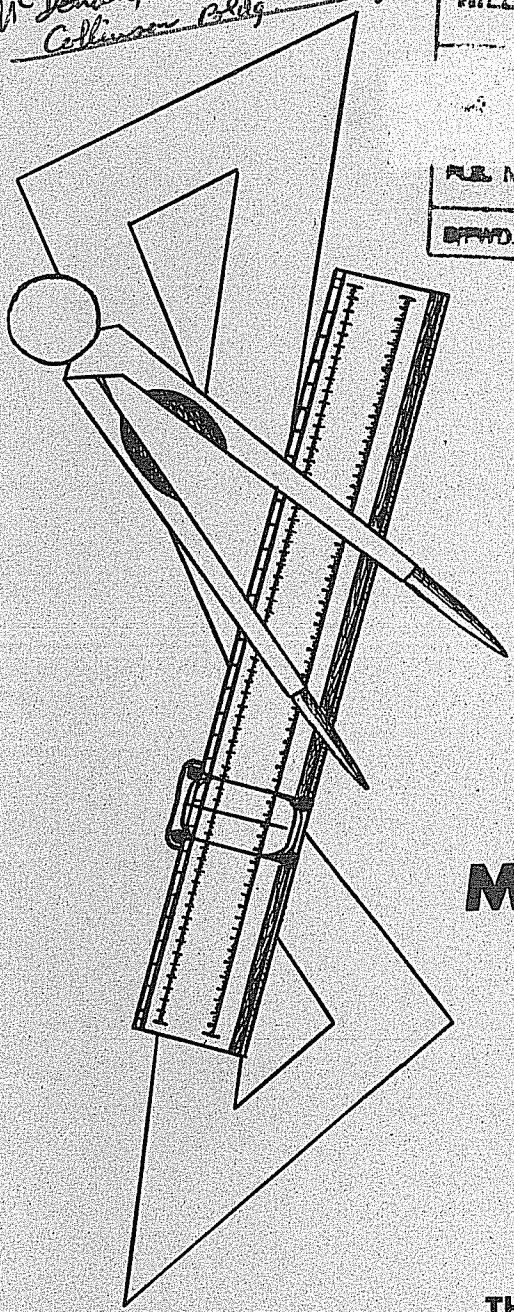
McDonald Room Library
Collinson Bldg.

HILLFIELD-STRATHALLAN COLLEGES

PUB. NOTE

BIND.



VOLUME 6
NUMBER 3
MAY 1968

ONTARIO MATHEMATICS GAZETTE

sponsored by
THE SAMUEL BEATTY FUND
THE ONTARIO MATHEMATICS COMMISSION
THE ONTARIO ASSOCIATION OF TEACHERS OF MATHEMATICS

ONTARIO MATHEMATICS GAZETTE

EDITORIAL BOARD

Editor-in-Chief	Dr. D.H. Crawford, Queen's University.
Research Editors	Mr. A.W. Harris, Althouse College of Education, University of Western Ontario.
	and
	Dr. N.M. Rice, Queen's University.
Business Manager	Mr. J.W. Fencott, R.H. King Collegiate Institute, Scarborough.

Additional copies of the Gazette may be obtained from:

The Secretary,
Ontario Mathematics Commission,
1260 Bay Street,
TORONTO, Ontario.

TABLE OF CONTENTS

Editorial	i
A Teacher Workshop Relating to the Four-Year Programme, by J.N.C. Sharp	1
THE number line, by T. Sampson	12
Varied Approaches to the Solution of a Problem in Geometric Construction, by F.C. Asbury	14
Personal Impressions of the 1968 OATM Convention, by D.H. Crawford	22
Grade 13 Mathematics Essays, by W. Bisset	25
The Empty Column	32
The Development of Modern Algebra, by Douglas Iles	36
Acknowledgement	45

EDITORIAL

Once more, spring is here, but with a difference. There are no Departmentally administered Grade 13 Examinations! As one sample of the effects of this emancipation, we print in this issue an article on Mathematical Essays which originated with its author's belief that "a major part of mathematical education should be through the device of researching a topic".

With this point of view your editor is in general agreement, and it is interesting to note that the workshop approach currently in vogue lower down the school (and illustrated by another article) is really based on the very same belief.

Somewhat the same line is to be found in the addresses of both Dr. Elliott and Dr. Davis at the recent OATM Convention, while the implications of this pupil-centred approach for the teacher formed the theme of Garth Kaye's speech there.

This then seems to be the main lesson for us to learn as teachers - how to structure the environment so as to maximally involve the student in active learning. It may sound easy, but to those of us who are trying, it is much more challenging and demanding than a teacher-controlled and centred situation. In fact, it is a life-long task, and we must move from where we are now, each trying to adopt his methods and approaches more in this direction,

while not forgetting that there still is a place for class teaching, and for exposition. Each of us is unique, and what may be good teaching for you may not be for me, and vice versa.

Let us be humble, and be willing to learn from each other - and above all, let us never stagnate!

A Teacher Workshop Relating to the Four-Year Programme

by J.N.C. Sharp

(Editor's note: A one-day workshop, organized by Mr. J.N.C. Sharp, Co-ordinator of Mathematics for the Board of Education of Etobicoke, was held in February 1968. The following is an account of the workshop; for further information, contact Mr. Sharp, c/o Instructional Services Centre

Fourteen Activities in all were involved, in which teachers acted as students, carrying out the projects in various groups. Activities 4, 10, 11 are described in some detail; the others are summarized briefly.)

Details of Selected Activities

Activity 4 Graphing

Project 1. Materials: Thermometer, electric kettle, container and water, watch with sweep second hand.

Method:
1. Boil water
2. Pour it into the container
3. Observe and record the temperature of the water as it colls, using one minute intervals.

Problems:
1. Draw a graph illustrating the relationship between temperature and time elapsed.
2. Should the graph be a continuous curve? Explain.

Project 2. Materials: 20 sheets of 1/2" square paper.

Method: Make an addition square of the numbers 1 to 15 inclusive. Mark all points that have a sum of 7.

Problems:

1. Do you recognize the pattern formed? If the numbers in the bottom row are on the x-axis, and those in the left column on the y-axis, what relationship exists between the x and y values of the points you have marked?

2. Repeat (1) for points whose sum is 15.

3. Are the above graphs continuous? Why?

Project 3. Materials: 20 sheets of 1/2" square paper.

Method:

(1) Make a multiplication square of numbers 1 to 25

(2) Mark all points that have a product of 24.

Problems:

(1) Do you recognize the pattern formed?

(2) If the numbers in the bottom row are on the x-axis, and those in the left column on the y-axis, what relationship exists between the x and y values of the points you have marked?

(3) Repeat for the points that have a product of 36. Do you obtain the same pattern? What is the relationship this time?

(4) Are the above graphs continuous? Why?

Project 4. Materials: Pencil and paper.

Method: Make a table showing the natural numbers from 1 to 10 inclusive with their respective squares.

Problems:

1. Draw the graph illustrating the relationship between a number and its square.
2. Should the graph be a continuous curve? Explain.
3. If a general point on the curve is (a,b) express a in terms of b and b in terms of a.
4. Read the value of 3.5^2 from the graph. Check by multiplication.
5. Read the value of $\sqrt{30}$ from the graph. Check by multiplication.

Project 5. Materials: Stop watch, marble, and four to six foot piece of drapery track.

Method: Raising one end of the drapery track by means of books, form an incline.

Problems:

1. Draw the graph showing the time required for the marble to go down the incline for various heights - 3", 6", 9", 12", 15".
2. Should the graph be a continuous curve?
3. Read from the graph the time for a height of 10".

Project 6. Materials: Ruler, pencil, paper.

Method: Draw figures with 4 sides, 5 sides, 6 sides, 7 sides, 8 sides, 9 sides and for each draw its diagonals.

Problems:

1. Draw the graph of the number of diagonals for each of the figures.

2. Should the graph be a continuous curve? Explain.

3. From the graph determine the number of diagonals for a figure with 11 dies.

Project 7. Materials: 64 cubes (more if possible), graph paper.

Method: Use small cubes to make successively larger cubes.

Problems:

1. How many variables are there in completing a new cube?

2. Draw a graph to show relationship between size and number of cubes required?

3. Is this a continuous graph? Why?

4. Compare surface area to volume and draw a graph. Is this continuous?

5. Compare surface area to weight (assume blocks weigh one ounce each) and graph.

6. Can you use the above to explain why a mouse can fall 100 feet and not be hurt while a human cannot.

Activity 10 Square Root

Project 1

The algorithm most commonly used involves dividing the number into pairs of digits, and doubling the trial divisor.

I	II	III	IV	V	VI
N	N^2	N	N^2	N	N^2
1	1	10	100	100	10,000
2	4	11	121	101	10,201
3	9	12	144	102	10,404
4	16	.	.	.	.
5	25	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
9	81	99	9,801	999	998,001

(a) How many digits are in the numbers in columns I, III, V ?

(b) How many digits are in the numbers in columns II, IV, VI ?

(c) If the N^2 column has a number with 7 digits, how many digits are in its square root?

(d) What is the relationship between the number of digits in a whole number and the number of digits in its square?

(e) What does this suggest relative to grouping the digits in N^2 to find the square root N ?

Find the side of a square whose area is 625 square units.

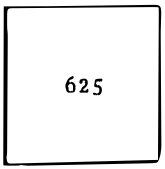


Fig. I

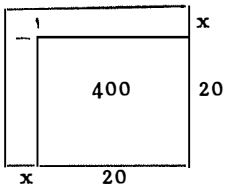


Fig. II

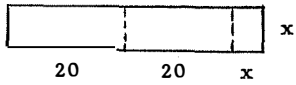


Fig. III

Method: Estimate the side of the square to be 20. Construct a square with side of 20 units as in Fig. II, with the remaining segment of lengths x units. Rearrange the outside strip of Fig. II to form the rectangle of Fig. III.

Problem: (a) What is the approximate length of the rectangle?

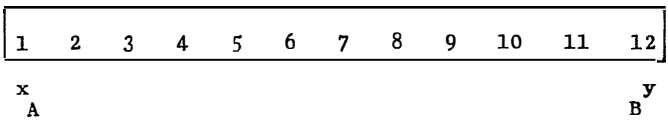
(b) What is the area of the rectangle?

(c) What is the approximate width of the rectangle?

(d) Using this value for x , calculate the side of the square. Check your result.

(e) Use the same construction to find the side of a square whose area is 1,089 square units.

Project 2. To find the square root of 12.



Method: A and B are two counters.
Move A from 1 to 12 .
Move B from 12 to 1 .
For each position 2 that A takes, B must be at the number y such that $xy = 12$.

Problem: (a) When A is at 2, 3, 4, where is B ?

(b) When A is at 8, 9, 10, where is B ?

(c) At what point is A and B together?

(d) What is the approximate square root of 12 ?

(e) Check your answer using a desk calculator.

Activity 11 The Pendulum

Project 1. Materials: Stop watch, string, weight.

Method: Form a pendulum with a string about 20 inches long. A swing consists of a trip back and forth. As accurately as possible, count the number of swings the pendulum makes for the intervals of time listed in the table.

Seconds	Swings
15	
30	
45	
60	
75	
90	
105	
120	
135	

Graph the ordered pairs of the form (number of seconds, number of swings). Is it meaningful to join the points? What geometric figure is formed?

Project 2. Materials: Stop watch, string, weight, ruler.

Method I: Form a pendulum with strings of various lengths and count the number of swings as accurately as possible to fill in the following table:

Length of Pendulum	Number of Seconds	Number of Swings
30 cm.	30	
40 cm.	30	
50 cm.	30	
60 cm.	30	
70 cm.	30	
80 cm.	30	
90 cm.	30	
100 cm.	30	

Method II: Graph the ordered pairs of the form (length of pendulum in cm., number of swings).

Connect each point with the origin. Label each line segment with the appropriate length of the pendulum in centimeters.

Can you determine a method of drawing a line segment to represent one swing per second? If so, draw it in another color.

How long should a pendulum be to be used for slicing off one second intervals?

Method III: Construct a table using the data above to record the time required for one swing and the square root of the length of the pendulum.

Graph the ordered pairs (time for one swing, square root of length of pendulum).

Describe the result of joining the points you have just graphed. What is your conclusion?

A straight line suggests a linear equation.

What equation is suggested by your graph?

Summary of Other Activities

Activity 1. Participants viewed a single concept film loop (from the series produced by the National Film Board, with consultation by Prof. D.H. Crawford of Queen's University). Assignment sheets contained questions pertaining to the topic of the film.

Activity 2. Multiplication of Integers. The model for this topic was a picture of an integer line and a cardboard or plastic arrow to serve as a marker. The rule of operation was that the unit directed segment may be slid or moved only in the I-line, and only in the direction of the arrow. To distinguish between positive and negative numbers, it was necessary to give the marker a rotation of 180 degrees.

Activity 3. Mensuration. Participants determined volumes of cylinders and cones by filling appropriate containers with grain. They determined the surface areas of cylinders and cones by opening the surfaces.

Activity 5. Inverse Proportion. In one project the participants were given pairs of connected gears, and they compared the ratio of teeth on the gears to the turning ratio of the gears. In another project they compared ratios of diameters of two pulleys connected by a belt to the turning ratio.

Activity 6. Geometry. The Theorem of Pythagoras was developed by rearranging pre-cut squares. Another project involved the Pythagorean relation developed from the angle in a semi-circle so that the two concepts were actually discovered.

Activity 7. The participants viewed film strips obtained from F.O.M. Film Strip of Canada, and then worked through question sheets.

Activity 8. The participants used stop watch, ruler, and string to determine the speed of a small battery-powered engine on a circular track. The speed was measured with the engine pulling various weights.

Activity 9. Multiplication of Integers. There were several projects in this activity, one concerned with constructing multiplication tables, and another having a toy car running back and forth along the number line.

Activity 12. Trigonometry. The participants "discovered" the Pythagorean theorem by fitting pieces together, constructed partial trigonometric tables, and graphed some trig functions.

Activity 13. Ratio. There were several projects, such as: measuring the height of successive bounces of a ball; measuring volume, surface area, and dimensions of various solids; comparing the number of beans in various jars with the weights of the jars; measuring certain lengths in inches and centimeters.

Activity 14. Division of Rational Numbers. The participants, working in small groups, worked through a series of discovery exercises.

THE number line*

by T. Sampson, Head, Dept. of Mathematics,
Thomas A. Blakelock High School, Oakville.

Why do the authors of text-books insist on labelling number lines with N, I, R, etc.? All number lines are the same.

Suppose a student had just finished plotting 2, 3 and 4 correctly on an N-line using the Domain N for the question. If you pointed out to him that the question said to use Domain Q, would you then insist that he now draw a Q-line or would you permit him to mark on the N-line the $2\frac{1}{2}$ and $3\frac{1}{2}$ which he had previously omitted (changing the label, of course, from N to Q). Other than the label what IS the difference between an N-line and an I-line or a Q-line? If you tell me that the I-line extends indefinitely in both directions whereas the N-line extends only to the right, then, in the name of all that is precise, why do you not call it the N-ray instead of the N-line?

When it comes to plotting in two dimensions, I have only seen one text which correctly illustrated N X N, using dots instead of lines. And it had two errors.

* Submitted in response to the editor's plea for teachers to write in about their pet peeves.

It referred to the N X N plane (a plane with points one unit apart?) and it included the point (0,0) in N X N as origin. (Should you label the first point on the x-axis with an 0 for origin and also with 1 for the scale?) .

Why not accept the reality that graphs differ according to whether the Domain is N, W, Q, I or R but that they are all plotted on the real number line or the real number plane. Stop that absurd practice of labelling N-line, I-line, N X N, R X R, etc.

Varied Approaches to the Solution of a Problem in Geometric Construction

by

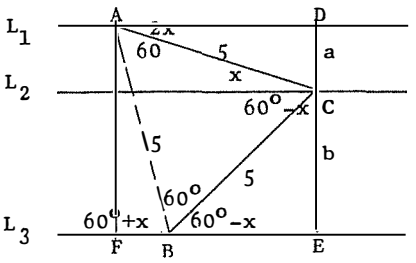
F.C. Asbury, 11 Kingsgarden Rd., Toronto

The following notes are offered in the belief that a variety of methods in the solution of the same problem often provides a rewarding mathematical experience.

The problem itself is not particularly difficult.

"To construct an equilateral triangle with vertices one in each of three given parallel lines."

I. A trigonometric analysis:



Given lines L_1 L_2 L_3 as shown. Suppose ABC the required triangle

$$\sin CBE = \frac{b}{5} = \sin(60^o-x)$$

$$\sin ABE = \frac{a+b}{5} = \sin(60^o+x)$$

Then $\frac{\sin(60-x)}{\sin(60+x)} = \frac{b}{a+b}$ i.e. $\frac{\sin 60 \cos x - \cos 60 \sin x}{\sin 60 \cos x + \cos 60 \sin x} = \frac{b}{a+b}$

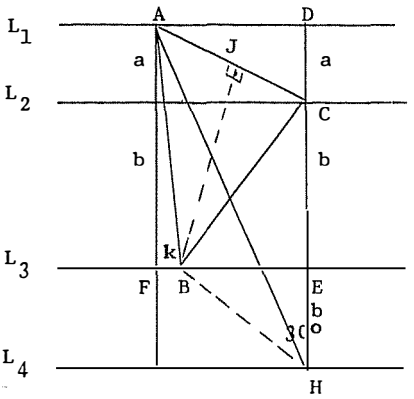
$\therefore$ by subtraction and addition of terms of the ratios

$$\frac{2 \cos 60 \sin x}{2 \sin 60 \cos x} = \frac{a}{a+2b} \quad \text{i.e.} \quad \tan x = \frac{a}{\frac{a+2b}{\sqrt{3}}}$$

Hence, in Δ ADC AD is evidently $\frac{a+2b}{\sqrt{3}}$.

This suggests the use of a 30^o, 60^o, 90^o, triangle with side opposite 60^o equal to $\underline{a+2b}$.

II. <u>A geometric construction based on I, together with an algebraic proof:</u>



Draw parallel L_4 distant $\underline{b}$ from L_3 and remote from L_2 . Construct $< DHA = 30^o$ as shown. Join AC and construct rt bisector JB to meet L_3 in B . AB = CB. Then ABC is required Δ . Let distance FB = k and use Δ's BCE, AFB .

From Δ HDA, since HD = a+2b $\therefore$ AD = $\frac{a+2b}{\sqrt{3}}$.

Since AB = CB $\quad \therefore (a+b)^2+k^2 = b^2 + (\frac{a+2b}{\sqrt{3}} - K)^2$

whence $k = \frac{b-a}{\sqrt{3}}$

Then $BC^2 = b^2 + (\frac{a+2b}{\sqrt{3}} - \frac{b-a}{\sqrt{3}})^2 = \frac{4}{3}(a^2+ab+b^2)$

$\therefore$ AB = BC = AC and Δ is equilateral.

III. A Euclidean proof suggested by formula $\frac{a}{\frac{a+2b}{\sqrt{3}}}$ as obtained in I above.

With above diagram and construction.

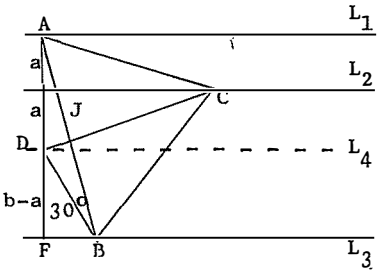
Since B is on rt bisector of AC $\therefore$ BA = BC

Also since B is on rt bisector of CH $\therefore$ BC = BH

Hence circle with centre B and radius BA, passes through A,C,H

$\therefore \quad \angle ABC = 2 \angle AHC = 60^o$, etc.

IV. A second Euclidean proof suggested by formula $k = \frac{b-a}{\sqrt{3}}$ in II above.



Draw L_4 distant a from L_2 and on opp. side from L_1, meeting $\perp$ AF at D

At D construct $\angle FDB = 30^\circ$

Join AB and draw its right bisector JC to meet L_2 in C.

Then ABC is the required equilateral Δ.

Join CD.

Since C is on right bisector of AD, AC = DC.

Since C is on right bisector of AB, AC = BC.

Hence a circle with centre C passes through A,D,B and radius CA.

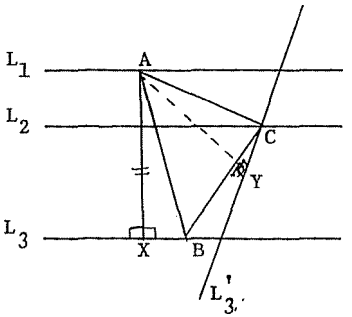
Let P be any point on this circle in major arc AB.

Then $\angle FDB$ = ext $\angle$ of cyclic quad APBD = $\angle APB$

$$= \frac{1}{2} \angle ACB$$

But $\angle FDB = 30^\circ$ $\therefore$ $\angle ACB = 60^\circ$, etc.

V. A proof by a Rotation method. (Courtesy of Prof. Ross Housberger.)

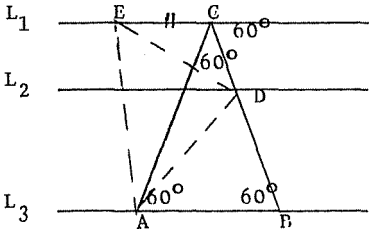


with any point A in L_1 as centre rotate the figure through 60^o . In this transformation we seek the image of L_3 (make $\angle XAY = 60^o$
$AY = AX$)

The image of vertex B is clearly the vertex C . Yet it must remain in L_3' the image of L_3 .

Hence C , the image of B , is the intersection of L_3' and L_2 . Whence the construction of required Δ ABC .

VI. A short and elegant Euclidean proof. (Courtesy of Mr. David Smith.)



Take any point A in L_3 . Construct 60^o angles at A and C to form the equilateral triangle ACB .

Let CB cut L_2 at D . Then AD is one side of the required triangle.

Proof: Make CE = BD

Then in Δ's ECA, DBA $\begin{cases} EC = DB \\ CA = BA \\ \angle ECA = 60^o = \angle DBA \end{cases}$

Hence EA = DA and $\angle$ EAC = $\angle$ DAB

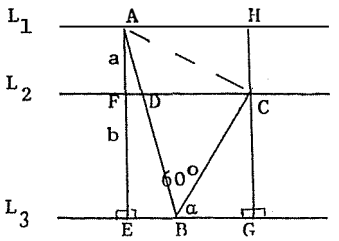
Also $\angle EAD = \angle EAC + \angle CAD$

$= \angle DAB + \angle CAD = \angle CAB = 60^o$

Hence Δ EAD is equilateral.

VII. A proof through an indirect construction approach.

Let the given parallels be L_1 L_2 L_3 as before. Suppose ABC the required equilateral triangle.

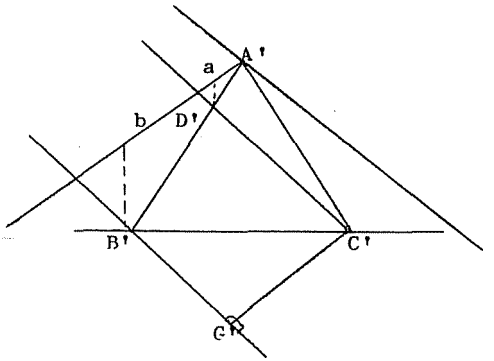


Draw $\perp$'s AFE, HCG as shown.

Call $\angle CBG = \alpha$

Then $\frac{AD}{DB} = \frac{AF}{FE} = \frac{a}{b}$ a known ratio.

Construct any equilateral Δ A'B'C'



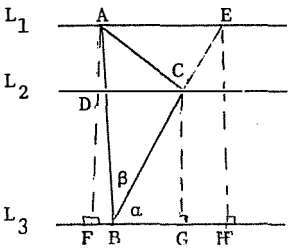
Divide A'B' internally at D' in the ratio $\frac{a}{b}$.

Join D'C' and draw $\parallel$'s through A' and B' to get $\angle C'B'G' = \alpha$

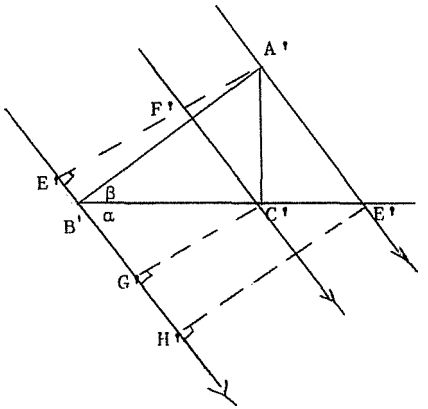
Whence the following construction for a synthetic proof. Take any point B in line L_3 and make $\angle GBC = \alpha$ to obtain point C in L_2. Make $\angle BCA = 60^0$ to obtain A in line L_1. Proof by $\parallel$ | Δ's.

VIII. This same method may be used to advantage in solving the following more general problem.

Find points, one in each of three given parallel lines which shall be the vertices of a triangle similar to a given triangle.



Let A'B'C' be the given Δ and L_1 L_2 L_3, with spacing a,b, the given $\parallel$ lines as before.



Divide A'B' internally at D' in the ratio a:b. Join D'C' and draw $\parallel$'s A'E' and B'H' as shown. Draw $\perp$'s A'F'E', C'G', E'H' at B in the original figure construct

$\angle HBC = \angle H'B'C'$ (α) to obtain point C in L_2.

Also construct $\angle CBA = \angle C'B'A'$ (β) to obtain point A in L_1. Then ABC is the required triangle.

Proof: $$\frac{BC}{B'C'} = \frac{CG}{C'G'} = \frac{b}{b'}$$

$$\frac{BA}{B'A'} = \frac{AF}{A'F'} = \frac{a+b}{a'+b'}$$

But since $\frac{a}{b} = \frac{a'}{b'}$

$\therefore \frac{b}{a+b} = \frac{b'}{a'+b'}$

and $\frac{b}{b'} = \frac{a+b}{a'+b'}$

$\therefore \frac{BC}{B'C'} = \frac{BA'}{B'A'}$

Also

$\angle CBA = \angle C'B'A'$ (β)

Hence $\Delta CBA \| | \Delta C'B'A'$

PERSONAL IMPRESSIONS OF THE 1968 OATM CONVENTION

by D.H. Crawford, Queen's University

(Editor's Note: In default of finding a suitable reporter, the editor has written this article himself. He apologises for gross omissions of other significant speeches and/or events.)

This year's convention appeared every bit as much of a success as previous ones, even although the keynote speaker, Dr. Irving Adler, was prevented from attending because of the illness of his wife.

Events began with the Early Bird Reception on the Monday evening, and closed with visits to IBM on Wednesday afternoon. The plenary sessions were well attended and rewarding; the numerous seminars catered well for different interests and needs, and the ladies programme proved very popular (though a suggestion was heard that it would be well to know in advance what theatre attractions, concerts, etc. were available). The school and book exhibits also attracted considerable attention, and favourable comment. Finally, mention should be made of the exceptional hospitality of IBM, both in sponsoring an excellent luncheon on the Wednesday, and in again arranging visits to see computers in operation.

Dr. H.A. (Andy) Elliott deputised for Dr. Adler at the first session. He used written material provided by

Dr. Adler as a springboard for an outline of what he himself conceived to be a modern school geometry programme - one based essentially on the ideas of translation, rotation, and reflection. With the help of excellent diagrams displayed via an overhead projector he illustrated how these various transformations were involved in tiling the plane. Students in the elementary school could readily experiment with triangular and other polygonal cut outs to find what kind of movements and shapes enable various translations to occur.

At more advanced levels, he showed how these ideas could be integrated with the concepts of group, vector, and matrices, and the overall effect was that geometry treated as the study of transformations should prove to be a most exciting and fruitful study.

It would be invidious to single out for particular mention any of the three alternative sessions provided on the Tuesday afternoon, and so this brief glimpse of the convention closes with reference to other events - the banquet, the concluding address by Dr. Robert Davis of the Syracuse University-Webster College Madison Project on the Wednesday afternoon.

The banquet meal was excellent, and the after dinner "performance" of Miss Angela Armitt proved quite literally to be a howling success. Her unusual figure, and devastating Irish humour, together with her deployment

of male resources in the form of Rev. John Egsgard nearly brought the roof down! (as well as revealing some unusual sights!)

At the final plenary session, Dr. Davis, deputising at very short notice for Dr. Adler, talked informally about behavioural objectives, a systems approach and what he called negative sophistication. He first spoke of the work of Eleanor Duckworth who in evaluating the African Science Study developed objectives not in terms of concepts and their understanding but via four credo statements, i.e., statements about different levels of beliefs. Defining objectives in this way enables the teacher to explore operationally how well a child really understands an idea or relationship in practice. If we are not careful, ritual may be accepted for reality. He went on to note that so far we had turned computers into poor teachers, but quoted from Education '78, an article in the Globe and Mail which predicted how students will then go to school "to pursue their own interests and to follow up their own ideas in a free environment" using almost unlimited informational resources, individual advice, and helpful dialogue. Finally, Dr. Davis identified three stages of development - reality, mental internalisation, and reflectivity (analogous to those of Piaget) and argued against proceeding to unnecessary levels of sophistication, e.g., using sets in the Primary Grades.

GRADE 13 MATHEMATICS ESSAYS

by

W. Bisset, Head, Department of Mathematics,
Don Mills Collegiate Institute, North York.

It is my feeling that a major part of mathematical education should be through the device "researching a topic". This is being initiated with our Grade 13 students this year in the form of a mathematics essay. A list of 22 essay topics has been prepared under the following ground rules.

"On the following sheets is a set of topics on some phase of mathematics that you have studied this year. You are required to do an "in depth" treatment of the topic using the listed questions as your theme. Each student enrolled in Math A is required to do exactly one essay , selected from the topics in the Math A or Math A and B sections. Each student enrolled in Math A and B is required to do exactly 2 essays, one selected from all topics and one selected from Math A and B or Math B topics.

The student is encouraged to present neat, well labelled diagrams as well as applications of his topic. He is also encouraged to advance the topic by the presentation of ideas beyond those suggested in the questions.

Assessment of each student's essay will be based on

(i) Mathematical content (ii) Clarity of presentation (iii) Oral examination on the topic.

Each student is required to make duplicate copies of his essay (use carbon paper), one of which he will retain in order to study for the oral examination. It is suggested that the essay be typed in duplicate (make a friend in the commercial dept.). The submitted copy will not be returned.

Assistance on all topics is available. Your teachers will not solve these problems for you, however they will suggest lines of attack, check solution attempts and suggest possible reference material.

The oral examinations will be held in the month of May. It is expected that your teachers will want to see a rough draft of your essay sometime in early April. Grade 13 Mathematics classes are cancelled for two weeks in March (immediately after the Winter vacation).

Deadline Date: Monday, May 6th, 1968 - be on time, there will be penalities for lateness.

* The virtues of brevity are urged upon the candidates."

Following are 3 Example Topics:

MATH A

TOPIC - Circular Functional values of $\frac{\pi}{10}$ (18^{o}) and Golden Section.

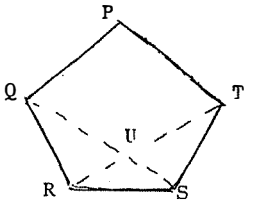
"Geometry has two great treasures: one is the Theorem of Pythagoras; the other the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel." J. Kepler

1. Find (a) $\sin 18^\circ$ (as an exact number value)
   (b) $\cos 18^\circ$
   (c) $\tan 18^\circ$

2. Given (1) ΔABC , in which $AB = AC$
   (2) $\angle B = \angle C = 2\ \angle A$
   (3) D is a point in AB so that $AD = BC$

   Prove (1) $\Delta ABC \sim \Delta CDB$
   (2) D is the "golden section" of AB
   (3) If AE bisects $\angle A$ and E is on BC then $\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$

3. Find $\sin 9^\circ$, $\sin 36^\circ$, $\sin 54^\circ$, $\sin 72^\circ$.

4. 

   Let P Q R S T be a regular pentagon with each side 1 unit .

   Let QU:US = τ (tau = τ)

   Show $\tau = (\sqrt{5}+1)/2$

5. Discuss the use of the golden section in art and architecture.

MATH A AND B

TOPIC - AFFINE TRANSFORMATIONS

"As Euclid collected his material from various sources, it is not surprising that we can extract from the Elements two self contained geometries that differ in their logical foundation, their primitive concepts and axioms. They are known as 'absolute' geometry and 'affine' geometry."

Coxeter

A plane transformation $(x,y) \to (X,Y)$

defined by $\begin{cases} X = lx + my + n \\ Y = px + qy + r \end{cases}$ $l,m,n,p,q,r \in R$

is called an Affine transformation.

1. Find x in terms of X,Y
   and y in terms of X,Y .

   What condition must be true about $pm - lq$?

2. Consider the affine transformation $X = x + y - 1$
   $Y = x - y + 2$

   Find the image of each of the following:

   (i) $(4, -1)$

   (ii) $3x - y + 1 = 0$

   (iii) $x^2 + y^2 = 4$

   (iv) $xy = 1$

3. Could the transformation, "rotation of 45^o followed by a translation $(2, -3)$" be considered as an affine transformation?

   Illustrate your answer by rewriting this transformation in the form of an affine transformation. (See No. 1).

4. Show that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is transformed into another conic under the affine transformation in (2).

   Is $ab - h^2$ invariant under this transformation?

TOPIC: PROBABILITY - BINOMIAL DISTRIBUTION

"There is possibly no branch of mathematics at once so interesting, so bewildering, and of so great practical importance as the theory of probabilities."

E.W. Davis
American Mathematical Society

$$(q + p)^5 = q^5 + 5q^4p + 10q^3p^2 + 10q^2p^3 + 5p^4 + p^5;\ p+q = 1$$

1. (a) For the following values of p, plot the distribution "curve"

$x \to P(x)$ where $x \in \{0,1,2,3,4,5\}$

x = no. of "successes"

$p = 0, .25, .5, .75, 1$ (use same graph)

(b) On a different graph plot the same distribution "curves"

where $x \in \{x \in Re \mid 0 \leq x \leq 5\}$

Use the results of 1(a) and your intuition to "fill in" the curves.

2. (a) The "mode" of a probability distribution is the event having maximum probability.

Show that the mode μ of a binomial distribution is $[(n+1)p]$ where $[k]$ is the largest integer less than or equal to k.

(b) Calculate the "mode" for each distribution in 1(a).

3. Suppose someone has a coin which he claims is biased. Also suppose the coin looks so respectable that you feel the probability for a head must be close to 1/2. You decide to test the coin statistically by tossing it 5 times and making a decision. You decide to accept his claim if either 5 heads or 5 tails occur. Otherwise you will be content with the "hypothesis" that the probability of a head is close to 1/2. Let "p" stand for the true probability of a head.

If $p = 1/2$, what is the probability that his claim will be accepted?

If $p = 3/4$, what is the probability that his claim will be accepted?

Calculate the probability of accepting his claim as a function of p. This function is called the "power function" for the test. Graph this function.

4. 3 Dice (Amber, Blue, Chartreuse) are rolled to determine each of the coefficients of the expression $Ax^2 + Bx + C$.

(a) On a single roll determine the probability that the roots of $Ax^2 + Bx + C = 0$ are equal.

(b) On a series of 10 successive rolls determine the probability that the roots are complex at least 8 times.

The complete list of essay topics (21 in all) is a compilation of problems associated with each topic. The source of problems has been mainly recent textbooks and examination papers and I sincerely thank the authors from whom I have borrowed.

The list of problems in each topic are to form the skeleton of the essay. It was felt that the students are too inexperienced in this sort of project now and would need some guidelines. As the students become more familiar with this kind of report their latitude will be increased.

It is my hope that in the near future, reports of this sort will be done by students from Grades 10 - 13 in each term.

THE EMPTY COLUMN*

A Parable About a 'New Notation' of Long Ago

Many years ago a Roman civil engineer, who was a high official in Alexandria, was approached by a young Arabian mathematician with an idea which the Easterner believed would be of much value to the Roman Government in their road-building, navigating, tax-collecting, and census-taking activities. As the Arabian explained in his manuscript, he had discovered a new type of notation for number writing, which was inspired from some Hindu inscriptions.

The Roman official presumably studied this manuscript very carefully for several hours, then wrote the following reply:

> Your courier brought your proposal at a time when my duties were light, so fortunately I have had the opportunity to study it carefully, and am glad to be able to submit these detailed comments. Your new notation may have a number of merits, as you claim, but it is doubtful whether it ever would be of any practical value to the Roman Empire. Even if authorized by the Emperor himself, as a proposal of this magnitude would have to be, it would be vigorously opposed by the populace, principally

* Supplied by Dr. K.E. Iverson of IBM Corporation.

because those who had to use it would not sympathize with your radical ideas. Our scribes complain loudly that they have too many letters in the Roman Alphabet as it is, and now you propose these ten additional symbols of your number system, namely

1, 2, 3, 4, 5, 6, 7, 8, 9, and your 0 .

It is clear that your 1-mark has the same meaning as our mark-I, but since this mark-I already is a well-established character, why is there any need for yours?

Then you explain that last circle-mark, like our letter O, as representing "an empty column," or meaning nothing. If it means nothing, what is the purpose of writing it? I cannot see that it is serving any useful purpose; but to make sure, I asked my assistant to read this section, and he drew the same conclusion.

You say that the number 01 means the same as just 1 . This is an intolerable ambiguity and could not be permitted in any legal Roman Documents. Your notation has other ambiguities which seem even worse: You explain that the mark-1 means ONE, yet on the very same page you show it to mean TEN in 10, and one HUNDRED in your 100. If my official duties had not been light while reading this, I would have stopped here; you must realize that examiners will not pay much attention to material containing such obvious errors.

Further on, you claim that your system of enumeration is much simpler than with Roman Numerals. I regret to advise that I have examined this point very carefully and must conclude otherwise. For example, counting up to FIVE, you require five new symbols whereas we Romans accomplish this with just two old ones, the mark I and the mark-V. At first sight the combination IV (meaning ONE before FIVE) for four may seem less direct than the old IIII, but note that this alert representation involves LESS EFFORT, and that gain is the conquering principle of the Empire.

Counting up to twenty (the commonest counting range among the populace), you require ten symbols whereas we now need but three: the I, V, and X. Note particularly the pictorial suggestiveness of the V as half of the X . Moreover, it is pictorially evident that XX means ten-and-ten, and this seems much preferred over your 20. These pictorial associations are very important to the lower classes, for as the African sys, "Picture tells thousand words."

You claim that your numbers as a whole are briefer than the Roman Numerals, but this is not made evident in your proofs. Even if true, it is doubtful that this would mean much to the welfare of the Empire, since numbers account for only a small fraction of the written records; and in any case there are plenty of slaves with

plenty of time to do this work.

When you attempt to show that you can manipulate these numbers much more readily than Roman Numerals, your explanations are particularly bad and obscure. For example, you show in one addition that 2 and 3 equal 5 , yet in the case which you write as:

$$\begin{array}{r} 79 \\ +16 \\ \hline 95 \end{array}$$

this indicates that 9 and 6 also equal 5 . How can this be? While that is not clear, it is evident that the other part is in error, for we know that 7 and 1 equal 8 , not 9 .

Your so-called "repeating and dividing" tables also require much more explanation, and possibly correction of errors. I can see that your "Nine Times" Table gives sets which add up to nine, namely

18 27 36 45 54 63 72 81 and 90

but I see no such useful correlation in the "Seven Times" table, for example. Since we have SEVEN, not nine, days in the Roman week, it seems far more important to have a system that gives more sensible combinations in this "Seven Times" table.

All in all, I would advise you to forget this overly ambitious proposal, return to your sand piles, and leave the number writing to the official Census Takers and Tax Collectors. I am sure that they give these matters a great deal more thought than you or I can.

The Development of Modern Algebra.*

by Douglas Iles

Modern algebra, with its study of abstract axiomatic systems such as groups, rings, and fields, did not suddenly appear overnight, but gradually developed as investigators sought new tools for attacking their problems.

The search for roots of polynomial equations is a good case in point. The formula for the solutions of a general quadratic equation had been known from antiquity, and in the 16th century Italian mathematicians had discovered formulas for the roots of general 3rd and 4th order equations (formulas which made essential use of complex numbers even when all the roots were real), but it wasn't until the 19th century, when he introduced the notion of permutation groups, that Galois was able to show that there was no general formula for the solutions of 5th degree equations. It was immediately apparent that this new notion of a "group" was very important, and mathematicians began to study it in its own right.

---

* Based on a term paper written while a student at Queen's University.

Similarly, Cantor (about 1875) first introduced "sets" in some very controversial papers dealing with different magnitudes of infinity. Gradually the language of set theory came to be recognized as a very convenient language for the precise expression of many diverse concepts in mathematics.

Another milestone in 19th century mathematical development was Wm. Rowan Hamilton's creation of quaternions. This is the algebra which does for rotations in a plane. This great Irishman racked his brains for fifteen years in an effort to create a suitable algebra for geometry, optical mechanics and other parts of physics.

What steps led Hamilton to quaternions? First, he stripped imaginary numbers of their mystery. If $i = \sqrt{-1}$, a complex number is of the form $a + bi$, $a,b \in$ Field. Hamilton regarded $a + bi$ as an ordered couple (a,b) . He imposed definitions of sum and product as if the laws of field algebra did in fact hold for them. If consistency of the system defined by the field postulates could be proven, it would follow that complex numbers and the rules by which they are combined would also be consistent. In his theory of complex numbers

$$(a,b) + (c,d) = (a+b,c+d)$$
$$(a,b)\,(c,d) = (ac-bd,ad+bc) .$$

The 0,1 of the field corresponds to (0,0), (1,0) . With these definitions Hamilton's couples satisfy all stated field postulates. These complex numbers can be represented by a vector in a plane. In 3-dimensional space how were vectors to be multiplied together? It was natural to suppose that the commutative law for multiplication would carry over to this generalization of complex numbers, representing rotation in 3-dimensional space. His great discovery in 1843 was an algebra in which the commutative law did not hold. In the algebra of quaternions $A \times B = -B \times A$. These quaternions were of the form (a,b,c,d) and could be represented in Vector notation as $(a,b,c,d) = a1 + b\underline{i} + c\underline{j} + d\underline{k}$. The multiplication table for quaternion units indicates their non-commutativity.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	j
j	j	-k	-1	i
k	k	j	-i	-1

Further contributions to the development of modern algebra were made in the 19th century by A. Cayley. His theory of algebraic invariants originated in a simple observation, the earliest instance of which, appears in the work of Lagrange. Neither Lagrange nor George Boole who did later work on the subject, realized that this

observation might form the germ of a vast theory. The observation is --

For $ax^2 + 2bx + c = 0$ to have two equal roots, $b^2 - ac$ must equal zero. Replace x in terms of y obtained from the transformation $y = \frac{px+q}{rx+s}$.

Solving for x, $x = \frac{q-sy}{ry-p}$.

We have the new equation $Ay^2 + 2By + C = 0$.

Carrying out the algebra

$$A = as^2 - 2bsr + cr^2$$

$$B = -aqs + b(qr+sp) - cpr$$

$$C = aq^2 - 2bpq + cp^2 .$$

This results in $B^2 - AC = (ps-qr)^2(b^2-ac)$

b^2-ac is the discriminant of the quadratic equation in x

B^2-AC is the discriminant of the quadratic equation in y.

Boole found consequently that the discriminant of every equation when its x is replaced by the related y, comes back unchanged except for a factor depending only on the coefficients of the transformation. He found other invariant expressions also but could determine no general method. In 1845 Cayley entered the field with his memoir, "On the theory of Linear Transformations". He set about to find uniform methods which would give him all invariant expressions of the kind described.

In addition to his work on algebraic invariants, Cayley made other outstanding contributions. Perhaps the most significant one was his invention of matrices. This originated in a memoir of 1858 and grew directly out of out of simple observations on the way in which the transformations (linear) of the theory of algebraic invariants are combined. Take the transformation

$$y \to \frac{px+q}{rx+s} \ , \ x \to \frac{Pz+Q}{Rz+S} \quad (+ \text{ means "is replaced by"}) \ .$$

The second transformation is applied to the x in the first

$$y \to \frac{(pP+qR)z + (pQ+qS)}{(rP+sR)z + (rQ+sS)} \ .$$

Now take the co-efficients of the three transformations and write them in square arrays

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} \ , \ \begin{vmatrix} P & Q \\ R & S \end{vmatrix} \ , \ \begin{vmatrix} pP+qR & pQ+qS \\ rP+sR & rQ+sS \end{vmatrix} \ ,$$

The result of performing the first two transformations successively, could have been written down by the following rule of multiplication

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} \ \begin{vmatrix} P & Q \\ R & S \end{vmatrix} \ = \ \begin{vmatrix} pP+qR & pQ+qS \\ rP+sR & rQ+sS \end{vmatrix}$$

where the rows of the array on the right are obtained in an obvious way, by applying rows of 1st array on the left onto columns of 2nd array.

Such arrays are called matrices. Cayley developed their algebra from a few simple postulates.

Postulate (i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$ iff $a = A$, $b = B$, $c = C$, $d = D$.

(ii) their sum is $\begin{vmatrix} a+A & b+B \\ c+C & d+D \end{vmatrix}$

(iii) $m\begin{vmatrix} ab \\ cd \end{vmatrix} = \begin{vmatrix} ma & mb \\ mc & md \end{vmatrix}$

Multiplication is not commutative except for special kinds of matrices.

The modern conception of algebra really began with the 19th century British "reformers". Algebra was understood as algebra i.e. as the abstract development of the consequences of a set of postulates, without necessarily any interpretation or application to numbers. The foremost of these "reformers" was George Boole. He reduced logic to an extremely easy and simple type of algebra. The postulates of this Boolean Algebra can be given an interpreation consistent with classical logic. Problems in logic can be translated into logical equations, solved by the devices of algebra and then retranslated into logical data.

Boole's "Mathematical Analysis of Logic" was published in 1848. The reduction of certain types of logical and mathematical reasoning to symbolism gathered speed until 1910, when it culminated in the appearance of

"Principia Mathematica" by Whitehead and Russell. The (10) postulates of the Algebra of logic evolved from the pioneering work of the 19th century and from them evolved a whole series of theorems. The striking differences between this algebra and a field arise largely from theorems which state --

$a \oplus a = a$ and $a \odot a = a$

replacing $\oplus$ by $+$, and $\odot$ by $\cdot$.

we see that $a + a = a$ i.e. $2a = a$ and $aa = a$

i.e. $a^2 = a$ for every a in the set.

So in this Boolean algebra of two elements $0,1$, the coefficients and exponents in polynomials are linked to the numbers $0,1$, with $oa = 0$, $a^o = 1$.

Any x in this algebra is idempotent ($x^2 = x$). This is characteristic of a commutative ring with a unit element (to be defined) which is a Boolean algebra.

By weakening the field postulates, algebraists developed the concept of a ring. This generalization of a field is consistent because the postulates of the original system are consistent. A ring is a field from which commutativity for multiplication has been dropped, along with an identity and inverse for multiplication.

The theory of Boolean Algebras is an important section in the wider domain of the theory of partial order which has developed during the last 20 or 30 years. Building on Dedekind's early work were such mathematicians as Emmy

Noether and B.L. van der Waerden. They stressed the importance of an application, in algebra, of general properties of those partially ordered sets which are now called lattices and of which Boolean algebra forms a special kind. The development of lattice theory is typical of the tendency to attain the highest possible degree of generality by introducing concepts of increasing abstractness. This theory of lattices was implicit in Boole's Algebra of sets. Analogies between Boolean algebra and properties of -- the Greatest Common Denominator (GCD) and the Lowest Common Multiple (LCM) -- were noted by mathematical logicians as early as 1920. These were not followed up. It was only in the mid 1930's that lattice theory really got under way, mainly in the work of G. Birkhoff and O. Ore.

As we have seen, the real developments in modern abstract algebra can be traced back at least as far as the early 19th century. These developments have come into view however only in the 1920's. The earlier theories, such as those of matrices and algebraic invariants have found increasingly profitable applications in recent science. Modern algebra contains a great deal more that has not yet passed into scientific circulation. It remains for science to bring it to the forefront.

BIBLIOGRAPHY

Bell, E.T., Mathematics, Queen and Servant of Science
Men of Mathematics

Beth, E.W., The Foundations of Mathematics

Newman, J.R., The World of Mathematics

Schaaf, W. (Editor), Mathematics, Our Great Heritage

Eves, H. and Newsom, C.V., An Introduction to the Foundations and Fundamental Concepts of Mathematics.

ACKNOWLEDGEMENT

The O.M.C. and O.A.T.M. wish to acknowledge with thanks a donation of $200.00 from the National Council of Teachers of Mathematics towards the cost of publishing the Ontario Mathematics Gazette.

.....

For the convenience of prospective members of the N.C.T.M., an application form appears below:

MEMBERSHIP BLANK

The National Council of Teachers of Mathematics
1201 Sixteenth Street, N.W., Washington, D.C. 20036

I hereby apply for membership in the National Council of Teachers of Mathematics and enclose $5.00 annual membership dues. (If you wish to receive both journals, remit $8.00. For Canadian mailing, add 25¢ for each journal; for foreign mailing, add 50¢.)

Name (Please print)_____
Last name First name Initial

Mailing Address_____
Street and number City

_____
State or Province ZIP Code

☐ New Membership
☐ Renewal

Please check journal(s) desired: ☐ Mathematics Teacher ☐ Arithmetic Teacher. I also enclose ☐ 50¢ additional fee for an individual subscription to the Mathematics Student Journal.

(Payment must accompany application. Make checks payable to the N.C.T.M.)