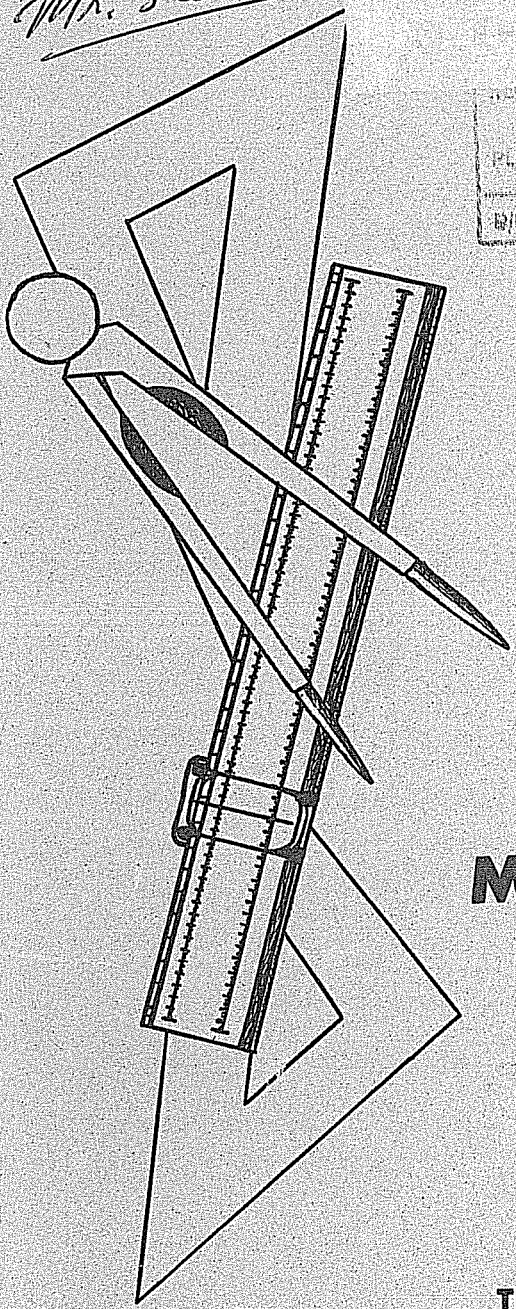
Mr. Steel



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ONTARIO MATHEMATICS GAZETTE

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E D I T O R I A L

Perhaps this editorial should begin with an apology. This issue, like the last one, is behind schedule. We do want to produce a first-class magazine, and to do so we need contributions from you out there who, after all, are our raison d'être. The contributions haven't been too plentiful, so take a deep breath and write an article, be it short or long, or even a letter to stir us up!

Incidentally, we are printing two letters to the editor, both about the workshop approach to elementary mathematics. Are there no High School Teachers who feel moved to write a blast to the Editor on this, or some other outrage, or even send a word of encouragement?

To be more serious, this issue is built round two long articles. The first discusses calculus and topology in an Exponentially refreshing manner, while the second considers a mysterious mathematical box. The article, if not the box, is decidedly meaty, while the box is undecidable. The style is delightfully non-conformist. Would you like the indispensable author to elucidate further? Let us know.

Between these, we are glad to print three articles aimed specifically at what goes on in the classroom. You may not agree completely with some of the categorical statements expressed in the first two, yet each represents an admirable depth of thinking about the more practical aspects of mathematics teaching. The third is essentially an appetiser. Surely every enterprising mathematics teacher should have a copy of this OISE publication.

Finally, there are one or two innovations in the rest of the material. Should we persevere with them? Over to you, and all the best for 1968.

D.H.C.

EXPLORATION AND INVENTION IN MATHEMATICS

by

G.F.D. Duff, University of Toronto

PART II of a 2-Part Article based on Expo Junior Science Lecture Series given in Montreal on August 5th, 1967.

Now we shall journey on to "the calculus" or the differential calculus, the science of continuous variation of numerical quantities. In mathematics, we have found that we work toward more general concepts by combining and recombining earlier ideas. In school we spend much time learning the properties of the system R of real numbers. Then we come to a new concept, that of a function, for which whole sets of numbers are used.

For example, suppose that at each hour of the day we measure the outdoor temperature in Fahrenheit degrees. We may obtain a table such as

Time (hours	1	2	3	4	5
Temperature (F)	70	72	74	77	80

To each time there corresponds a temperature reading. The set of these numbers can be graphed.

For every intermediate time a temperature can also be read off, if we draw a smooth curve through the given points.

The entire curve is said to determine a <u>function</u> f .

Thus the temperature function contains or embodies all the information provided by all of the readings at various times. To emphasize that each reading of temperature involves a correspondence of a specific time with a certain temperature reading we write

$$f: t \to T = f(t)$$

Here $f(t)$ ("f at t") is the value of the function f at the (numerical) instant t.

Of course there are many different functions that are useful in mathematics, just as there are many possible runs of temperature variation throughout the afternoon, and many other relations that could be graphed.

Before we continue with our function describing a pleasant summer afternoon I want to answer a question that many of you are probably asking. Did I say that the list of numbers 70, 72, 74, 77, 78 describes a function? A few minutes ago such a list was called a vector. Why the change?

If you did ask yourself this question, then you have done very well, for it is an excellent question. We <u>can</u> call this list a vector, and we <u>can</u> go on with this idea, and eventually we would again come to "functional analysis" or "20th century calculus", in which we use a lot of geometrical language, such as "point", "vector", or "space" to talk about functions.

Here and now, however, this would be too much for us to undertake. So let us return to the idea of a function, and its graph, and ask some other questions, preferably awkward questions. If we were asked to describe what is happening on the particular day these readings were taken, we would answer, "It is getting warmer". More technically, as t increases, T also

increases. It is when we begin to ask "how rapidly is this increase going on" that we come to the basic idea of the "differential calculus".

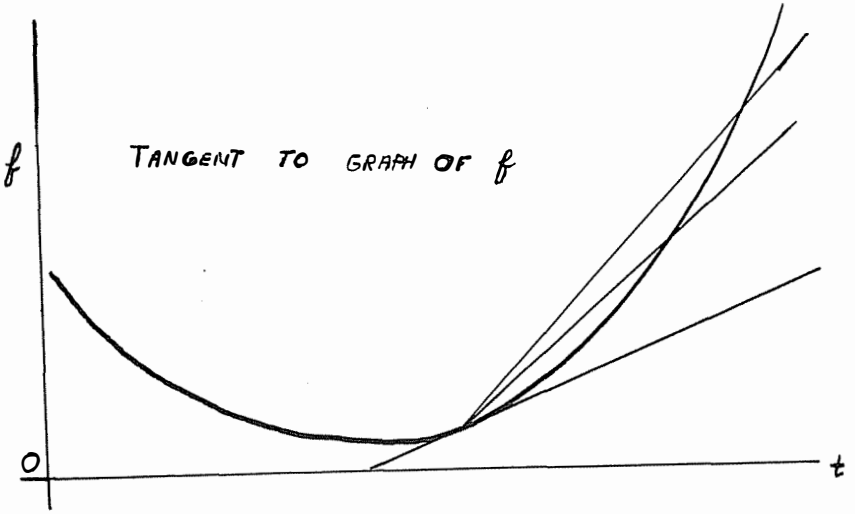
To be specific, we could surely say that at 2 o'clock, the temperature is rising at 2 degrees per hour. At 4 o'clock it is apparently rising at about 3 degrees per hour. Indeed, there is a rate of increase of temperature with respect to time at every instant of time. To see how we may measure it, let us examine more closely a small portion of the graph, as in the upper diagram in Figure 2.

Join two points of the graph by a chord (or segment of a straight line). The chord has a slope, or steepness, which tells the rate of increase approximately. To find a precise value for the rate increase, however, we must draw the tangent, or line touching the curve at time t, and measure its slope. Ideally we imagine that we could measure this slope with perfect accuracy, although in real life only an approximate and somewhat inaccurate measurement could be obtained graphically.

But now we have determined a new function - the rate of increase function. There is a value of the tangent slope, or rate of increase, that corresponds to each instant of time! Since this defines a new function "derived" from the given function it is called the derivative of that function. The symbol f', or the symbol $D_{f'}$, are often used to denote this new function.

Let us prolong the graphs of f and of f' over a longer time interval. In the late afternoon the temperature begins to fall. Thus its rate of change takes on negative numerical values. Let us comment on the way the features of the two graphs correspond.

In fact let us do a little "roller coaster" mathematics, as in Figure 3.


f
TANGENT TO GRAPH OF f
0
t

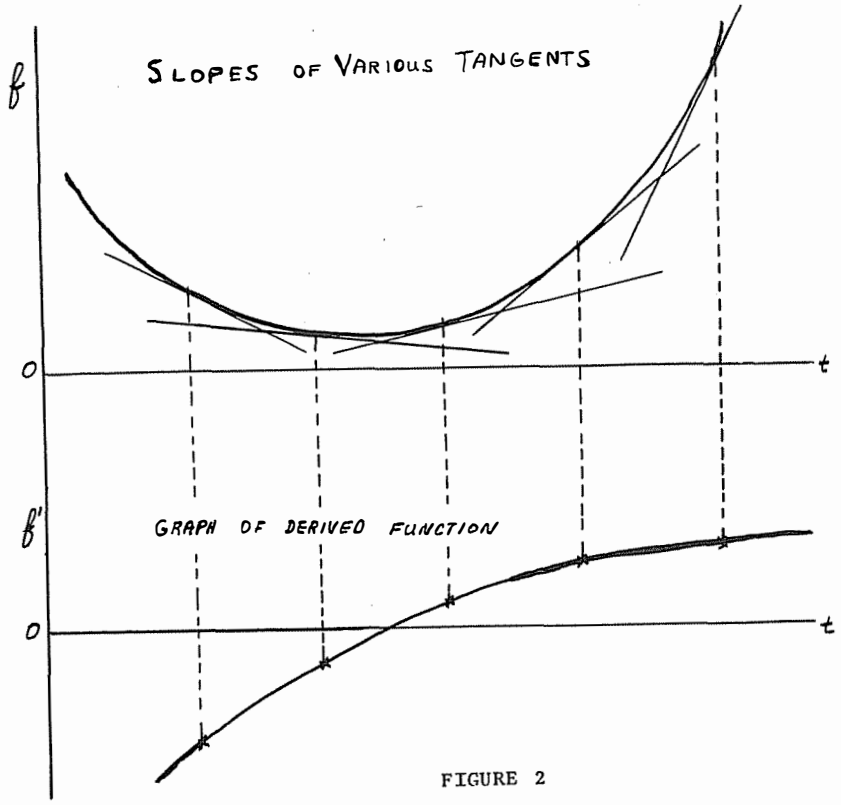

f
SLOPES OF VARIOUS TANGENTS
0
t
f'
GRAPH OF DERIVED FUNCTION
0
t

FIGURE 2

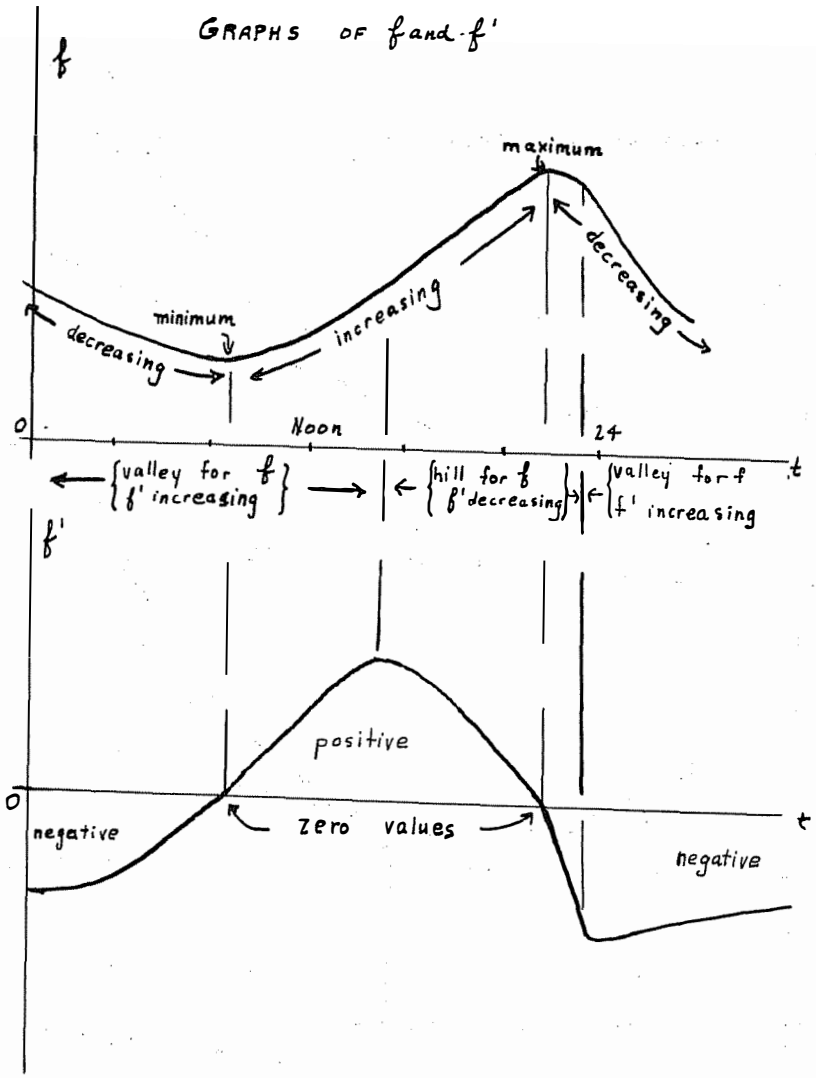

GRAPHS OF f and f'
f
maximum
minimum
decreasing
increasing
decreasing
0
Noon
24
t
valley for f
f' increasing
hill for f
f' decreasing
valley for f
f' increasing
f'
positive
0
negative
zero values
negative
t

FIGURE 3

f	increases	f'	positive
f	decreases	f'	negative
f	has a maximum, and neither increases nor decreases	f'	has value zero
f	has a minimum, and neither increases nor decreases	f'	has value zero
f	has a graph bending down to right: "hill"	f'	decreases
f	has a graph bending upward to right: "valley"	f'	increases
f	has a graph that does not bend up or down: "inflection"	f'	has a maximum or a minimum

If you like freehand drawing, then take some large sheets of paper, sketch some curves and plot their derivative curves or "rate of increase" curves. Enjoy yourself by making the curves high, wide and handsome.

Imagine you are a motorist on an open highway, and plot the readings of your mileage meter. What is the significance of the derived function? The velocity! Velocity is rate of change of position with respect to time! Thus the speedometer is a machine that automatically reads off the rate of change of position. The speedometer is, as we say, an automatic deriving mechanism; it "differentiates" or "derives" the distance travelled with respect to time. What significance has the curve of rate of change of velocity with respect to time? It is the acceleration.

In symbols, $v = Ds$

and $a = Dv = D^2s$.

In Figure 4 is shown the history of a motor trip to

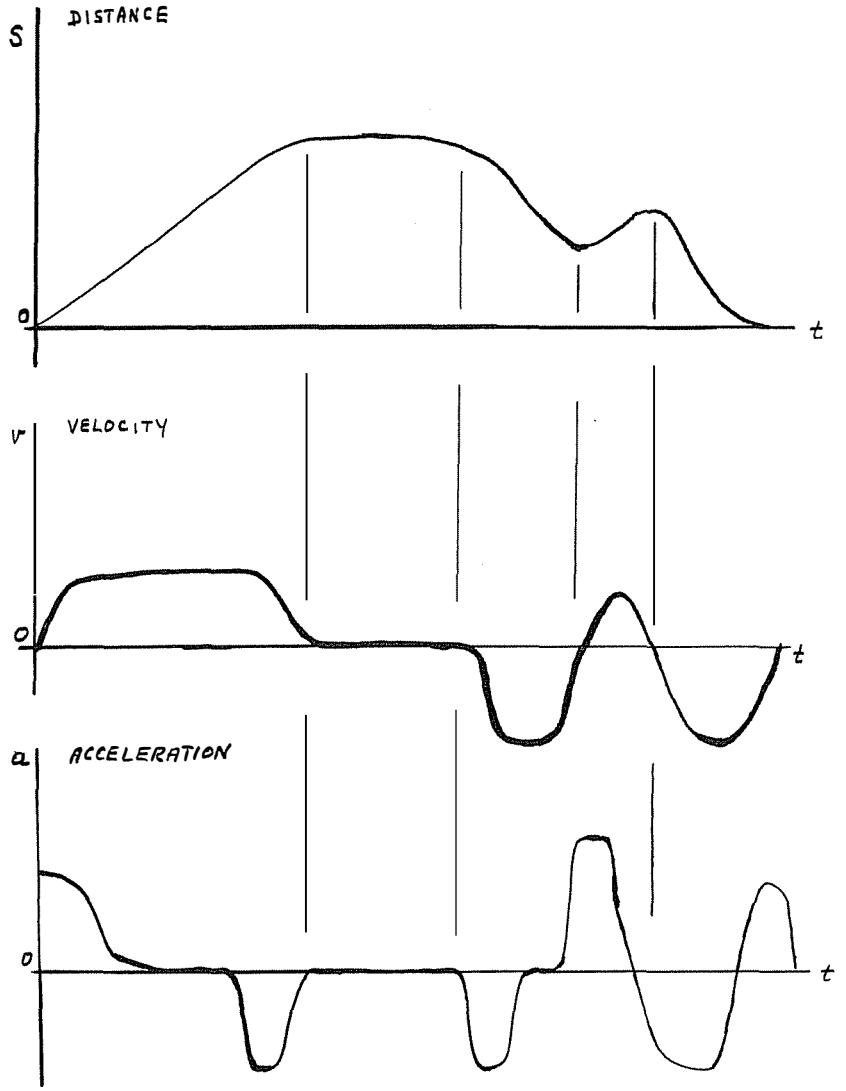


FIGURE 4

Expo, with graphs of the distance from the starting point, the velocity, and the acceleration. Notice the gradual start, the steady speed, and the application of the brakes before arrival. Notice that while the car is in the Expo parking lot, the velocity and acceleration are both zero. On the return trip the velocity is counted as negative, since the distance is decreasing. Suddenly we realize that some souvenirs have been forgotten. We start back again, but soon someone finds them under a coat in the back seat, and we continue our interrupted return journey. Observe how the various changes of speed and direction appear on the graphs of velocity and acceleration. Acceleration or deceleration occur in fairly short intervals, during which the velocity curve rises or falls, and the position curve bends. The distance travelled to the destination is represented on the velocity curve by area = the area between the curve and the t-axis. Since the total return distance is equal to the total going distance, the sum of the two areas above the t-axis is equal to the sum of the two areas below.

The process of reaching a desired value of distance by putting together values of its rate of change, the velocity, is the opposite process to derivation. This opposite or inverse process is called integration and the study of the two operations of derivation and integration constitutes the calculus.

Many interesting and useful applications motivated the study of calculus from the time it was invented by Newton, by Leibnitz and others three hundred years ago. **Fascinating** also is the way in which, by a clever use of functions and formulas, results of perfect numerical precision can be obtained in an apparently miraculous way! But again we must pass on to another topic, and leave you to explore this aspect of calculus for yourselves.

We now arrive at Topology, - the continuous variation of non-numerical quantities.

We all know that Geometry began long ago with the study of the measurement of land - hence the name Geo (earth) and metric (measure). By Geography or Geology we likewise mean the "drawing" or describing of the earth, and we all know that in geography or geology much use is made of maps. However, the qualitative description of the land is known as topography; in topography the various features of a geographic region are discussed in relation to one another. Similarly, there is a branch of mathematics, a very important and active branch, in which the qualitative relationships of points, lines, curves, regions and geometric sets are discussed. This is topology, and I would now like to describe to you in an elementary way some of the leading ideas and examples of topology.

Here is a short descriptive passage that contains many words used by a topologist. "The set of places occupied by Expo is in the neighbourhood of Montreal. The main components of the exposition are located on two separated islands that are surrounded by the St. Lawrence River. The visitor to Expo can travel continuously over the two islands on the minirail, which forms a closed loop having neither a beginning nor an end. The islands are connected to the mainland by bridges, as shown on the map of Expo.

This description involves several topological ideas, such as: connected set of points, neighbourhood, continuous, separate, enclose, boundary and map.

Let me give you another everyday example of a topological nature. Imagine a set of metal rings that are looped together to form a chain as in Figure 5. What is the difference between this set and a second set of links that are

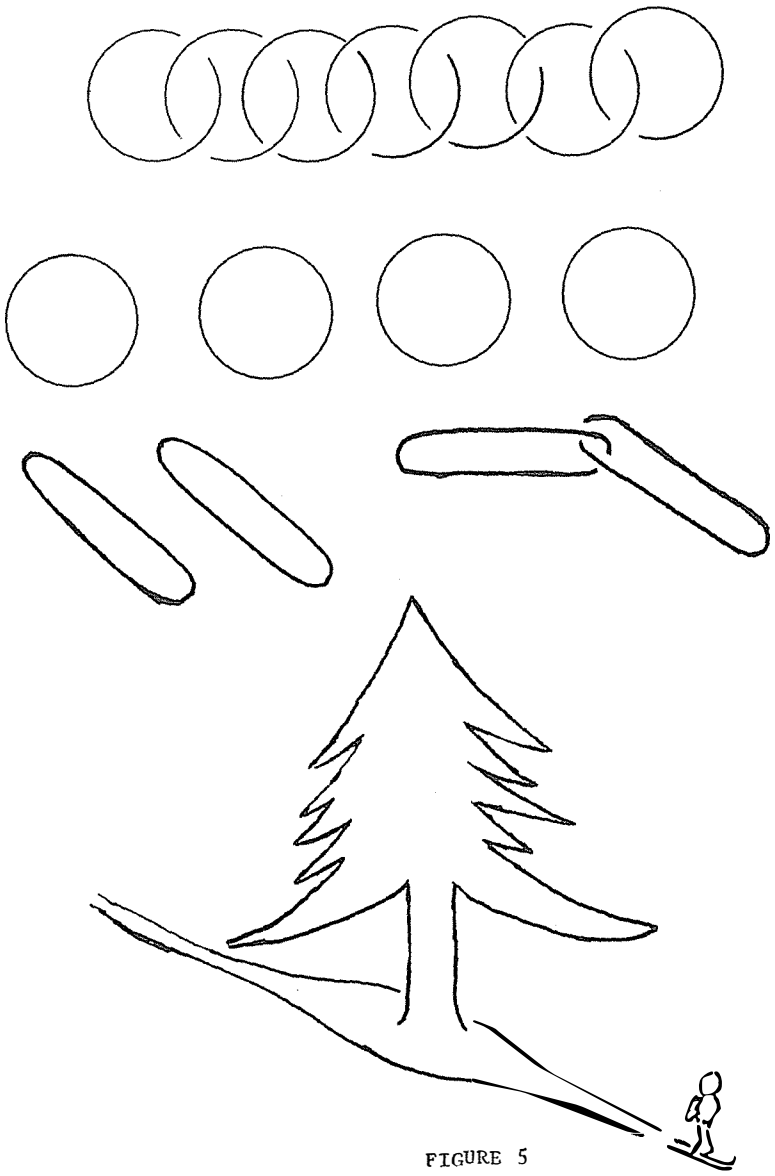


FIGURE 5

all separate from each other? Clearly it is not a difference in the links themselves, for they can be made identical. It is a difference in their mutual spatial relationship.

Suppose I was to give you two rubber bands linked together as a chain (see Figure 5 again). Now rubber bands come from the factory not linked, like the second set of metal loops just mentioned. You would know that in order to link them I must cut one of the bands, loop it through the other and then rejoin the severed ends with invisible rubber cement. Indeed the two linked rubber bands are a sort of a joke, just as the second set of separated metal links are a joke - and a very poor joke for the sailor who wishes to use such a chain to hold his anchor.

The topologist describes such jokes with a straight face, however, and calls them homology classes.

Perhaps you remember the cartoon of a skier travelling down a snowy hillside, the two tracks of his skis through the snow passing one on each side of a very large tree. Nothing impossible or ridiculous about this - just an example of a non-zero homology class, if we assume that the tree reaches up to the sky.

However the basic concept on which topology is founded appears in the example of the severed rubber band. The cut band is no longer continuous. Indeed rubber bands or rubber sheets show very well the kinds of motions, or deformations or stretchings, that can be made in a continuous fashion, with cutting forbidden. Consequently topology is popularly described as "rubber sheet geometry". Those of you who saw the films shown this morning will recall that two shapes related through a continuous deformation are called homeomorphic. In the popular mind, the most important fact of topology may be the celebrated homeomorphism of the donut and the teacup.

However, this is only part of the story, because in

topology we really take the continuous motions of the donut or of the rubber sheet, more or less for granted, and concentrate instead on the jokes - that is, the different possible relationships that cannot be reached one from another by a continuous motion.

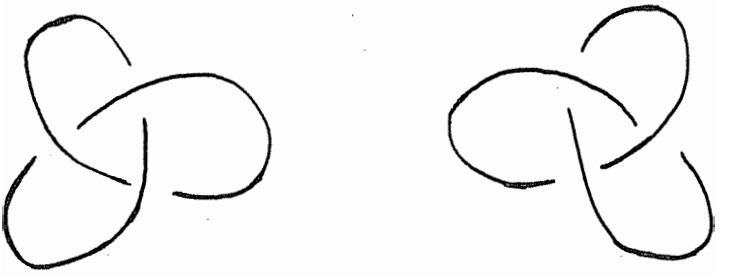
The rubber band, or a single metal link of a chain, is itself a simple closed curve. Such a curve may be also called a knot - it is like a piece of string that has been knotted and then had the two ends spliced to keep the knot from unravelling. Here, in Figure 6, is another example of a knot- the trefoil knot, and you see that the two trefoil knots shown are mirror images of each other. If there is a continuous motion, or mapping, that carries one such curve into the other, the two are topologically equivalent. In the case of the trefoil knot, there is no continuous mapping that carries the left hand knot into the right hand one. Thus the two knots are essentially distinct.

More complicated linkings are also possible. The famous Borromean rings, for example, are so related that if any one of the three rings is removed, the remaining two are unlinked; but the three together are essentially and symmetrically linked.

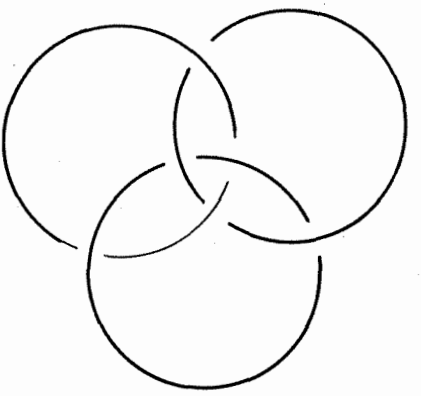
Of course we can envisage closed curves that are both knotted and linked. You can see that the nature of a knot depends on the way the curve is embedded in 3-dimensional space, while the nature of the linking of two closed curves depends on the way their positions are related to each other. It has been found that there are algebraic systems, called groups, which are well adapted to describe the various knots and links that can occur.

Since this is all rather abstract, one must be very good at picturing or visualizing to be an expert topologist. It is interesting that there is one famous

TREFOIL KNOTS



BORROMEAN RINGS



KNOT AND MULTIPLE LINK

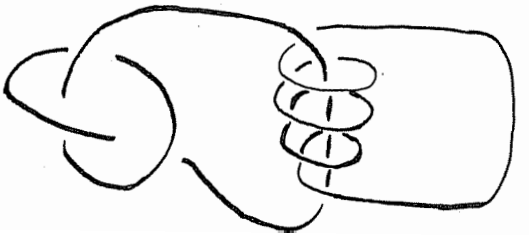


FIGURE 6

topologist, the Soviet mathematician L.S. Pontryagin, who has been blind from childhood.

Now let us consider another example of a topological "joke".

The Blowhard Tire Company (an organization I have just invented) makes inner tubes for automobile tires, and these inner tubes have a special trademark: two purple stripes (see Figure 7). One purple stripe is on the inside (where the compressed air is) and it goes right around the inside over the full length of the circumference of the tire. The second purple stripe goes around the outside like a single loop of a tire chain. From this you can visualize that if the rubber of the inner tube were removed, the two purple stripes would be linked like the pair of doctored-up rubber bands.

One day a topologist proved that the inner tube could be turned inside out (this is true) through the air valve by a continuous motion. So he turned his inner tube inside out continuously by this means. Then the first purple stripe around the circumference came to the outside, while the second purple stripe around the short "tire chain" loop went to the inside. If, and I repeat this, if the two strips resumed their approximate positions, then you can visualize that they are no longer linked, but are now separated like two genuine rubber bands.

What happened? What did we assume that was not justified? Or is this a genuine paradox? Perhaps you can still find out for yourselves even if I tell you that we got the homology classes of purple stripes confused.

However our guided tour of topology should now turn to more serious things. I would like to explore with

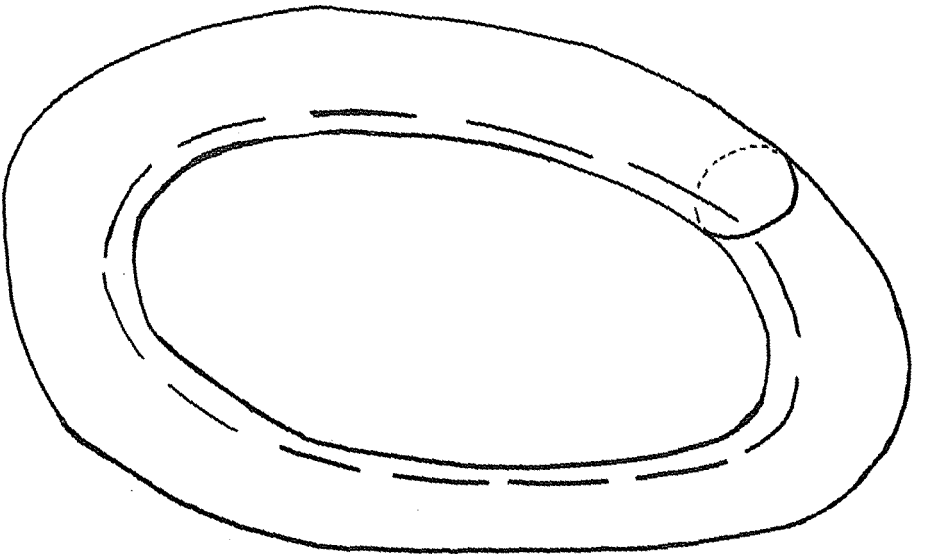


FIGURE 7

you an important, perhaps obvious, but yet rather subtle theorem that involves the idea of a boundary.

The concept of a boundary is a good example of an everyday idea that has found its mathematical place in topology. In everyday life we often do not recognize the generality of this idea, for surfaces, closed curves or edges, and ends, all form parts of various types of boundaries.

If the skin of an orange were mathematically thin, we would say that the skin is the boundary, or surface, of the orange. The orange itself is a three-dimensional solid, and its boundary is a two-dimensional surface. The surface itself is called a two-dimensional sphere.

However, "boundary" is a very general idea that applies to objects of various dimensions as Figure 8 suggests. Thus Canada, on the map, is a two-dimensional country. But the boundary of Canada with the United States of America is a boundary line. This line is one-dimensional. If we take the entire boundary of mainland Canada including its ocean frontiers, we have a continuous line that forms a closed curve like an irregular circle. To a topologist, the boundary is a one-dimensional sphere or simple closed curve.

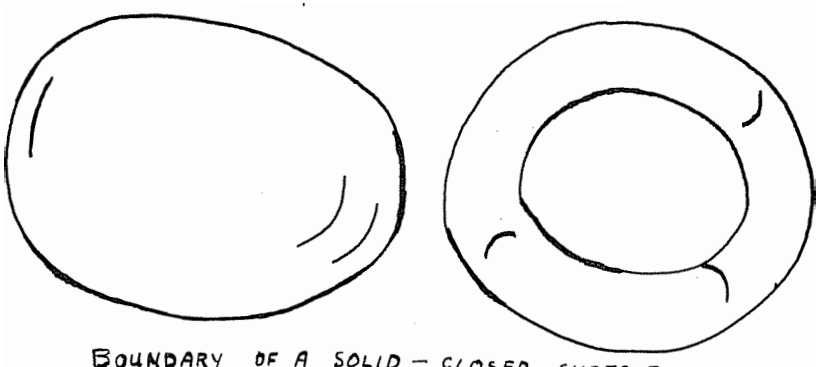
Then let us take a one-dimensional object such as a line segment. The boundary of a line segment consists of its two ends - two points forming a set of dimension zero.

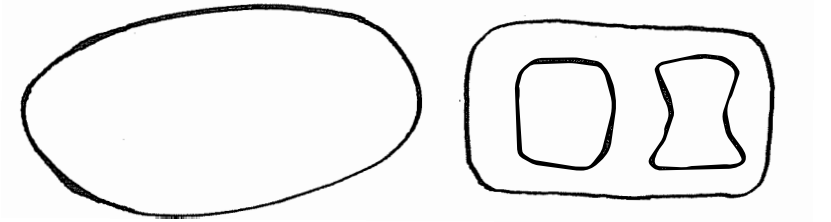
Thus

$$\dim \text{ (boundary)} = \dim \text{ (object)} - 1$$

This is one theorem of topology.

But the theorem I am after is much more striking than this! What is the boundary of a one-dimensional loop, or circle, such as a rubber band? But it has no boundary.


BOUNDARY OF A SOLID – CLOSED SURFACE


BOUNDARY OF AN AREA - CLOSED CURVE(S)

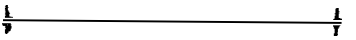

BOUNDARY OF A SEGMENT – TWO POINTS

FIGURE 8

We say, its boundary is zero. Note that a circle is itself the boundary of a two-dimensional disc, namely, the region, say, on our sheet of paper, within the circle. When Expo began, this statement about circles was illustrated by the minirail loop. Some passengers went around and around, and they never came to the boundary because there was no boundary! Finally the ticket collectors installed some artificial boundary points.

I should make clear that when I speak of a boundary, I mean an entire, or complete, boundary of some object, and not just a part of it. Thus the entire full circle is the boundary of a disc. But the circle itself has zero boundary.

Are there any surfaces that are boundaries? Yes, for example, the entire skin of the orange, or the complete Blowhard Company inner tube, which is the boundary of the compressed air in the tire, provided the air valve is not leaking. But the orange skin itself has zero boundary! And likewise we see that the surface of the inner tube has zero boundary! For if there was any such boundary, the air could escape around it!

Thus we are led to conjecture as follows:

If an object is itself a boundary of something else, then its own boundary is zero; or, "the boundary of a boundary is zero".

This is the theorem I want: it can be proved! Try it on all sorts of objects!

Denote the operation of "locating the boundary" by b. Then the theorem says that

$$b(b) = b^2 = 0 \ .$$

Some of the ancient navigators, many centuries ago,

refused to sail around the world because they were afraid of "falling off the edge of the world". If they had known this theorem, they would have realized that since our geographical world is the boundary of that solid sphere, the earth, it can have no boundary itself. The world of geography is said to be finite (200,000,000 square miles) but unbounded (that is, having boundary zero.) Thus the ancient navigators could never have fallen off the edge, because there was no edge.

Let me end this account by asking a modern day version of this same question that involves topology with astronomy and cosmology. "Does the entire universe have a boundary?" If so, what is it? Will we ever fall off it? Possibly, some day, one of you will be able to help give an answer to such questions.

****

In the far Canadian Arctic it is so cold that everything contracts a great deal, and a special value of pi is needed to take account of this contraction. It is called Eskimo pi .

GRADE 13 SEMINARS IN MATHEMATICS

by

Don Pickersgill,
Port Dover Composite School.

General Organization

In the school year 1966-67, the number of credits required for a Secondary School Honour Graduation Diploma was reduced to seven. At the same time, the school was re-organized on a 6-day, 48 period cycle.

It was decided that the time allotment for the subjects of Grade 13 would be as follows:

Introduction to Analysis -	8 periods per cycle
Algebra, English, French, Latin -	7 periods per cycle
Physics, Chemistry, Biology, History, Geography -	6 periods per cycle

This meant that in a program consisting of English, French, Latin, Analysis, a student would be taking 29 periods per cycle, and that in a program consisting of English, Analysis, Algebra, Physics, Chemistry, a student would be taking 34 periods per cycle.

The extra 14 to 19 periods were welcomed as a chance for students of Grade 13 to pursue individual study and research at a time when the library and other facilities were open to them.

In order to direct the research, it was decided to schedule seminar periods on the timetables of both students and teachers. These periods were allotted on the basis of one seminar period to every 8 students in a subject so that there were, for example, three French seminars for 25 students and two Geography seminars for 11 students.

Seminar periods were set up in such a way that they

fulfilled the following requirements:

(a) That at least one seminar per subject was open to every Grade 13 student.

(b) That no student was scheduled into more than one seminar per subject per week.

(c) That no seminar group be larger than 10.

(d) That every seminar in French, Science and Geography be held in the teacher's class room.

(e) That all other seminars be held in the teacher's class room when possible. To make the above arrangements possible, it was necessary to discover the number of students in each subject who did not take each other subject. As an example, by timetable inspection, it was found that seven Analysis students did not take Chemistry, 10 Analysis students did not take Biology, and 8 Analysis students did not take French. It was possible, then, to schedule one Analysis seminar opposite classes in each of Chemistry, Biology and French.

Mathematics Seminars

At the first mathematics seminar of the year the students were presented with sheets of ground rules and a list of over 100 topics and references. The English department supplied a list of rules on the format of written reports. There are approximately four hundred books in the school which are suitable for Grade 13 mathematics reports. It was possible to add many new books to the mathematics section of the library by simply submitting a list of desired titles, authors, publishers, and prices to the librarian. The format of a seminar period should provide a welcome change from normal class room routines. Students should be encouraged to discuss

their topics with each other in an informal way. The enthusiasm generated by this practice is contagious! Students who have never had the opportunity and training to do individual research in mathematics are bewildered by it all at first. As preparation, the students read sections from various references pertaining to their topic. The emphasis in the written reports must be on applications, so that they are marked on their own thinking. The ideal situation would be for the students to make up their own theorems and applications. In practice, they need some definite questions to answer on the topic they have read. Creativity and individuality develop gradually as practice and encouragement from the instructor build up their self-confidence.

The topics which the students choose may be either related to what has been done in class or completely separate, but must not be identical to work on the course. Each student is responsible for a written report at the beginning of October, November, December, February, March and April. The minimum length of each report is 3 or 4 pages, but most students seem to get involved and do much more. Most students find that they spend at least as much time working on these at home as they do in seminar periods. During the second semester, the student and instructor choose the written report which is most likely to prove stimulating for the rest of the class. The student is then responsible for preparing and delivering this report orally to the rest of his seminar group, in April or May. The construction of original visual aids is particularly encouraged in this connection.

The instructor acts mainly as an adviser, helping the student with difficulties, and providing interesting leads on the one hand, while refusing to do all of the thinking for the student on the other. It is essential that the instructor spend 10 or 15 minutes discussing each report with the student, emphasizing the strong points and showing

how to avoid the weaknesses in the future. By providing topics and references on varying levels of difficulty, it is hoped that we can provide a stimulating experience for every student.

The following is a short list of topics and references. Most of these books contain questions suitable for getting the student started in writing his report.

1. Topology

Patterson, Topology, Oliver and Boyd

Lietzmann, Visual Topology, Chatto and Windus

Jones, Elementary Concepts of Mathematics, MacMillan

NCTM 28th Yearbook, Enrichment Mathematics for High School

Moore, Elementary General Topology, Prentice Hall

Meserve and Sobel, Mathematics for Secondary School Teachers, Prentice Hall

Dynkin and Uspenski, Multicolour Problems, Heath

Life Science Library, Mathematics

2. Solid Analytical Geometry

Schaum's Outline Series, Theory and Problems of Solid Analytical Geometry

Stanton, Fryer, Algebra and Vector Geometry, Holt, Rinehart and Winston

McCrea, Analytical Geometry of Three Dimensions, Oliver and Boyd

3. Continued Fractions

NCTM, An Introduction to Continued Fractions

4. Curves

Lockwood, A Book of Curves, Cambridge

Johnson, Curves in Space, McGraw-Hill

Row, Geometric Exercises in Paper Folding, Dover

Shanks et al, Pre-Calculus Mathematics, Addison Wesley

Life Science Library, Mathematics

Cohn-Vossen, Geometry and the Imagination, Chelsea

5. History of Computing

Johnson, Computing Devices, McGraw-Hill

Life Science Library, Mathematics

6. Boolean Algebra

Allendoerfer and Oakley, Principles of Mathematics, McGraw-Hill

Glicksman and Ruderman, Fundamentals for Advanced Mathematics, Holt, Rinehart and Winston

Richardson, Fundamentals for Advanced Mathematics, MacMillan

Arnold, Logic and Boolean Algebra, Prentice Hall

7. Circuit Design

Allendoerfer and Oakley, Principles of Mathematics, McGraw-Hill

Glicksman and Ruderman, Fundamentals for Advanced Mathematics, Holt, Rinehart and Winston

Richardson, Fundamentals of Mathematics, MacMillan

8. Linear Programming

Rutledge, Cairns, Mathematics for Business Analysis, Holt, Rinehart and Winston

Schuster, Elementary Vector Geometry, Wiley

NCTM 28th Yearbook, Enrichment Mathematics for High School

Glicksman, An Introduction to Linear Programming and the Theory of Games, Wiley

Barsov, What is Linear Programming?, Heath

9. Newtonian Approximations

Fisher, Ziebur, Calculus and Analytic Geometry, Prentice Hall

10. Fibonacci Numbers

Vorobyov, The Fibonacci Numbers, Heath

Life Science Library, Mathematics

11. Special Functions

Allendoerfer and Oakley, Principles of Mathematics, McGraw-Hill

Glicksman and Ruderman, Fundamentals for Advanced Mathematics, Holt, Rinehart and Winston

12. Cardinality

Yarnelle, An Introduction to Transfinite Mathematics, Heath

Young, Infinity and Beyond, Ginn.

13. Annuities and Bond Values

Petrie et al, Algebra, A Senior Course, Copp Clark

14. Inverse Circular Functions

Shanks et al, Pre-Calculus Mathematics, Addison Wesley

Rickey, Cole, Plane Trigonometry, Holt, Rinehart and Winston

15. More Difficult Problems on Induction

Stanton, Fryer, Topics in Modern Mathematics, Prentice Hall

Sominski, The Method of Mathematical Induction, Heath

Youse, Mathematical Induction, Prentice Hall

16. Determinants

Stanton, Fryer, Algebra and Vector Geometry, Holt, Rinehart and Winston

Margulis, Systems of Linear Equations, MacMillan

Aitken, Determinants and Matrices, Oliver and Boyd

17. Groups

Grossman,Magnus, Groups and Their Graphs, Singer

18. Probability

Dynken and Uspenski, Random Walks, Heath

Niven, Mathematics of Choice, Singer

Ball, Mathematical Recreations and Essays, MacMillan

19. Logic

Suppes, Introduction to Logic, Van Nostrand

Allendoerfer and Oakley, Principles of Mathematics, McGraw-Hill

Richardson, Fundamentals of Mathematics, MacMillan

20. Geometry

Smogorzhevskii, The Ruler in Geometrical Constructions, Blaisdell

Kostovskii, Geometrical Constructions Using Compasses Only, Blaisdell

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It is amazing, yet true, that over 90% of the cars built in Detroit since 1958 are less than 10 years old.

SOME COMMENTS ON TEACHING TECHNIQUES

by

W.G. Anglin,
Ontario Department of Education.

Introduction

The opportunity which has been available to me over the past year or so to visit and observe mathematics teachers at work has been one which I feel has greatly broadened my horizons and the base from which I view teaching. I have had a chance to view the techniques used by the extremely competent, conscientious and imaginative teacher and to compare them with those of the unimaginative plodder who manages to get things done with little or no life and inspiration.

This article is one in which I will try to point out some of the things which I have observed and which I feel can help to put more life into the presentation of material. In certain cases I will try to illustrate my points by examples from the courses presently in use in the schools. The observations made and the opinions expressed are entirely my own and in no way represent an 'official point of view'.

Gaining the Confidence and Respect of the Student

One of the keys in the teacher-student relationship is a feeling on the part of the student of both confidence in and respect for the teacher. Confidence is fostered when the student is aware that the teacher has an adequate background in the subject matter and comes to class prepared to teach it. Students are very perceptive and can

quickly spot a teacher who is unsure of the course content or who is on shaky ground because he has not adequately prepared the lesson for the day. This is not meant to suggest that a teacher must be fluent in all the varied branches of mathematics. It is not an admission of incompetence to say "I don't know", but too many "I don't know" answers about course content can certainly shake student confidence.

Respect will readily be gained from the students if they feel that they are respected by the teacher. This is particularly true of the senior students in a four year program. The teacher who has sincere respect and liking for these students will find working with them extremely rewarding. In many instances they show more maturity than their counterparts in a five year program. A little attention paid to some of the students' extra-curricular interests can pay tremendous dividends in class.

Some Do and Don'ts of Questioning

The questioning technique of many teachers is one thing which keeps them from communicating with their students in a clear and interesting way. There are several situations in this connection on which I will elaborate.

If a question is worth asking then the time is well-spent getting an answer to it, even if the answer is incorrect. Some teachers will ask a question, then another, then another and sometimes will do so without getting any of them answered. A student giving an incorrect answer should not be cut off in mid-answer or simply told 'No' without explanation. The student who gives an incorrect answer certainly thinks that what he is saying is correct and feels that he deserves some explanation, either from the teacher or from other students, of where his reasoning is incorrect. (I am assuming here that the reader will

see for himself the distinction between refusing to accept that $3 \times 4 = 8$ without explanation and refusing to accept that $(a+b)(a+b) = a^2+b^2$. The student in the lower grades giving the latter answer deserves explanation of where he is in error although in the senior grades this is as bad as saying $3 \times 4 = 8$.)

Some teachers have a tendency to stop a student as soon as it is evident, (to the teacher), that he is giving a correct answer. It may be that the teacher is the only other one in the room who knows that the answer will be correct and the steps necessary to obtain it. The answers to questions should be given for the benefit of the students, not the teacher.

When a student is obviously guessing at the answer to a question and happens to guess correctly, the answer should not be accepted until the student has been asked to justify it. For example, if a Grade 9 student who is asked for the sign of the product $(-3)(-4)(2)(1)(-1)$ says "positive .... negative" and then, from the look on the teacher's face, says confidently "negative" he should be asked "Why?". He apparently does not know why.

Some teachers will name the student who is to answer a question before asking it. For example, "Margaret ... what are the roots of the quadratic equation $x^2 - 7x+12 = 0$?". Once the students become aware that this is the pattern of the teacher's questioning only the student named will be listening to the question and attempting to answer it. If the question is asked, and then a little 'thinking time' is provided before naming a student, everyone will be trying to formulate an answer to it.

Too often the teacher starts a question with, "Can anyone tell me ....?", "Do you know ....?", or "Do you remember ...?" Questions of this sort are a tacit admission by the teacher that his students may not know their work, and perhaps, are not even responsible for knowing it.

Such questions give the students an "out". They present little, if any, challenge to a class. If the questions are reworded, "Tell me ....", or "What is ...?", then the student must display his knowledge or lack of it to himself, to his class, and to his teacher. The student who can answer a question correctly will and likes to do so. The student who cannot answer demonstrates that he needs help.(along with others, perhaps) or that he has been negligent. Questions, like those reworded above, put the burden of responsibility precisely where it belongs - on the students' shoulders. Moreover, such questions will give the teacher more opportunity of testing orally and of discovering whether or not previously taught concepts have been fully grasped.

Speed and Thoroughness of Presentation

There is a tendency on the part of many beginning teachers to present work to their classes much too quickly and to assume that 'you only need to be told once' when presenting material. This is probably a result of only recently leaving the university atmosphere where the lecture method is the standard. Anyone with elementary or secondary teaching experience knows that one exposure does not ensure complete mastery. The addition of fractions is something which must be reviewed every time there is a need for it, regardless of grade level. (This was just as true of Grade 13 students from a 'traditional' program as it is true of Grade 13 students from a 'new' program.)

Many teachers seem to assume that their students are unable to generalize from several particular cases, for example, in Grade 12 there is no need for a teacher to define for a class an axis of symmetry. By discussion in the introductory lession the definition can be arrived at by the end of the class. By asking the class how to place a mirror on a rectangle, equilateral triangle, square,

regular pentagon, regular hexagon, etc. so that the image in the mirror completes the figure the idea of axis of symmetry can be illustrated and defined by the class. To put the definition on the board at the start of the period and say that "$(x,-y)$ is the image of (x,y) when the x-axis is the "axis of symmetry" is to rob the students of any interest in, or feeling for, an 'axis of symmetry'.

I have tried, in this article, to give an indication of some of the factors which I feel rob a teacher of the necessary communication with his class. A mastery of subject matter is not, in itself, sufficient for success as a mathematics teacher. It is a necessary condition but along with it must go the ability to communicate with the students and gain their respect and confidence.

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JOKE:

The following story is true: A student was asked, "Does the sequence $\{\frac{n}{n^2+1}\}$ have a limit as $n \to \infty$?" He computed correctly that $\frac{n}{n^2+1} \to 0$ as $n \to \infty$, and concluded "therefore there is no limit".

INTERMEDIATE GEOMETRY: SOME LESSONS AND SUGGESTIONS

by

D. Boyle*

Sir Wilfred Laurier Secondary School, London

This material, based on the "K-13 GEOMETRY REPORT", has been prepared by OISE for enrichment and experimental purposes in Grades 7 to 10. This is not a "new" geometry course, but rather, some interesting topics have been presented so that a class can follow the topics as far as possible, limited only by their ability to grasp the concepts. The selection of topics from the original K-13 Report was based on numerous factors; however, the deciding factor was - "Is experimental material readily available for the classroom teacher?". Thus some topics are similar to others already developed elsewhere (e.g., The School Mathematics Project).

Some of the topics included are:-

Paper Folding. This is suggested as an approach to symmetry and at the same time as an introduction to some basic geometrical ideas. For example, given a paper triangle, by folding the edges on top of each other in pairs the concurrence of the angle bisectors may be illustrated. Similarly, the concurrence of the medians, altitudes, and

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* Chairman, K-13 Geometry writing team, London, July, 1967.

right-bisectors may be shown. With a large triangle "Euler's Line" is demonstrable.

Tiling. A plane covered with congruent tiles can lead to the exploration of many geometrical concepts - Congruency, Parallel lines, Midpoint Theorem, Regular Hexagons, etc.

The approach to Euclidean geometry is based upon the construction and then examination of simple diagrams. For example, constructing the inscribed circle and the three escribed circles of a triangle can involve angle bisectors, altitudes, tangents, dilatations, etc. This section finishes with a discussion of the construction and properties of the Nine-Point Circle.

A different approach to formal geometry is suggested by a short section, Some Key Results of Euclidean Geometry. These results are displayed in a very logical development on two pages!

Solid Geometry is introduced with the problem of representing a 3-D object on a plane without too much distortion. Reference is made to the Renaissance painters, their attempts to produce realistic paintings, the problems that they had with perspectivity, and how their work helped lay the foundations of projective geometry.

Geometry on a Sphere provides an example of a geometry with no parallel straight lines, triangles whose angle measures add to more than 180^{o}, etc. This section also provides a link between geography and mathematics.

Coordinate Geometry is introduced using Transformations. For example, the point $(3,2)$ is mapped onto $(-3,2)$, by means of a reflection in the Y-axis. The point $(3,2)$ is mapped onto $(-3,-2)$, by means of a reflection in the Y-axis,

followed by a reflection in the x-axis.

These are just a few of the 22 topics covered by the report. Other topics include "Pick's Theorem", Construction of Conic Sections, Vector Geometry, Trigonometric Graphs, Polar Coordinates.

Any teacher using the material is requested to notify OISE of the sections used, at what grade level the material used, and a brief commentary on the results of the work.

Further information can be obtained from:

Coordinator of Development
The Ontario Institute for Studies in Education

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WHAT GÖDEL SAW IN HILBERT'S BOX*

by

Don C. Ferguson, Queen's University.

Today I should like to tell you about some of the results that have been obtained in the area of mathematics known as mathematical logic. In particular, I have in mind some of the contributions that were made by David Hilbert and Kurt Gödel.

Let me begin with Hilbert. His existence was established in 1862 by Herr and Frau Hilbert, as recorded in the little German town of Königsberg. Most authorities would have us believe that the parent Hilberts began this most successful research project at some time in 1861. Hilbert lived longly, not departing from the scene until 1943, and while he was with us he did some marvelous things. He probably did some God-awful things too, but I have not been apprised of such activities. In his prime Hilbert was recognized by his contempraries as a mathematician of the first order and he also exerted an unusually great influence on the mathematical research of his day. Yet in his early years his behaviour was observed to be quite ordinary. As an infant he did not prove any interesting theorems, instead, it has been noted that he was content to suck his thumb, or whatever else happened to be handy.

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* Based on a talk to High School students given at Queen's Math Weekends, Fall 1967.

In later years a profusion of papers was to follow in various fields of mathematics. Though his research was widespread, it went deep and it was highly original. The areas that he contributed to most influentially are algebra, geometry, analysis and logic (I wonder what else is left). In your courses in mathematics at the high school you have undoubtedly had some contact with Euclidean geometry, to which Hilbert made significant improvements as illustrated in hiw now famous Foundations of Geometry, and your acquaintance with algebra and analysis has come (or is yet to come) in the form of trigonometry and calculus. Now although Hilbert made lasting contributions to algebra and analysis (e.g. we do not refer to Hilbert Spaces in deference to Henri Poincaré), today I would like to tell you something about his impact on mathematical logic, a topic with which you may have had less contact.

One of Hilbert's most cherished projects was to build a very special box which I will refer to hereafter as Hilbert's Box. Here it is.

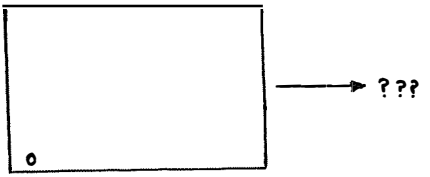


Hilbert's Box

So let us now turn our attention to this marvelous box, and try to obtain some sort of idea of what Hilbert put in his box and what he wanted the box to do.

First of all, I'd like you to think of Hilbert's Box as some sort of mystical machine which is equipped exteriorly with a single button, and the only thing you are allowed to do to the machine is press the button.

In other words, in no way can you interfere with the internal make-up of the machine. If you have the courage to press the button, the machine will start to work (you'll be able to detect this by the vulgar noises it will begin to produce). But pay no attention to the noises, what you should look for is what comes out of the machine.



Hilbert's Box

Mind you, there is no guarantee that anything will ever come out of the machine. None the less, keep your eyes open in case something does emerge. If something is ejected, be happy, for a monumental thing has happened; but keep your wits about you, remain calm, for it may be 10^{17} years (or longer) before the next emergence from the box takes place.

Well, back to more specific information about what is in the box. To this end, I must speak to you about what is known as a

formal system or formal logic ,

which, for brevity, I shall simply call a logic. So what then is a logic? Well roughly speaking a logic is made up of four integral parts, and here they are:

1° alphabet consists of a finite number of distinguishable symbols.

2^{o} there is given a finite number of explicit rules by means of which any finite string of symbols from the alphabet can be differentiated as being either meaningful or meaningless, and those which are judged as meaningful are called formulas or assertions.

3^{o} there is specified a finite number of formulas, each of which is thereafter referred to as an axiom.

4^{o} there is explicitly given a finite number of rules of inference each of which is finitistic in nature.

That is all. Briefly then, a logic consists of

1^{o} alphabet

2^{o} formulas

3^{o} axioms

4^{o} rules of inference.

There are two aspects of a logic which I want you to notice, namely that a logic is

5^{o} finitistic,

6^{o} deterministic.

The finitistic nature is clearly referred to by the several occurrences of the word "finite" in 1^{o}, 2^{o}, 3^{o} and 4^{o}. In addition, when in 4^{o} I said that a rule of inference was "finitistic", I meant that such a rule of inference is applicable to a finite number of formulas only and that the result of applying a single rule of inference yields a single formula. The deterministic nature of a logic is directly referred to by the use of such words and phrases as "distinguishable", "explicit", "differentiated",

"specified", "explicitly given". Actually I think that I can best explain what I mean by finitistic and deterministic by means of particular examples. But before giving such examples, there are two further aspects of a logic that are of singular importance. I refer to the notions of

7^{o} proof,

8^{o} theorem.

In fact, one of the main purposes of introducing logics in mathematics is that in such a setting the notion of proof (and hence of theorem) can be greatly clarified. Here then is the clarification. A <u>proof</u> in a logic consists of a finite string of formulas in the logic, say

$$F_1, F_2, \ldots, F_n$$

which satisfy the conditions (where j is one of $1,2,\ldots,n$)

i) F_j is an axiom,

or ii) if $j \geq 2$, then F_j can be obtained from $F_1, F_2, \ldots, F_{j-1}$ by means of a single application of a single rule of inference.

When the above conditions hold we say that

$F_1, F_2, \ldots, F_n$ is a proof of the <u>theorem</u> F_n;

i.e. the last formula in a proof is called a theorem.

Now for some examples. First let me describe a whole class of logics of a particularly simple nature. These logics are known as:

Post Logics

1° the alphabet consists of but two symbols, namely:

| (called "post") and

b (called "blank")

2° the formulas are arbitrary in the sense that any finite string of posts and blanks is a formula; e.g.

|, b, |b, b||, |bb||b| are formulas.

3° the axioms are restricted in the sense that there can be only one axiom, but this axiom can be any formula whatsoever.

4° the rules of inference are known as productions and are of a particularly simple nature. If α,β and x are formulas, then by:

$$P: \alpha x \rightarrow x\beta \qquad \text{or} \qquad \alpha x \overset{P}{\rightarrow} x\beta$$

we describe the production P as follows:

> The production P can act on any formula of the form αx and in so acting produces the formula $x\beta$.

In other words, the production P allows one to obtain the conclusion $x\beta$ from the premise αx . The formulas α,β are called the prefix, affix of P (respectively).

This completes the description of Post logics in general. One should note that the alphabet and formulas of any Post logic are the same, and hence to specify a particular Post logic one need only specify the axiom and the productions.

Example 1 (A Post logic)

The axiom: $|b|$

The productions: there are to be three P,Q,R given by

$$P: ||x \to x| \; ; \quad Q: b|| \to x|| \; ; \quad R: bx \to x$$

where x is any formula.

Upon noting that $|b|$ (the axiom) is not the prefix of any one of the productions, we see that the only theorem is the axiom $|b|$. Mind you, there are many proofs of this theorem. Here are three proofs of the axiom:

i) $|b|$

ii) $|b| \; , \; |b| \; , \; |b|$

iii) $|b| \; , \; |b| \; , \; |b| \; , \; |b| \; , \; |b| \; , \; |b| \; , \; |b| \; .$

Example 2 (Another Post Logic)

The axiom: $b|bbb|b|$

The productions: there are to be two P,Q given by:

$$P: |bbx \to x| \; ; \quad Q: bx \to x$$

We will now prove some theorems for you.

Theorem 1 $|bbb|b|$

Proof. We claim that

$$* \quad b|bbb|b| \; , \; |bbb|b|$$

is in fact a proof of the theorem in question. This follows since the first formula in * is the axiom and:

$$Q: b|bbb|b| \to |bbb|b| \; .$$

Theorem 2 $b|b||$

Proof. We claim that

$$\# \quad b|bbb|b| \ , \ |bbb|b| \ , \ b|b||$$

is a proof of this theorem. We have already (in the proof of Theorem 1) explained why the first two formulas in # can occur in a proof.

Finally we note:

$$P: \ |bbb|b| \rightarrow b|b||$$

and hence # is a proof of its last formula.

Theorem 3 $|b||$

Proof. We claim that:

$$\$ \quad b|bbb|b| \ , \ |bbb|b| \ , \ b|b|| \ , \ |b||$$

is a proof for this theorem. We have already explained (in the proof of Theorem 2) why the first three formulas in $ can occur in a proof. Finally we note that

$$Q: \ b|b|| \rightarrow |b||$$

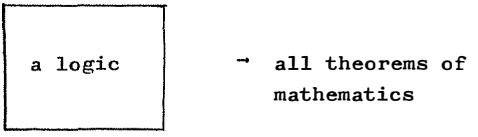
and hence $ is a proof of its last formula.

This completes our theorem proving for this logic, for we claim that there are no other theorems (other than the axiom), by inspection.

I will give no further examples of Post logics, nor of more general logics; but let me remind you what I hoped you would note from the above examples. Namely, you can actually check my alleged proofs to see that they are indeed proofs, and this is the way we want matters to be in any logic. Thus given any logic and an "alleged" proof of a formula in that logic, the validity of the alleged proof can be checked mechanically. Thus there can be no

controversy over theorems that are produced in a logic, you examine the alleged proof step-by-step to see if the various steps in the proof are permissible by means of the given rules of inference, and because of the finitistic and deterministic nature of a logic this can really be done. Consequently, even someone as inherently stupid as a digital computer could check proofs in a particular logic.

Well, of the many things that Hilbert required of his box, he required that it be a logic. What else? Hilbert wanted the axioms and rules of inference that he put into the box to be sufficiently powerful that out of his box would flow the known theorems of mathematics.



Hilbert's Box

Thus he contemplated sitting back and watching what came out of his box. Of course, once a theorem came out of the box, whether known or unknown, there could be no doubt of its validity (provided one was willing to believe in the axioms and rules of inference that were built into the box). So Hilbert's box was, so to speak, to replace the activity that we refer to as mathematics. To be sure, it was to be a great box.

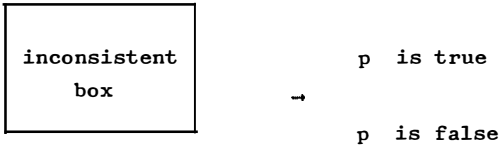
Now of the many things that one might desire of a box, Hilbert realized that he must at least have a consistent box. So what then is a consistent box? It is one that is constructed in such a way that it never produces a theorem of the form

p is true ,

as well as a theorem of the form:

p is false .

In other words, an inconsistent box would be one which would produce such theorems



Somehow it is clear that the best thing to do with an inconsistent box is to throw it away. However, not only did Hilbert want his box to be consistent, but he even wanted his box to be so shrewd as to be able to prove this. Another way of interpreting this would be: If we watched the theorems that came out of the box long enough, then eventually out would come the theorem:

"I am consistent."

Of course, it is conceivable that a box be consistent without being wise enough to be aware of its consistency.

So Hilbert was faced with two problems which at first glance seem quite monumental:

1° build a box from which all of contemporary mathematics will emerge,

2° build a consistent box.

Well Hilbert and his boys (known as the Hilbert or formalist school of mathematics) made significant advances on the first problem 1° , they were able to build a box from which number theory could be seen to spew forth, and, of course, the ultimate box would have to be capable

of at least this. After all, if you expect to be able to prove all known theorems, you had better be able to prove some theorems about real numbers. How does one get real numbers? One constructs them from rational numbers! How does one get rational numbers? One constructs them from the integers, and number theory is a study of some of the deeper properties of the integers; divisibility, primes, prime factorizations, etc. So they were on the right track, success was but around the corner, and so it was time to check the box for consistency. Here they failed and miserably so. Mind you, they were honest, hard working chaps and they exerted great effort but the consistency problem would not yield to their efforts.

About this time there enters on the scene, the second figure in our little soap opera in the form of Kurt Gödel. In 1931 at the tender age of 25, Gödel made some remarkable discoveries concerning Hilbert's box, discoveries that brought him almost instant fame (well--at least among mathematicians---well--at least among logicians). So I should now like to tell you something of what it was that Gödel saw in Hilbert's box. In some sense, what Gödel saw essentially smashed Hilbert's box all to hell, and all future boxes that Hilbert and the boys might have attempted to build. Specifically he proved:

Gödel's Inconsistency Theorem. Dear David, if your box will spew out number theory, then it will never produce the theorem:

"I am consistent."

I think the significance of the Inconsistency Theorem is fairly clear. If Hilbert wanted a box which could zap out number theory for him, then the box would not be able to establish its own consistency. In other

words, Hilbert had been too greedy, his box could not be built. Mind you, the Inconsistency Theorem does not deny the possibility that there exist consistent logics which can produce number theory, but only that such logics will not be aware of their consistency.

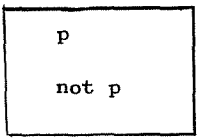
In order to tell you of Gödel's second success, let us take a moment off to conceive of another attribute which a box may or may not possess. This is the notion of a box being undecidable. Such a box would have the property that it contained a formula p , yet the box would never eject a theorem of the form

" p is true"

nor a theorem of the form:

" p is false."

Such a formula or assertion p is called an undecidable assertion for the box. Another way of thinking of this is that if the box contains the notion of negation (usually expressed by "it is false that ...." or "not..."), then neither "p" nor "not p" will ever get out of the box; i.e. they are imprisoned in the box.



"p" as well as "not p" are imprisoned in the box

an undecidable box

By the way, recall those teeny, weeny logics that I called Post Logics! Remember how pitifully simple they were! Well now, it just so happens that some of those

little devils form undecidable boxes (mind you, not the examples I gave). Perhaps we are now ready for

Gödel's Undecidability Theorem. There is even worse news David, if your box is consistent and can spew out number theory, then it is undecidable.

In the remaining time I'd like to discuss the Undecidability Theorem with you for it has implications both mathematical and philosophical. First let us consider the philosophical impact of the theorem.

To this end, let us consider number theory for a moment. Number theory abounds with problems which have not been solved, but which can be simply stated, such as the Twin-Prime-Pair Hypothesis (abbreviated TPPH). Recall that a prime is a positive integer p which is at least two (i.e. $p \geq 2$) and whose only positive divisors are 1 and p itself. Hence the primes in increasing order are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots.$$

and this succession goes on forever for we know that there are infinitely many primes (the Greeks knew this too). We now say that an ordered pair (m,n) of positive integers is a twin-prime-pair if and only if m,n are primes and $m+2 = n$. As examples of twin-prime-pairs we would have then

$$(3,5)\ ,\ (5,7)\ ,\ (17,19)\ ,\ (29,31)\ .$$

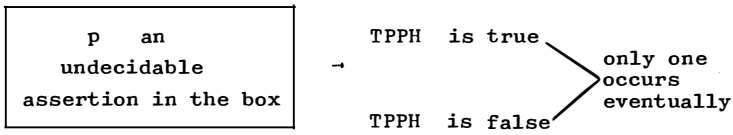
We can now state the TPPH, namely:

TPPH: There are infinitely many twin-prime-pairs!

The question arises as to whether TPPH is true or false! No one knows the answer! Yet although the answer to this problem is not known, the classical mathematician is convinced that there are only the two possibilities:

"TPPH is true" or "TPPH is false".

The above attitude is almost universally accepted and is quite independent of anything to do with boxes, but is an integral part of the classical attitude toward mathematics. But at this point, let us do some wishful thinking. Suppose we were able to build a box which would decide this problem for us (i.e. would prove or refute the TPPH) and was consistent!!! Even though this fantastic box would solve our problem about twin-prime-pairs (if we watched long enough), the Undecidability Theorem says that there would still be a meaningful assertion p which is in the box and is undecidable for the box.



Box which solves the TPPH

In other words, building fancier and fancier boxes which will solve successively deeper problems (even if we could do this) will never free us from the arduous labor of invoking our ingenuity, because the creation of such marvelous boxes simply create further assertions which, though intuitively meaningful, cannot be solved by the box. So even if we were to ignore the consistency

problem, there is still no hope of constructing a box from which all of contemporary mathematics will slowly but surely spew forth. In other words, as far as mathematics is concerned the machines are not going to take over. Whether or not the public likes it, mathematicians will remain indispensable. This assurance of our inherent usefulness to the further development of mathematics is most comforting, and we owe such a marvelous revelation to none other than Kurt Gödel.

So much for the philosophical point of view. What about the mathematics involved? Well obviously mathematicians no longer try and build Hilbert boxes! But in order to better understand the Undecidability Theorem there is little doubt but that we should get a firmer grip on the question: "What sort of logic was in the box?" Actually, when Gödel stated the above theorems he did refer to the logic in the box (you can be sure that Big Daddy Gödel knew what was in there). Allow me to paraphrase what he said as follows:

Gödel's First Amendment. I say there David, if you allow recursive functions in your box, then my two theorems follow.

You might expect that Hilbert would have simply replied: "Ach du Himmel Kurt, zen gekeepen your recursiven functionen out from mein box Ja!" Well Hilbert did not make the above remark, for remember that he wanted his box to be able to spew out number theory, and of the many indignities that Gödel heaped on him there was

Gödel's Second Amendment. Forsooth and furthermore: If the logic in your box will spew out number theory, then it will spew out recursive functions; and conversely,

if the logic will handle recursive functions, then it can produce number theory.

Alas! Hilbert had no choice, whether he liked it or not, he essentially had recursive functions in his box. Consequently we can gain insight into the mathematical nature of the logic in Hilbert's box, if we can gain some insight into the notion of a recursive function. The surprise is that recursive functions can be comprehended and appreciated with relative case, whereas if I were to describe the logic in Hilbert's box, I'd never get past the description, for it really was quite a sophisticated logic. In other words, in the time remaining, I feel that I have a fighting chance to tell you what a recursive function is, but not a hope in hell of revealing the nature of the logic in Hilbert's box.

So here is my little tale about recursive functions. Once upon a time, long before the advent of Gödel, the notion of

an effectively computable function

had come to the attention of many creative mathematicians. Effective computability is an intuitive notion which more or less means that you can really and truly compute the functional values for such a function; i.e. if the function is f and I give you a member x of its domain, then after a finite length of time you would return to me and say "Here it is Ferg, just what you wanted f(x)!" Well Gödel's notion of a recursive function was his version of what an effectively computable function should be, but, instead of leaving matters at the vague fuzzy level, he gave a precise definition. His definition follows.

First let N denote the set of non-negative integers, i.e.

$$N = \{0,1,2,3,4,\ldots\} ;$$

let N^2 denote all ordered pairs of non-negative integers, N^3 denotes all ordered triples of non-negative integers, and so on. Thus

$$4 \in N \ , \ (9,0) \in N^2, \ (1,3,10^{10}) \in N^3, \ (1,0,8,24,16,4) \in N^6,$$

and so forth.

First of all a recursive function f is restricted to be a function of the type

$$f: N^k \to N$$

i.e. the domain of f is N^k (the set of all k-tuples of non-negative integers) and the range of f is a subset of N (i.e. f is N-valued). Now the idea is this. One starts by stipulating that a large family of functions are recursive by fiat - this is sort of reminiscent of the business of adopting axioms for a logic. Next one accepts two constructive processes as being such that when they are applied to recursive functions they produce recursive functions. More specifically:

The Initial Functions

Each function of the following type is recursive:

1^o the successor function $\sigma: N \to N$ given by

$$\sigma(n) = n+1 \ .$$

2^o the identity function $\pi: N \to N$ given by

$$\pi(n) = n \ .$$

3^o all constant functions; thus if $f: N^5 \to N$ is defined by

$$f(i,j,k,l,m) = 31 \quad \text{for all} \quad i,j,k,l,m \in N$$

then f is a constant function.

4^o all the selector functions on N^2, N^3, N^4,... On N^2 there are two selector functions $\pi_1: N^2 \to N$ and $\pi_2: N^2 \to N$ as given by

$$\pi_1(m,n) = m \ , \ \pi_2(m,n) = n \ .$$

In general, on N^k there are k selectors.

So the above gives us an initial supply of ready-made recursive functions, and, in fact, functions which should cause you no great surprises. So you've really known recursive functions all along. Note that it is rather clear that the initial functions are effectively computable; for if they are not, then who is? Now we turn to the construction processes which will permit the formation of further recursive functions.

The Substitution Construction

We give but a special case. Suppose that

$$f: N^3 \to N \ ; \quad g: N^2 \to N \ , \ h: N^2 \to N \ , \ \phi: N^2 \to N$$

are all recursive, then the function

$$f[g,h,\phi]: N^2 \to N$$

which is defined by

$$(f[g,h,\phi])\ (m,n) = f(g(m,n),\ h(m,n),\ \phi(m,n))$$

is recursive.

The Recursion Construction.

We give but a special case, a recursion of order 2. Suppose $g: N \to N$ and $h: N^3 \to N$ are recursive, then there is a unique recursive function $f: N^2 \to N$ satisfying:

$$f(m,0) = g(m)$$

$$f(m,\sigma_n) = h(m,n, f(m,n))$$

Finally we can define what we mean by a recursive function in general. A function f of the type $f: N^k \to N$ is recursive if and only if it can be shown that f is obtainable from the initial functions by means of a finite number of applications of the constructions of substitution and recursion. So you see the idea is not nearly as impressive as it sounds. Also please note that it is intuitively clear that the constructions of substitution and recursion preserve the notion of effective computability; i.e. if f,g,h,ϕ of the substitution construction are effectively computable, then so is $f[g,h,\phi]$; and if g,h of the recursion construction are effectively computable, then so is f. Consequently every recursive function has the heuristic property of being effectively computable. There is not much time left to say much about recursive functions, but allow me the privilege of a tiny theorem to show you that it really is easy to prove things about recursive functions.

Theorem: the addition function $+: N^2 \to N$ is a recursive function.

Proof. Recall that the basic properties of addition on N are:

$$\begin{cases} m + 0 = m \ , \\ m + (n+1) = (m+n) + 1 \ , \end{cases}$$

where $m,n \in N$. Now let us put the above in more functional looking notation by writing $+(m,n)$ instead of $m+n$. We obtain

$$+(m,0) = m = \pi(m) ,$$
$$+(m,\sigma_n) = +(m,n) + 1 = \sigma(+(m,n)) .$$

Upon examining the above and comparing it with the enunciation of the recursion construction, we see that we've almost expressed $+$ in terms of recursive functions and the recursion construction. Pressing on in this direction, consider the function $f: N^3 \to N$ defined by

$$f(m,n,k) = k+1 = \sigma(k) .$$

We claim that f is recursive. Let π_3 be the third selector on N^3 ; i.e. $\pi_3: N^3 \to N$ and

$$\pi_3(m,n,k) = k .$$

Note:

$$f(m,n,k) = \sigma(k) = \sigma(\pi_3(m,n,k))$$

and so

$$f = \sigma[\pi_3] .$$

As σ, π_3 are initial (and therefore recursive) we conclude from the substitution construction that f is recursive. Furthermore,

$$f(m,n, +(m,n)) = +(m,n) + 1 = +(m, \sigma_n)$$

and so we have:

$$+(m,0) = \pi(m)$$

$$+(m, \sigma_n) = f(m,n, +(m,n)) \ .$$

We see then that $+$ can be obtained from the recursive functions π and f by a recursion of order 2 and so $+$ is itself recursive. This completes the proof.

There are almost tears in my eyes as I now admit that it is time to shut my mouth and discontinue spreading the gospel about recursive functions. But before doing so I maintain that there remain many more wondrous tales to be told about these recursive chaps. I'll fade with a question:

"Is every effectively computable function recursive?"

Thank you for your kind attention.

References

[1] David Hilbert and his Mathematical Work by Hermann Weyl, Bulletin of the American Mathematical Society, 50 (1944), 612-654.

[2] On Formally Undecidable Propositions of Principia Mathematica and Related Systems by Kurt Gödel (translation by Bernard Meltzer with an introduction by Professor Braithwaite), Oliver and Boyd, Edinburgh and London.

ONTARIO MATHEMATICS COMMISSION NEWS

Did you know

(1) that the current President is Dr. G.F.D. Duff, the author of the first article in this issue, and shortly to become Chairman, Department of Mathematics, University of Toronto?

Dr. Duff has rendered yeoman service to Mathematics Education in Ontario and Canada in numerous capacities - as author, Curriculum Committee member, Chairman of the K-13 Geometry Study Report Committee, etc., etc.

(2) that the Advisory Committee on Elementary Education met with Dr. Z.P. Dienes on Saturday, January 29th in Toronto?

Dr. Dienes is a mathematician turned psychologist and mathematics learning specialist. Since hitting the headlines 8 or 9 years ago with his work on structural materials and his book "Building Up Mathematics" (published by Hutchinsons, London, England), he has worked with Jerome Bruner at Harvard, spawned very successful Projects on Learning Mathematics in the Pacific, and is now in Canada at the University of Sherbrooke, P.Q. One of his books was reviewed in an earlier issue of the Gazette by A.W. Harris.

Dr. Dienes' intellectual power, vast energy, and ability to identify completely with children as they participate in mathematical experiences he has devised have to be seen to be believed.

(3) that an Advisory Committee on Colleges of Arts and Technology, chaired by Mr. A.I. Vertesi of Mohawk

Valley C.A.T. (formerly Hamilton College of Technology) has been working for nearly a year now.

(4) that there is now a liaison sub-committee bridging the gap between the Elementary and Secondary Committees. Its Chairman is Mr. David Skoyles, Mathematics Coordinator, of Windsor.

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ONTARIO ASSOCIATION OF TEACHERS OF MATHEMATICS

You can see the big news for yourself on the opposite page, and read about it below. To register now, get in touch with

Mr. W. Kierstead

Dear Colleague,

After much preparation, the Executive and Councillors of the O.A.T.M. are now ready to release the details of our Seventy-Seventh Annual Meeting.

The location will be the Inn-on-the-Park in Toronto, which has so successfully accomodated us for the past two years. The convention opens with registration on Monday, March 18th, at 7:00 p.m., and continues until Wednesday afternoon March 20th.


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There are many exciting things that could be said about the programme for our convention, but you need only read the attached outline to realize the value to be gained by listening to the speakers listed, and by joining in the group sessions. Included in this list are the names of many people, active in the field of mathematics in Ontario, who will speak on topics of interest to teachers of mathematics at any level - elementary, secondary and post-secondary.

Our main speaker is Dr. Irving Adler, formerly of Columbia University, and presently a professor at North Bennington College, Vermont. He has a wide background of teaching experience at the secondary and post-secondary level. He is well known as an author, having written some 40 books by himself and an additional 30 books in collaboration with his wife. We look forward to listening to a man who has written books on so many topics.

We urge you to plan now to attend our convention and to put up at the Inn. If enough rooms are rented our convention costs will be greatly reduced.

Reservation forms for rooms at the Inn are being mailed across the Province, but if you do not have a form don't hesitate to write directly to: The Inn-on-the-Park, 1100 Eglinton Ave. E., Don Mills (Toront) and please state that you are associated with O.A.T.M.

Registration forms for the convention are also being distributed and we would appreciate their use, but if there is a shortage in your school please use the tear-off portion at the bottom of this letter.

Register now, and plan to start off the convention programme by attending the Monday evening reception where you will have an opportunity to meet and talk with the programme speakers as well as all of your mathematics-teaching friends.

D.A. Fox,
Publicity Representative.

ONTARIO ASSOCIATION OF TEACHERS OF MATHEMATICS

77TH ANNUAL CONVENTION PROGRAM

MARCH 18, 19, 20, 1968 - INN ON THE PARK (Toronto)

Convention Office — Conference Room 3

M O N D A Y

7:00 p.m.	Registration - Main Lobby
8:00	Early Bird Reception - Park Ballroom
	Host - Dr. D.C. Russell, Honorary President, O.A.T.M.

T U E S D A Y

8:00 a.m.	Registration - Centennial Ballroom - Crush Lobby
9:00 - 9:15	Opening Session - Centennial Ballroom - Salons C and D
	Mr. L.D. Auckland, President, O.A.T.M.
9:15 - 10:15	General Session - Centennial Ballroom - Salons C and D
	What Geometry Should be Taught in Schools - Dr. Irving Adler, Author
10:45-11:45	Group Sessions
12:00-1:30	Lunch
1:45-2:45	General Sessions
	Centennial Ballroom - Salon A
	Demonstration Lesson - Primary Miss Janet Jorden, Allandale Heights, Public School, Barrie
	Wilket Creek Room
	A Secondary School Topic Mr. Douglas Potvin, Loyola College, Montreal.
	Trillium Room
	From Euler to Archimedes (Networks and Regular Solids), Dr. D.C. Russell, Department of Mathematics, York University
2:45-3:15	Exhibits - Park Ballroom Math.Fair - Park Ballroom Foyer

3:15-3:30	Canadian Association of Math. Teachers Father J.C. Egsgard, President
3:30-4:30	General Session - Centennial Ballroom - Salons C and D The Role of the C.A.A.T. - Mr. Jack Porter, President, Sheridan, C.A.A.T.
7:00 p.m.	Banquet - Centennial Ballroom - Salons A and B Chairman - Mr. L.D. Auckland Speaker - Miss Angela Armitt, University of Western Ontario

W E D N E S D A Y

9:00 - 9:15	Business meeting - Centennial Ballroom - Salons A and B
9:15 - 10:15	General Session A New Role for the Teacher - G.A. Kaye, Associate Supt., Ontario Dept. of Education.
10:15-10:45	Exhibits - Park Ballroom Math.Fair - Park Ballroom Foyer
10:45-11:45	Group Sessions
12:00-1:30	Luncheon (compliments of I.B.M.) Centennial Ballroom - Salons C and D Chairman - J.R. MacLean, Assistant Superintendent, Ontario Department of Education.
1:45-2:45	General Session The Group Concept in the Curriculum of the Future - Dr. Irving Adler Presentation of Awards (Math. Fair)
3:00	Tour

G R O U P S E S S I O N S

TUESDAY - 10:45 - 11:45	WEDNESDAY - 10:45 - 11:45

1. Centennial Ballroom - Salon A

   Measurement Without Formulae - R.E. Shadbolt, Ontario Department of Education, Ottawa

2. Centennial Ballroom - Salon B
Workshop Approach to Discovery - K.W. Kenney, Althouse College of Education, London.

3. Conference Room 2
Solids, Shapes, Numbers - J. Ryan, St. Patrick's College, Ottawa.

4. The Talbot Room
Mathematics in Colleges of Applied Arts and Technology - A.I. Vertesi, Mohawk College, Hamilton.

5. The Sherwood Room
Mathematics in Business and Commerce - Mrs. Mary St. Louis, Garson-Falconbridge Secondary School, Garson, Ontario.

6. The Wilket Creek Room
Seminars in Grade 13 - Mrs. Jean Leppard, Vaughan Rd. Collegiate Institute, Toronto.

7. Conference Room 1
Tests and Testing in the Intermediate Division - Dr. Dorothy Horne, Ontario Institute for Studies in Education.

8. The Edwards Room
Computer Science - Glen Bonham, Assistant Superintendent, Curriculum Division, Ontario Department of Education,

PROGRAM FOR WIVES OF DELEGATES

MONDAY

Evening	Reception

TUESDAY

Morning	Shopping Tour
Afternoon	Theatre Party (tentative)
Evening	Banquet

WEDNESDAY

Morning	Demonstration of Flower Arranging and/or Fashion Show (Inn-on-the-Park)
Afternoon	Tour

(For the information of the ladies, the Inn-on-the-Park employs excellent hair sylists.)

CANADIAN ASSOCIATION OF MATHEMATICS TEACHERS (C.A.M.T.)

by

D.A. Fox,
Publicity Representative, O.A.T.M.

On December 8, 9, 1967 the C.A.M.T. held an Invitational Meeting at the Chateau Laurier Hotel in Ottawa. The purpose of the meeting was to discuss Mathematics Curricula in Canadian Schools and the constitution, program etc. of C.A.M.T.

The 40 participants included delegates from each Canadian province who represented provincial teachers organizations and/or provincial curriculum departments. Ontario was well represented by Father J.C. Egsgard and Prof. A.W. Harris (Ontario Mathematics Commission); Messrs. L.D. Auckland and D.A. Fox (Ont.Assoc. of Teachers of Mathematics); and Messrs. G.A. Kaye, G.C. Bonham, J.R. MacLean and G.A.Scroggie (Ont. Dept. of Education).

The mathematics curricula across Canada were outlined by these provincial representatives. This provided further stimulus for a continuing exchange of ideas between provinces.

Before the meeting ended a constitution and program for the C.A.M.T. were decided upon. The delegates unanimously elected, as president, Father John C. Egsgard of St. Michael's College H.S. Toronto.

It should be noted that Father Egsgard will be talking briefly about the C.A.M.T. at the Annual Convention of the O.A.T.M. on March 18, 19, 20 at the Inn-on-the-Park, Toronto.

LETTERS TO THE EDITOR

Dear Sir:

Thank you very much for your note concerning my subscription to the Ontario Mathematics Magazine. It is clear to me why I appreciate its publication: as long as there are individuals (such as you) devoted to mathematics and the teaching of mathematics willing to take the time and patience to help put the magazine together, teachers in Ontario (such as I) will continue to benefit from the sharing of new ideas and fresh perspectives.

There are, of course, occasions when I am outraged by some of the contributions published, or distressed by the points of view presented. That, too, is to the good: for that is what such a publication should do.

The type of thing that outrages me is "The Mathematics K-6 Project 1966-67" (Oct. '67). I'm afraid I cannot grasp how the classroom teacher becomes "but the guide while the pupils (are) the real discoverers and doers"... learning ... doing ... discovering ... semantics: it all rather sounds like Maoist double-talk, frankly. I find distressing the emphasis placed on invention, in Mathematics; to me, it seems evolution would be nearer the mark... on p.21 (Oct. '67) the facsimile "testimonials" bemuse...on p. 16, the "Interview" depress... I don't know why.

Best wishes, to you, and to the Gazette.

Yours sincerely,

Roy A. Magda.
Appleby College,
Oakville, Ont.

Dear Sir:

With a class of 41 in Gr. 7 and 8, I have been unable to find time to write comments on the articles in Ontario Mathematics Gazette, until now.

When I read some of the articles and noted the materials suggested, I contrasted the situation in my school with that of e.g. Sudbury. Here we have a crowded room with six rows of fastened-down seats and no hope of anything else for some time to come. Yet my pupils are intensely interested in math. and most subjects. Math. presents a challenge to them and they work hard and usually successfully, with the problems presented. It is impossible to work in small groups but we do work as a class group in each grade.

Working in small groups may be very useful in certain situations but there is always the danger that one child will lead the group and do the thinking for the others, especially in a grade where there is a wide range of ability among the students. Believe me I have given this a good test and went back to the class group where each child does his own thinking and planning with individual help when needed.

Moving forward at each pupil's individual speed will not work in my situation either. It led to frustration for many children when they got behind and worried about it. However, longer and more difficult assignments for the quicker students, and shorter ones for the slow learners but all on the same subject matter is successful here.

Geometry has become so interesting that the pupils often spend spare time working out shapes etc. Here groups are successful and fun.

I see no waning of interest in my children and we have been using the new math. throughout the school for four years now. We have difficulties sometimes but somehow they are overcome and the children feel a sense

of victory when this happens.

The early number facts in addition etc. must not be neglected. They must be learned by constant meaningful use, otherwise frustration in later grades will be the result.....

Yours truly,

(mrs.) E. Verna McKagne,
Woodville Public School

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PROBLEMS AND SOLUTIONS

Send problems and solutions to:

The Problems Editor,
Ontario Mathematics Gazette,
c/o Department of Mathematics,
Queen's University,
Kingston, Ontario.

PROBLEM NO. 6 (M.P. Closs and E.L. Cohen, University of Ottawa)

(Editor's note: We received the following poem from the authors. The problem is - what does it mean?)

SMOTHS

Three hundred smoths in a languid proon
Were slamping in pairs of three
When a mathematician with a scoon
Insisted on being a referee.

The smoths consent came with ease
But the proon was not so quick
And the mathematician with a squeeze
Shalted the proon'till it was sick.

Now the proon jibbled at a sector
And leemed at the smoths wed-a-wed
Which hit the scoon along a vector
From which cause it now was dead.

This sad fate need not have been
Had only the scoon known the fact
That slamping in threes must not be seen
As a self-contradictory act.

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BOOK REVIEW

Geometry: Kindergarten to Grade Thirteen.* Report of the K-13 Geometry Committee of the Development Division. Curriculum Series, No. 2, Toronto: The Ontario Institute for Studies in Education, 1967. Pp. viii, 119.

If this report had nothing else to commend it, the quality of its writing would justify at least a first reading. In several passages in particular, one finds a remarkably lucid treatment of essentially technical material, laid on with an adroit appeal to analogy and even, occasionally, to subtle humor. In addition, the authors have mercifully spared us that super-abundance of tiresome platitudes which makes deadly reading of so many curriculum reports.

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* Reproduced by permission from the Ontario Journal of Educational Research, 10, 1 (Autumn, 1967), 67-70.

Geometry: Kindergarten to Grade Thirteen has, however, far more than mere literary merit. A glance at the roster of committee members and consultants and a rapid overview of their work suggests that questions of role, approach, and required emphasis in respect to geometry in Ontario schools were studied here more painstakingly than ever before. It is a report, therefore, to which serious attention must be paid.

The reader who is also a proponent of educational research will find himself in agreement with many of the general comments and conclusions. For example, the committee calls for a more sophisticated use of diagnostic tests by teachers, tests which would attempt to measure performance at levels higher than that of rote memorization. And while no specific mention was made of the possible use of a taxonomy for this purpose, such an approach was at least hinted at, and the principle approved, in various sections.

Equally congenial are the sober comments on the likely success of offering instruction to a group of students with a wide range of mathematical aptitudes and achievements; here and elsewhere in the report, the committee shows great concern for the time-honored problem of individual differences. One might single out also the stated desire for testing, revision, and evaluation prior to large-scale involvement of teachers and students, and the recommendation for close liaison with OISE and other bodies "so that appropriate basic research, field testing, evaluation and implementation may be adequately coordinated" (p. 108; italics in original).

Passing over many other worthwile observations, we come to what, in this reviewer's mind, is the most significant aspect of the report. In brief, the committee has attempted to base its general plan for the sequencing of curriculum experiences directly upon psychological theory. Consequently

it recommends the child's geometric experience consist of three general phases which correspond generally to Piaget's stages of intuitive thinking, concrete operations, and formal operations and which embrace Grades K to 6, 7 to 10, and 11 to 13 respectively.

A dilemma arises at this point. On the one hand, educational psychologists have berated curriculum builders for several years for ignoring Piaget and so, to be consistent, we should applaud attempts to utilize his theory. On the other hand, this reviewer feels obliged to quarrel with the report for precisely this reason. The somewhat embarrassing fact is that many educational psychologists have changed their minds -- primarily in light of the research results of the past five years -- and no longer believe that Piaget offers a kind of time clock for the pacing of curriculum content. Admittedly this is still a controverisal matter and those who would implement the committee's proposals should read Sullivan's[1] recent exhaustive analysis of the evidence pertaining to this problem.

In this writer's opinion, however, sufficient evidence has been brought forward to require the conclusion that the appearance of Piaget's psychological "benchmarks" (concrete thinking, formal thinking) can be hastened to a much greater degree than is assumed by Piaget-oriented curriculum builders. It may be, for example, that children (as a result of their normal interaction with the environment) typically must attain an age of fifteen years or thereabouts before they can cope with problems whose solution Piaget sees as marking entrance into the stage of

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1 Edmund V. Sullivan, Piaget and the School Curriculum: A Critical Appraisal, Bulletin No.2 (Toronto: The Ontario Institute for Studies in Education, 1967, in press).

formal operations; however, this writer is convinced, not without evidence, that there is nothing in these tasks which is beyond the capabilities of considerably younger children (seven to ten years), provided that we are sufficiently ingenious in analyzing the skills required and in providing training to ensure that these skills are mastered.

What then can be said conclusively about a possible limit in accelerating the appearance of Piaget's stages? Very little, in fact, on the basis of empirical evidence, except for the truism that the amount of acceleration is likely to be proportional roughly to the additional effort (and cost) involved in attempting to achieve it. Consequently it would seem that the Cambridge Mathematics Report[2] was not demonstrably in error when it said, in respect to its own pacing proposals, that "the investigations of Piaget...do not justify any conclusion relevant to our task."

In view of the foregoing, Geometry: Kindergarten to Grade Thirteen takes, in this writer's opinion, too pessimistic a view as to what children could accomplish, given a sufficiently resourceful pedagogy. For example, one must conclude from the general comments and details of the curricula proposed that the committee believe that young children cannot understand the notion of logical necessity (that is, that given certain premises, certain conclusions must follow); that even in the secondary stage (Grades 7 to 10) any approach to proof in euclidean geometry should assume only a capacity for "informal" or "intuitive" reasoning; and that the serious study of the axiomatic bases of the various geometries must await the tertiary stage.

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2 Goals for School Mathematics, Cambridge Conference on School Mathematics (Boston: Houghton Mifflin, 1963).

Discussion of these contentions is inhibited by the fact that the report does not adequately define or postulate any psychological mechanism for some of the key concepts involved. However, in this writer's view, the understanding of logical necessity occurs when a child possesses a set of noncontradictory implicit responses which allow him to reproduce, internally, the sequence of actions involved in moving from the premises to the conclusion. For example, a child understands that from the premises $A > B$, $B > D$, must follow the conclusion $A > D$, because he is able to perform an implicit set of ordering responses which arrange unequivocally the three elements in a particular way. This same child, between the ages of four to seven, say, also knows that a different set of premises (for example, $D > A$, $A > B$) leads to a different conclusion. The understanding that one set of assumptions leads to euclidean geometry, another to a noneuclidean geometry, would seem to involve the same psychological processes as these elementary problems, differing only in the complexity of the set of initial premises which must be mentally represented. Moreover, these processes, accompanied by an overriding teachable "strategy" (essentially a set of rules for generating sequences of implicit responses), enables students of ages seven to ten to solve the "original" problems of euclidean geometry.

This is not an argument that young children should be engaged in the study of the precisely formulated axiomatic systems of geometry, but rather that a distinction be made between what they cannot do, and what for other reasons they should not do. Surely we should reject any attempt to invoke some kind of mysterious "formalism", advanced by vague reference to Piaget, unless such processes are substantiated by a clear behavioral formulation of their attributes, a formulation which makes clear how they are distinguished from

simpler processes.

Two other Piaget orientations cause some misgivings. First, although the term "discovery learning" is carefully avoided, it comes through that the committee favors an approach where the child, working with concrete material, is encouraged to arrive at his own generalizations. This view is so moderately stated here that perhaps no cautionary note need be sounded; however, in the event that any reader of the report should conclude that this matter is settled (and there appears to be a growing uncritical acceptance of the discovery method in Ontario), we should mention again Sullivan's contention that Piaget's theory per se does not favor any particular pedagogy, and Ausubel's[3] critical review of the empirical evidence on the efficacy of that method as compared to reception learning (that is, a method in which the teacher enunciates generalizations and provides examples). Ausubel's review does not unequivocally favor discovery learning.

Finally, the tendency to limit vocabulary severely in the early years and generally to downgrade the importance of language training seems related to the Piaget notion that the mental structures grow through a process in which language is a correlated, but not causal, variable. It need be only mentioned here that this is a contentious issue, and that some argue that early language training could not only facilitate the processes of classification and concept development but also could accelerate the appearance of the "bench-marks" themselves. The controversy is currently "hot" and one should at least acknowledge both its sides, justifying any position taken by an attempted refutation of the opposite point of view.

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3 D.P. Ausubel, The Psychology of Meaningful Verbal Learning (New York: Grune and Stratton, 1963), chap. 7.

The present writer has one recommendation to add to the list proposed by the committee; it is that the experimental tryouts of the proposed geometry sequences be paralleled with a set of small scale projects in which "task analysis" is used to see what speed and level of mathematics learning children can actually attain, given no a priori Piaget prescriptions concerning appropriate staging. Initially at least, such experiments would involve small groups of children and, therefore, a relatively high per pupil cost, and it will be claimed that such evidence offers no guide as to what the school -- employing mass instructional methods -- can be expected to accomplish. This claim is true, but the school should have always before it a concept of what is possible, given resources and pedagogical ingenuity. It should never be allowed the comfortable self-deception that the present ceiling in learning is the result of some kind of natural law.

In conclusion, Geometry: Kindergarten to Grade Thirteen is a document well worth reading for mathematics educators and those interested in the problems of learning. Moreover, if the controversies alluded to in this review are followed up by both subject matter specialists and educational psychologists, this report could provide a meeting ground for a more profitable kind of interaction between these groups than has existed in the past.

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