This appendix continues the article "*Notakto*—My Favourite Mathematical Game for K–12 Classes" by Peter Gustainis that appears in Gazette 64(3) beginning on page 44.

Appendix: An Intuitive Guide to Two-Board *Notakto*

In order to guarantee victory with two boards, you need to play second. If you play first, your opponent must make a mistake in order for you to win. Unlike one board, there are many different strategies to win at two-board *Notakto*, if you start with the advantage (by going second). Here is the simplest I have found:

Let's say you (the teacher) are playing second against a student. Regardless of the student's first move, on your first move, go in the centre on the other board. In Figure 7, the student plays on the right board, so you would play centre on the left. If the student plays on right again, and you have the chance to eliminate that board on your second move, do so. Now they are effectively playing second on the remaining board, with the centre "X" already played, and you can use one-board *Notakto* strategy to win.

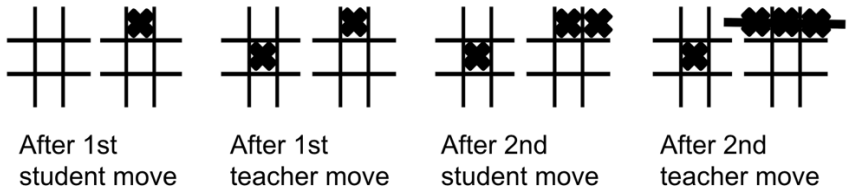


Figure 1: *Simplifying a two-board game of* Notakto *to one board, while keeping the advantage for the teacher*

If the student plays on the right board again, but not in a way where you can eliminate it, you need to prevent them from eliminating it. You play on the right board to create the shape shown on the far right of Figure 8 (this shape will always be possible). This prevents the student from eliminating the far-right board. Now, if they play on the right again for their third move, you will always be able to eliminate the board, and you have a winning position on the remaining board, as discussed.

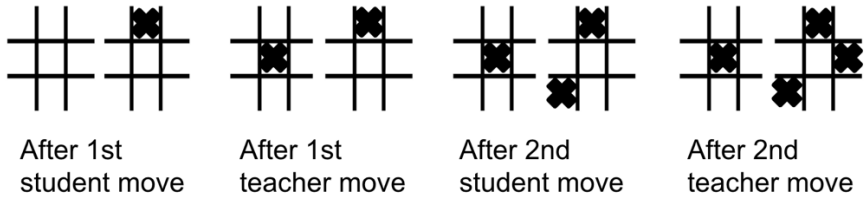


Figure 2: *Preventing the student from eliminating the right board after teacher's second move*

If the student, for their third move, instead decides to play on the left board, things get a bit tricky (see Figure 9). You cannot eliminate the left board, or you lose the advantage. Instead, play on the left board in a way that guarantees endstate 3. Again, if they play on the right board, you eliminate it and win. Instead, if they play on the left board for their fourth move, you

respond by creating endstate 3. Now, if they play on the right board, you eliminate it and win immediately.

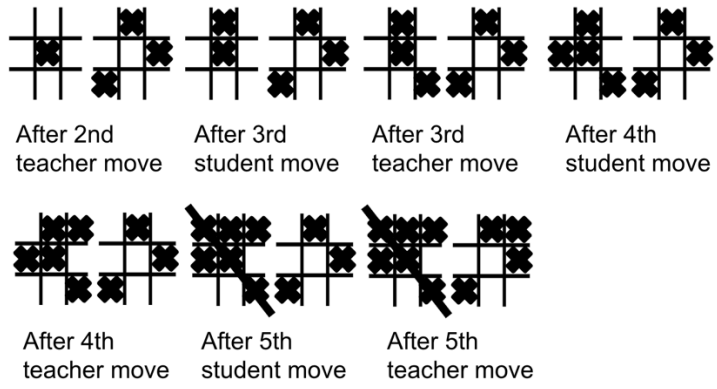


Figure 3: *Teacher in winning position after five moves*

If they eliminate the left board on their fourth or fifth move, you play on the right board in the corner between the two "X"'s on adjacent sides (in this case, in the top-right corner). This is the only winning move because it is the only move that guarantees endstate 6, which will lead to you winning because you played the fourth "X", and endstate 6 has six "X"'s.

Going back to the student's second move, if they decide to play on the left board (the one with the centre you played), play in the centre of the other board (see Figure 10).

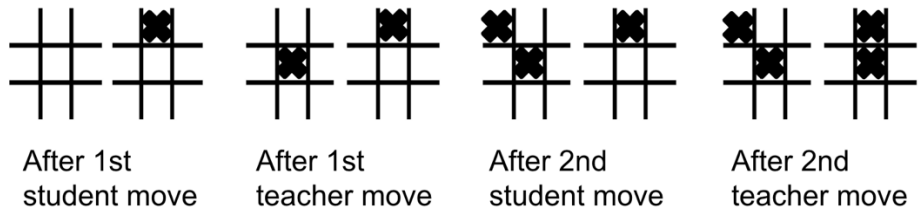


Figure 4: *After two moves by each player, both boards have two "X"'s.*

If the student decides to eliminate a board, you are in a winning position on the remaining board, using the one-board *Notakto* strategy.

If they do not eliminate a board on their next move, play on the same board they do, making a move that moves toward endstate 3 (Figure 11). For your third move, you play middle-left on the right board (bottom-right also works). Middle-right on the right board would lose the advantage because the student could eliminate the left board on their move, and you would lose (since the right board is now an endstate).

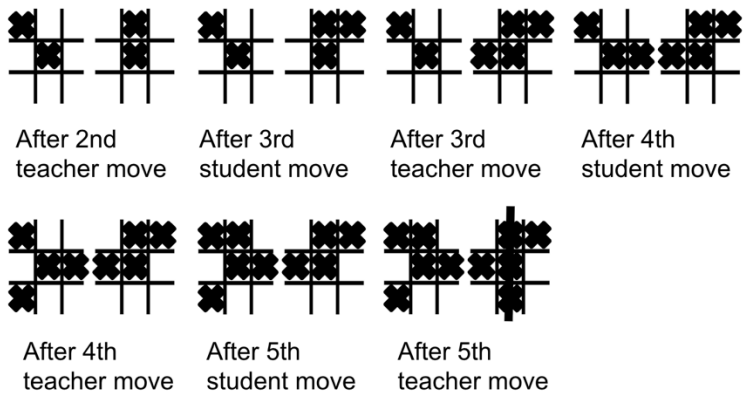


Figure 5: *Teacher wins after five moves*

From here, if the student eliminates either board, you win using one-board strategy. If they play on the right board, completing the endstate, eliminate the left board and you win. If they play on the left board (see Figure 11, 4th student move), again you play on that board continuing toward endstate 3 (in Figure 11, 4th teacher move; top-right and bottom-middle also work). Once more, if the student eliminates either board, you win. If not, they must make endstate 3, and you eliminate the other board and win (see Figure 11, 5th teacher move).

The explanation above covers all possibilities and paths to victory if you start with the advantage. Try a few games against the app, going second, to try this approach for yourself. If you play first instead, you can't win unless your opponent makes a mistake. As mentioned earlier, there are many different strategies to win at two-board *Notakto*, so just because an opponent deviates from what you see above does not necessarily mean that it's a mistake. To determine if an opponent made a mistake in general, see the appendix, "Complete Solution to *N*-Board *Notakto*."

Since the full solution is quite complicated, below are some simpler, practical tips to challenge opponents when you don't start with the advantage:

- The longer you make the game, the more chances you give your opponents to make a mistake! Avoid eliminating boards, and prevent your opponents from eliminating boards by making elimination a losing move on their turn.
- Look for boards that are an even number of moves from an endstate, or already an endstate. These are called type "*a*" boards (Figure 12 – explained in "Complete Solution to *N*-Board *Notakto*" appendix). If you can eliminate the other board and be left with a type "*a*" board, then do so. You will be winning on the remaining board.
- Mix up your moves between games, as this can throw off your opponent's pattern recognition. There are many different branching paths in two-board *Notakto* (and even more for three boards and above). If an opponent is used to seeing only one or two of those branches, they might not know how to respond.

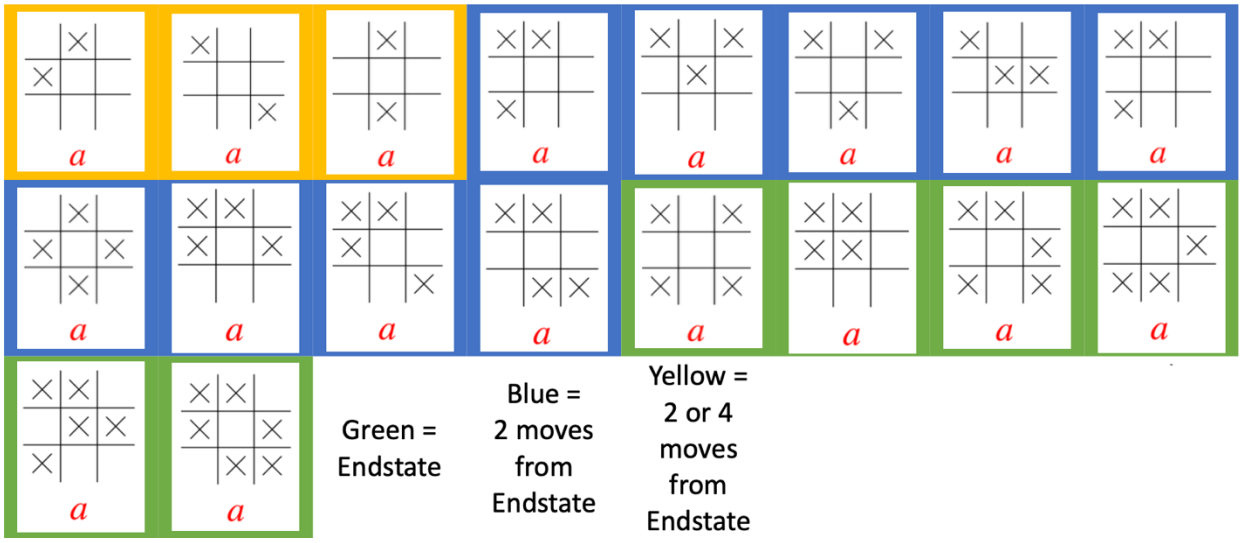


Figure 6: *All boards of type "*a*"*

A good way to practise for playing against an opponent who starts with the advantage is to play on the app on a lower difficulty (easy or medium), and give the computer the advantage to start. Watch for a mistake, then take the advantage and win. This is valuable preparation for playing *Notakto* in your class, or for students playing in a *Notakto* match (see "Extensions and Explorations" appendix for match rules).

Appendix: Complete Solution to *N*-Board *Notakto*

The full solution to *Notakto* is outlined in an article by Plambeck & Whitehead (2013), but it is very technical and can be difficult to follow. A more accessible explanation, with lots of examples, can be found in a three-video series on YouTube (Numberphile, 2015 & 2016), but it glosses over some important details. I would encourage you to watch the videos after reading this appendix. Below is my synthesis of the article and videos, combined with some of my own practical insights, using the full solution to win at *Notakto*. I give my students links to the videos and article, if they are interested, but I usually don't walk them through the analysis I show below. It's a lot to digest, and doesn't give an intuitive understanding of the game. It also spoils the fun of discovering the patterns for themselves, or deciphering the information on their own!

In *Notakto*, there are nine locations on a board, either blank or with an "X," which means for any given board, there are $2^9 = 512$ different possible board states. However, many of these are isomorphic, either a rotation or reflection of another board state, thus effectively the same. There turns out to be 102 non-isomorphic (unique) board states (see Figure 13). For each one, a board value is assigned, in terms of variables or the number 1. This will be the basis of the analysis of any set of *Notakto* boards, called a game state.

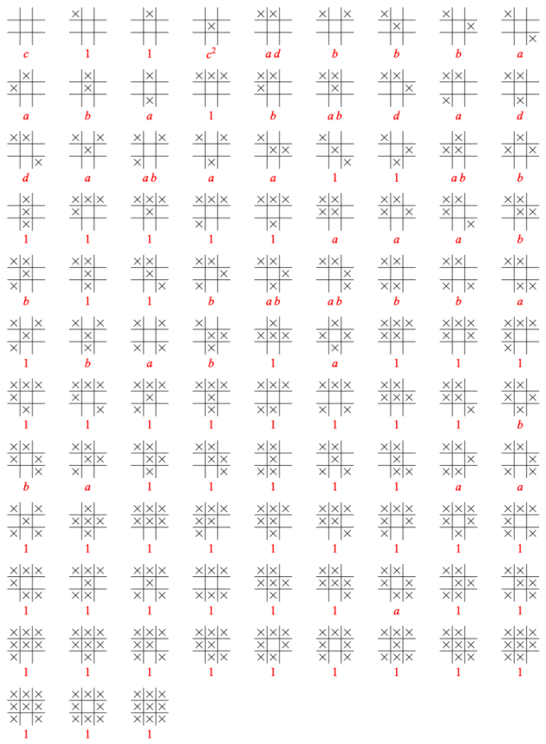


Figure 7: *All 102 non-isomorphic (unique)* Notakto *board states, with board values (Plambeck & Whitehead, 2013)*

As an example, consider the six-board *Notakto* game shown in Figure 14. To analyze the game state, multiply the board values together to get:
$1 \cdot c \cdot ab \cdot 1 \cdot c^2 \cdot ad = a^2bc^3d.$

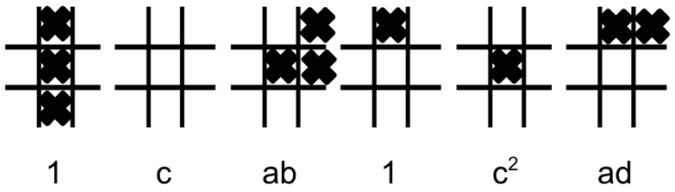


Figure 8: *An example of a six-board* Notakto *game, with each board value labelled*

To simplify the expression a^2bc^3d, use the following seven reduction rules:

1) $a^2 = 1$ 2) $b^3 = b$ 3) $b^2c = c$ 4) $c^3 = ac^2$
5) $b^2d = d$ 6) $cd = ad$ 7) $d^2 = c^2$

Use rule 1 to simplify $a^2bc^3d = bc^3d$.
Then use rule 6 three times to turn all c's to a's: $bc^3d = a^3bd$.
Then finally, use rule 1 again to get the simplified answer (game value): $a^3bd = abd$.

If it's your move and the game value is: a, b^2, bc, or c^2 (key game values), then you are losing. If not, there is at least one move that will make it so you are winning, by turning the current game value into a key game value. If that is the case, you have the advantage. A player with the advantage can retain the advantage, with best play, regardless of the moves their opponent makes.

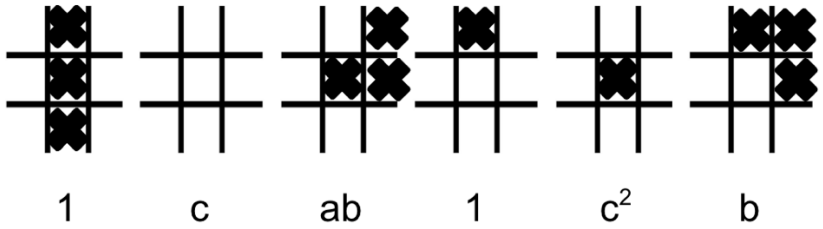


Figure 9: *An example of a six-board* Notakto *game with game value* c^2

In the six-board example from Figure 14, the game value isn't a key game value. That means if it was your move, there is at least one move that will turn the game value into a key game value. For example, placing an "X" on the far-right board in the middle-right position (Figure 15) turns that board value into "b." Now, the game value can be determined.
Multiply values together: $1 \cdot c \cdot ab \cdot 1 \cdot c^2 \cdot b = ab^2c^3$.
Use rule 4: $ab^2c^3 = a^2b^2c^2$.
Use rule 1: $a^2b^2c^2 = b^2c^2$.
Use rule 3: $b^2c^2 = c^2$.

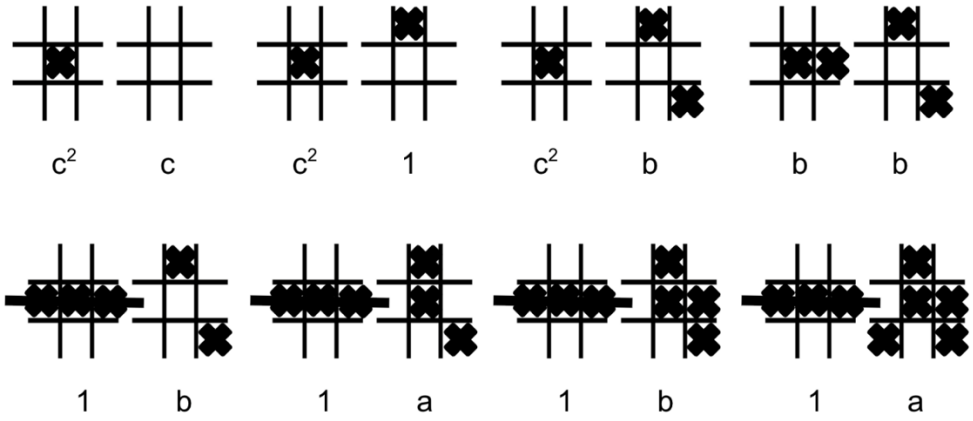


Figure 10: *Example of* Notakto *game with two boards (from Figure 6), with board values*

Figure 16 reviews the two-board game from Figure 6, using this framework. Put yourself in the position of the teacher (the second player). In this game, the student played first with an "X" in the centre of the left board. It's your move and the game value is: $c^2 \cdot c = c^3 = ac^2$, using rule 4. Since this isn't a key game values (a, b^2, bc, or c^2), there is at least one move that turns the game value into one of those values, which means you have the advantage. One such move is

shown above, turning the game value into $c^2 \cdot 1 = c^2$. After the student's next move, the game value is bc^2, which can't be simplified and isn't a key game value, so you can find a winning move. After your next move, the game value is b^2, so you still have the advantage. After the student eliminates the left board, you can turn the right board into an a, with the advantage. Finally, after the student's fourth move, you make the right board into an endstate and you win!

As you could imagine, this gets complicated as you add more boards. It appears that in order to use this framework, you need to memorize, or have a cheat sheet, for all 102 board values and the reduction rules. Then, you need to be able to calculate the game value, sometimes several times, while trying to figure out which moves will give you a key game value!

However, it turns out that you can bypass a lot of the headache with some simple tricks, and by playing in a way that keeps things simple for you. With a bit of practice on the app, you can be an expert on up to three boards, without much trouble.

First, start with the 102 board states. If you look closely, the majority are eliminated boards, which all have a board value of 1, which means it doesn't affect the game value. There are only 46 remaining board states (Figure 17), and most follow a simple pattern, or aren't very important.

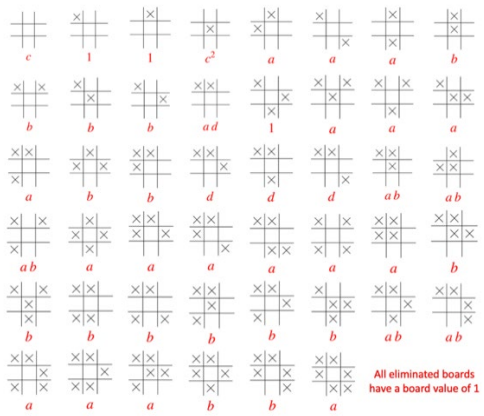


Figure 11: *Board states and values for all 46 non-eliminated boards, ordered by number of "X"s*

Notice that all endstates have a board value of "a" (see green board states in Figure 18). Further, all boards with three or four "X"'s with a board value of "*a*" are two moves away from an endstate. In fact, all boards with a value of "*a*" can be made into an endstate with an even number of moves, and this can be done regardless of what moves are made in between by an opponent. This means for any board that has a value of "*a*" and isn't an endstate, if your opponent plays on that board, and doesn't eliminate it, you can make another move on your turn that turns the board value back into "*a*." Therefore, during a game where you have the advantage, any move your opponent makes on an "*a*" board that doesn't eliminate it, you can cancel out by turning it back into an "*a*", returning the game value to what it was earlier, which is winning for you.

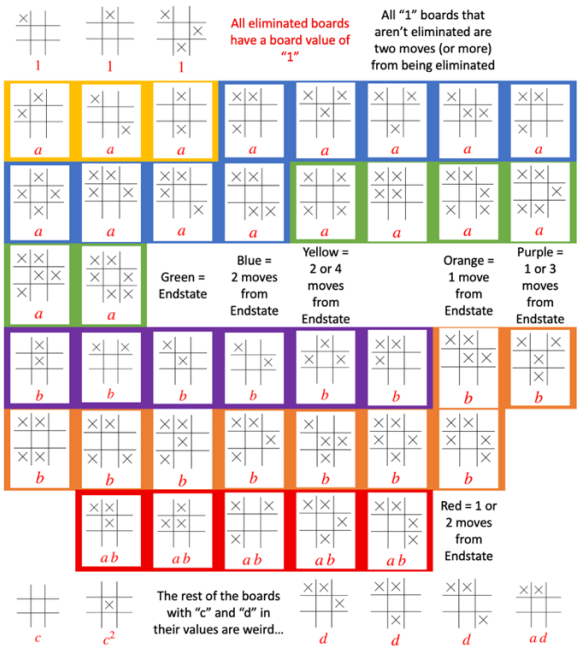


Figure 12: *All board states, organized by board value and colour-coded*

For any move on a blue board that doesn't eliminate it, there is a move you can make to turn it into a green board. For any move on a yellow board not eliminating it, there is a move you can make to turn it into a blue or green board. Crucially, there is no single move on an "*a*" board that turns it into another "*a*" board. This is true for all boards: if you make a move on a board, its board value always changes.

A similar pattern exists for the "*b*" boards. The ones in orange are all exactly one move away from an endstate, so any move that doesn't eliminate it will turn it into a green board. For any move on a purple board, it will either turn it into a green board, or you can make a move afterwards to turn it into an orange board. The "ab" boards are hybrid of "*a*" and "*b*," where each board is exactly one or two moves away from an endstate, if not eliminated prematurely. This means your opponent's move will determine whether the board is like an "*a*" or a "*b*." The rest of the boards have some patterns to relate their board state and board value, but they are less clear-cut.

For the "1" boards that aren't just eliminated boards, they are at least two moves from being eliminated. This means that any move by your opponent on a "1" board can be turned back into a "1" on your turn, potentially by eliminating the board. However, this is also true for the "c^2" board. It is at least two moves from being eliminated, so the pattern doesn't quite hold in general. The "*c*" and "c^2" boards are enigmatic, and I cannot fully explain how they relate to one another. Similarly for the four boards with "*d*" in their board value (the last four in Figure 18), they don't follow a neat pattern that I can explain comprehensively. The only pattern I can see in the last

four boards: each can become "*a*", "*b*," or "*ab*" after one move, so they could become "*a*" or "*b*" after two moves, depending on your opponent's move.

If you have the advantage, you will likely never need to make a move that leaves a board in one of the four states with "*d*" in their value. There will likely be an easier alternative by making a move on the "*d*"-valued board, like in Figure 15. If you see one of the four "*d*" boards after an opponent's move when they had the advantage, they probably made a mistake, and you can make a move on that board to seize the advantage!

I have created a "cheat sheet" with Figures 17 and 18, along with the reduction rules and key game values (bit.ly/NotaktoCheatSheet). Feel free to use this when practising against the app, or even while facilitating in class, but I would encourage you not to share it with students.

I know this might feel daunting, but with a bit of practice this is very manageable. If you are only playing with three or fewer boards, the game value is easy to figure out, and you often don't even need the reduction rules. Since I suggested avoiding the "*d*" boards, rules 5, 6, and 7 aren't necessary either. With the patterns I've outlined, you can identify the green, blue, orange, and red boards, without having to memorize them. In my experience, it is helpful to memorize the yellow, purple, and uncoloured boards, but this isn't too onerous with a bit of practice, or a quick glance at the cheat sheet.

At this point, try a few games against the app and you should be well prepared to run this in your class. You don't need to be an expert when you start, but what I've shared should give you what you need to challenge your students, and you will improve as you play more.

Appendix: Extensions and Explorations

At the request of some enthusiastic students, I organized tournaments, where students from any of my classes, and even some from outside my class who learned from their friends, participated. Students had the chance to compete against one another to win a big prize for their class.

Notakto Match Rules: Flip a coin to decide on player 1. Player 1 chooses the number of boards (up to six). Player 2 decides whether they would like to go first or second, so they get the advantage to start. Play out the game until there is a winner. For the next game, player 2 chooses the number of boards, while player 1 decides whether they would like to go first or second. The match ends when one player wins twice in a row, once starting with, and once starting without, the advantage. This could go on indefinitely with perfect play, but in practice, this rarely happens. After a previously specified amount of time (5–10 minutes), or number of games (10–20 games), the match is a draw. Depending on the number of players, you could organize a round robin or knockout tournament to decide the winner of the prize!

For two or more boards, you can play a modified version of *Notakto*, which is trickier for the advanced players who have mastered up to six boards. It's called *No-Centre Notakto*, and the player with the advantage (first player for an odd number of boards, second player for an even

number of boards) cannot play an “X” in any centre spot for the first few moves. The other player can play whatever they want, with a special exception for three boards. This adds an extra dimension to the game, but a skilled player can still always win if they start with the advantage, though it’s trickier. The list below shows the minimum number of moves that the player with the advantage cannot play in any centre, but a fun challenge is to see how many moves you can go without going in the centre, while still keeping the advantage. The number of moves below guarantees the player that starts with the advantage can keep it, with best play.

No-Centre Notakto

1 Board: Not possible.
2 Boards: Second player cannot play a centre on their first move.
3 Boards: First player cannot play a centre on their first move.
(On their first move, second player cannot play on the same board directly across from the first “X” placed by first player. If they do, the only move to keep the advantage is to eliminate that board by going in the middle.)
4 Boards: Second player cannot play a centre on their first two moves.
5 Boards: First player cannot play a centre on their first two moves.
6 Boards: Second player cannot play a centre on their first three moves.

I like to use the No-Centre variation in *Notakto* tournaments with sufficiently advanced players. The player who chooses the number of boards also chooses the variant, regular or No-Centre, when playing a *Notakto* match.

To connect *Notakto* to curricular topics, you can keep track of how many moves it takes to finish a game of *Notakto*, with a certain number of boards, then consider the statistics of the data after many games. A good challenge is to figure out the shortest, or longest, possible game of *Notakto*, with a certain number of boards.

A fun challenge for advanced players is to try to play the longest game possible, while still winning by the end. Often, if you have the advantage and can eliminate a board, then it’s a good idea because the game becomes simpler. Trying to play the longest game possible, by keeping boards active as long as possible, is a fun challenge. This can also be combined with trying to play as many moves as possible, without going in the centre for a double challenge.

For interested players with some coding experience, creating their own version of the app is a fantastic challenge. A good starting point is to create the game on the computer so two humans can take turns and play against one another (on one board to start, then with two or more boards). Next, add in an option to play against the computer, which plays to get the advantage if possible. In the past, I have allowed students who create their own computer program to use it for the three-board *Notakto* challenge I outlined earlier. A basic version of the *Notakto* game I made in Python is available at www.bit.ly/NotaktoCode.

For further coding exploration and challenge, you can create multiple difficulties by having the computer make incorrect moves with some frequency, if it has the advantage, like in the app. In the app, if the computer doesn't have the advantage, it plays a move at random. Therefore, you could make a "Super" difficulty level by programming the computer to avoid eliminating boards when they don't have the advantage, or preventing the human player from eliminating boards by making it a losing move after the computer move, in an effort to prolong the game. This could even be accomplished by applying machine learning to the game.

Finally, if you want to learn more about the mathematics behind these games, and how the board values and reduction rules were figured out, take a look at the paper referenced earlier (Plambeck & Whitehead, 2013). While the article assumes an understanding of misère game theory, it provides references on page 2, which you can look at if you want to review this theory. Many of these other papers look at other variants, including larger boards like 4x4, 5x5, etc. (George & Janowski, 2016).