8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Abacus

Volume 57 • Number 2 • December 2018

COMPARISONS IN NUMBER SENSE AND NUMERATION

TABLE OF CONTENTS

Abacus Editor Greetings 1
RESEARCH SUMMARY 2
- Comparisons in Number Sense and Numeration
LINKS TO MANIPULATIVES . . . 3
- Number Line Balance Scale
LET'S DO MATH: 4-5
- Comparing Whole and Decimal (Bansho Plan)
LINKS TO LITERATURE 6
- *Just a Little Bit* by Ann Tompern
LET'S DO MATH: 6
- Comparing Game Points Problems
LET'S DO MATH: 7
- Runners' Times Problems
NEXT STEPS FOR YOUR PROFESSIONAL LEARNING 8
- Application to Your Classroom
- Suggested Readings

A Publication of the OAME/AOEM

ABACUS CO-EDITORS



MARY LOU KESTELL works on K to 12 mathematics professional learning at the provincial level. She has spent 40+ years working in all aspects of Ontario mathematics education and is a past-president of both OAME and OMCA. Recently MaryLou has been working with Knowledgehook and TVO on ILC Online Interactive Refreshes.



KATHY KUBOTA-ZARIVNIJ currently focuses on K to 6 mathematics professional learning at the provincial level. She is a long-time OAME board member and OAME past president. In relation to her work in mathematics education, she uses complexity thinking to interpret mathematics teaching and learning for students and teachers.

In our daily life, there are many occasions for which we make qualitative and quantitative comparisons in order to arrive at conclusions and to make decisions. Consider these three different cell phone plans.

- What comparisons could be made? (e.g., qualitative -unlimited messaging, voice mail, call display; quantitative -activation fee, cost per month, cost of cell phone, amount of data)
- Which combinations of comparisons are most compelling? (cost only; cost and wireless features)



Cell Phone Plan A	Cell Phone Plan B	Cell Phone Plan C
• $60 per month • 2-year plan • unlimited minutes • unlimited messaging (text and picture), call display, call waiting • data 1 GB • activation fee $15 • cell phone $590	• $115 per month • 2-year plan • unlimited Canada-wide and U.S. calling • unlimited texting and video, caller ID, voicemail • data 1 GB • activation fee $25 • cell phone $419.99	• $120 per month • 2-year plan • unlimited Canada-wide calling • unlimited texting, picture and video, call display, voicemail, call waiting • 2 500 call forwarding minutes • data 1 GB + bonus 1 GB • activation fee $25 • cell phone (32 GB) $199

In this issue, the Research Summary and contextual problems (solved with different methods and models of representation) focus on absolute and relative comparisons of whole and decimal numbers using different and equivalent number forms within problem solving contexts. The problems and reasoning tasks in this issue are designed for use within a three-part problem-solving lesson and/or as ten-minute mini-lessons.

RESEARCH SUMMARY – COMPARISONS IN NUMBER SENSE AND NUMERATION

The purpose of comparing two or more quantities is to determine the relationship between them. Two or more quantities could be equivalent (=). One quantity could be greater than (>) or less than (<) another quantity. Some generalizations for comparing numbers include:

- the greater the number of digits in a whole number, the greater is a whole number
- the greater the number of digits in a decimal number, the lesser is a decimal number
- if two whole numbers have the same number of digits, then the whole number with the greater digit for the greater number unit is the greater whole number.

Numbers are associated with units, such as object units (e.g., marbles, people), number units (e.g., thousands, ones, hundredths, thirds), measurement (length, area, temperature). To compare two or more numbers, their units must be the same. The selection of a unit makes it possible to use numbers in comparing quantities. For number comparisons, the units of objects (e.g., marbles, game cards) and number units (e.g., thousands, tens, hundredths) must be the same.

These relations may be expressed as either absolute or relative differences.

Absolute Number Comparisons

The *absolute difference* comparison determines how close two number values are to each other. Basic comparison models focus on “more than” and “less than” models.

Variations of "More Than" Model
___ more than 18 is 21
3 more than ___ is 21
___ is 3 more than 18
18 is 3 less than ___
21 is ___ than 18

Variations of “Less Than” Model
___ less than 18 is 13
5 less than ___ is 13
___ is 5 less than 18
13 is 5 less than ___
13 is ___ less than 18

An absolute difference of two quantities implies the use of the subtraction operation. Carpenter et. al. (2015) identified three types of comparison problems:

- *compare difference unknown* → (28 – 15 = ☐; ☐ = 13)
  Jan has 15 green marbles and 28 brown marbles. How many more brown marbles than green marbles does Jan have?
- *compare quantity unknown* → (15 + 13 = ☐; ☐ = 28)
  Jan has 15 green marbles and some brown marbles. She has 13 more brown marbles than green ones. How many brown marbles does Jan have?
- *compare referent unknown* → (28 – 13 = ☐; ☐ = 15)
  Jan has 28 brown marbles. She has 13 more brown marbles than green ones. How many brown marbles does Jan have?

How are the equations different? 4 is 2 more than 2. 4 is 2 times as many as 2.

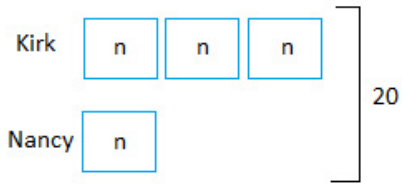
These are examples of two types of reasoning. The first uses absolute comparison, which refers to a quantity by itself, without respect to its relation to other quantities. In contrast, the second uses relative comparison, which compares that quantity to the originals to see how they relate to one another. An **absolute difference** is a subtraction; a **relative difference** is multiplicative or a ratio.

Relative Number Comparisons

Relative change and relative difference are used to compare two quantities while taking into account the quantity of the things being compared; that is the multiplicative relationship between two quantities. Relative or multiplicative comparison problems involve a comparison of two numbers in which one is described as a multiple of the other or how many times larger one number is than the other number. The number quantifying this relationship is not an identifiable quantity Therefore, a relative comparison is expressed as a ratio and is a unitless number.

What could the equation or description be for this bar model showing a relative comparison of cards?

- Kathy has 3 times as many cards as Nancy and together they have 20 cards.
- 20 = (3 times as many as 5) + 5
- 4 x *n* = 20, so *n* = 5
- 15 cards : 5 cards (part-part ratio)
- 15 cards : 20 cards, 5 cards: 20 cards (part-whole ratio)

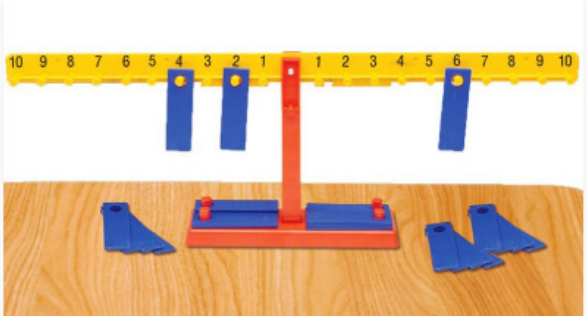


Therefore, the comparison is based on one set being a particular multiplier of the other. The underlying question for multiplicative comparison problems is what factor would multiply one quantity in order to result in the other. A **ratio** is a numerical comparison of 2 or more quantities in the same units, which indicates their relative sizes. In a ratio of two numbers, the first value is termed the *ancedent* and the second number is the *consequent*.

LINKS TO MANIPULATIVES – NUMBER LINE BALANCE SCALE

When students compare numbers, a number line balance scale can be used to test and prove their hypotheses. For example, 4 and 2 is “equal” to 6.

By comparing numbers on the left and right of the balance fulcrum, students can prove whether number equations are true (expressions are equal) or false (expressions are not equal).





True or False?

9 + 7 + 4 = 3 + 5 + 6 + 7

Identify number equivalences.

9 = 6 + 3; 7 = 7 and the remaining

Number comparison is 4 < 5

So, 9 + 7 + 4 = 3 + 5 + 6 7 is false and 9 + 7 + 4 < 3 + 5 + 6 + 7, so the right side is heavier.

LESSON LEARNING GOALS
Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem) and After (Practice) parts of the lesson

Grade 5	Grade 6	Grade 7
-represent, compare and order whole and decimal numbers from 0.01 to 100 000 -demonstrate an understanding of place value in whole and decimal numbers from 0.01 to 100 000	-represent, compare and order whole and decimal numbers from 0.001 to 1 000 000 -demonstrate an understanding of place value in whole and decimal numbers from 0.001 to 1 000 000	-represent, compare and order decimals to hundredth -evaluate expressions that involve whole numbers and decimals, including expressions that contain brackets, using order of operations;

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

True or False Problem 1

Is this equation true or false? How do you know?
4 000 + 300 + 20 + 5 + 0.08 + 0.006
= 3 200.043 + 1 083.022 + 42.001 + 3.02

Solution 1 *–Compare Sums of Expressions*
4 000 + 300 + 20 + 5 + 0.08 + 0.006 = 4 325.086
3 200.043 + 1 083.022 + 42.001 + 3.02 = 4 283.065 + 45.021
= 4 283.065 + 17 + 28.021 = 4 300.086 + 28
= 4 300.086 + 25 + 3 = 4 325.086 + 3
False – the number expressions are not equal
4 325.086 < 4 328.086

inequality *greater than, >*

Solution 2 *–Compare Same Number Units*
4 000 = 3 000 + 1 000
300 = 200 + 83 + 17 (42-25=17)
25 < 25 + 3
0.08 = 0.04 + 0.02 + 0.02
0.006 = 0.005 + 0.001
False – right expression is 3 more than the left expression

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

True or False Problem 2

Is this equation true or false? How do you know?
320 312.131 + 230 411.012 + 101 722.103 + 21 500.041
= 331 812.143 + 220 421.012 + 120 202.101 + 1 510.031

What info are you using to solve the problem?
- two number expressions to compare
- true means that the number expressions are equal
- false means that the number expressions are not equal ≠, where a number expression is greater than > or less than < the other number expression

AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Equivalence of Number Expressions *-exchange the quantity of same number units between numbers in an expression with no change to the sum of the number expression*

equations *-shows equal value of 2 expressions and/or numbers*

Solution 1
left side expression
320 312.131
230 411.012
101 722.103
21 500.041
673 945.287

right side expression
331 812.143
220 421.012
120 202.101
1 510.031
673 945.287

True – both number expressions have a sum of 673 945.287
So, the number expressions are equal or equivalent.

Compare Sums of Number Expressions

Solution 2
left side expression
320 312.131
230 411.012
101 722.103
21 500.041

right side expression
switch digits
331 812.143 (switch digits)
220 421.012
120 202.101
1 510.031

320 312.131 (equal)
231 921.012 (switch digits)
120 202.101
1 510.043

320 312.131 (equal)
230 411.012 (equal)
121 302.101 (switch digits)
1 920.043

320 312.131 (equal)
230 411.012 (equal)
101 722.103 (equal)
21 500.041 (result)

True – reorganize right side expression to be the same as left side.

Compare Composed Same Numbers

Any number can be represented by quantity and number unit. e.g., 320 312.131 ***(standard form)***
expanded form *-> 300 000 + 20 000 + 300 + 10 + 2 + 0.1 + 0.03 + 0.001*
expanded factor form *-> (3x100 000) + (2x10 000) + (3x100) + (1x10) + (1x0.1) + (3x0.01) + (1x0.001)*
equivalent expanded factor form *-> from 9 to 4 number units (32x10 000) + (31x10) + (11x0.1) + (31x0.001)*

Equivalence of Number Expressions *- compare the quantity of combined same number units (thousands, ones and thousandths) to check for equivalence*

e.g., the quantity of the number units 100 000s, 10 000s and 1 000s are combined as 1 000s

320 312.131 with 3 number units
300 000 + 20 000 = 320 x 1 000
312 = 312 x 1
0.131 = 131 x 0.001

Equivalence of Number Expressions *- compare the quantity of each number unit from 100 000s to 0.001 to check for equivalence*

Solution 3
320 312.131 + 230 411.012
+ 101 722.103 + 21 500.041
= 331 812.143 +220 421.012
+ 120 202.101 + 1 510.031

left expression	*right expression*
600 000 =	(3+2+1) x100 000
70 000 =	(3+2+2)x10 000
2 000 =	(1+1) x 1 000
19 00 =	(8+4+2+5) x 100
40 =	(1+2+1) x 10
5 =	(2+1+2) x 1
0.2 =	(1+1) x 0.1
0.08 =	(4+1+3) x 0.01
0.007 =	(3+2+1+1)x 0.001

Compared the quantity of each individual number unit from 100 000 to 0.001.

Compare Individual Same Number Units

Solution 4

	Left	Right
X 1 000 *(thousands)*	320	331
	230	220
	101	120
	21	1
equal	(+10+20)	(+11+19)
X 1 *(ones)*	312	812
	411	421
	722	202
	+500	+510
equal	(+520)	(+500+10+10)
X0.001 *(thousandths)*	131	143
	12	12
	103	101
	41	31
equal	(+2+10)	(+12)

True – left expression is equal to right expression.

Compare Combined Same Number Units

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-

Highlights/Summary

Compare whole and decimal numbers from 0.001 to 1 000 000 by comparing:
- composed same numbers by exchanging the quantity of same number units
- comparing the quantities of individual same number units *(e.g., 320 312.131 has 9 individual number units)*
- comparing quantities of combined same number units (*e.g., 320 312.131 can be represented as 3 same number units: thousands, ones and thousandths)*

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

True or False Problem 3

Is this equation true or false? Why?
340 312.131 + 230 411.011 + 101 722.108 + 21 590.041
= 331 812.143 + 240 421.012 + 120 272.105 + 1 530.031

Solution 1
x 0.001 ->
(131+11+108+41)
= (143+12+105+31)

x 1 ->
(312+411+722+590)
= (812+421+272+530)

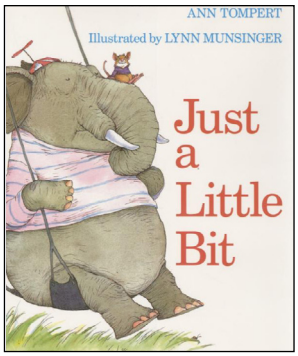
x 1 000->
(340+230+101+21)
= (331+240+120+1)
True – expressions are equal

Solution 2
340 312.131
230 411.011
101 722.108
21 590.041
694 035.291

331 812.143
240 421.012
120 272.105
1 530.031
694 035.291
True – expressions have same sum

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.



LINKS TO LITERATURE – *JUST A LITTLE BIT* BY ANN TOMPERT

Elephant climbed up on one side of a see saw and mouse climbed on the other side. Elephant plunged to the ground as Mouse flew into the air. There was no balance! One by one their friends, like giraffe, zebra and lion climbed onto Mouse's side of the see saw, and they push down as hard as they can to get Elephant off the ground. Just as they're about to give up, a small brown beetle flies down from the sky and perches atop Mouse's nose, adding the needed weight to move that see saw.

What mathematics could explain this situation?

- If the relative size of each animal is identified in relation to the elephant and other animals, then a numeric comparative explanation of the elephant to the other animals can be made.
- If the absolute size of each animal is identified, then the sum of each animal's mass can be compared to the mass of the elephant.

LET'S DO MATH – COMPARING GAME POINTS

Before (Getting Started) – Problem 1

Joe and Joan picked 3 cards each and recorded their points on a number line balance scale.

Why is the scale not balanced?



Solution 1	*Solution 2*
Left → 2 and on the right → 2, so balanced or equal. Left → 6 + 4 = 10 and on the right, 4 + 5 = 9. Left side is 1 greater than the right side. Joe has 1 more point, so the scale is not balanced.	Left side total = 6 + 4 + 2 = 12 points Right side total = 2 + 4 + 5 = 11 points 12 > 11 so the left side is 1 point greater than the right side and so it is heavier; so the scale is not balanced.

During (Working On It) –Comparing Game Points Problem 2

If these game points were recorded on a balance scale, would the scale be balanced?

Joe's cards				Joan's cards			
5	35	25	10	30	15	25	5

After (Consolidation)

Anticipating Student Responses:

Solution 1 – Comparing Sums	*Solution 2 – Comparing*
Joe's total = 5 + 35 + 25 + 10 = 75 Joan's total = 30 + 15 + 25 + 5 = 75 Their scores are the same. If the points were recorded on the balance scale, the scale would be balanced.	Same/Different 5 = 5 and 35 > 30, 25 = 25 and 10 < 15 5 = 5 (5 more for left side, 5 more for right side) The scale will be balanced.

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first for student discussion followed by solution 2?

- *Solution 1* –get sum of 2 addition expressions are compared (operational thinking)
- *Solution 2* – comparing each number for equivalence or inequality (comparative thinking)

LET'S DO MATH – RUNNERS' TIMES

Before (Getting Started) – Runners' Times Problem 1

Eight runners (A to H) are ready to compete in a 400 m relay race. Here are their times for 100 m runs.



A	10.44 m/s	E	8.895 m/s
B	8.82 m/s	F	9.5 m/s
C	9 m/s	G	10.74 m/s
D	9.8 m/s	H	10.063 m/s

Who are the four fastest runners?

Solution 1 -

- The fastest times are the lower numbers with 8 m/s and 9 m/s as the leading digit. B 8.82, C 9, D 9.8, E 8.895, F 9.5.
- Write each number to nearest thousandths B 8.820, C 9.000, D 9.800, E 8.895 and F 9.500 m/s
- In order from least to greatest m/s B 8.820, E 8.895, C 9.000 and F 9.500

B, C, E, and F are the fastest runners.

Solution 2 -

- Ignore the times with 10 m/s or more. Consider 8.82, 9, 9.8, 8.895, 9.5 m/s.
- The two fastest are 8.820 and 8.895 m/s
- The next two fastest are 9.0 and 9.5 m/s

B, E, C, and F are the four fastest runners.

During (Working On It) -Making Two Competitive Teams Problem 2

How would you sort the 8 runners into two teams for the Track and Field meet?

After (Consolidation)

Anticipating Student Responses

Solution 1

The fastest runners' total times →
8.820 + 8.895 + 9.00 + 9.500 = 36.215 m/s

The slower runners' total times →
10.440 + 10.740 + 10.063 + 9.800 = 41.043 m/s

Two teams with a difference of 5 m/s in running rates.

Team 1: B, C, E, and F

Team 2: A, D, G, and H

Solution 2 -

I would put the 2 fastest on Team 1 and the next two fastest on Team 2. Then I'd put the 4th and 5th runners on Team 1 and the 5th and 6th on Team 2.

Team 1: 8.820 + 8.895 + 10.063 + 9.800 = 36.958 m/s

Team 2: 9.00 + 9.500 + 10.440 + 10.740 = 39.68 m/s

This makes two teams with a difference of only 2.722 m/s in their running rates.

Team 1: B, C, A, and D

Team 2: E, F, G, and H

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first, then solutions 2 and 3 for student discussion?

- *Solution 1* - comparing by unequal (fastest and slowest) running times to maximize the difference in running times
- *Solution 2* - comparing by creating running teams with about equal running times to minimize the difference in running times

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how number comparison problems develop students' understanding of the relationship between the quantity of same units. In this issue, same units included object units, number units (e.g., hundred thousands, ten, thousandths).
- To suit your students' learning needs, select and/or design number comparison problems or reasoning tasks that based on grade specific curriculum expectations in the strand, Number Sense and Numeration. In order to design developmentally appropriate and challenge learning tasks, consult mathematics learning trajectories for number comparison concepts, such as the Ontario number sense and numeration curriculum expectations across the grade and published mathematics learning trajectories (e.g., Carpenter et al, 2015).
- Design absolute and relative number comparison problems and reasoning tasks that develop students' experience with different applications in real life situations, such as explaining
- Practise noticing the different comparison methods (operational thinking, comparative thinking by individual and combined units) and using related mathematical language that your students and you use to explain the comparisons (equivalent, greater than, less than, double, triple, halve). Consider the mathematical reasons why your students choose to use or not these different comparison methods to solve these problems. Is the mathematics inherent in the problems related to previously experienced and/or learned mathematics or is it not within their realm of experience?
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit the details of different methods for absolute and relative number comparisons.

SUGGESTED READINGS

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

- Ban Har, Yeap. (2010). *Bar modelling: A problem-solving tool.* Singapore: Marshall Cavendish Education.
- Carpenter, T.P., Fennema, E., Franke, M.L., Levi, L., and Empson, S.B. (2015). *Children's mathematics: Cognitively guided instruction,* 2e. Portsmouth, NH: Heineman.
- Carpenter, T.P., Franke, M. L. and Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.*
- Ontario Ministry of Education. (2005). *Ontario Curriculum, Mathematics, Grades 1 to 8*. Toronto: Queen's Printer for Ontario.
- Small, Marian. (2015). *Building proportional reasoning: Across grades and math strands, K-8*. Toronto: Nelson Education.
- Small, Marian. (2010). *Big ideas from Dr. Small: Grades K-3.* Toronto: Nelson Education.
- Small, Marian. (2010). *Big ideas from Dr. Small: Grades 4-8.* Toronto: Nelson Education.

CLOSING

Our next issue series will focus on absolute and relative comparisons in measurement. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a three-part problem-solving lesson design framework.

CALL FOR *ABACUS* SUBMISSIONS

Send your teaching strategy ideas, problems and student solutions to the *Abacus* Co-Editors via email