students decide which. They then present their findings, showing a short video clip from the movie and explaining the trigonometry. This activity can be done individually, in pairs, or in groups.

(Caution: There is a BBC series called *Trigonometry*,

which appears to be about polygamy, and not mathematical trigonometry.)

Stairmaster

Students measure various flights of stairs, either in school or out of school, to obtain measurements for riser height (rise) and tread depth (run). Computing the slope of each stair flight gives the tangent of the degree of inclination for the stairs. Using an inverse tangent gives the angle in degrees. Students can then pose and answer questions such as: Are all stairs of similar inclination? Is there a range of inclinations for most stairs? Why do you think the inclination of the stairs is fairly uniform? What factors might go into the design of flights of stairs?

Extensions:

- Group data or class data can be entered into Desmos, and linear regression used to find the average slope, and subsequently the average angle of elevation of flights of stairs.
- Scatter plots can be used to identify outliers; students may pose possible reasons for these outliers.
- Class data may be sorted into home, apartment, commercial buildings to identify significant differences, if any; students may then pose possible reasons for these differences.

What's Your Grade?

Working in groups, students use the Internet to find highway signs that indicate the grades of hills, such as 5% grade, 8% grade, and so on. Students then interpret their findings, compute the angles of inclination of these hills, choose a presentation format for the data in a table, graph, or diagram, and respond to questions such as:

- What is the greatest angle of elevation that your group found?
- Why do you think this maximum exists?
- What could be done to change this maximum?
- Why are train tracks limited to a 3% elevation?

Students can choose how to present their results. They may choose to use technology, such as PowerPoint or Prezi, or print and post their results on large chart paper and share via a gallery walk.

Going Around in Circles

This group activity (Irvine, 2019) involves a graphing calculator and a motion detector. While graphing calculators have lost their place in mathematics instruction (Irvine, 2021), they remain one of the only ways to access data from motion detectors. In this activity, students walk around a hula hoop while facing a motion detector. The resulting graph is sinusoidal (Figure 3). This is a great activity to deepen student understanding of the relationship between circular motion and sinusoidal graphs, as well as fertile ground for "What If?" analysis. The activity can also highlight the use of the scientific method in mathematics and enhancement of higher-order thinking skills (Irvine & Fitzpatrick, in preparation).

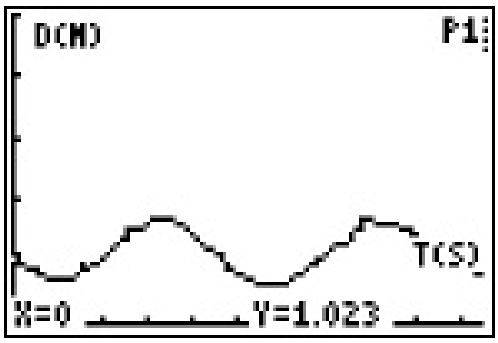


Figure 3: Graph for Going Around in Circles activity

While this activity used older technology that is available in most schools, teachers may wish to use more modern technologies such as MicroBits®, which would allow for the integration of coding into the activity.

The Adventures of Wanda Worm

This entertaining activity (Irvine, 2019) is based on the article "Spaghetti Sine Curves" from *The Mathematics Teacher* (Ozgun-Koca et al., 2013). Wanda digs her way around a partially buried soup can. Students simulate her position by constructing angles at 15˚ intervals around a circle, and break off spaghetti pieces for the height at each angle. The spaghetti pieces are then taped to a horizontal line marked in 15˚ intervals, to produce a sine curve (Figure 4). Done in groups, the activity reinforces the relationship between circular motion and sinusoidal graphs, and deepens student understanding. The angles may be measured in radians at $\frac{\pi}{12}$ intervals, if the teacher wishes. For more details on this activity, see Irvine (2019).

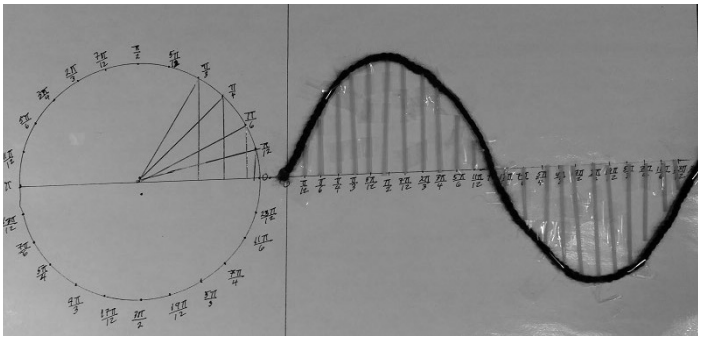


Figure 4: Spaghetti sine curve

Trigonometry Fair

Modelled after the science-fair concept, this showcase combines some of the activities described above. Students present their favourite activity, usually on poster board, and explain the trigonometry involved to their classmates or to students from other classes. The presentations range from students actually composing computerized music to explanations of how students measured inaccessible heights, using clinometers, and many other projects. Students exhibited great pride in showcasing and explaining their work. This activity can also be used for assessment.

Conclusion

Student engagement is critical for success. Conner and Pope (2013) found that student engagement declines over the course of a student's time in school, and that "by upper high school, 40% to 60% of students are disengaged" (p. 1427). We cannot allow this pattern to continue. Mathematics is a subject that one learns by doing. As teachers, we must engage and motivate our students to succeed. The tasks described in this article illustrate some of the ways that we can engage students in the study of trigonometry, resulting in a greater understanding of concepts and more intrinsic motivation, which has been shown to be a key factor in lifelong learning (Ryan & Deci, 2000). These activities do not require a great deal of preparation on the teacher's part, yet the resulting student engagement and motivation far outweigh the teacher's investment in time, and demonstrably increase student success.

References

Archambault, I., Janosz, M., & Chouinard, R. (2012). Teacher beliefs as predictors of adolescents' cognitive engagement and achievement in mathematics. *Journal of Educational Research, 105*(5), 319–328. doi.org/10.1080/00220671.2011.629694

Conner, J.O., & Pope, D. (2013). Not just robo-students: Why full engagement matters and how schools can promote it. *Journal of Youth and Adolescence, 42*(9), 1426–1442. doi.org/10.1007/s10964-013-9948-y

Irvine, J. (2019). Opportunities to address metacognition and higher-order thinking skills when we teach trigonometry. *Ontario Mathematics Gazette, 58*(2), 45–48.

Irvine, J. (2020). The history of mathematics—Whose history is it? *Ontario Mathematics Gazette, 59*(1), 40–45.

Irvine, J. (2021). Blast from the past: The rise and fall of the hand-held calculator in mathematics education. *Ontario Mathematics Gazette, 60*(2), 7–9.

Irvine, J., & Fitzpatrick, S. (in preparation). Hula hoop hypotheses.

Moyer, J.C., Robison, V., & Cai, J. (2018). Attitudes of high-school students taught using traditional and reform mathematics curricula in middle school: A retrospective analysis. *Educational Studies in Mathematics, 98*(1), 115–134. doi.org/10.1007/s10649-018-9809-4

Ouweneel, E., Schaufeli, W.B., & LeBlanc, P.M. (2013). Believe, and you will achieve: Changes over time in self-efficacy, engagement, and performance. *Applied Psychology: Health and Well-Being, 5*(2), 225–247. doi.org/10.1111/aphw.12008

Ozgun-Koca, S.A., Edwards, M.T., & Meagher, M. (2013). Spaghetti sine curves. *Mathematics Teacher Learning and Teaching, 107*(3), 180–187. doi.org://10.5951/mathteacher.107.3.0180

Reeve, J., & Lee, W. (2014). Students' classroom engagement produces longitudinal changes in classroom motivation. *Journal of Educational Psychology, 106*(2), 527–540. doi.org/10.1037/a0034934

Ryan, R.M., & Deci, E.L. (2000). Self-determination theory and the facilitation of intrinsic motivation, social development, and well-being. *American Psychologist, 55*(1), 68–78. doi.org/10.1037/0003-066X.55.1.68

Scherrer, V., & Preckel, F. (2019). Development of motivational variables and self-esteem during the school career: A meta-analysis of longitudinal studies. *Review of Educational Research, 89*(2), 211–258. doi.org/10.3102/00346543188 19127

Schussler, D.L. (2009). Beyond content: How teachers manage classrooms to facilitate intellectual engagement for disengaged students. *Theory Into Practice, 48*(2), 114–121. doi.org/10.1080/00405840902776376

Shernoff, D.J., Csikszentmihalyi, M., Schneider, B., & Shernoff, E.S. (2003). Student engagement in high school classrooms from the perspective of flow theory. *School Psychology Quarterly, 18*(2), 158–176. doi.org/10.1521/scpq.18.2.158.21860

Smith, J.P., III, & Star, J.R. (2007). Expanding the notion of impact of K–12 standards-based mathematics and reform calculus programs. *Journal for Research in Mathematics Education, 38*(1), 3–34. www.jstor.org/stable/30034926 ▲

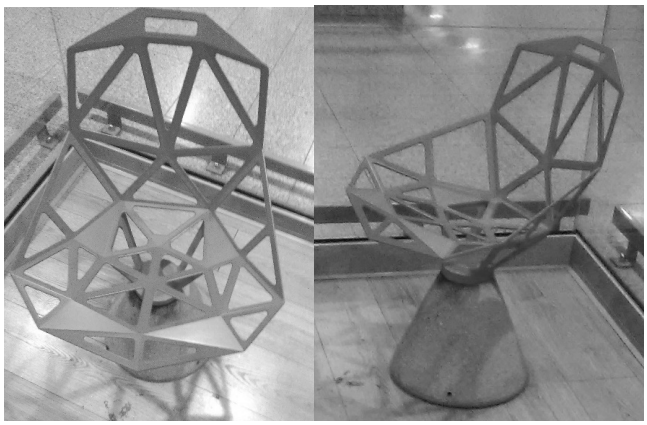
Geometric Chair?

Below are two photos of a chair in the train station in Rouen, France.

How many congruent triangles are used in the design of this chair?

How many distinct, but similar, triangles are used?

Two quadrilaterals appear in the design. How could they be turned into triangles? If they were turned into triangles, would the triangles be congruent or similar to any other triangles?



Notakto — My Favourite Mathematical Game for K–12 Classes



PETER GUSTAINIS
pgustainis@bss.on.ca

Peter is a secondary math and physics teacher, currently working at The Bishop Strachan School. He enjoys exploring connections between mathematics and other fields of study, such as science, art, and technology.

Why is *Notakto* my favourite mathematical game? First, it is simple enough to play with a Kindergarten student, but with enough depth to perplex and fascinate students all the way to Grade 12 and beyond! Second, it has no numbers, but as you play the game, you will encounter fundamental mathematical concepts such as geometry, symmetry, and logical reasoning. It makes an engaging and challenging lead-in activity before any unit on geometry or shapes. Finally, it is a lot of fun for students and teachers alike, so much so, that I have organized tournaments at my school for interested students, at their request! It's an amazing activity for the first week of your course, accessible and interesting for every student. It makes a fantastic non-curricular task for developing collaborative problem-solving skills, and it will get your students thinking mathematically right away (without even realizing it)!

The Game of *Notakto*

The rules for *Notakto* are simple. *Notakto* is exactly the same as Tic-Tac-Toe (Noughts and Crosses), but with two important rule changes:

1) Both players play with "X"'s only.
2) The first player to make a 3-in-a-row loses.

I often dedicate a full class to *Notakto* at the start of the year as an exercise to get my students practising their collaborative problem-solving skills. To explain *Notakto* to students, start by asking for a student volunteer to play a game with you while the rest of the class observes. Let them choose if they want to go first or second. Figure 1 shows how a game with a student might go if they choose to go first.

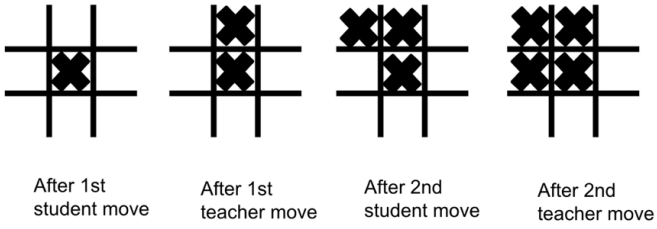


Figure 1: *Example of* Notakto *game with one board, with student going first, where teacher wins*

Notice that after four moves, the student cannot make another move without making a 3-in-a-row. This means the teacher wins. The final board in Figure 1 is called an *endstate*, where the board doesn't have a 3-in-a-row, but any additional "X" is guaranteed to make a 3-in-a-row. Every game of *Notakto* will end in an endstate, and the endstate is very similar to checkmate in chess. It turns out there are six unique endstates, not counting rotations and reflections. You can tell the students this, or leave it a mystery. Here are some good questions to prompt student thinking about endstates:

- Do you think this is the only endstate possible?
- How many different endstates do you think are possible?
  - You could also discuss rotations and reflections of the endstate above and whether or not they count as "different."
- Do you think all of them have the same number of "X"'s? Could they have more? Fewer?

Afterwards, create groups of three and challenge the students to figure out a strategy that guarantees victory every time. In order to complete the challenge, every member of their group must beat you, the teacher, on their own, with no help from teammates during the game. They can prepare and strategize beforehand, but once the game starts, they are on their own. For each game, they get to choose whether they want to go first or second. To guarantee victory, they need to go first, but don't share that yet. A good suggestion for students to get started is to play as many games as possible, swapping up who goes first or second, but not erasing boards afterwards. The more they play, the more they will notice important patterns and find more of the endstates. This works well on vertical non-permanent surfaces so that groups can see what strategies other groups are trying, and endstates they have found.

I suggest giving at least 5 minutes for play before accepting any challenges from groups. If a group is unsuccessful in their challenge, tell them they can try again after all the other groups have had a chance, or after a couple of minutes, if no other group wants to challenge.

Once you get good at the game, you can easily play against two or three groups at the same time to speed things up.

If a group is successful, tell them that this was just the first challenge, and that the next challenge is a bit trickier. It involves playing the game with two boards, at the same time! Rules for multi-board *Notakto* will be explained later. Students can start playing two-board *Notakto* to try to figure out a winning strategy, but it's much more complicated. I tell students that if any group can beat me at two-board *Notakto*, even just once, they each win a small prize. After explaining the rules to the first group, any other successful groups are pointed in their direction to get the rules explanation. Depending on the age of the group, and how quickly the correct strategy spreads around the room, figuring out how to win at one-board *Notakto* takes 15 to 25 minutes with Grades 8–12. I have only played *Notakto* with Grades 8–12, but I imagine it would work similarly with younger grades, adapting as needed.

Before I show you how to win at one-board *Notakto*, step away and try the game for yourself. It's best if you have someone to play against, but even if you don't, see if you can find all six unique endstates before reading on!

How to Win at One-Board *Notakto*

After 15 to 25 minutes, or once at least half of the groups have successfully completed the first challenge, bring the class together again. Ask another student volunteer, one from a group who completed the challenge, to play against you for the class to observe. Figure 2 shows how a game with a student might go if they choose to go first and win.

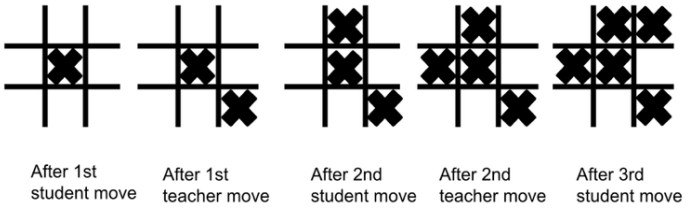


Figure 2: *Example of* Notakto *game with one board, with student going first, where student wins*

While playing the game, prompt the student to explain their thinking for each move. They will know that going first in the centre is important, but they probably won't know why. The reason is related to the six possible endstates (see Figure 3). Pause and see if any group has all six endstates. Often no single group will, but all six will be present somewhere around the room. Ask students to think about how many endstates are possible if you include rotations and reflections. This is a great way to introduce the idea of rotation and reflection symmetry (e.g., endstate 2 has four-fold rotation symmetry, and endstate 6 has two-fold rotation symmetry; endstate 4 has reflection symmetry about a vertical central line, while endstate 5 has reflection symmetry about a top-right to bottom-left diagonal line). There are twenty-three in total: four for endstate 1, one for endstate 2, eight for endstate 3, four for endstate 4, four for endstate 5, and two for endstate 6. You could offer a prize to a group who can list out all the possible endstates, including rotations and reflections.

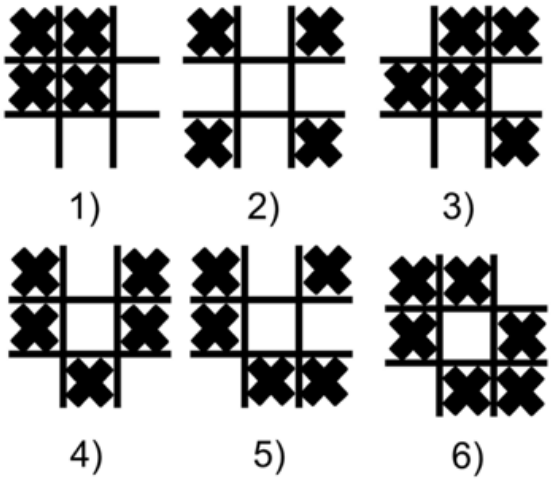


Figure 3: *All six unique endstates for* Notakto

Back to the game in Figure 2: by playing in the centre on the first move, the only possible endstates are 1 and 3. This drastically simplifies the game, since now it can only end one of two ways. Notice endstate 1 has four "X"'s, and endstate 3 has five "X"'s. This means that the first player (the student in this case) will win if the game ends in endstate 3. Before making your first move, prompt students to think about how many possible moves you have. Many will say eight, since there are eight spots available after the middle is taken. However, there are really only two possible moves, side or corner, since all sides and corners are effectively the same.

To guarantee endstate 3, on their second move, the student can make the shape shown in Figure 2, "After 2nd student move." Endstate 1 is now no longer possible, and endstate 3 is unavoidable, guaranteeing their victory. Figure 4 shows all the possible paths to victory, if the first player plays in the centre to start (not including rotations and reflections).

When playing, prompt students to think about whether there were other moves that also worked. Often, these moves will share some symmetry with one another, but not always. For example, for the student's second move in Figure 2, they could have played top-middle or middle-left, since these both produce the same shape, but rotated.

Alternatively, on their second move the student could have played in an adjacent corner, top right or bottom left. This creates a different shape (not a rotation or reflection of the board after the second student move in Figure 2), but still guaranteeing endstate 3 (see the right side of Figure 4).

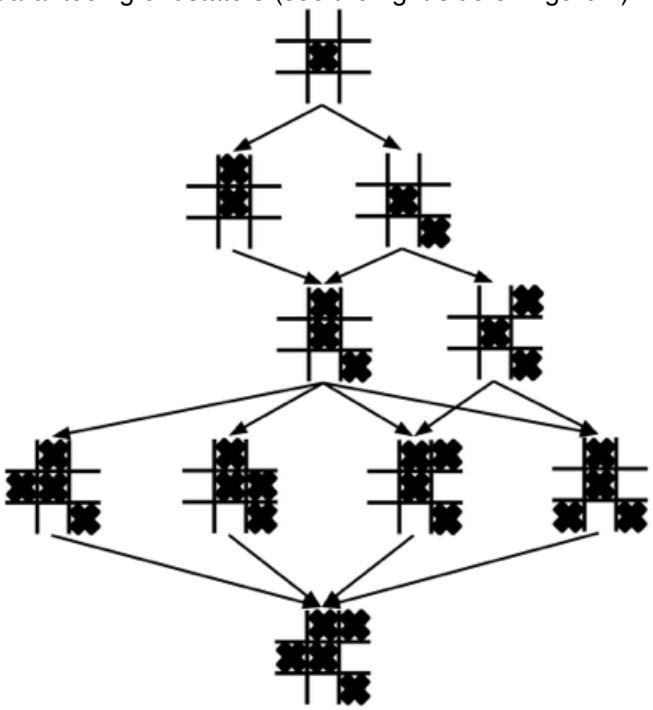


Figure 4: *Tree diagram of the winning paths for first player when they play in the centre to start*

If the first player doesn't go in the middle first, the second player can always win by simply mirroring the first player's move on the board (see Figure 5 below). This will result in endstate 2 or 6, both of which have an even number of "X"'s, so the second player will win. You can explain this strategy to students directly, or challenge them to find it on their own. This turns out to be very important for *Notakto* with more than one board. Afterwards, show the whole class the rules for two-board *Notakto*.

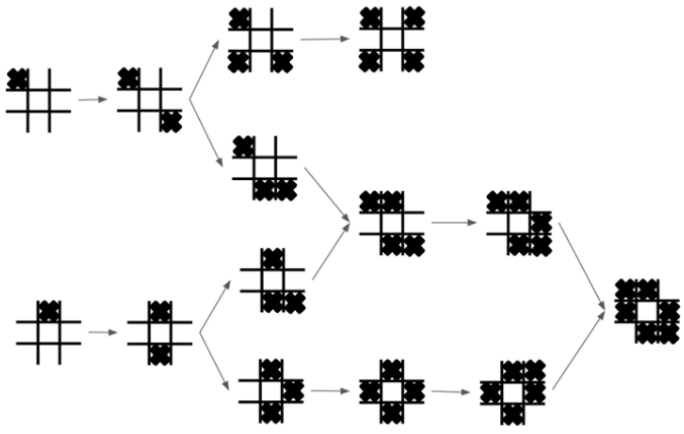


Figure 5: *Tree diagram of winning path for second player, if first player does not play in the centre to start*

Rules for *Notakto* with Two or More Boards

On each player's turn, they can put an "X" on any board. Play continues as normal, but when a 3-in-a-row occurs, the board gets eliminated and play continues on the remaining boards. This keeps going until there is only one board remaining, and then whoever makes 3-in-a-row on that board loses, as normal. In this way, any game of three-board *Notakto* will always eventually become a game of two-board *Notakto*, which will then become a game of one-board *Notakto*. That means that in order to master three-board *Notakto*, you will need to master two- and one-board *Notakto* first. It may be helpful to remind students of this, who try to rush to two-board *Notakto*, without fully understanding one-board *Notakto* first.

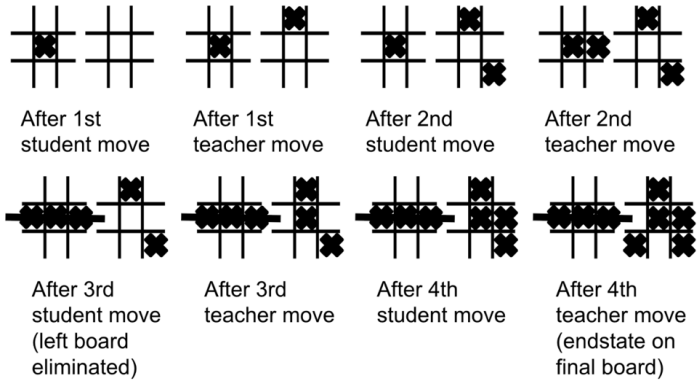


Figure 6: *Example of* Notakto *game with two boards, with teacher going second, where teacher wins*

In Figure 6 on the student's third move, they decide to eliminate the left board, which means the game finishes on the right board. Unfortunately, this is a bad move because the teacher can turn the remaining board into a familiar board state that guarantees endstate 3, meaning the teacher wins, as they will place the fifth "X" in the endstate.

After explaining the rules to the students, send them back to their groups to play and, when ready, to challenge you for a prize! You will likely have a couple of groups beat you, but they will often struggle to do so consistently. Encourage them to keep playing and looking for patterns, or move on to three-board *Notakto*, if they think they have two-board mastered. At some point, give them the hint that for two-board *Notakto*, you want to go second; for three-board, you want to go first; for four-board second; etc. If you start first with one board, or second with two boards, etc., you say you have "the advantage" to start.

As a final challenge to the class, I tell them any student or group of students can challenge me anytime for the rest of the year to three-board *Notakto* for a chance to win a big prize for the entire class (only once per class). The rules:

- You can only challenge me once per day.
- You must beat me three times in a row (all on the same day).
- You can choose whether you go first or second each time (they should choose first).
- You can't have any notes, papers, or reference sheets.

I leave the challenge open indefinitely until one person or group from each class beats me! To help them practise (outside of class time, since this requires a smartphone), there is a free app for iOS and Android, where they can play *Notakto* against a computer (Counterwave Games, 2024). There are multiple difficulty levels, but maximum difficulty is best to test your strategy (on lower difficulty, the computer will make incorrect moves with some frequency).

Going Beyond One-Board *Notakto*

Once you get to two or more boards for *Notakto*, all the different branching possibilities become quite complicated to explain exhaustively. However, it is well within the grasp of enthusiastic Grade 8 students to uncover for themselves, even up to six or more boards. It is likely possible for younger students as well, though I don't have experience working with them. The patterns will emerge organically from play with one another, and especially with the app. For example, they might notice that playing in the centre, if possible, is a good idea because it drastically reduces the number of endstates for that board, and it is easier to eliminate later on to simplify the game.

However, this learning process does take a lot of trial and error. Mastering up to three or more boards will take some out-of-class time and patience from students. As a teacher, it is very helpful (though not strictly necessary) to be proficient at the game up to two or three boards. This allows you to facilitate the challenges I have outlined previously. In the appendix (available on the OAME website in the members only section under "Member Links \ Files and Links"), I provide an intuitive guide to two-board *Notakto*. The intuitive solution for three or more boards follows a similar pattern.

Also in the appendix, I share the full solution to the game for any number of boards. The full solution is complicated, but I have included some tips and tricks to make it easier to implement in practice. If you want an overview of the full solution, there is a three-video series on YouTube (Numberphile, 2015 & 2016). The full solution to *Notakto* is also outlined in an article by Plambeck & Whitehead (2013), but it is very technical and can be difficult to follow.

The final appendix section includes some ideas for how you can take *Notakto* even further, including *Notakto* match rules, *Notakto* variants, and other ways to engage students, such as tips on how to create computer code that plays *Notakto*.

I hope you enjoy playing *Notakto* with your students, and show it to your friends and colleagues. It's my favourite mathematical game, and it still is after more than seven years of play. It's an amazing introductory activity to get all your students engaged and collaborating at the start of the year. It provides a low-floor, high-ceiling problem to get them thinking co-operatively and mathematically, and it will fascinate students well beyond the one day in class. It's also an excellent activity for any unit related to geometry, symmetry, logical reasoning, or visual–spatial thinking.

I am always delighted when I get the chance to engage with someone about *Notakto*. I'd love to hear about how it goes in your classes, or if you have any questions about the game.

References

Counterwave Games. (2024). *Notakto*. www.counterwavegames.com/notakto/

George, W., & Janowski, J.E. (2016). *Group actions on winning games of super tic-tac-toe.* arXiv preprint. www.arxiv.org/abs/1606.04779

Numberphile. (2015, November 26). *Tic-tac-toe (with Xs only) – Numberphile* [Video]. YouTube. www.youtube.com/watch?v=ktPvjr1tiKk

Numberphile. (2015, December 1). *Secrets of 3-board tic-tac-toe – Numberphile* [Video]. YouTube. www.youtube.com/watch?v=h09XU8t8eUM

Numberphile. (2016, January 7). *Notakto (extra footage) – Numberphile* [Video]. YouTube. www.youtube.com/watch?v=v494yltz_7I

Plambeck, T.E., & Whitehead, G. (2013). The secrets of notakto: Winning at X-only tic-tac-toe. *Recreational Mathematics Magazine, 1,* 49–54. arXiv preprint. www.arxiv.org/abs/1301.1672 ▲

Wordsearch Packing?

What size is the smallest wordsearch for linked consecutive numbers?

Examples where random letters can be placed in empty cells:

Zero–Two

Z	E	R	O
		W	N
	T		E

Four–Ten

X	I	S	
	T	E	N
F	I	V	E
O		E	I
U		N	G
R		I	H
		N	T

▲ 2025–2026 OAME BOARD

OAME WEBSITE
www.oame.on.ca

DIRECT GENERAL INQUIRIES TO: EXECUTIVE DIRECTORS

Fred and Lynda Ferneyhough
4751 Mack Ave.
Plympton-Wyoming, ON N0N 1J6
Business Phone: 647-466-0674
EDs@oame.on.ca

PRESIDENT

Kit Luce
Barrie, ON
kit.luce@oame.on.ca

PRESIDENT ELECT

Mark Chubb
St. Catharines, ON
mark.chubb@oame.on.ca

VICE-PRESIDENTS

Cal Armstrong
Burlington, ON
cal.armstrong@oame.on.ca

Jennifer Clark
Bowmanville, ON
jennifer.clark@oame.on.ca

EMERITUS

Paul Alves
Georgetown, ON
paul.alves@oame.on.ca

CHAPTERS & REPRESENTATIVES

CHAMP: Giovanni Bruccoleri
Mississauga, ON
giovanni.bruccoleri@oame.on.ca

GOLDEN: Jessica Drake
St. Ann's, ON
jessica.drake@oame.on.ca

GVMA: Glen McMillan
Guelph, ON
glen.mcmillan@oame.on.ca

ISOMA: Cynthia Shi
Oakville, ON
cynthia.shi@oame.on.ca

MAC^2: Stephanie Rogers
Bracebridge, ON
stephanie.rogers@oame.on.ca

NOMA: Heather Theijsmeijer
Kagawong, ON
heather.theijsmeijer@oame.on.ca

NWOAME: Janet Fuchek
Thunder Bay, ON
janet.fuchek@oame.on.ca

O$_3$4ME: Teresa Weselake
McNab Braeside, ON
teresa.weselake@oame.on.ca

PRMA: Meghan Waybrant
Oshawa, ON
meghan.waybrant@oame.on.ca

QSLMA: Tanis Lattanzio
Kingston, ON
tanis.lattanzio@oame.on.ca

SAME: Nkiru Azza
Scarborough, ON
nkiru.azza@oame.on.ca

SWOAME: Elizabeth Pearsall
Kingsville, ON
elizabeth.pearsall@oame.on.ca

TEAMS: Alvarine Aldridge
Toronto, ON
alvarine.aldridge@oame.on.ca

WOMA: Brian McBain
Sarnia, ON
brian.mcbain@oame.on.ca

Y^4 MA: Gina Micomonaco
Richmond Hill, ON
gina.micomonaco@oame.on.ca

DIRECTORS – TERM EXPIRING 2026

Elli Weisdorf
Toronto, ON
elli.weisdorf@oame.on.ca

Lisa Rossiter-Thornton
Toronto, ON
lisa.rossitor-thornton@oame.on.ca

DIRECTORS – TERM EXPIRING 2027

Olive Creary-Satchell
Toronto, ON
olive.creary-satchell@oame.on.ca

Michael Frankfort
Richmond Hill, ON
michael.frankfort@oame.on.ca

Jennifer Holm
Kitchener, ON
jennifer.holm@oame.on.ca

DIRECTORS – TERM EXPIRING 2028

Rosanna Mortillaro
Toronto, ON
rosanna.mortillaro@oame.on.ca

Mariarosa Schniering
Newmarket, ON
mariarosa.schniering@oame.on.ca

Cameron Redsell-Montgomerie
Toronto, ON
cameron.redsell-montgomerie
@oame.on.ca

NCTM REPRESENTATIVE

Gerard Lewis
Toronto, ON
gerard.lewis@oame.on.ca

WEBSITE CO-ORDINATOR

Greg Clarke
Orillia, ON
web@oame.on.ca

GAZETTE EDITOR

Timothy Sibbald
North Bay, ON
tim.sibbald@oame.on.ca

ABACUS CO-EDITORS

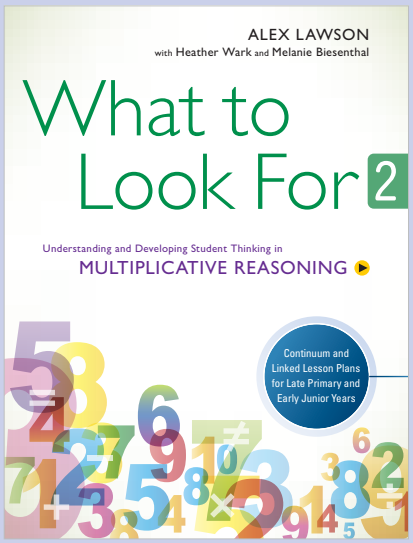
Marc Husband
Toronto, ON
marc.husband@oame.on.ca

Tina Rapke
Toronto, ON
tina.rapke@oame.on.ca

Cristina De Simone
Mississauga, ON
cristina.desimone@oame.on.ca

Pearson

Look What's New!



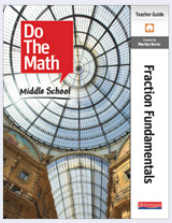
Now Available!

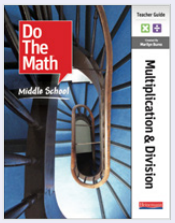
Resources for Intervention

Do The Math®

Now available for Grades 1–5 and 6–8

Flexible, classroom-tested instruction for building numerical reasoning and confidence.

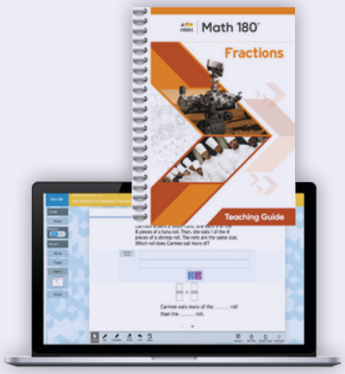




Grades 3–12

HMH Math 180®

Delivers evidence-based instruction and assessments to propel students to grade-level proficiency.



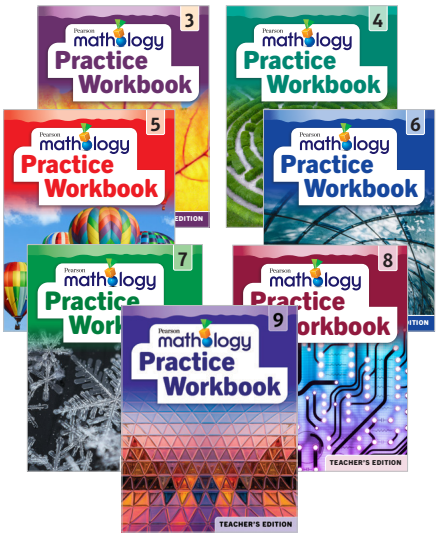
NEW

mathology

Now up to Grade 9









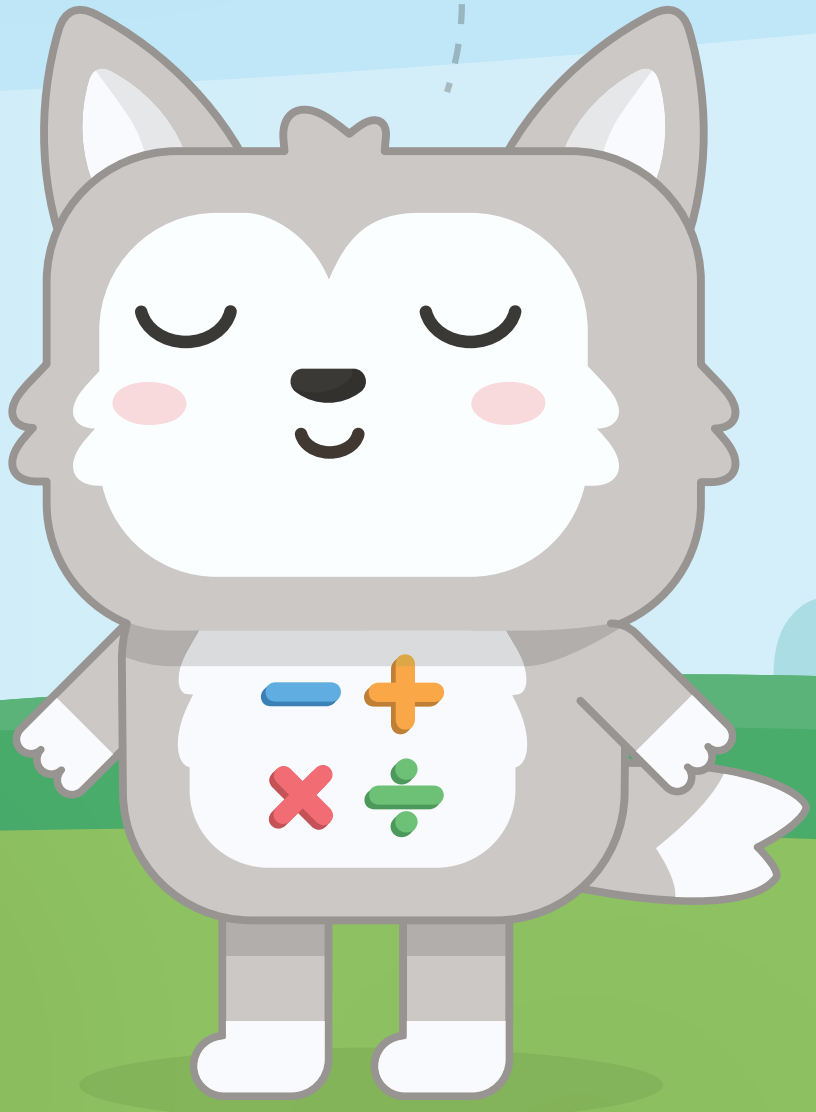
matific

Adaptive Responsive On-line Practice



Supporting math success in early years

Vretta | Carleton University



To schedule a demo or request complimentary access to pilot the research-backed numeracy screener in your class or school, email **info@vretta.com**.

www.vretta.com