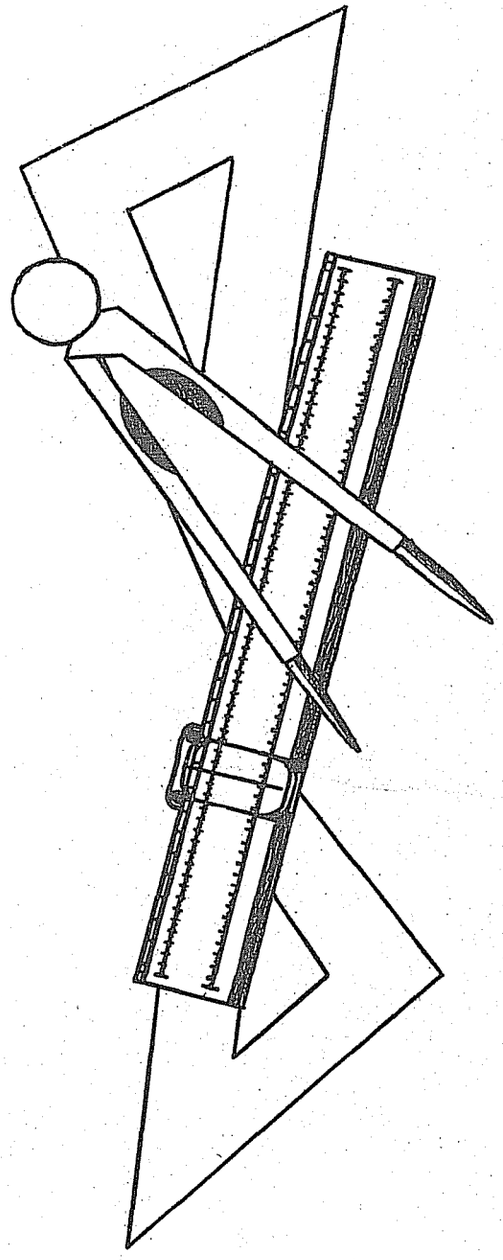
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COMPUTING MACHINES AND THE BRAIN

Hans Heinig, McMaster University

A psychologist once approached an eminent mathematician in Chicago some thirty years ago and asked that he submit to a Rorschach test in order to study the process of invention and discovery in the brain. The mathematician consented and was required to determine what the specific blots on a number of cards were. Unable to give an answer immediately, the mathematician went home to look over the cards once more. He was clearly troubled not to be able to give an answer and for a week he thought very much about the cards, not being able to make anything of them. One evening the mathematician phoned the psychologist friend excitedly and exclaimed "I've got it, I know now what is on the cards you gave me, they are ink blots".

Such an answer is clearly not the expected one from the majority of subjects. In fact such an answer might be given by a machine, say a computer. To be sure, a "thinking" computer.

But what could one reasonably call a thinking machine? Assume an experimenter is communicating by teletype with each of two rooms, one containing a computer the other a man. If after exchanging a string of messages the experimenter is unable to decide in which room the computer or the man is, then one may say the computer is thinking.

In recent years a number of controversies arose about the role of the computer in society and its influence on the life of the individual. Can a computer for example replace the brain of a person and how does each function. These questions are still with us. In this note we wish to discuss some aspects of these questions and look at some learning processes of the human brain and that of a mathematical machine.

Let us give some examples. In the case of certain games such as tic-tac-toe it is likely that a computer could be programmed - and I am sure a great number of highschool students could - to play faster and with optimum strategy than any human. Here the game is completely determined.

In the case of the game of chess N. Wiener pointed out that a computer could be programmed to play the game with an individual. But if this individual is a fair player the computer may lose. Nevertheless, if the computer is permitted to "learn" from the previous game and play the same person again, the chances for the computer to win the next game is higher. Indeed if the process is continued, the computer will eventually win every game with that player.

Other situations such as to automate theorem proving in Euclidean geometry have so far not been successful although the art of proving theorems of this kind is easily learned by highschool students. In Cartesian geometry

on the other hand some progress has been made. Analytic geometry is more algebraic - less visual and the more mechanical non-visual techniques of proofs lend themselves easier to programming.

The human brain may well be considered a mathematical machine but it is more than that. Programming a machine involves the definition and analysis of language. Mathematically, a language is simply defined as a set of strings of symbols such as letters, digits, blank spaces, punctuation marks, etc. The study of mathematical linguistics is a branch of combinatorics and applied algebra; words and strings of sentences detached from their meaning are studied. It is clear that elements of mathematical linguistics can be computerized, but is this also possible in the case of the natural human language? I think not. The average person distinguishes various modes of words, say hearing words, uttering words, reading and writing words. The mathematician does not consider words heard by the ear, voiced by the larynx or recognized by the eye. Neither do mathematicians ask in which way words and sentences are transmitted to the brain.

Human language is clearly more subtle than the expressions of the computer. The many misunderstandings attest to that. Ambiguities and innuendos - so often dangerous and destructive to individuals and society - cannot be explained

mathematically. Mechanical translation for example from French into English had some limited success and certainly aids skilled translators. However, without the human element its practical utility is doubtful as the number of misconceptions arising from the dialogue of our own two cultures attests.

These are some of the discrete aspects of computer operations. How about continuous mathematics as encountered in calculus or classical analysis? The brain then functions similar to the analogue computer. Of course much discussion and study pertains to the question of the nature of the continuous. In the study of the real numbers the gaps between the discrete and the continuous can be easily illustrated. It appears moreover that the human brain, like hybrid computers, has at least two qualitatively different modes of thinking. Both are rooted in our behavioural techniques for counting and measuring. The human neural network appear to involve many continuous processes like chemical inhibition which are much more varied and subtle than those of binary networks of the "on" and "off" firings of the axons.

The associate kind of thinking, such as memorization, rephrasing and reconstructing has been considered to be essentially digital (Turing). But this appears doubtful. Word manipulation involves hearing, which is basically continuous. As early as the middle of the last century

investigators tried to analyse the mechanism of the human ear. Then, the mathematician Bernhard Riemann for example developed a mathematical model to explain functions of the ear. However, it appears that the soundwave transmission to the cochlea is much more difficult than was assumed earlier and even today defies quantitative analysis. Further, it is unclear physiologically how we achieve the sensation of stereophonic sound. Perhaps the comparison of the pressure functions of each ear involves synchronization and integration between the cerebral hemispheres.

Reading is also a process which involves analogue and digital processes.

As mentioned above the axons or conductors of our nervous system act essentially digitally in the "all or none" fashion. However, the emitter or synapse and the receiver in the brain are clearly functioning in a continuous fashion. Thus the smoothly varying stimuli of sound, sight and touch are pleasing while the staccato of sound or the flickering of light is doubtless unpleasant.

From what we have said above it appears that the brain is a hybrid computer relying both on the discrete and continuous.

Let us look at some of the learning processes of the young child. When a child is about 6 months to two years, it is subjected to an infinite number of apparently disordered stimuli from its surrounding. The brain has to establish consistent patterns which evolve into strings

of consistent patterns and finally recognition and comprehension is achieved. Together with this process of learning, the brain establishes special concepts and must coordinate its own body with its surroundings.

After two or three years the child's mathematical thinking deals with arithmetic and it must learn to count. As with words this is done by repeating sounds and by the age of three most children mimic sequences well enough to say "one, two, three, four, five" as easily as "trick or treat", or "abracadabra". By playing various games, children somewhat older learn to count quite easily up to a hundred or more. However up to this point, there seems no relationship between tactile perception and the counting. But counting candies or cookies or spoons and forks, that is, counting a set of objects, requires more skill. Here each element of the set of objects is paired with successive ordinal numbers and the child needs to remember which elements have been counted. At first the counting is facilitated by touching the elements and later this is done with the aid of visual perception alone. To recognize that the cardinality of the objects is independent of the order in which the elements are counted comes still later. This process involves inductive reasoning. Only after grasping this principle does the child really understand the meaning of cardinal numbers.

The psychologist Jean Piaget estimates that this normally happens at the age of seven. The addition and multiplication tables are at first also memorized, and later retained by mnemonics. Subtraction and division follow until the concepts are comprehended. Similarly, fractions, decimals and negative numbers. Traditionally mastery of accurate intricate symbolic calculations were emphasized in tests. Perhaps to some extent a deeper understanding of the nature of number and their relations was thereby effected. But of course a computer can do such calculations more accurately, faster and often cheaper. However, to operate a computer will give little or no insight into the properties of numbers. Although its results may. To learn how to operate a computer may sometimes be desirable but it is surely duller than performing the most tedious of mechanical calculations in the traditional way.

Previously we pointed out that a chess playing computer, if allowed to revise its moves, can "learn". John von Neumann showed mathematically not long ago, that under reasonable hypotheses any three computers of various degrees of complexity can even build a computer "offspring" of higher complexity than any of its "parents". It may even be possible that in the future, thinking machines have an existence independent of man.

There are examples - albeit not very sophisticated ones - that in a given (tesselation) space in which there are sufficiently many parts of a machine it may be able to

assemble its own parts and reproduce. A machine to harvest the sea of metals and minerals or perhaps pollutants by means of "machine trees" would fall in this category. This is of course still a dream of the engineer. Actually certain crystals are of this nature. These are, however, self reproducing machines which do not belong to the class of thinking machines.

It is certain that thinking machines can be built - and have been built - which approximate certain particular functions of a brain. It may in fact be faultless. However to build machines which simulate the brain in all its modes with all its versatility and adaptability seems impossible.

References:

Garrett Birkhoff; Mathematics and Psychology; SIAM Review; vol. 11, (4), 1969, pp. 429-469.

Norbert Wiener; Cybernetics; MIT Press 1965.

REPORT ON PRE-REQUISITES IN MATHEMATICS BEFORE ENTERING COMMUNITY COLLEGES

L. Johnson, Northern College of Applied Arts and Technology, South Porcupine

Most teachers of Mathematics in the Community Colleges are concerned at the lack of mathematical ability of students entering the first year business and technical courses. From enquiries made from mathematics departments in the colleges, some basic comments have arisen.

1) That students have great difficulty in interpreting literal expressions and in problem solving.
2) The students are unable to apply mathematics to other subject areas.
3) That the concepts taught in High School are not really understood.

From these comments it may appear that High School students are not shown any practical use for their mathematics.

From replies received from various colleges, it appears that the main areas of weakness lie in the following topics -

1) Transposition of variables
2) Removing Brackets
3) Operations with fractional and negative exponents
4) The four basic operations in Algebra
5) Evaluation of algebraic expressions
6) Solving of equations containing one, two or three unknowns

7) Quadratic Functions and related problems

8) Distance and slope of a straight line

9) Basic formulas of the conics and tangents to a circle

10) Use of the Slide Rule

11) The practical aspects of trigonometry

12) The Sine and Cosine Law

13) The simple reciprocal and Pythagorean relationships.

The topics listed are already included in the High School curriculum.

Comments from various Mathematics Departments -

"Moreover they should basically know computational skills with whole numbers and fractions."

F. H. Seca,

Confederation.

"However, particularly as it is a structured subject, I cannot help but feel it is impossible to overemphasize the adage - Quality not Quantity."

Norman Gray,

Sheridan.

"We would like students entering St. Lawrence College to be well grounded in computational and algebraic skills."

S. W. Nibbey,

St. Lawrence, Cornwall.

"And while students have been exposed to such elementary topics as transposition of variables, removing brackets, operations with fractional and negative exponents, etc.,

they have little understanding of these processes and certinaly no facility in handling them."

J. W. Husband,

Georgian.

At least two of the Community Colleges are attempting to solve the problem of lack of Mathematical ability by setting up a liaison committee between High School and College Mathematics Departments. This may well be the answer to the problem if close liasion can be maintained and through this liaison a better understanding of what mathematical skills are required to enable the student to make a success of further education on entering the College system.

Johnson Michael Committee

Report of subcommittee on

technological aids of the

committee for CAAT's of the

O.M.C.

CRITICAL EVALUATION OF A MATHEMATICS ACHIEVEMENT TEST

Arthur Roberts, St. Clair CAAT, Windsor

Introduction:

An evaluation was recently made of a Mathematics Achievement Test* currently in use in the School of Technology at St. Clair College. This test has been used in the past by a number of the "old" Institutes of Technology and it has now been adopted by at least five of the current CAAT's. At St. Clair College the test is administered to all freshman technology students at the time of entrance to the college. The purpose of the test is diagnostic; that is, those students who score low on the test (approximately the bottom one-third percentile) are required to participate in a remedial mathematics program in order to correct their deficiencies.

The evaluation was conducted under the following format:

1. Test reliability.
2. Test validity.
3. Individual item analysis.
4. Comparison of the predictive validity of the Mathematics Achievement Test versus other methods.

Summary of Results:

The analysis produced the following results:

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*Test of Knowledge of General Mathematics, Department of Mathematics, Hamilton Institute of Technology, Hamilton, Ontario.

1. The reliability of the test was found to be approximately 0.80 (Max = 1.0). This indicates that reasonably consistent results can be obtained with repeated use of the test because of its inherent stability.

2. The relationship between scores on the test and first year G.P.A. was found to be 0.42 (Max = 1.0). This means that the validity of the test for G.P.A is somewhat low. However, this does not make the test non-functional for prediction since most research has shown that validity coefficients of this type fall in the range of 0.40 to 0.60.

3. From the item analysis, it was found that 41 out of the 90 items for the test could be classified as unacceptable from a degree of difficulty or discriminative ability standpoint.

4. The predictive validity of the Mathematics Achievement Test compared as follows with that obtained from two other sources:

Method	Validity Coefficient
Mathematics Achievement Test	0.42
EPSAT[1] (total score)	0.36
EPSAT (Parts I,II & IV - Math. Sections)	0.53
High School Subject Average	0.64

[1] Moore, B., Lapp, C. J., and Griffin, H., Engineering and Physical Science Aptitude Test, The Psychological Corporation, New York, N. Y.

Details of Evaluation

1. Reliability

Three procedures were used in calculating relability coefficients. These were:

(a) Test - retest method.

(b) Split - half technique.

(c) Method of rational equivalence (Kuber-Richardson formulas[1]).

The following is a summary of these results:(see appendix for sample calculations).

Procedure	Formula Used	$\sqrt{tt}$
1. Test - Retest	PPM (Pearson Product Moment)[2]	0.267 (non significant)
	Spearman ρ[1]	0.265 (N.S.)
2. Split - Half	PPM + Spearman Brown	0.811
	Rulon's Diff[1].	0.811
3. Rational-Equivalence	KR 20	0.810
	KR 21	0.800

[1] Richardson, M. W. and Kuber, G. F., "The Calculation of Test Reliability Coefficients Based Upon the Method of Rational Equivalence", Journal of Educational Psychology, 1939, 30, 681-687.

[2] Garrett, H. E., Statistics in Psychology and Education, 6th ed., New York: McKay, 1966.

The first thing to note from this data is the consistency of the results. Excluding the test-retest results (which were influenced by "practice effect"), all other methods of calculation produced a reliability coefficient of approximately 0.80. This indicates that any of these formulas can be used with equal confidence, and, of course, it also indicates that the test has reasonable reliability.

The low reliability coefficient obtained from the test-retest procedure can be directly attributed to "practice effect". That is, the gain in score after the test was re-administered was not the same for all individuals since they did not retain information or acquire basic mathematical skills at the same rate in the interim between the two tests. In fact, the "practice effect" is extremely exaggerated in this instance since four months were allowed to pass before the examinees were retested.

2. Validity

All validity coefficients were calculated using the Spearman "ρ" formula shown in example 1.(b) in the appendix. The results of these calculations are outlined in the appendix.

It was decided to use G.P.A. as the criterion for validity calculations rather than the students' final mathematics grade since the ultimate purpose of the test is to select students for remedial work so that they can improve their chances of succeeding in first year technology.

3. Item Analysis

(a) Item Difficulty

The difficulty of an item can be determined in several ways[1]:

(1) by the judgment of people who rank the items in order of difficulty, (2) by how quickly the item can be solved and (3) by the number of examinees in the group who get the item right. This latter method is usually the standard method of determining item difficulty. With this method the proportion (p) passing a given item is calculated to obtain an index of item difficulty.

According to most authors, the range of item difficulty should be from 0.20 to 0.90 although the majority of the items should be clustered around 0.50 since this value provides the sharpest discrimination among the examinees. (A few items passed by greater than 90% can be included for psychological effect).

For the test in question, 21 of 90 items fell outside the suggested range of difficulty and the majority of these were below 0.15.

(b) Item Validity Index (Discrimination)

The validity index of an item (i.e., its discriminative power), is determined by the extent to which a given item discrimates amongst examinees who differ sharply in the

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[1]Garrett, op. cit., pp. 365-368.

function measured by the test. The validity index in this instance was determined by biserial correlation which gives the correlation of an item with the total score on a test. Using this procedure, two extreme groups (the upper 27% and the lower 27% of the examinees) were set up and the proportion of correct responses for an item in each group was then determined. The biserial correlation between the two groups for a given time was then found by making use of Flanagan's table[1]. This was considered to be the validity index for the item.

If the item is properly discriminating, the validity index should be positive (i.e., a greater proportion of the upper 27% get the item right than the lower 27%), and, according to most authors, the index should be greater than 0.20. For this particular test, 33 of 90 items had validity indices less than 0.20 and 26 of these 33 were zero or less than zero.

In summation then, it was learned that 21 of 90 items were "unacceptable" in terms of degree of difficulty and 33 of 90 were unacceptable in discriminative ability. It was also discovered that only 13 items were common to both categories. Thus 41 (33 + 21 - 13) of 90 items were found to be unacceptable on the test in terms of either the degree of difficulty or discriminative power. These items are underlined in the appendix.

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[1]Tate, M. W., Statistics in Education, New York: The MacMillan Co., 1955, p. 364.

Discussion of Results

The results indicate that the test is in need of a major overhaul. This could involve three courses:

(a) A revision of the unacceptable items listed in the appendix.

(b) An elimination of these items to shorten the test and give more students a chance of completing the test in the allotted time.

(c) The replacement of this test with another suitable standardized test.

I tend to favour the last course of action and I would recommend that a committee be set up to develop and/or acquire a test that could be used by all the C.A.A.T.'s to screen out technology students in need of remedial work in mathematics. It would also be the responsibility of this committee to develop adequate provincial norms.

In the interim, the Mathematics Achievement Test should still be used because of the relevence of the local norms that have been developed from past usage.

APPENDIX

SAMPLE CALCULATIONS

REMEDIAL GROUP

1. RELIABILITY (TEST-RETEST)

(a) PEARSON PRODUCT MOMENT COEFFICIENT (P.P.M.)

$$R_{xy} = \frac{N\Sigma XY - [\Sigma X \Sigma Y]}{\sqrt{[N\Sigma X^2 - (\Sigma X)^2]\ [N\Sigma Y^2 - (\Sigma Y)^2]}}$$

$$R_{xy} = \frac{30\ (39{,}561) - [(893)(1316)]}{\sqrt{[30\ (28{,}170) - (893)^2]\ [30\ (59{,}062) - (1316)^2]}}$$

$$= \frac{1{,}186{,}830 - 1{,}175{,}188}{\sqrt{(845{,}100 - 797{,}449)\ \ (1{,}771{,}860 - 1{,}731{,}856)}}$$

$$= \frac{11{,}642}{\sqrt{(47{,}651)\ \ (40{,}004)}}$$

$$= \frac{11{,}642}{\sqrt{1{,}906{,}230{,}604}}$$

$$= \frac{11{,}642}{43{,}660}$$

R_{xy} = 0.267.

1. RELIABILITY (TEST-RETEST) (cont'd)

(b) SPEARMAN

$$\bar{\rho}_{xy} = 1 - \frac{6\Sigma D^2}{N(N^2 - 1)}$$

$$= 1 - \frac{6\ (2855.50)}{30\ (900 - 1)}$$

$$= 1 - \frac{17133.00}{26970}$$

$$= 1 - 0.635$$

$$\rho_{xy} = 0.265$$

2. RELIABILITY - REMEDIAL GROUP USING SPLIT-HALF METHOD

(a) Using Rulon's Formula:

$$r = 1 - \frac{s_d^2}{s_t^2}$$

Where $s_d^2 = \frac{\sum d^2}{N} - \left(\frac{\sum d}{N}\right)^2$

$$s_t^2 = \frac{\sum (X + Y)^2}{N} - \left(\frac{\sum (X + Y)}{N}\right)^2$$

Thus $s_d^2 = \frac{266}{30} - \left(\frac{-20}{30}\right)^2 = 8.422$

$$s_t^2 = \frac{59062}{30} - \left(\frac{1316}{30}\right)^2 = 44.507$$

$$r = 1 - \frac{8.422}{44.507} = 0.811$$

(b) Using Spearman-Brown Prophecy Formula:

From PPM Formula $r_{\frac{1}{2}t} = 0.682$

(Half test)

For total test:

$$r_{tt} = \frac{2 r_{\frac{1}{2}t}}{1 + r_{\frac{1}{2}t}}$$

$$r_{tt} = \frac{2 (0.682)}{1 + 0.682}$$

$$= 0.811$$

CONTROL GROUP

1. RELIABILITY USING KUDER-RICHARDSON FORMULAS

(a) FORMULA 20

$$r_{tt} = \frac{n}{n-1}\left[\frac{\sigma_t^2 - \Sigma pq}{\sigma_t^2}\right]$$

$$= \frac{90}{90-1}\left[\frac{(8.79)^2 - 15.313}{(8.79)^2}\right]$$

$$= 0.810$$

(b) FORMULA 21

$$r_{tt} = \frac{n}{n-1}\left[1 - \left(\frac{Mx - Mx^2/n}{\sigma_t^2}\right)\right]$$

$$= \frac{90}{89}\left[1 - \left(\frac{40.34 - 40.34^2/90}{8.79^2}\right)\right]$$

$$= 0.800$$

TABLE I - RELIABILITY CALCULATION USING REMEDIAL GROUP AND TEST - RETEST METHODS

Student	Raw Scores		Rank Order		PPM			Spearman ρ	
	First Test	Rewrite	First Test	Rewrite	X^2	Y^2	XY	$D = R_2 - R_1$	D^2
	X	Y	R_1	R_2					
1	44	45	1	14.5	1936	2025	1980	13.5	182.25
2	40	50	2	6.0	1600	2500	2000	4.0	16.00
3	38	55	3	2.5	1444	3025	2090	-0.5	0.25
4	34	46	4	12.0	1156	2116	1564	8.0	64.00
5	33	48	6.5	8.5	1089	2304	1584	2.0	4.00
6	33	44	6.5	16.5	1089	1936	1452	10.0	100.00
7	33	43	6.5	18.5	1089	1849	1419	12.0	144.00
8	33	35	6.5	27.0	1089	1225	1155	20.5	420.25
9	31	56	11.0	1.0	961	3136	1736	-10.0	100.00
10	31	50	11.0	6.0	961	2500	1550	-5.0	25.00
11	31	41	11.0	21.0	961	1681	1271	10.0	100.00
12	31	40	11.0	22.5	961	1600	1240	11.5	132.25
13	31	37	11.0	25.0	961	1369	1147	14.0	196.00
14	30	55	15.5	2.5	900	3025	1650	-13.0	169.00
15	30	51	15.5	4.0	900	2601	1530	-11.0	121.00
16	30	46	15.5	12.0	900	2116	1380	-3.5	12.25
17	30	45	15.5	14.5	900	2025	1350	-1.5	2.25
18	29	50	19.0	6.0	841	2500	1450	-13.0	169.00
19	29	48	19.0	8.5	841	2304	1392	-10.5	110.25
20	29	40	19.0	22.5	841	1600	1160	3.5	12.25
21	28	31	21.0	29.5	784	961	868	8.5	72.25
22	27	44	23.0	16.5	729	1936	1188	-6.5	42.25
23	27	43	23.0	18.5	729	1849	1161	-4.5	20.25
24	27	38	23.0	24.0	729	1444	1026	1.0	1.00
25	26	47	25.5	10.0	676	2209	1222	-15.5	240.25
26	26	31	25.5	29.5	676	961	806	4.0	16.00
27	24	33	27.0	28.0	576	1089	792	1.0	1.00
28	20	42	28.0	20.0	400	1764	840	-8.0	64.00
29	19	46	29.5	12.0	361	2116	874	-17.5	306.25
30	19	36	29.5	26.0	361	1296	684	-3.5	12.25
N = 30	$\sum X = 893$	$\sum X = 1316$			$\Sigma X^2 = 28170$	$\Sigma Y^2 = 59062$	$\sum XY = 39561$		$\Sigma D^2 = 2855.50$
	$\overline{X} = 29.76$	$\overline{Y} = 43.86$							
	$SD_x = 7.28$	$SD_y = 6.67$			$R_{xy} = 0.267$ (N.S. $R_{0.05} = 0.361$ d.f. = 28				$\rho_{xy} = 0.26$ (N.S. $\rho_{0.05} = 0.361$ d.f. = 28

TABLE II - RELIABILITY CALCULATION USING REMEDIAL GROUP AND SPLIT-HALF METHOD

			Rulon's Formula				PPM		
Student	Scores Odd X	Scores Even Y	$d = (X - Y)$	d^2	(X+Y)	$(X+Y)^2$	X^2	Y^2	XY
1	28	28	0	0	56	3136	784	784	784
2	27	28	-1	1	55	3025	729	784	756
3	27	28	-1	1	55	3025	729	784	756
4	27	24	3	9	51	2601	729	576	648
5	24	26	-2	4	50	2500	576	676	624
6	25	25	0	0	50	2500	625	625	625
7	25	25	0	0	50	2500	625	625	625
8	22	26	-4	16	48	2304	484	676	572
9	23	25	-2	4	48	2304	529	625	575
10	21	26	-5	25	47	2209	441	676	546
11	22	24	-2	4	46	2116	484	576	528
12	20	26	-6	36	46	2116	400	676	520
13	24	22	2	4	46	2116	571	484	528
14	23	22	1	1	45	2025	529	484	506
15	22	23	-1	1	45	2025	484	529	506
16	20	24	-4	16	44	1936	400	576	480
17	21	23	-2	4	44	1936	441	529	483
18	22	21	1	1	43	1849	484	441	462
19	24	19	5	25	43	1849	576	361	456
20	21	21	0	0	42	1764	441	441	441
21	24	17	7	49	41	1681	576	289	408
22	20	20	0	0	40	1600	400	400	400
23	22	18	4	16	40	1600	484	324	396
24	19	19	0	0	38	1444	361	361	361
25	19	18	1	1	37	1369	361	324	342
26	17	19	-2	4	36	1296	289	361	323
27	15	20	-5	25	35	1225	225	400	300
28	16	17	-1	1	33	1089	256	289	272
29	14	17	-3	9	31	961	196	289	238
30	14	17	-3	9	31	961	196	289	238
TOTALS	648	668	-20	266	1316	59062	14410	15254	14699
			$r_{tt} = 0.811$				$r_{\frac{1}{2}t} = 0.682$ $r_{tt} = 0.811$		

TABLE III - DIFFICULTY INDEX AND VALIDITY INDEX FOR CONTROL GROUP (1 of 3)

N = 29 $\quad \overline{X} = 40.34 \quad$ S.D. = 8.79

Question	Number Right (Max=29)	Proportion Right (p) Difficulty Index	Proportion Wrong (q = 1-p)	(pq)	%Right Top 27%	%Right Lower 27%	Validity Index
1	27	0.93	0.07	0.065	87	67	0.08
2	16	0.55	0.45	0.248	62	62	0.00
3	21	0.72	0.28	0.202	87	50	0.42
4	21	0.72	0.28	0.202	62	62	0.00
5	13	0.44	0.56	0.246	75	23	0.48
6	23	0.79	0.21	0.166	87	87	0.00
7	28	0.96	0.04	0.038	100	100	0.00
8	18	0.62	0.38	0.236	75	62	0.14
9	24	0.82	0.18	0.148	87	87	0.00
10	18	0.62	0.38	0.236	87	25	0.60
11	18	0.62	0.38	0.236	87	50	0.42
12	10	0.34	0.66	0.224	50	25	0.26
13	18	0.62	0.38	0.236	100	62	0.61
14	18	0.62	0.38	0.236	62	62	0.00
15	13	0.44	0.56	0.246	75	12	0.64
16	17	0.58	0.42	0.244	75	50	0.26
17	15	0.51	0.49	0.250	75	12	0.64
18	25	0.86	0.14	0.120	87	87	0.00
19	18	0.62	0.38	0.236	100	23	0.79
20	25	0.86	0.14	0.120	73	87	0.18
21	25	0.86	0.14	0.120	100	75	0.51
22	26	0.89	0.11	0.098	100	87	0.37
23	24	0.82	0.18	0.148	100	67	0.61
24	16	0.55	0.45	0.248	73	25	0.48
25	12	0.41	0.59	0.242	62	25	0.37
26	8	0.27	0.73	0.197	37	25	0.10
27	11	0.37	0.63	0.233	62	25	0.37
28	13	0.44	0.56	0.246	75	12	0.64
29	5	0.17	0.83	0.141	37	0	0.60
30	4	0.13	0.87	0.113	12	0	0.33

TABLE III - (cont'd) (2 of 3)

N = 29 $\bar{X}$ = 40.34 S.D. = 8.79

Question	Number Right (Max=29)	Proportion Right (p) Difficulty Index	Proportion Wrong (q = 1-p)	(pq)	% Right Top 27%	% Right Lower 27%	Validity Index
31	25	0.86	0.14	0.120	100	75	0.51
32	20	0.68	0.32	0.218	75	50	0.26
33	21	0.72	0.28	0.202	75	62	0.14
34	11	0.37	0.63	0.233	50	25	0.26
35	12	0.41	0.59	0.242	100	62	0.61
36	24	0.82	0.18	0.148	100	37	0.73
37	21	0.72	0.28	0.202	75	62	0.14
38	12	0.41	0.59	0.242	75	12	0.60
39	17	0.58	0.42	0.244	100	25	0.79
40	15	0.51	0.49	0.250	62	25	0.37
41	15	0.51	0.49	0.250	87	25	0.60
42	22	0.75	0.25	0.188	100	50	0.68
43	13	0.44	0.56	0.246	75	0	0.80
44	6	0.20	0.80	0.160	12	12	0.00
45	7	0.24	0.76	0.182	75	37	0.37
46	25	0.86	0.14	0.120	100	75	0.51
47	10	0.34	0.66	0.224	37	0	0.58
48	19	0.65	0.35	0.228	87	25	0.60
49	12	0.41	0.59	0.242	50	50	0.00
50	5	0.17	0.83	0.141	25	0	0.51
51	10	0.34	0.66	0.224	50	12	0.18
52	9	0.31	0.69	0.214	50	25	0.26
53	8	0.27	0.73	0.197	25	12	0.18
54	16	0.55	0.45	0.248	62	37	0.25
55	8	0.31	0.69	0.214	25	12	0.18
56	23	0.79	0.21	0.166	87	62	0.31
57	26	0.89	0.11	0.098	100	100	0.00
58	16	0.55	0.45	0.248	62	62	0.00
59	23	0.79	0.21	0.166	100	62	0.61
60	10	0.34	0.66	0.224	25	37	-0.14

TABLE III - (cont'd) (3 of 3)

N = 29 $\bar{X}$ = 40.34 S.D. = 8.79

Question	Number Right (Max=29)	Proportion Right (p) Difficulty Index	Proportion Wrong (q = 1-p)	(pq)	% Right Top 27%	% Right Lower 27%	Validity Index
61	14	0.48	0.52	0.250	75	50	0.26
62	9	0.31	0.69	0.214	62	0	0.73
63	9	0.31	0.69	0.214	75	25	0.45
64	0	0	1.00	0.000	0	0	0.00
65	0	0	1.00	0.000	0	0	0.00
66	13	0.44	0.56	0.246	62	25	0.37
67	6	0.20	0.80	0.160	12	25	-0.18
68	7	0.24	0.76	0.182	25	12	0.18
69	1	0.03	0.97	0.029	0	0	0.00
70	6	0.20	0.80	0.160	25	12	0.18
71	23	0.79	0.21	0.166	100	50	0.68
72	5	0.17	0.83	0.141	25	12	0.18
73	0	0	1.00	0.000	0	0	0.00
74	14	0.48	0.52	0.250	62	12	0.51
75	1	0.03	0.97	0.029	12	0	0.37
76	16	0.55	0.45	0.248	62	50	0.13
77	5	0.17	0.83	0.141	0	0	0.00
78	8	0.31	0.69	0.214	25	50	-0.26
79	8	0.31	0.69	0.214	25	25	0.00
80	3	0.10	0.90	0.090	25	12	0.18
81	11	0.37	0.63	0.233	37	25	0.14
82	3	0.10	0.90	0.090	25	0	0.50
83	2	0.06	0.94	0.056	0	12	-0.33
84	1	0.03	0.97	0.029	0	0	0.00
85	2	0.06	0.94	0.056	12	0	0.33
86	5	0.17	0.83	0.141	12	12	0.00
87	3	0.10	0.90	0.090	25	0	0.50
88	1	0.03	0.97	0.029	0	12	-0.33
89	1	0.03	0.97	0.029	0	0	0.00
90	2	0.06	0.94	0.056	0	0	0.00
				Σ_{pq} = 15.313			

TABLE IV TABLE OF DATA FOR VALIDITY CALCULATIONS: RANDOM SAMPLE FROM FIRST YEAR GENERAL TECHNOLOGY

Student No.	"Criterion" Final Exam GPA	Rank	Gen. Math Mark	Rank	Grade 12 Average	EPSAT Part Scores I	II	III	IV	V	VI	EPSAT Combined Scores Total	I,II, III,IV, VI	I,II, IV	III,VI
1	4.00	1	60	2		23.75	8.75	11.00	4.00	25.25	1.00	73.75	48.50	36.50	12.00
2	3.90	2	36	23.5		17.75	6.50	33.00	8.75	34.00	19.50	119.50	85.50	33.00	52.50
3	3.84	3	37	21.5	67.0	22.50	8.75	27.00	0.00	29.25	17.50	115.00	85.75	41.25	44.50
4	3.78	4	38	19.5											
5	3.70	5	59	3	77.3	25.00	7.50	31.00	7.50	11.27	16.00	98.27	87.00	40.00	47.00
6	3.66	6	36	23.5	73.6	20.00	7.50	11.00	7.50	17.25	8.50	71.75	54.50	35.00	19.50
7	3.63	7	57	4											
8	3.52	8	49	7		19.00	7.50	9.00	0.00	13.75	11.50	60.75	47.00	26.50	20.50
9	3.50	9	30	32.5		20.00	5.25	15.00	3.75	19.25	19.00	82.25	63.00	29.00	34.00
10	3.38	10	45	11.5	77.0	23.75	7.50	17.00	5.00	24.25	10.00	87.50	63.25	36.25	27.00
11	3.28	11	41	17	68.1	20.00	6.75	6.00	1.75	11.75	7.50	53.75	42.00	28.50	13.50
12	3.02	12.5	41	17	75.1	23.75	5.00	13.00	8.75	15.75	17.50	83.75	68.00	37.50	30.50
13	3.02	12.5	29	34	72.3	25.00	8.75	19.00	7.50	15.50	20.50	96.25	80.75	41.25	39.50
14	2.91	14	34	27.5	69.0	15.25	5.25	-5.00	2.75	-0.50	0.00	17.75	18.25	23.25	-5.00
15	2.77	15	44	13.5	72.4	14.50	5.75	7.00	4.75	15.75	12.50	60.25	44.50	25.00	19.50
16	2.75	16	45	11.5	76.1	17.50	5.00	21.00	2.50	21.75	8.50	76.25	54.50	25.00	29.50
17	2.72	17	46	9.5	74.3										
18	2.62	18.5	34	27.5		20.00	3.75	29.00	3.75	17.75	16.00	90.25	72.50	27.50	45.00
19	2.62	18.5	71	1		21.25	8.75	17.00	8.75	25.50	17.50	98.75	73.25	38.75	34.50
20	2.61	20	32	30.5	63.3	18.75	8.75	24.00	5.00	26.75	14.50	97.75	71.00	32.50	38.50
21	2.58	21	46	9.5	73.0										
22	2.45	22	41	17		14.25	3.75	14.00	3.75	19.50	11.50	66.75	47.25	21.75	25.50
23	2.42	23	38	19.5		12.50	7.75	20.00	3.75	8.00	11.50	63.50	55.50	24.00	31.50
24	2.41	24	44	13.5	65.3	18.75	8.75	25.00	3.00	18.50	19.00	93.00	74.50	30.50	44.00
25	2.36	25	35	25.5	70.0	17.00	6.25	18.00	5.00	16.75	14.50	77.50	60.75	28.25	32.50
26	2.34	26	32	30.5		22.50	7.50	23.00	6.25	15.00	19.00	93.25	78.25	36.25	42.00
27	2.30	27	48	8		22.50	6.25	26.00	7.50	26.00	16.00	104.25	78.25	36.25	42.00
28	2.27	28	55	5		22.50	7.50	13.00	5.00	16.75	10.00	74.75	58.00	35.00	23.00
29	2.19	29	30	32.5	66.8	21.25	2.25	21.00	0.75	23.00	17.50	85.75	62.75	24.25	38.50
30	2.16	30	28	35.5	70.2	18.75	2.50	13.00	1.25	9.25	7.00	51.75	42.50	22.50	20.00
31	2.00	31	37	21.5	65.9	18.75	5.00	9.00	1.25	20.50	10.00	64.50	44.00	25.00	19.00
32	1.70	32	28	35.5	65.7	20.25	7.00	11.00	2.75	19.50	10.00	70.50	51.00	30.00	21.00
33	1.67	33	25	39	66.3	13.75	1.25	-3.00	-1.25	21.00	7.00	38.75	17.75	13.75	4.00
34	1.64	34	42	15											
35	1.59	35	33	29	60.6	15.00	0.00	4.00	-1.25	10.75	10.00	38.50	27.75	13.75	14.00
36	1.42	36	27	37	67.0	16.50	3.50	18.00	-0.25	16.50	2.50	56.75	40.25	19.75	20.50
37	1.40	37.5	21	40		13.75	7.50	10.00	2.75	10.75	4.00	48.75	38.00	24.00	14.00
38	1.40	37.5	35	25.5	69.6	17.75	3.25	19.00	5.00	18.00	16.00	79.00	61.00	26.00	35.00
39	1.34	39	51	6	56.5	22.75	7.50	20.00	6.25	23.00	11.50	91.00	68.00	36.50	31.50
40	0.93	40	26	38											
Validity Coefficient			0.42		0.64	0.41	0.46	0.21	0.46	-0.08	0.19	0.36	.37	0.53	0.16

OPEN-BOOK EVALUATION

Douglas Ailles, Kipling Collegiate Institute, Etobicoke

There exists a changing slant on the degree of freedom for students to choose courses and areas of study. This new deemphasis on formal examinations coupled with an increased importance of day to day progress (as I believe it should be), requires the teacher to know his students, their interests, and their level of achievement. Consequently, he must gauge the degree of difficulty and content of his course in mathematics so that it will be meaningful to all members. Mathematics is a cumulative process, but then there are so many diversified fields and distinct avenues to explore. Why not encourage the student to discover that mathematics is not as clear cut as the text presupposes? Can he create a problem for which there is no unique solution? Is he aware that there is more than one acceptable solution to a problem? Does he realize that a problem may not have a single concise answer, but will differ according to the variables involved?

Open-book assignments are one way in which teachers can meet and foster the individual needs of students. Some questions to consider:

What is the nature of these assignments?

Are they related to the course of study?

Are they based on current topics discussed in class or are they co-related?

How long should the assignment be?

What duration of time should be allotted to complete the assignment?

How are late assignments treated?

What is the method of evaluation?

Of what personal gain are these assignments to the student?

Do open-book assignments have any validity with relation to students' progress and gradings? At the beginning of the school year I indicated to my grade nine and ten classes, that throughout the year they would be given some mathematics assignments. They were to be completed outside of class, with a minimum of direction, using any source of reference available. Some students are mark conscious and inquired as to the weight given to their solutions. It is now the policy in many schools for examinations to comprise fifty percent of the student's mark and the term mark also counts for fifty percent. In the majority of cases this fifty percent term work has usually been arrived at by testing. I told the class that the assignments could count for a considerable part of this fifty percent. This was necessary to instil a motivation for those who were mark orientated. In reality though these marks were never considered in this light as they would affect the final outcome of the study. Twenty assignments were given throughout the year to each student. The assignments

were in most cases co-related to the programme in class but the nature of the assignments was quite distinct. They were designed to be intriguing and challenging to the students.

Sample Problems

The cold water tap in a bath tub can fill the tub with water in 6 minutes and 40 seconds. The hot tap can fill this tub in exactly 8 minutes. The tub when filled, will empty in 13 minutes and 20 seconds when the stopper is removed. How long will it take to fill the tub if both faucets are going full blast and the stopper is out?

A bug sits on the outer edge of a long playing record that plays 15 minutes. If he does not move his initial position, how many feet does he travel while the record is playing once. Show all information in order to work this problem. Part marks will be given.

In one sack, there are two apples; in another sack, there are two oranges. In a third sack, there is an apple and an orange. Each sack is incorrectly labelled AA, OO, AO. By looking at just one piece of fruit from one of the sacks, you are able to tell what is in each sack. How can you do this?

Each assignment was graded out of ten, recorded and returned as soon as possible. The students were usually given between one and two weeks to prepare these solutions. Part marks were allotted for a correct line of reasoning. Research assignments were given and students soon discovered that a variety of answers were acceptable. The problem of copying was minimized by insisting that completed solutions be submitted before the deadline.

At the end of the year these marks were totaled and compared to the actual years mark earned in mathematics for each individual. Out of a range of one hundred and thirty-one participants in the study (some were transferred), correlation between the marks received on open-book assignments and the year's mark was .681 approximately equal to .7. In fact with the aid of sampling theory, the regression line to estimate the student's year mark from my research is estimated $Y = .5831X + 27.72$.

This study suggests that you can be ninety-five percent sure that the correlation of the students' assignments mark compared to the final year mark will lie within the range $.59 < P < .8$ for a bivariate normal population.

I feel that open-book assignments are one phase of valid means for evaluating a mathematics student. Using this method as a teaching aid, teachers could readily formulate a fairly accurate impression of classes and help those students who need extra assistance.

If a student received 64% on a series of assignments, then you could predict his mark.

$$Y = (.5831)64 + 27.72 = 65.$$

On the other hand if a student received perfect in all of his assignments to date i.e. 100%, then, $Y = (.5831)100 + 27.72 = 86$. Then his year mark would be in the range of 85%. That is, if the student maintains his same level of achievement, progresses equally well on tests and examinations as he does for daily progress, then the coreelation between the two sets of marks should be quite high. In some instances however this is not the case.

Some students put forth an extra effort to prepare for tests and hence a slight discrepancy from the projected mark to actual mark may exist. On the year's work, moreover who is to say that a student is worth 84 marks in a particular subject. Is not 83 or 85 or even 80 a just assessment? This is why some schools are reluctant to assign an actual mark but rather a range of grading, varying by ten marks at most. Is this not more reasonable? In this case a projection with 95% confidence seems a reasonable assessment of the student.

At the outset I hypothesized that there was a correlation between evaluation open-book assignments and student test gradings. Thus I conclude that open-book assignments form another method in conjunction with established methods to evaluate students.

The comments from the students themselves indicated to me that they actually looked forward to mathematical assignments with zest and enthusiasm. "Why don't they put questions like this in text books? I worked four hours straight on this research problem, before I got it!" Quite a statement from a group of students who regarded mathematics as a dead subject for a number of years, isn't it!

HAVE LIBRARY - IT TRAVELS

John Girhiny, Department of Mathematics, University of Guelph

Yes, O.M.C. has a library that travels.

At this point you might be asking "What is O.M.C.?" O.M.C. is the Ontario Mathematics Commission. The members of O.M.C. represent all areas and levels of mathematics education in Ontario. There are elementary teachers, secondary teachers, university professors, community college instructors as well as department heads, superintendents, administrators, and teachers' college masters who are members. The commission is also geographically representative within the province. Without going into detail, the commission is basically concerned with quality mathematics education. If you are interested in learning more about O.M.C., what it has done, is doing and hopes to do please write to Ontario Mathematics Commission, 1260 Bay Street, Toronto 5, Ontario. If you are concerned about mathematics education in elementary schools and wish to learn how O.M.C. might help please write to the present chairman of the Elementary Advisory Committee: John Girhiny, Department of Mathematics, University of Guelph, Guelph, Ontario.

The Library

The Ontario Mathematics Commission has up-dated its Travelling Library. It consists of two sections.

A Teachers' Reference Section. This could be very helpful to any Board of Education wishing to choose basic

books in establishing a resource center, as well as those who would like to enrich one already in existence. It is an excellent opportunity to examine for a short period of time a good number of reference books.

A Pupils' Text and Teachers' Manuals Section. This could assist teachers by acquainting them with texts and manuals other than the ones they maybe using. It also provides the mathematics or classroom teachers with an excellent opportunity to obtain different ideas and views for their mathematics lessons. This section contains elementary texts only. We hope to organize one for use in secondary schools as material becomes available.

This Library is now ready for circulation. Any Board of Education, school principal, or a member of the teaching staff who assumes the responsibility, as well as any organization connected with education can obtain it on request. The requests should be address to

Mr. Sarto Lefebvre,
St. Anne's School,
235 Albert Street,
North Bay, Ontario.

The costs to the borrower is the cost of transportation both ways. The Library is shipped collect and is to be returned pre-paid. The cost is approximately $7.00 each way, but may vary, depending on the distance and means of transportation.

To enable us to provide this service for as many teachers as possible, we suggest that it be kept no longer than two weeks. If, for a special reason, more time is required, we will be pleased to make the necessary arrangements.

Note: In requesting the library please specify if you want Section 1, Section 2, or both.

SECTION 1

TEACHER'S REFERENCE

1.	Trends in Elementary School Mathematics	Lloyd Scott (Rand-McNally)	W. J. Gage Limited 1500 Birchmount Rd., Scarborough, Ont.
2.	Evaluating Pupils' Understanding of Arithmetic	H. Dutton Wilbur	Prentice-Hall of Canada 1870 Birchmount Rd., Scarborough, Ont.
3.	What is Number?	D. L. Mumford	Ontario Mathematics Commission 1260 Bay Street Toronto.
4.	Geometry		Ontario Institute for Studies In Education 252 Bloor St., W., Toronto.
5.	I Do and I Understand	Nuffield Math Project Chambers and Murray	Longmans Canada Ltd., 55 Barber Greene Dr., Don Mills, Ont.
6.	The Arithmetic Teacher	Marguerite Brydegaard	National Council of Teachers 1201 Sixteenth St. N.W., Washington, D.C. 20036.
7.	Basic Systems	J. Stanley (W.H. Sadlier)	Palm Publishers 501 Yonge St., Toronto.
8.	Your Child and Mathematics		Palm Publishers
9.	Background to Geometry	G. F. Marris (Basil Blackwell Oxford)	Copp Clark Publishing 517 Wellington St. W., Toronto.
10.	Computation and Structure	Chambers and Murray	Longmans Canada Limited 55 Barber Green Dr., Don Mills, Ont.
11.	Some Basic Laws of Mathematics	J. S. Friis (Basil Blackwell)	Copp Clark Publishing
12.	Groups	J. S. Friis (Basil Blackwell)	Copp Clark Publishing

13.	Mathematics for Elementary School Teachers		National Council of Teachers of Mathematics.
14.	Foundations of Elementary School Arithmetic	(Scott Foresmen)	McGraw Hillof Canada Ltd. 330 Progress Avenue Scarborough, Ont.
15.	Charting the Course for Arithmetic		McGraw Hill of Canada
16.	Notes on Mathematics in Primary Schools	Association of Teachers of Mathematics	Cambridge University Press 70 Bond Street Toronto.
17.	A. Workshop Approach to Mathematics	R. A. S. Pethen	Macmillan Co. of Canada 70 Bond St., Toronto.
18.	Geometry in the Classroom, New Concepts and Methods	H. A. Elliott- J. R. Maclean- J. M. Jorden	Holt Rinehart Winston Canada Ltd., 833 Oxford St., Etobicoke, Ont.
19.	Mathematics Enrichment	R. D. Ripley G. E. Tait	Copp Clark Publishing
20.	Understanding the Metric System	David Munroe Miller (Allyn and Bacon)	Macmillan Co. of Canada
21.	Retracing Elementary Mathematics	L. Henkin- W. N. Smith- V. J. Varrneau- M. J. Walsh	Macmillan Co. of Canada
22.	Some Lessons in Mathematics (A Handbook on the Teaching of Modern Mathematics)	Association of Teachers of Mathematics	Cambridge University Press
23.	The Creative Use ofMathematics in the Junior School	L. G. W. Sealy (Basil Blackwell)	Copp Clark Publishing
24.	Intermediate Mathematics Methodology		Ontario Institute for Studies in Education
25.	Objectives for Mathematics Learning (Some ideas for the Teacher)		Ontario Institute for Studies in Education

26.	Adventures in Graphing	W. H. Glenn- D. A. Johnson	McGraw Hill of Canada
27.	Regular Polytopes		Macmillan Co. of Canada
28.	Figures Are Fun Books 1, 2, 3, 4, 5, and Teacher's Manual	J. A. H. Hunter	Copp Clark Publishing
29.	The Shapes of Things	L. G. W. Sealey (Basil Blackwell)	Copp Clark Publishing
30.	Mathematical Forum	J. Murray	W. and R. Chambers
31.	Properties of Operation with Numbers		National Council of Teachers of Mathematics
32.	Programmed Arithmetic Drill Books 1-2-3 Book 1 - Grades 3-6 Book 2 - Grades 4-7 Book 3 - Grades 5-9	D. W. Menear	Copp Clark Publishing
33.	Think of a Number	G. A. Moss (Basil Blackwell)	Copp Clark Publishing
34.	Arithmetic Drill Problems (Books 1-2-3-4)	D. W. Menear	Ryerson Press 299 Queen St. W., Toronto.
35.	Modern Mathematics, Introductory Concepts and Their Implications	A. B. Evenson	W. J. Gage Limited
36.	Experiences in Mathematical Discovery (Book 1)		National Council of Teachers of Mathematics
37.	Pattern and Power of Mathematics Books (1-2-3-4)	A. J. Moakes P. D. Croom C. Phillips	Macmillan Co. of Canada
38.	School Mathematics Project - Teacher's Guide (Books 1-2)		Cambridge University Press
39.	Expanding Mathematics Matrices	J. D. Friis	Copp Clark Publishing

SECTION 2

PUPILS' TEXTS AND TEACHERS' MANUALS

Remedial Teaching

1)	Programmed Arithmetic Drill Book 2 - Grades 4-7 Book 3 - Grades 5-9	D. W. Menear	Copp Clark Publishing Company (1962) 517 Wellington St. W. Toronto.
2)	Arithmetic Drill Problems ($1.25 each) Book 1 - Grades 3-6 - (1965) Book 2 - Grades 4-9 - (1965) Book 3 - Grades 5-9 - (1963) Book 4 - Grades 6-9 - (1965)	D. W. Menear	The Ryerson Press 299 Queen St. W., Toronto.

Specific Books on Measurement:

Mathematics and Measuring Books 1 - Measuring Length 2 - Measuring Weight 3 - Measuring Time 95 cents each. 4 - Measuring Quantity 5 - Measuring Area 6 - Measuring Volume	P. W. Cordin	Macmillan Co. of Canada Limited 70 Bond Street, Toronto.

(The above books are good guides for the teaching of measurements).

Discovery Approach

1)	Mathematics Through Discovery Books: Pathway to Mathematics - $1.40 ea. Book 1 - Primary (1967) 1.00 " 2 - Junior (1967) 1.00 " 3 - Intermediate (1967) 1.00 " Teacher's Book (1968) 1.90 "	Dora E. Whittaker	ClarkeIrwin and Co. Ltd., 791 St. Clair Ave. W., Toronto.
2)	Nuffield Mathematics Project Your Child and Mathematics W. H. Cockcroft Pictorial Representation Mathematics Begin Computation and Structure (2) Computation and Structure (3)	Nuffield Foundation	Longmans Canada Ltd. 55 Barber Greene Rd., Don Mills, Ont.

Nuffield Mathematics Project Cont'd.

I Do, And I Understand - (This is the first book published in this series. It is a guide to the Nuffield Approach).

Mathematical Forum - An anthology of the first five issues of the Bulletin of the Nuffield Math. Teaching Project edited by D. E. Mansfield.

3) Working with Mathematics - Hanwell-Maclean-Elliott-Jorden
HoltRinehart Winston of Canada Limited
833 Oxford Street
Etobicoke, Ont.

$2.25 (3 Teacher's edition copies)(1968)
$1.25 (1 pupil edition copy(1967)

4) Let's Work It Out — G. F. Morris — William Clowes and Sons Limited, London, England(1965)

Activity - Experience-Discussion Approach

Thinking Through Mathematics-Series- — Thomas Nelson and Son Canada, 81 Curlew Drive, Don Mills, Ont.

a) My Wonder Know Book - K. O. Carson and L. U. Lindsay
Teacher's Edition

b) Books -1-2-3-(Pupils') - K. O. Carson
Teacher's Guide - 1-2-3 - J. M. Barrett

c) Books -4-5-6-)Pupils') - K. O. Carson
Teacher's Guide-4-5-6- - M. M. Armstrong

5) O. T. F. Assignment Cards on Area Measurement — Ontario Teachers' Federation, 1260 Bay St., Toronto.

6) Mathex Series. - Nelson and Sawyer General Editors
Published by Encyloopedia Britannica, 151 Bloor St. W., Toronto.

Primary Level		Junior Level	
Matching and Graphing	Book 1	Graphing and Probability	Book 6
Numeration	Book 2	Numeration	Book 7
Operations	Book 3	Operationand Problem Solving	Book 8
Geometry	Book 4	Geometry	Book 9
Measurement and Estimation	Book 5	Measurement	Book 10

Each of the above series consists of a Teacher's Resource Book and Pupils' Pages.

Mostly Teacher Centered - Books

1) Elementary Mathematics Series (1967-68)

McGraw-Hill Co. of Canada Limited
330 Progress Ave., Agincourt, Ont.

a) Primer (Teacher's Edition)	Herbert F. Spitzer
b) Book 1 - Teacher's Edition	Mary O. Folsom
c) Book 2 - Teacher's Edition Pupils'	
d) Book 3 - Teacher's Edition	
e) Book 4 - Pupils'	Herbert F. Spitzer
f) Book 5 - Pupils'	J. Houston Banks
g) Book 6 - Pupils'	Paul C. Burns
h) Book 7 - Teacher's Edition	Mary V. Kahrs
i) Book 8 - Teacher's Edition	Mary O. Folsom

2) Contemporary Mathematics Series (1968)

William H. Sadlier
Chicago, Ill., U. S. A.

a) 1b - Teacher's Edition	
2a - Teacher's Edition	
2b - Pupils' Teacher's Edition	
3 - Pupils' Teacher's Edition	Stanley J. Bezuska
4 - Pupils' Teacher's Edition	William H. McDonnell
5 - Pupils' Teacher's Edition	
6 - Pupils' Teacher's Edition	
7 - Pupils' Teacher's Edition	Helen K. Halliday
8 - Pupils' Teacher's Edition	Bertha O. Weiss
	Miriam S. Newman
	Charlie G. Goode
	Manuel and Key prepared by Joseph Gehringer

3) The School Mathematics Project (1966)

MacMillan Co. of Canada Limited
70 Bond Street, Toronto.

a) Book 1 - Pupils' (Grade 7)
Teacher's Guide
b) Book 2 - Teacher's Guide (Grade 8)

4)	Fundamentals of Mathematics Series (1966) a) First Course (Grade 7) Pupils' " " Teacher's Manual b) Second Course (Grade 8) Pupils' " " Teacher's Manual c) Fundamentals of Maths - (8 +)	Edwin I. Stein	Macmillan Co. of Canada Limited 70 Bond Street, Toronto.
5)	Mathematics - A New Approach (Gr. 7) Book 1 - Pupils' Book 1 - Teacher's	D. E. Mansfield and D. Thompson	Clarke Irwin and Co. Limited 791 St. Clair Ave. W., Toronto.
6)	Modern Mathematics (1967) a) Part I b) Part II — Junior High	H. E. Parr J. R. Shelley	Clarke Irwin and Co.
7)	Elementary School Mathematics (2) Grade 8 - Teacher's (1968)	Eicholz-Martin-Brumfiel-Shanks	Addison-Wesley (Canada) Limited 57 Gervais Drive Don Mills, Ont.
8)	Exploring Mathematics-Book 1 (Gr. 7) (1968)	Kennedy-Jameson-Johnson	Holt Rinehart Winston of Canada Limited 833 Oxford St., Etobicoke, Ont.

Mathematics for Fun

1)	Figures Are Fun a) Books 1 b) " 2 c) " 3 d) " 4 e) " 5 f) Teacher's Manual	J. A. H. Hunter	Copp Clark Publishing Company 517 Wellington St. W., Toronto.
2)	Think of a Number (1968) Book 1	G. A. Moss	A. T. Brooke and Son 18 St. Clements Oxford, England.

Geometry

a) The Shape of Things (1967)	L. G. W. Sealey	A. R Mawbray and Co , Limited Oxford, England
b) Background to Geometry Book 1	G F Morris	William Glowes and Sons Ltd., London, England.
c) Geometric Experiences (Models)		Copp Clark Publishing Limited 517 Wellington St. W., Toronto

Enrichment for Grades 7 and 8

a) Mathematics Enrichment (1966)	R. D. Ripley - George E. Tait	Copp Clark Publishing
b) Understanding the Metric System	David Munroe Miller	Macmillan Co of Canada 70 Bond Street, Toronto.

High School Level

1) Introduction To Number System (1968)	Spooner and Mentler	Prentice-Hall of Canada 1870 Birchmount Ave., Scarborough, Ont.
2) Mathematics a New Approach (1965)	D. E. Mansfield and D. Thompson	Clarke Irwin and Co. 791 St. Clair Ave. W., Toronto.
Book 2 Pupils'		
Book 2 Teacher's		
Book 3 - Pupils'		
Book 3 - Teacher's		
Book 4 - Pupils'		
Book 4 - Teacher's		
3) Pattern and Power of Mathematics (1967)	A. J. Moakes- P D. Croome- T. C. Philips	Macmillan Co. of Canada 70 Bond Street, Toronto
Book 1		
Book 2		
Book 3		
4) Practice Mathematics	Oikle and Osborne	Copp Clark Publishing 517 Wellington St. W., Toronto.
Book 1		
Book 2		
Book 3		
Book 4		

5) Refresher Mathematics (1965) Pupils' Teacher's	Stein	Macmillan Co. of Canada 70 Bond Street, Toronto.
6) Refresher Workbook in Arithmetic (1968)	Stein	Macmillan Co. of Canada
7) Contemporary Mathematics Basic Systems - Pupils' Basic Systems - Teacher's	Stanley J. Bezuszka	W. H. Sadlier Chicago, Illinois.
8) Expanding Mathematics Series (1966) Book 3 - Matrices	J. S. Friis	Adlard and Son Ltd., Dorking, Surrey England.
9) Modern Mathematics - Introductory Concepts and Their Implications (1967)	A. B. Evenson	W. J. Gage Limited 1500 Birchmount Rd., Scarborough, Ont.
10) Modern Mathematics - Part One " " Part Two (1967)	H. E. Parr - J. R. Shelly	Clarke Irwin and Co., 791 St. Clair Ave. W., Toronto, Ont.

MATHEMATICAL NOTES

INDIRECT METHOD OF PROOF

R. Singh, Arthur D. H. School, Arthur, Ontario

Fundamental concepts of logic and patterns of inference are being introduced in high school mathematics in Ontario at the intermediate level. One pattern of reasoning, that even the senior students don't seem to understand, is the indirect method of proof. In this article, we explain briefly the nature of this method and then give examples to consolidate the type of reasoning envisaged by it.

Indirect method of proof is frequently considered synonymous with "Reductio ad Absurdum" i.e. Proof by Contradiction. However an analysis of this method reveals that it involves three distinct patterns of reasoning:

(a) Proof by Elimination.

(b) Proof by Contraposition.

(c) Reductio ad Absurdum.

We explain these one by one. In doing this, we make use of some ideas of elementary logic.

(a) Proof by Elimination. In this type, we enumerate all the possibilities and then eliminate all but one of them i.e. to say we prove by valid reasoning that all the possibilities but one are false. This type of reasoning is quite straightforward and is readily expressible in logic terminology as follows:

$(p \vee q \vee r \vee \ldots\ldots\ldots \vee t) \wedge (\sim q \wedge \sim r \wedge \ldots\ldots\ldots \wedge \sim t) \rightarrow p$.

However we should be careful about two things in the application of this technique: (i) that all the possibilities, irrespective of the conditions of the problem, have been taken into consideration. A standard mistake of this type is the non-enumeration of the possibilities 4, 5 and 6 in the proof of the theorem of example 1; (ii) that the possibilities are being eliminated by valid reasoning. As an illustration of this type of reasoning, we prove the following theorem of Euclidean geometry.

Example 1. Prove that* "If two parallel straight lines are intersected by a transversal, then the sum of the measures of the interior angles lying on the same side of the transversal is equal to 180^{o}."

Proof. The following possible cases arise:

1. The sum of the measures of the interior angles lying on the same side of the transversal is $> 180^{o}$.

2. The sum of the measures of the interior angles lying on the same side of the transversal is $< 180^{o}$.

3. The sum of the measures of the interior angles lying on the same side of the transversal is $= 180^{o}$.

4. The sum of the measures of the interior angles lying on one side (of the transversal) is $> 180^{o}$ and on the other side is $< 180^{o}$.

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*In future, we omit the phrase "Prove that". However all examples are meant to be proved.

5. The sum of the measures of the interior angles lying on one side is $> 180^o$ and on the other side is $= 180^o$.

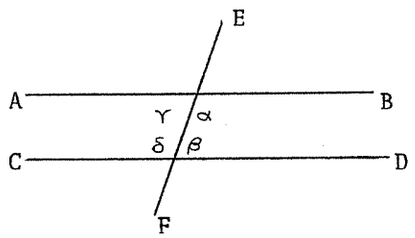
6. The sum of the measures of the interior angles lying on one side is $< 180^o$ and on the other side is $=180^o$.

Let us consider these cases one by one:

1. In this case we have: $m_\angle \alpha + m_\angle \beta > 180^o$.

and $m_\angle \gamma + m_\angle \delta > 180^o$.

Therefore we have: $m_\angle \alpha + m_\angle \gamma + m_\angle \beta + m_\angle \delta > 360^o$.



This is a contradiction, because the sum of the measures of two straight angles is $= 360^o$. Thus the first possibility is eliminated.

2,5,6. By a reasoning similar to the above, we eliminate these cases.

4. This case cannot occur because of Euclid's Parallel Postulate*.

- The straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ being parallel.

Therefore the remaining possibility 3 is true.#

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*"If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the side on which are the angles less than two right angles."

(b) Proof by Contraposition. This technique is very useful when the direct proofs become tedious, lengthy or difficult. However it is applicable to implications only. Here we prove the contrapositive of the given statement. We make use of the logical equivalence: $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$. This equivalence is immediate from the following truth table:

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Now we give some examples of this type of reasoning.

Example 2. If x^2 is an odd number, then x is also odd; $x \in N$.

Solution. Instead we prove its contrapositive: If x is an even (not odd) number, then x^2 is also even; $x \in N$. Proof: Since x is even, therefore $x = 2n$, $n \in N$. Now $x^2 = (2n)^2 = 4n^2 = 2(2n^2)$, which is even.#

Example 3. If two straight lines are distinct, then they have at most one common point



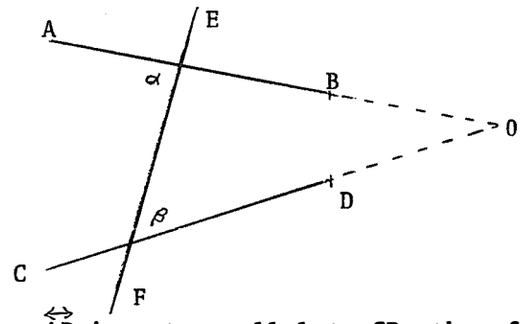
Solution. In this case also we prove its contrapositive: If two straight lines have more than one common point, then

they are not distinct.

Proof: This follows immediately from Hilbert's axiom: "There exists exactly one straight line containing two points."#

Example 4. If two straight lines are intersected by a transversal such that the alternate interior angles are congruent, then the lines are parallel.

Solution. The contrapositive of the above theorem is: If two straight lines are not parallel, then the alternate interior angles formed by the intersection of a transversal with them are not congruent.



Proof: Since $\overleftrightarrow{AB}$ is not parallel to CD, therefore they meet in some point 0 on one side of the transversal $\overleftrightarrow{EF}$. By the "Exterior Angle Theorem", $m_{\angle}\alpha > m_{\angle}\beta$. Thus the alternate interior angles are not congruent.#

(c) Reductio ad Absurdum. This is the most familiar form of indirect proof and has been in use since the Greek era of mathematics. In this case we use the logical equivalences: (i) $p \leftrightarrow \sim(\sim p)$ and (ii) $(p \rightarrow q) \leftrightarrow \sim(p \wedge \sim q)$. We

depict these logical equivalences in the following truth tables:

(i)

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

(ii)

p	q	$p \to q$	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	F	T

In this technique, we start by assuming the negation of the conclusion and then end up in a contradiction by proceeding directly. As is clear from above, this technique is equally applicable to implications and non-implications. First of all, we discuss a few examples of the non-implication type, which we select from algebra, arithmetic, geometry and set theory. We start with a standard example, which can be found in all the text books. However we give a relatively different proof of this result.

Example 5. $\sqrt{2}$ is an irrational number. Then $\sqrt{2} = \frac{a}{b}$, $a, b \in I$, $b \neq 0$ and $(a, b) = 1$. We may note that $a \neq 0$. From the previous equation, we have: $a^2 = 2b^2$. Since the square of any integer always ends in 0,1,4,5,6, or 9, therefore $2b^2$ ends in 0,2 or 8. Thus a^2 ends in 0, 2 or 8. But a^2 cannot end in 2 or 8, therefore a^2 ends in 0. This is only possible if $(a, b) = 5n$, $n \in I$, $n \neq 0$. But this contradicts that $(a, b) = 1$.#

Example 6. $\log_{10}2$ is irrational.

Solution. Assume that $\log_{10}2$ is rational.

Then $\log_{10}2 = \frac{a}{b}$, $a, b \in I, b \neq 0$. Therefore, we have:

$$(10)^{\frac{a}{b}} = 2$$

i.e. $10^a = 2^b$, which is impossible.#

Example 7. The number of primes is infinite.

Solution. Assume that the number of primes is finite. More specifically, let 2,3,5,7,11..., p be the aggregate of primes such that p is the largest of them. Let q = 2.3.5.7.11...p+1.

Then q is not divisible by 2,3,5,...p. It is therefore either prime or divisible by a prime between p and q. In either case, we have found a prime greater then p. This contradicts the fact that p is the largest prime.#

Example 8. $\phi \subseteq A$, $\forall$ set A.

Solution. Assume that $\phi \not\subseteq A$. Then there exists at least one element x such that $x \in \phi$ and $x \notin A$. But this is contradiction, because ϕ is the null set.#

Example 9. A circle cannot be squared by Euclidean tools. (straight edge, compass)

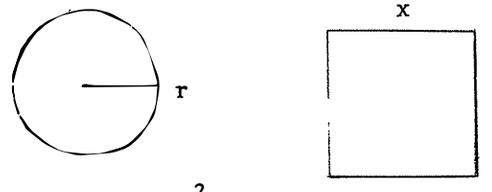
In proving this result, we need the following definitions and lemmas.

Definition 1. A number is called algebraic if it is the root of some equation with rational coefficients.

Definition 2. A number which is not algebraic is called transcendental.

Lemma 1. Any magnitude that can be constructed from unit length by Euclidean tools is an algebraic number.

Lemma 2. π and $\sqrt{\pi}$ are transcendental numbers.



Solution example 9. Let πr^2 denote the area of the circle (to be squared). Let x denote the side of the square such that $\pi r^2 = x^2$. Let us further assume that x is algebraic i.e. to say it has been constructed by Euclidean tools. Taking r to be the unit length, the above equation reduces to: $\pi = x^2$ i.e. $x = \sqrt{\pi}$, which shows that x is transcendental. This contradiction establishes the truth of our result.#

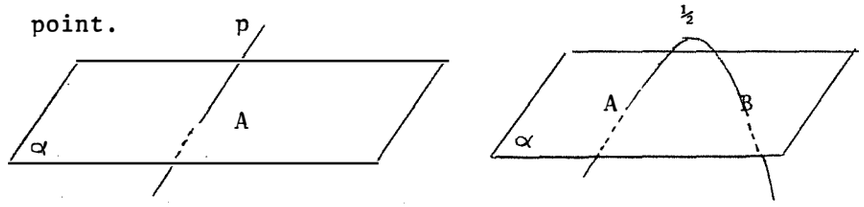
Other familiar examples of the non-implication type are the "Uniqueness Theorems". We give two examples of this type- one from algebra and one from geometry.

Example 10. In a group, the identity is unique.

Solution: Let e and e' be two identity elements of some group (G, *), such that $e \neq e'$. By group axioms, e (being the identity) has the property: $e*a = a = a*e \ \forall \ a \in G$. In particular, $e*e' = e' = e'*e$. By a similar reasoning, we have: $e'*e = e = e*e'$. Therefore $e = e'$, which is a contradiction

Thus the identity is unique in any group.#

Example 11. If a straight line, not lying on a plane, intersects it, then their common section contains exactly one point.



Proof: Let p be the straight line which intersects the plane α but does not lie on it. If possible, let their common section contain two points A and B. Since A and B lie on p as well as α, therefore the straight line $\overleftrightarrow{AB}$ (p) lies on α. This is a contradiction. Thus their common section contains exactly one point.#

Now we discuss some examples of the application of this technique to conditional statements.

Example 12. If x^2 is an odd number, then x is also odd; $x \in N$. (cf. ex. 2)

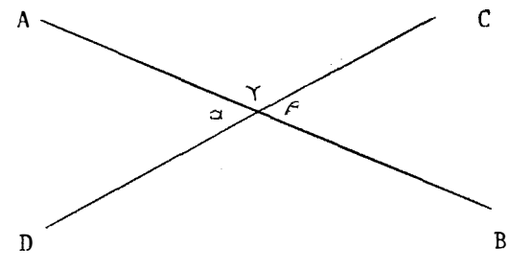
Solution. Assume that x is not odd i.e. to say x is even. Therefore $x = 2n$, $n \in N$. Now $x^2 = (2n)^2 = 4n^2 = 2(2n^2)$, which is even. This is a contradiction.#

Example 13. If $ab = 0$, then $a = 0$ or $b = 0$; $a, b \in R$.

Solution. Assume that $a \neq 0$ and $b \neq 0$. (Recall $\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$. Now $a \neq 0$ and $ab = 0$ imply that $b = 0$, which is a contradiction.#

Example 14. If two straight lines intersect, then vertically

opposite angles are congruent.(≡)



Proof: Assume $\angle\alpha \neq \angle\beta$. Therefore $m\angle\alpha \neq m\angle\beta$.
Thus $m\angle\alpha + m\angle\gamma \neq m\angle\beta + m\angle\gamma$, which means that two straight angles are not congruent. This is a contradiction.#

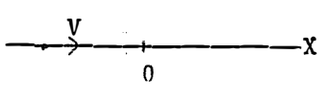
From the variety of examples, we can see that it is a very powerful method of proof. A resourceful teacher can find more problems for its application. The logic involved in it can be explained intuitively and the students convinced of its soundness. Although logic is introduced formally in Grade X yet there is a great lack of its explicit use in high school mathematics. This method affords a very good opportunity for the logic we teach to be put into practice.

ACCELERATION PROBLEMS OF THE CALCULUS AS HANDLED BY INEQUALITIES

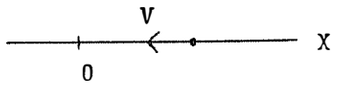
Paul Cronin, Port Colborne High School

One of the most interesting problems arising in the Grade 13 Mathematics A acceleration unit is as follows: if a particle moves on the X axis according to the law $x = t^3 - 5t^2 + 3t$, find the intervals of time during which (i) the particle is approaching the origin, (ii) the acceleration is directed towards the origin, (iii) the speed is increasing. There are several ways of handling this problem (none of them easy) and a rather neat one is through a consideration of the following principles: (i) if the particle is simply given to be approaching the origin then nothing is known about the state of its acceleration, and it could be to the left or right of the origin (we exclude the case where it is at the origin as then the question of whether or not it is approaching, the origin is meaningless). Hence, letting the velocity of the particle be V, the following two configurations are possible:

(a) $X < 0$



(b) $X > 0$



If velocity to the right is taken as positive and to the left as negative then in either (a) or (b) the product of X and V is negative. Thus, the condition for a particle approaching the origin is:

$$\boxed{X \quad V < 0}$$

(ii) If nothing is given except the acceleration of the particle is directed towards the origin, then nothing is known about the nature of its velocity and it could be to the left or right of the origin. Hence, letting the acceleration of the particle be a, the following configurations are possible:

(a) $X < 0$ (b) $X > 0$



If acceleration to right is taken to be positive and to the left as negative then in either (a) or (b) the product of a and X is negative. Hence the condition for the acceleration of a particle to be directed towards the origin is:

$$\boxed{a \quad X < 0}$$

(iii) given that speed (magnitude of the velocity) is increasing then, regardless of the position of the particle, the acceleration must be "helping" the speed and hence the velocity and acceleration must have the same direction. Hence their product will always be positive and the condition for increasing speed is thus:

$$\boxed{a \quad V > 0}$$

Now applying these conditions to the original situation, wherein $x = t^3 - 5t^2 + 3t$, and hence $V = 3t^2 - 10t + 3$ and

$a = 6t - 10$, we have the condition:

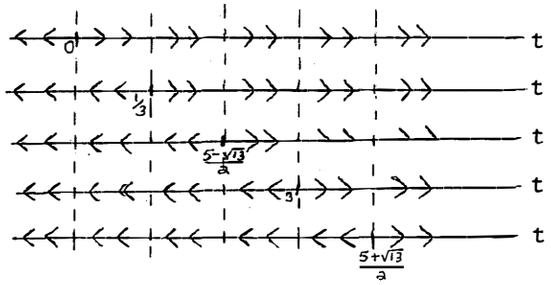
(i) the particle approaching the origin as $XV < 0$ or

$$(t^3 - 5t^2 + 3t)(3t^2 - 10t + 3) < 0$$

i.e. $$t(t^2 - 5t + 3)(3t - 1)(t - 3) < 0$$

or $$t\left(t - \frac{5 + \sqrt{13}}{2}\right)\left(t - \frac{5 - \sqrt{13}}{2}\right)(3t - 1)(t - 3) < 0$$

and denoting on separate t lines the intervals where each of five factors is positive or negative by $>$ or $<$ respectively, we have



Now the product of the five factors can be negative only if:

(a) all five factors are negative - this only happens for $t < 0$

(b) three of the five factors are negative - this only happens for

$$\frac{1}{3} < t < \frac{5 - \sqrt{13}}{2}$$

(c) one of the five factors is negative - this only happens for

$$3 < t < \frac{5 + \sqrt{13}}{2} .$$

Thus the particle is approaching the origin for $t < 0$, $\frac{1}{3} < t < \frac{5\sqrt{13}}{2}$, $3 < t < \frac{5 + \sqrt{13}}{2}$.

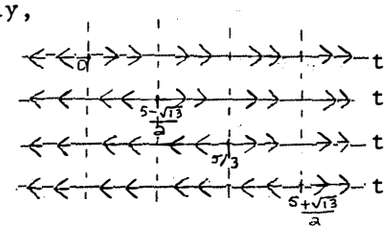
(ii) for the acceleration of the particle to be approaching the origin we have $aX < 0$ or $(6t - 10)(t^3 - 5t^2 + 3t) < 0$

i.e. $2(3t - 5)t(t^2 - 5t^2 + 3t) < 0$

$$2t(3t - 5)(t - \frac{5 + \sqrt{13}}{2}\quad (t - \frac{5 - \sqrt{13}}{2}) \quad < 0$$

$$\text{or } t(3t - 5)(t - \frac{5 + \sqrt{13}}{2})(t - \frac{5 - 13}{2}) < 0$$

and again denoting on separate t lines the intervals where each of the four factors is positive or negative by $>$ or $<$ respectively,



Now the product of the four factors can be negative only if:

(a) three of the factors are negative - this only happens for $0 < t < \frac{5 - \sqrt{13}}{2}$

(b) one of the factors is negative - this only happens for $\frac{5}{3} < t < \frac{5 + \sqrt{13}}{2}$.

Thus the acceleration of the particle is approaching the origin for

$0 < t < \frac{5 - \sqrt{13}}{2}$. and $\frac{5}{3} < t < \frac{5 + \sqrt{13}}{2}$.

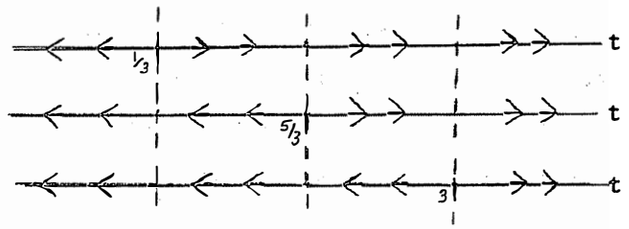
(iii) increasing speed is $aV > 0$ or

$(6t - 10)(3t^2 - 10t + 3) > 0$

i.e. $2(3t - 5)(3t - 1)(t - 3) > 0$

or $(3t - 5)(3t - 1)(t - 3) > 0$

and denoting again on separate t lines the interval where each of the 3 factors is positive or negative we have:



Now for the product of the three factors to be positive, it could be that:

(a) one factor is positive-this occurs for $\frac{1}{3} < t < \frac{5}{3}$

(b) all three factors are positive - this occurs for $t > 3$.

Hence the speed is increasing for $\frac{1}{3} < t < \frac{5}{3}$ and $t > 3$.

Similar "theorems" can easily be worked out for other cases, such as acceleration directed away from the origin being $aX > 0$, etc. It is left to the reader to contrast this approach with the more traditional graphical solution.

n	$\pi(n)$	Approx. Percentage
10	4	40
100	25	25
1,000	168	16.8
10,000	1,229	12.3
100,000	9,592	9.6
10^6	78,498	7.8
10^7	664,579	6.6
10^8	5,761,445	5.8
10^9	50,847.534	5.1
10^{10}	455,052,511	4.6

Column 3 seems to be approaching zero as a limit which supports the thesis.

Theorem 2A

There are lots and lots of primes, and they are very evenly distributed.

Proof. (a là Dirichlet)

Consider three classes of number (those that have a remainder of 1, 2, 0, upon division by 3).

I 1, 4, 7, 10, 13, 16, 19, ...

II 2, 5, 8, 11, 14, 17, 20

III 3, 6, 9, 12, 15, 18, 21, ...

There is only one prime in III.

How are the primes distributed between list I, II? (i.e., are they evenly distributed?)

Now consider eight classes of number (those that have a remainder of 1, 2, 3 ..., 0 upon division by 8).

I 1, 9, 17, 25, 33, 41, ...

II 2, 10, 18, 26, 34, 42, ...

III 3, 11, 19, 27, 35, 43, ...

IV 4, 12, 20, 28, 36, 44, ...

V 5, 13, 21, 29, 37, 45, ...

VI 6, 14, 22, 30, 38, 46, ...

VII 7, 15, 23, 31, 39, 47, ...

VIII 8, 16, 24, 32, 40, 48, ...

There are no primes in IV, VI, VIII and only one in II.

Dirichlet proved that for any modulus n, each list will have either zero, one or an infinitude of primes and that this infinitude will be the same in each list.

Theorem 2.B:

There aren't very many primes and they are unevenly distributed.

Proof:

See 3.

Theorem 3.A:

Primes almost never occur.

Refer to the table in Theorem 1.B.

Note that $\pi(n) < \frac{n}{\log n}$

and that, even though $\frac{n}{\log n} - \pi(n) \nrightarrow 0$, the percentage difference does $\rightarrow 0$.

$\therefore \pi(n)$ is asymptotic to $\frac{n}{\log n}$

and so $\frac{\pi(n)}{n}$ is asymptotic to $\frac{1}{\log n} \longrightarrow 0$.

Theorem 3.B:

Primes always occur.

If we choose an arbitrary number for the period of the

inverse of a prime, we can always find the prime (the speaker claimed!). e.g., If the period is 6, then the prime is 7 because $\frac{1}{7} = .\overline{142857}$ which has a period of 6.

Theorem 3.C:

There is just the right number of primes.

Consider 1, 2, 3, 4, 5, 6, 7, 8 ... .

This is a multiplicative semi-group (whatever that is!).

If we delete 1, we have all prime or composite numbers and we still have a multiplicative semi-group.

However, if we replace 1, and delete 2, we note that all previous primes are still primes but we now have some new primes, (4, 6, 8, 10, 14, ...). In fact, all numbers of the form 2p where 2 is a prime are also primes.

We now have a problem with arithmetic. For example, 36 doesn't have a unique prime factorization.

e.g. $\frac{4}{36} = \frac{4}{6.6} = \frac{4}{36}$

OR

$$\frac{4}{36} = \frac{4}{3.3.4} = \frac{1}{3.3} = \frac{1}{9}$$

PROBLEMS AND SOLUTIONS

Edited by W. Fraser

Please send solutions and problems you would like to suggest for inclusion in this section to Professor W. Fraser, Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario.

The only problem presented in 1970 for which no solution has been submitted is problem 70.01. The inequality that occurs in the problem relates to the approximation of an irrational number by a rational. Knowing that $\sqrt{2}$ is irrational, and that q and p are integers, can a positive lower bound be assigned to $|q^2 - 2p^2|$, and if so, what is it?

Problems

Problem 71.01

Describe a square so that its sides pass through four given coplanar points located so that no point is inside the triangle formed by the other three.

Problem 71.02 (Proposed by Mr. E. Evans, Guelph, Ont.)

The transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \quad (bc - ad \neq 0)$$

changes the graph of a conic with centre at the origin into

another conic also with centre at the origin. However, if the transformation is applied to the parabola $y^2 = 4px$, the graph that results is another parabola, but in general the new vertex will not be at the origin.

(1) Prove the last statement above.

(2) Find, in terms of a, b, c and d the condition that

(i) the vertex remains at the origin;

(ii) the length of the latus rectum remains unchanged.

(3) Find a set of integers a, b, c, d such that only condition 2 (ii) is satisfied.

Solutions

Shortly after manuscript for the November issue was prepared, solutions to problem 70.04 were received from R. Koffeman, Westmount S. S., Hamilton, and J. D. Lammin, Fergus. In addition solutions to problem 70.05 were submitted by G. W. Rogers, East York C. I., Toronto and J. D. Lammin.

Problem 70.06, September 1970

A solution to this problem was submitted by J. D. Lammin, Fergus. The solution presented below involves similar principles but is rather different for the sake of brevity.

Given $y = x^3 - ax - b, \qquad 0 < a < 3, \qquad 0 \leqslant x \leqslant 1.$

$y' = 3x^2 - a$

$y'' = 6x.$

Thus we have a maximum at $x = 1$, the value of the maximum being $1 - a - b$, we have a minimum when $x^2 = \frac{a}{3}$, the value of the minimum being $-\left(\frac{2a\sqrt{a}}{3\sqrt{3}} + b\right)$, and a maximum at $x = 0$, the value of the maximum being $-b$.

In order that the two maxima and the minimum have equal absolute values, we must have the signs of the maxima opposite to that for the minimum. Hence we have the two relations

$$1 - a - b = -b,$$

$$\frac{2a\sqrt{a}}{3\sqrt{3}} + b = -b.$$

These relations imply $a = 1$, $b = -\frac{1}{3\sqrt{3}}$.

Now assume that there is a pair (a, b) such that $\max_{0 \leq x \leq 1} \left| x^3 - ax - b \right|$ is less than $\max_{0 \leq x \leq 1} \left| x^3 - x + \frac{1}{3\sqrt{3}} \right|$.

If we consider the difference $(x^3 - ax - b) - (x^3 - x + \frac{1}{3\sqrt{3}}) =$ $x(1 - a) - (b + \frac{1}{3\sqrt{3}})$, this difference by the assumption would have to be negative at $x = 1$, positive at $x = \frac{1}{\sqrt{3}}$, and negative again at $x = 0$. This implies that we would require a linear expression to be zero at two different places, which is impossible unless it is identically zero. Hence

there can be no such pair (a, b) and $\max_{0 \leq x \leq 1} |x^3 - x + \frac{1}{3\sqrt{3}}| \leq \max_{0 \leq x \leq 1} |x^3 - ax - b|$ for all real a, b.

Problem 70.07

Solutions to problem 70.07 have been submitted by R. Singh, Arthur D. H. S., D. Fletcher, Wiarton D. H. S., and T. J. Griffiths, North Lambton S. S. The difficulty in this problem is that of designing a systematic method of counting which will not miss any cases, or inadvertently produce a duplication. One method proposed was to determine the coefficient of x^{100} in the product

$$(1 + x + \ldots + x^{100})(1 + x^5 + \ldots + x^{100})(1 + x^{10} + \ldots + x^{100})$$
$$(1 + x^{25} + \ldots + x^{100}) \ldots (1 + x^{50} + x^{100}).$$

The following iterative procedure is typical for a problem of this kind. Represent methods of counting out n cents according to the coins used by the following scheme:

C_n; only cents

F_n; cents and nickels

D_n; cents, nickels and dimes

Q_n; cents, nickels, dimes and quarters

H_n; cents, nickels, dimes, quarters and half dollars.

It is readily verified that $H_n = O_n + H_{n-50}$

$$O_n = D_n + Q_{n-25}$$

$$D_n = F_n + D_{n-10}$$

$$F_n = C_n + F_{n-5}$$

$$C_n = 1 \text{ for all } n.$$

It is also readily checked that these relations remain valid if we take

$H_0 = O_0 = D_0 = F_0 = C_0 = 1$, and $H_n = Q_n = D_n = F_n = C_n = 0$ if $n < 0$.

The values of H, O, D, F, C can be built repetitively until we find $H_{100} = 292$.

K-13 Mathematics: Some non-Geometric Aspects, Part I: Statistics and Probability, Curriculum series 6; OISE, 1970, 57 pp., $2.50.

This report is part of the continuing effort of the Ontario Mathematics Commission to improve school mathematics curriculum. Its aim is to focus on the content of mathematics curriculum and on organization and teaching method in order to make "mathematics relevant to modern thought". The report makes a considerable number of recommendations. The teacher education recommendation should be valuable to the teachers, the Boards of Education and the Universities alike. The suggested curriculum for a Probability-Statistics programme containing many illustrative examples is well thought out and should seriously be considered by educators for implementation in the secondary schools. The authors make it quite clear that they advance only a suggested programme, nevertheless, the core and enrichment part of the curriculum should be a guide when programmes of this kind are designed.

If there is anything in the report which deserves some criticism it is statements which need amplification as they seem to distract from the central theme. For example without telling the reader the essential differences between mathematics and applied mathematics, the statement that "Applied

mathematics involves mental processes and abilities that are quite distinct from central concepts of mathematics" is confusing. Further amplification is needed when it is asserted that "a student's progress in the understanding and use of new concepts and his progress in applying these concepts already learned should not be confused".

It would be amiss to review this report without mentioning its bibliography. It is one of the most extensive I have seen for some time and should indeed be of great value when studying this document.

H. P. Heinig,
McMaster University.

Elementary Mathematics 3, with Commentary, by A. B. Evenson, and W. C. McClure, assisted by Roger F. Hendren and Joan Routledge, W. J. Gage Ltd., Toronto, 1966, 286 pp. and 121 pp.

Here we find an ideal text for primary school at Grade 3 level. The many full sized illustrations are just delightful and the examples, exercises and review test should captivate the students interest and hold it. The well-spaced easy-to-read type and the step by step development of the material should make the book a pleasure to read. The review tests and exercises at the end of the sections will make the student and teacher aware of the progress which has been made. There are also recap and new information areas.

From a teaching point of view the content is ideal and the small type captions at the bottom of pages may aid the teacher in reviewing appropriate sections. In addition the Teachers Commentary contains useful discussion on the text under the headings "Objectives", "Comments" and "Lesson Suggestions". Last but not least I wish to mention the many solutions to some of the more difficult exercises which are so valuable to the busy teacher.

Doris Nancekiewill,
Woodward Public School.

Mathaction 4, by W. W. Bates, G. Brown, Joanne Grodzinski, Barbara Luftman, Copp Clark Publishing Co., Toronto, 70 pp., 1969.

Mathaction 4 is an activities program with a workbook format. It is clearly printed, with no errors that I could see and was designed to be used with any modern mathematics text book. As the number in the title suggests, it is at the fourth year level of elementary education.

It contains 16 rounds of activities with 4 parts to each round. Each round presents a logic activity, a number activity and a games activity. A list of alternate activities is also provided. All game descriptions, activity cards and pages, which can be reproduced as stencils, are included for each round. Materials have been tested in classrooms and are said to have been successful. Most materials for the

games are inexpensive or can be improvised. The authors claim that no background experience is necessary but I hesitate to accept this statement, completely.

Some of the topics investigated are:

1. Numbers
   - pairing and multiplication
   - magic squares
   - number patterns
   - equivalent fractions
   - weighing objects
   - machine mathematics
2. Geometry
   - naming shapes
   - graphing
   - area and perimeter
   - protractor compass
   - life geometry
3. Logic
   - one difference - "two-difference" games with Logiblocs (logical thinking)
   - symmetry, reflection
   - visual discrimination
4. Games
   - to reinforce number facts -(Mathopoly, Mathdrag)
   - operations
   - naming points
   - operations with integers.

I have used some of this material in my classroom, and feel it is worthwhile. The children enjoy the work and consider the sessions set aside for it, as playtime.

If one is looking for extra activities to enhance the mathematics program, this book could be the answer.

Marie Myers,
Bertha Shaw Public School.
South Porcupine, Ont.

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Please list the catalogue numbers for these books.

NOTES ON MEETINGS

The N.C.T.M. Conference in Montreal, November 5-7

some comments by Jack LeSage, Program Consultant, Region #7

One of the high school math clubs had a very interesting project. They were developing materials (games, etc.) to provide drill in arithmetic and geometry for elementary school children.

This seems to me to have several advantages. One, of course, is that this should initiate interest among some students in our profession and provide pre-service training for those already interested in teaching. Another advantage is that, as the students pass these on to the elementary school teachers and, hopefully, try them with the children, a real spirit of co-operation at the classroom level should develop between the elementary and secondary schools.

The club is the math club of Holy Name High School, 3120 Laurier East, Montreal, P.O. The staff sponsor is Sister Vera Hoelscher.

The N.C.T.M. Conference in Winnipeg, October 15-17

Comments abstracted from the Saskatchewan Mathematics Teachers Society News Letter, November, 1970

Dr. W. W. Sawyer. "Education is extremely unlike what goes

on in many schools to-day." These were the words of the critical and radical part of Dr. W. W. Sawyer. His cautious half however, warned the one thousand seven hundred and nine convention guests not to throw away what is good just for the sake of change. Education has a great tendency for oscillation. History shows frequent incidents of the band-wagon effect.

The thrust of Dr. Sawyer's key-note address and that of the many speakers to follow seemed to be of focussing attention on the student as a learner and on teaching as the creating of learning experiences. Dr. Sawyer suggested that we treat all students as great men in a little way. This to him implied continuous progress and individualization rather than the lock-step method and the complete reliance on the oral method of instruction. Students learn much from books or from friends outside the school and educators must learn to take advantage of this.

Continuous progress implies continuous evaluation. It becomes imperative for the educator to know the progress of each student as an individual. A plan must be developed whereby the skills and knowledge requirements of a program are articulated and then assessed and recorded on an individual basis. The crash systems employed to serve the ends of certain segments of society have not been done in the interest of students. He indicated children should not fail, that there should be continuous progress in the activity-oriented process. Bad teaching damages souls

and so we must be humanistic in our approach remembering that mathematics is a basic necessity for all students.

Jim McLean. Jim McLean, Assistant Superintendent of Curriculum for Ontario, used as his topic "Freedom to Learn". He stated his philosophy as "School is for Children". He maintained school slows down the natural development process of children because we cramp their curiosity. There should be an environment for change and the teacher is the key agent in that change. The three R's are Respect, Relevance, Responsibility.

Film. A film shown on the open-space school which is not organized for the convenience of the custodian. Resources are accessible and relevant. Children choose the process after the teacher organizes the recources and environment for learning. Involve the children! Vertical streaming allows the older children to tutor the younger. He claimed the system develops responsible behaviour and better learning. When questioned, Mr. McLean indicated that he felt there was no need to learn anything until the student felt he required that skill (like number facts). There's a need to involve parents and community to ensure maximum growth. "A child who is learning cannot fail".

MEETINGS

Ontario Association of Teachers of Mathematics

80th Annual Convention

March 21, 22, 23, 1971

Sheraton Brock Hotel

Niagara Falls, Ontario

Program Highlights

SUNDAY:
- Earlybird Reception, refreshments

MONDAY:
- Keynote speaker - Dr. Z. Usiskin
- 14 sessions dealing with Mathematics Education from kindergarten to college details on next page)
- Hospitality for the wives - including fashion show and luncheon at the famous Skylon, to be followed by shopping tour at the Boutique Centre.
- Banquet featuring Dr. Lola May as guest speaker
- Tour of Niagara Falls night life

TUESDAY:
- Keynote speaker - Mr. A. Dayman
- 7 sessions dealing with Mathematics Education from kindergarten to college (details on next page)
- Luncheon and speaker sponsored by I.B.M. (tickets free and available when registering)

EXHIBITS: Displays of learning materials in Mathematics by the leading suppliers of texts, equipment and hardware.

MATH FAIR: A display of projects in Mathematics submitted by schools across the province.

ADVANCE REGISTRATION: For advance registration and further information, contact:

Mr. Alex Prokich

Program Details

SUNDAY:
- 7:00 p.m. Registration, Main Lobby
- 8:00 p.m. Earlybird Reception

MONDAY:
- 8:00 a.m. Registration, Main Lobby
- 9:15 a.m. Keynote Address: "Have We Gone Far Enough With the New Mathematics?" - Dr. Z. Usiskin
- 10:15 a.m. Exhibits, Math Fair
- 10:45 a.m. Group Sessions
  - "Topics in the Theory of Numbers", - Mr. R. Honsberger
  - "Senior Division Mathematics", - Mr. M. Liebovitz
  - "Geometry, K-8", - Mr. John Del Grande
  - "Use of Enrichment Materials" - Mr. Ron Ripley
  - "Mathematics to Interest the Slow Learner", Mr. J. Milliken
- 1:45 p.m. General Sessions
  - "Mathematics and the Child in Elementary School", - Mrs. A. Schermann
  - "Mathematics in the Community College and Secondary Schools", Panel discussion
  - "Applications of Transformations to Secondary School Topics", - Dr. Z. Usiskin
- 2:45 p.m. Exhibits, Math Fair
- 3:15 p.m. General Sessions
  - "Motivating Math Students Through Various Media", Mr. F. Ebos
  - "Mathematics in the Non-Graded Elementary School", - Mr. A. Prokich
  - "Maintaining Number Skills in the Elementary School", - Miss M. Sewell
  - "Secondary School Mathematics", - Dr. K. Fryer
  - "The Role of O.A.T.M. in Ontario Mathematics Education", - Mr. D. Fox

	7:00 p.m.	Banquet - Dr. Lola May as guest speaker
TUESDAY:	9:00 a.m.	Keynote Address: "Development of Elementary School Mathematics", - Mr. A. Dayman
	10:00 a.m.	Exhibits - Math Fair
	10:30 a.m.	Repeat Group Sessions from Monday
	11:30 a.m.	O.A.T.M. Business Meeting
	12:30 p.m.	I.B.M. Luncheon

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O.A.T.M.

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