▲ What's the Problem? Getting to the Root of the Problem

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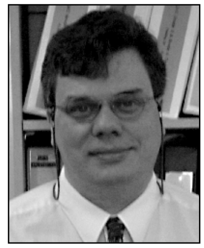

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Welcome back, problem solvers! In the September column, I mentioned that I am trying to share some problems with specific curriculum ties. The problem that I left you with was finding a rational approximation to $\sqrt{2}$. I had envisioned, depending on the age and sophistication of the student, that there would be elements of square roots, decimals, fractions, equations, sequences, and patterns that would come up, as well as lines and slopes. Let's see what can be done with this problem.

As a starting point, it might be a good idea to bring out a calculator to find out what $\sqrt{2}$ looks like as a decimal. Calculating we get $\sqrt{2} = 1.41421356...$

From the decimal digits of $\sqrt{2}$, we can get the following approximations to $\sqrt{2}$.

$$\frac{14}{10} = \frac{7}{5} \qquad \frac{141}{100} \qquad \frac{1414}{1000} = \frac{707}{500}$$

We can see that at each stage, the denominator of the lowest term fractions are growing by factors of 2, 5, or 10. Students could first consider why this is true. Then they might notice that if the denominators are $5n$, the numerators are slight adjustments to $7n$, making the ratio close to $\frac{7}{5}$, but incorporating additional precision. For example, 100 = 5 x 20, but 141 = 7 x 20 + 1.

There are other ways to think about $\sqrt{2}$ as well. For example, if $\frac{a}{b}$ is a rational approximation for $\sqrt{2}$, then a^2 is about $2b^2$. Thus we can find approximations to $\sqrt{2}$ by looking for perfect squares that are almost double other perfect squares.

Collection 1

	n	n_sqrd	dbl_n_sqrd
=	caseindex	n^2	$2n^2$
1	1	1	2
2	2	4	8
3	3	9	18
4	4	16	32
5	5	25	50
6	6	36	72
7	7	49	98
8	8	64	128
9	9	81	162
10	10	100	200

Collection 1

	n	n_sqrd	dbl_n_sqrd
=	caseindex	n^2	$2n^2$
11	11	121	242
12	12	144	288
13	13	169	338
14	14	196	392
15	15	225	450
16	16	256	512
17	17	289	578
18	18	324	648
19	19	361	722
20	20	400	800

I created the above tables using Fathom. From the tables, we see some good candidates for *a* and *b*. Since

$2 \times 2^2 + 1 = 3^2$ $\quad$ $2 \times 5^2 - 1 = 7^2$

$2 \times 7^2 + 2 = 10^2$ $\quad$ $2 \times 12^2 + 1 = 17^2$

candidates for $\frac{a}{b}$ are:

Rational Approximation	Decimal Equivalent	Approximate Difference from $\sqrt{2}$
$\frac{3}{2}$	1.5	8.58×10^{-2}
$\frac{7}{5}$	1.4	1.42×10^{-2}
$\frac{10}{7}$	$1.\overline{428571}$	1.44×10^{-2}
$\frac{17}{12}$	$1.41\overline{6}$	2.45×10^{-3}

Notice that the second entry was exactly the first approximation we got from looking at the decimal digits of $\sqrt{2}$. Also, other than the third entry, the approximations get better. The third came from when double a square was 2 away from another square, whereas all the others were off from a square by only 1. Notice, too, that the last entry is a better approximation than $\frac{141}{100}$, even though it is using much smaller numbers.

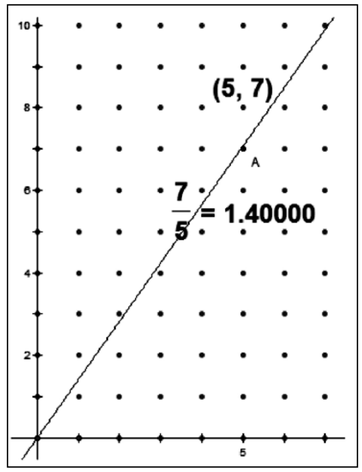
Let's concentrate on when double a square is one away from another square. If you extended your table, you would find that the first five approximations that come from these numbers are

$$\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$$

The last two are included, since $2 \times 29^2 = 41^2 + 1$ and $2 \times 70^2 = 99^2 - 1$.

The last entry gives an approximation that matches $\sqrt{2}$ to four decimal places! If you take a close look at the fractions above, the numerators and denominators each form a sequence with a fairly straightforward recurrence relation. See if you can find it. You can check my wiki for the answer. (**Hint:** Similar to Fibonacci.)

Let's go off on a different path. Consider the graph of the line with equation $y = \sqrt{2}x$. It has slope $\sqrt{2}$, which we can approximate with the slope formula using points that are almost on the line. We know for a fact that (0,0) is on the line, so then if (*x*,*y*) is a point close to the line, then $\frac{y}{x} \bullet \sqrt{2}$. The graph below, created using the Geometer's Sketchpad, shows one of the approximations.



The sketch, along with directions for how to create it, are also on my wiki.

Last of all; let's perform a little algebraic sleight of hand. Since we are trying to approximate $\sqrt{2}$, then if $x = \sqrt{2}$, $x^2 = 2$. Thus,

$$x^2 + x = x + 2$$

$$x(x+1) = x + 1 + 1$$

$$\frac{x(x+1)}{x+1} = \frac{x+1}{x+1} + \frac{1}{x+1}$$

$$x = 1 + \frac{1}{x+1}$$

If we consider this as a recurrence relationship, that is

$$x_n = 1 + \frac{1}{x_{n-1} + 1}$$

using any positive number as our first term, the sequence converges to $\sqrt{2}$. As a matter of fact, the only values that won't converge are ones where $x_n = -1$ for some *n*. It turns out these values are those listed below.

$$-1, -\frac{3}{2}, -\frac{7}{5}, -\frac{17}{12}, -\frac{41}{29}, -\frac{99}{70}, \ldots$$

Hmmmmm…. Interesting!

Suppose we took the equation $x = 1 + \frac{1}{x+1}$, which is satisfied by both $\sqrt{2}$ and $-\sqrt{2}$, and substituted it into itself, we would get

$$
\begin{aligned}
x &= 1+\frac{1}{x+1} \\
&= 1+\frac{1}{1+1+\frac{1}{x+1}} \\
&= 1+\frac{1}{2+\frac{1}{x+1}}
\end{aligned}
$$

If we continued this process *ad infinitum*, we would get

$$
\sqrt{2} = 1+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+...}}}}
$$

which is called the simple continued fraction expansion of $\sqrt{2}$. If you end the fraction at one of the twos and work the fraction back to an improper fraction, it will be one of the values from our list.

OK, now it is time for your homework. A square is constructed on an 11 by 11 pin geoboard. What are all the possible areas for the square? Until next time, happy problem solving! ▲