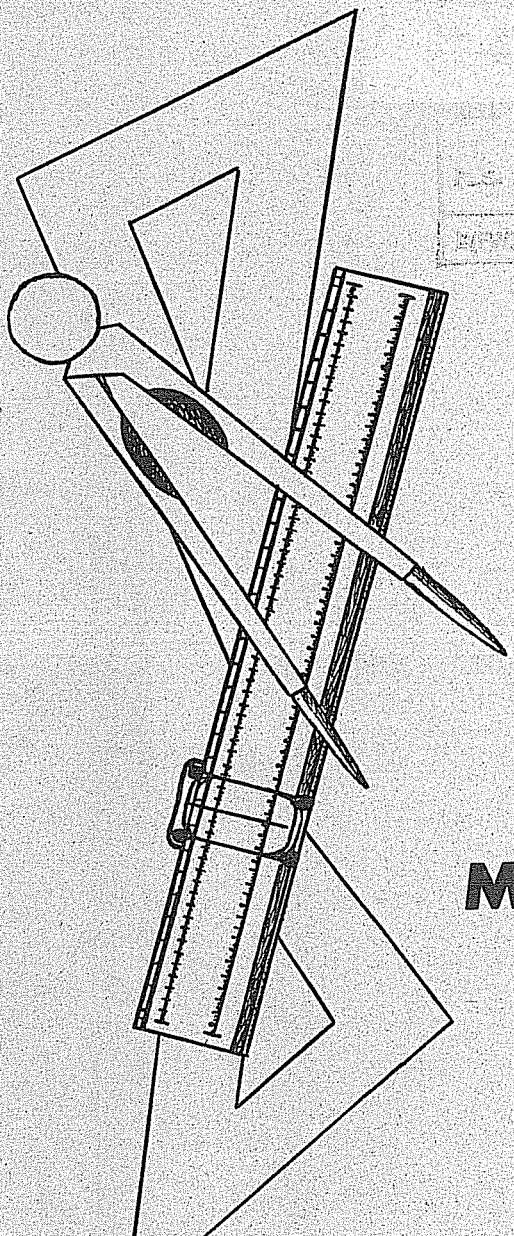


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# ONTARIO MATHEMATICS GAZETTE

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ONTARIO MATHEMATICS GAZETTE

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# EDITORIAL

This issue of the Gazette has no particular theme. I hope you will find its articles interesting and constructive.

I have just finished reading an extraordinarily absorbing and moving book, called HOW CHILDREN FAIL, by John Holt (Delta) 1964. The author has a rare gift for probing the thoughts of school children as they struggle with numbers and addition facts. How much do we as teachers know, or for that matter, care, about the thought processes of our charges?

As we embark on a new school year, rounding out our Centennial projects, perhaps we should consider how to teach the new Mathematics courses. Classroom organization, teaching methods, teacher training, courses of study, textbooks, evaluation, and soon, - these are all facets of the problem of providing our students with the best possible Mathematics education. These items really cannot be dealt with separately as they are inextricable.

This is all a roundabout way of leading up to a few comments on current problems. The first I shall deal with is the organization of children into classes for the learning of Mathematics. It is first on the list because the children are our whole "raison d'etre". If it is true that children learn best by doing, in a relaxed atmosphere, in a laboratory type situation, often working in groups, then our schools should be organized to provide that kind of learning climate. The Department of Education deserves a great deal of credit for bringing Miss Edith Riggs to Ontario to conduct her workshops. I wish all of you could have attended one. Let's hope that significant changes in approach, both at the elementary and secondary levels, follow. (Why not in the universities too?)

With regard to teacher training, I welcome the announcement by the Minister, alheit delayed, that the emergency summer courses will he abolished next year. In this I take issue with Dr. Stanton, who hecause of his undeniable talents as a teacher, as well as a mathematician. deserves to be listened to. My main point in this regard is the feeling that every prospective teacher requires, as a hare minimum, ten weeks of practice teaching under competent guidance. The present scheme of paying full salaries to university graduates without any classroom experience is completely inadequate. The format of the College of Education year for prospective secondary school teachers is, of course, open to change. However, we should not reduce the amount of professional training required just because we disagree on some details of the method of training. That argument carried over to our pupils would lead to the abolition of all mathematics teaching from kindergarten up!

There is not space in this editorial to deal with all of the issues of the day; let's look at textbooks for a moment (ugh!). Have our authors gone mad? Are they being paid for their tomes by the pound? Do they realize that for a student to read a 500 page book and work a few of the problems he would need to be doing mathematics 12 hours a day, 200 days a year? Why can't we apply some of that mathematical precision and conciseness to our dialogue with the student reader of mathematics?

My last blast, before I subside into obscurity again, after an interesting year at the helm of the Mathematics Commission, is about examinations. I am glad the Grade 13 examinations have been abolished, although I fear they will he replaced by something worse. We are altogether too examination conscious in our system and it was time for a change. However, if this results in an increased use of multiple choice tests in the name of "standards" and "uniformity", we may wish for the good old days when many a Grade 13 student learned a lot by working old exam papers with the help of competent interested teachers - even though the examination

day itself taught him nothing. Incidentally, I thought the Grade 13 Analysis paper was a disaster. What a way to end a tradition - with a loud raspberry!

I should like to close on a more optimistic note. The Ontario Teachers' Federation deserves the highest praise and commendation by all mathematics teachers. At a time when the Federation has many urgent demands on its resources, it has decided to increase its grant to the Ontario Mathematics Commission by nearly 40% from $9,000 to $12,500. The Executive and members of O.M.C. are most grateful for this display of confidence. We hope to apply much of the budget increase to a study of the Intermediate Division, beginning with Grades 7 and 8. We need your help and suggestions. Write to the Gazette or the Secretary of O.M.C.

Best wishes for a successful year.

A.W.H.

# EXPLORATION AND INVENTION IN MATHEMATICS

by

G.F.D. Duff, University of Toronto

PART I of a 2-Part Article based on Expo Junior Science Lecture Series given in Montreal on August 5th, 1967.

I should like to talk to you about certain aspects of mathematics, and the way it is regarded by people who engage in doing mathematics, illustrated by some particular facts and situations typical of modern mathematical activity. From your work in school you must already have some very definite impressions of the nature of this fascinating subject, and from a visit to Expo 67 you must be impressed by the immense wealth of geometrical variety in the many architectural forms of the Expo pavilions, the sculpture, and the decorations.

I wonder whether mathematics has ever intrigued you because of its classical virtues of precision and logical perfection - the virtues attributed for so many centuries, not altogether correctly, to the Geometry of the Greeks, and especially to Euclid. I wonder whether you have ever been spellbound by the austere beauty of the basic concepts of Geometry, and by the force and majesty of the many impressive theorems arising from them. I wonder, also, if perhaps your interest in mathematics has to some extent a more practical foundation, such as the use of it in the worlds of science, engineering or commerce, or even (perish the thought) of its use in the passing of examinations! No doubt some combination of these motives acts upon us all when we undertake mathematical work or study. What I wish to emphasize in my talk today is related to the way we might look upon the mathematical ideas we work with. I want to stress that all such mathematical ideas and concepts are the result of the creative invention of some one or more people. I want to show you that at the present time, and no doubt in the future too, the flow

of new ideas and concepts in mathematics is increasing and continually finding new applications. I want to suggest to you, even if I cannot demonstrate it as well as I would like to, that the role of art in mathematics is today also a flourishing one, no less than in music or painting, and with at least as much impact, in the long run, upon our culture. I am sure, however, that a tour of Expo 67 will quickly suggest to you the important role of mathematics, and especially of informal geometry, upon culture and art themselves.

How many of you have taken courses in what is popularly called 'New Mathematics'? To my mind, many of the new courses convey very well some of the ideas I have just tried to express. You learn that these are new ideas, about new and different things, and new kinds of reasoning and combinations of the ideas. Actually most of these ideas are not strictly new now, for they were invented for the most part in the nineteenth century. 'Newer' mathematics would really be a better title. But many of you will meet 20th century mathematics in the university. Some day some of you will learn some 21st century mathematics, or even invent some. Despite the real difficulty of keeping up to date, however, you are all very fortunate that at a young age, when you still have great flexibility of outlook and much inventiveness, you are seeing a vastly wider variety of mathematical concepts. Whereas the ideal of teaching mathematics was once "rigour", now it also includes "vigour". In this respect the teaching of mathematics has returned to a more intimate relation with its applications.

If we are to be vigorous mathematicians, we have to look at the whole world around us as a source of mathematical ideas. We have to make the building of mathematical forms, or theories, or "models", an activity that flows naturally from an inquiring state of mind. If we are to enjoy such enterprises we must, however, be prepared to accept some disappointments with a smile, and be ready to learn from adversity. For nearly all of our bright ideas will turn

out less well than we had hoped. Never mind, because this happens to all who adventure in any field of science. Let us instead adopt the very wise saying of Lawrence Sterne's Sentimental Journey: "Tis more blessed to travel hopefully than to arrive." For our own hopeful journey this afternoon, we shall stop at three or four places, to admire the scenery and perhaps get a good view, or a new view, of some prominent feature of the mathematical landscape.

At our first stopping place, we shall discuss certain Number Systems. In modern mathematics, we do not just do new calculations. Rather we often invent and use entirely new systems or structures in which new kinds of calculations can be done. Let us explore such a system based on the arithmetic of the 12 hour clock.

Thus (writing noon hour as zero) we see that (interpreting in hours)

| | |
|---|---|
| $2 + 2 = 4$ | 2 hours after 2 is 4 |
| $3 + 6 = 9$ | 6 hours after 3 is 9 or vice versa |

and so on, as usual. But if a sum exceeds 12, we reckon only its excess over 12:

| | |
|---|---|
| $5 + 8 = 1$ | 8 hours after 5 is 1, or 5 hours after 8 is 1 |
| $9 + 7 = 4$ | 7 hours after 9 is 4, or 9 hours after 7 is 4 |
| $10 + 11 = 9$ | 11 hours after 10 is 9, or 10 hours after 11 is 9 |

Subtraction (or periods of hours of time) is also easy to do if no "Carry over" appears.

| | |
|---|---|
| $9 - 6 = 3$ | (e.g. 6 hours before 9 is 3) |
| $11 - 4 = 7$ | (e.g. 4 hours before 11 is 7). |

However, if a multiple of 12 appears, we again discard it:

| | |
|---|---|
| $2 - 4 = -2 = 10$ | (four hours before 2 is 10; the -2 is replaced by 10, adding 12). |

$3 - 8 = 7$ (8 hours before 3 is 7).

A complete system of addition and subtraction is possible, with well-defined answers. Those who like axioms can try to invent a set of axioms to fit this "clock arithmetic" model, which is sometimes called "arithmetic modulo 12".

With real numbers, we can also multiply and divide. Here, if we multiply $a$ and $b$, we obtain an integer $n$: $a \times b = n$. In figure 1 is shown the table of multiples of 4 and their reduction modulo 12.

Observe the pattern of the right-hand column, only the numbers 0,4,8 appears, each 4 times in cyclic order. Now can we reverse this operation? Can we divide? If $a$ and $c$ are given, can we find an $n$ such that

$$a \times n = c \ ?$$

For example, is there an $n$ such that

$$4 \times n = 7 \ ?$$

Clearly not, as 7 would appear in the right-hand column if there was! Also, is there an $n$ such that

$$4 \times n = 8 \ ?$$

Yes - there are four such $n$'s, namely 2, 5, 8, 11. This is a feast or famine situation, and it means that <u>division</u> can not be usefully defined in this system. Such a system is called a <u>ring</u> -- you can see that it does not have all the properties of division that we are accustomed to when we use real numbers.

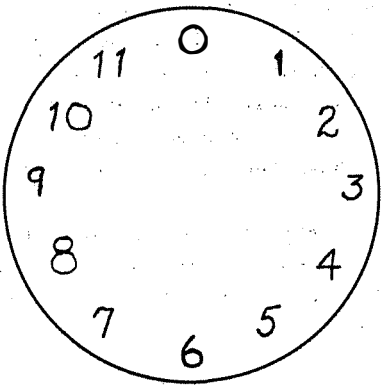
However, let me change things a little. Let us defy misfortune by setting up a clock with 13 hours on its dial. In other words, let us cast out all multiples of 13 instead of multiples of 12.

That is,

$$6 + 7 = 0 \ , \quad \begin{array}{l} 4 + 7 = 11 \\ 9 + 6 = 2 \\ 11 + 12 = 10 \end{array} \quad \begin{array}{l} 2 - 7 = 8 \\ 5 - 9 = 9 \end{array}$$

Can we multiply on the 13 hour clock? The multiples of 4 are given in the figure.

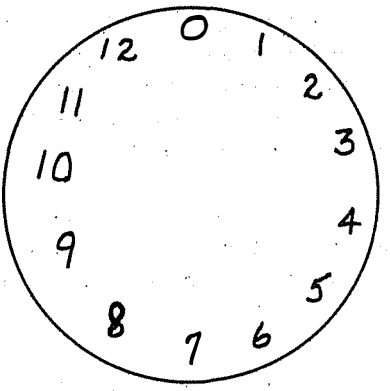
$R_{12}$




modulo 12

4 x 0 = 0
4 x 1 = 4
4 x 2 = 8
4 x 3 = 12 = 0
4 x 4 = 12 + 4 = 4
4 x 5 = 12 + 8 = 8
4 x 6 = 12 + 12 = 0
4 x 7 = 24 + 4 = 4
4 x 8 = 24 + 8 = 8
4 x 9 = 24 + 12 = 0
4 x10 = 36 + 4 = 4
4 x11 = 36 + 8 = 8

$F_{13}$




modulo 13

4 x 0 = 0
4 x 1 = 4
4 x 2 = 8
4 x 3 = 12
4 x 4 = 13 + 3 = 3
4 x 5 = 13 + 7 = 7
4 x 6 = 13 + 11=11
4 x 7 = 26 + 2 = 2
4 x 8 = 26 + 6 = 6
4 x 9 = 26 + 10=10
4 x10 = 39 + 1 = 1
4 x11 = 39 + 5 = 5
4 x12 = 39 + 9 = 9

Figure 1

What numbers appear in the right-hand column? (We may call this the range of the operation of multiplication by 4). All the numbers of the system appear! Moreover each number appears just once.

Thus division can be defined successfully:

If $4 \times n = 3$ then $n = 4$

$4 \times n = 9$ $n = 12$

$4 \times n = 7$ $n = 5$ , and so on.

The system of clock arithmetic modulo 13 has all the usual arithmetic properties, and is called a finite field. It can be shown that a field is so obtained if the number of hours on the clock is 2, 3, 5, 7, ... a prime.

Thus you have seen, in these examples, several concepts that are important in the theory of number systems, which is an important branch of modern algebra. There are many different systems, or structures, in algebra, and a lot of work has been done in describing and classifying them.

Perhaps you may say: but what use are they? Sometimes we cannot answer this question directly. You see, we have decided to "travel hopefully". However, here is an example of the use of division in a system modulo 12 or 13.

At an Expo restaurant hamburger rolls come in boxes of 12, and are being served 4 to a tray. If I have 7 odd ones to start, can I arrange such a number of trays as will come out to a whole number of trays and boxes? Let the number of trays be $x$ . Then the number of hamburgers is $4x$ , so that

$$4x = 12n + 7$$
$$= 7 \text{ (modulo 12)}$$

Since $4 \times x = 7 \pmod{12}$ has no solution, we cannot do this.

However, if the boxes contain a baker's dozen or 13, we can arrange to come out even, since $4 \times x = 7 \pmod{13}$ has the solution $x = 5$ . Thus 5 trays should be used.

At the second stop of our journey we shall discuss vectors. You have probably heard of a vector as a directed

line segment, and have a mental picture of an arrow flying in midair. I shall bring you down to earth, however, with a quite different, though equivalent, invention.

Let us think of a list of numbers: thus $\begin{matrix}3\\5\\2\end{matrix}$.

Such a column of numbers may easily appear on a shopping list, which is a list of quantities desired, or ordered:

3 oranges
5 apples
2 bananas.

If we adopt the understanding that the various fruits are listed in this order, we can write the numbers as a column vector:

$$V = \begin{matrix}3\\5\\2\end{matrix}$$

If one person has the order vector

$$V = \begin{matrix}3\\5\\2\end{matrix}$$

and a second person has the order vector

$$W = \begin{matrix}4\\6\\7\end{matrix}$$

then a sum of these orders is certainly found by addition of the respective components

$$V + W = \begin{matrix}3\\5\\2\end{matrix} + \begin{matrix}4\\6\\7\end{matrix} = \begin{matrix}3+4\\5+6\\2+7\end{matrix} = \begin{matrix}7\\11\\9\end{matrix}$$

If the vectors V and W are represented by arrows from the origin to the points with Cartesian coordinates (3, 5, 2) and (4, 6, 7) respectively, then V + W is represented by the "vector sum" (7, 11, 9) , which is formed as the diagonal of a certain parallelogram, or the third side of a certain triangle.

Back at the supermarket we now learn that the prices are: oranges 4 cents each, apples 3 cents, and bananas 5 cents.

Let us write these unit costs in a row

$$[4 \quad 3 \quad 5]$$

and call this a row vector.

What happens when we approach the cash register with our order, represented by the vector

$$V = \begin{matrix} 3 \\ 5 \\ 2 \end{matrix}$$

Does the girl who operates the cash register know that the "list" is represented by a vector? Perhaps not, but she knows what to calculate! The total bill is

$$\begin{aligned} & 3 \text{ oranges at } 4 \text{ cents} \\ + \quad & 5 \text{ apples at } 3 \text{ cents} \\ + \quad & 2 \text{ bananas at } 5 \text{ cents} \\ \text{or} \quad & 3 \times 4 + 5 \times 3 + 2 \times 5 \\ = \quad & 12 + 15 + 10 \\ = \quad & 37 \text{ (cents)} \end{aligned}$$

Combining the respective components of the order vector and cost vector, she has calculated a single number. This number is known as the dot product, of the cost vector C and the order vector V , and is denoted by

$$C \cdot V$$

Thus $C \cdot V = [4 \ 3 \ 5] \cdot \begin{matrix} 3 \\ 5 \\ 2 \end{matrix} = 37$ .

These vectors have 3 components, or, as we sometimes say, are vectors of 3 dimensions.

Actually Miss "Dot Product", the cashier, uses a cost vector having hundreds of components. She has a list,

| | |
|---|---|
| alphabet soup | 17 |
| apples | 3 |
| asparagus | 23 |
| . | |
| . | |
| . | |

and the list contains hundreds, perhaps several thousand entries. Thus the cost vector has a very large dimension! much larger than the three dimensions of our geometrical world!

Do you have the appetite to explore a space of several thousand dimensions? Mathematicians have found that there is, so to speak, a lot of room in such spaces. They have even explored spaces of vectors having infinitely many dimensions. In these very roomy spaces the dot product $C \cdot V$ of two vectors is still computed the same way, by multiplying the corresponding components and adding them to the bill. As a matter of fact, mathematicians have found that, if this bill becomes infinitely large, it is mathematically unsatisfactory, as well as financially disastrous!

If we were to pursue this line of thought far enough, we would arrive at a subject known as "Functional Analysis", and which could also be called "20th century calculus". However we have gone far enough for one outing and I suggest we return now to another idea which comes from algebra and is also related to vectors.

A vector is a list. Sometimes we have several lists, such as the shopping lists of Mrs. Black and Mrs. Green, and it is econvenient to put them in a "box".

| | Mrs. B. | Mrs. G. | |
|---|---|---|---|
| | 3 | 6 | oranges |
| M = | 5 | 2 | apples |
| | 2 | 8 | bananas |

Such a "box" is called a matrix. This matrix has 3 rows and 2 columns. Actually a column vector is just a matrix

having only one column!

Mrs. Black and Mrs. Green go to the supermarket regularly twice a week, each time with the same shopping list. On one Monday, the prices were 4, 3, 5 as we said before. But on Thursday the prices had gone up! In fact, the increases were 1, 1, 2 . We illustrate with the price matrix

$$P = \begin{matrix} 4 & 3 & 5 \\ 5 & 4 & 7 \end{matrix}$$

where the top row shows "before" and the bottom row "after".

Let us tabulate the various bills; for convenience we list the matrices again.

| | | | | B. | G | | B. | G. |
|---|---|---|---|---|---|---|---|---|
| Monday | 4 | 3 | 5 | 3 | 6 | | 37 | 70 |
| Thursday | 5 | 4 | 7 | 5 | 2 | = | 49 | 94 |
| | | | | 2 | 8 | | | |

The bills for Mrs. Black and Mrs. Green, on Monday and Thursday, are as shown, and we have written them in the natural position for such a scheme. Of course each of these totals is a dot product! Business for Miss Dot Product has been very busy indeed! But the result is a new matrix of two rows and two columns; let us call it T . We shall write

$$PM = T$$

and proudly declare that we have learned how to multiply matrices.

This kind of multiplication is very useful, and important, in pure mathematics, as well as in more practical affairs. In the design of aircraft, or in economic forecasting, very large matrices are sometimes used. Automatic

computers are very good at multiplying matrices, for this involves much arithmetic. Perhaps the most significant point is that by inventing the new concept of a matrix, and operations of addition and multiplication for it, we can now carry on our arithmetic on a "wholesale" basis rather than a "retail" one.

*********

# AN INTRODUCTION TO NON-DECIMAL NUMERALS

by

John Runstedtler, Kitchener Waterloo C.V.S.

The elementary student has been using decimal numerals for computation ever since grade one. It is not surprising, therefore, that by the time he gets to grade seven the decimal system is so familiar to him that he cannot appreciate its power and elegance. Hence, the topic of non-decimal number systems has been included in the revised grade seven and eight curriculum to give the student some insight into the structure of unfamiliar yet analogous number systems and, as a result, a better understanding of his own decimal system. In this article the story of how a number system developed on the mythical planet Esab -4 introduces the student to base 4 numerals.

From the beginning of their history, the Esabs raised cattle for a living. Numerals were required to represent the number of cattle a farmer owned. One stick was used to represent a cow but as the size of herds increased, this symbolism became unwieldy and difficult to read. For example, the idea of eight cattle was represented by |||||||| . Even the invention of paper and pencil could not overcome the drawbacks of this notation.

So one of the more ingenious Esabs developed a more readable symbolism by replacing each of the previous tally numerals with a code numeral.

| tally | code | tally | code |
|---|---|---|---|
| \| | 1 | \|\|\|\|\| | 7 |
| \|\| | T | \|\|\|\|\|\| | ᔕ |
| \|\|\| | Δ | \|\|\|\|\|\|\| | ᒥ |
| \|\|\|\| | □ | \|\|\|\|\|\|\|\| | E |
| | | ⋮ | ⋮ |

The teacher will have to invent further code numerals, perhaps up as far as thirty, if the students are to be assigned an exercise at this stage in the lesson. Would such an exercise help them compare the merits and deficiencies of both the tally and code systems? It may be wise to avoid using two code symbols to form a new code symbol, for example, E□ which could be interpreted as twelve, eighty four, or thirty two (eight fours), etc. If the student suggests such a numeral as E□ then the teacher will have to decide whether to pursue the suggestion with all the possible complications or just tell the student that this is not how the Esabs wrote their numbers. The interpretation of E□ as eighty four might spoil the logical development of the lesson by anticipating a base system too soon.

Since each number had a unique symbol, it was easy to recognize but difficult to add or subtract. An improvement was necessary. The Esabs were very superstitious people and believed that the four moons which circled their planet influenced the future. "Four" became a magic number. They revised the old tally notation by grouping sticks or strokes in bundles of four. For example, seven cattle were represented by 𝍸 ||| . But as the size of herds increased, even this new symbolism became too clumsy. For example, the idea of thirty cattle had to be recorded as 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 || .

Up to this time, too many cattle had been dying in the fields on cold winter days so the Esab farmers began building barns to protect their herds. Again the magic number four played an important part. Every barn contained four stalls and each stall held four cattle. An Esab ranch consisted of four farms, each farm having four barns. The Esabs decided to use a white stick or stroke ($|_w$) to represent a cow, a red stick ($1_r$) a stall of four cattle, and a blue stick ($1_b$) a barn of sixteen cattle. Now thirty could be represented by the numeral $|_b X_r X_w$ .

The subscript "r" in $\text{X}_r$ would not be necessary in class since the students could write X with red pencil or represent it by a bundle of three red coloured sticks tied with an elastic. The teacher can draw out some important ideas from the students here. For example, "what number is represented by $\text{X}_r\text{X}_w|_b$ ? are the numbers $|_b\text{X}_r\text{X}_w$ and $|_r\text{X}_w\text{X}_b$ the same? what number is equivalent to $\text{X}_w$ ? do the Esabs require the symbol $\text{X}_w$ ? "An exercise could be assigned which would challenge the students to add and subtract these unusual numbers.

But carrying three boxes of coloured sticks to the weekly cattle market proved too cumbersome, so the Esabs decided to use only sticks of one colour.

The students will likely suggest how to solve the problem of using only one colour by agreeing to write the "cows digit" first, followed by the "stalls digit" and the "barns digit". Perhaps this will help illustrate just how arbitrarily our own decimal numerals have been constructed. Again a short exercise on adding and subtracting a few of these numbers might be appropriate. The student will see the need for a zero numeral as a placeholder at this time.

For many years the Esabs used their number system to solve the everyday problems of quantity, size, profit, and loss very efficiently. No one, however, had thought about the possibility of constructing any other number system until one day a cosmonaut from earth landed on Esab -4. The first question the Esabs asked him was "How far away is your planet, Earth?" When the cosmonaut wrote down the answer 75,000 light years the Esabs were confused because they had never seen such strange looking numbers. Likewise the earthman was puzzled when the Esabs wrote the numberal | X X to explain how many planets were in their solar system.

So the Esabs and earthman realized they would have to sit down and compare their number systems in order to communicate intelligibly. Each showed the other what symbol he would use to represent the idea of one cow, two cows, three cows, etc. and soon they understood each others systems. "For example," said the cosmonaut, "I would write that there are 27 planets in your solar system, so 27 is the same number as | X ¥ ".

Have the student explain why these numerals represent the same number idea. This is the natural time to introduce the idea "base".

The Esabs thought the earth-digits 0,1,2,...,9 were easier to write so they asked the cosmonaut if they could borrow some of them. The earthman suggested "0" and "1" be left as they were, "X" be replaced by "2" , and "¥" be changed to "3" . Now the Esabs rewrote the numeral | X ¥ as 1 2 3 but the cosmonaut still used the symbol 27 . To make sure both symbols would be interpreted as "twenty seven", they agreed the base should be written as a subscript, for dxample, $1\ 2\ 3_4$ and $27_{10}$ .

The student should be allowed to omit the subscript "10" from "$27_{10}$" . An occasional oral challenge will remind the stuuent that base 10 is understood when "27" is written.

For the next few days the Esabs and the Earthman practised converting from base 4 to base 10 numerals as well as adding and subtracting in both base 4 and base 10 .

It may be valuable to have the students check their base 4 addition by converting the numbers to decimal form and seeing if the addition in base 10 produces an equivalent answer.

eg.

$$\begin{array}{rcl} 13_4 & \rightarrow & 7_{10} \\ +\ 11_4 & \rightarrow & 5_{10} \\ \hline 30_4 & \leftarrow & 12_{10} \end{array}$$

As a follow up to this introduction to base 4, other non-decimal bases might be studied by dividing the class up into small groups where each group represents inhabitants of a different planet with a different base system. Spacemen could be sent from planet to planet learning how to handle each other's number systems.

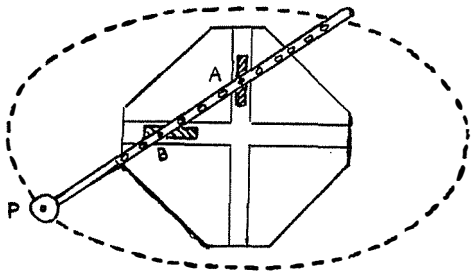
It is hoped that this story will help the student enjoy the topic of non-decimal number systems.

# THE TRAMMEL OF ARCHIMEDES

by

Robert J. Schmidt
Preston High School

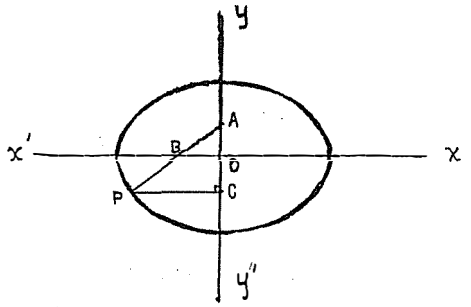
A device which can be used for drawing ellipses is known as the Trammel of Archimedes or an ellipsograph. The device consists of a square piece of wood with two grooves cut out along the diagonals.




The grooves are made so that two sliders, A and B, attached to a long arm are free to slip along the grooves. To the end of the arm is attached a pencil or a piece of chalk, P, which will trace out the ellipse as the arm rotates about the central block guided by the constantly changing positions of the sliders in the grooves. By changing the position of the two sliders on the arm, ellipses of different eccentricity can be drawn.

DERIVATION OF THE EQUATION:

Let the two perpendicular grooves be taken as the coordinate axes. Let ABP represent a typical position of the arm.




Let P have coordinates $(x,y)$.
Let AP and BP have lengths a and b respectively.
Draw PC perpendicular to the y-axis.
Since $\Delta ABO$ is similar to $\Delta APC$,

therefore $AB/AP = AO/AC$ and $AB/AP = BO/PC$

Hence $(a-b)/a = AO/(AO + y)$ $(a-b)/a = BO/x$

Therefore $AO = y(a-b)/b$ $BO = x(a-b)/a$

But $AO^2 + BO^2 = AB^2$

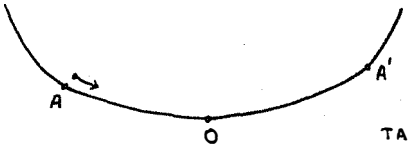
Therefore $y^2(a-b)^2/b^2 + x^2(a-b)^2/a^2 = (a-b)^2$

$$y^2/b^2 + x^2/a^2 = 1$$

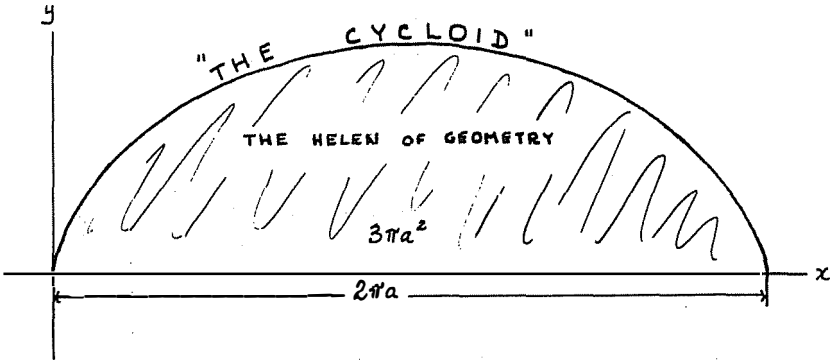
Therefore the equation of the locus of P is $b^2x^2 + a^2y^2 = a^2b^2$.

CYCLOID

THE HELEN OF GEOMETRY




TAUTOCHRONE



# CYCLOID

## THE HELEN OF GEOMETRY

by

Frank J. Rachick,
Ingersoll District Collegiate Institute

DEFINITION: Cycloid: the plane locus of a point which is fixed on the circumference of a circle, as the circle rolls upon a straight line (e.g. the path described by a point on the rim of a wheel).

The discovery of the cycloid seems to be due to Charles Bouvelles who in a geometry published in Paris in 1501 refers to this curve in connection with the problem of squaring a circle.

Galileo (1564-1642) who gave this curve the name cycloid, valued the cycloid for the graceful form it would give to arches in Architecture. He ascertained its area by weighing paper figures of the cycloid against that of the generating circle and found the area to be nearly three times the area of the circle. A mathematical determination was made by Evangelista Torricelli (1608-1647), a pupil of Galileo, who in 1644 showed that the area is exactly three times the area of the generating circle by the use of early integration methods.

Pascal (1623-1662) who was kept awake one night by a toothache turned his thoughts to the ROULETTE (the name Pascal gave to the Cycloid; Roberval (1602-1675) gave it the name TROCHOID - both these terms now have a slightly different meaning.

ROULETTE: if a curve rolls, without slipping, along another, fixed, curve, any point or line which moves with the rolling curve describes a roulette.

TROCHOID: has the same meaning as roulette. It is used more particularly for roulettes traced by points carried by a circle rolling on a fixed circle.

To Pascal's surprise his teeth stopped aching. He regarded this as a sign from Heaven and worked incessantly for eight days during which time he completed a tolerably full account of the geometry of the cycloid.

He determined the area of a section produced by a line parallel to the base, the volume generated by it revolving around its base, or around the axis and finally the centres of gravity of these volumes and also by half these volumes cut by planes of symmetry. Before publishing these results in 1658 he sent to all Mathematicians a challenge offering prizes for the first two solutions of these problems. Only John Wallis (1616-1703) and A. LaLouvere competed for them. La Louvere was quite unequal to the task; Wallis, being pressed for time, made numerous mistakes - neither got the prize. Pascal then published his own solutions which produced a great sensation among scientific men. Wallis also published his results with the errors corrected.

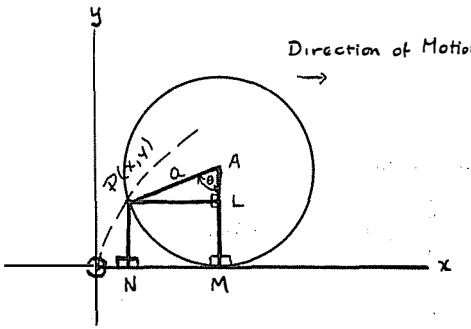
Huygens, Wren and Fermat solved some of these problems but did not compete in the contest. The chief discoveries of Christopher Wren (1632-1723), the celebrated architect of St. Paul's Cathedral in London, were the rectification of a cycloidal arc and the determination of its centre of gravity.

Huygens (1629-1695), interested in pendulum clocks, wanted to find the curve along which a bob might swing, such that the period would be exactly the same whether the bob traversed a long or a short arc. The curve Huygen found was the cycloid. If the curve is inverted and an object, placed on its surface and allowed to slide along this curve under the action of gravity, Huygen proved that no matter at which point the mass starts it will reach its lowest point in the same time and will go on to reach the

symmetrically located point in that same time. The property discovered here is the Tautochrone. Huygen wasn't interested in the property the cycloid showed to sliding bodies but was concerned with the designing of a clock. If he could get the bob of a pendulum to follow such a curve under the action of gravity, then he would have the essential device for a truly accurate clock. Huygens found the answer through further mathematical work on the cycloid.

Bernoulli Brothers, in 1697, Jacob I (1654-1705) and Johanne I (1667-1748) made the discovery that the Brachistochrome Property is a characteristic of the cycloid (the Brachistochrome Property: the curve down which a smooth particle will most quickly move under gravity from one given point to another). That the cycloid is also a tautochrome struck Johannes I as something wonderful and admirable: "With justice we may admire Huygens because he first discovered that a particle falls on a cycloid in the same time always, no matter what the starting point may be. But you will be petrified and astonished when I say that exactly this same cycloid, the tautochrome of Huygens, is the Brachistochrome we are seeking".

Development of the Parametric Equations of the CYCLOID.




A - Centre of circle with radius a .

AP makes an angle θ with the vertical

O is the origin where P is in contact with the ground.

P(x,y) is any point on the locus.

$x = OM - NM$
$= a\theta - a \sin \theta$
$x = a(\theta - \sin \theta)$

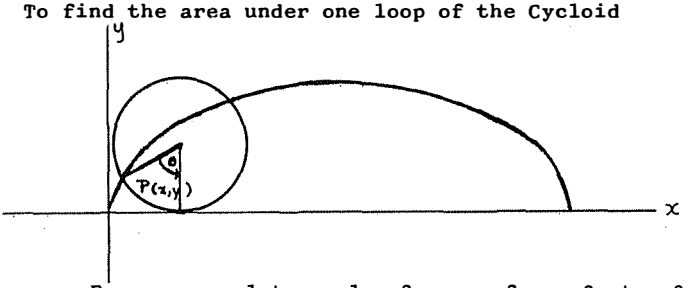
$y = AM - AL$
$= a - a \cos \theta$
$y = a(1 - \cos \theta)$

the Parametric Equations of the cycloid are:

$$x = a(\theta - \sin\theta)$$

$$y = a(1 - \cos\theta)$$

To find the area under one loop of the Cycloid




For one complete cycle $\theta$ goes from 0 to $2\pi$ .

$\therefore$ For the area under one Branch $0 \leq \theta \leq 2\pi$ .

$x = a(\theta - \sin\theta)$ $\quad$ $y = a(1 - \cos\theta)$

$\therefore$ the area $= \int_{\theta 20}^{\theta=2\pi} y\,dx$

$$= \int_{\theta=0}^{\theta=2\pi} y \frac{dx}{d\theta}\cdot d\theta$$

$$= \int_0^{2\pi} a(1-\cos\theta).a(1-\cos\theta)d\theta$$

$$= a^2 \int_0^{2\pi} [1-2\cos\theta + \cos^2\theta]d\theta$$

$$= a^2[\theta - 2\sin\theta]_0^{2\pi} + a^2\int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta)d\theta$$

$$= a^2(2\pi) + \frac{a^2}{2}[\theta + \frac{1}{2}\sin 2\theta]_0^{2\pi}$$

$$= 2\pi a^2 + \frac{a^2}{2}[2\pi]$$

$$= 2\pi a^2 + \pi a^2$$

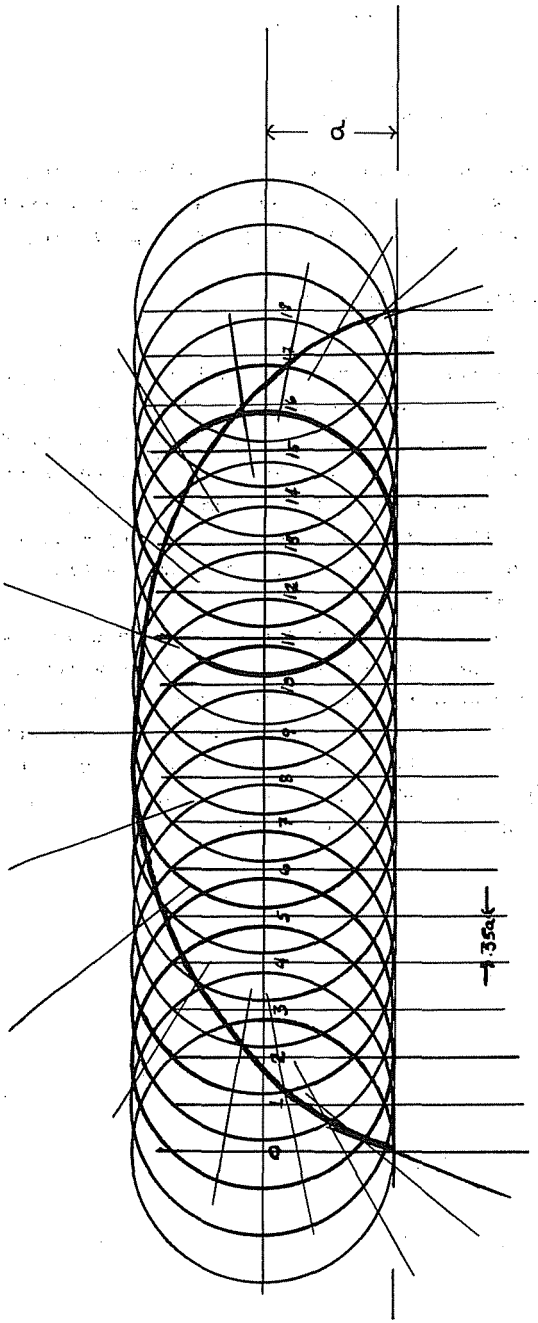
$$\text{Area} = 3\pi a^2$$

But circle has radius a $\therefore$ Area of generating circle is $\pi a^2$ and we can see that the area under the cycloid is three times the area of the generating circle.

Construction of a Cycloid:

Draw two lines "a" units apart. Draw a series of vertical lines crossing the previous lines at intervals of $\frac{2\pi a}{18}$ which is approximately .35a . (The 18 in the denominator indicates the points will be plotted for $20^\circ$ intervals of $\theta$ $0^\circ \leq \theta \leq 360^\circ$). Label these points 0,1,2,... as in the diagram. Draw radii at multiples of $20^\circ$ to the vertical, the angles being measured in a direction away from motion. The construction is seen on the next page.

The cycloid "on account of its singular beauty, elegant properties and the endless rows which it stirred up between quarrelsome mathematicians challenging one another to solve this or that problem in connection with it, the cycloid has been called "The Helen of Geometry" after the Graeco-Trojan lady whose mere face is said to have "launched a thousand ships.""


d
1.35d

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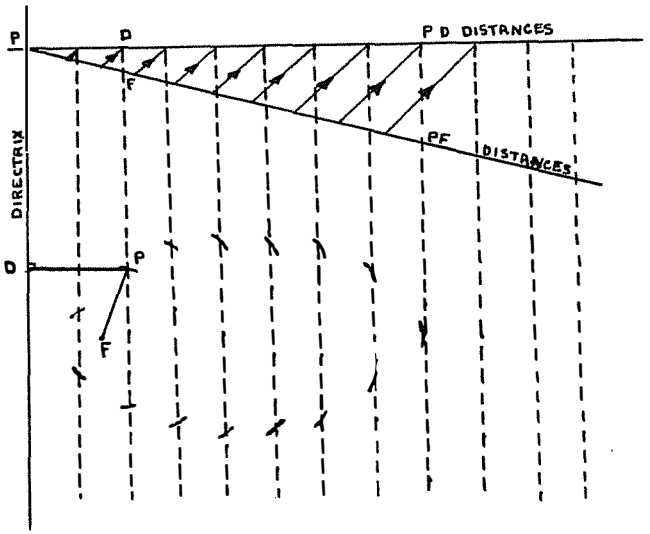
# CONSTRUCTION OF THE CONICS USING THE FOCUS-DIRECTRIX DEFINITION

by

D.G. Steeves, A.N. Myer Secondary School, Niagara Falls

The following construction is suggested as a method of introducing the focus-directrix definition of the conics. This construction allows the students to discover the role e plays in the definition.

A pair of compasses, straight edge, pencil, and lined paper are the only tools necessary.




$\frac{PF}{PD} = e < 1$

CONSTRUCTION: Place the lined paper so that the lines dotted in diagram are vertical. Select one line as the directrix and a point as the focus. Construct a family of similar triangles with the vertex of the family on the directrix and one side perpendicular to the dotted lines.

PF distances can be picked off with compasses. With centre at F , the point of intersection with the corresponding PD distance can be readily found on the appropriate dotted line. This gives a point on the locus of the conic.

By changing the slope of the family of parallel line segments (indicated in the diagram), one gets a parabola (family of isosceles triangles) or hyperbola.

# GRADE XII - 5 YEAR COURSE: A COMMENTARY

by

D.R. Campbell and G.B. Lawson, Althouse College of Education

ED. NOTE: The authors have both taught this course. One to an average A and S class, the other to an S.T. and T. class.

The new Grade XII course has now been in operation for two years and we are presenting a few thoughts on this program in the hope that other teachers will submit their views before an updating revision of curriculum S 12-B takes place.

Generally speaking, it is felt that the content of the new course is a great improvement. The new program provides a variety of topics and integrates the work in algebra, trigonometry, synthetic and analytic geometry. The increased emphasis on graphing throughout is excellent.

The chief criticism is that while the material is very suitable for the above-average student, there is too much content for those whose ability is average or below-average. This latter group understands the majority of topics when presented, but lacks the time to assimilate them and their retention span is short. It is probable that the weaker students would benefit more from the four-year program in mathematics.

The concept of the spiral approach which is inherent in the new courses of study has advantages, one being the introduction of concepts at an elementary level enabling the student to 'live with' the idea and master new notation before attempting a more intensive study. One disadvantage, which is apparent in Grade 12, is the failure to allow sufficient time for a review of concepts which were treated in an introductory manner in the previous grade. Trigonometry and analytic geometry are cases in point. Although the curriculum allows four weeks for the unit on trigonometry in

Grade 11, it must be realized that almost a year lapses before the study is resumed in Grade 12 and the average student has retained very little of the content. It is difficult to do justice in the time allotted to the work indicated in Unit 5 of S-12B.

The present course provides an excellent experience for the student with ability in mathematics and these students have little difficulty with the prescribed content. Unfortunately, too many students in the five-year stream lack real ability in mathematics and there is no indication at present to suggest that these students will not continue to remain in the five year stream. Until some change is made at the administrative level, the curriculum will have to reflect this fact and be designed so that teachers can pay more than lip service to the prescribed content.

When the new courses in Grades 9 and 10 are established, the majority of the work in Unit 1 could be eliminated, or at least reviewed in a shorter time, thus leaving time for other portions of the work. The analytic geometry of the straight line could be reviewed in the section on the circle, giving a longer time to this topic.

In the unit on logarithmic functions, the section on computation using graphs is quite unsatisfactory and could be omitted. It has been suggested by those engaged in teaching mathematics at the post-secondary level that logarithms are no longer used in many computations and that less time should be spent in logarithmic calculations and more time on the theory of logarithms. This poses a real problem. If students do not spend time in computational practice with logarithms, they do not have an opportunity to become familiar with logarithmic notation and the theory of logarithms will prove rather difficult. The use of the slide rule is now included in the four-year Grade 10 course and it is suggested that if it is not to be included in the five-year program at an earlier grade, then it should appear in the Grade 12 work. Students should be encouraged to use

the slide rule for computations in mathematics and science classes.

Unit 3, the quadratic function, is a good section of work combining algebraic and graphical methods. The problems in some texts tend to be too difficult, particularly in equations solved like quadratics, and in maximum and minimum questions. Teachers must select the problems from these sections with care and not attempt too many difficult problems with the whole class.

In Unit 4, the circle, some texts have tried to link synthetic and analytic methods but this has not proven very successful. This may be due, in part, to the fact that the students are required to learn a number of new properties concerned with the synthetic geometry of the circle and the correlation of methods becomes confusing. We believe that the present content in synthetic geometry is satisfactory, provided that the students are not expected to do difficult deductions. The analytic geometry of the straight line could be reviewed here, leading to the analytic geometry of the circle. Since the course of study does not include the solution of linear-quadratic systems, analytic proofs of the chord properties must be confined to simple particular cases. None of the topics seem to be too difficult for the average student.

Trigonometric functions is a topic that is popular with both teachers and students, and is most useful in later work. As was stated earlier, more time is required to cover this section adequately. If calculating machines or slide rules are available, the computational work in the problems on the sine and cosine laws could be handled more easily and some time saved.

Unit 6, sequences and series, is interesting and within the scope of the students. In the course revision it is hoped that extra time could be allotted to this topic so that annuities might be adequately covered.

In conclusion, we believe that the Grade XII course in Mathematics is a good one. However, a number of difficulties

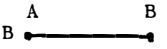
arose because teachers and text books became overly ambitious in their depth of treatment of certain topics. The introduction of topics at a lower level should provide the extra time needed by eliminating Unit 1. Also, as teachers become more familiar with the content, they will know where the pitfalls are and will devise ways of overcoming them. It is hoped that the next revision of S-12-B will not add new topics without a corresponding reduction and will bear in mind that sufficient time must be left for review in all topics.

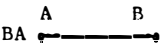
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# ORIENTED POLYGONS

by

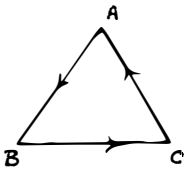
Ernest R. Ranucci
State University of New York, Albany, New York

The line segment AB 

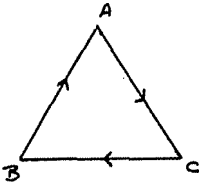
and the line segment BA 

may be represented by the same configuration. However a walk from A to B, followed by a walk from B back to A would result in a net effective change of zero. We arbitrarily agree to consider the distance from A to B as positive and the distance from B to A as negative. Thus $\overline{AB} + \overline{BA} = 0$.

The triangle with vertices A, B, and C, may be designated in six different ways:

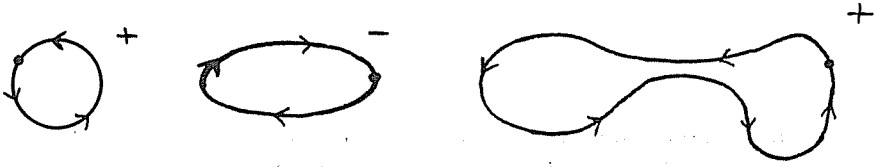



| | |
|---|---|
| ABC | CBA |
| BCA | ACB |
| CAB | BAC |

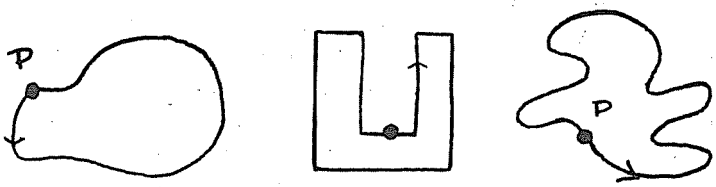



In the first three cases, we proceed from one vertex to another in a counter-clockwise manner.  In line with the rather stupid historical precedent, we agree to consider this counter-clockwise rotation as positive. In the second three cases, we proceed in a clock-wise manner  and consider the rotation negative. If we add the notion of orientation to perimeter - an aspect of geometry usually considered to be independent of sign - perimeter may be either positive or negative. Thus, if the perimeter of $\Delta ABC$ is traced in a clock-wise manner $(C \to B \to A)$, then reversed $(A \to B \to C)$ the resulting combination of movements may indicate a net effective

perimeter of zero. In a similar manner, perimeters of simple polygons other than triangles may be positive or negative. Configurations need not be rectilinear; the perimeters of each of the following are oriented:




Simple closed curves like those below divide the plane into three regions. Points on the plane are either inside the curves, outside the curves or on the curves. For convenience, rectilinear configurations are also considered as curves. If an initial point such as P is marked, the curve can be traversed in one continuous circuit.




Non-simple curves like those below are self-intersecting. In a complete traverse of such curves, certain points will be included more than once.

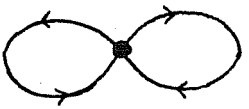


Fig. 1

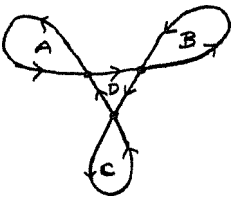



Fig. 2

If we consider oriented perimeter, that of Fig. 1 would be considered zero - if the left curve were congruent to that at the right. In Fig. 2 the perimeter, considered as the sum of four separate perimeters, means that the oriented perimeter of A is positive, that of B positive, that of C positive and that of D negative. The effective perimeter of the entire configuration would be the algebraic sum of the perimeters of the four oriented regions.

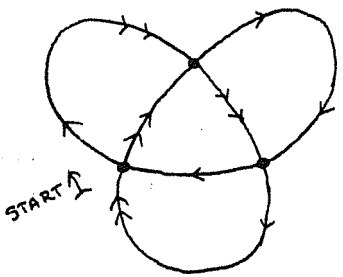
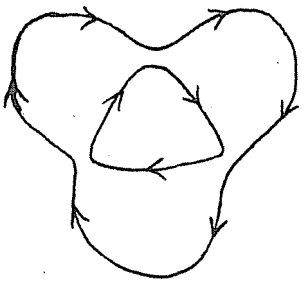



Fig. 3

What are we to do in a case like that of Fig. 3 where the tracing of the complex curve results in inclusion of a common area? This is unlike that of Fig. 2. The solution suggested by Clifford calls for re-drawing of the figure in such a way that the arrows of orientation flow into those of similar type. That is, clockwise types



join clockwise types; counter-clockwise types join counter-clockwise types. The resulting configuration now consists of a simple closed curve with negative perimeter superimposed on a larger one with negative perimeter. The effective perimeter is now the sum of the two separate negative perimeters.

If the configuration were to be traced in a different way (Fig. 4) the oriented perimeter might be different, even though the figures were congruent. This is to be expected; oriented segments AB and BA were, after all, considered to be equal in absolute value only. So with other perimeters. Thus, the effective perimeter in Fig. 4 would be the algebraic sum of that of the larger negative curve and that of the smaller positive curve.

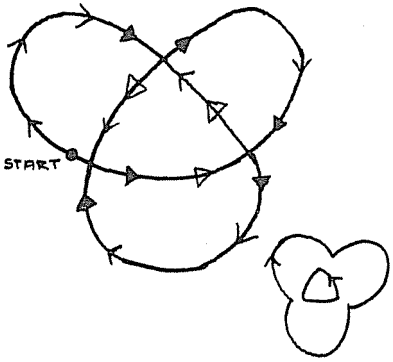
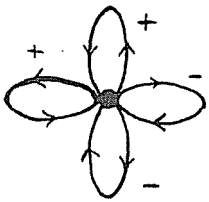
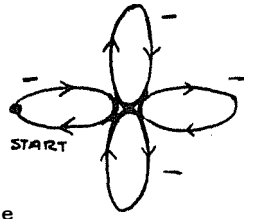



Fig. 4

What about areas? Here again, areas generated in a clockwise manner will be termed negative. Those generated in a counter-clockwise manner will be termed positive. The effective area of the figure at the left is zero in accordance with the orientation of the four separate cusps.



The effective area of the figure at the right is negative and four times that of any one cusp. Had the direction of generation been reversed, the area would have been termed positive.




When self-intersecting configurations are considered (Clifford calls these "tangles"), the rules for the handling of complex perimeters still function. Consider Fig. 5. The numbers refer to the areas of the simple closed curves in which they are placed. At every node or point of intersection, redraw the figure, at least symbolically, in such a manner that like oriented arrows follow each other.

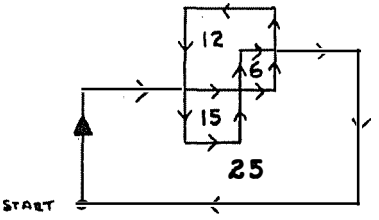
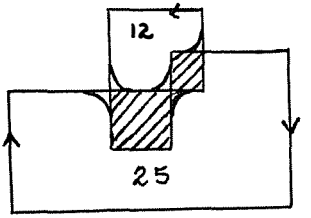
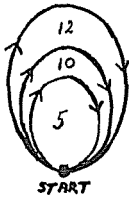



Fig. 5

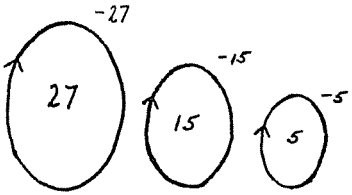
The new figure which emerges at the right is seen to be two simple, closed curves. The effective area is the algebraic sum of $(^{-}25)$ and $(^{+}12)$ or -13 square units. Interestingly enough, the polygons with area 15 and 6 have nothing to do with the final result. If these regions are analyzed, it will be seen that they are "bites" out of a larger region, with opposite orientation to that of the larger region.

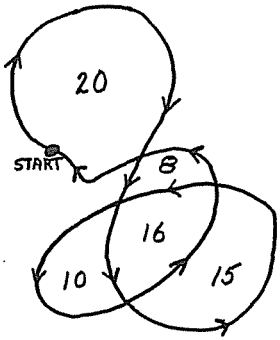


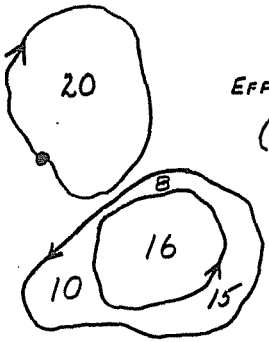

An analysis of several tangles follows:



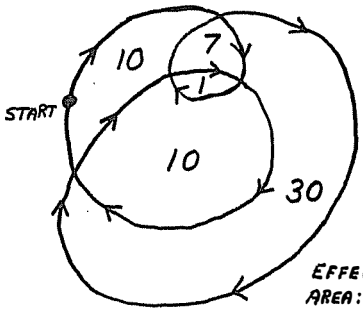

EFFECTIVE AREA:
-27 sq. units.

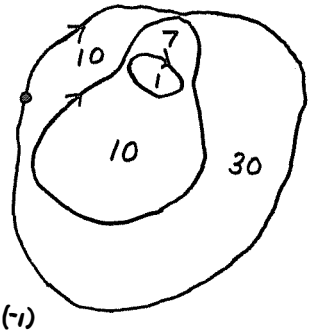


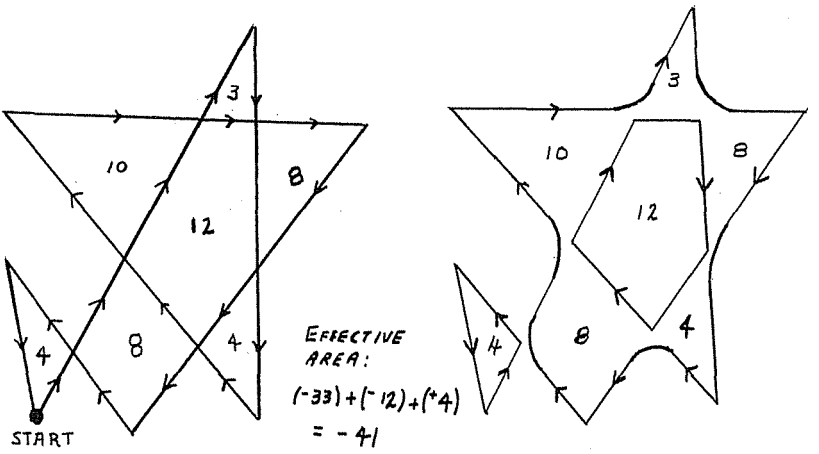




EFFECTIVE AREA:
$(^{-}20) + (^{+}33) + (^{+}16)$
$= {}^{+}29$




EFFECTIVE AREA:
$(-40) + (-17) + (-1)$
$= -58$




Orientation may also be applied to the generation of angles.

Angle ABC , where ray BC is terminal,

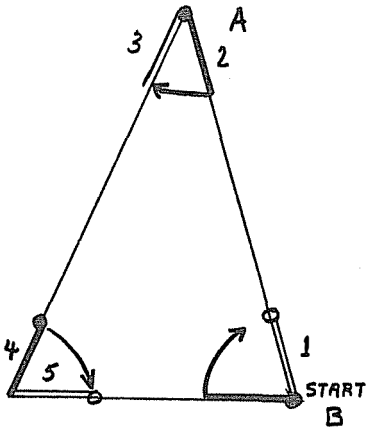
is normally considered positive. Angle CBA

where ray BA is considered

terminal, is negative.

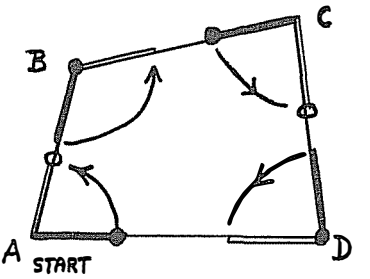
Martin Gardner[2] gives an interesting exercise in computing the sum of the angles of various polygons. In many of these exercises, orientation must be considered.

Place a matchstick in position 1 of $\Delta ABC$ at the right. The orientation of the tip of the match is important. Slide (translate) the matchstick to position 2, then rotate it to position 3. Slide it to position 4, then rotate it to position 5. Slide it to B , then rotate it to its original position. It ends up reversely oriented. This means that the sum of the three angles at A, B, and C, must be either $180^{\circ}$ (or a multiple thereof.) A careful check of its intermediate positions will indicate that the rotationwas actually $180^{\circ}$ .

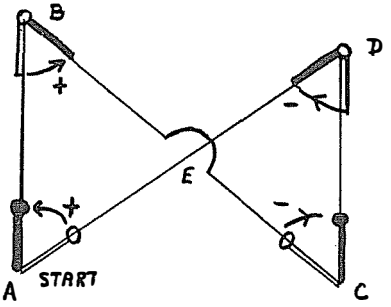



Examination will show that the angles were all generated in a counter-clockwise or negative sense.

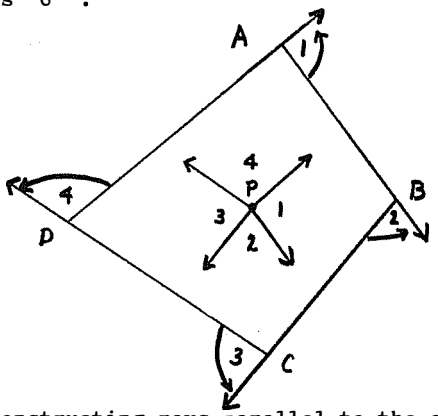
For polygon ABCD , the sum turns out to be $360^{\circ}$ (or $0^{\circ}$ or a multiple of $360^{\circ}$). Here the terminal position of the matchstick is exactly what it was at the beginning. The angles were all generated in a positive sense. Here again, analysis will indicate that the matchstick actually rotated $360^{\circ}$. We may indicate that this is $= +360^{\circ}$ .




A re-entrant polygon like ABCD , although unorthodox, may be analyzed via the matchstick approach. Vertex E is "accidental". It is formed by polygonal path A*B*C*D*A but is not to be considered a vertex of the polygon. The matchstick analysis shows that angles A and

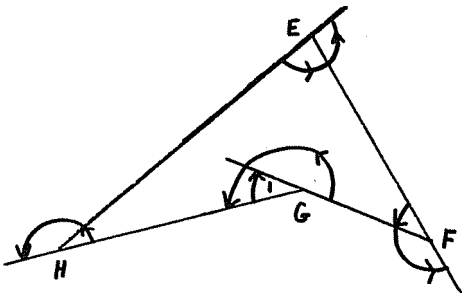


and B were generated positively. Angles C and D were generated negatively. The matchstick ends up oriented as it was in the beginning. What is more, analysis will show that there was no real rotation. The positive and negative phases of the rotations "undid" each other. About all we can say is that the sum of angles A and B equals that of C and D . If point E is considered a vertex, Euclidean geometry indicates that the sum of angles A and B must equal that of C and D . (Vertical angles are equal; the sum of the angles of a triangle is $180^{o}$). The effective sum of the oriented angles of the re-entrant quadrilateral is $0^{o}$ .




Orientation of angles is important in finding the sum of the exterior angles of a re-entrant polygon. For polygon ABCD and for any convex polygon, the sum of the exterior angles will always be $360^{o}$ . This may be established by selecting some convenient point P and constructing rays parallel to the sides of the original polygon. Since the sum of the angles about P is $360^{o}$, the sum of the exterior angles must also be $360^{o}$. (Angles whose sides are respectively parallel are, normally, equal).

Re-entrant polygons such as EFGH , however, cause certain problems. The exterior angles at E,F and H , are formed, as usual, by prolonging sides of the original polygon. What are we to do at G ? Here the exterior angle formed

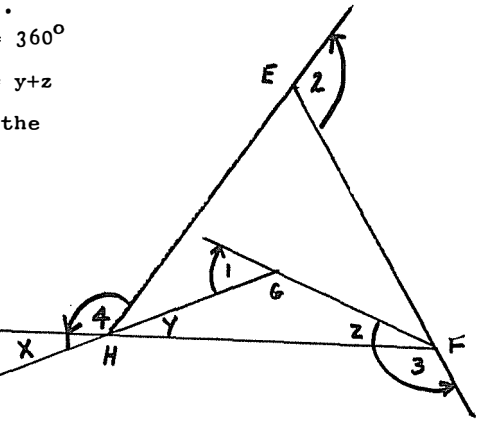


may be one of several possibilities. Let us first prolong FG as in the other three cases. Exterior angles are, by definition, supplementary to adjacent interior angles. Since reflex angle G is oriented positively and angle 1 (part of angle G ) is oriented negatively, the sum of the two is still $180^{o}$ . It would, therefore, be logical to consider angle 1 the exterior angle at G . In other words, an exterior angle may be in the interior of a non-convex polygon.... In order to compute the sum of the exterior angles of the polygon it will be necessary to transform the polygon slightly.

Connect points F and H .

In $\Delta EFH$ , $\angle 2 + \angle 3 + \angle 4 = 360^{o}$

and $\angle 1 = y+z$




In Quad.EFGH , the sum of the exterior angles is:

$2 + (z + 3) + (4 + x) + 1$

or:

$2 + (z + 3) + (4 + y) + 1$

$= 2 + 3 + 4 + y + z + 1$

But angle 1 is oriented negatively and angles y and z are oriented positively.

Therefore: $\angle 2 + \angle 3 + \angle 4 = 360^{o}$ . In a similar manner it can be shown that the sum of the exterior angles of any polygon, convex or non-convex, will always be $360^{o}$ .

To summarize: The invention of the system whereby real numbers could be either greater than zero or less than zero enriched the field of algebra by expanding the possible areas of investigation. In like manner, the addition of orientation to the field of Euclidean geometry offers vast areas for fruitful investigation.

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# MOTION GEOMETRY*

by

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This lecture began a year ago in Calgary; I didn't finish and asked to come down here and give the rest. At a school where I taught for several years, we once had a performance at a morning assembly by a student orchestra, the nineteenth century version of "rock and roll", I think. The principal, when thanking the group, suggested that when they had learned another piece they might come back and entertain us again.

Unfortunately I have not learned a new piece; this will be variations on last year's theme: "geometry a-go-go". I hope my facetious choice of words does not imply that the topic is a passing fancy or fad.

## Definition

Motion geometry, or transformation geometry, is usually attributed to Felix Klein, who said that we ought to organize school geometry on the basis of invariances under groups of transformations.

A geometrical transformation is a correspondence of geometric objects. We shall restrict ourselves to the plane in this discussion, and to the definition:

A geometrical transformation is a one-one matching of points, i.e. a mapping of the whole plane onto itself. When we study geometry from this point of view, we look for properties of figures which are unchanged under a given transformation.

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* Based on a talk given at the Summer Meeting of the N.C.T.M, Montreal, August 1967.

The introduction of this topic involves the so-called rigid motions, or isometries, which preserve distance and angle: reflection, translation, rotation and their products or combinations.

Why Motion Geometry?

Poincaré said that if there were no solid bodies in nature, there would be no Geometry. To him the concept of group was central to this, for "Geometry is the study of the group of motions of solid bodies." Belief in Poincaré's views provides one with two excellent reasons for using motion geometry: first, it tends to point out forcefully some of the most important geometrical properties, strongly relating them to the physical world and three dimensional space; second, when the pupil is first introduced to geometry, he is at a stage of learning when the handling, moving and comparing of real objects in an intuitive, constructive manner provides him with the best learning experience. Motion geometry is ideally suited to provide the learning vehicle.

Euclid's organization of geometry, which influenced mathematicians for centuries, was largely on the basis of logical presentation, with very little, if any, regard being given to learning. The use of motion geometry will bring us to the same conclusions and theorems, but the organization will be more suited to the pupil's stages of learning.

A third, no less important reason, is the way this approach brings about unity with algebra, as the notion of relation and function is now brought to the fore in geometry. Thus one important structure of Mathematics is now shown to be pervasive. In this regard, although the concept of group is not my topic today, motion geometry does afford a simple interesting example of group structure.

Deductive Geometry

In my talk at Calgary I showed some examples of the

rigid motions treated graphically and intuitively. Many of the examples I used could be introduced into the junior high school programme.

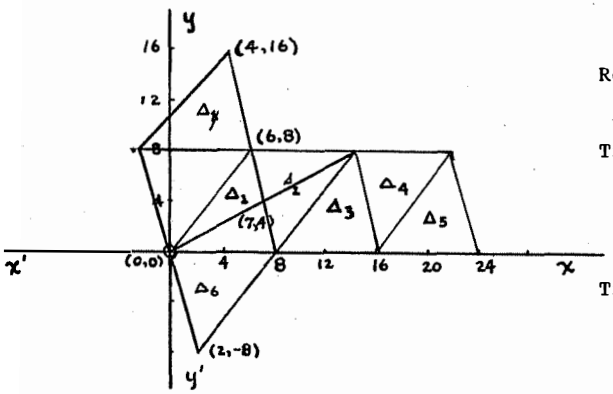
However, as the child progresses from this level into the teens, we do expect him to learn the meaning of proof, and to see the possibilities of the axiomatic development of a subject. The purpose of my talk today is to examine the possibility of introducing certain parts of geometry deductively, on the basis of transformations, and if possible, to derive them from a common source, so that properties acquired at one stage may be applied at a later stage.

Background to Deductive Motion Geometry

Theories of learning would seem to indicate that children in their pre-teens should learn the properties of geometrical figures by looking at whole figures and solids, feeling them, moving them, fitting them together, to learn about their shapes and symmetries. It would be better to study special cases before general ones. Thus we begin with the square, rectangle, rhombus, parallelogram, kite, equilateral and isosceles triangle, and such solids as the cube, rectangular solid, and so on.

Among other things we note the number of ways these objects can be fitted back into their positions, the axes of symmetry, ways they can be folded or dissected. An interesting topic is "tiling the plane": determining if it is possible to fill the plane or space with a given figure or solid, and working out the motions that will match each image with the original object.

Some of these topics lend themselves nicely to graphical treatment, where simple integral coordinates can be used to define typical positions.




Reflection in (7,4):

$\Delta_1 \rightarrow \Delta_2$

Reflection in (4,0):

$\Delta_1 \rightarrow \Delta_6$

Translation (8,0):

$\Delta_1 \rightarrow \Delta_3$

$(x,y) \rightarrow (x+8,y)$

Translation (-2,8):

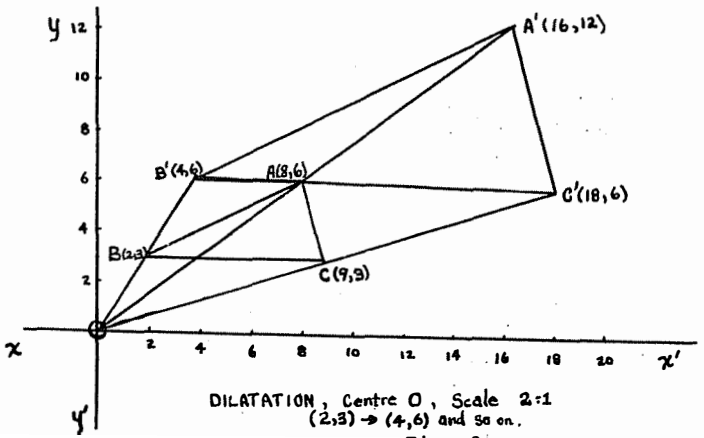
$\Delta_1 \rightarrow \Delta_7$

$(x,y) \rightarrow (x-2,y+8)$

Fig. 1

Starting with a triangle, the plane can be filled by translations, half turns (reflections in points) and their products.

GRAPHICAL TREATMENT of DILATATIONS




DILATATION, Centre O, Scale 2:1
(2,3) → (4,6) and so on.

Fig. 2

Many of the most sophisticated theorems of geometry can first be realized intuitively by such experiments. In the diagram above the triangles are placed so that the areas may be readily calculated.

Deductive Motion Geometry

There is no unique best set of definitions and basic assumptions with which to begin the axiomatic treatment of any subject. My choice is based on Max Jeger's book, Transformation Geometry. I propose to deal with reflection in a line, reflection in a point, translation and rotation, examining some of the axioms and methods of proof.

The definition of reflection will be a rather loose one - really as an undefined term - reflection will be characterized by our assumptions about it.

Reflection

Given the plane and a line s in the plane, reflection is a "flipping over" of the plane, lining up the fixed line s with itself, point by point.

As you can see my first diagram does not characterize the motion well, because we have no imprint of the original position of the plane. Now, with the second diagram, you can "see" by my actions the turning over of the plane. However, there are no distinguishable features to indicate what has happened. The marking of a point on the original plane will help in this respect.

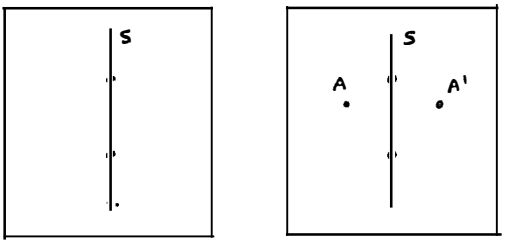



Fig. 3

I shall designate this transformation by $M_s$ ("mirror image in the line s"). What are the basic assumptions and obvious properties?

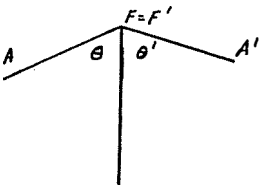
(i) Every point of the plane has an image, and we can reverse the mapping: $M_s(A) = A'$ and $M_s(A') = A$ .

(ii) Points and their images are on opposite sides of the line s , except for

(iii) points on line s , which remain fixed.

(iv) By the "definition" given, it is clear that straight lines are preserved;

(v) distances and angles are invariant; however a figure and its image have opposite orientation.




In all cases $\theta = \theta'$ ; thus if F is collinear with A and A' $\theta = \theta' = 90^{\circ}$ .

Fig. 4

(vi) This leads to the important conclusion that s is the perpendicular bisector of AA' .

As a matter of fact, we might define a right angle as the angle formed by reflection of a point in a line.

(viii) Lines perpendicular to s are fixed, but not point by point.

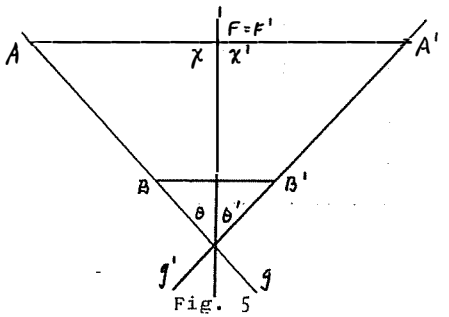



Fig. 5

Incidentally, the introduction to this topic can be handled graphically.

## Products of Reflections

We shall now consider the operation of combining two reflections.

Definition: Parallel lines are lines perpendicular to the same line.

The discussion may be related to physical reflection in a mirror.

(i) Two parallel mirrors

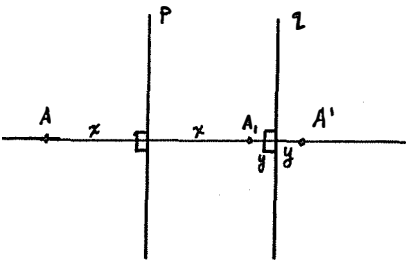



Fig. 6

Important conclusion:
$AA' = 2x + 2y$ .

$M_p(A) = A_1$

$M_q(A_1) = A'$

$M_q(M_p(A)) = A'$

Note that if p and q are a fixed distance apart, $AA'$ is fixed.

(ii) Intersecting mirrors

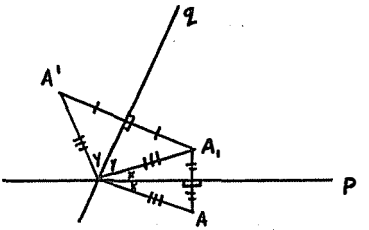



Fig. 7

Important result:
$\angle AOA' = 2x + 2y$

Note that if the angle between p and q is given, $\angle AOA'$ is fixed.

We note that the various line segments and angles marked equal are known to be so by the basic properties of reflection.

(iii) p perpendicular to q

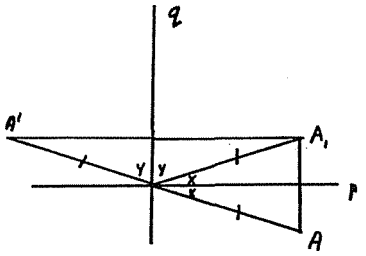



Fig. 8

In this special case $2x + 2y = 180^{o}$ and AOA' is a straight line.

This is used to define a useful new transformation, the half turn, or reflection in a point.

Reflection Problems

In physical reflection we usually consider the mirror to be silvered on one side. In geometry we can have reflection on both sides of the line.

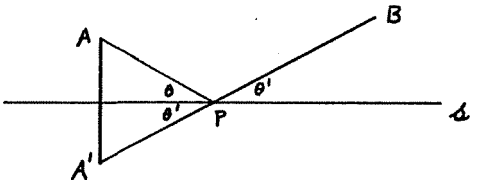



Fig. 9

The law of reflection: angle of incidence = angle of reflection, is easily proved by assuming Fermat's Principle (of least time ).

AP = A'P by preservation of distance

$\theta = \theta'$ " " " angle

Then vertically opposite angles are equal.

$\because$ A'PB is the minimum distance between A' and B,

$\therefore$ it follows that AP+PB is the minimum distance.

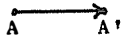
Problem:

Given: A line $s$ passing through $X$ and $Y$, and two points $A$ and $B$ on the same side of $s$.

Required: to find a point $P$ in $s$ such that $\angle APX = \angle BPY$

Translation

A translation is a transformation which slides the plane a given distance in a given direction. All points have the same displacement.




A point or object moves from $A$ to $A'$.

We say the object has been displaced. We can represent a displacement by a directed line segment or arrow.

Equivalent Directed Line Segments: $\overline{PQ} \cong \overline{AB}$ iff the segments are parallel, of equal length and of the same sense.




We think of vector as an infinite set of these equivalent "arrows".




$AB = \{PQ \mid PQ \cong AB\}$, where $AB$ is read as "the vector $AB$".

Fig. 11

In summation, a vector is an equivalence class of these directed line segments, and may be represented by any element of the set.

You will recall that the product of two reflections gave a rotation angle twice the angle between the two lines.

$$x + y = 30^\circ \Leftrightarrow 2x + 2y = 60^\circ .$$

THEOREM: A rotation $\theta$ about a given point $O$ can be expressed as the product of two reflections in lines through $O$, such that the angle between them is $\frac{1}{2}\theta$.

Compositions of Rotations

These are simple if the centre of rotation is common. The following case is more interesting.

Given: $R_1(O_1, 60^\circ)$ and $R_2(O_2, 90^\circ)$

Required to Find: $R = R_2R_1$ .

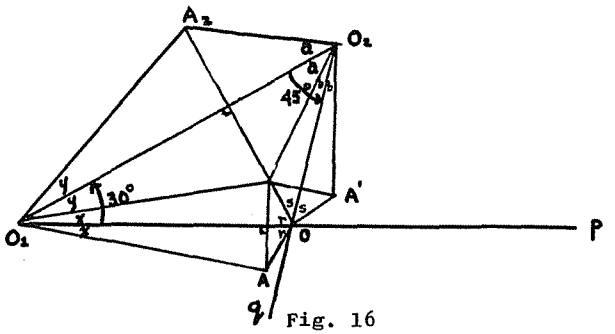


Fig. 16

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$x + y = 30^\circ \Leftrightarrow 2x + 2y = 60^\circ$ .

$R_1(A) = A_2$

$a+b = 45^\circ \Leftrightarrow 2a + 2b = 90^\circ$

$R_2(A_2) = A'$

But since by reflection $OA = OA_1 = OA'$ , $A'$ can be obtained from $A$ by a counterclockwise rotation about $O$ .

$$r + s = 180^o - (30^o + 45^o)$$

$$= 105^o$$

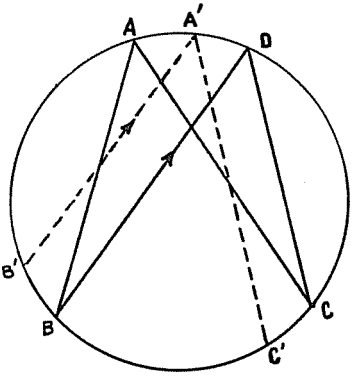
$$\therefore \therefore 2r + 2s = 210^o$$

$$\therefore \angle AOA' = 150^o = 60^o + 90^o .$$

THEOREM: The product of any two rotations is a rotation. The centre of the rotation may be obtained as shown above, and the angle of rotation is the algebraic sum of the given angles.

Deduction on Rotations

Angles in the same segment of a circle have equal measure. The proof given below depends on the assumption that angles with co-directional arms have equal measure.



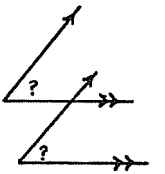
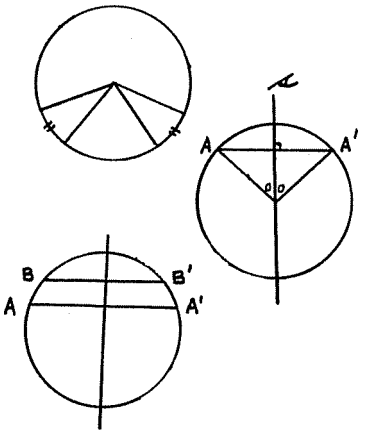




Fig. 17

Rotate $\angle BAC$ about the centre of the circle until the image of AB is parallel to BD . Then it is easy to prove $C'A' \| CD$ , and thus it follows that $\angle A = \angle A' = \angle D$ .

What are the basic assumptions, beyond the one mentioned above?




(i) Equal angles at the centre subtend equal arcs at the circumference. (Recall Allendoerfer's article in The Mathematics Teacher, Feb. 1965).

(ii) Under reflection, since angles are preserved, arc AX = arc XA'

(iii) Parallel chords have a common axis of symmetry.

$\therefore$ arcAB = A'B'

Fig. 18

The proof sketched above depends on these assumptions.

Summary

This has been a hasty survey of the possibilities of setting up a deductive system in motion geometry. I have not discussed the different products of transformations which form groups under composition. The background of informal geometry required to precede such a development needs careful study. We have not looked at enlargements, similarities, or the affine transformations in general.

It would appear that it is possible to have a simple derivation of the rigid motions from a common source. The order of development used here was reflection in a line, reflection in a point, translation, rotation.

I hope that this paper has given some idea of how deductive geometry might be based on "motions".

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Mathematics Department, Carleton University
Mathematics Department, University of Ottawa
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## PROBLEMS AND SOLUTIONS

Send problems and solutions to:

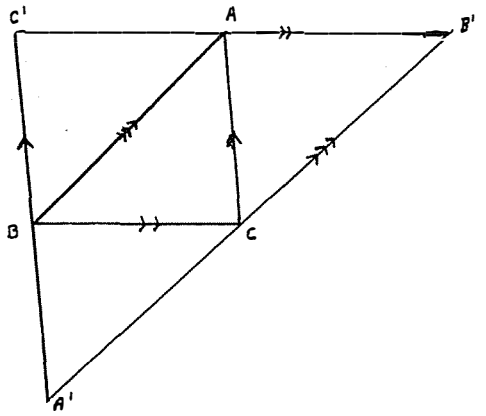
The Problems Editor,
Ontario Mathematics Gazette,
c/o Department of Mathematics,
Queen's University,
Kingston, Ontario.

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PROBLEM NO. 5 (A. W. Harris, Althouse College of Education)

Given that the right bisectors of the sides of any triangle are concurrent, prove that the altitudes of any triangle are concurrent.

The proof given below is suggested in a problem in Coxeter's Introduction to Geometry, page 17.



Given: $\Delta ABC$

Required: to prove that the altitudes of $\Delta ABC$ are concurrent.

Proof: Draw through A,B and C lines parallel to BC, CA and AB respectively, meeting in A', B' and C' .

It is easy to prove that A,B and C are respectively the midpoints of the sides B'C', C'A' and A'B' of $\Delta A'B'C'$ . The right bisectors of the sides of $\Delta A'B'C'$ can be proved concurrent. But these are also the altitudes of $\Delta ABC$ .

Thus, knowing how to prove the right bisectors of the sides of a triangle concurrent, we have a method of proving the altitudes of a triangle concurrent.

What is your proof of this theorem?

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SOLUTION TO PROBLEM NO. 2 (March 1967); M.R. Nanjappa, Sharbot Lake High School, Sharbot Lake.

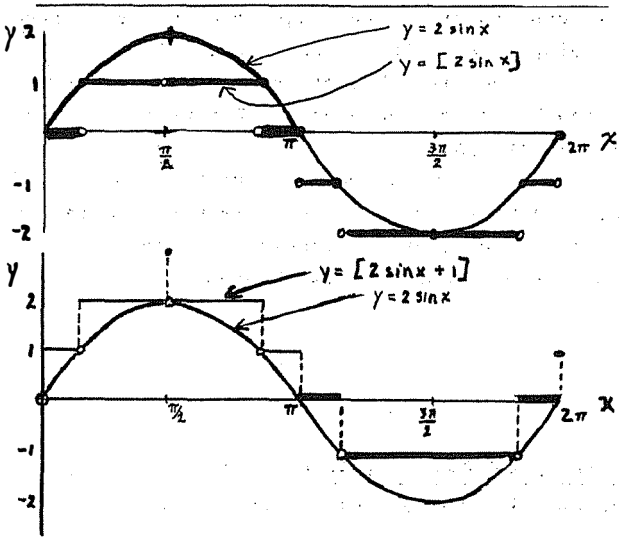
Correction: $[\frac{11}{3}] = 3$ instead of $[\frac{11}{3}] = 2$ .

Solution:

a) Graphs of $y = 2 \sin x$ and $y = [2 \sin x]$ ; $0 \leq x \leq 2\pi$.

| x | 2 sinx | [2 sinx] | x | 2 sinx | [2 sinx] |
|---|---|---|---|---|---|
| 0 | 0 | 0 | $\frac{3\pi}{5}$ | 1.9022 | 1 |
| $\frac{\pi}{10}$ | .6180 | 0 | $\frac{2}{3}\pi$ | 1.7320 | 1 |
| $\frac{\pi}{6}$ | 1.0000 | 1 | $\frac{3}{4}\pi$ | 1.4142 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| $\frac{\pi}{4}$ | 1.4142 | 1 | $\frac{\pi}{6}\pi$ | 1.0000 | 1 |
| $\frac{\pi}{3}$ | 1.6320 | 1 | | | |
| $\frac{\pi}{2}$ | 2.0000 | 2 | $\pi$ | 0 | 0 |




The number of isolated points on the graph of $y = [2 \sin x]$ are 9 .

b) Consider $f(x) = 2 \sin x$ , and let

$$y = [2 \sin x + 1]$$

The graph of the function $y = [2 \sin x + 1]$ satisfies all the requirements of the problem.

∴ The required "steps and stairs" function is

$$y = [2 \sin x + 1]$$

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SOLUTION TO PROBLEM NO. 3 (May 1967); R.D. Mackintosh, Riverdale Collegiate, Toronto.

This is an "area" proof. From diagram IV, M, I, N are collinear if $\Delta MNP = \Delta MIP + \Delta PIN$ .

Let $\Delta MNP = \phi$ , $\Delta MIP = \alpha$ , $\Delta PIN = \beta$

Hence it is required to prove that $\phi = \alpha + \beta$

From diagram II, $\phi + \delta = \Delta MNC = \frac{1}{2}\Delta MAC = \frac{1}{4}\Delta ABC$ since a median of a triangle bisects its area.

$$\therefore \phi = \frac{\Delta ABC}{4} - \delta \qquad (1)$$

From diagram I

$$\text{Fig. } AIPC = 2\beta + 2\delta$$

From diagram III, fig. $AIPC = x + 2z$

$$\therefore 2\beta + 2\delta = x + 2z \qquad \therefore \beta = \frac{x}{2} + z - \delta$$

From diagram III

$$\alpha + z = \frac{1}{2}(y + z) \quad \text{(median IM of } \Delta BIC)$$

$$\therefore \alpha = \frac{y+z}{2} - z = \frac{y-z}{2}$$

$$\therefore \alpha+\beta = \frac{y-z}{2} + \frac{x}{2} + z - \delta = \frac{x + y + z}{2} - \delta$$

But from diagram III $\Delta ABC = x + 2y + 2z$

$$\therefore \frac{x+y+z}{2} = \frac{\Delta ABC}{4}$$

$$\therefore \alpha+\beta = \frac{\Delta ABC}{4} - \delta \qquad (2)$$

From (1) and (2) $\therefore \phi = \alpha + \beta$

$$\therefore \Delta MNP = \Delta MIP + \Delta PIN$$

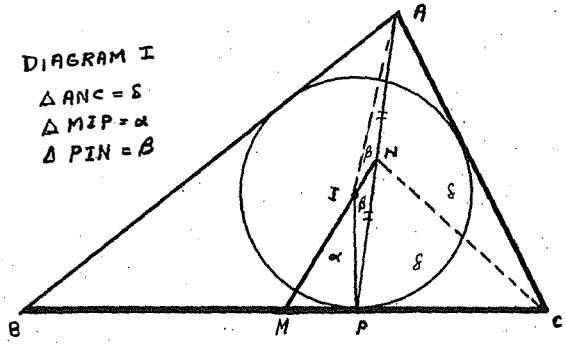
$\therefore$ M,I,N are collinear

DIAGRAM I

$\Delta ANC = \delta$

$\Delta MIP = \alpha$

$\Delta PIN = \beta$




$BM = MC$

$AN = NP$

PROVE N, I, M are collinear

DIAGRAM II

$\Delta ANC = \delta$

$\Delta MNP = \phi$

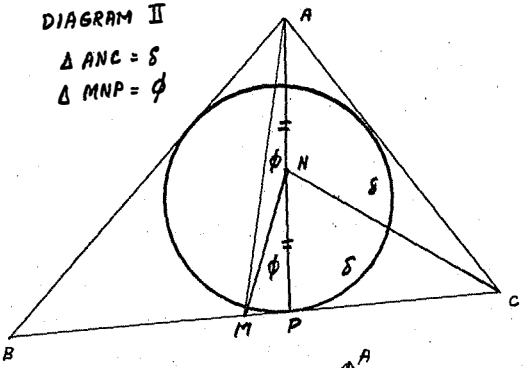
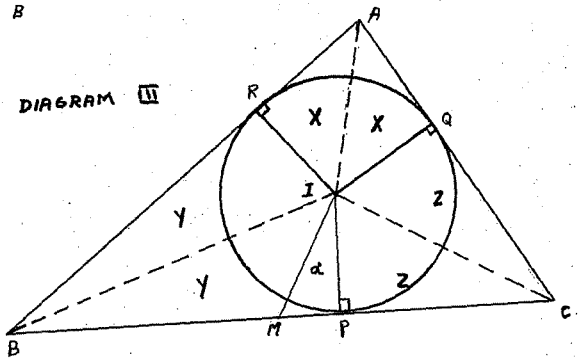



DIAGRAM III

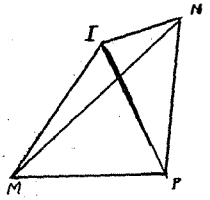



$X = \Delta AIQ = \Delta AIR$

$Y = \Delta BRI = \Delta BIP$

$Z = \Delta IQC = \Delta IPC$

DIAGRAM IV



ANOTHER SOLUTION TO PROBLEM NO. 3 (May, 1967);

F.C. Ashbury, Toronto.

Theorem: In any triangle ABC , the sides of the pedal triangle of the incentre meet the non-corresponing sides of the medial triangle (which is the pedal triangle of the circumcentre) in six points which lie in pairs on the internal bisectors of the angles A,B,C, and also lie in pairs on the circles with diameters BC, CA, AB .

Further, the sides of the pedal triangle of the incentre meet the corresponding sides of the pedal triangle of the circumcentre in three points which are collinear, in pairs, with the opposite vertices of ABC .

In $\Delta ABC$ , (see accompanying diagram), $M_1M_2M_3$ are the mid-points of BC, CA, AB respectively. $P_1P_2P_3$ are the points of contact of the inscribed circle.

Let the sides of $\Delta P_1P_2P_3$ and those of $\Delta M_1M_2M_3$ , produced as necessary, meet in $R_1,\ldots, R_9$ as shown.

1) Bearing in mind that $BP_1 = BP_3 = s-b$ , etc.

and that $M_3M_2 \| BC$

with $M_3M_2 = \frac{1}{2}BC = \frac{a}{2}$ , etc.

the angles similarly marked are readily shown equal.

2) Since $M_3R_1 = M_3M_1 - R_1M_1 = M_3M_1 - P_1M_1$

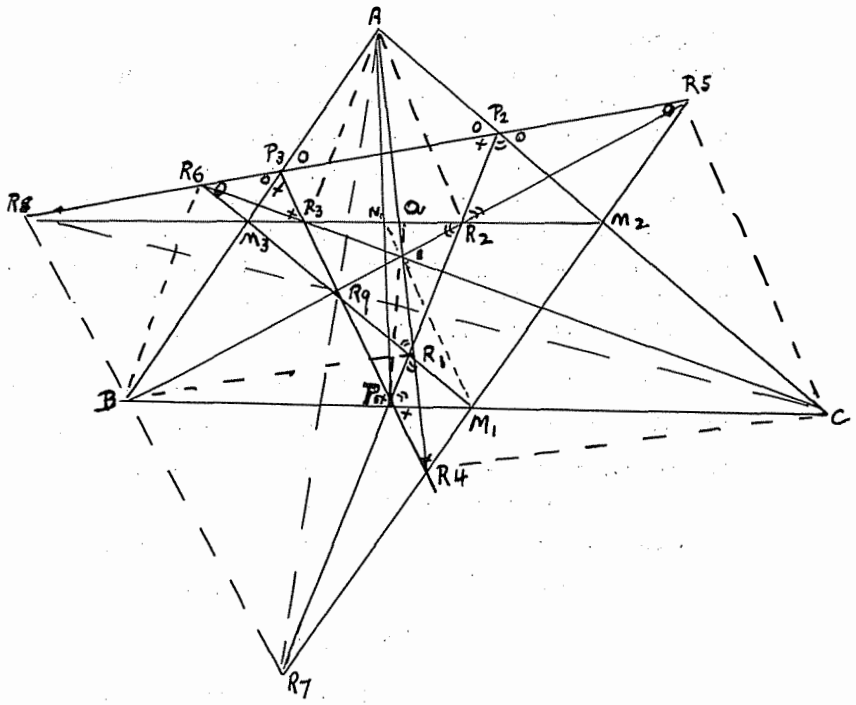
$$= \frac{b}{2} - (\frac{a}{2} - \overline{s-b}) = \frac{c}{2} = M_3A = M_3B$$

$$\therefore \angle M_3AR_1 = \angle M_3 R_1 A$$

$$\text{i.e. ext. } \angle BM_3R_1 = 2 \angle M_3 AR_1$$

$$\text{but } \angle BM_3R_1 = \angle BAC$$

Hence $AR_1$ bisects the interior vertical angle at A and, further, the circle with diameter AB passes through $R_1$ .




3) Similarly $M_2R_3 = M_2M_3 - R_3M_3 = M_2M_3 - P_3M_3$

$$= \frac{a}{2} - (\frac{c}{2} - \overline{s-a}) = \frac{b}{2} = M_2A = M_2C$$

Hence $CR_3$ bisects the vertical angle at $C$ and the circle with $AB$ as diameter passes through $R_3$.

4) Also, in $\Delta M_3R_1R_2$

$$\angle M_3R_2R_1 = \angle M_3R_1R_2$$

$$\therefore M_3R_2 = M_3R_1 = M_3B = \frac{c}{2}$$

Hence the circle on AB as diameter passes through $R_2$ and since $\angle M_3 R_2 B = \angle M_3 B R_2$

$$\therefore \angle M_3 B R_2 = \frac{1}{2} \text{ ext. } \angle A M_3 R_2 = \frac{B}{2}$$

and $BR_2$ bisects $\angle B$ .

Since the angle bisectors meet at the incentre I , it follows that AR, $BR_2$ and $CR_3$ are concurrent in I .

It also follows from the above that $BR_1 \perp AR_1$ , the bisector of interior $\angle A$ (and that $AR_2 \perp BR_2$ , the bisector of $\angle B$).

5) Thus the perpendicular from B on the bisector of the interior vertical angle at A passes through $R_1$ , the intersection of $M_3M_1$ and $P_2P_1$ . A similar condition exists at $R_2 \ldots R_6$ .

In the case of $R_5$ , for example,

$$M_1R_5 = M_1M_2 + M_2R_5 = M_1M_2 + M_2P_2$$

$$= \frac{c}{2} + (\frac{b}{2} - \overline{s-a}) = \frac{a}{2} = M_1B = M_1C \text{ , etc.}$$

Hence B, $R_2$,$R_5$ are collinear.

Similarly for A, $R_1$, $R_4$ and for C, $R_3$, $R_6$ .

The above results may be summarized as follows:

The sides of $\Delta P_1P_2P_3$ meet the <u>non-corresponding</u> sides of $\Delta M_1M_2M_3$ in six points $R_1$ to $R_6$ which lie in pairs on the interval bisectors of the angles A,B,C and also lie in pairs on the circles with diameters BC, CA, AB . Hence the general enunciation (para 1) above.

6) Now let the pairs of <u>corresponding</u> sides of the pedal triangles $P_1P_2P_3$ , $M_1M_2M_3$ meet in $R_7$,$R_8$,$R_9$ .

In $\Delta$'s $R_8P_3R_9 = R_8R_5C$ ,

$\angle$ $R_8P_3R_9 = \angle R_8R_5C$

Now $\frac{R_8P_3}{R_8R_5} = \frac{P_3M_3}{R_5M_2} = \frac{P_3M_3}{P_2M_2} = \frac{\frac{c}{2} - \overline{s-a}}{\frac{b}{2} - \overline{s-a}} = \frac{a-b}{a-c}$

Again, $\Delta\ P_3M_3R_9 \parallel \Delta\ R_4M_1R_9$

$$\therefore \frac{P_3R_9}{R_9R_4} = \frac{P_3M_3}{M_1R_4} \quad \text{i.e.} \quad \frac{P_3R_9}{P_3R_9 + R_9R_4} = \frac{P_3M_3}{P_3M_3 + M_1R_4}$$

$$\text{i.e.} \quad \frac{P_3R_9}{P_3R_4} = \frac{P_3M_3}{P_3M_3 + M_1P_1}$$

$$\text{i.e.} \quad \frac{P_3R_9}{P_3R_4} = \frac{\frac{c}{2} - \overline{s-a}}{\frac{c}{2} - \overline{s-a} + \frac{a}{2} - \overline{s-b}} = \frac{a-b}{a-c}$$

Hence $P_3R_9 = \frac{a-b}{a-c} \cdot P_3R_4$

and $\frac{P_3R_9}{R_5C} = \frac{a-b}{a-c} \cdot \frac{P_3R_4}{R_5C} = \frac{a-b}{a-c}$ $\because$ $P_3R_4 = R_5C$ in $\parallel^m P_3R_5CR_4$

$$\therefore \frac{P_3R_9}{R_5C} = \frac{R_8P_3}{R_8R_5} \quad \text{or} \quad \frac{R_8P_3}{P_3R_9} = \frac{R_8R_5}{R_5C}$$

Hence the containing sides of equal angles are proportional and $R_8$, $R_9$, C are collinear .

Similarly $\Delta$'s $AR_3R_9$, $R_9R_4R_7$ may be used to prove A, $R_9$,$R_7$ collinear

and $\Delta$'s $R_8P_3B$ , $R_8R_5R_7$ , to prove $R_8BR_7$ collinear. Hence para 2 of the General Enunciation.

7) Proceeding to Problem No. 3.

$N_1$ is the midpoint of $AP_1$ and therefore lies in $M_3M_2$ . Join $P_1$ to 1 , the incentre, and produce to meet $M_3M_2$ in Q . Then $QP_1 \perp M_3M_2$ and also $\perp$ BC.

Draw AX $\perp$ BC .

Area $\Delta ABC = \frac{1}{2} AX.BC = QP_1.a$

also $\Delta ABC = s.1P_1$

$$\therefore \frac{QP_1}{1P_1} = \frac{s}{a}$$

$$\therefore \frac{QP_1 - 1P_1}{1P_1} = \frac{s-a}{a}$$

$$\therefore \frac{Q1}{1P_1} = \frac{s-a}{a}$$

Hence $$\frac{R_2 1}{1B} = \frac{Q1}{1P_1} = \frac{s-a}{a}$$

But $$\frac{N_1R_2}{M_1B} = \frac{M_3R_2 - M_3N_1}{M_1B}$$

$$= \frac{\frac{c}{2} - \frac{1}{2} BP_1}{\frac{a}{2}} = \frac{\frac{c}{2} - \frac{s-b}{2}}{\frac{a}{2}}$$

etc. $= \frac{s-a}{a}$

Hence $$\frac{R_2 1}{1B} = \frac{N_1R_2}{M_1B}$$

and since $\angle N_1R_2 1 = \angle 1BM_1$

$\therefore N_1, 1, M,$ are collinear.

8) In an analogous fashion the pedal triangle of the centre of each escribed circle of ABC also furnishes nine points of intersection with the sides of the medial triangle. Of these points, in each case, two coincide respectively with two of the original six (for the pedal triangle of the incentre) while all six lie in pairs on one or other of the circles with sides of ABC as diameters; the remaining three are collinear, in pairs, with a vertex of ABC .

As a result, each circle described on a side of ABC as diameter contains four "double" R-points. These "double" points are the extremities of diameters which are diagonals of circumscribed rectangles and which lie in the sides of the medial triangle. Also, each angle bisector, whether interior or exterior, contains two double R-points.

(Editor's note: Mr. Asbury adds "... while the theorem is probably of no special importance, it may have considerable interest as a supplementary exercise for the abler student. At any rate, the construction is an exacting test of accuracy and neatness, the proof furnishes a pretty thorough review of the geometry of inscribed, escribed and circumscribed circles of the triangle, and the diagram provides a wealth of illustrations of sets of concyclic and collinear points and of concurrent lines.").

SOLUTION TO PROBLEM NO. 4 (May, 1967);

K. Dennis, Courtice Secondary School, Bowmanville.

Professor Harris' hat-check problem is an interesting one to generalize. Consider the following three problems in which $n$ hats are returned at random to $n$ men.

Let $P_n(j)$ be the probability that exactly $j$ of the $n$ men get their own hats and $P_n(j+)$ the probability that at least $j$ men get their own, $j \in$ No.

Find: (a) the probability, $P_n(o)$, that no man gets his own hat

(b) the probability, $P_n(m)$, that exactly $m$ men get their own

(c) the probability, $P_n(m+)$, that at least $m$ get their own hats.

The solutions to these three problems can be developed from the theorem:

$$P_n(1+) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots (-1)^{n+1} \frac{1}{n!} \qquad ---(1)$$

The proof of this result is instructive and can be found in Introduction to Finite Mathematics by Kemeny, Snell, Thompson, pp. 140-142.

Solution to (a):

Since either at least one man gets his own hat or no man does and not both $P_n(o) + P_n(1+) = 1$

$$P_n(o) = 1 - P_n(1+)$$

and using (1), $P_n(o) = 1 - (1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots (-1)^{n-1} \frac{1}{n!})$

$$\text{If } n > 1,\ P_n(o) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots (-1)^n \frac{1}{n!} \qquad ---(2)$$

$$\text{If } n = 1,\ P_n(o) = 0$$

Then if $n = 4$, we have Professor Harris' problem

and $\overset{5}{P}(o) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} = \frac{44}{120} = \frac{11}{30}$

It is easy to show that $n = 1$ gives $P(o) = 1$, $n = 2$ gives $\overset{2}{P}(o) = \frac{1}{2}$, $n = 3$ gives $\overset{3}{P}(o) = \frac{1}{3}$, but the surprising result is that for $n > 3$, $\overset{n}{P}(o) \doteq .37$ and is essentially independent of $n$.

This is shown by putting $x = -1$ in the expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

and getting $e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots (-1)^n \frac{1}{n!} + \ldots$

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots (-1)^n \frac{1}{n!} + \ldots$$

Comparing with the R.S. of (2) shows that as $n \to \infty$, $\overset{n}{P}(o) \to \frac{1}{e} \doteq .368$.

Solution to (b):

The probability that exactly $m$ men get their own hats is

$$\overset{n}{P}(m) = \frac{\text{number of ways of giving exactly } m \text{ men their own hats}}{\text{total number of ways of giving hats}}$$

The denominator is simply $n!$.
The numerator is the number of ways of choosing a group of $m$ men, which is $C(n,m)$, times the number of ways of giving the members of the group their own hats, which is 1, times the number of ways of distributing the remaining $(n-m)$ hats among the remaining $(n-m)$ men so that no man gets his own. Call this last number $x$.
Then the probability that none of the $(n-m)$ men gets his own hat is

$$^{n-m}P(o) = \frac{x}{(n-m)!}$$

$$x = (n-m)! \; {}^{n-m}P(o)$$

$$^{n}P(m) = \frac{C(n,m) \times 1 \times x}{n!}$$

$$= \frac{C(n,m)(n-m)! \; {}^{n-m}P(o)}{n!}$$

$$= \frac{n! \; (n-m)! \; {}^{n-m}P(o)}{(n-m)! \; m! \; n!}$$

$$\therefore \; {}^{n}P(m) = \frac{1}{m!} \; {}^{n-m}P(o)$$

or substituting for $^{n-m}P(o)$ from (2)

$$\text{If } (n-m) > 1, \; {}^{n}P(m) = \frac{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots (-1)^{n} \frac{1}{(n-m)!}}{m!} \quad ---(3)$$

$$\text{If } (n-m) = 1, \; {}^{n}P(m) = o$$

If $(n-m)$ is small, say $(n-m) \leq 3$, the calculation is simple, For $(n-m) > 3$ the approximation from Solution to (a) can be used with little error.

$$^{n}P(m) \doteq \frac{1}{em!} \quad --- (3a)$$

Solution to (c):

Because the events are mutually exclusive

$$^{n}P(m+) = {}^{n}Pm + {}^{n}P(m+1) + {}^{n}P(m+2) + \ldots + {}^{n}P(n) \quad ---(4)$$

and the values on the right can be calculated from formula (3).

Problem 11(b) on p. 168 of Algebra 13 can be done this way and the answer is $\frac{11}{120}$ .

(Editor's notes: 1. Mr. Dennis suggests the following additional problem: if n hats are returned at random, what is the most likely number of men to get their own hats?

2. Comments on and solutions to problem No. 4 were also received from Prof. S. Clark, University of Saskatchewan; and from E. Evans, Guelph, Ontario.)

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## BOOK REVIEW

Title: Mathematics in Western Culture, by Morris Kline. Published 1953 by Oxford University Press, New York.

Professor Kline has undertaken a gigantic task - that of tracing the influence of mathematics and mathematicians on the society of their time, from Biblical days until the present. He states his theme boldly: "Mathematics has been so much a part of civilization and cultures that many historians see mirrored in the mathematics of an age the characteristics of the other chief works of that age." Having established the theme its justification requires probing into the fields of art, architecture, astronomy, science, literature, psychology, theology, politics and economics, as well as mathematics. The author makes many interesting points, but I fear the scope is too much for any one man, no matter how learned. He is most at home in the allied fields of mathematics and physics, and in fact makes a very good case for his thesis stated in another work that the former can be most effectively taught by means of illustrations from the latter. For example, a student who had a knowledge of only elementary algebra could make a good start towards understanding the fundamentals of differential calculus by reading the section on seventeenth century cosmologies.

The mathematics dealt with at the beginning of the book is mainly in the field of geometry and here Professor Kline is at his best. He presents a number of well known theorems clearly but tersely, where they are relevant and in a manner which would interest and stimulate a high school student. I found this to be the most satisfactory part of Mathematics in Western Culture. A long section dealing with a mathematical analysis of the use of perspective in Renaissance art, on the contrary, seemed technical for anyone but a serious student of drawing; to me

it was contrived and quite bloodless. The chapter following, dealing with projections and map making is much shorter and much more convincing as an illustration of the book's theme. Inasmuch as projective geometry is the mother of some of our "new" geometries, any student could read this part of the work with profit.

Mathematically the central sections of the book are disappointing. That the Age of Reason was one of the most fruitful intellectually in man's history is undisputed, and Professor Kline gives a quite thorough account of the principal philosophers. His examples of the mathematics on which these philosophies supposedly rested are however too sketchy by comparison, for only the fundamental ideas of differential calculus are touched upon. He does note, in all fairness, that not all seventeenth century intellectuals were enamored of mathematics. Samuel Butler writes scathingly of a contemporary:

> In Mathematics he was greater
> Than Tycho Brahe or Erra Pater:
> Forhe, by Geometrick scale,
> Could take the size of Pots of Ale;
> Resolve by Signs and Tangents Streight,
> If Bread or Butter wanted weight;
> And wisely tell what hour o'th'day
> The Clock doth strike, by Algebra.

Newton, on the other hand, could write in all seriousness: "To make this (solar) system ... required a cause ... very skilled in mechanics and geometry". Fortunately Newton's reputation does not rest on his theology.

In dealing with the underlying mathematics in music the author again takes what to me is an extreme position. Sine functions are examined in great detail in connection with sound waves; the mathematics of the even tempered scale and the structure of traditional harmony is explored; Fourier's theorem is stated (though not proved). All this

results in the statement: "It is theoretically possible to play Beethoven's Ninth Symphony entirely with tuning forks." In this day of electronic music we know how much more there is to satisfying music than the production of sinusoidal sound waves.

Another curious use of mathematics occurs in the arithmetic morality of Jeremy Bentham. Professor Kline cites an example:

"If two men have $1,000. each and $500. is taken from one and given to the other, less pleasure is gained by the act than pain is incurred, for the recipient's wealth is increased by one third while the loser loses one half".

In the latter half of the book the author deals with the basis of statistics and probability and their applications in science industry and philosophy, the geometries of Lobachevsky and Riemann, and on to a moderately clear exposition of Einstein's basic premises in the special and general theories of relativity. The section on paradoxes of the infinite could be read as either light recreation or as matter for deep consideration, depending on your inclinations. Either way there are some fascinating thoughts. There is included an ingenious proof to Zeno's paradox of Achilles and the tortoise, or if you like Rufus and Matilda in the buggy.

Here and there throughout the book there are insights into the character of some of the influential mathematicians, and this is not the least of its values. The account of Keppler's formulation of his matrimonial equation and its success is hilarious; Bentham was certainly less successful as a mathematical lover. Keppler, unlike Bentham, had the good sense to make the mathematics his servant and not his master. The Pythagoreans also, in their own devious ways, used their geometries and their number theories to prove what they found otherwise convenient to believe; to be forbidden to "sit on a quart pot" wouldn't really put much of a crimp in their social lives. They unfortunately went too far, and when they used the properties of the integers to keep women out of their organization they brought on their

own heads the wrath of exactly half of the population. There is a moral here for all of us.

Taken as a whole this book affords a good survey of most of the mathematics likely to have been encountered by a senior high school student, and not too much beyond. But it is not a text book. Rather it is a book which could most profitably be read in a leisurely manner by a mature person with an interest in things mathematical. Such a person might not agree with all of Professor Kline's conclusions, but he would have an appreciation of mathematics as a dynamic discipline which has played an ever increasing role in the development of the world as we know it today.

A. Glass,
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