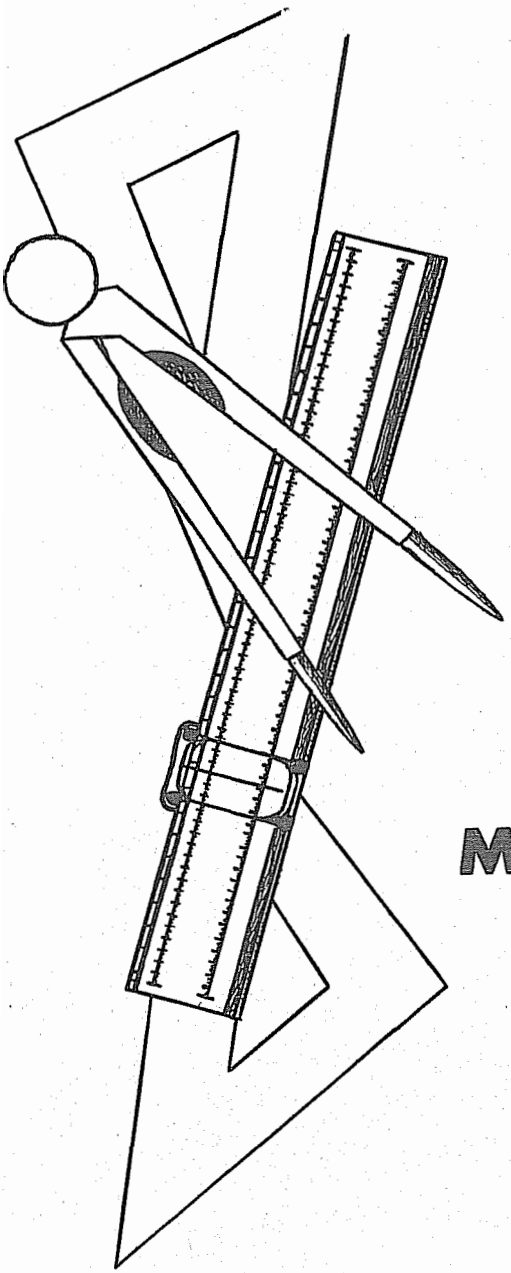


VOLUME 8
NUMBER 1
NOVEMBER 1969

ONTARIO MATHEMATICS GAZETTE

sponsored by
THE SAMUEL BEATTY FUND
THE ONTARIO MATHEMATICS COMMISSION
THE ONTARIO ASSOCIATION OF
TEACHERS OF MATHEMATICS

ONTARIO MATHEMATICS GAZETTE

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EDITORIAL

This issue of the Gazette is the last one to appear in the decade 1960-69. It is therefore fitting to have as a central article a written version of the lecture given by Dr. George Duff at the Tenth Anniversary Meeting of the Ontario Mathematics Commission.

Ontario has came a long way in school mathematics in the last ten years. The initial revamping of the high school curriculum to be modern in content and point of view was followed by a focus of attention on the elementary and less academic students, with increasing emphasis on the need to consider not just relevant content but appropriate ways of communicating it. At the same time, other more general forces have been operating on the school system, which have resulted in considerable experimentation with school organization, timetabling etc., in a general move towards the so-called ungraded school and individualization of instruction.

As Dr. Duff points out, the advent of the ungraded school program poses special problems for mathematics teaching because of the highly sequential nature of learning in mathematics. In his view, with the availability of much more varied textual and other relevant materials, the mathematics teacher, operating now in a much freer administrative and organisational atmosphere, will now need to exercise much greater professional

judgment and initiative. He sees the Ontario Mathematics Commission as having both the capacity and the responsibility of helping teachers cope with these emerging challenges.

Two of the articles in the present issue seem to serve such purposes. One deals with an experiment now being made in a particular high school to provide a variety of programs with considerable responsibility for learning devolving on the student. The other represents an attempt to place the approach to learning known as "The Mathematics Workshop" in proper perspective. To me, both are significant contributions to constructive thinking about school mathematics, and should be read and pondered over.

Also in this issue are two interesting articles, which show the outward and inward faces of mathematics! As the article on tangency indicates, one can find mathematical problems to solve in many engineering situations. One of the harder tasks confronting the mathematics teacher is to foster the ability of students to identify as well as solve such problems. Here is a fertile field for individual or group projects in applied mathematics for the senior high school student.

The second article of the pair discusses the distinctions between "the rationals" and "the reals". The exposition is clear and forceful, and reading of the article by senior students is highly recommended both as an overview exercise in helping them see the significance of the distinction and also as a stimulating introduction to careful thinking in analysis.

We who try to serve you, the classroom teacher by bringing to your notice what we think are important issues for you to reflect on, need your support and encouragement just as much in the seventies as in the sixties. Please let us have your comments, reactions and suggestions so that we can work better together to improve the learning and teaching of mathematics in schools.

Merry Christmas, and a Happy entry into 1970.

D. H. C.

"HERE COME THE SEVENTIES"*

by

Professor G.F.D. Duff,
Head, Department of Mathematics,
University of Toronto

As the Ontario Mathematics Commission is now celebrating its tenth anniversary, it is appropriate to review its affairs and I would therefore like to speak of the past history and the future prospects of the Commission.

During the last decade there has been a very profound transformation of the nature of school mathematics. This new movement in school mathematics had its origin about fifteen years ago, although there had been even before that some pioneers whose voices had gone unheard. About 1955 one became aware that there were a number of new approaches to the teaching of certain parts of mathematics in schools, particularly to the introduction of such topics as Set Theory. These were being introduced in a number of places, not actually in Ontario, but chiefly in the United States and Europe. The new movement gathered speed within a fairly short time so that by 1958 or 1959 many people were aware that a time of change was impending. Now, in Ontario there had been a mathematics curriculum that had not, it seemed, undergone any substantial adjustment for several decades. It is true there had been changes of detail, many of which were in the nature of removing certain topics from the curriculum because they were no longer carrying their weight. These were not, however, always replaced by new topics

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* A lecture on school mathematics delivered to the tenth anniversry meeting of the Ontario Mathematics Commission September 20, 1969.

which were of more contemporary interest. Indeed there was a situation at that time that was entirely ready for change.

The Ontario Mathematics Commission had its beginning in 1959 when a group of teachers and university mathematicians including Father Faught, who is here today, and Dr. Coleman and the late Howard Mulligan, persuaded the Ontario Teachers' Federation to reactivate an old committee that they had lying handy and to convene a special committee on high school mathematics. This resulted in the conference held at Lakefield ten years ago in August 1959, and shortly afterwards the Mathematics Commission was established. So I would date the effective beginning of the Commission as in 1959 and that means that it has now existed for ten years and it has covered the period of the sixties.

We have here today a number of persons who have served as Presidents of the commission: Father Faught, Dr. Coleman, Mr.Norman Sharp, Professor Harris, Dr. Elliott whom I have not seen yet though I understand he will be here today. It seems to me interesting and remarkable that the OMC has throughout this ten year period succeeded in mobilizing a combined group of people who are interested in school mathematics and who themselves come from a wide variety of educational institutions. They have come, in fact, from the Teachers' Colleges, and the Colleges of Education, from Colleges of Applied Arts and Technology and from Universities. From my own experience with the Commission, I would say that the fruitful work of the Commission has been done as a result of very genuine and effective interaction of these groups of people in the working committees that were established. In Ontario we have a group of people, taking them as a part of the mathematics teaching professions as a whole, who are capable of continuing this activity at a level which

should keep Ontario in a strong position, well in the forefront of mathematical education.

The problems that the Commission faces now are twofold: first, they are financial as was indicated at the business meeting. I do not myself believe that the financial problems are the most essential here. It is rather another aspect of the situation which brings the Commission now to a "point of inflection", really a point of looking around to see what is the proper course to chart now. This is that there has been a dramatic change in school mathematics that is now well established in the schools and the question is "what comes next?" May I remind you that the new curriculum in the high schools in Ontario began about 1962 in grade 9 and worked upward and has been in effect in grade 13 for some three or four years now. There has been a second wave of curricular revision in the secondary schools and there have been very substantial changes both instituted and impending in the elementary school mathematics program. It takes ten or tweleve years for a revision of the school curriculum to trace its way upwards from kindergarten to grade 13, perhaps I should say it takes 14 years, but in fact I think it takes less, because as has been seen in this process, the system jumped the gun and the changes were instituted simultaneously at several levels. However, there is now a situation in which many teachers have been preoccupied for several years with adjusting to the new circumstances an adjustment which for many was extremely difficult, without any doubt, difficult and arduous.

One finds it very hard to recapture or go back to the frame of mind that we were all in say ten years ago and see how great the curricular changes have been during that time. There is now an atmos-

phere of much greater freedom and flexibility, not only in the teaching of mathematics, but of course in other school subjects as well. The ungraded school program is on its way in and this poses very special problems for mathematics teaching. Indeed, it seems to me, that it poses problems of such a nature that the Commission will have a very important task in clarifying and elucidating some of the resulting problems from the viewpoint of the mathematical profession.

Throughout these ten years the main financial support of the Commission has come from the Ontario Teachers' Federation. We have been told today the rather ominous news that the Teachers' Federation may reconsider the magnitude of their financial support for the Commission. Although it is not surprising that they may reconsider, I hope the Teachers' Federation does not greatly reduce the grant that they have made to the Commission. The Teachers' Federation has shown an outstanding example of professional leadership in its support of the Mathematics Commission. One might perhaps ask why "mathematics" as, apart from the Commission, the Teachers' Federation does support the Ontario Education Research Council, but not other subject areas specifically, and yet mathematics certainly led the way during the last few years in bringing in very basic and important changes. We now confront problems which are more acute in mathematics, it seems to me, than they are in any other subject. These problems arise, of course, from the highly sequential nature of learning in mathematics.

At the beginning of the period I have mentioned, the situation for courses in school mathematics was that the Ontario Department of Education set the curricula which were interpreted by inspectors and by a small number of other persons in the Department and they were

administered more or less uniformly in the schools. There has been a very substantial change in this administrative picture. The new atmosphere of flexibility, freedom and innovation has extended, even in the formal sense, to many of the curricula, the number of optional topics is very much greater, there is in some sense of the advanced courses, an almost complete freedom for teachers to design the course that they will give. The Department of Education no longer assigns the same priority to the keeping-up of the year to year advancement of these changes in the second wave of the reform in the high schools. If I remember correctly, the grades 11 and 12 curricula for this second change are now overdue by about two years and there is a gap or interpolation between grades 10 and 13 which does in fact call for a teacher to supply considerable independent interpretation. Now if this were an administrative oversight of the Department it would be regrettable. But I think it is in fact an adjustment to circumstances. There is some danger in attempting to prescribe curricula in mathematics too precisely any longer. The opportunities for giving specially designed work to particular students, or particular groups of students, is now much greater than before. There is now a much larger supply of teaching materials, textbooks and other sources of information about school mathematics available to a teacher, and hence a much wider domain from which to choose his material. He has, or will have, an atmosphere of tailoring his courses to his students and to particular parts of his class to the greatest possible extent. This has always been true to a limited extent but I think it will be even more true in future and therefore he will be called upon to exercise much greater professional judgment and initiative. At the same time it is difficult for a large administrative structure to provide him with precise up-to-date course outlines. It seems to me that in this fact lies the greatest service that the Commission can

render in the next years, namely, in assembling and making available information on relevant changes in school mathematics programs.

Such catalogues need not be in the precise form of curricula but in the sense "here is a set of things so organized that it will be relatively easy for particular choices to be made from it". They may be a set of courses at slightly different levels, some of which may be suitable in one circumstance, and some in another. I am quite conscious of this hierarchial aspect of mathematics because of the changes that have just been introduced at the University of Toronto in the undergraduate program. For many years we were very conservative on the matter of allowing flexibility to students in the choice of their courses. Indeed, I personally felt for many years that this was unfortunate in general and especially unfortunate in mathematics, because a number of students got trapped between the honour and general levels. We now have a program of several different levels of courses and students may transfer from one to another of these either downward or upward with the greatest possible freedom. The next year or two will show us how well this works. I believe that it will result in a very much better experience for a large number of intermediate students. I think we in the University of Toronto at least, were behind the schools in this particular respect up to the present moment. I hope we are now up with them and have temporarily gone a pace ahead but I am sure that similar types of flexibility are going to be very important over the forthcoming years. Such programs require considerable delicacy of choice in the selection and presentation of classroom mathematics work.

In the Commission there has not yet been a full adaptation

to the situation envisioned by the Hall-Dennis Report because we have been preoccupied with pointing out the one feature of that Report which carries with it a great peril for mathematics. You cannot learn mathematics entirely by inspiration - you have to put in some perspiration! You have to do things in a certain order. I hope we have made this point clear and will now be able to go on to the next step which is how to interpret their recommendations about course material so as to give a student the greatest possible number choices as he goes through his program from year to year. To maintain curricular tables which permit such freedom of choice, would be, as I have indicated, a very worthy task for the Commission. Such tables or outline would be in the nature of advice or suggestion, they would not be formal curricula. The original recommendations made to the Department of Education some years ago by the Commission, were of the same kind and they were adopted in a remarkably short time.

However, we have to face the fact that the "great leap forward" in school mathematics was a result of a number of circumstances that are not repeated today. The importance of education in technology, the space competition, the over-ripeness of the earlier curriculum in mathematics and the great pressure from professional mathematicians enhanced the impulse for the adoption of a larger amount of mathematics having 20th century spirit in the school curriculum. These conditions are not present today to the same extent that they were some years ago.

However, it is time that we took another look to see whether there should be another substantial round of changes. This may be obscured for the next little while by the effects that accompany the current move of student unrest and the "liberation" of the school curriculum in

general. Perhaps we even have to face the fact that we have gone so far in loosening the administrative restraints in teaching school mathematics that we should now decide where to stop. The work of setting up curricular tables or systems of choices, which the Commission I think alone in Ontario is in a position to do without any administrative restraint, will show us where that point is.

There will also be some technical changes in school mathematics that ought to come in within the next few years. I shall not attempt to outline these in any comprehensive way, but just mention two that I regard as significant. I think that one of the most important of these, already well underway, lies in bringing about a closer relationship between school mathematics and computer science. Computer programming and simple student computer languages are already being taught in a number of classes. One can foresee that eventually there will be a kind of merger of standard mathematical notation and computer programming language notation. This should probably come about in the time of our present students and already substantial steps have been taken in this direction. A corollary of this surely is that the teaching of arithmetic, algebra and computer programming itself in school should be adapted to such an ultimate merger from the very earliest and most elementary level. I think that the computer language teaching situation is not yet in balance with the needs of society. There was for example a recent discussion in the public press of the fact that the universities are all teaching Fortran while the jobs are in business and industry where they need COBOL. I think it is up to an organization like the Mathematics Commission to try and foresee situations like that and to work towards their resolution. While I am on this point, I would like

to say one more thing about the subject of computer science which should have such a close relationship with school mathematics. One suspects that the computer program work will turn out to be the hardest taskmaster or disciplinarian that mathematics students have ever had. In the past we have complained a great deal about the fact that students could not do arithmetic, that they could not do algebra, and that they could not do this, that or the other, accurately. I think that much of the burden of correction and criticism in this respect will, in future, be carried by the computer itself,and I hope that teachers will feel that this is a constructive development.

Another topic that has appeared recently in advanced mathematics that may before long be reflected in school mathematics, is that of mathematical linguistics. I myself do not have a precise notion of how this is going to develop, but I think it is in the cards that soon it will be exerting definite influence on school mathematics programs. Again, there will be a definite issue of designing these programs to enhance the training of a student in both mathematics and in the use of language.

These developments reflect a very basic change in the way in which mathematics is being used in society. This change is certainly apparent in the sciences. Ten or twenty years ago the big customer was physical science, particularly physics and chemistry. However, there has been more recently a definite switch of emphasis towards the social sciences and even the humanities. We are now, in university, training substantial numbers of students who intend to go into the social sciences in high quality mathematics courses. It impressed me a great deal that students coming into university, and I presume therefore also students

in high schools, seemed to have sensed that in the world of the near future, these subjects are going to have an increasing importance. This is perhaps a very natural response to the overcrowded conditions of our contemporary world - the issues of social organizations and their adjustment and reaction to contemporary conditions. It means that the nature of mathematics will be influenced too and that the mathematics of organizational complexity will take a more prominent position. Both computer science in its practical and theoretical aspects, and mathematical linguistics are instances of this tendency.

In Canada as a whole we have a good start on such developments. It is perhaps worth pointing out that in the past, both in Ontario and in Canada, in the recent past, we have been emphasizing what is called pure research work to a much greater extent than in other Western Countries. Approximately two-thirds of the research work done in Canada is on pure science and only one-third on applied science, whereas in Britain, United States and Europe, the reverse ratio is the case. Perhaps we in Ontario should be conscious of this because I think we have come to depend upon these other countries for our applied science and have perhaps therefore tended to give primary emphasis to pure mathematics, thus perhaps we have over-emphasized it. At the same time, because of the changing nature of mathematics and the development of these new disciplines and applications, the topics that are going to be thought of as applied mathematics or useful mathematics, applicable mathematics, in the future, are going to be very different from those which have been so used in the past.

Now I would like to say a word about the possible ways in which I think the Commission may adjust itself to these changing circum-

stances. One can foresee a situation in which the Department of Education is taking, as I said, a less prominent role in curriculum development and especially in the detailed supervision of teaching. The new county boards will, on the other hand, be taking a more prominent role and the coordinators of mathematics in the larger city boards and those who will be doing this work in the county boards, will undoubtedly have a very influential position in the school mathematics picture in Ontario. Indeed, I think their initiative will probably de decisive for actual implementation of courses and the Commission must, I think, adapt itself to working very closely with this group. Now another direction in which the Commission can have and does have some influence is through the Ontario Association of Teachers of Mathematics. I foresee three possible ways in which, if the financial picture does not improve, the Commission might adjust itself. One is by forming a closer association with the OATM. Already these two organizations are partners in the publication of the Gazette and it seems to me that together there would still be the organizational basis for the task that the Commission ought to continue doing.

Another possible direction which ought to be considered, is that the Commission should enlarge its horizons from the provincial level to the national level. There is an Association of Mathematics Teachers of Canada formed two or three years ago, which at the present time is like the Commission in the sense that it is a federation of organizations. It does not have individual members but is rather a federation, in which respect it is like the Commission, and it may be that one resolution of the situation we are entering now and one which could be of considerable service elsewhere than in Ontario, would be for a closer relationship with the CAMT.

There is a third possibility which I mentioned before in the Commission meetings and which is not excluded by either of these other two. In mathematics education, we have through the Commission established an organization in which the university mathematics and the school teachers in mathematics have been able to work together effectively. However, there are still other groups of people who have a strong interest in mathematics, namely, the commercial, industrial and financial mathematicians. The list of these is really quite impressive. Canada,is, for example, the home of one of the largest actuarial associations in the world, there is a very large computing and data processing society, a vigorous operational research society, and there are other groups in industry as well. There is about to be formed an Applied Mechanics Association which has some contacts with the engineering profession. When these groups are taken all together they make just as much a part of the overall mathematical fraternity as the educational groups. Perhaps it is time that some organization took the initiative of forming a federation in which the various needs of these groups are reflected. I think that the Mathematics Commission is in as good a position to do this at the present time as any organization. The structure that might in this way be formed would resemble the Institute of Mathematics and Its Applications that was established in Great Britain some five or six years ago. It is an organization in which all these groups, including particularly the educational group, have a part. I am not sure that Ontario is ready for an organization like this, though I believe that perhaps Canada is and I am perhaps hoping that some of the speakers later today will give their views on this issue. I think we are moving into a time when the effective use of mathematics in the world at large, is going to be just as important, perhaps even somewhat more

important to us, than it has been in the past, and this is how as a professional group we ought to respond to that situation.

One other aspect of the development of mathematics as a whole that I would like to mention now is that I think there is going to be, perhaps is already, a moving together of the nature of school mathematics teaching and university mathematics teaching. Many of you have, no doubt, some experience and knowledge of the student movement in universities in which not only the things that are taught but the way in which they are taught, are coming under scrutiny. I think the university mathematicians are about to be marked for their school teaching prowess and I think that on the other hand that the teachers not only are, but have been, under a very substantial influence to update their knowledge of mathematics and the contemporary happenings in the school mathematics world. Thus, the thrust of these two forces is drawing these two groups closer together. I think the Mathematics Commission has been well in the forefront of this: I certainly have found the work of the Commission to be of great interest and significance, not only in school mathematics, but also for the work that I have to do in my own job and I think that as an organization, the Commission should take some pride in this fact. I think it has been in advance of the times in this respect and believe that if it, as a group, is able to react to some of these changes that we have been discussing, it may continue in advance of the times and give leadership in the school mathematics world.

A NEW PROGRAM IN SCHOOL MATHEMATICS*

by
Bob Wigle,
Head, Department of Mathematics,
Aldershot High School

In September, 1969, the Mathematics Department in Aldershot High School introduced a new program in Mathematics. The program is designed to recognize and accommodate the wide ranges of learning rates, of interests, and of academic goals found among High School students today. It is hoped, too, that the program will encourage students to be more self-reliant, more cooperative with each other, and more thorough in their studies.

Based on the materials written in support of the School Mathematics Project in Britain, the program consists of 8 courses of 12 or 13 units each. Four of the courses are designated General, and four, Advanced. The distinction is in the degree of abstraction involved. Successful completion of any 8 units in a course earns credit for that course. Each student progresses at his own rate on the material of his choice. Of course, his choice may be severely limited by his own goals. The two most advanced courses correspond in content and difficulty to the present Grade 13 courses in Analysis and Algebra. A student aiming at completing these will find many prerequisites among the preceding 6 courses.

Students electing to take Mathematics will have the equivalent of 5 periods per cycle in their first and second years and either 5 or 10 periods per cycle in each of the next three years. The number of credits (one per course) which a student may accumulate in a year is limited only by his ability and ambition.

*Editor's Note: Questions concerning this programme will be welcomed. They should be addressed to Mr. Wigle, and discussion of them may be featured in subsequent issues of the Gazette.

The program will place heavy demands on the teachers. For example, as the first year progresses a teacher may find he has students studying units in G1, G2 and A1 in the same class. It is in this situation that we hope self-reliance and cooperation will develop.

Evaluation of student progress is on an individual basis, and at the request of the student. A cumulative record is held both by the student and his teacher. A comprehensive examination in each course will be administrated once annually and will form part of the evaluation information for each student taking the course for credit.

Will the project succeed? Will the students emerging from it be better people? better mathematicians? Will the teachers be able to meet the demands of the program over the long haul? The answers to these questions all lie in the future.

Outline of Courses

The eight courses outlined are intended to provide a student with a program in Mathematics which is interesting, challenging, and relevant.

Each individual course offers considerable flexibility, and there is some variation permitted in the order in which the courses may be taken. The flexibility should meet the requirements of students who:

(a) plan no further study of Mathematics beyond the Secondary School;

(b) intend to enrol in a College of Applied Arts and Technology;

(c) intend to continue the study of Mathematics at the University level.

IMPORTANT:

Prerequisites for the study of topics in the later courses are indicated by the small letters. A student who plans to continue the study of Mathematics beyond Secondary School should select his topics in each course carefully in order to meet the necessary prerequisites. However, it is always possible to take the time to pick up a unit from a previous course if this becomes necessary or desirable.

A student is permitted to progress at his own rate. He may work as an individual or as a member of a group. He should maintain close contact with the teacher to

(i) obtain assistance with difficulties,
(ii) check his rate of progress,
(iii) check the depth of study required.

A student should keep a notebook (as distinct from an exercise book), in which he records and summarizes significant results as they are found.

Evaluation:

On completion of a unit of work the student should arrange with the teacher to take a test on that unit. (Upon completion of several units the student should be prepared to answer questions on any of them.)

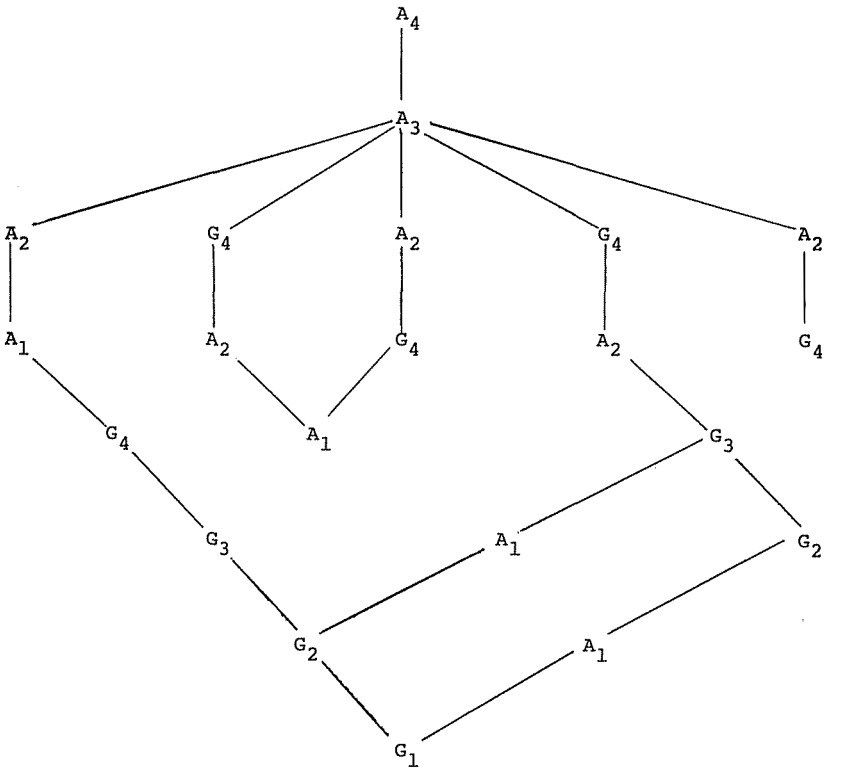
A comprehensive examination will be written by all students sometime late in the year. One purpose of this examination is to provide the student with the experience of preparing for, and writing formal examinations, since these still form one of the important methods of evaluation in post-secondary institutions.

Reports and Records:

Both the student and the teacher will have duplicate copies of the student's Mathematics Record. The student's copy is attached. This record is a continuous report to the student and will form the basis of any other report regarding the progress of the student.

Mathematics Course Outlines

(based on School Mathematics Project)



1. The chart above indicates the various routes by which a student may progress, at his own rate, to his desired goal in Secondary School Mathematics.

2. G indicates a course at the general level, and A indicates a course at an advanced level.

Aldershot High School Mathematics Department

Outline of General Mathematics Courses.

Credit for one of the courses outlined is awarded to a student who completes any 8 units in that course.

Course Unit	G_1	G_2	G_3	G_4
1	Coordinates 1/3*	Similarity and Enlargement 2/3	Probability 3/1	Logarithms 4/4
2	Fractions 1/4	Order and Punctuation 2/4	Isometries 3/2	Searching for Functions 4/8
3	Angle 1/5	Reflection and Rotation 2/5	Matrices 3/3	Coordinates in 3 dimensions 4/9
4	Sequences and Relations 1/7	Translation and Vectors 2/7	The Circle 3/5	Structure and Equations 4/10
5	Decimal Fractions 1/9	Relations and Functions 2/8	3-dimensional Geometry	Geometry: Conclusions from data 4/14
6	Area 1/10	Slide Rule 2/9	Linear Programming 3/8	Computation 4/15
7	Linear Relations 1/11	Solids 2/10	Waves 3/9	Areas and Graphs 5/1
8	Bread-and-butter arithmetic 1/14	Ratio and Proportion 2/11	Functions and Equations 3/10	Trigonometry 5/3
9	Surveying 1/15	Trigonometry 2/12	Shearing 3/12	Plans and Elevations 5/5
10	Topology 2/1	Equations and Orderings 2/13	Computers and Programming 3/14	Practical Arithmetic 5/9
11	Statistics 2/2	Pythagoras's Theorem 2/14	Solutions of Equations and Orderings 4/2	Computation 5/14
12	A Project	A Project	Trigonometry 4/3	A Project

*References such as this are to books developed by the School Mathematics Project. Thus, 1/3 means School Mathematics Project Book 1, Ch. 3.

Outline of Advanced Mathematics Courses

Credit for one of the courses outlined is awarded to a student who completes any 8 units in that course.

Course Unit	A_1	A_2	A_3	A_4
1	A new look at Arithmetic 1/1	Matrices and transformations 4/1	Number systems and Algebraic forms Adv/3	Structure Adv/1
2	Sets 1/2	Isometries 4/5	Functions Adv/4	Flow diagrams, N, and Induction Adv/2
3	Number patterns 1/6	Thinking Statistically 4/6	Graphs Adv/5	Vectors Adv/9
4	Polygons and Polyhedra 1/8	Networks 4/7	Indices and Logarithms Adv/6	Groups Adv/10
5	Symmetry 1/13	Proportion and rates of change 4/11	Derivatives Adv/7	Sigma notation & finite series Adv/11
6	Number patterns 2/6	Vector Geometry 4/12	Circular Functions Adv/8	Position and Spread Adv/12
7	Rates of change 3/4	Probability 4/13	Further Trigonometry Adv/14	Further Vectors Adv/13
8	Networks 3/6	Quadratic Functions 5/4	The Quadratic Function Adv/15	Probability Adv/20
9	Identity and Inverse 3/11	Transformations and groups 5/6	Techniques of Differentiation Adv/17	Linear equations Adv/21
10	Statistics 3/13	Invariants 5/7	Area Adv/22	Binomial Prob. F'ns and Binomial Theorem Adv/25
11	Loci and Envelopes 3/15	Statistics 5/8	Techniques of Integration Adv/23	Probability Parameters Adv/26
12	A Project	Linear Programming 5/10	Introduction to Differential Equations Adv/27	Complex numbers Adv/34
13			Polynomial Equations Adv/35	Linear Dependence Adv/42

AFTER THE MATHEMATICS WORKSHOP - WHAT NEXT?

by

Eldon R. Pipher

Program Consultant in Mathematics,
Niagara Region

The title of this paper which was presented in one of the seminars at the O.A.T.M. meeting of March, 1969 is deliberately ambiguous. It lends itself to at least three possible interpretations:

1) A Mathematics Workshop! What (will they think of) Next?

2) You've had a mathematics workshop on Monday and Tuesday. What will you do on Wednesday?

3) Are mathematics workshops a passing phase? A band wagon? If so, what will be the next emphasis?

To some extent I hope to make comments upon all of these questions, briefly of course, since I don't wish to take more than seven minutes of your reading time!

What has led to the rise of workshop approaches in Ontario.

For many years the predominant method of presenting a new concept to children has been a teacher directed, developmental type of approach. The entire class proceeded through a pre-determined sequence of steps leading to a generalization, hopefully discovered by the students.

Professor Zoltan Dienes refers to this method as the "Toscannini Approach". The maestro selects the tune and decides how it is to be played,indicates precisely when the players begin and end, calls on various members to perform from time to time, and describes very definitely how the orchestra members are to do their work.

The "virtuoso" method of teaching mathematics makes certain basic assumptions. It assumes that the teacher always knows the best sequence of steps, that the same sequential development is appropriate for all pupils, that the problem of the lesson is equally challenging to everyone, that all pupils are ready on a particular day, and that all children learn at more or less the same speed.

In addition to challenging these assumptions many teachers have asked, "While this approach may be a discovery method, how many children actually discover the new generalization for themselves? How many children simply listen to one of their peers state the generalization?"

The previous description is, of course, a somewhat unfair caricature of the developmental mathematics lesson. Many teachers have built into this method numerous variations which move toward overcoming the problem of individual differences. I have never said, and do not now say, that there is not a place for a well planned developmental lesson. Its use varies with the type of content, the facilities available, and the maturational level of the learners.

What I <u>am</u> saying is that a workshop technique does offer an alternative which might be used much more frequently than it is at the present time.

<u>Various Types of Workshops</u>

Workshop methods are numerous and vary from class to class. This is the way it should be since the teacher must always attempt to adapt any technique to the needs, interests, and capabilities of her students.

Sometimes children work in groups and move as a group from activity to activity. Or children work at a specific problem in groups but move individually to a new activity. In some workshops children select activity cards from a central source and then work individually, in pairs, or in small groups depending upon the problem chosen or on their ability to get along with others.

A mathematics workshop does not include the presenting of the same problem to the entire class but, by its approach, permits the children to work on their own in an attempt to find a solution.

An Example of One Workshop Technique

An estimation activity illustrates the approach just mentioned. Each child is given a sheet of paper on which there has been typed randomly nothing but "6's". (I use 6 because it is my favourite number, was considered perfect by the Greeks, and has the additional feature that when you invert the page you have 9's for those children who like variety.)

Because of the work involved I find it preferable to have the children work in groups of four.

Each child makes a "wild guess" at the number of 6's on the sheet and records it, being allowed a few seconds only to do so. His group selects one of the childrens' estimates for their groups guess. These are recorded in a table on the blackboard.

The guesses of the children always prove quite interesting. Although there are slightly more than 4000 sixes on the sheet I use, estimates vary from about 100 to 3,000,000 in a typical junior class. Some estimates are rounded off; others are attempts at being very precise.

The children are then told to work together to come up with a "more careful guess". A minimum of instructions is given, the children being told that the answer is to be an approximation,only that they may cut the paper, draw lines on it, obtain extra copies if needed etc.

It is interesting to see the methods devised by the pupils. Some divide the paper into 1 inch squares, other fold the sheet into eighths or sixteenths; some count the sixes, others only the spaces in selected rows.

When the time is up the groups present their findings reporting both their estimate and the methods used. A discussion follows where children evaluate each others' methods and raise questions which have not been answered.

Under the guidance of the teacher many concepts and skills can be developed which arise from the activities. These include:

- estimation on the basis of random sampling
- application of the size of numbers
- rounding off numerals
- reading and writing numerals
- area (page divided into 1" squares)
- linear measurement (so many to an inch)
- computation - (considerable practice in addition, subtracting, multiplication and division occurs during the workshop)
- negative and positive integers (where children compare the "wild guesses" with the more careful estimates)
- average
- fractions

Using the concept of estimation on the basis of random sampling children can use skills arising from it to estimate the number of beans in a jar or box, bricks in the school building, blades of grass

in the school yard, words in a book, sheets of paper in a pile, holes in the ceiling tiles, coins needed to pave the classroom floor, and so on.

The main point I wish to emphasize in the description of the estimation activity is this. After the children have had sufficient opportunity to explore activities and make discoveries it is essential that the teacher ensures that they see some significance in what they have found out, understand relationships between the new concept or concepts and those previously learned, and are able to apply the newly learned skills to new situations. A workshop is not an end in itself - it is a method of developing concepts and skills.

Another Way of Integrating Workshops with Concept Development

The description of the workshop on estimation illustrated how activities involving a single problem can lead to many related concepts capable of being developed further at the discretion of the teacher. Another approach is where a teacher intends to teach a single concept but prior to the lesson sets up a workshop of activities which at first sometimes seem unrelated. Such an example is found in the topic of factoring.

Prior to the lesson the teacher might set up activities such as these:

Station One

The children select a constant number of square units and determine how many rectangles (whole number units only) can be constructed. A table of results is made.

Station Two

Using Cuisenaire rods the children see how many trains of the same coloured cars be made showing the length of 24, 36, 37, 28 etc.. The results are recorded.

Station Three

Using a multiplication matrix or table (to 10x10) the children record, the number of times certain numbers appear on the table.

No. of Occurrences	Numbers
0	11, 13, 17 .....
1	
2	2, 3, 5, 7, ...
3	4, 9 .....
4	6, 10 .....
etc.	

Station Four

Pupils play "Guess My Number". One child selects a number greater than 10 and less than 100. The other children try to guess the number by asking only, "What is the remainder if I divide your number by N?" (N is any whole number less than 10). The object is to guess the number in as few questions as possible.

After the children have completed the activities, which may take more than a "period", the teacher might use the experiences as a background for a lesson on factoring. She might begin by telling the children that all of the activities which they did in the workshop have something in common. Challenging them to discover what the concept is she might develop the idea of factoring in a very meaningful way.

The common element in both of these descriptions of workshops is clearly this. From the experiences of the children the teacher helps develop the underlying concepts; the activities are not left dangling as ends in themselves.

After the Mathematics Workshop - What Next?

Finally I would like to return to the third interpretation of my title. Are mathematics workshops simply part of the pendulum swing? Will they go the way of the Cuisenaire rods?* Will we soon say, "Now that mathematics workshops are no longer in vogue, what will we do next?"

The answer, I think, depends on us. If we take an all-or-nothing approach to methodology in mathematics, that is, using workshops exclusively on the one hand or on the other hand never permitting children to explore ideas by themselves we will soon be asking the final question of the previous paragraph.

However if we recognize that teachers need a variety of techniques, that large group lessons and group workshop activities complement rather than contradict each other, and that not all children reach an optimum learning experience in the same type of activity then the best advantages of workshops will not be lost.

We must also recognize that engaging in an activity does not guarantee that the concept involved is perceived by the child or is necessarily meaningful to him. Help from the teacher is needed in order that the child derive maximum value from the experience.

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* Have the Cuisenaire rods outlived their usefulness? Let's hear your views! Editor

And let us be alert to side effects which can develop, when teachers try methods which are new, at least to them. When workshops are conducted under chaotic conditions, when children have a sense of freedom but no sense of responsibility it may be that young people will emerge who are great discoverers and individualists but horrible people to live with.

We need always to remember this. When a teacher, with her own background knowledge and understanding of her pupils, decides what concept or skill needs to be learned she has an obligation to consider carefully the method which will work best in the particular situation at hand. No technique is so good that it is adequate for all types of children, for all ages and levels of maturation, with all types, and under the guidance of all teachers. Variety is still the spice of good mathematics teaching.

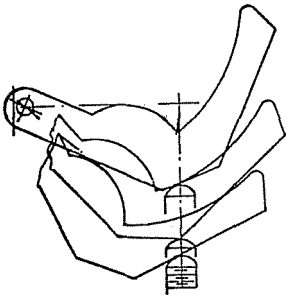
TANGENCY PROBLEMS RELATED TO A V-BLOCK

by

M. K. Straka

Curriculum Supervisor,
Ontario Department of Education

Adjustable Pivoted V-Block Compensates Arc Motion*



V-block in the range of positions supported by jack screw.

Normally you would expect that a V-block pivoted off to one side and raised or lowered to maintain the centre height of round parts of various diameters would shift the parts sideways.

This V-block has no such shift. The curve of the V-surfaces compensates for the arc motion in adjustment.

The dimensions given are in decimals. If your block is to be large they might be inches or if it is to be of medium size a centimeter scale could be used in the layout.

Contact surfaces used to locate work are shown in heavy lines.

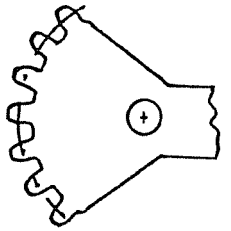
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* This is a reproduction of an article taken from an unidentified trade journal, and forms the data for the subsequent discussion of tangency.

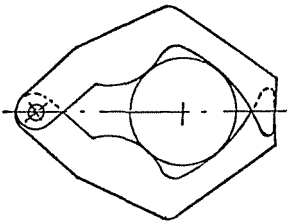
With the proportions given the location will be close for a range of circles from 2.8 to 13.2 units in diameter.

The centre line (horizontal) through the smallest circle of the range and the pivot point is tangent to the 3.95 radius. For the largest the V-block swings down by the 40 degree indicated on the dimensioned diagram.

The weight of the part may be supported in a number of ways. V-block may have a toothed segment and be adjusted by worm or may have a jack screw under it as suggested in the figures.



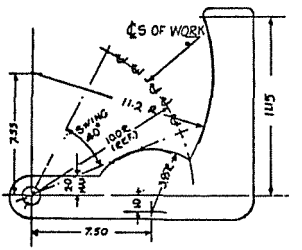
Teeth segment and worm may be used to adjust the V-block.

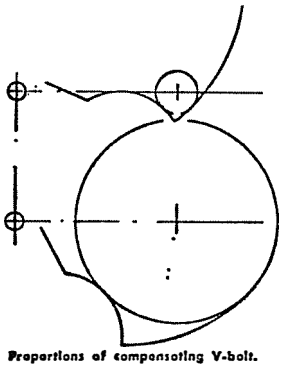


V-jaws which locate work independently of its diameter.

The outer end of the block might be hung from an eye-bolt linkage or tightening up of the pivot screw may hold the block in position.

The blocks may be combined as a pair to locate and retain a part in a pair of jaws as indicated by one of the sketches.





Proportions of compensating V-bolt.

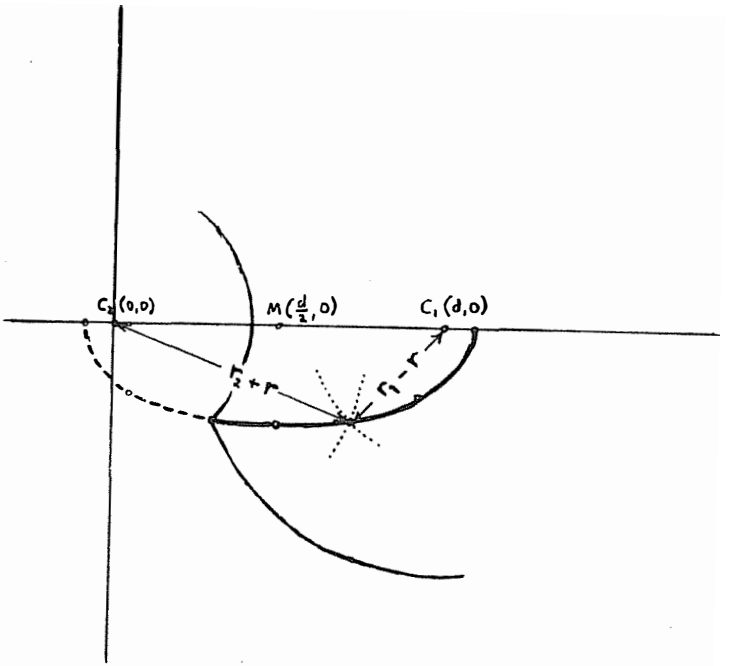
The foregoing article reproduced from a (unfortunately unidentified) trade journal provides a host of interesting material for instructions in analytical geometry.

. . . . . .

From the article one is inclined to assume that the locus of the centres of the workpiece is a circle. The given limits for the cross-sectional radii seem to be caused by mechanical considerations. Furthermore it is not evident whether the V-block with the given dimensions is the only working design or other dimensions are possible.

Translated into mathematical terms the problem investigates the tangency conditions of 3 circles, 2 contour circles of the V-block (fixed radii r_1 and r_2) and the cross-section of the workpiece (variable radius r). The locus of the centres of the latter is required. Choosing C_2 as the origin and C_1 on the x-axis, the choice of r_1, r_2 and d will determine the shape of the V-block, provided $r_1 + r_2 > d$. The required locus is found as the intersection of circles with centres in C_1 and C_2 and respective radii of r_1-r and r_2+r. (See fig. 1.) From this it follows that the locus must be an ellipse, whose parameters are required.

Figure 1



The major axis $2a = r_1 - r + r_2 + r = r_1 + r_2$

The centre of the ellipse is at $M(\frac{d}{2}, 0)$.

The parametric form of the ellipse may be written as

$$x^2 + y^2 = (r_2 - r)^2$$

$$(x - d)^2 + y^2 = (r_1 + r)^2$$

Elimination of y yields $X = \frac{d}{2} + \frac{1}{2d}\left[r_2^2 - r_1^2 - 2r(r_1 + r_2)\right]$

For $x = \frac{d}{2}$, $r = \frac{1}{2}(r_2 - r_1)$, and the half minor axis is computed as $b = y(\frac{d}{2}) = \frac{1}{2}\sqrt{(r_1 + r_2)^2 - d^2}$

The equation of the ellipse can now be written:

$$\frac{(x - \frac{d}{2})^2}{\frac{1}{4}(r_1 + r_2)^2} + \frac{y^2}{\frac{1}{4}\left[(r_1 + r_2)^2 - d^2\right]} = 1$$

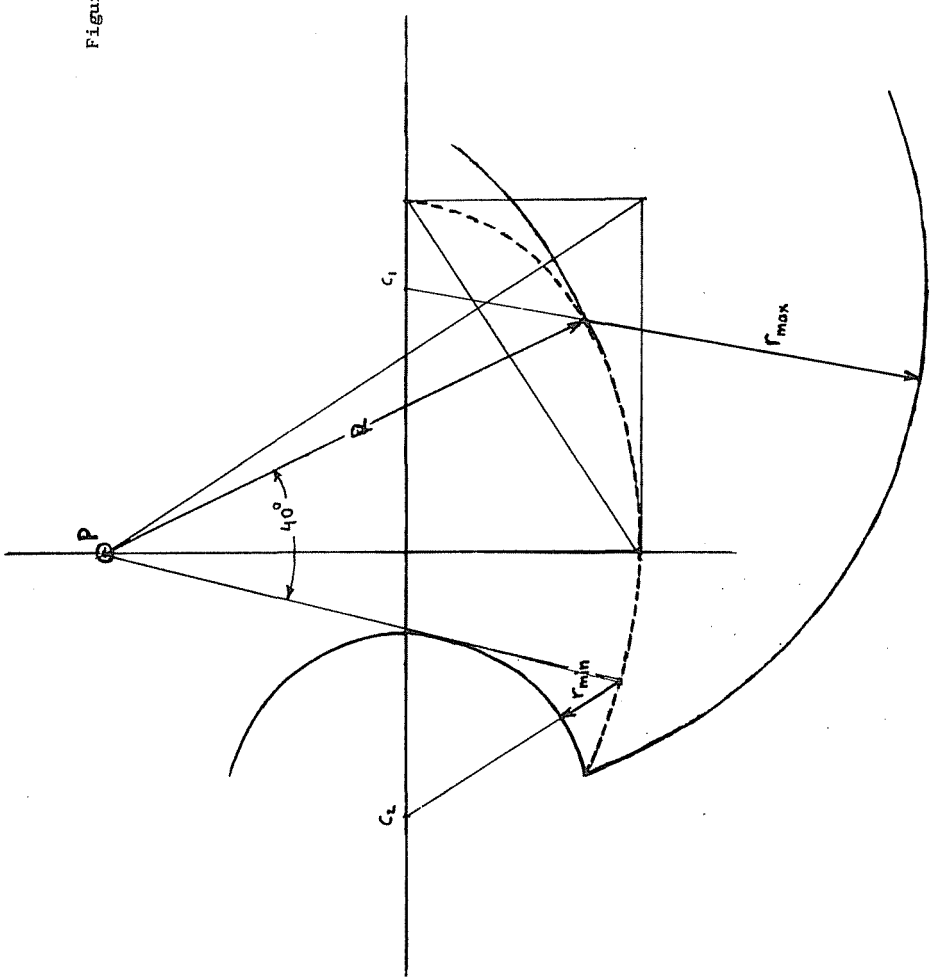
A discussion of the significance of the two elliptic arc sections (solid and dotted in fig. 1) follows naturally from the general solution.

One may read off the parametric form that if $r = 0$, the centre is the point of intersection of the given circles.

It is now evident that only a certain part of the elliptic arc, preferably the section in the neighbourhood of $x = \frac{d}{2}$ may be approximated by a circle. The admissable error in positioning the centre of the workpiece will determine the range of permissible radii (or the swing angle of the jaws).

For design purposes the position of the pivot has to be determined. It is the centre of the circle of best fit at the vertex of the locus ellipse. The centre (P) can be found by the well known construction (see fig.2) using the fact that the radius of curvature $R = \frac{a^2}{b}$ at this point.

Figure 2



Several interesting problems can be solved. One is the actual design of a V-block, for a given range of radii and a given tolerance for the displacement of the centre. The solution offers an opportunity to demonstrate a typical engineering approach of the cut-and-try type, including the design of a computing procedure using mechanical or electronic computing aids.

Other questions are: What is the consequence of the design decision $r_1 = r_2$? What is the radius of the biggest piece the jaws can hold? Is it possible to choose the design parameters such that the locus ellipse becomes a circle? What effect has the relation $r_1 \gg r_2$ or $r_2 \gg r_1$ on the usefulness of the V-block? Is the condition $r_1 + r_2$ d necessary?

A certain design may be taken to a workshop and manufactured. The unit would give rise to some more interesting questions on measurement.

Example (using the parameters given in the original article):

$r_1 = 11.2$, $r_2 = 3.95$ and d = 11.38 (computed)

(see fig. 2)

Ellipse parameters: $a = \frac{1}{2}(r_1 = r_2) = 7.575 = 7.57$

$b = \frac{1}{2}\sqrt{(r_1 + r_2)^2 - d^2} = 5.00$

Radius of curvature $R = \frac{a^2}{b} = 11.47 = 11.5$

(Note discrepancy with given R = 10.0)

From fig. 2 using the 40° swing the limiting radii r_{min} and r_{max} are measured to be 1.4 and 7.4. (The computation of these radii is a veritable assignment in itself.)

It is hoped that this short article will demonstrate the versatility and merits of applied problems in the teaching of mathematics and that many more will be discovered and used.

Sources of additional problems

Math. problems from Industry - Chrysler Corporation.

Solutions to Math. problems from Industry - Chrysler Corp.

Illustrative Problems in Engineering - General Electric.

The author wishes to acknowledge the aid of Mr. D. H. Craighead who discovered the problem.

CARDINALITY AND COMPLETENESS IN THE SET OF REAL NUMBERS

by J. E. Hogarth
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We have seen the need for numbers other than the rationals in order, for example, to have a number whose square is 2. Such numbers are of course called irrationals. We know that the irrationals have a special decimal representation, and that together with the rationals form a new number system called "The Reals". The real number system has all the essential properties of the rationals; i.e.

(a) The field properties are satisfied. This is why in algebra we often do not need to be aware of whether we are dealing with the rational number system or the real number system. The rules are the same.

(b) The real numbers are ordered in the same way as the rationals. This is why the reals can be represented by points on a number line, as can the rationals. We say, for example, that $a > b$ if and only if a lies to the right of b.

What then are the vital distinctions between the rational number system and the real number system, except that the former

is contained in the latter? There are two such distinctions.

1. The rationals are countable; the reals are uncountable

This is a way of saying that the reals constitute a "larger" set than do the rationals. But we must be careful as to what we really mean by "larger". For finite sets, where our intuition is reliable and where our experience applies, we would naturally say that set A is larger than set B if $B \subset A$ and $B \not\simeq A$. But this is a special case, even for finite sets. Suppose, for example, A and B have no elements in common. Then we could resort to a more general idea, namely that A is larger than B if B can be put into one-to-one correspondence with some proper subset of A. For finite sets this would be a satisfactory and consistent way of deciding which of two sets is "larger".

For infinite sets (where our intuition can often mislead us) the approach given above falls apart. An infinite set is defined to be one which can be put into one-to-one correspondence with a proper subset of itself. (e.g. the natural numbers are in one-to-one correspondence with the even natural numbers.) We would then have the unnatural situation of an infinite set being "larger" than itself.

What we would like is some way of talking about the "largeness" or "size" of a set - (the proper word is cardinality) -

so that all sets can be ordered according to their "size". The usual notions of order would imply such properties as 'If A is larger than B then B is not larger than A', 'If A is larger than B and B is larger than C, then A is larger than C', 'A is not larger than itself', and 'either A is larger than B, the same size as B, or smaller than B'. These properties are automatically satisfied for finite sets, where we identify size with the number of elements in the set.

For infinite sets, we can start by agreeing that all infinite sets are larger than any finite set. (Any definition we use must give this result.) The crucial question for infinite sets is this: Are all infinite sets necessarily equal in "size"? This of course will depend on how we define "size"; but - more to the point - is it possible to define "size" in such a way as to distinguish certain infinite sets?

The answer is yes. We say that A is "larger" than B if there is a one-to-one correspondence between B and some subset of A, but there is no one-to-one correspondence between A and any subset of B.

Sets which can be put into one-to-one correspondence with each other are said to have the same "size" or cardinality.

This definition permits sets to be ordered according to their "size", or cardinality.

A set which has the same "size" as the natural numbers is said to be countable. No infinite set is "smaller" than the set of natural numbers. An infinite set which is "larger" than the set of natural numbers is said to be uncountable. (Not all uncountable sets have the same "size".)

Using the above definition the following surprising and unintuitive results can be proved:

(a) The set of all rationals is countable.

(b) The set of all reals is uncountable.

(c) The set of real numbers between 0 and 1 is uncountable. (The same is true of the set of real numbers in any finite interval, no matter how small.)

See if we can prove (a) and (c)

If one accepts the above conclusions, it is perhaps even more difficult to comprehend the fact that between every pair of reals, no matter how close, is an infinite set of rationals.

2. The set of real numbers has the property of completeness; the set of rationals does not.

The completeness property of the set of real numbers has the illusion of simplicity. Intuitively it is simply that there is a one-to-one correspondence between the set of real numbers and the points on a (geometric) line. This is a simple enough idea, but the illusion arises from thinking that something

has been said about the real numbers. It will be recalled that the geometric line was assumed to have the property of having its points in one-to-one correspondence with the real numbers (and ordered in the same way). Nothing is accomplished by describing a property of the real numbers in terms of a property of another object (the geometric line) which in turn was described in terms of the real numbers. However, for those who do not aspire to a deep understanding of the calculus it is best to accept the geometric line as an intuitive concept, and to relate the real numbers to the points on such a line. (For such the discussion ends here.) The more ambitious student should somehow realize that in fact the shoe is on the other foot. Modern (analytic) geometry is based on the real number system, rather than the other way around. What is needed then is some way of describing the completeness property in terms of the real numbers themselves. The final description may seem unrewarding and not to the point, but it is the basis of the entire subject of calculus.

First, we will return momentarily to geometry. It is pertinent to ask why we use the real number line in geometry rather than the rational number line. (i.e. a line whose points are in one-to-one correspondence with the rational numbers). The answer lies in *why* we use a number line at all in geometry. One essential reason for this is to be able to define distance

between two points; in the simplest (one dimensional) case it is the absolute value of the difference between the numbers corresponding to the two points. Now if we only were to consider a single line, the rationals would do as well as the reals for this purpose. It is when we go to two (or more) dimensional geometry that we would run into difficulties. If we used the rational number line, then applying the Pythagorean Theorem there would be no way for example to describe the length of the hypotenuse of a right angled triangle whose other sides are of length one. Thus we could not have a distance between certain pairs of points in the plane. A consistent two dimensional geometry involving distance is just not possible using the rational number line. That the problem does not arise with the set of real numbers is because of its completeness property.

It is important to bear in mind the following facts about the real number system.

The real number system is a mathematical entity which can be constructed (in the mathematical sense) without using the geometrical considerations from which its need arose. However, any understanding of the real number system is both abstract and profound. In fact, to quote Courant and Robbins, there is no way for our intuition to distinguish between rational and irrational numbers. In summary, to provide a model for

geometry we needed a number system with all the "nice properties", (i.e. field and order) of the rational number system, but for which there is a square root for every positive element. Such number systems are possible, in particular the real number system, but this system is much "larger" and much more complex than might be expected. In order to acquire a fringe benefit we must utilize a monster.

Now to get back to the completeness property of the real number system.

First, we must discuss what is meant by a "cut" in an ordered number system. (The discussion applies to both the rationals and the reals.) It is possible (in many different ways) to divide the set of numbers into two subsets with the following properties:

(a) Every number in one subset (called the lower subset) is less than each number in the other subset (called the upper subset). It follows that the two subsets have no members in common.

(b) Every number is in <u>one</u> of the subsets. For example the subsets $\{x \mid x \leq 0\}$ and $\{x \mid x > 0\}$ would have the required properties. Such a division into two subsets is called a cut.

We now examine more closely the two subsets in the example given above. The lower subset, $\{x|x \leq 0\}$ has a largest number, i.e. 0. The upper subset however does not contain a smallest number. For if it had a smallest number a, then $0 < \frac{a}{2} < a$, so that $\frac{a}{2}$ is in the upper subset and is less than a, which is a contradiction.

Now we can define the completeness property of the real numbers.

Completeness property: For every cut in the real numbers, either the lower subset contains a greatest number or the upper subset contains a smallest number.

A fascinating but completely unintuitive fact may be stated here. The set of all possible cuts in the rationals can be put into one-to-one correspondence with the reals.* Thus the rationals are countable but the cuts in the rationals are uncountable.

We will now demonstrate by example that the rationals do not have the completeness property. Consider the cut in the rationals described by the subsets $\{x|x < 0 \text{ or } x^2 \leq 2\}$ and $\{x|x > 0 \text{ and } x^2 > 2\}$. The upper subset has no smallest rational. Also since there is no rational whose square is 2 the lower subset has no greatest rational.

---

*A more dramatic statement would be "the set of all possible cuts in the rationals is the reals!"

If we use the same prescription for a cut in the reals, again the upper subset has no smallest real number. Therefore, using the completeness property, there must be a greatest real number in the lower subset. For this to be true there must be a real number whose square is 2, (i.e. for which the equality in the condition $x^2 \geq 2$ applies).

Similar arguments can be used to show that the completeness property of the reals assures the existence of a real square root for any positive real number. Thus the real number line (because of the completeness property) is a suitable model for geometry and is consistent with the Pythagorean Theorem.

PROBLEM NO. 9 (Vol. 7, No. 3, 1969, p.214)

Tangents to a parabola at B and C meet at A. Prove that (a) the area bounded by the parabola and the chord BC is two-thirds that of the triangle ABC, (b) the line joining the midpoints of AB and AC is tangent to the parabola at the midpoint of the line.

SOLUTION by M. K. Straka

Let the parabola assume the position $y = kx^2$ and choose 2 arbitrary points $B(-b, kb^2)$ and $C(c, kc^2)$. The tangents in these points with the respective equations

$$y = -2kbx - kb^2$$

$$y = 2kcx - kc^2$$

intersect in $A\left(\frac{c-b}{2}, -kbc\right)$

The equation of the line through B_1C_1 say l, is

$$y = kx(c - b) + kbc$$

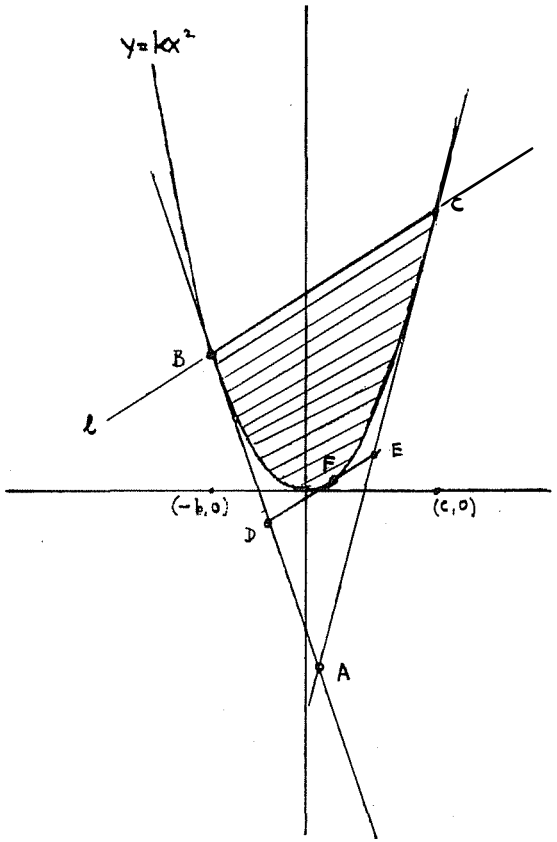
The area between parabola and chord, A_1:

$$A_1 = \int_{-b}^{c} \int_{kx^2}^{kx(c-b)+kbc} dy\, dx = \frac{1}{6}K\,(b+c)^3$$

The area of the triangle ABC, A_2 :

$$A_2 = \tfrac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ \frac{c-b}{2} & c & -b \\ -kbc & kc^2 & kb^2 \end{vmatrix} = \tfrac{1}{4}k(b + c)^3$$

$$\boxed{A_1 = \frac{2}{3} A_2}$$


y=kx²
C
B
l
F
E
(-b,0)
(c,0)
D
A

The co-ordinates of the midpoints are:

$$E\left[\frac{3c-b}{4}\ ,\ -\frac{kc}{2}(b-c)\right]\ ,\quad D\left[\frac{c-3b}{4}\ ,\ \frac{kb}{2}(b-c)\right]$$

The line through these points has the equation

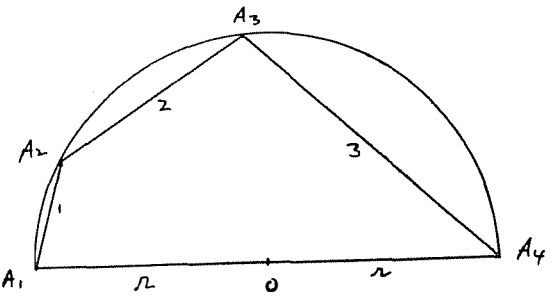
$$y = k(c-b)x - k\left(\frac{c-b}{2}\right)^2 .$$

(As seen from similar triangles, slope BC = slope DE). Intersection with $y = kx^2$ yields only 1 point of intersection (tangency): $F\left[\frac{c-b}{2}\ ,\ k\left(\frac{c-b}{2}\right)^2\right]$

It can readily be shown that F is the midpoint of DE.

Similar solutions, but avoiding double integration, were received from R. A. MacDonald and E. Evans. Mr. Evans also observed that similar results are true for the circle, and also for the ellipse, at least for certain chords. He suggests, as a supplementary problem, to consider what happens in the case of the ellipse for general chords.

PROBLEM NO. 10 (R. D. Alexander)



(a) Three connected chords A_1A_2, A_2A_3, A_3A_4 are inscribed in a semi-circle with diameter A_1A_4. Given that $A_1A_2 = 1$, $A_2A_3 = 2$, $A_3A_4 = 3$, determine the radius of the semi-circle.

(b) Generalize (a) for chords of arbitrary length.

(c) Generalize (b) for N chords.

THE GELFAND CLUB OF ONTARIO

1. Announcement. This notice announces the formation of a new mathematics club in Ontario, named in honour of I. M. Gelfand, Professor of Mathematics at the University of Moscow, U. S. S. R. This Club will seek to make modest contributions to the teaching of mathematics at all levels.

During 1969-1970 lists of problems will be distributed monthly by the Gelfand Club and high school students will be invited to submit solutions. There will be no prizes, medals, examinations or competitions.

Anyone who would like to participate in the work of the Gelfand Club is cordially invited to write to Professor Israel Halperin, Department of Mathematics, University of Toronto, Toronto, Montario.

2. Who is Gelfand? Israel M. Gelfand is one of the leading mathematicians of our time; his works in functional analysis, group representations, quantum mechanics and biology have made his name known throughout the scientific world.

In 1964 Professor Gelfand organized the Mathematics Correspondence School to encourage talented high school students; thousands of students scattered across the Soviet Union corresponded with graduate students and professors at Moscow University using

pamphlets written for them by Professor Gelfand and collaborators. These pamphlets are being translated into English and two are now available from the General Publishing Co., Ltd., 30 Lesmill Rd., Don Mills, Ontario.

3. Invitation to Ontario high school students. The Gelfand Club will distribute on the first of each month beginning October 1, 1969, lists of definitions, exercises and problems. You are invited to submit solutions to the problems.

You will find it useful to try the exercises and you may use them without proof in your solutions for the problems (refer to an exercise by quoting its number). But please do not submit solutions for the exercises.

You can consult teachers, friends and books but please submit solutions which are entirely original work by you, only.

The problems are not easy, even for professors. But with hard thinking and a little luck you may find solutions. However, if you are unable to solve any of the problems do not be the least bit down-hearted (on the other hand, if you do not find the exercises or problems interesting you should perhaps consider carefully before choosing research in mathematics for your career). Whenever you do solve a problem try your hand at generalizing the problem.

Use sheets of standard size, $8\frac{1}{2}$"x 11". Write on one side of sheet only. Number your sheets in the upper right corner and put your initials in the upper left corner. Use an ordinary paper clip to hold the sheets together and put the sheets unfolded in a large envelope (with a return address) addressed to:

Professor Israel Halperin,
Department of Mathematics,
University of Toronto,
Toronto, 181, Ontario.

and mail not later than the 20-th of the month. Please enclose a second such large envelope (folded) addressed to yourself and stamped with the same amount of postage. This second envelope will be used to send you the problems of the next month and to return your solutions. Please send such envelopes if you wish to receive the problems, even though you are not submitting solutions, or if you are not a high school student.

Type or write very clearly, use short paragraphs and double spacing, display formulae and calculations in the centre of separate lines and do everything you can to make your presentation a pleasure to read. If you fail to do this, volunteers to read your work probably will not be available.

In each communication please give your name, age, school and grade.

4. Sample of definitions, exercises and problems.

D1. Suppose a and b are natural numbers (that is, a and b are both from the set $1, 2, 3, \ldots$). We say a divides b, and write $a|b$, to mean:

$$ac = b, \text{ for some natural number } c.$$

D2. A natural number p is called a prime if the two conditions hold:

(i) $p > 1$,

(ii) $a|p = a = 1$ or $a = p$.

E1. If p is a prime and a,b are natural numbers and $p|ab$, then either $p|a$ or $p|b$.

E2. If a is any natural number and $a > 1$ then a can be expressed as a product of powers of primes in one and only one way:

$$a = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

with r some natural number, $n_1, \ldots, n_r$ natural numbers and $p_1, \ldots, p_r$ primes such that $p_1 < p_2 < \cdots < p_r$.

P1. Suppose that a given rectangle has sides of length a,b respectively and suppose than an unlimited supply of rectangular

tiles are available, each tile of sides m and 1 respectively (a, b, m are natural numbers and all lengths are in terms of some fixed unit of length). You are asked to cover the given rectangle using such tiles, each tile to be placed with its sides parallel to the sides of the given rectangle, but without overlapping and without cutting tiles. Show that this cannot be done unless $m|a$ or $m|b$.

P2. Suppose that $\alpha_1, \ldots, \alpha_m$ are complex numbers and that they satisfy the conditions:

(i) $|\alpha_i| \leq 1$ for all i,

(ii) $\alpha_1 + \ldots + \alpha_m = 0$.

Show that by re-ordering these complex numbers it is possible to satisfy $m - 2$ inequalities:

$$|\alpha_1 + \alpha_2| < 5$$

$$|\alpha_1 + \alpha_2 + \alpha_3| < 5$$

$$\ldots$$

$$\ldots$$

$$\ldots$$

$$|\alpha_1 + \ldots + \alpha_{m-1}| < 5$$

(of course, even after the re-ordering $|\alpha_1| \leq 1$, $|\alpha_1 + \ldots + \alpha_m| = 0$ will hold, by hypothesis). Interpret this result geometrically in the plane of complex numbers.

THE SAMUEL BEATTY FUND

Following Dr. Samuel Beatty's retirement in 1952 as Head of the Mathematics Department and Dean of the Faculty of Arts, University of Toronto, a number of graduates of Mathematics and Physics, of the University of Toronto, organized a scholarship fund to recognize the great contribution that Dr. Beatty made to the education of students of Mathematics and Physics. The primary purpose of the fund was and is "to promote and encourage the study of mathematics in the said University and throughout the Province of Ontario."

ONTARIO MATHEMATICS COMMISSION

TRAVELLING LIBRARIES

A second Travelling Library available for display at educational meetings has been sponsored by the Advisory Committee for Elementary Education of the Ontario Mathematics Commission.

Each library consists of several dozen books supplied by the Canadian Text Book Publishers' Council, National Council of Teachers of Mathematics, Ontario Institute for Studies in Education, individual publishers and others interested in Mathematics.

For further information about the libraries address inquiries as follows:

Elementary	Secondary
Mr. E. R. Pipher, Program Consultant, Region 6, Ontario Department of Education	Prof. G.B. Lawson Althouse College of Education 1137 Western Road, London, Ont.

The only expense to the borrower of either library is the payment of express charges both ways.

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1969 - 1970

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