8 PAGE INSERT ELEMENTARY MATH

ONTARIO ASSOCIATION FOR MATHEMATICS EDUCATION

Abacus

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COMPARISONS IN MEASUREMENT

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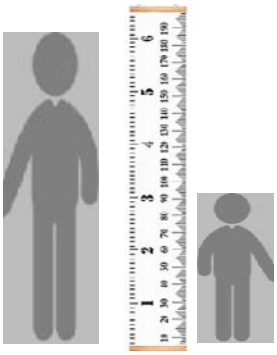
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KATHY KUBOTA-ZARIVNIJ currently focuses on K to 6 mathematics professional learning at the provincial level. She is a long-time OAME board member and OAME past president. In relation to her work in mathematics education, she uses complexity thinking to interpret mathematics teaching and learning for students and teachers.



Is it possible that a child's height at 2 years old may predict the child's height as an adult? Some say that a child will reach about 50% of her adult height at only 2 years of age. Also, an 11-year-old child will weigh about 50% of his adult weight.

A different method to predict the height of a child in adulthood is the combining the mother's height (cm) and the father's height (cm), adding 13 cm for boys and subtracting 13 cm for girls, and then dividing the result by two.

- What measurements could be taken to prove or disprove these statements about predicting the adult height? Mass? Length?
- Which measurement comparisons are most relevant: absolute (using subtraction of height of adult and child) or relative (multiple of the child height in relation to the adult height)?

In this issue, the Research Summary and contextual problems (solved with different methods and models of representation) focus on absolute and relative comparisons of measurements (length, area, mass, time and volume). The problems and reasoning tasks in this issue are designed for use within a three-part problem-solving lesson structure.

RESEARCH SUMMARY – COMPARISONS IN MEASUREMENT

When is it useful to make an absolute measurement comparison or a relative measurement comparison? Here is a list of examples that show absolute and relative measurement comparisons.

Measurement	Absolute Comparison	Relative Comparison
Length of a blue whale (25 m), height of giraffe (5 m)	25 m – 5 m = 15 m blue whale is 15 m longer than the height of the giraffe.	5 m x 5 = 25 m Blue whale is 5 times greater in length than the giraffe.
Area of the Rogers Centre (32 100 m^2), Scotiabank Arena (61 780.5 m^2)	61 780.5 m^2 – 32 100 m^2 = 19 680.5 m^2 Scotia Bank arena is 19 680.5 m^2 greater than the area Rogers Centre.	61 780.5 m^2 ÷ 2 ~ 32 100 m^2 or 32 100 m^2 x 2 ~ 61 780.5 m^2 Scotia Bank arena is about ½ the area of the Rogers Centre.
Volume of Lake Ontario (1 640 km^3), Lake Huron (3 543 km^3)	3 543 km^3 – 1 640 km^3 = 1 903 km^3 Lake Huron has a greater volume of 1 903 km^3 than Lake Ontario.	1 640 km^3 x 2 ~ 3 543 km^3 or 3 543 km^3 ÷ 2 ~ 1 640 km^3 Lake Huron is about 2X greater in volume than Lake Ontario.
Mass of yellow school bus (13 607 kg), minivan (2 000 kg)	13 607 kg – 2 000 kg = 11 607 kg The mass of a school bus is 11 607 kg greater than a minivan.	13 607 kg bus ÷ 2 000 kg MV~ 6 The mass of a school bus is 6X greater than a minivan.
Life span of gorilla (g = 40 years), dog (d = 10 years)	40 years – 10 years = 30 years Life span of a dog is 30 years less than a gorilla.	40 years g ÷ 10 years d = 4 Life span of a dog is 4 times less than a gorilla.

Solving Comparative Measurement Problems

A pail and a pitcher contain 2500 mL of water altogether. The pail contains 9 times as much water as the pitcher. How much water does the pail and pitcher contain individually?

Solution 1	*Solution 2*
1 pitcher + 9 pitcher = 10 pitchers	pail = pitcher x 9
1 pail = 9 pitchers	pail + pitcher = 2500 mL
10 pitchers = 2 500 mL (total water)	(pitcher x 9) + pitcher = 2 500 mL
2 500 mL ÷ 10 pitchers = 250 mL for 1 pitcher	10 pitcher x __ = 2 500 mL = 10 pitcher x 250 mL

LINKS TO MANIPULATIVES – "THE MEASURE OF THINGS"

(**https://www.bluebulbprojects.com/measureofthings/default.php**)

It is a search engine website that identifies comparative or relative measurements of physical quantities. You can find out how much, how long, how many, how far, how large, how tall, how high, or how heavy "something" is in relation to equivalent measures for quantities of different units, such as length, time, height, area, volume, mass and rates, such as speed. Take a physical quantity, like the volume 20 cubic centimetres and see how it compares to a list of familiar objects. Using this tool, provides indirect measurement comparisons.

So, 20 cubic centimetres is about the size of 1 marshmallow, 2 times as big as a gumball, ½ as big as a golf ball, 3/10 as big as a chicken's egg, 1/5 as big as a hockey puck and 1/10 as big as a tennis ball or baseball, 1/18 as big as an ostrich egg. These relative comparisons can be tested out through direct measurement comparisons using the objects identified.

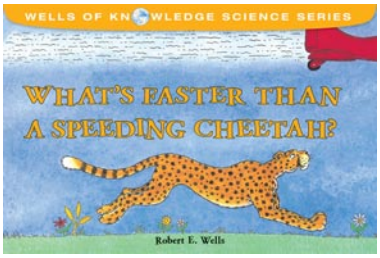
To get a relative comparison of a specific object, research the measurements of the object, like African Bush elephant.

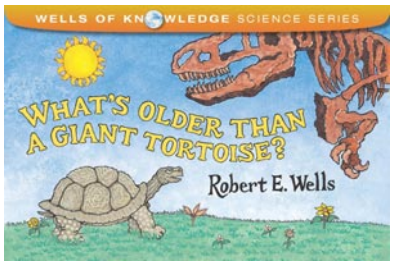
Attribute	Actual Measurement	Comparative Measurement
mass	6 000 kg	0.9 x 1 tyrannosaurus rex 1.6 x 1 hippopotamus
lifespan	70 years	0.3 x 1 giant tortoise's 750 000 x 1 basketball game
population	300 000 elephants	number of elephants is an object (counting unit not a measurement unit)
water intake	189.271 litres	30 x 1 basketball (size 7) 832 x 1 cup of water

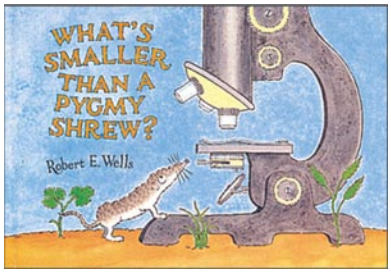
LINKS TO LITERATURE – BY ROBERT E. WELLS

Relative comparisons are made to a referent or the measurement to which other measurements are compared. The title of each book identifies the referent: size of the blue whale, running speed of the cheetah, age of the giant tortoise, size of the pygmy shrew. Let's compare the relative size of the mass and life span for each animal in relation to the African bush elephant (referent).









Attribute	elephant	blue whale	cheetah	giant tortoise	pygmy shrew
life span	70 years	110 years	12 years	100 years	1 year
mass	6 000 kg	140 000 kg	50 kg	417 kg	0.04 kg

Make ***absolute comparisons*** by solving problems, such as "How much less is the mass of the cheetah in comparison to the elephant?"

- The absolute difference between the mass of the elephant and cheetah is
  6 000 kg elephant – 50 kg cheetah = 5 550 kg more elephant.
- The elephant has a mass that is 5 550 kg greater than the cheetah.

Make relative comparisons by solving problems, such as "About how many times less is the mass of the cheetah in comparison to the elephant?"

- The relative difference between the mass of the cheetah in comparison to the elephant is
  6 000 kg elephant = 50 kg cheetah x multiplier = 50 kg cheetah x 120
- The cheetah's mass is about 120 times less than the mass of an elephant.

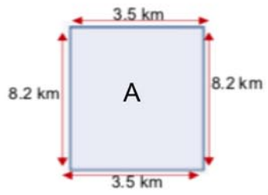
LESSON LEARNING GOALS
Identify the curriculum expectations (MOE, 2005) grade-before and grade-after the grade specific learning goal, in order to:
- see the development of mathematical knowledge, skills, and strategies across grade levels
- determine the mathematical focus of the problems chosen for Before (activation), During (solving problem) and After (Practice) parts of the lesson

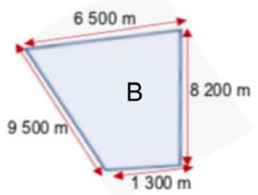
-determine, through investigation, the relationship between the side lengths of a rectangle and its perimeter and area	-solve problems requiring conversion from kilometres to metres -solve problems requiring the estimation and calculation of perimeters of rectangles and area	-construct a rectangle, a square, a triangle and a parallelogram using a variety of tools given the area and/or perimeter

BEFORE (Getting Started) – 5 to 10 minutes
- Activating students' mathematical knowledge and experience using a prompt or problem, related to the math in the lesson problem.
- Includes student responses to highlight ideas and strategies.

Perimeter Problem 1

Which field needs the greatest amount of fencing? And by how much?





Solution 1 –
(A) (3.5 +3.5 +8.2 +8.2) km
=(7.0 + 16.4) = 23.4 km
(B) (6.5 +8.2 + 9.5 +1.3) km
= (24+1.5) km = 25.5 km
Difference (A) and (B)
(25.5 – 23.4) km = 2.1 km (B)
Field B - 2.1 km more fencing

Solution 2 –
Hypothesis (A) < (B) in km
(3.5 x2) + (8.2 x2) + ___
= (6.5 +8.2 +9.5 + 1.3) km
(23.4 + ___) km = 25.5 km
(23.4 + 2.1) km = 25.5 km

Field B - 2.1 km more fencing than Field A

DURING (Working on It) – 15 to 20 minutes
- Understanding the problem – Teacher asks, "What information from the problem are we using to solve it? Explain." Teacher records in a list below the problem, the information that students identify.
- Students solve the problem on chart paper (landscape) with markers (visible for class discussion) in pairs or small groups.
- Teacher circulates to record different student solutions, in addition to the ones the teacher anticipated.

Perimeter Problem 2

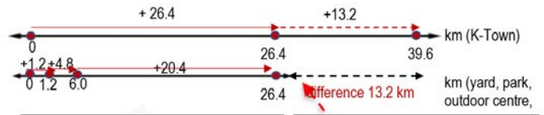
The school yard perimeter is 1 200 m. The park perimeter is 3.6 km more than the school yard and 15 600 m less than the outdoor centre perimeter. The sum of school yard, park and outdoor centre perimeters is 13 200 m less than the M-Town's perimeter. What is the perimeter of M-Town?

What info are you using to solve the problem?
- perimeter (school yard, y) = 1 200 m
- perimeter (park, p) = 3.6 km + school yard = perimeter (outdoor centre, oc) – 15 600 m
- perimeter of M-Town is unknown

AFTER (Consolidation) – 20 to 25 minutes
- Teacher selects 2 or more solutions for class analysis and discussion in a sequence (1st, 2nd, 3rd, etc.) based on mathematical relationships between the solutions and the lesson learning goal.
- Students (authors) explain and discuss their solutions with their classmates; co-construction of success criteria towards the lesson learning goal
- During whole class discussion, solutions are organized (often re-organized) to show mathematical elaboration from one solution to the next and towards lesson learning goal.
- Teacher mathematically annotates (math terms, math symbols, labelled diagrams, concise explanations) on and around solutions to make mathematical ideas, strategies, and models of representation explicit.

Equivalent Measurement Units
-Use same measurement units to calculate perimeter (metres (m) or kilometres (km))

1 km = 1 000 m because
1 000 m = 1 000 m x 0.001 km /1 m = 1 km
1 m = 0.001 km because
1 km ÷ 1000 parts = 1 part = 1 m



Solution 1
1 000 m = 1 km
P(y)=1 200 m =1.2 km
P(p)=(1.2+3.6) km =4.8 km
15 600 m =15.6 km
P(oc) - P(p) =15.6 km
P(oc) - 4.8 km =15.6 km
P(oc) =(4.8 +15.6) km =
20.4 km
Sum = P(y) + P(p) + P(oc)
= (1.2 +4.8+20.4) km
= (25 +1.4) = 26.4 km

P(M-Town) - sum =13.2 km
P(M-Town) - 26.4 =13.2 km
P(M-Town) = 26.4 +13.2
= 39.6 km

Subtraction as Adding On

Solution 2
P(M-Town) – [(P(y) + P(p) + P(oc)] = 13.2 km

Comparing P(K-Town) to P(y+p+oc) to see a difference, 13.2 km



- What's the same?
26.4 km is P(y+p+oc)
- What's the difference?
13.2 km

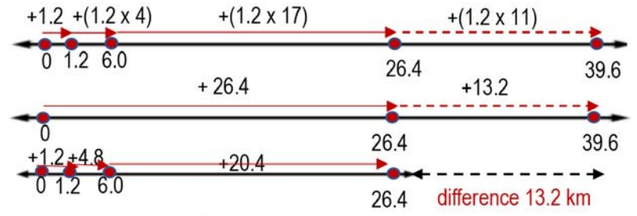
P(M-Town) = same + diff
= (26.4 + 13.2) km
= 39.6 km

Subtraction As Comparison

Absolute Comparisons *– determines how close two number values are to each other, in order to consider whether two numbers are considered matches (same); relative difference is a subtraction*

Relative Comparisons *– determines the numerical relationships between two quantities while taking into account the sizes of things being compared, multiplicatively to a referent; relative difference is a ratio (multiplicative comparison of 2 quantities)*

The referent is 1.2 m, so each perimeter measurement is described as a multiple of 1.2 m.



Solution 3
1 000 m=1 km, 1200 =1.2 km
Multiples of 12 tenths
(1.2 = 12x0.1)

P(y) = 1.2 km
P(p) =(1.2 +3.6) km
= (1.2 x4) km = 4.8 m
P(oc) = (1.2 +3.6 + 15.6) km
=[1.2 x (1+3+13)] =(1.2 x17) km

P(K-Town) = P(y) + P(p) + P(oc) + 13.2 km

P(K-Town) in km
=1.2+(1.2x4)+(1.2x17)+(1.2x11)
=1.2 x (1 +4 + 17 + 11)
= 1.2 x 33
= (12 x 0.1) x 33
= (12 x 11 x 3) x 0.1
= [(12 x 11 x 3 x 0.1
= (132 x 3) x 0.1
= 396 x 0.1 (396 tenths)
= 39.6 km

Multiplicative Relationships

Solution 4
P(M-Town)->sum perimeters of the school yard, park, outdoor centre and 13 200 m

1 200 m = 1.2 km (referent)

All of the perimeters are a multiple of 12 tenths of 1 km
1.2 x **3** = 36; 1.2 x **13** = 15.6
1.2 x **11** = 13.2

P(M-Town) = 1.2 + (1.2 + 3.6) + (1.2+3.6+15.6) + 13.2
=1.2 km x(1+1+3+1+3+13+11)
= 1.2 km x 33 (33 sets of 1.2)
= [(1.2 x 30) + (1.2 x 3)] km
= [(12x30x0.1) + (12x3x0.1)] km
= [(360+36) x 0.1] km
= 396 x0.1 km (396 tenths)
= 39.6 km

One Referent and One Equation

AFTER (Highlights/Summary) – 5 minutes
- Teacher and students revisit the student solutions for key ideas, strategies, and models of representation that are related to the lesson learning goal (anticipated success criteria)
- Teacher lists key ideas, strategies, and models of representation so the students can see how the mathematical details from their solutions relate explicitly to the lesson learning goal; summary of co-constructed success criteria

Highlights/Summary

Perimeter is measured as the sum of the side lengths of quadrilaterals.
Compare the perimeter of quadrilaterals (e.g., school yard, park, sports centre) using:
- *absolute comparison – using subtraction (adding on)*
- *relative comparison – using multiplicative relationships in relation to a referent measurement (to which other measurements are compared);*

AFTER (Practice) – 5 to 10 minutes
- Teacher chooses 2 or 3 problems, similar to the lesson problem for student to solve in pairs as a scaffold and individually.
- Problems could vary by number (choice, size), problem contexts, or what is unknown or needs to be solved
- Students are asked to solve these problems using a strategy different from the one they used for the lesson problem.

Perimeter Problem 3

The perimeter of A-Town is greater than M-Town because it has double the sports centre and 2 400 m more that the park. How much greater is A Town that M-Town?

Solution 1
P(A-Town) = P(M-Town) + P(p) + 2.4 km

P(A-Town) in km
= 39.6 + 4.8 +2.4
= [(39+4+2)x1] +1.8
= 45 + 1.8
= 46.8 km

P(A-Town) > P(M-Town)
(46.8+39.6) km
>39.6km

Solution 2
Referent of 1.2 km
P(A-Town)
= 1.2 + [2 x(1.2 + 3.6)] + (1.2+3.6+15.6+2.4) + 13.2
= 1.2 m x (1+8+19+11)
= 1.2 m x 39
= (1.2 m x 40) – 1.2 m
= 48.0 m – 1.2 m
= 46.8 m
P(A-Town) > P(M-Town)

Note: Mathematical *annotations* include mathematical vocabulary, symbols, elaborations of mathematical details from solutions, labels describing the method/strategy and questions to further thinking. All *annotations* are records of students' mathematical discussion.

Note: A classroom board is longer proportionally than these 2 pages. Due to the space constraints on these pages, the mathematical *annotations* are recorded above the solutions with arrows, rather than on and around the solutions.

Grade 5	Grade 5	Grade 6

LET'S DO MATH – COMPARING AREA

What is the area of the placemat. Measure using the square grid



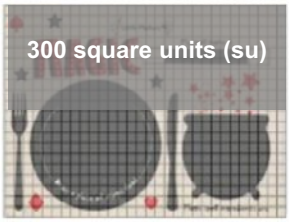
Solution 1

The placemat is covered by 20 rows of 30 squares in each row.

20 rows x 30 squares
= (2 x 10) x 30 = 2 x (10 x 30) su
= 2 x 300 su = 600 square units

30 square units In 1 row

20 rows

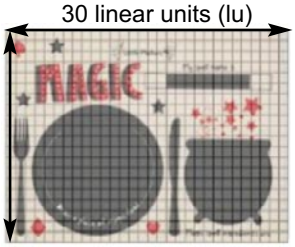


Solution 2

Area (placemat) = length x width
= 20 linear units x 30 linear units
= (2 x 10) lu x (3 x 10) lu
= [(2 x 3) x (10 x 10)] square units
= (6 x 100) = 600 square units

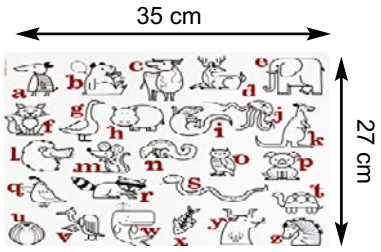
30 linear units (lu)

20 linear units (lu)

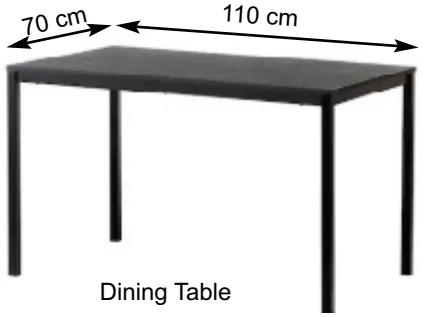


During (Working On It) – Placemat Problem 2

Compare the area of the placemat and the surface area of the table.



Placemat



Dining Table

After (Consolidation)

Anticipating Student Responses:

Solution 1

Area (1 placemat) = length x width
= 35 cm x 27 cm = 875 cm^2

Area (1 table surface)
= length x width
= 70 cm x 110 cm
= (7 x 10) x (11 x 10) cm^2
= (7 x 11) x (10 x 10) cm^2
= (77 x 100) cm^2 = 7700 cm^2

Absolute Comparison (Difference Between Area of Table Surface and Placemat)
= 7700 cm^2 – 875 cm^2 = 6825 cm^2
Table surface has 6825 cm^2 more area than the area of 1 placemat.

Solution 2

Area (1 placemat) = length x width
= 35 cm x 27 cm = 875 cm^2

Area (1 table surface)
= length x width
= 70 cm x 110 cm
=(7x10) x (11x10) cm^2
= (7x11) x (10 x 10) cm^2
= (77 x 100) cm^2
= 7700 cm^2

Relative Comparison (Ratio of Areas))

$\frac{7700}{875}$ = 7700 cm^2 table 875 cm^2 placemat
= 8.8 placemats/1 table

Area (1 table surface) = 8.8 x Area (1 placemat)
8 placemats will fit on the table surface.

Coordinating Discussion for Student Learning:

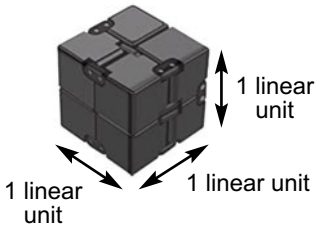
Why might solution 1 be chosen first for student discussion followed by solution 2?

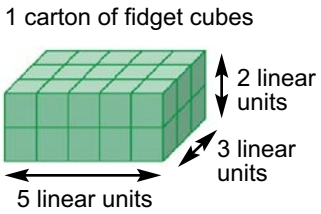
- *Solution 1* - shows the absolute difference between the two areas
- *Solution 2* - shows the multiplicative relationship between the two areas

LET'S DO MATH–ºCOMPARING VOLUME

Before (Getting Started)
– Packaging Problem 1

- Each fidget cube is wrapped in a box, which is 1 cubic unit.
- One carton contains many fidget cube boxes.
- How many fidget cube boxes fit into 1 carton?





Solution 1

Fidget boxes are organized in 2 layers of 3 rows by 5 boxes in each row

Top layer is 5 rows x 3 cube boxes = 15 cube boxes

Two layers x 15 cube boxes = 30 cube boxes

1 carton holds 30 cube boxes.

Solution 2

1 fidget cube box has volume of 1 cubic unit.

Volume (1 carton) = (length x width) x height
= (5 linear units x 3 linear units) x 2 linear units
= 15 square units x 2 linear units = 30 cubic units

1 cube box has volume of 1 cubic units, so for 30 cubic units means there are 30 cube boxes.

During (Working On It) – Problem 2

The package needs to include double the original number of cube boxes in a carton.
How should the dimensions of the carton change?

After (Consolidation) *Anticipating Student Responses*

Solution 1

- Double the length, width and height and get a carton double the size of the original carton
- Top layer would be (5 x 2) x (3 x 2) = 60 cube boxes and there would be 4 layers.
- I'm already at 60 cube boxes, which is double 30 boxes. So, something is wrong.

Solution 2

- 1 carton has 30 boxes, so add another 30 boxes to make a total of 60 boxes.
- It is like putting 2 cartons on top of one another, so that the height increases from 2 linear units to 4 linear units.
- The carton dimensions are 5 linear units x 3 linear units x (2 linear units + 2 linear units) = (15 x 4) = 60 cubes

Solution 3

I think I could double the length, or the width, **OR** the height and get a rectangular prism double the size of the original rectangular prism.

V (original rectangular prism) = 5 x 3 x 2 = 30 cubic units
V (double length) = 10 x 3 x 2 = 60
V (double width) = 5 x 6 x 2 = 60
V (double height) = 5 x 3 x 4 = 60

Coordinating Discussion for Student Learning:

Why might solution 1 be chosen first, then solutions 2 and 3 for student discussion?

- *Solution 1* - error - double the cubes means double all three linear dimensions
- *Solution 2* - absolute comparison - adding the original carton on top of the original
- *Solution 3* - relative comparison – doubling only one of the 3 linear dimensions

NEXT STEPS FOR YOUR PROFESSIONAL LEARNING

Application to Your Classroom

- Consider how measurement comparison problems develop students' understanding of the relationship between the quantity of same measurement units. In this issue, same measurement units included length, area, volume, time and mass.
- To suit your students' learning needs, select and/or design measurement omparison problems or reasoning tasks that based on grade specific curriculum expectations in the strand, Measurement. In order to design developmentally appropriate and challenge learning tasks, consult mathematics learning trajectories for measurement comparison concepts, such as the Ontario measurement curriculum expectations across the grade and published mathematics learning trajectories. (e.g., Battista (2004, 2011); Clements and Sarama (2009, 2014))
- Decide if the choices of measurement are related to familiar contexts or are they not within their realm of experience? Design absolute and relative measurement comparison problems and reasoning tasks that develop students' experience with different applications in real life situations, such as comparisons of masses, life span, heights, water intake, wildlife and urban areas.
- Practise noticing different comparison methods: absolute difference (using subtraction); relative (ratio, multiplicative reasoning) and using related mathematical language that your students and you use to explain the comparisons (greater than, less than, as many times as).
- Think ahead of the kinds of mathematical annotations you will record on and around the student solutions to make explicit the details of absolute and relative measurement comparisons.

Suggested Readings

Some readings used in the Research Summary and in the development of the problems and solutions are listed below.

- Battista, M.T. (2004). Applying cognition-based assessment to elementary school students' development of under- standing of area and volume measurement. *Mathematical Thinking and Learning, 6*(2), 185–204.
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CLOSING

Our next issue series will focus on absolute and relative comparisons in patterning and algebra and data management. As well, the next issue will outline the ways that teachers need to know these notions for teaching, as well as, the ways that teachers can use these mathematical ideas and pedagogical strategies within a three-part problem-solving lesson design framework.

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