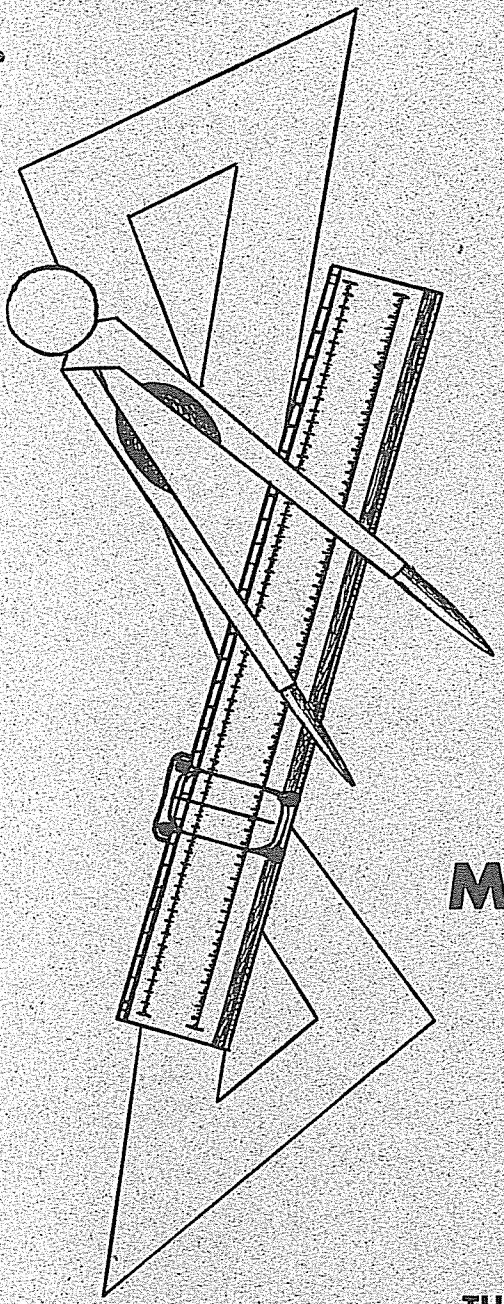


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E D I T O R I A L

1968 was a year of sadness, and frustration. It seemed that man had learned little from history. War, civil disobedience and unrest and assassination were in the forefront. Yet the year closed with a scientific and human achievement of the first order, the circling of the moon.

What have these remarks to do with the teaching of mathematics? Perhaps they will serve to remind us that science and technology in themselves are sterile, and that it is how we view them and use them that counts most. If in our teaching of mathematics we cannot impart concern and goodwill for the students in our charge, what have we really accomplished?

Unlike the last issue, which concentrated on a special theme, this issue ranges over the whole set of aspects of school mathematics from pedagogy and psychology to content and philosophy.

There is much material for thought and further study provided in the numerous references at the end of the first two articles. The reader will recognise the flavour of 'modern mathematics' in the first, yet hopefully will see in this interesting and clear exposition of an area of increasing importance a sound extension of traditional high school geometry and algebra. Why not introduce a modicum of experiences in convexity to your senior students this spring?

The second article is scholarly in nature, and represents a serious scientific attempt to come to grips with the validity of the

much-vaunted discovery method. It cites an imposing array of authorities, and will require careful reading. Of what significance is it for the classroom teacher of mathematics, you may ask? Well, one answer is the age-old truth that there is no panacea in education. Inovations have their place, but we must learn to see each in perspective, and strive to investigate it with our present knowledge and skills. I hope that you will be stimulated to read some of the work referred to, and so become more aware of the subtleties of the learning-teaching situation.

Should the article on conjugate radicals be classed as light relief? Why not? It is written in a refreshing vein, revealing the kind of sensitivity and humanity which I had in mind in my opening paragraphs. Let's put some imagination into our teaching!

The closing article is a long one, admittedly. Yet it is well worth reading. It puts a point of view cogently, arguing basically that one danger of the modern mathematics movement is arrogance and egotism. Personally I have some sympathy with the author, as I am still somewhat sceptical of the extent of abstract mathematics in the present Ontario High School Curriculum! It is interesting to speculate as to what mathematics students will be studying in 1979, and which students will be studying what.

Am I being reactionary? Possibly, possibly not. I favour experiment, yet in all honesty there has been little controlled experiment of the general effect of the changes introduced into high school mathematics since 1962. Perhaps University freshman are more acceptable to University Professors. What about the average Grade 12 or Garde 13 graduate? What mathematics does he know or really understand? How much 'modern' mathematics

is it desirable or necessary for him to understand?

As we move towards the 1970's with the pressures of the space race, of automation, and of educational technology, what changes will there be in high school mathematics? This is no time to relax, and think that the job is done! We must keep testing our goals in intelligent ways, both intuitively and scientifically, seeking ever to have the best interests of our students at heart.

A final word - it is heart-warming to note the increasing interest of the Canadian Mathematics Congress in school mathematics - as witnessed by the two announcements we are printing. I welcome this interest and concern and trust that it will bear fruit.

And so - good thinking and teaching in 1969.

D.H.C.

CONVEXITY

B.J. Kirby, Queen's University

The study of convexity has long been, implicitely or explicitely, of major importance in mathematics : it dates back to the time of Euclid and yet remains today a field of active and important research. Much of synthetic geometry and of elementary calculus is concerned with properties of convex figures and convex functions. Besides being a study pursued for its own sake by contemporary mathematics, it plays a fundamental role in many branches of modern applied mathematics - linear and nonlinear programming, decision and game theory, automatic control theory, and the theory of group representations and of positive operators, for instance.

Because of its many fundamental contributions to both pure and applied mathematics, convexity is thus an excellent supplementary topic for high school students. And it is very accessible to students of varying abilities and knowledge, from those with some slight Eudidean geometry to those who are familiar with linear spaces and calculus. It is quite possible that students at all levels of their studies can enjoy and even make original contributions to, this field.

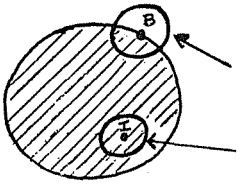
Basic Properties of Convex Sets

Definition A set C of points is convex if C includes all points on the line segment joining any two points of C.

Formally, C is convex iff

$\underline{a}, \underline{b} \in C \implies \mu \underline{a} + (1-\mu)\,\underline{b} \in C$ for $0 \leq \mu \leq 1$ since $\mu \underline{a} + (1-\mu)\,\underline{b}$ represents all points on the line segment joining the points $\underline{a}$ and $\underline{b}$, where μ takes on values in $[0,1]$

For example, the following are plane convex sets,



every such circle centred at B contains points in the set and points not in the set.

this circle, centred at I, contains only points of the set.

We will call a convex figure a convex set in R^2 together with all its boundary points. (The restriction to two dimensions is not essential to many of the results which follow.)

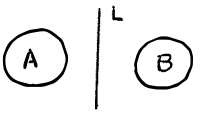
In R^2 the definition of a convex set can be given in terms of coordinates as follows:

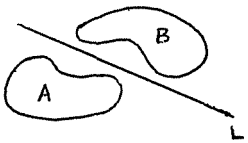
$$\underline{a} = (a_1, a_2) \text{ and } \underline{b} = (b_1, b_2) \in C$$

$$\Rightarrow \mu(a_1, a_2) + (1 - \mu)(b_1, b_2) \in C \Leftrightarrow (\mu a_1, t(1 - \mu) b_1, \mu a_2 + (1 - \mu) b_2) \in C$$

Separating and Supporting Lines

A line is called a separating line of 2 sets if one of the sets is entirely on one side of the line and the other set is on the other side of the line. For instance, in the diagrams below L separates the sets A and B





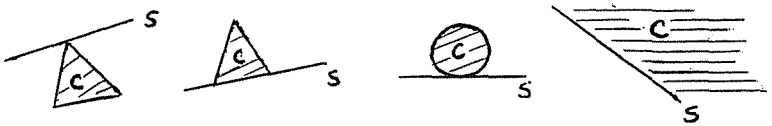
but obviously the sets below cannot be separated by a line.



Now what characterizes sets which can be separated in this way? It turns out that if the two sets are convex, then they can always be separated; to be precise, we have the theorem:

If C_1 and C_2 are disjoint convex figures, then there is a line which separates them.

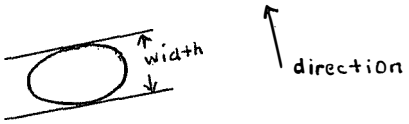
The correctness of this theorem is intuitively clear but a vigorous proof is quite difficult. The other important line connected with a convex set is the supporting line. A line S is called a supporting line if it passes through at least one boundary point of C and if the entire figure lies on one side of the line. For example, in the diagrams below S is a supporting line of C.



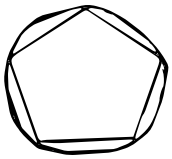
It should be observed that in any direction there can be drawn exactly two parallel supporting lines to any bounded convex figure.

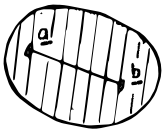
Convex Figures of Constant Width

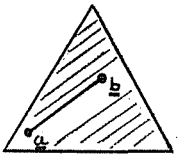
The width of a (bounded) convex figure in a given direction is the distance between a pair of supporting lines of the figures perpendicular to this direction:

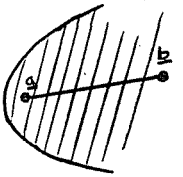


Of particular fascination in this connection are sets of constant width, i.e. convex sets which have the same width in every direction. The simplest of these, of course, is the circle; an interesting family of such curves can be generated by starting with polygons with an odd number of sides. For example

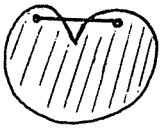




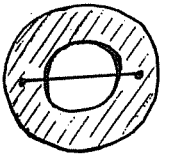




whereas



and



are not convex.

In 3 dimensions, a solid ball or a solid cube are convex sets whereas a doughnut or a barball on the surface of a sphere are not.

In one dimension an interval is a convex set and, in fact, is the only convex set.

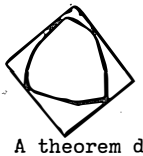
In order to discuss further properties of convex sets it is necessary to recall a few definitions. A plane set C is <u>bounded</u> if there is a circle which contains C as a subset. Examples of bounded sets are a line segment or a circle or an ellipse whereas a straight line or a parabola are unbounded sets - one cannot draw a circle "about" those latter figures.

A point $x \in C$ is called a <u>boundary point</u> of C if every circle (of positive radius) with centre C contains points in C and points not in C. In contrast with this, a point $x \in C$ is an <u>interior point</u> of C if there is a circle with centre C containing only points of C. In the diagram below the point B is a boundary point and the point I is an interior point of C.

Perhaps the most interesting is the Reuleaux triangle.



It can be shown that a square can always be circumscribed about a convex figure of constant width so that a side of the square has any given direction. For instance, in the case of the Reuleaux triangle.



An interesting application of this fact is to the boring of a square hole!

A theorem due to Barbier shows that all figures of constant width h have the same length π h; thus can be checked easily for the circle and for those figures of constant width generated by polygons, as mentioned above. Finally, of all figures of constant width h, the circle has the largest area and the Reuleaux triangle the smallest.

Convex Combinations

The previous sections might be termed "synthetic". There are a number of properties of convex sets which are more amenable to an algebraic approach; they provide good illustrations of elementary vector algebra.

A convex combination of a finite number of points $\underline{x}_1, \ldots, \underline{x}_m$ is defined as

$$\underline{x} = \sum_{i=1}^{m} \mu_i \underline{x}_i \quad \text{when} \quad \mu_i \geq 0 \text{ for all } i = 1, \ldots, m$$

$$\text{and} \quad \sum_{i=1}^{m} \mu_i = 1$$

An essential property of such convex combinations is given by the following

Theorem The set of all convex combinations of a finite number of points $\underline{x}_1, \ldots, \underline{x}_m$ is convex, i.e.

$$X = \left\{ \underline{x} : \underline{x} = \sum_1^m \mu_i \underline{x}_i ;\ \mu_i \geq 0,\ \sum_1^m \mu_i = 1 \right\} \quad \text{is convex.}$$

The following simple proof is quite typical of those encountered in the subject.

Proof: Suppose $\underline{v}$ and $\underline{w}$ are any two elements of the set X. Then, by definition, $\underline{v} = \sum_1^m \mu_i' \underline{x}_i$ where $\mu_i' \geq 0$ and $\sum_1^m \mu_i' = 1$

and $\underline{w} = \sum_1^m \mu_i'' \underline{x}_i$ where $\mu_i'' \geq 0$ and $\sum_1^m \mu_i'' = 1$

In order to prove that X is convex, we must show that

$$\lambda \underline{w} + (1 - \lambda) \underline{v}$$

is also in X for any λ such that $0 \leq \lambda \leq 1$.

Now $\lambda \underline{w} + (1 - \lambda) \underline{v} = \lambda \sum \mu_i'' \underline{x}_i + (1 - \lambda) \sum \mu_i \underline{x}_i$

$$= \sum \left[\lambda \mu_i'' + (1 - \lambda) \mu_i' \right] \underline{x}_i$$

But $\lambda \mu_i'' + (1 - \lambda) \mu_i' \geq 0 \because \lambda \mu_i'' \geq 0$ and $(1 - \lambda) \mu_i' \geq 0$

Also, $\sum \left[\lambda \mu_i'' + (1 - \lambda) \mu_i' \right] = \lambda \sum \mu_i'' + (1 - \lambda) \sum \mu_i'$

$$= \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1.$$

$\therefore \lambda \underline{w} + (1 - \lambda) \underline{v}$ can be written $\sum v_i \underline{x}_i$

Where $v_i = \lambda \mu_i'' + (1 - \lambda) \mu_i'$, $i = 1, \ldots, m$

and $v_i \geq 0$ for all $i = 1, \ldots, m$ and $\sum v_i = 1$

Thus $\lambda \underline{w} + (1 - \lambda) \underline{v} \in X$

$\therefore$ X is convex.

Convex Functions

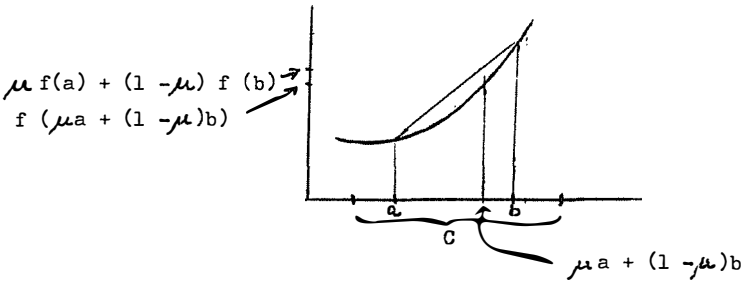
Besides its use in the term "convex set" the word "convex" also occurs in the term "convex function". The relation between the two uses is provided by the theorem following the definition.

Definition If C is a convex set and if f is a function mapping C into the real numbers, then f is convex in C iff

$$\underline{a}, \underline{b} \in C \Rightarrow f(\mu \underline{a} + (1-\mu)\underline{b}) \leq \mu f(\underline{a}) + (1-\mu) f(\underline{b})$$

for all μ such that $0 \leq \mu \leq 1$.

In the special case when C is a one-dimensional set (and therefore an interval) the definition can be illustrated as follows:



Examples of convex functions are $f : x \rightarrow Ax + b$, $f : x \rightarrow x^n$, n odd positive integer,

The relationship between convex sets and convex functions is given by the

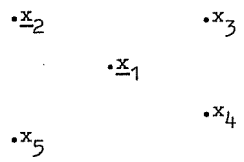
Theorem: Suppose f is a real-valued function defined on an interval $C \subset R$.

Then f is convex iff the set

$$T = \{(x,z) : x \in C,\ z \in R \text{ and } z \geq f(x)\}$$

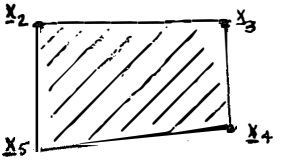
is convex.

Example Given the five points $\underline{x}_1, \ldots, \underline{x}_5$ shown below



then the set of all convex combinations of these five points is the following polygon and its interior which is, of course, convex.

Diagram



Often we wish to find the "smallest" convex set containing a given set A; this smallest set is called the convex hull of A. If A has a finite number of points its convex hull is just the polygon given by the convex combinations of its points.

You might enjoy proving the following fundamental theorem : any bounded convex set with a finite number of vertices is the convex hull of its vertices, i.e. any point in the set can be written as a convex combination of its vertices.

One of the most immediate application of these ideas is to linear programming. A basic result here is that any bounded convex set has vertices in every supporting line. This implies that if a linear programming problem has a feasible solution, at least one feasible solution will be a boundary point.

The proof follows almost immediately from the definitions.

The usefulness of convex functions in mathematics is very great. Their importance stems from the fact that, in general, it is only to such functions that optimization procedures can be applied. That is, nearly all of the many methods for finding the maximum (or minimum) of a function involve the convexity of the function in an essential way. Since such optomization problems are assuming an increasingly dominant place in modern applied mathematics, so too is the role of convex functions and convex sets.

ANNOTATED REFERENCES ON CONVEXITY

1. "Convex Figures and Polyhedra", L.A. Lyusternik, D.C. Heath and Company, Boston, U.S.A. (1966) (Translated from Russian)

   A thorough exposition of the geometric aspects of convex sets in two and three demensions, all of which is accessible to senior high school students.

2. "Convex Figures", I.M. Yaglem and V.F. Boltyanskii, Holt, Rinehart and Winston, N.Y. (1961) (Translated from Russian)

   This book presents many interesting and important properties of convex sets in the form of problems with full solutions given. It is of about the same difficulty as Lyusternik's book.

3. "Linear Algebra", G. Hadley Addison-Wesley, Reading, U.S.A. (1961)

   This contains a chapter on convex sets, besides chapters on vectors, matrices and linear transformations; it is written for first year university students and is particularly suitable as a preparation for studying linear programming.

4. "Convexity", H.G. Eggleston Cambridge University Press, London (1958)

5. "Convex Sets", F.A. Valentine, McGraw-Hill Book Company, Toronto (1964)

   These two books are written at an advanced undergraduate level, and require some knowledge of topology and functional analysis.

6. "Convexity" American Math. Society, 1963

   This is a report on a Symposium and contains many new results and applications of current research with the subject.

7. C.U.P.M. Geometry Conference, Proceedings, Part 1 Mathematical Association of America (1967), Box 1024, Berkeley, California 94701, U.S.A.

   These proceedings contain lectures on convexity by Branko Grinbaum and Vitor Klee. They argue that convexity should be the central idea of the college course in geometry and provide many stimulating pedogogical and technical insights.

TOWARD A PROPER BALANCE BETWEEN DISCOVERY AND TELLING

By Dr. Chester D. Carlow
Curriculum Department
The Ontario Institute for Studies in Education

The increasing emphasis upon discovery learning in recent years has been closely tied to these pervasive educational developments:

1. The freeing of the learner from authoritarian control,
2. the availability of an increasing variety of educational materials, and
3. the increasing emphasis upon structure in curriculum.

The movement toward freeing the learner from authoritarianism is merely one aspect of the many-sided, but erratic movement toward political and economic independence for men and women. In education, the movement has consisted of growing freedom from strict classroom orderliness on the one hand and rote, repetitive drill on the other.

It is interesting to note that the movement toward greater freedom has not, of itself, been sufficient to spark the current emphasis upon discovery learning. The latter had to await the advent of a wide variety of materials and sufficient structure in the curriculum. It didn't make sense in mathematics, for example, to mount a program of discovery learning when the teacher was providing makeshift explanations, without appropriate concrete experiences, in terms of unrelated concepts with very limited applications. In such a 'bag of tricks' approach, the student was ill equipped to make interesting or worthwhile discoveries. And if, by chance, he were to discover that the correct procedure for dividing with fractions was to "invert the divisor and multiply" this discovery, because it does not recur frequently elsewhere, would be of little use in making further

discoveries. Today, with the availability of carefully developed educational materials at lower grade levels and of carefully developed systems of interrelated widely applicable concepts and principles at higher grade levels, a sustained program of discovery learning becomes more feasible throughout the school years. The question arises as to what extent the discovery method should now replace the expository or lecture method as the predominant method of teaching and learning. Some appear to believe that the discovery method should become predominant (Jerome Bruner, Hall-Dennis). Others believe that the discovery method should be employed only moderately (David Ausubel), or to a limited extent (Lee Cronbach).

Those who advocate a major role for discovery learning maintain that the method of discovery is the natural and preferred way of learning for man (Torrence, 1963), that it leads to better retention and transfer (Gagné, 1965, P. 168), that it has mystical motivating powers (Kersh, 1964) and that it requires no practice or consolidation beyond discovery (Hendrix, 1947). The Hall-Dennis Report, while warning that "no one factor, no one method, no one endearing human characteristic can be seized as a magic wand which will transform children into life-long learners and adventurers" (P.49), nevertheless leaves little doubt that the "complete freedom to discover, evaluate, think and decide (P.69) under the guidance of the teacher is to be preferred to the deadening routine (Page 69), the forceful instruction in facts and skills (P.70), the blind memorization (P.71) which one associated with tranditional expository teaching. (Hall-Dennis, 1968).

In marked contrast, David Ausubel deplores what he considers to be the grossly exaggerated claims of the proponents of discovery and finds the predominant use of the discovery method too time consuming. To Ausubel,

> "the most unique attribute of human culture, which distinguishes it from every other kind of social organization in the animal kingdom, is precisely the fact that the accumulated discoveries of millenia can be transmitted to each succeeding generation in the course of childhood and youth, and need not be discovered anew by each generation". (Ausubel, 1968. P.475)

Thus, Ausubel acknowledges the usefulness of the discovery method when learning problem-solving skills or during the early stages of learning any abstract subject matter, especially prior to adolescence. (Ausubel, 1968, P. 472, 73). Once the learner has acquired the necessary concepts and the ability to think abstractly, beginning in junior high school, on the other hand, verbal exposition become the most efficient way of learning. (Ausubel 1968, P.86) The student, without the aid of concrete experiences, can proceed directly to abstract understanding that is "qualitatively superior to the intuitive level in terms of generality, clarity, precision, and explicitness". (Ausubel, 1968, P.87) For this reason, Ausubel would have the student, after Junior High, primarily engaged in learning the same basic subject matter which the scientist learned in his student days. "If he is ever to discover, he must first learn; and he cannot learn adequately by pretending he is a junior scientist." (Ausubel, 1968, P.487)

In support of approximately the same thesis, Bernard Friedlander has observed that discoveries by children, prior to age eleven or twelve, may form closed, idiosyncratic systems because the learners at that age

are unaware of restrictions that logic imposes on possibility and reality. (Friedlander, 1965) From a practical point of view, Friedlander further points out that in a classroom, not all children will learn equally well through the discovery method. Thus, the teacher may overestimate the advantages of discovery for the few who are doing the discovering, while ... underestimating the cost in loss of communication and bleakness for the many who don't ...". (Friedlander, 1965)

Lee Cronbach is

> ... impressed by the possibility that some experience in discovering principles in a field of knowledge will radically alter the relation that the learner perceives between himself and the knowledge, and his way of behaving when he forgets a solution or encounters an unprecedented problem. I offer the hypothesis, however, that this can be accomplished by devoting only a small fraction of the instructional time to inductive procedures. (Cronbach, 1965)

In order to obtain a clear explanation of the vast difference between those who strongly endorse from those who urge caution with respect to discovery learning one must again turn to Ausubel. The discovery enthusiasts, in his view, have depicted expository teaching in terms of its worst abuses rather than its essential characteristics. They have incorrectly overgeneralized the conditions under which discovery is appropriate to all conditions, and have erroneously identified "discovery learning" with "meaningful learning", while assuming that all expository learning must be dull and rote. (Ausubel, 1968. P.P.84, 85)

The last mentioned identification has been fostered, undoubtedly, by a combination of two forces. On the one hand, there is an association between the lecture method and the previous classroom environment of monotonous drill. On the other hand there has been a certain halo attached to discovery in Western Culture for the past 2,000 years or more.

From the time Archimedes ran stark naked through the streets of ancient Syracuse shouting "Eureka!", after discovering the principle of buouancy, to recent discoveries related to the atom bomb and space travel, discovery has been associated with freedom, excitement, and success. It goes without saying that the classroom variety of discovery, involving lesser minds and reduced experiential background, will not at the same time have both the freedom and the success. As aptly phrased by Ausubel, " ...What happens to Archimedes junior's motivation and self-confidence if, after seventeen immersions in the tub he has merely succeeded in getting himself soaking wet!" (Ausubel, 1968, P.494) If, on the other hand, the teacher guides the procedure in such a way that discovery becomes inevitable,then the result may be cook-book kind of discovery which is as dull and rote as the worst examples of the expository method. Learning through exposition, moreover, need not be authoritarian, formalistic, passive, or dull. If the exposition is preceded by the appropriate devlopment of background concepts, if the material is well sequenced so that important relationships and distinctions are perceived, if the student is disposed to rework the new material critically and relate it non-arbitrarily to things he already knows rather than accept it on faith, then such learning may be active, interesting and meaningful. (Ausubel, 1968, P.85, 88, 491)

A review of research tends to support the more selective use of discovery teaching-learning advocated by Ausubel, Friedlander, and Cronbach. For example, discovery of generalizations is superior to memorization without understanding (McConnell, 1934), but not superior to expository learning with understanding (Forgus and Swartz, 1957). The inductive or discovery method promotes greater motivation on short, interesting learning

tasks (Kersh, 1958, 1962), but not on longer tasks (Karle, 1960; Cronbach, 1965) expecially when the amount of exposure to the material (Krumboltz and Yabroff, 1965) and the motivation due to a sense of unfinished task (Craig, 1965) is the same for the expository and the discovery groups.

Hendrix found that early verbalization of discoveries impedes transfer for some students (Hendrix, 1961), while Gagne and Smith found that students who verbalized general principles during problem solving practice with instructions did better than those who did not verbalize. (Gagne and Smith, 1962). Verbalizations apparently encourage them to think more clearly about their procedures or to give better self-guidance.

Several investigators, using classroom materials, found no significant difference between the didactic or expository method and discovery. (Nichols, 1956; Wolfe, 1963; Krumboltz and Yarbroff, 1965).

Studies showing some degree of superiority of didactic over discovery procedures were conducted by Wittrock (1962), Krumboltz and Yarbroff (1965), Judd (1908), and Braddeley (1963). Kersh found that expository teaching results in greater initial learning than does discovery (Kersh, 1964). In a well known early study, Judd found that subjects given verbal training in refraction of light did better than those not given verbal training in hitting targets with darts at different depths of water (Judd, 1908). Braddeley found that having the correct answer given after failure to unscramble anagrams, results in better retention than discovery of the correct answer (Braddeley, 1963).

There are several reasons for the discrepancies among research findings on discovery learning. Among these are poor controls on the

amount of exposure to the learning materials, the enthusiasm and skill of teachers with different methods, the degree of feeling by the learner that he has been selected for a different or special kind of treatment, (resulting in differential Hawthorne effect), and the feeling by the learner that the task is unfinished (resulting in a differential Zeigarnik effect). In addition, there is no agreed upon definition of discovery learning. Many procedures from simple first observations and easy small-step linear programming to very difficult autonomous discovery are given the same label. Little wonder that the results of research on discovery teaching and learning vary widely and often conflict.

Perhaps the most serious cause of discrepancies among studies of discovery learning is that they have consisted of attempts to make comparisons with other methods prior to adequate evidence of the conditions under which the various types of discovery learning are successful. Thus, one might conpare a long-established and standardized expository method with a relatively new type of discovery method whose characteristics are unknown.

It was to help correct this defect that the present author conducted an experiment on variables <u>within</u> guided discovery learning. The learning task consisted of fifteen generalizations on probability and was of such difficulty that the average number of hints required for discovery per generalization was in excess of three. The sample was selected randomly from "successful" ninth grade college preparatory boys. Efforts were made to control for communication among learners as well as for all the variables mentioned above. (Carlow, 1967)

Each student was worked with individually to insure that he alone made each discovery. The question or problem was posed and hints, always in the form of questions, were provided with flexibility according to the perceived needs of the students.

After four learning sessions on successive afternoons, the learners were randomly assigned to practice sessions in which they could solve problems using the generalizations which they had discovered. When they encountered difficulty they might receive hints referring them to previous work, but no new information.

A retention test was given three weeks after the week of learning and supervised practice.

The results of the experiment were as follows:

The practice sessions, as predicted, made significant differences in the retention scores. The mean retention score of those who had two practice sessions was approximately twice the mean retention score of those who had no practice session, while those who received one practice session fall about half way between.

Of the personality factors, intelligence quotient (I.Q.) and conceptual level were each positively correlated (approx. +.40) with good hint, time, and retention scores. Submissiveness had a significant negative correlation of approximately the same magnitude. (Conceptual level is defined here as a personality factor in which higher conceptual levels are associated with lower stereopathy and greater flexibility and creativity in the face of complex and changing problem situations. Submissiveness is defined as the need for a high level of control bases upon social conformity and other directedness such as humility, getting along with others, and keeping one's place).

Contrary to expectations,there were no significant correlations between intellectual interests or motivation and hint, time, or retention scores.

The experimental findings with, perhaps, the most important implications for arriving at a proper balance between the discovery and expository methods of teaching and learning came as a result of observing patterns of scores after the fact. Hence, no statistical significance is attached to the patterns. One merely speculates and raises questions for further research.

Four clear patterns of response were noted:

1. A "reflective" group who were the best discoverers under the conditions of the experiment. They saw relations quickly and were careful to understand thoroughly before going to new problems. They had moderately good hint and time scores followed in three weeks with excellent retention scores.
2. An "impulsive" group who made the discoveries as quickly as the reflective group. Their retention scores, however, were the lowest of any group. It was hypothesized that they did not exercise self-criticism to check their insights and see the relations to other discoveries before proceeding to a new problem.
3. A "safety oriented" group who took the longest to make discoveries, but did moderately well on the retention test. They seemed afraid to make guesses and unable to see relationships. Once arrived at a conclusion, however, they memorized it well and were able to recall it three weeks later. With a less challenging task, they might have obtained higher performance scores than those of any other group.

4. A "normal group" with normal personality profiles. Their submissiveness scores were clearly between those of group 1 and group 2. Other scores did not differ significantly from the norm.

The above findings, even though highly tentative and undoubtedly inaccurate, have far-reaching implications for the achievement of a proper balance between discovery and telling. For, although further investigations may indicate that other groupings with different characteristics may be more useful than the one outlined above, one can know in advance that some groups with distinctive learning characteristics will emerge. One can imagine that different kinds of discovery and different strategies of telling will be "right" for each such group. Then the question will not simply be whether the discovery method is better or worse than the expository method. Rather, one will ask "what kind of discovery, with what degree of teacher guidance, with what kind of learner, under what conditions?"

Thus, the movement toward a proper balance between discovery and telling will occur in specific situations only and to the degree that the over-simplified and sentimental fantasies surrounding these methods give way to reliable and objective information.

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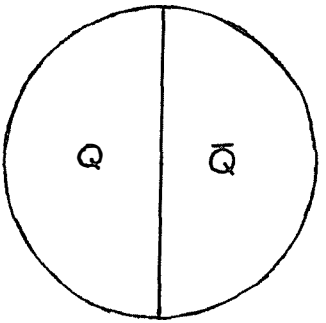
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CONJUGATE RADICALS

by Thomas Wolder

(Chelmsford Valley High School)

In the presentation of this concept in the Grade XI course and in the newly proposed Grade X course there has always been some vagueness as to how this concept can be unified to a Mathematics course which stresses Number Properties and Sets. If this idea of conjugate radicals be embodied in the general principle of Number Sets we must return to the two basic subsets of Real Numbers. Let this set of numbers be represented by a Venn Diagram in the following manners.



It is apparent therefore that the Real Number set is divided into two entities which do not overlap. $Q \cap \bar{Q} = \emptyset$. In other words Q (the set of Rational Numbers) and $\bar{Q}$ (the set of Irrational Numbers) are disjoint and therefore have no elements in common. Also it may be discovered that $Q \cup \bar{Q} = R$ or every Real Number is either Rational or Irrational but not both. Now let us consider the division line between Q and $\bar{Q}$ to be an impregnable wall which only has one gate to let travellers through. The guard has been instructed to let only those with correct travel permits through, and they may only pass from Irrational to Rational land.

Now in the land of Irrationals-mainly due to an internal crisis - the transcendental quantities such as π (pi) have been prevented from ever obtaining travel permits, not even in combination with other pi's such as ; π^2, etc. The privilege has only been granted to worthy Irrational Citizens such as the simple square roots $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$ and the nobler binomial square roots such as $\sqrt{5} - \sqrt{2}$. Even these special priviledged groups must be accompanied when passing through the gate. The quantity that must be chosen to accompany these chosen few is called a conjugate.

Who are these conjugates? A conjugate must be available in the set of irrational numbers since they cannot pass out of irrational land until they have found their conjugate. So this condition seems reasonable. Thus simple square roots have simple conjugates; e.g. $\sqrt{2}$ has $\sqrt{2}$, $\sqrt{7}$ has $\sqrt{7}$, and so on.
The nobler square root expressions have conjugates almost like themselves.

The next point to stress would be the correct binary operation that must be employed in order to be able to pass into the land of Rational numbers. It can be easily proven that addition and subtraction just will not suffice. Therefore only division and multiplication could be used to pass through that gate. Now it could be stated that the guard who permits travellers through prefers the operation of multiplication. Thus $\sqrt{2}\sqrt{2} = 2$ which is a member of the set of Rational numbers.

With the nobler quantities such as $(\sqrt{7} - \sqrt{5})$ it can be shown that by using the FOIL* rule of binomial multiplication that the

* (first, outer, inner, last)

result of $(\sqrt{7} - \sqrt{5})(\sqrt{7} + \sqrt{5})$ will be $\sqrt{49} - \sqrt{35} + \sqrt{35} - \sqrt{25} = 7 - 5 = 2$. Therefore again the result is a member of the set of Rationals.

In a subsequent discussion it can be shown that in no way can rational numbers be combined to form irrational numbers. Therefore it is a one-way path from Q to $\bar{Q}$ which can only be effected through the operation of conjugation.

TRIANGLE OF PERMUTED VALUES

by Colin Swan
Windsor

In studying permuations and combinations I noted that there existed no conventional table for permuted values to correspond to Pascal's Triangle for combined values. It is therefore, my intention to present in this paper such a table of values and to develop its properties.

The table I shall deal with, and which I shall call the 'Triangle of Permuted Values' or the 'Permutation Triangle', is arranged on the same set of axes as Pascal's Triangle with the value of ${}_nP_r$ being found by a cross reference, the same as that used to obtain ${}_nC_r$. It should be noted that on the following table there appears an area which is undefined as r becomes greater than n.

TRIANGLE OF PERMUTED VALUES

n \ r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	2								
3	1	3	6	6							
4	1	4	12	24	24						
5	1	5	20	60	120	120					
6	1	6	30	120	360	720	720				
7	1	7	42	210	840	2520	5040				
8	1	8	56	336	1680	6720	20160	40320	40320		
9	1	9	72	504	3024	15120	60480	181440	362880	362880	
10	1	10	90	720	5040	30240	151200	604800	1814400	3628800	3628800

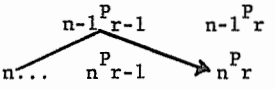
In a logical progression let us first consider the methods of forming the table.

METHOD I $\quad {}_{n}P_{r} = n\,({}_{n-1}P_{r-1})$

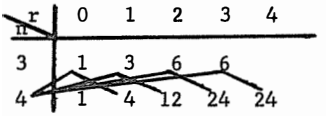
Proof: L.S. $= \dfrac{n!}{(n-r)!}$ R.S. $= \dfrac{n(n-1)!}{n-1-(n-1)!}$

$= \dfrac{n!}{(n-r)!}$ Q.E.D.

It should be noted at this point that the methods presented are as complicated on an algebraic basis as in ${}_{n}P_{r}$. However, on a practical basis these three methods simplify the formation considerably. The following diagram illustrates method one.



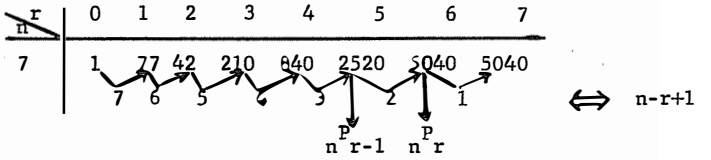
Example 1



METHOD II $\quad {}_{n}P_{r} = {}_{n}P_{r-1}(n-r+1)$

The importance of this method is that any required row of the table may be obtained from knowing the value of n without having to know the values of the previous row as is required in method one.

Example 2



The obvious simplicity of this method is that for any row, you begin with

${}_nP_o = 1$, multiply it by n and then multiply this value by n-1. For sucessive values multiply the value of ${}_nP_r$ just obtained by a number, one less than that used to obtain the last value of ${}_nP_r$ with the minimum value of the multiplier being one.

The proof for this method and for method three follows directly from that used for method one.

METHOD III $\quad {}_nP_r = r({}_{n-1}P_{r-1}) + {}_{n-1}P_r$

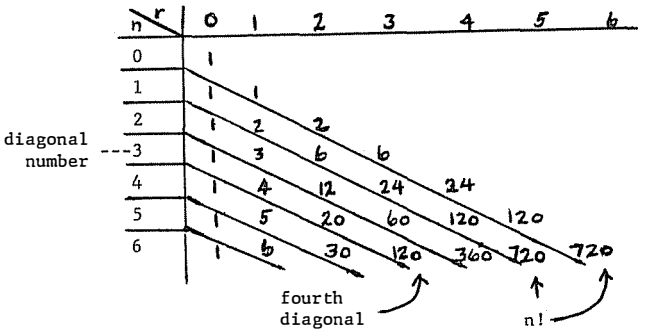
It is this method which is analogous to that used by Pascal. The only, but significant, difference is that in this method the term including (n-1, r-1) is first multiplied by r before being added to the term containing (n-1, r). We shall make use of this fact later in a closer comparison.

As a final note, re methods of formation of the table, each of the three methods presented is dependent upon the knowledge that ${}_nP_o = 1$. After asserting this fact one may proceed to form the table as described above.

PROPERTIES OF THE PERMUTATION TRIANGLE

Observing the completed table, the most obvious property is that the axes expand infinitely in both directions. Secondly we should note that the first two diagonals along the hypotenuse of the table are equal to n! (N.B. - The first diagonal is called the 'zero diagonal' and each sucessive diagonal has a number one less than the actual number of digonals to that point. e.g.: the fifth diagonal is called 'number four diagonal' with the four being taken from the vertical axes of the table.)

Example 3



It follows directly that the values found in one diagonal are arrived at by dividing the values of the previous diagonal by the number of the desired diagonal and starting with the value in the previous diagonal which is in the same horizontal row as the required diagonal number. Since we now have a sequence of numbers the next logical step is to define them as a sequence. It is found that each diagonal forms a recursive sequence of the following general form:

$a_n = na_{n-1}$ given a_{n-1} where n is taken from the horizontal column in which a_n is found. One must, in this case, remember which diagonal is being developed as n in the recursive formula is not directly related to the diagonal number.

Example 4

Consider the fourth diagonal (number three diagonal) given:-

$$a_{n-1} = a_3 = 1$$

then $a_n = a_4$

$= 4(1)$

$= 4$

$a_{n-1} = a_4 = 4$

$a_n = a_5$

$= 5(4)$

$= 20$

This recursive sequence follows directly from the first formulation method for the Permutation Triangle.

Let us now proceed one step further by considering the horizontal and vertical columns as series which leads us directly to the sum of the series.

Partial Sum of a Horizontal Column

$$\sum_{r=1}^{x} P(n,r) = n + n \sum_{r=1}^{x-1} P(n-1,r)$$

where x is the number of terms to be summed and n is a constant taken from the vertical axis.

Proof: L.S. $= \sum_{r=1}^{x} P(n,r)$

$$= {}_nP_1 + {}_nP_2 + {}_nP_3 + \ldots + {}_nP_x$$

BUT: ${}_nP_r = n({}_{n-1}P_{r-1})$

$$= n({}_{n-1}P_o + {}_{n-1}P_1 + {}_{n-1}P_2 + \ldots + {}_{n-1}P_{x-1})$$

$$= n + (n^2-n) + n({}_{n-1}P_2) + \ldots + n({}_{n-1}P_{x-1})$$

$$= n^2 + n({}_{n-1}P_2) + n({}_{n-1}P_3) + \ldots + n({}_{n-1}P_{x-1})$$

R.S. $= n + n \sum_{r=1}^{x-1} P(n-1,r)$

$$= n + n({}_{n-1}P_1 + {}_{n-1}P_2 + \ldots + {}_{n-1}P_{x-1})$$

$$= n + (n^2-n) + n({}_{n-1}P_2) + \ldots + n({}_{n-1}P_{x-1})$$

$$= n^2 + n({}_{n-1}P_2) + n({}_{n-1}P_3) + \ldots + n({}_{n-1}P_{x-1})$$

Therefore L.S. = R.S.

Example 5

To find the sum of the first three terms of row n=5 (not including ${}_5P_o$) :

$$\sum_{r=1}^{3} {}_5P_r = 5 + 5 \sum_{r=1}^{2} {}_4P_r$$

$$= 5 + 5({}_4P_1 + {}_4P_2)$$

$$= 5 + 5(4 + 12)$$

$$= 85$$

To explain this algebraic form it may be said that to find the sum of x terms in a horizontal row, sum the previous row to one term less, multiply that sum by n of the desired row and add n to the resultant product.
It follows directly that the total sum of a horizontal column is:

$$\sum_{r=o}^{n} P(n,r) = 1 + n \sum_{r=o}^{n-1} P(n-1,r)$$

The proof of this equation follows directly from the proof for the partial sum with the restriction that the equation is first rearranged as follows:

$$\left(\sum_{r=o}^{n} {}_nP_r \right) - 1 = n \sum_{r=o}^{n-1} {}_{n-1}P_r$$

Example 6

To find the total sum of row n=5 (including ${}_5P_o$):

$$\sum_{r=o}^{5} {}_5P_r = 1 + 5 \sum_{r=o}^{4} {}_4P_r$$

$$= 1 + 5({}_4P_o + {}_4P_1 + {}_4P_2 + {}_4P_3 + {}_4P_4)$$

$$= 1 + 5(1 + 4 + 12 + 24 + 24)$$

$$= 326$$

In short, to find the total sum of a horizontal row, sum the previous row, multiply that sum by n of the required row and add one.

Now let us consider the sum on a Vertical Column. It should be noted that unlike the horizontal row, the vertical column extends without bound and therefore no total sum is possible. It then follows directly that:

Form I $$\sum_{n=r}^{x} {}_nP_r = \frac{1}{r+1} \prod_{i=1}^{r+1} z + i - 2$$

where $x = z + r - 2$ and r is a constant

Form II $$\sum_{n=r}^{z+r-2} {}_nP_r = \frac{1}{r+1} \prod_{i=1}^{r+1} z + i - 2$$

Form III $$\sum_{n=r}^{x} {}_nP_r = \frac{1}{r+1} \prod_{i=1}^{r+1} x - r + i$$

We shall use the most simplified form of the equation in the proof.

R.T.P. $$\sum_{n=r}^{x} {}_nP_r = \frac{1}{r+1} \prod_{i=1}^{r+1} x - r + i$$

Let r be some integer.

Since r is the lest possible value of x, let x = r.

L.S. = r!

R.S. $$= \frac{1}{r+1} \prod_{i=1}^{r+1} i$$

$$= r!$$

Therefore the equation is true for the least possible value of x

Assume the equation to be true for x = k, and prove it for x = k + 1

R.T.P. for x = k + 1

Now, $$\sum_{n=r}^{k+1} {}_nP_r = \left[\frac{1}{r+1} \prod_{i=1}^{r+1} k-r+i \right] + \frac{(k+1)!}{(k-r+1)!}$$

And we must prove that

$$\text{L.S.} = \frac{(k+2)!}{(r+1)(k-r+1)!}$$

$$\text{R.S.} = \frac{(k+1)!}{(r+1)(k-r)!} + \frac{(k+1)!}{(k-r+1)!}$$

$$= \frac{(k+1)!}{(k-r+1)!}\left[1 + \frac{k-r+1}{r+1}\right]$$

$$= \frac{(k+1)!}{(k-r+1)!} \times \frac{k+2}{r+1}$$

$$= \frac{(k+2)!}{(r+1)\ (k-r+1)!} \qquad \text{Q.E.D.}$$

Since the third form relates directly to the values on the table let us consider it first by using the following sum.

$$\sum_{n=3}^{7} {}_nP_3 = \frac{1}{4} \prod_{i=1}^{4} (7-3)+i$$

In this form x=7 indicates the limit of the sum is ${}_7P_3$ but does not indicate how many terms are to be summed. The number of terms for this example is 5 with the first term of the series being ${}_3P_3$. However, if we consider zero as the first term and ${}_nP_n$ as the second term of the series, the equation sums six terms.

We may obtain 6 by x-r+2 which, if we refer back to the condition given for the first form of the equation, is the expression for z. Therefore, if a given number of terms is to be summed the second form is used (remembering that z includes zero as the first term of the series), and if the terms ${}_nP_r$ are to be summed to a given value of n then the third form is used.

Example 7

Find the sum of the first 8 terms of the column r=4.

Note: z = 9 since zero is included as the first term of the series but is not part of the actual table.

Therefore: $\sum_{n=4}^{9+(4-2)} {}^{n}P_4 = \frac{1}{5} \prod_{i=1}^{5} 9 + i - 2$ (Form II)

R.S. $= \frac{(7+1)\ (7+2)\ (7+3)\ (7+4)\ (7+5)}{5}$

$= \frac{8.9.10.11.12}{5}$

$= 19008$

Therefore the sum of the first 8 terms of r=4 is 19008

Example 8

Find the sum of the terms to n=11 of the column r=4

$\sum_{n=4}^{11} {}^{n}P_4 = \frac{1}{5} \prod_{i=1}^{5} 11 + i - 4$ (Form III)

R.S. $= \frac{(7+1)\ (7+2)\ (7+3)\ (7+4)\ (7+5)}{5}$

$= 19008$

As a final note on the properties of the table, I have omitted the following since I have been unable to discover their formulae, and therefore present the following for possible solution by the reader:

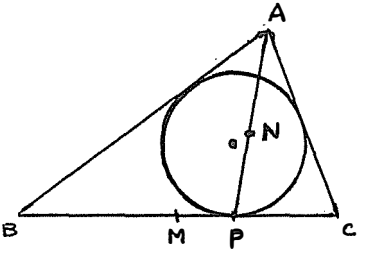
- sum on a diagonal
- general term of the row series
- general term of the column series
- sum of the row series in terms of n and r

Reference: 'Triangle of Permuted Values'
- C. L. Swan
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page 62

I should like to acknowledge the assistance of Dr. J. H. Blackwell, Head, Applied Mathematics, University of Western Ontario, in the determination of the proof for the sum on a vertical column.

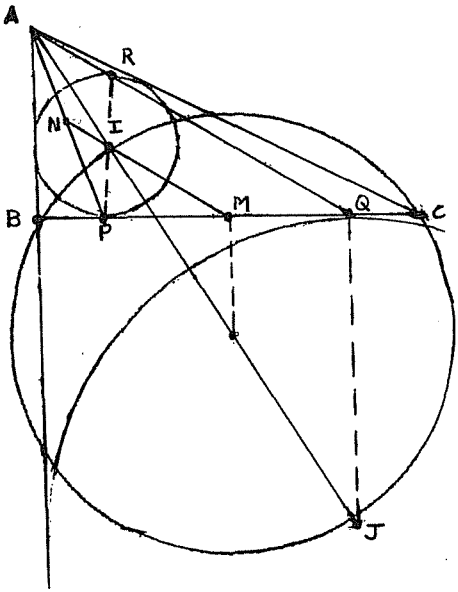
PROBLEM NO. 3 (Vol. 5, No. 3, May 1967, p.33)

In the diagram I is the center of the inscribed circle, N is the midpoint of AP and M is the midpoint of BC. Prove that N, I, M are collinear.



SOLUTION by Mr. A. W. Walker, Toronto, Ont.

The incircle of triangle ABC with centre I touches the side BC at P, the excircle with centre J on line AI touches BC at Q. Because A is a homothetic centre of these two circles, the line AQ passes through the R diametrally opposite to P on the incircle. Angles IBJ and ICJ being 90^{o} each, the circle on diameter IJ passes through B and C, and the foot of the perpendicular from its centre onto BC is therefore the common midpoint M of BC and PQ. Then the midpoints N, I, M of PA, PR, PQ are collinear on a line parallel to ARQ.



REMARKS

(a) The area method (OMG, vol. 6, no. 1, p.65) is a very powerful one for studying the geometry of the triangle when made more systematic by introducing homogeneous areal coordinates, popular 100 years ago but now relatively unfamiliar. The areas of triangles BIC, CIA, AIB being proportional to the side lengths a, b, c, of reference triangle ABC, we say that the point I has areal coordinates (a,b,c); likewise, we have A(1,0,0), M (0,1,1), P(0,s-c,s-b), N (a,s-c, s-b), and the determinant of the coordinates of N.I.M vanishes for collinearity.

(b) By affine projection we have the following generalization of Prob. 3: if P is the contact point with side BC of any conic touching all three sides of triangle ABC, the centre of the conic lies on the line through the midpoints of AP and BC.

(c) Result (b) is, in turn, a degenerate case of a still more general

theorem (Newton, Gauss, Chasles): the midpoints of the three diagonals of a quadrilateral are collinear with the centre of any member of the family of conics touching its four sides. Applying this to "quadrilateral" ABPC, where P is any point on line BC, we have (b).

(d) In the very interesting article by Mr. F. C. Ashbury (OMG, vol. 6, no. 1, p.67), the properties established for the meets of non-corresponding sides of the medial triangle and the pedal triangle of an incentre or excentre will also be found on p.39, 48 of a lengthy paper by Mackay on the properties of the angle bisectors of a triangle (Proc. Edinburgh Math. Society, series 1, vol. 13, 1895). His other result has the following projective generalization: the meets of the corresponding sides of any pair of triangles T_1 and T_2, each inscribed in and perspective with a triangle T, are the vertices of a triangle circumscribed to T and perspective with each of triangles T, T_1 and T_2. (Quarterly Journal of Math., series 1, vol. 4, 1861, p.153). This self-dual figure has many striking properties and has been studied by several writers; see, e.g. R. A. Johnson, Advanced Euclidean Geometry, p.261 for a partial list of results, stated (without proof) for the case where T_1 and T_2 are the medial and "orthic" triangles of T. See also Nouvelles Annales de Math., (series 2), vol. 4, 1865, p.178 and 522; (series 3), vol. 5, 1886, p.304; vol. 10, 1891, prob.' supp., p.13 and 37; vol. 14, 1895, p.514.

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1968 - 1969

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ONTARIO ASSOCIATION OF TEACHERS OF MATHEMATICS

SUNDAY, MARCH 16, SKYLINE HOTEL

7:00 p.m. Registration: Convention Court

8:00 p.m. Early Bird Reception: Provincial Room.
Host: Dr. K. O. May

MONDAY, MARCH 17, SKYLINE HOTEL

8:00 a.m. Registration: Convention Court.
Exhibits and Math. Fair: International Room A & B

9:00 a.m. Opening Session: International Room C,
Mr. J. R. MacLean.

9:15 a.m. General Session: International Room C,
Dr. E. Snapper, Dartmouth College, Hanover, N.H.:
THE EUCLIDEAN GROUP.

10:15 a.m. Exhibits and Math. Fair: International Room A & B

10:45 a.m. Group Sessions

11:45 a.m. Lunch; Exhibits and Math. Fair: International
Room A & B

1:45 p.m. Dr. J. Hogarth, Queen's University: SCHOOL MATHEMATICS
& THE HALL-DENNIS-REPORT, Provincial Room.

Mr. W. W. Bates, Director of Mathematics, Toronto:
BALANCED PROGRAM IN MATHEMATICS, Starlite Room.

2:45 p.m. Exhibits and Math. Fair: International Room C,
Mr. G. Scroggie, Assistant Superintendent,
Curriculum Section, Department of Education:
THE EVOLUTION OF CURRICULUM.

7:00 p.m. Banquet - International Room C. Chairman -
Mr. J. R. MacLean.
Speaker - Dr. T. S. Webster, Dean of Student
Affairs, Queen's University.

9:00 p.m. Dancing - The Cloud Room

TUESDAY, MARCH 18, SKYLINE HOTEL

8:00 a.m. Registration: Convention Court.
Exhibits and Math. Fair: International Room A & B

9:00 a.m. Business Meeting: International Room C.

9:15 a.m. General Session: International Room C,
Dr. E. Snapper - THE AFFINE GROUP.

10:15 a.m. Exhibits and Math. Fair: International Room A & B

10:45 a.m. Group Sessions.

12:00 noon Luncheon (Compliments of I.B.M.) - International Room C. Chairman - Mr. D. W. Alexander.
Speaker - Dr. K. E. Iverson, Manager of Algorithm, Analysis, Thomas J. Watson Research Centre, Yorktown Heights, N.Y.

FILMS

MONDAY AND TUESDAY - A selection of films will be shown. Paris Room

GROUP SESSIONS

MONDAY AND TUESDAY - 10:45 a.m.

1. Mr. W. Anglin, Program Consultant, Ontario Department of Education: MAKING 4-YEAR MATHEMATICS INTERESTING, Monday: International Room C, Tuesday: Continental Room.

2. Mr. F. Ebos, Education Supervisor, ETV Branch, Ontario Department of Education: USING TELEVISION IN THE TEACHING OF MATHEMATICS, Starlite Room.

3. Mr. V. Mathewson, West Hill Secondary School, Owen Sound: A STUDENT INVOLVEMENT PROGRAM, New York Room.

4. Mr. E. Pipher, Hamilton Teachers' College: AFTER THE MATHEMATICS WORKSHOP - WHAT NEXT?, Paris Room.

5. Mr. H. L. Ridge, Coordinator of Mathematics, York: DESK CALCULATORS IN THE SECONDARY SCHOOL, Provincial Room.

6. Dr. E. Snapper, Dartmouth College: DISCUSSION OF GENERAL SESSION ADDRESS.

FIRST CANADIAN MATHEMATICS OLYMPIAD

For several years, there have been plans to institute a national competition in Mathematics at the pre-University level. This competition will supplement competitions which are currently being held in each of the Provinces. Recently, the International Nickel Company has offered funds to be used by the Canadian Mathematical Congress for furthering Mathematics education at this level. The Canadian Mathematical Congress has, in addition, contributed more funds, and has arranged for the

FIRST ANNUAL CANADIAN MATHEMATICS OLYMPIAD

to be held for the first time in 1969. First information about details of this competition are spelled out in the following paragraphs.

The Olympiad will be a national competition, open to not more than 187 students selected across Canada. Allocations have been made to the various Provinces, and the quota for the Province of Ontario is a number of students not to exceed 50.

The students must have completed their pre-University mathematics education. Thus, in Ontario, the competition is open to invited students from Grade XIII or earlier.

Candidates from the Province of Ontario will be selected by a Provincial Committee which has been set up under the Chairmanship of Professor R. G. Stanton (York) and comprising Professor George Duff (Toronto), Professor John Coleman (Queen's), Professor Kenneth Fryer (Waterloo). Selection will be largely based on the results of the 1969 Mathematics Association of American contest, sponsored in the Province

of Ontario by the Acturial profession. However, the Committee will also look at the very top results obtained on the 1967 Ontario Junior Mathematics Contest and at top scores obtained by Grade XII students in the 1968 competitions of the Mathematics Association of America and the Ontario Senior Mathematics Problems Competition.

The first prize is an award of $6,000, donated by the International Nickel Company of Canada, to be given in yearly installments of $1,500 if the top student studies at a Canadian university and if his standing continues to be satisfactory. There are then four prizes of $500 each to the next four highest ranking students and five prizes of $200 to the next five highest ranking students. There are 20 further certificates for honourable mention.

If the contestant with highest standing decides not to enter a Canadian university, then he will receive one of the $500 prizes and the $6,000 award will be offered to the student with the next highest standing.

There is a proviso that no more than three of the nine prizes from second to tenth prize may be won by students from the same Province. This imposes a maximum of four prizes (first prize and three others) from any one Province.

The Contest has been set by a Committee of the Canadian Mathematical Congress centered at the University of Victoria (Victoria, British Columbia). The Contest does not involve mathematics beyond the level of what is usually called "pre-Calculus Mathematics"; thus, it does not involve Calculus, Vectors and Matrices, Probability, and topics of this nature.

The list of nominees for the Province of Ontario will be sent to the Department of Mathematics at the University of Victoria early in May. In cases when an invitation is extended to Ontario students to write the examination, the examination will be mailed to the supervising teacher for each student early in May of 1969. The examination will be written on Wednesday, May 21, 1969, and will be of three hours duration. The schools are being asked to schedule it from 9.00 a.m. to 12.00 a.m. It is hoped that this will not cause any problems, since it will be exceptionally rare if more than one or two students from any particular school are invited to write (the quota for our Province being not more than 50).

The supervising teacher must mail the completed examination booklet in a stamped, addressed envelope which will be provided for that purpose. Mailing must take place within 24 hours of the date of the examination, namely by May22, 1969.

The marking of the examination will be completed at the University of Victoria by May 30, 1969, and the results will be announced as soon as possible. Papers which arrive after May 30 will not be eligible for awards.

Ontario students who are being invited to participate in this First Canadian Mathematical Olympiad will be informed of the fact as soon as possible after the results of the 1969 M.A.A. Contest for Ontario are available.

CANADIAN MATHEMATICAL CONGRESS

MATHEMATICAL BOOK LIST - LISTE DE LIVRES DE MATHEMATIQUES

These bibliographies were prepared by the members of the Mathematical Education Committee of the Canadian Mathematical Congress. They are strongly recommended for the purpose of helping high school students and teachers who are forming mathematical reading lists or libraries or for students' leisure reading.

The book lists are available in both French and English. However, the French and English are quite different - one is not the translation of the other!

Copies of the list may be obtained from the SECRETARIAT at nine cents a copy plus postage (postage on a single copy is five cents)

CANADIAN MATHEMATICAL CONGRESS
SECRETARIAT: 985 SHERBROOKE STREET WEST,
MONTREAL.

WHY MATHEMATICS*

by C. Lanczos

Dublin Institute for Advanced Studies,
Ireland

It seems a weatherbeaten subject and yet has something fascinating about it because it has many facets and can be illuminated from many angles. Why Mathematics? One can study a subject for a long time without asking any questions, but sooner or later the question of motivation is bound to crop up. When I went to school, the parents and teachers urged us lazy-bones to be particularly diligent with our homework in mathematics, on the pretext that it'sharpens the mind'. Whatever this word meant, it had good propaganda value. Many years later, when I was teaching at Purdue University in the town of Lafayette, I came in touch with many excellent people in all walks of life, - lawyers, doctors, ministers, business men - all people, to whom the attribute 'sharp mind' seemed to apply. I made a kind of inconspicous correlation study by trying to find out what benefit they had from their early mathematical training. In the majority of cases I got answers like: 'Mathematics? I dreaded it. I still remember in nightmares the fruitless hours spent on my home work. How I navigated through without disaster, is still a miracle to me'. I came to the conclusion that the 'sharp mind' hypothesis may be slightly exaggerated.

The famous Academy of Plato in Athens had the celebrated inscription on its doors: 'No one ignorant of geometry may enter here'. Had we asked an educated Greek of that period, why Plato has chosen this slogan, he would have had no difficulty in explaining the motivation. Geometry was simply the

*Reprinted by kind permission of the Editor Newsletter of the Irish Mathematics Teachers' Association, Dublin, Ireland

science, which showed us how to make statements about the objective order of things, how to find out something about the world in which we live, and generally, how to discipline our mind and get an education, which goes beyond the narrow confines of our primitive desires.

If today an educational institution put up a similar sign, it would have to do considerable explaining what in fact is meant. First of all, the term 'geometry' would require explaining. In the two thousand years which have elapsed since Plato, our social and scientific outlook has changed beyond measure. In Plato's time to be 'educated' has a rather well-circumscribed meaning. The State was composed of the citizens of Athens, essentially a middle-class society, augmented by the slaves who did all the menial tasks. We have put away with slavery (at least on paper). We have today a highly stratified society, with extreme specialisation. We face therefore a much more complex situation. The little boy who asked Euclid what he will gain, if he memorises all these lemmas and corollaries and theorems of geometry, elicited the laughter of Euclid, who called the janitor of the school and directed him to give the boy a penny, 'because', he said, 'he wants to get out something from the study of geometry'. We cannot laugh if such a question is asked. We must be able to answer the student, who inquires about the advantages and disadvantages of taking mathematics as a subject. And even if we dislike the too pragmatic question: 'What do I gain by the study of this subject?', the student of today will sooner or later ask this question, and we must be able to answer him.

The new feature of the present day situation is that mathematics is no longer a mere subject of education, but a regular profession, in fact a highly recognized profession. As a person may become a teacher, or an engineer, or a physician, he may also become a mathematician. Up to our days mathematics was pursued as a free play of the mind, as a form of liberal art, whereas today it is a regular bread and butter occupation. This change affected our educational system profoundly. The new phenomenon is the evolution of pressure groups who with great noise and persuasiveness try to modify or transform the educational aims which existed even a generation ago.

'Euclid must go' is the famous (or infamous) title, under which one of the leading advocates of the so-called 'modern school of mathematics' attacked the traditional school curriculum. Here it was no longer a question of re-fashioning university education on the advanced level. What the universities do on their graduate program, is their internal problem. They can establish any number of modern courses if they feel that the professional mathematician should get an up-to-date education in matters that certain leading figures of the profession consider as of primary importance. But the aggressive talk entitled 'Euclid must go' is a new phenomenon, because here a professional mathematician wants to go down to the grassroots and take a hand in shaping the policies of school education which up to now was not influenced by the clamor of modernism, being based on values which held out for hundreds of years.

Mathematics as a school subject was for hundreds of years an essentially cultural subject; the student studied mathematics because it played such a decisive role in that scientific and technological evolution,

that has changed our patterns of life so profoundly. It was not a question that every important discovery of the last three hundred years should of necessity be included. In fact the elementary and secondary schools considered it their main task to introduce the student to the general principles of quantitative and abstract thinking by showing the power of demonstration and developing the principles of inductive and deductive reasoning.

Allow me to quote a great name, who received a completely traditional education in mathematics, which nevertheless impressed him profoundly. I would like to quote a few paragraphs from the Autobiographical Notes, with which the great physicist Einstein enriched the volume published in honour of his seventieth birthday *). Here is how he remembers one of his early intellectual experiences (following the wondrous observations with a compass, which he once received as a present from his father):

> 'At the age of 12 I experienced a second wonder of a totally different nature: in a little book dealing with Euclidian plane geometry which came into my hands at the beginning of a school year. Here were assertions, as for example the intersection of the three altitudes of a triangle in one point, which - by no means evident - could nevertheless be proved with such certainty that any doubt appeared to be out of question. This lucidity and certainty made an indescribable impression upon me. That the axioms had to be accepted unproved, did not disturb me. Generally I was perfectly

*) Albert Einstein: Philosopher-Scientist
edited by P.A.Schilipp, (Library of Living Philosophers
Evanston, III., 1949)

satisfied if I could make the deductions from propositions, the validity appeared evident to me. For example, I remember that an uncle of mine told me about the Pythagorean theorem before the holy geometry booklet had come in my hands. After much effort I succeeded in 'proving' this theorem, on the basis of the similarity of triangles; in doing so it seemed 'evident' to me that the ratios of the sides of the right-angled triangle would have to be completely determined by one of the acute angles. Only something which did not in similar fashion seem to be 'evident',appeared to me in need of any proof at all.

It thus appeared that it was possible to get certain knowledge of the objects of experience by means of thinking, this 'miracle' rested on an error. Nevertheless, for anyone who experiences it for the first time, it is marvellous enough that man is capable at all to reach such degree of certainty and purity in mere thinking that the Greeks demonstrated to us for the first time in their geometry'.

This statment of Einstein is interesting from more than one viewpoint. He refers to the logical 'error' to believe that certain postulates are self-evident. But this error is of minor significance compared with the tremendous feat that our mind, after accepting certain apparently evident postulates, can from then on by deductive reasoning arrive at results which are in agreement with the facts of the objective

world. It was Einstein who demonstrated that the geometry of nature does not follow the postulates of Euclidean geometry. Yet he would have been the last to advocate that 'Euclid must go', because he considered the experience of the miracle that our mind is able to explore the world of objective things, more important by far than the changing patterns of our physical or mathematical theories. What we know, or think we know, about the world of reality, is constantly changing but what remains unchanged is the conviction that the world of natural phenomena is governed by exact and inexorable laws, which can be formulated through the language of mathematics. It was Einstein, who condensed his scientific credo in the words: 'The most incomprehensible thing about the world is that it is comprehensible.' The last years of Einstein's life were spent in incessant ponderings on the all-comprehensive basis of the physical world, the 'word law' which should include everything, in a grandiose amplification of the postulates established by Euclid. Is it not possible that Einstein would have never departed from the pragmatic trend of his younger years, (leaving his monumental 'Theory of General Relativity' undiscovered), had he not been vitally influenced by the deductive feature of Greek Geometry? NB

We can quote a similar experience, attributed to Pascal, the famous philiosopher and mathematician, who was about the same age, when a textbook of geometry came in his hand, which gave a bare outline of the elementary aspects of the subject. The young boy filled in all the details and in a short time was not only on the frontier, but in his sixteenth year presented to the French Academy a new and fundamental theorem in projective geometry, of exceptional ingenuity.

Could we imagine that if Euclid's postulates have been replaced by the much more perfect 'Foundations of Geometry' by Hilbert, these budding geniuses would have drawn the same inspiration for their soaring imagination? I once attended a refresher course for secondary school teachers, in which the organiser chose as reading material Hilbert's 'Foundations of Geometry'. During the course he once asked the participants, how many of them could draw inspiration from this set of postulates; not a single hand went up. Undoubtedly here is a neat and watertight package which leaves no loopholes. Take it out in a thousand years from the mothballs, and it is still perfect. But the mechanism becomes so heavy that the technical details smother the eagle flight of the imagination. Somehow the inherent magic peters out of the subject, by making it too soberly logical. Everything is neatly packed away and unassailable, but it is so surgically sterilised that all life departs from it. I am reminded of those spick-and-span hotels in America, where you find cellophane wrappers around every drinking glass, with the inscription: 'It's sanitized', and there is a strip across the toilet seat: 'Sanitized, - for your protection', which is all very fine, but then I think of little children playing in the Dublin back streets, they are not sanitized, not even overly clean, and certainly not surgically sterilized, the play is sometimes rough and tumble, but if one of them falls and gets hurt, you don't have to run immediately to the doctor to give him an anti-tetanus injection. And they enjoy themselves tremendously, whereas their counterparts in the sanitized glass boxes sit around neatly with a wan smile on their lips and if you ask them: 'How do you do?' they answer 'Thank you Sir, I am fine'. Hilbert's axiomatic system is certainly sanitized, while the

Euclidean postulates are full of holes and the bacteria swarm around unchecked. But is it not astonishing that nevertheless in all the hundreds of years in which people worked in Euclidean geometry, not a single serious error was made?

I still remember the fascination I got from my second year algebra course, when our home problem was: 'At what moment will the two hands of a watch coincide after four o'clock?' It was a problem taken from actual life. One could ask Daddy to take out his watch and make the actual experiment, bringing the two hands together after 4 o'clock. But how proud one felt that one's answer, of course, had absolute accuracy, whereas the actual reading would hardly go beyond one quarter of a minute. Moreoever, the answer was obtained by mathematical reasoning and thus had to be correct, the physical experiment served merely to check up on the answer. And there were so many similar instances, whereas the problems constructed in point sets and intersections and unions have an anaemic quality, because they seem far removed from reality. To sever mathematics from reality and make it to a purely logical game, may appeal to some students but in all probability would suffer greatly by its apparent lack of red blood cells.

Of course, the teacher is frequently handicapped by a given syllabus which has to be followed, whether he likes it or not. But even then it is necessary that he should be aware of the inherent limitations of his subject and see things in the proper perspective. Certainly he should not lose track of the difference which exists between the professional and the cultural aspects of mathematics. The young mathematician, who wants to go out in life and claims that he has no chance of succeeding as he is not familiar with the slang of modern

mathematics since otherwise he cannot write papers and cannot succeed in his academic career, has certainly a valid point to his defence. But the question is, whether the professional motivation should enter on an educational level, which caters to a much larger group of students.

During a recent scientific colloquium in a provincial university of France I had a chance to investigate, what happened to the great French analysts of the last century, Darboux, Picard, Jordan,and others. Who are their successors? The information received was that the field of classical analysis is entirely deserted, they just have no successors. I inquired whether the Bourbaki people (the advocates of the modern trend) are so particularly popular in France. The answer was, that there are plenty of people in France who dislike them but unfortunately the scientific life of France is highly centralized and it so happens that a Bourbaki man is appointed to all the leading positions, and one who is not in line with them is tucked away in some provincial institution, where he cannot do much harm. Under such circumstances we can well imagine that the traditionlists have little chance and even the elementary school curriculum will be affected. Similar conditions probably exist in the United States.

In the meantime we may well ask, whether our set of values has really changed so radically since the time of Plato. Certainly Einstein's inspiration from the study of geometry was in no way different from the inspiration of the great Greek geometers who were likewise enthused by the possibility of understanding the laws of the physical world through our mind. The emphasis here is on the objective world order which we try to understand through the constructions of mathematics, rather than seeing

in mathematics a logical exercise of the human mind and nothing else. The advocates of modern mathematics argue that by making mathematics to a logical game they can include much more than the world of physical reality, which may represent a rather special, perhaps not even very interesting form of mathematics, characterised by too great normality. They can also point to the fact that the number of mathematicians living today by far outstrips the total number of mathematicians who lived through all the ages, up to our times. Furthermore, they can add the sad fact that the volume of mathematical papers published in the last fifteen years surpasses the entire output of mathematical literature up to that time.

The naive optimist, who believes in 'progress' and is convinced that today is better than yesterday and in ten years time the world will be infintely better off than it is today,will come to the conclusion that mathematics (and more generally all the exact sciences) started only about twenty years ago, while all the predecessors must have walked around in a kind of limbo of half-digested and improperly conceived ideas.

On the other side of the ledger is the curious remark made once more by Einstein in the same 'Autobiographical Notes'. He tells us that after graduation from the university he was so fed up with learning that for a full year he did not take a scientific book in his hand. What he found so revolting was that one had to take so many courses in which one was not interested. Today the same compulsion is infinitely heavier and there is little doubt that a genius of the type of Einstein would be out in the cold, under the present climate of scientific education. The false pretense for ever heavier school schedules fails to recognise that the human brain is of limited capacity and unable to digest adequately more

than its share. In a recent comment on mathematical preparation an educator wanted to characterise our backwardness by the following statement: 'Is it not astonishing that a person graduating in mathematics today knows hardly more than what Euler knew already at the end of the eighteenth century?'. On its face value this sounds as a convincing argument. Yet it misses the point completely. Personally I would not hesitate not only to graduate with first class honours, but to give the Ph.D. (and with 'summa cum laude'), without asking any further questions, to anybody who knew only one quarter of what Euler knew, provided that he knew it in the way in which Euler knew it. To harass the student with constantly increasing demands to his memory will hardly solve any problems. The important thing is not to exercise his memory, but rather to develop enthusiasm for his subject and a thorough understanding of the many channels, through which the human mind attempted to gain gradual clarity in its struggle toward the absolute.

Once more I would like to return to the Greeks. They coined a remarkable word, which is full or portent and implication: 'Cosmos'. With the word 'Cosmos' they wanted to designate the entire universe, but as a well-ordered and fundamentally coherent entity, which is founded on eternal and unalterable laws. This is in fact a tremendous idea, which can give us consolation. The existentialists claim that when we are born, we are thrown into a cold world,in which the individual is lost and his problem is to find his place in a vast expanse, which is indifferent to his woes, and can offer him no hope that he will ever get anywhere. This feeling of hopelessness does not hold, if we accept

the Greek idea of the Cosmos, in which the individual is not lost, because he is part of a grand design, in which every part fits the whole and belongs to it with inevitable necessity. If we think of it, is not this idea of the Cosmos the basis of our entire civilization, not only of the sciences but of the arts as well, and even of religion? We may express the same idea in a hundred different languages, in a hundred different shapes, but deep down we feel that there is something wonderful in the fact that we do not live in a haphazard world, in which things happen by blind accident and meaningless chance, but in a world created according to some marvellously intricate but basically coherent scheme.

Since the early days of the Greek nature philosophers the human mind never lost its desire to come to terms with the mysteries of the universe. And the peculiar fact holds that from the beginning it was the quantitative aspect of nature, expressible in the symbol world of mathematics, which gave the greatest promise that some day we may succeed. The ancient Greeks erected an edifice which has served as a model of an exact science ever since. When Newton erected his own system of natural philosophy, he followed the patterns of Euclidean geometry. Basically the dream remained the same. We try to understand the basic principles of the physical world. The Greeks considered geometry as a divine occupation because God manifested His spirit in the laws of geometry. Newton discovered a set of equations which had a divine quality, because it seemed that they include everything. The veneration in which Newton was held, was based on the assumption that his equations

are the final equations of the universe, being the correct formulation of the laws of mechanics, coupled with the belief that in the final analysis everything is reducible to mechanics. We see that the dream of the old Einstein to create a 'unified field theory' had its predecessors. Einstein returned once more to the geometrical method of the Greeks, but now experimenting with a much more advanced mathematical model of geometry than Euclid. And even in contemporary physics many outstanding physicists believe that the bewildering variety of elementary particles shows a remarkable pattern, which indicates a basic unity. Whatever the final picture may become, we have no doubts that it will not be conceivable without a great deal of high-powered mathematics.

But parallel to this 'cosmic orientation' we observe in our days a peculiarly different development. The emphasis has shifted from the universal to the individual, from the 'it' to the 'I'. As an eminent physicist of recent times expressed it: 'We are not interested in the universe but in the game that we play with the universe'. The dangers associated with such a situation are obvious. The agressiveness of our historical era is perhaps a direct consequence of this shift of emphasis. It is well possible that the exaggerated importance given to modern mathematics is in itself a mere symptom of the general pattern. If one reads the Nobel lectures given by recent Nobel Laureates and compares them with those of earlier days, one is struck by a peculiar change in tone. How different in the majority of cases is the attitude of these lectures now, compared with the attitude that prevailed in the twenties, when men like Laue, Planck, or Einstein gave an account of their work.

How much more modesty was at home then, how little boasting of one's own importance. Today the emphasis is on the wonderful things that 'I' have accomplished.

I cannot help thinking that the attempted new curriculum in the field of 'modern mathematics' will contribute to an alienation from the cosmic viewpoint and still augment the already high tensions which can be observed in our time, caused by the constantly increasing demands of an egocentered population that has lost sight of the eternal and inviolable laws, to which we all are of necessity submitted. By emphasising the pure logic of modern mathematics, the human mind withdraws into itself and loses touch with that fundamental substratum that the Bible expresses with the magnificent words:

'Underneath are the everlasting arms' ; (Deut. 33:27)

There is no question that in view of the vastly expanded technology the applications of mathematics have multiplied. But the fields of applied mathematics remained in close contact with the traditional edifice, which is strong enough to serve as the basic substructure. Naturally, the methods of mathematics continue to develop and it would be foolish to deny our epoch the privilege of important new contributions. But to hail our times as the originator of an entirely new science, which need not bother with the past and has the right to construct everything from scratch, betrays a dangerous short-sightedness which can lead to a dissolution of mathematical research into an empty play with words.

If we are satisified that mathematics should be pursued as a mere logical game, similar to chess or bridge, then Modern Mathematics can serve as an adequate groundwork. But if we accept the thesis, demonstrated

by the historical evolution, that mathematics is the inevitable foundation of the natural sciences, then we have to look further. Nature does not confront us with point sets but with quantities. And inexorably, sooner or later, the problem of the continuum will crop up, with which the human mind grappled for thousands of years. Should the student simply ignore these age-old struggles and jump without precedents to a new way of thinking, which has now allegedly solved all difficulties? We can only pray that such a state of finality will never occur. We will go on pondering and meditating, the great mysteries still ahead of us, we will continue to err and stumble on the way, and if we win a little victory, we will be jubilant and thankful, without claiming, however, that we have done something that can eliminate the contribution of all the millenia before us.